NTNU
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Improvements within online decision support in a retail context

Estimation of customer's preferences over
freshness of a product in an inventory model
with complete upwards and downwards
substitution

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Problem Description

The topic of this master thesis derives from the specialization project titled *Improvements on inventory management of perishables through information sharing* that focused in assessing and investigating the value of a specific information in the replenishment decision making of perishable products in a retail context. The established background including the acquired knowledge and experience was intended to be used to investigate improvements within online decision support in a retail context of perishable products focused in handling data uncertainty through application of maximum likelihood estimation and the expectation-maximization algorithm. The background and these tools are used in this master thesis in an inventory model within a stochastic context with complete upwards and downwards substitution regarding the remaining shelf-life of products focusing in the estimation of customer's preferences over freshness of a demanded product.

Summary

This master thesis aims at the estimation of customers' preferences regarding the remaining shelf-life of products available on shelves of a store in a retail context. The research questions focused on how such preferences can be modeled within a stochastic inventory system, how parameters of the demand of such model can be estimated and what is the impact on profit, fill-rate and waste of this model in relation to other similar models.

Therefore, an inventory model considering stochastic demand both on quantity and whether the demand is satisfied by the newest items and the oldest items was developed. Perfect downwards and upwards substitution was considered in the inventory model. In this master thesis, upward substitution means that the excess demand for new items can be satisfied by old items and downward substitution means that the excess demand for old items can be satisfied by new items. In addition, the demand was considered to be censored by stock-level.

The inventory model developed with a basic base stock replenishment policy generated the data that were used for the estimations of referred preferences which were performed by the adoption of the expectation-maximization algorithm with the support of the maximum likelihood estimation method. To calculate the impact of the estimations on profits, fill-rates and waste; the inventory model with a stock-age dependent replenishment policy was adopted in a design of experiment to compare the use of the estimations acquired from different scenarios. The different scenarios corresponded to variation on mean total demand and how the demand was distributed over the remaining shelf-life of products available including the considered most relevant distribution found in the literature.

Among other results, the outcomes showed mainly that in a scenario with both delivery time and ordering frequency equals to 1 period, the estimations resulted in accurate results. The use of estimation did not show any outstanding general results in relation to other depletion policies. However, the application of the estimations with a stock-age dependent replenishment policy indicated that it has potential to achieve higher improvements on the replenishment of inventory upon the correct calibration and most probably considering higher delivery time and ordering frequency.

Preface

This master thesis is the closure of a two year master's degree program in Global Manufacturing Management with specialization in purchasing hosted in the Norwegian University of Science and Technology (NTNU). It was inspired by a specialization project which also had its foundations grounded in inventory management of perishable products in a retail context.

In the specialization project, I felt simultaneously keen and challenged. Coming from an engineering background and a specialization in computational fluid dynamics, the word simulation was not just familiar but it enticed me at the moment that I had to decide the scope of this project. Knowing that I could immerse myself in such topic from a management perspective really excited me since my management experience over my professional career also captivated my professional passions toward administrative improvements through organization, planning, execution, etc. However once I started working on the project, I realized that the level of complexity was higher than the expected. It had been more than five years since I published my last paper on simulation of turbomachinery. Moreover, the modeling and simulation of physical phenomena of fluid flow did not share a lot of their methods, concepts, practicalities, interests and evidently authors with inventory modeling and simulation. There was a lot to learn, to research and to work.

The challenge did not scare me even though the time seemed short and the workload seemed heavy. The result was presented as an outcome of an attempt to balance the planning, learning and execution to the schedule and requirements. Moreover, I had presented it still eager to do a lot more on the theme.

Later after Christmas of 2017 passed, it was time to choose the theme for the master thesis and my eagerness drove as much naturally as it could drive my initiatives towards the same inventory management of perishable products in a retail context theme. As a result of a series of meetings that counted with the input of my supervisors and Relex Solutions AS, it was decided that I would investigate the possibility to estimate customers' preferences regarding the remaining shelf-life of products available on shelves at retail stores, a proposal with a fascinating relevance but full of intricacies. The quality of the specialization project gave me the ambition to do more on the theme and the confidence that more could be done. Nevertheless, the complexity of the proposal nourished the respect and humbleness required by the research of such intricacies.

Rapidly, it was perceived that a lot more should be learned. In addition to the inventory simulations and all statistical theory associated with it, more statistical theory had to be revised now on the estimation of censored data in a dynamic and stochastic process. Once again, the challenge was accepted. The result is presented as an outcome of a lot of learning, working and satisfaction of watching the results of such complex modeling and programming turning into a coherent and enlightening outcome. Withal, another interesting outcome also derived from all this work. Furthermore, my eagerness and enticement to do more within this field is even more vibrant.

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Abbreviations

CDF	=	Cumulative distribution function
DOE	=	Design of experiment
EM	=	Expectation-maximization
FIFO	=	First in first out
FAO	=	Department of the Food and Agriculture Organizatio
LIFO	=	Last in last out
IDH	=	Indice de Dveloppement Humain (French: Human Development Index)
IFPR	=	International Food Policy Research Institute
MLE	=	Maximum likelihood estimator
NOK	=	Norwegian krone
PDF	=	Probability density function
PMF	=	Probability mass function
POS	=	Point of sales
RFID	=	Radio-Frequency IDentification
SIRO	=	Service in random order
SKU	=	Stock keeping unit
vs.	=	Versus

Table of Notation

Notation	Section	Definition
t	2, 1	Specific period usually in the index of a variable indicating the value of the variable at period t
x_t	2.1	On hand inventory at period t
q_t	2.1	Quantity of purchased items at period t
s_t	2.1	Quantity of sold items at period t
d_t	2.1	Quantity of demanded items at period t
w_t	2.1	Quantity of wasted items due to perishability t
π_t	2.1	Profit at period t
c_s	2.1	Selling price for each unit of the item
c_u	2.1	Unit cost for each item purchased
c_h	2.1	Unit holding cost for each in stock
c_w	2.1	Unit cost of wasted items
c_w	2.1	Unit cost of wasted items
m	2.1.3	Remaining shelf life of the item usually in the index of a variable correspondent to items with remaining shelf m
M	2.1.3	Maximum shelf life of the items in an inventory system
$\varphi_P(\cdot)$	4.1	Probability mass function of the Poisson distribution
Λ	4.1	Mean demand
$\Phi_P(\cdot)$	4.1	Cumulative distribution function for the Poisson distribution
D_t	4.1	idd. random variable denoting total demand at period t
$\Gamma(\cdot)$	4.1	Gamma function
p	4.1	probability of customers picking up the oldest item available
a	4.1	index correspondent to the oldest items in stock also called product a
λ_a	4.1	mean demand correspondent to the oldest items in stock
$D_{a,t}$	4.1	idd. random variable denoting demand for the oldest items at period t
$d_{a,t}$	4.1	Quantity of the demand for the oldest items at period t
$\varphi_a(\cdot)$	4.1	Probability mass function of the demand for the oldest items
$\varphi_B(\cdot)$	4.1	Probability mass function of the Binomial distribution
b	4.1	index correspondent to the newest items in stock also called product b
λ_b	4.1	mean demand correspondent to the newest items in stock
$D_{b,t}$	4.1	idd. random variable denoting demand for the newest items at period t
$d_{b,t}$	4.1	Quantity of the demand for the newest items at period t
$\varphi_b(\cdot)$	4.1	Probability mass function of the demand for the newest items
τ	4.1	ordering frequency
L	4.1	delivery time
X_t	4.1	Total quantity of items in stock considering all shelf-lives
T	4.1	Number of periods or the maximum period f the inventory system
$\mathbb{Z}^+$	4.1	Set of integers above zero
$\mathbb{Z}^*$	4.1	Set of positive integers including zero
$\mathbf{x_t}$	4.1	Vector of ending inventory
$\xi(\cdot)$	4.1	Function which represents the balance equation
$\pi_t(\cdot)$	4.1	Function which represents the profit at period t
$r_{1,t}$	4.2	Realized sales or censored demand for the items with remaining shelf-life equals to 1
$r_{2,t}$	4.2	Realized sales or censored demand for the items with remaining shelf-life equals to 2
$\mathcal{L}(\cdot)$	4.2	Likelihood function
$\mathbf{r_t}$	4.2	vector of realized sales at period t
$\mathbf{r}$	4.2	Set $\{\mathbf{r_1}, \mathbf{r_2}, ... \mathbf{r_t}, \mathbf{r_t} + \mathbf{1}, ... \mathbf{r_T}\}$
$\boldsymbol{\varphi_c}$	4.2	Set of the PMFs of the censored observations
λ_m	4.2	Mean demand vector
$\hat{\lambda}_m$	4.2	Maximum likelihood estimator for the mean demand vector
$\varphi_{ca}(\cdot)$	4.2	censored distribution for product a
$\varphi_{cb}(\cdot)$	4.2	censored distribution for product b
$\boldsymbol{\delta}$	4.2	Kronecker delta
Θ_{AB}	4.2	State space of the set $\{r_{1,t}, r_{2,t}\}$ correspondent to the event AB
$\Theta_{\overline{AB}}$	4.2	State space of the set $\{r_{1,t}, r_{2,t}\}$ correspondent to the event $\overline{AB}$
$\Theta_{\bar{A}B}$	4.2	State space of the set $\{r_{1,t}, r_{2,t}\}$ correspondent to the event $\bar{A}B$
$\Theta_{A\bar{B}}$	4.2	State space of the set $\{r_{1,t}, r_{2,t}\}$ correspondent to the event $A\bar{B}$
I	4.2	Fixes number of inventory simulations
$E(\cdot)$	4.2	Expected value
$P(A|B)$	4.2	Probability of event A Conditional to event B
$\varphi_{\overline{AB}}(\cdot)$	4.2	conditional joint probability distribution of demands for event $\overline{AB}$
$\varphi_{\bar{A}B}(\cdot)$	4.2	Conditional joint probability distribution of demands for event $\bar{A}B$
$\varphi_{A\bar{B}}(\cdot)$	4.2	Conditional joint probability distribution of demands for event $A\bar{B}$
S	4.2	Base stock quantity
ζ	4.3.3	Fill-rate

Chapter 1

Introduction

This master thesis is focused on inventory management of perishable items in the retail industry environment considering uncertainties in both demand and the customer preferences with regard to remaining shelf-life of the same perishable products. It basically addresses the challenge of estimating of how products are purchased in relation to their remaining shelf-life.

This chapter discusses the motivations for this assessment along with the background that can be used to justify the relevance of the study. Research questions are identified to provide a clear focus. Then, a path to the resolution of these questions is given in an framework designed for this project.

1.1 Motivation

The motivation to study the problem and research questions proposed in this master thesis originated from cooperation efforts between the industrial economics and technology management department from the Norwegian University of Science and Technology (NTNU) and a company called Relex Solutions AS. Relex is a company dedicated to support customers in retail businesses with SaaS (Software as a service) solutions for unified retail planning, enabling cross-functional optimization of retails core processes: merchandising, supply chain and store operations. Among other services, Relex provide solutions to issues related to forecasting and replenishment optimization.

In addition, Montojos (2017) which focused on researching improvements on inventory management of perishables through information sharing mainly assessing the value of information of the remaining shelf-life of perishable products to the customers prior to purchasing also inspired this master thesis. This work modeled part of the theoretical background and structure used hereby and it was also used as a starting point for the collaboration aforementioned. Once the technical background was formed and the study was concluded, a natural next step towards the formulation of this master thesis was to question what could be done in the field that had not been extensively investigated. The outcome of the work was shared with Relex which presented some of their aspirations that were

pertinent to the formed theoretical background. Consequently, the need for investigating customer preferences regarding remaining shelf-life of perishable products was elected the most relevant and promising object of study given the context.

A meeting log with a summary of each meeting held to discuss about this thesis is presented at Appendix A at the end of this document. This meeting log reports from initial discussions about the potential topics to discussions about the methodology adopted hereby. The main proposal was first to search in the literature different methods that models customer preferences with regards to the remaining shelf-life of products and assess them against real data. However, the complications expected for the data acquaintance and quality checking of data were considered hindrance which could require much waiting resulting in a potential long delay on the delivery of this master thesis. That is how the research took the turn towards a more fundamental research on the estimation of such preferences.

Besides Relex's aspirations and the challenges for the data acquaintance, the fact that no literature aiming at the estimation of such preferences was the main driven of this master thesis. As a rule, customers' preferences regarding the remaining shelf-life of the demanded products are set up following a FIFO (First in first out) policy in the inventory model when inventory of perishable products in a retail environment are simulated. It happens because the stores use different strategies to sell the oldest products first and avoid waste. However, some authors use LIFO or random policies as an extension of their researches such as Ferguson and Ketzenberg (2006) who assessed the value of information of the age of products in a retail context considering FIFO, LIFO and a pre-established random issuing policy. Lowalekar et al. (2016) modeled an order-up-to-level policy with random issuing of the items from inventory for the blood supply in the case of blood banks. Although it allegedly didn't provid proofs of the randomness of the issuing of the items, it provided some empirical support for it. However, it didn't study the veracity of the random issuing policy that was elected for the model which was a trivial uniform distribution (Units of all ages were equally likely to be issued).

On the other hand, some other concerns that stimulate optimization of inventory management of perishable products in general have also naturally motivated this research. As cited in Montojos (2017), these concerns emerge from numerous contexts. Economic concerns such as to increase profitability; environmental, social and political concerns embodied in the matter of waste reduction; commercial concerns associated with stock-outs; which are discussed and illustrated in a more extensive reach in Montojos (2017); are some of the motivators of this research.

To sum up, about 88 million tons of food are wasted annually in the EU and the costs associated with it are estimated in 143 billion euros (Stenmarck et al., 2016). A relatively recent article from the news agency The Guardian claimed that Americans throw away almost as much food as they eat (Goldenberg, 2016). Meanwhile, it is claimed in a report from the United States Department of Agriculture (USDA) that an estimated 12.3 percent of American households were food insecure at least some time during the year in 2016, meaning they lacked access to enough food for an active, healthy life for all household members (Coleman-Jensen et al., 2017). By performing just few mental calculations, it is not difficult to realize that most probably by using part of the rough 50 % of food waste in America, it would be enough to alleviate the struggle for food in it. Even the costs with

food distribution could be possibly covered by the savings resulted from waste reduction.

This analysis could be extended by studying the data from the Global Hunger Indexes (`http://ghi.ifpri.org/#`) provided periodically by the International Food Policy Research Institute (IFPRI). Although this index shows most alarming scores for Africa and South Asia, which just by itself provides ground for the ethical issues on food waste, even in Europe which has some of the highest IDHs on the globe there is food insecurity.

On the environment impact of waste, the Natural Resources Management and Environment Department of the Food and Agriculture Organization (FAO) from the United Nations sponsored a report called Food Wastaged Footprint Impacts on Natural Resources (Jan et al., 2013). This report points out negative environmental impacts on land use, water footprint, biodiversity and carbon footprint, widespread issues strongly indicated to be connected to the greenhouse effect and global warming.

On the commercial issues related to stock-outs, besides the direct economic impact that sock-outs cause, there are other impacts related to customer loyalty. According to Emmelhainz et al. (1991), retailers may lose 14 % of consumers of the missing products depending on the interests from customers that are not fulfilled. In other words, stock-outs may represent not just non-realized sales of the not satisfied demand but also non-realized sales of future demand that could be realized and could even reduce waste rate.

1.2 Background

In addition to the motives that propel this master thesis to seek the estimation of customers' preferences with regards to the remaining shelf-life of perishable products, this investigation finds ground in the technology developments that enables the assessment of the efficacy of this estimation. First of all, the developments on information system technologies have allowed a gradually tighter coupling of information about the overall process and the actual process status itself (Decker et al., 2008). Figure 1.1 illustrates well how this happens over different technologies.
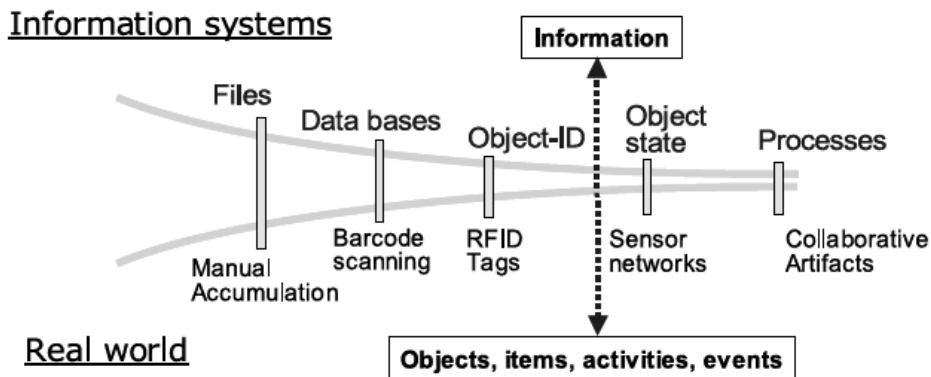

Figure 1.1: Bridging the gap between the real world and information systems Fleisch et al. (2005).

At first, the manual accumulation of information and filling systems comprise a long distance between the real world and the information stored. Then the barcode, which was considered the state-of-art for electronic identification of goods in supply chains in (Decker et al., 2008), shortens this distance. A barcode is usually attached to the good, it contains a limited amount of information, and must be scanned through a line-of-sight close to the bars in order to pass the information further (Ramanathan et al., 2014).

The RFID technology shortened the distance even more since this is a radio-based identification technology which does not require line-of-sight to be scanned and allows identification of single items within a box of items. Location tracking and tracing are possible as far as the infrastructure of RFID readers is deployed (Decker et al., 2003). In addition, RFID tags carry large data capabilities increasing the quantity of information that can be stored.

The wireless sensor networks is an upcoming technology which uses embedded sensors aiming at achieving collaborative features that can be tailored to the requirements of the industrial environment in which it operates. It is still under development and it is not adopted in the logistics of perishable goods from the retail industry in a large scale as the barcodes. But this is also a reality and may soon be enabling a even more coupled and interconnected supply chain (Decker et al., 2008).

In association to the data capacity and information coupling, other technologies may add to the reach of the supporting on the estimation of customer preferences such as the increasing capabilities of retail video analytics. Video analytics can now provide people counting data with a quite high degree of precision and accuracy in challenging environments such as crowded places for example due to all technological advancement as indicated in Merad et al. (2010); Ryan et al. (2010); Hou and Pang (2011); and aid the uncovering of customer behavior by for example recognizing buying events as in Popa et al. (2011). This kind of business intelligence information provides retailers with valuable insight that allows them to optimize merchandising/marketing and improve the customer experience thereby hopefully boosting profits (Connell et al., 2013).

Extrapolating these deliberations about information coupling and other technological advancements to the potential applications that they may have in the support to estimation of customers' preferences, it is possible to infer that there is a technological background that indeed enables the assessment of the efficacy of such estimations. Moreover, these technologies are becoming more and more available. They have been applied in different supply chains increasingly (Bhattacharya et al., 2008) and their benefits have been treated as a widespread truth both in the academia and industry.

For example, compared to barcode experiences the usability of RFID technology influences positively the adoption of this technology (Ramanathan et al., 2014) since it has the potential to provide process freedom by reducing labor requirements and real-time visibility across the supply chain especially in retailing and logistics (Angeles, 2005). Furthermore, the benefits of RFID lead to increased supply chain efficiency for short shelf life products (Krkkinen, 2003). RFID may allow new level of inter-connectivity among businesses by improving supply chain data flow through mass serialization and granularity of data Schuster et al. (2004). Moreover in relation to video analytics, there are already a number of commercial deployments of computer vision systems in the retail environment (Connell et al., 2013).

A fast screening of available literature or a brief search after the state-of-art information system technologies that have been applied to the retail industry shows that the industry has been encountering multiple utilities for these technologies and is in fact adopting them as reported in Martinez et al. (2004) and Nath et al. (2006). These technologies are a reality in the retail industry and there is a clear tendency that they will be continuously improved and will be adopted even more. Therefore, any research that seeks improvements that may be captured by these applications are relevant from the usability perspective and has ground to be worked out.

1.3   Problem statement and research questions

Based on the motivation and background, the following problem statement supported by three research questions in the the retail context of fresh products with fixed shelf-lives is proposed.

How can customer preferences with regards to the freshness of demanded products available on the shelves be estimated in an inventory model with complete upwards and downwards substitution in a retail context?

1. **How can customer preferences of products' freshness be modeled within a stochastic inventory system?**

2. **How can the parameters of the demand of such a model be estimated?**

3. **What is the impact on profit, fill-rate and waste of this model in relation to similar models which consider differently the preferences of products' freshness?**

In the context in which the inventory studied hereby is modeled, upward substitution means that the excess demand for new items can be satisfied by old items and downward substitution means that the excess demand for old items can be satisfied by new items. The customers' preferences mentioned in the research question are related to their preference for the oldest or newest items which guide the demand for respectively oldest and newest items available to be purchased.

The problem statement and the supporting research questions are the backbone of this specialization project which is used to guide the blueprint of this document. Following, a project framework is provided with the intent to indicate the path which this master thesis trailed to address the problem statement and answer the research questions.

1.4   Thesis framework

After the presentation of the motivation, background the problem statement and the research equations in addition to this framework in chapter 2, a literature review is provided and structured in two sections. The first one, section 2.1, has an extensive review of relevant literature that was used to acquire domain of the subject and to understand the extent of the ramifications from inventory management research. This review is a general review

and provide many references as examples of the subtopics with the intent to illustrate the aspects that are considered in the inventory management research and mainly in inventory modeling. The second section, section 2.2, is more detailed and presents the base literature about random depletion policies used in this research. At last, section 2.3 presents a brief historical review of how censored demand has been estimated according to available literature.

Later in chapter 3 the main methods and theories used in this master thesis are presented, reviewed and discussed. Some theoretical considerations that are the theoretical foundation of this thesis are presented in section 3.1. A brief explanation about the theoretical approach of the literature review is outlined in section 3.2. At last, a general overview of specific theories and methods used throughout this master thesis to address the problem statement and answer the research questions are presented in further sections of this chapter. Section 3.4 contains an overview of the main concepts within probability theory; section 3.5 reviews and presents the maximum likelihood method; section 3.6 also reviews and presents the Expectation-Maximization algorithm, section 3.7 presents the Monte Carlo method and 3.8 provides fundamental theory about replenishment policies in inventory management and presents the replenishment policies utilized throughout the master thesis.

Chapter 4 provides all main considerations of the modeling applied to answer the problem statements and research questions. This chapter supported by chapter 5 was structured to systematically answer in a sequence each research question. The objective of the thesis focuses on the estimation of customer preferences with regard to the remaining shelf-life of available inventory in a retail context. The first research question is related to how these preferences can be modeled. Therefore, section 4.1 presents the model used to represent this preferences in a stochastic inventory system in a retail context. The second research question aims at the estimation of the parameters associated with such preferences. Then, section 4.2 shows how the parameters of the demand model which portrays the customer preferences are estimated. Finally, the research question three focuses on the impact of the estimations in the inventory system with regard to profit, fill-rate and waste. Section 4.3 outlines the design of experiment and modeling used to answer this research question. It basically presents the framework with some supplements that are used to perform the analysis required to answer the third research question.

The third research question is then answered in chapter 5. Section 5.1 indicates the main findings on the estimation of the parameters and section 5.2 presents the outcomes of a series of experiment which adopted the estimated parameters associated with stock-age dependent replenishment policies to evaluate their impact in the inventory system.

Then, a discussion chapter provides a summary of the main findings. It also indicates how and to what extent the problem statement and the research questions were answered. Furthermore, it provides the limitations of this project in addition to suggestions for further research. At last, a conclusion chapter provides a more straightforward answer to the problem statement giving a closure to the project.

Chapter 2

Literature Review

This chapter focuses on the review of the literature also used as the background of this master thesis. The literature used was revised for two purposes: Acquiring enough knowledge of inventory management, control and modeling; find the most relevant findings considering random depletion policy considering the remaining shelf-life of the depleted products for inventory management of perishables; and acquire enough knowledge of estimation of censored demand in similar contexts as the one adopted in this master thesis in addition to present the most relevant work developed about it present in available literature.

Therefore, this chapter is structured in three sections. Section 2.1 presents an extensive review of relevant literature that was used to acquire the knowledge necessary to understand the extent of the ramifications from inventory management research. Section 2.2 presents the base literature used in this thesis considering random depletion policies that was used as supporting tool in the analysis. Finally, section 2.3 outlines the considered most relevant literature related to the estimation of censored demand.

2.1 Inventory management

Inventory management has been extensively studied in management science and its related fields such as economics, business and engineering over the years, decades and even a century considering the publication from Harris (1913) dated in 1913 and its outstanding contribution on inventory control. Harris (1913), which is one of the cornerstones of inventory control research, used deterministic modeling and some approximations with regard to periodic quantity in stock to calculate the ordering quantity which minimizes inventory holding costs and ordering costs. Erlenkotter (1990) provided an interesting review of the early literature on this model and its evolution over the years until the 90s. Arrow and Harris (1951) also developed a fundamental work on inventory control considering uncertainty models with stochastic demand.

Among the researches in the field, it is possible to find different approaches taking into consideration different aspects and aiming usually at the same objectives: Increasing profitability, reducing stock-outs and reducing waste; as already mentioned in the introduction,

in section 1.1. These variations happen mainly due to the dynamic environment of the industry and complexity of the mathematical models. In order to estimate better ordering policies on the modeling of an inventory problem, one must for example answer questions such as: Is the model handling one or multiple items? Do the items perish or not over time? How long does it take to perish? How are these items issued from stock (the oldest ones first or the newest first)? Does it even perish? How can the demand be estimated? These and other questions make the pool of knowledge oriented towards decision making just in inventory control an intertwined set of ramifications that makes the task to have an overview of it complex and tedious.

In an attempt to shed light on it, the following equations are presented as starting point for these ramifications. Later, terms and concepts associated to these equations are used to categorize the subject and review the literature in a structured organization which is detailed in figures **Fig. 2.1** and **Fig. 2.2**. Different approaches, equations, notations and methods are used to model and solve problems related to inventory management. The approach used in this first section of the literature review has an illustrative objective to present the considered most relevant concepts to understand how inventory management of perishables is handled by academic literature. Some of these concepts were simplified to fit to the goal, but they may be treated from a more flexible and complex perspective depending on the application.

The inventory balance equation or the one period transfer equation, which is given by

$$x_{t+1} = x_t + q_t - s_t - w_t, \tag{2.1}$$

reproduces the progression of the inventory over time in terms of quantity of items in stock. This equation has conservative characteristics and stipulates that the total quantity of a specific item in inventory (or on hand inventory) at a period $t + 1$, x_{t+1}, is in balance with the on hand inventory, x_t, the quantity of purchased or produced items, q_t, plus the quantity of sold items, s_t, and the lost inventory, due to perishability or other characteristics that limit the shelf life of a product, w_t, at the previous period t. Any replenishment policy or any inventory model is subjected to this or a similar balance equation. The argument t describes the evolution of the system, usually discretely, over time.

The quantity of sold items which depends on the demand, d_t, and the items available to fulfill it is given by

$$s_t = min[d_t, q_t + x_t]. \tag{2.2}$$

When the demand is higher than the quantity in stock and items received from production or purchasing, then the realized sales is the total number of items available at the respective period and there is non-realized sales. On the other hand, the remaining stock carried over to the next period if $q_t + x_t$ is higher than d_t. The number of items at remaining stock that comes to be perished, obsolete or that receive any kind of differentiated pricing due to advanced age at period t add up to w_t.

At last, the profit for each period, given by

$$\pi_t = c_s s_t - c_u q_t - c_h x_t - O_t + c_w w_t, \tag{2.3}$$

corresponds to the profits associated with the inventory at period t. Hereby, this equation is separated in three components:

1. **The realized revenue** - It corresponds to the term $c_s s_t$, where c_s is the selling price of each unit of product and s_t is still the total sales at period t.

2. **The realized expenses** - These are the costs incurred to maintain the inventory operational. It corresponds to the terms $c_u q_t$, $c_h x_t$ and O_t. c_u is the unit cost for each item purchased or produced. It can be the direct price charged by the supplier or the production costs associated to each unit produced such as material and labor costs which multiplied by the quantity of items purchased or produced, q_t, results in the total unit cost incurred at period t. Usually called holding cost, c_h is the sum of all costs associated to the storing of each unit in the inventory such as maintenance, inventory labor, insurance, among others. The unit holding cost multiplied by the on hand inventory, x_t, provides the total storage cost of these items. Finally, the O_t corresponds to the fixed costs associated to the ordering of the batch quantity of items to be purchased or produced in the period t. It can be the sum of costs related to transportation, administration, among others.

3. **The salvage value or disposal expenses** - The term $c_w w_t$ corresponds to the costs related to the lost of inventory. Here, w_t is the quantity of units which was not sold in the period t when it could be sold by its full price, m. c_w can be either a positive lower than m, making the term a salvage value and therefore a realized revenue, or negative, making it a disposal expense and thus a realized expense. This term is related to the perishability, seasonality or even obsolescence of the items that must be either completely discharged, for example food, blood, obsolete components and some chemicals after expiration dates, incurring a variety of expenses depending on the item in question; or it can be sold by a lower price than its full intended price, for example clothing after its design season and perishables close to the expiration dates. w can also be disregarded (equal to 0) for cases such as plane seats after plane departure and hotel rooms after midnight for instance.

The main objectives presented in literature as discussed in chapter 1; increasing the profitability, reducing stock-outs and reducing lost inventory or perished items; can be achieved by maximizing π_t in equation (2.3) and reducing w_t, identified in equations (2.1) and (2.3). These objectives are usually pursued by settling the decision making on the variable order quantity, q_t, and how and when exactly the items should be ordered to determine the ordering policy. In the literature, these objects are evaluated and the ordering policies are stipulated depending on how the other parameters from the presented equations are modeled and estimated. Furthermore, these parameters are modeled according to different aspects considered in the industry and according to mathematical simplifications to make the analyses feasible.

The following sections present an overview of the relevant industrial aspects and mathematical simplifications among other features that are presented in the literature. At first, an overview of the concepts connected to inventory balance equation, (2.1), is presented following the structure illustrated in **Fig. 2.1**. Later, an overview of the concepts associated with the profit equation, (2.3), is presented as structured in **Fig. 2.2**.

$$x_{t+1} = x_t + q_t - s_t - w_t$$

$$s_t = min[d_t, q_t + x_t]$$

Excess Demand

$$s_t = \begin{cases} min[d_t, q_t + x_t], & \text{if sales is lost} \\ d_t & \text{if sales is backlogged} \end{cases}$$

- Morton (1969
- Kaspi & Perry (1983)
- Joseph (1987)
- Mak (1987)
- Aull-Hyde (1996)
- Kalpakam & Shanti (2001)
- Chang & Dye (2001)

- Graves (1982)
- Liu & Cheung (1997)

Demand

- Macy (1945)
- Fortuin (1980)
- Schultz (1987)
- Petrović & Petrović (1992)
- Hedenstierna et al. (2017)
- Syntetos et al. (2009)

Planning Horizon

Deterministic

- Pierskalla & Roach (1972)
- Veinott (1960)

Stochastic

- Van Zyl (1963)
- Bulinskaya (1964)
- Nahmias & Pierskalla (1973)
- Fries (1975)
- Cohen 1976,
- Chazan & Gal (1977)
- Williams & Patuwo (1999, 2004)

Finite

- Arrow et al. (1958)
- Nahmias (1975, 1977,1978)
- Williams & Patuwo (1999, 2004)

Infinite

- Nahmias & Pierskalla (1973)
- Fires (1975)
- Brodheim er al. (1975)
- Chazan & Gal (1977)
- Kalpakam & Shanti (2000)
- Lian et al. (2005)

Issuing Policy

- Derman & Klein (1958)
- Pierskalla & Roach (1972)
- Pratacos (1979)
- Keilson & Seidmann (1990)
- Ferguson & Ketzenberg (2006)
- Haijema (2011)
- Cherkesly et al. (2015)
- Lowaleckar et al. 2016)

Life-time

- Van Zyl (1963)
- Nahmias (1975)
- Graves (1982)
- Nahmias (1982)

Multiple Items

- Veinott (1965)
- Ignall & Veinott (1969)
- McGillivray & Silver (1978)
- Bitran & Dasu (1992)
- Bassok et al. (1999)
- Shin et al (2015)

Multiple Locations

- Clark & Scarf (1960)
- Prastacos (1981)
- Chen (1998)
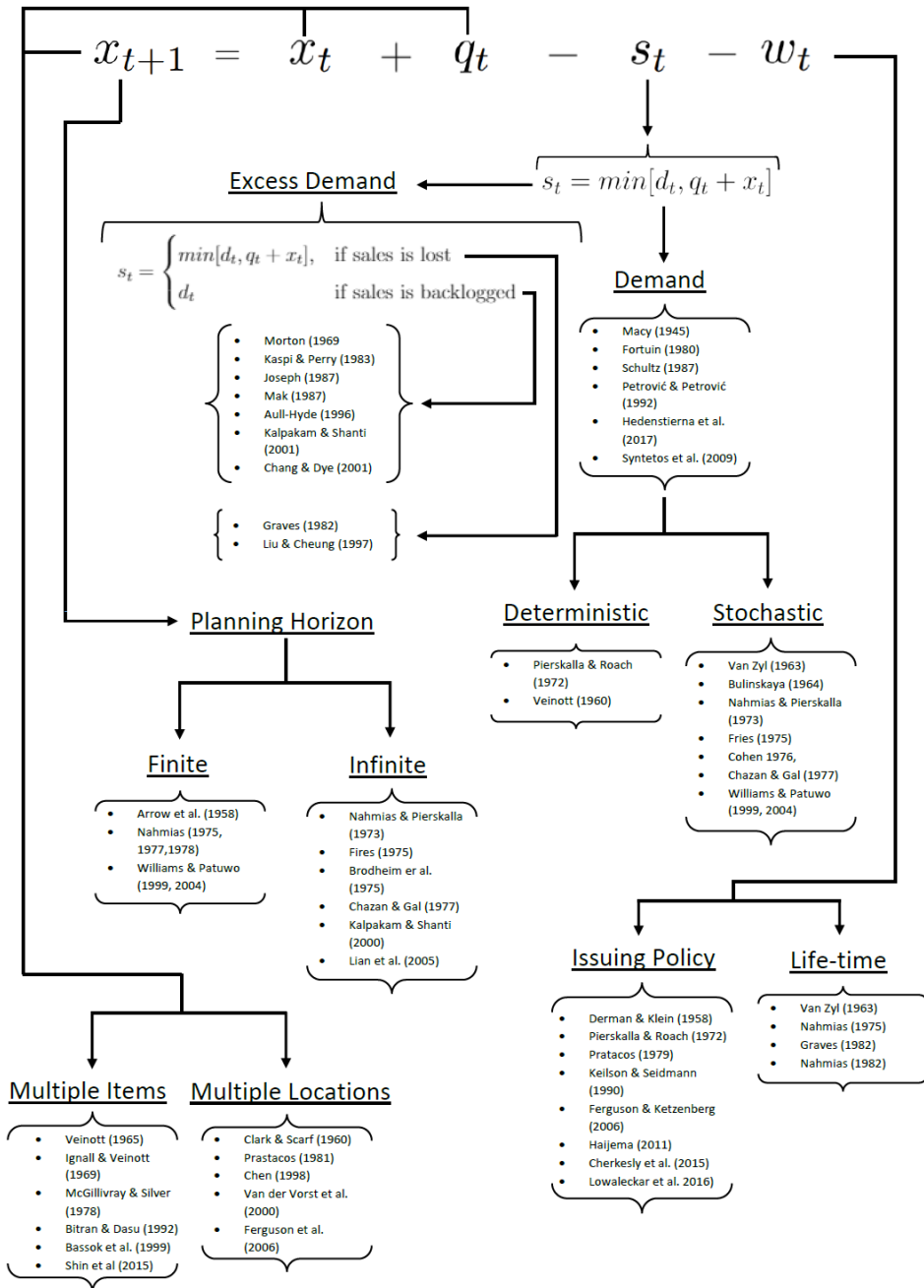- Van der Vorst et al. (2000)
- Ferguson et al. (2006)

Figure 2.1: Literature review structure for the inventory balance equation.

2.1.1 Sales and demand modelling

The aspects elected to be discussed first in this literature review are the ones related to the sales, term s_t. As it was indicated in equation (2.3), this term is the only term that certainly adds positively to the profitability of the model and, as expected, the more items sold the higher profits are incurred. Furthermore, equation (2.2) shows a direct relation to the demand, d_t or the items in stock plus the purchased or produced items, $q_t + x_t$, with a conditional on whether the sales or the term $q_t + x_t$ is lower.

Demand is uncertain and usually must be anticipated (Silver et al., 2016). It can also be separated into four different patterns considering period path: Trend, Seasonality, Random Variation and Cyclic (Arnold et al., 2008). These patterns may be determined and influenced by uncountable different aspects in a complex correlated relationship such as weather, holiday seasons, marketing characteristics, etc. There are several papers that focuses on determining and analyzing just the demand determinants of specific markets and products such as Dean and Meyer (1996) for new venture formations in manufacturing industries, Lanfranchi et al. (2014b) for high qualy food products and Lanfranchi et al. (2014a) for fish products in Messina for example.

Considering the anticipation of demand, although the field organizational forecasting was formally born only in the 1950s (Makridakis et al., 1998, p. III), this was a concern even before that. For instance, Macy (1945) discussed and reviewed in general terms selected problems encountered in procurement planning in the U.S. army during the second world war and indicated how the use of field research aided the solution of some of those problems by applying qualitative forecasting methods.

Later with the advancement of mathematical and statistical tools and methods, mathematical demand models started to be developed and analyzed. Fortuin (1980), for example, made a comparison of five popular probability functions in the field of stock-control models. Schultz (1987), studied variants of the Poisson model in forecasting for inventory control model. Petrovi and Petrovi (1992) worked with non-extrapolative approaches, such as reliability theory and expert systems in the development of models for advising on stocks of spare parts. Hedenstierna et al. (2017) developed an analytic approach for estimating total demand considering sales and footfall data aiming also at the demand that is not captured in historical review of pure sales data due to stock-outs. Syntetos et al. (2009) presented an extensive review of forecasting research focused on inventory management over the 50 years up to the date the paper was published. Among other conclusions, the paper states that Poisson distribution is a natural candidate for representing low demands. In fact, this is the main reason why Poisson distribution was also used to model demand in this thesis.

On the mathematical modeling of demand, two approaches are considered in inventory control modeling: Deterministic and Stochastic.

Deterministic

The deterministic approach, when demand distribution is considered known and certain, can be described as mathematically simpler than the stochastic approach since it does not require statistical modelling (Silver et al., 2016). For perishable items for instance, under fairly general conditions an optimal policy always orders in such a way that items never

perish Nahmias (1982). A similar trivial conclusion such as this cannot be stated about stochastic demand. On the other hand, the deterministic approach does not capture the uncertainties associated with the demand (Porteus, 2002).

When a certain problem is proposed in the literature, this problem is usually approached by the application of deterministic demand modeling and then its research is expanded for the stochastic modeling. Pierskalla and Roach (1972), for example, utilized the deterministic demand to study optimal issuing policies for blood and then extended the analyses to a stochastic approach. Veinott (1960) utilized the deterministic demand to work on three main problems: Determining an optimal ordering policy considering that the disposal and issuing policies are given, determining optimal ordering and disposal policies concerning the issuing policy is given, and determining optimal issuing and disposal policies considering the ordering policy is given. Later, all these problems were studied regarding stochastic demand from many different mathematical perspectives.

Stochastic

Opposing to the deterministic approach, the inventory control problems with stochastic modeling becomes quite complex leading to non-trivial solutions. For inventory control of perishable items for example, there are optimal solutions just for problems modeled with items having shelf life up to 2 periods after delivery. The well known newsvendor problem described by Arrow et al. (1958) covers the problem for a planning horizon and shelf life reduced to 1 period. Bulinskaya (1964) covered the same problem considering a probability for the case which the items perish upon delivery and the rest for the case which the items perish in 1 period.

Van Zyl (1963) and Nahmias and Pierskalla (1973) developed optimal policies for 2 periods shelf life items. The former considering non-age-dependent quantity of perishables charging ordering and stock-out costs and the last including out-dating costs. Fries (1975) analyzed optimal policies for items with shelf life over than 2 periods and as described by Nahmias (1982): Owing to the multidimensional state variable, the computation time using these models is quite long for shelf life over than 3 periods, making computation of an optimal policy impractical for a real problem. Therefore, many researchers turned to the more practical question of seeking effective heuristic policies that would be easy to define, easy to implement and close to optima Karaesmen et al. (2011).

Cohen (1976); Chazan and Gal (1977) and Nahmias (1977a) considered base stock policies that keep a constant order-up-to-level for total items in system, summed over all ages or, as in Brodheim et al. (1975), only new items in the system. Chazan and Gal (1977) treated the age distribution as a finite Markov chain and showed that expected out-dating was convex in the inventory level. Cohen (1976) used a transfer equation and an objective function of expected cost per period. Nahmias (1977a) used a similar model but also considered shelf life random.

Other heuristic analysis considering stochastic demand were performed by Nahmias (1977b) utilizing higher-order approximations, Nandakumar and Morton (1993) utilizing heuristic based on a periodic inventory problem in the framework of a newsboy problem and attempted to bound the various newsboy parameters. Williams and Patuwo (1999) Williams and Eddy Patuwo (2004) computed the optimal order quantity using the New-tonRaphson method which is known to converge quadratically in the neighborhood of a

root.

In the stochastic approach, demand is typically discretely modeled as independent identically distributed non-negative random variables, ξ_i, with distribution function, $\Phi(s)$ having a density function, $\phi(s) > 0$ for $s > 0$ and with finite mathematical expectation $E[\xi_i] = \mu$ as described by Bulinskaya (1964).

Excess demand

Excess demand happens when demand is higher than the on hand inventory and the received items ($d_t > q_t + x_t$). In this case, the non-covered demand, $d_t - (q_t + x_t)$, can be either backlogged (the customer will wait to purchase the item once it becomes available) or lost (the customer decides not to purchase the item or purchase it somewhere else).

As described by Nahmias (1982), when the inventory control is modeled considering backlogging, the system permits carrying of negative inventory. Then, the sales can be realized in future periods when new products arrive. Equation (2.2) describes the system considering lost sales. Both the order quantity, q_t, and the on hand inventory x_t may influence the variable sold items, s_t. Lost sales happens when $d_t > q_t + x_t$. Regarding backlogging, the system should be modeled replacing equation (2.2) following the respective conditionals from equation (2.4). In this case when $d_t > q_t + x_t$, the sold items, s_t, are equivalent to d_t and the on hand inventory at $t+1$, x_{t+1}, become negative. This means that the non-realized sales are carried to the future periods when they can be realized. These relations are given by

$$
s_t = \begin{cases} min[d_t, q_t + x_t], & \text{if sales is lost,} \\ d_t & \text{if sales is backlogged.} \end{cases} \tag{2.4}
$$

Morton (1969) did a very preliminary work on lagged optimal inventory not considering backlogging. On inventory management of perishable products, lost sales are also usually considered for blood bank inventories modeling as in Kaspi and Perry (1983). Kalpakam and Shanthi (2001) also studied a perishable inventory system with modified base stock policy and arbitrary processing times considering lost sales.

Backlogging is usually considered in inventory control in retailing such as in Joseph (1987). Aull-Hyde (1996) studied a backlog inventory model during restricted sale periods assuming that the sale price is not available.

In some cases, partial backlogging is considered in order to develop the model as most real as possible. One of the first papers considering partial backlogging was Mak (1987) by determining optimal production-inventory control policies for an inventory system with partial backlogging. For perishable items, Chang and Dye (2001) analyzed an inventory model with partial backlogging and permissible delay in payments.

Some researchers use a model for lost sales first and then extend the studies to a backlogging model due to trivial results or to compare the outcomes. Graves (1982) utilized a model considering lost sales to study the application of queuing theory to continuous perishable inventory systems and later also considered backlogging case since it was a simple extension. Liu and Cheung (1997) analyzed service constrained inventory models with random shelf lives and lead times considering partial backlogging, full backlogging and lost sales; and later compared the multiple cases.

2.1.2   Order quantity and on hand inventory

Both the order quantity, q_t and the on hand inventory, x_t, have a multiplicative direct relationship and may have an additive inverse relationship (it can either be positive or negative) to the profitability. In addition, these variables may also influence the number of sold items according to equation (2.2). In these regards, three main considerations are usually made in the modelling of inventory control problems: What is the planning horizon for the ordering policies and inventory control (the extension of the index t), if the system covers the modelling for single or multiple items and if the system covers the modelling for one or multiple inventory.

Planning horizon

The modelling of the planning horizon is related to the extension of the index t (the quantity of periods considered in the analysis). This has high relevance in the generalization of the policies analyzed and for obsolescence modelling. Obsolescence is usually modeled by assuming that the length of the planning horizon is random and is fundamentally different from perishability in that since once items become obsolete they are not reordered. This was the case in Brown et al. (1964) using a class of models for optimizing inventory costs considering stochastic obsolescence. David et al. (1997) using a deterministic demand model in a dynamic programming approach to analyze continuous-review versions of the classical obsolescence problem.

The newsvendor problem (see Arrow et al. (1958)), considers only one period as the horizon planning. Nahmias (1975, 1977b, 1978); Williams and Patuwo (1999); Williams and Eddy Patuwo (2004) already cited considered finite horizon. Nahmias and Pierskalla (1973); Fries (1975), also already cited, developed the optimal solution for the 2 period shelf life regarding first finite horizon and then generalized it for the infinite horizon case. Brodheim et al. (1975) which applied stock policies that keep a constant order-up-to-level for total items in system, summed over all ages with only new items in the system considering infinite horizon. Kalpakam and Shanthi (2000); Lian et al. (2005) also considered infinite horizon. The former analyzed an inventory system with Poisson demands stocking perishable items having constant hazard rates and the last used a discrete-time inventory with (s,S) policy model in which the stored items have a random common shelf life with a discrete phase-type distribution.

Single and multiple items

Inventory models for a system with multiple products can be extremely complex, even more when one or some of them have a fixed life time (Nahmias, 1982). Therefore, the typical problems not directly related to excess demand are modelled considering a single product inventory. In this case, only one variable such as the x_t is necessary to represent the on hand inventory in the period t. When excess demand is taken into consideration, substitution may become an interesting alternative to cover a larger share of the demand enabling higher profitability and less waste. Thence, more variables are necessary to cover the on hand inventory quantities for all products.

There is a broad field of study in inventory theory only on the matter of substitutability

of products and the inventory management of these products. According to Van Woensel et al. (2007), perishable products have for example a considerable high substitution rate. Kamakura and Russell (1989) found that substitution does not necessarily follow a symmetric pattern by inspecting the retail scanner data for one product category in the food sector over a determined period. Gruen et al. (2002) studied the distribution of out-of-stock responses over a large number of product categories. According to this work 45% of customers facing stock-out are willing to buy an substitute product and approximately 9% do not purchase anything to cover their demand. The remainders do seek other alternatives to complete their purchase such as purchasing in other stores. These results show some critical commercial aspect of stock-outs as discussed in the introduction, section 1.1.

Shin et al. (2015) worked on a comprehensive overview of substitution problems in operations management from three different perspectives assortment planning, inventory decision and capacity planning. On inventory management, Veinott (1965) did a fundamental contribution aiming at an optimal ordering policy that minimizes the expected discounted costs over an infinite time horizon for a multi-Product, dynamic, non-stationary inventory problem. This work was later generalized by Ignall and Veinott (1969).

Afterwards, further development of these works were performed for the most simple case considering two products such as McGillivray and Silver (1978) investigating the effects of substitutable demand on stocking control rules and the associated inventory and shortage costs, and such as Bitran and Dasu (1992) modeling the problem with stochastic assumptions to compare two approximation procedures. Bassok et al. (1999) considered up to three products in a single period inventory problem with proportional costs and revenues and full downward substitution.

On inventory management of perishables, inventory control of blood considering substitution was initially the main concern. Nahmias (1976) generalized the analyses of optimal ordering policies for a single-product inventory model from Nahmias (1975) to analyze a realistic blood bank case with a system consisting of one fixed life perishable product and one nonperishable, where the nonperishable may be substituted for the perishable. Later, (Deuermeyer, 1979, 1980) studied the one period problem for two products with finite shelf life. They also accounted production planning to the papers taking into consideration that one of the products was manufactured by one production process and the second product was manufactured by the same production process and an extra one. The optimal order policy was characterized by four regions in the space of two vectors representing the on hand inventory of the two products for different shelf lives and indicated four combinations of using the two processes or neither of them.

Single and multiple locations

The typical researches that are not focused on issues related to distribution of goods in a net of suppliers and inventory locations have their inventory systems modeled regarding the simpler scenario of single location and single supplier (product is delivered from one supplier to one inventory) as the problem has been presented so far. Although this modeling is simpler, it is not very realistic. Usually, in both blood banking and food management, goods are produced at a central facility and subsequently shipped to regional centers for distribution (Karaesmen et al., 2011). These systems which address distribution challenges are called multi-echelon or multi-location systems.

Similarly to the multiple items case, the multi-location or multi-echelon systems have to resort to more variables to represent the on hand inventory and the cost related to the introduction of different locations to the problem. Furthermore, the allocation decision becomes an objective of study in addition to the replenishment decisions as in Prastacos (1981) and Ferguson et al. (2006).

Clark and Scarf (1960), which was one of the first relevant papers covering the subject, modeled the a serial system (considering only one supplier) of two sequential installation. The on hand inventory at installation 1 was denoted by x_1; the stock to be delivered one period in the future by w_1; and the stock on hand at installation 1, plus on hand at installation 2, plus in transit from 2 to 1, by x_2 (i.e., x_2 is echelon 2 stock). The one-period costs at installation 1 was denoted by $L(x_1)$, and those at echelon 2 by $L(x_2)$. In addition, the unit shipping cost from 2 to 1 had to be accounted and was denoted by c_1.

Later the multi-echelon and location systems were considered to model systems of perishable items. Cohen and Pierskalla (1979) for example focused in obtaining optimal target inventory levels for a community blood center with multiple locations. In this paper, a two-echelon system with a central facility, 0, supplies two other facilities, 1 and 2. External demand occurs only at facilities 1 and 2. Inventory control must be modeled in all three locations and the on hand inventory variables have to account with the shelf life of the items.

Chen (1998) investigated the costs of restrictions related to stationary replenishment activities in multi-echeleton inventory. Van der Vorst et al. (2000) developed a discrete-event simulation model to analyze a fresh product supply chain with three echelons.

Another case that is also grounded in realistic aspects and increases the complexity for its modeling compered to the single-location cases is the multi-supplier system. On the problems considering multi-suppliers, Minner (2003) performed a comprehensive review of the theme. He describes the modelling challenge of it by listing the additional variables resulted from the multiple supplier scenario. Basically, there are multiple suppliers with different respective lead times, different also respective unit prices and an additional fixed setup cost for each order placement.

2.1.3 Lost inventory

The shelf life of a product can be either infinite or finite. Hereby, four types of products with finite shelf life are recognized: The ones which perish as in the blood bank inventories and food industry; the ones which can get obsolete as typically in the market for electronic components; the ones which can get out of fashion as in the clothing industry; and the ones that just don't fulfill their occupancy for hotel rooms and seats in airplanes in the tourism market.

When the items on hold in inventory have a finite shelf life, accounting the variable w_t in equation (2.3) is no longer necessary since the product can be stored infinitely without loosing its value and the problem becomes more trivial to work with. For products with finite shelf lives, besides the need to account the variable w_t in equation (2.3), other concerns that make the systems modeling more complex gain relevance. One important concern related to finite shelf life problems is how the shelf life is modeled and affects the parameters of the inventory system. Another concern is how the items are issued (or depleted) in relation to the shelf life of items on hand.

Shelf life

Nahmias (1982) considered two classification of perishability with regard to the shelf life: fixed shelf life and random shelf life. The former category includes those cases where the shelf life is known a priori. Shelf life is specified as a number of periods or a length of time independent of all other parameters of the system. In the later category, the shelf life is not fixed and follows a decay function or it is a random variable with a specified probability distribution. Van Zyl (1963) also called fixed shelf life products as age dependent and random shelf life products as age independent.

For fixed shelf life and in the discrete time single inventory model, the on hand inventory is represented by a vector **x** of on-hand inventories varying in age from 1 to $n - 1$ periods of a commodity with a shelf life of n periods (Nahmias, 1975). Doing this, all the non-consumed items from inventory with a specific age at period t (quantity $x_{t,j}$, t representing the time period and m; with $m = 0, ..., n$; the remaining shelf life of the item) will be added to the virtual inventory for the items with shelf life of one period less at $t + 1$. Hence, the system that is modeled by equations (2.1), (2.2) and (2.3) can be modeled by the following equations once fixed shelf life and discrete time are considered:

$$x_{t+1,m} = x_{t,m+1} + q_{t,m+1} - s_{t,m+1} - w_t, \tag{2.5}$$

$$\pi_t = \sum_{m=0}^{n} (c_s s_{t,j} - c_u q_{t,m} - c_h x_{t,m} - O_{t,m}) + c_w w_t \tag{2.6}$$

$$s_{t,j} = min[d_{t,m}, q_{t,m} + x_{t,m}] \tag{2.7}$$

$$w_t = max[0, d_{t,0} - x_{t,0}] \tag{2.8}$$

The demand for items with specific ages will depend on the issuing policy. Another important point on this model is the fact that in many real cases there is no control of the age of items in stock. These two points will be addressed later. Note that this is a system with two dimensions, this makes that the problem becomes naturally much more complex than the models which consider infinite shelf life.

For the continuous time single item inventory models, inventory quantity is defined as a function of the time t. In Graves (1982) for example, the inventory system is characterized as a Markov process with state variable $A(t)$ corresponding to the age of the oldest unit in inventory at time t and the replenishment process is considered constant; that is, new inventory is continuously produced at a constant rate of c units per time unit. Therefore, $cA(t)$ represents the amount of present inventory and the inventory control modeling becomes even more complex.

An example of random shelf life is described by Nahmias (1982). Suppose the shelf life of individual units is a random variable with a negative exponential distribution having parameter θ. $I(t)$ is the number of units surviving to time t exclusive of demand. Since each unit has probability of $e^{-\theta s}$ of surviving an additional s units of time, it follows that the number of units surviving to time $t + s$ is a binomial random variable with parameters $n = I(t)$ and $p = e^{-\theta s}$. It follows that the expected number of units surviving until time $t + s$ is np or $I(t)exp{-ds}$. Hence, we obtain the well known exponential decay process.

Depletion management

Another relevant matter that plays an important role in inventory modeling for perishable products is how the items leave the inventory to fulfill demand in relation to their remaining shelf-life. The depletion of the items may be done under the control of the inventory manager which may adopt a policy, often called issuing policy, that will most benefit the inventory management. The same depletion may also be done without full control of the inventory manager. Usually in grocery stores, the same perishable items with different remaining shelf life are often displayed for the customers. Even though the stores use different strategies to incentive the customers to pick up the items following the stores interests, the customers choice may still differ from these interests. For example, the stores may organize the milk cartons placing the items with the oldest remaining shelf-lives in front of the shelves and the newest items in the back of the shelves attempting to incentive the customers to pick up the oldest items. Opposing to this incentive, some customers may use some time to seek the available items with longest remaining shelf-life and pick up the items from the back of the shelf since in most of the cases such items have a higher utilization to customers. Others may not have time or even the interest to seek for the newest items and will pick up the first item available

The two most used depletion policies in inventory modeling are the FIFO (First-in-first-out) and LIFO (Last-in-first-out) policy. The FIFO policy is referred to the policy which the oldest item in stock is always issued first and the LIFO means that the newest item in stock is always issued first. Changing the depletion policy will change the whole dynamics of the inventory model, and as a consequence, the outcomes of the model will also vary.

The choice of the depletion policy will directly impact the calculations on the realized sales variable determined by equation (2.7), and as a consequence by the equations (2.5) and (2.6). For the FIFO policy for example and considering that all items are delivered with the same shelf life, the demand for the oldest items and for each period, represented hereby by $d_{t,0}$, should be set equal to the total demand in the beginning of the inventory control calculations. Afterwards, the inventory balance for the items with the same remaining shelf life should be calculated. If there is a remaining demand not covered by the oldest items in stock, the realized sales, equation (2.7), and the inventory balance, equation (2.5), will be calculated for the next shelf life level, represented hereby by $d_{t,1}$. The same calculations should be performed for all remaining shelf life levels, from $m = 0$ to $m = M$ considering that M is the maximum shelf life, so the complete inventory level for each age category can be figured out. Considering LIFO as depletion policy, similar calculation should be performed, but the starting point should be $m = M$ and ending point $m = 0$.

This inventory depletion management problem was first formally studied by Derman and Klein (1958). In this study, different conditions were given under which either LIFO or FIFO was an optimal issue policy. Later, Pierskalla and Roach (1972) concluded that for most of the objective functions considered in their work, the optimal depletion policy was FIFO. They considered three objective functions: To maximize the total current utility of the system, defined as the value of all demands satisfied in the past plus the value of items in stock at present; to minimize the total number of units backlogged at any given time or to minimize the total number of lost demands in the lost demand cases; and to minimize the total number of items reaching the last age category. Their results have shown that

FIFO was optimal for the last two objective functions, and it maximizes the utility of the system provided all excess demand is backlogged for the first objective function, although not optimal.

In the early 1990's, Keilson and Seidmann (1990) have performed a more extensive comparison of FIFO and LIFO policy considering stochastic demand. They used spoilage rate, mean age at delivery, expected time between stock-outs, service level, and mean on hand inventory level as performances characteristics to be evaluated. The FIFO policy showed to assure higher service level and longer times between stock-outs; lowering of mean age at delivery with lowering of demand rate; and whenever the sales price is age independent, FIFO can bring higher profits with lower supply rates. On the other hand, LIFO policy leads to lower age of items delivered and on hand inventories and a probable mode economical solution when the utility, and the sales price, of new items are higher than for older ones. But these two last conditions do not often happen in real cases.

Although, FIFO, is the policy which generally results in better performance and it is the most applied in inventory control research, it is often not realistic from a practical perspective that inventory depletion is not always controllable by a retailer as already mentioned. When control is left to customers, they are apt to select the freshest products first, even though that retailers often use systems to guide customers decisions towards FIFO policy, such as load-from-the-back shelving systems commonly used for dairy products (Ferguson and Ketzenberg, 2006).

Therefore, research on inventory management considering LIFO also has large relevance. Cohen (1976) analyzed evolution over time of the LIFO inventory stock age distribution in an environment of stochastic demand. Prastacos (1979) and Haijema (2011) examined the regional distribution problem for a perishable product considering LIFO depletion policy to satisfy the demand. Slightly out of inventory control problems, Cherkesly et al. (2015) have used the LIFO policy as a conditional to solve the classical pickup and delivery problem with time windows with population-base and meta-heuristic procedures.

Obviously, the depletion policies are not restricted to the LIFO and FIFO approaches since products may be issued at any age. Haijema (2011) was one of the first studying optimal depletion policy extending the analysis to policies other than the FIFO and LIFO duo. Using simulation combined with Markov Decision Processes, it was shown that the optimal dynamic depletion policy for short shelf-life products like platelets results in the same out-dates but lower shortage as compared to FIFO.

In the literature search of this master thesis, few studies considering the SIRO (service-in-random-order) policy were found and they are classified hereby according to how the randomness of the depletion policy was modeled. Pegels and Jelmert (1970) have used a hypothetical probability approach in which hypothetical transition probabilities were used in a Markovian approach to study a blood bank related problem. The probabilities were not based upom real case scenarios. Sapountzis (1985) and Lowalekar et al. (2016) have used an uniform distribution approach. The depletion policy applied considered that the items are issued randomly following a uniform distribution i.e. items from all ages have the same probability to be issued. Finally and in a more sophisticated approach, Vaughan (1994) and Ferguson and Ketzenberg (2006) have considered that the probability of an item in each age category being issued is equivalent to the proportion of total quantity of inventory represented by the quantity of items of the given age category. For example, if

20% of the units in inventory have a remaining shelf-life of three days, then a particular unit of demand has a probability of 0.20 of being satised with a unit of inventory with a remaining lifetime of three days. These studies constitute the base literature to this master thesis and are further reviewed in the second section of this chapter.

2.1.4 Costs and revenue

In equation 2.3, 5 variables related to costs or revenue are presented. Each term has more, less or no relevance depending on what it is the object of study and the goals intended with the inventory modelling.

$$\pi_t = c_s s_t - c_u q_t - O_t - c_h x_t + c_w w_t$$

Sales Price	Ordering Cost	Holding Cost	Salvage Value
• Kunreuther & Schrage (1973) • Smith (1975) • Cohen (1977) • Thomas (1970) • Adachi et al. (1999) • Elmaghraby & Keskinocak (2003) • Ferguson & Koenigsberg (2007)	• Fries (1975) • Cohen (1976) • Nahmias & Pierskalla (1973) • Nahmias (1975, 1977) • Nahmias (1978) • Kalpakam & Shanthi (2001) • Williams & Eddy Patuwo (2004) • Lian & Liu (1999) • Lian et al. (2005)	• Muhlemann & Valtis-Spanopoulos (1980) • Weiss (1982) • Giri & Chaudhuri (1998) • San-José et al. (2015)	• Fujiwara et al. (1997) • Pareek et al. (2009)

Figure 2.2: Literature review structure for costs and revenue.

Ordering costs

Represented hereby by the variable c, the unit ordering cost is present in most of the inventory control models such as in Fries (1975); Nahmias (1975); Cohen (1976); Nahmias (1977a); Williams and Eddy Patuwo (2004); Kalpakam and Shanthi (2001), already cited. This is a debt that is typically linearly withdrawn from the profit equation for each of the items that are ordered and received through purchasing or production when $x \geq 0$ unit, where $c > 0$. There are exceptions in which the unit cost is not considered in the inventory model. For example, Nahmias and Pierskalla (1973) studied the optimal policies for perishables which perished in two periods considering only costs for unsatisfied demand and deterioration. Lian et al. (2005) considered the discrete review model with fixed ordering cost, holding cost per unit per period, shortage costs per unit, and out-dating costs per unit.

The fixed ordering cost, r, can be related to the fixed set up costs for production necessary for the ordered products and/or the transportation, administrative matters, among other expenses incurred when the items are purchased. The presence of this cost also

changes completely the dynamics of how the profitability of the model will develop with the ordering sizes and replenishment policies since it makes the system nonlinear.

Nahmias (1978) was the first to analyze the perishable inventory problem with positive fixed ordering cost in addition to the standard unit cost described. In this paper, for the one-period problem, it was established that the structure of the optimal policy is the (s,S) only when the shelf life of the object is two; shelf lives of more than two periods have a more complex structure. Lian and Liu (1999) followed by Lian et al. (2005), already cited, have also performed the research considering fixed ordering cost.

Inventory holding cost

Inventory carrying cost or inventory holding cost, h, includes all expenses incurred by the firm derived from the volume of inventory carried. As inventory increases, so do these costs. They can be broken down into three categories: Capital costs invested in inventory; storage costs to cover the space, workers, and equipment necessary to maintain the inventory; and risk costs such as damage while inventory is held or moved and pilferage (Arnold et al., 2008, p. 354). This cost is usually included in the inventory modelling influencing linearly the profitability.

Some authors have given a different focus on inventory holding costs taking into consideration a more realistic approach and its impact on the inventory control model. For example, Muhlemann and Valtis-Spanopoulos (1980) considered in their inventory model that the average value of the inventory increases the cost of financing it, expressed as a percentage of its average value i.e. the cost represented by a percentage of the average value of the stock held is such that the percentage is a function of the average value of the stock held. They did it to realistically represent the opportunity cost of not having the capital available for other purposes or the interest payable on the loan that had to be raised at their inventory system.

Weiss (1982) considered an inventory model with constant demand rate, taking the holding cost as a nonlinear function of inventory and the length of time for which the item was held in stock respectively in a generalization of the economic order quantity model. San-Jos et al. (2015) performed a similar work but considering partial backlogging. At last, Giri and Chaudhuri (1998) performed the same work considering a deterioration rate per period.

Sales price

Hereby, m represents the unit price of the items on inventory, that means how much revenue will be generated for each item sold. When the literature focuses on investigating issues of inventory control systems not related to pricing such as optimal or improved replenishment policies, depletion policies, allocation challenges for multiple location systems, among others already mentioned; the sales price is modeled as a constant given by a fixed sales price or a margin on top of the unit cost.

Often, the sales price is not even considered since the goal of the inventory modeling investigation is set in minimizing the actual costs as in Nahmias (1976); Giri and Chaudhuri (1998) and Lian et al. (2005), already cited. However, pricing marketing actions such as pricing manipulation can drive consumer demand, phenomenon called price elasticity

of demand, which significantly influences operations management decisions in areas such as capacity planning and inventory control (Maddah et al., 2011).

Pricing has become one of the most widely studied topics in the operations management literature in the last decade (Karaesmen et al., 2011). Anyhow, some decades ago this issue was already considered in inventory modeling. Smith (1975) investigated optimal ordering and price policies regarding demand as deterministic and a function of price. Cohen (1977) also considered deterministic demand rates as a known function of the price of a unit to minimize the cost function production levels, but for an exponentially decaying product. Even earlier, other authors had already investigated decisions related to pricing policies considering deterministic demand as Thomas (1970) not considering backlogging in his model, and Kunreuther and Schrage (1973) aiming at determining pricing and ordering decisions, but considering price constant over the periods.

Later, the effect of selling prices started to be investigated with stochastic considerations. Adachi et al. (1999) investigated a perishable product inventory model with consideration to different selling prices of perishable commodities under stochastic demand discriminating selling prices by different shelf lives. Ferguson and Koenigsberg (2007) modeled a two period inventory system with stochastic demand and price discrimination of unsold aged items and investigated the competition between the cheaper aged items and the new products with higher prices but perceived better quality. Basically, this work showed that the second selling opportunity overcame the effect of cannibalizing sales of the second period new product. Elmaghraby and Keskinocak (2003) provided an extensive review and practices in dynamic pricing.

Salvage value

Salvage value is another term that is often included in the cost or profit equation in inventory modelling of perishable products. It is basically revenue that is incurred for each item that has passed its shelf life. The salvage value is small and should not exceed the shortage cost, otherwise the system tends to hold excess sub-products to obtain revenue from salvage (Fujiwara et al., 1997). Some researches consider salvage value due to realistic approximations and to investigate its impact.

Fujiwara et al. (1997) consider a specific inventory control problem for finite-shelf life fresh-meat-carcass in supermarkets including salvage value to establish optimal ordering and depletion policies aiming at profit maximization. Pareek et al. (2009) solved analytically a deterministic inventory model for time dependent deteriorating items accounting salvage value aiming at the minimization of total inventory cost.

2.2 Random Depletion Policies

The focus of this master thesis is the estimation of how perishable products are depleted in relation to their shelf-lives i.e. how the products that are sold at grocery stores are picked by customers considering remaining shelf-life. In the industry as claimed by Relex, inventory projections are calculated assuming FIFO depletion policy. However, this is not how customers choose their groceries in reality. Instead, customers typically prefer fresher products as much as stores build up their selling strategies to sell the oldest items

first. Hence, a realistic depletion policy of perishable products in a retail context takes place somewhere in between LIFO and FIFO policies in a random ordered. This type of depletion policy is called in some papers, such as in Ferguson and Ketzenberg (2006), SIRO (service-in-random-order) policy.

As point of departure for this work, a review of available literature focusing on how the academia and industry have contemplated the SIRO policy was performed. All relevant references found in this review comprise the base literature that was used to inspire and support the analysis of this thesis. The main objectives, features and findings for each reference are compiled in the table 2.1. In the following lines, these references are presented and discussed.

In the base literature, the references were divided into three categories related to how the randomness of the depletion policy is modeled which are listed and described below.

- **Hypothetical probability -** In this case the depletion policy was considered to follow hypothetical transition probabilities with no scientific background such as in Pegels and Jelmert (1970).

- **Uniform distribution -** The depletion policy considers that the items are issued randomly following a uniform distribution i.e. items from all ages have the same probability to be issued. This approach was applied by Sapountzis (1985) and Lowalekar et al. (2016).

- **Proportional distribution -** The probability of an item in each age category being issued is equivalent to the proportion of inventory of each age category in relation to the total inventory. For example, if 20% of the units in inventory have a remaining shelf-life of three days, then a particular unit of demand has a probability of 0.20 of being satised with a unit of inventory with a remaining lifetime of three days. Both Vaughan (1994) and Ferguson and Ketzenberg (2006) have implemented this approach in their work.

In Pegels and Jelmert (1970) which is the oldest reference found that included randomness in the issuing policy in its model, hypothetical transition probabilities of depleting the items were used in a Markovian approach to study a blood bank related problem. The objective of the model applications was to determine the effects of the issuing policies on average inventory levels, which determine blood shortage probabilities, and on the average age of blood at the time of transfusion. Depletion policies that issue fresher items with a higher probability than older ones are defined as modified LIFO policies, and depletion policies that issue older items with a higher probability than fresher ones are defined as modified FIFO policies. The probabilities used were not based upon real case scenarios.

Sapountzis (1985) studied the demand of a characteristic curve of a blood bank for a particular blood group, which expresses the probability of a unit of blood out-date as a function of the age of blood entering the blood bank. It was assumed a random issuing, or depletion, policy following a uniform distribution. The application of the model developed used a hospital context as a background. Using the collected data, a different set of probabilities were applied for the demand of different blood types and their parameters were estimated. Two probabilities distributions were used: A negative exponential distribution

which had its parameter estimated by application of the method of moments and the Erlangian distribution which had its parameter estimated by maximum likelihood estimator. The fitting of the curves were tested by applying the Kolmogorov-Smirnov criterion and the results were considered satisfactory for all blood types.

In this work, the characteristic curve was considered the best means to judge the performance of a blood bank due to the provision of expiry rates for every age of blood in the bank. Furthermore, the comparison of the efficiency between two blood banks, comparing the characteristic curves of the two banks for a specific blood group, was another application mentioned for the characteristic curve. At last, with the use of the model policies that were investigated for reducing the expires at a blood bank. It was shown that reduction of the first unreserved period by 30% has a low effect on the expires at the blood bank and consequently this policy was considered not worth to be implemented due to the high cost.

Lowalekar et al. (2016) also used random issuing policy of perishables in a blood bank context to model an inventory driven by an order-up-to-level replenishment policy under a periodic review setting. This paper aimed at optimum policy parameters for the model through the application of a gradient search-based heuristic. A real life application of the model was shown in the search of the optimum frequency and order-up-to-level.

Each unit had a xed lifetime. All the units which were not used during their xed useful life were discarded at the end of its shelf-life. A xed cost for every unit that was discarded was charged. A xed cost was charged for every review. A fixed ordering cost was charged for every receivable. All units received had maximum remaining shelf-life, that is, they arrived fresh. No back-order was considered. A xed cost of shortage was charged for every unsatisfied demand unit. A xed holding cost for each unit was also charged.

The approximate model and heuristic was considered capable to be used together to determine the optimal collection quantity and the optimal frequency of setting blood. The model conrmed the idea that the blood should not be collected beyond a certain level during donation periods. This is because the expected wastage increases rapidly beyond a certain point while the average shortage does not reduce signicantly, indicating that any additional unit beyond a certain level of collection will most likely be wasted. In addition, the model indicated that collecting blood frequently instead of in large quantities is the key to reduce the wastage. It was also noted that high values of the delivery cost would prompt the hospitals to order less frequently and in large sizes leading to wastage increase.

Vaughan (1994) worked out a model for determination of inventory ordering and outdate policy for a perishable item with random shelf-life considering of consumer-realized product expiration. It means that in the model, a base stock of a order-up-to-level policy and in which remaining shelf-life the items are out-dated were the decision variables. Demand was considered to be Poisson distributed and no backordering was considered. The depletion policy was considered to be random following the proportion of the quantity on stock respective to each shelf-life category. For example, if 20% of the units in inventory have a remaining shelf-life of three days, then a particular unit of demand has a probability of 0.20 of being satised with a unit of inventory with a remaining shelf-life of three days.

The main contribution of this research was to show the outdating policy decision under the conditions modeled is significantly impacted by the value of the base stock policy applied. Therefore, it was considered inappropriate to assign at first a value to the outdating dates priorly to determine the base stock which minimizes only the costs of carrying,

shortage and outdates considering the product has random useful shelf-life. In addition, the fixed-shelf-life perishable inventory model showed to have its most relevant application only when the product had truly a deterministic shelf-life, and it was less effective when the deterministic shelf-life was imposed as a policy upon a product with shelf-life that was actually random.

Ferguson and Ketzenberg (2006) considered a discrete infinite horizon inventory model for single product and single echelon to study the value of information of the age of products to be delivered prior to ordering for FIFO and LIFO depletion policies. A special analysis was also performed for a random depletion policy. In this case depletion of the items also happened following the proportion of the quantity on stock respective to each shelf-life category. Both demand and shelf-life of products to be delivered were stochastic. The costs used in the model; unit purchasing cost, inventory holding cost and revenue; were all linear and related to the quantities of items. All demand that was not fulfilled was lost, the order quantities were modeled as multiple of a fixed batch quantity and the lead-time for them was one period.

A periodic review and heuristics myopic replenishment policies which the order decision rested on whether sufficient stock existed in the current period that could carry over and minimize expected cost only in the next period was used and verified against optimal policies. Two scenarios were used for each depletion policy, the base-no-information-scenario considering that the information about the shelf-life of the items to be delivered were not known prior to the ordering and the information-scenario when this information was known and used on the ordering decisions.

The results showed that value of information was largely a function of the level of uncertainty the retailer experiences and the sensitivity of its costs to uncertainty. It was also shown the information sharing was generally more beneficial when demand was satisfied with a FIFO depletion policy than with a LIFO depletion policy. In addition, information sharing resulted in a net decrease in retailer replenishment orders due to a reduction in the amount of retailer out-dating and an increase in out-dating at the suppliers facility. At last, the value of information did demonstrate no sensitivity with respect to the order batch size. However, it was considered that such conclusion could be partially misled due to a restricted analysis on the evaluation of scenarios where the order batch size did not significantly exceed the mean demand rate.

Table 2.1: Base literature – Main references used in the literature review of random depletion policies.

Reference	Contribution	Depletion policy	Inventory model considerations	Remarks on depletion policy
Pegels and Jelmert (1970)	Determine the effects of the issuing policies on average inventory levels, which determine blood shortage probabilities, and on the average age of blood at the time it is transfuse.	Hypothetical probability	Demand and supply is stochastic and stationary over time; Depletion policy consisted in hypothetical modified FIFO and modified LIFO issuing policy were considered; The model was applied with theory of absorbing Markov chains is applied to several human-blood-issuing policies.	No theoretical or empirical background was used as evidences about the random depletion policies and its distributions.
Sapountzis (1985)	Studied the demand of a characteristic curve of a blood bank for a particular blood group, which expresses the probability of a unit of blood out-date as a function of the age of blood entering the blood bank.	Uniform distribution	Random depletion policy following a uniform distribution was applied; different set of probabilities were applied for the demand of different blood types; A negative exponential distribution which had its parameter estimated by application of the method of moments and the Erlangian distribution which had its parameter estimated by maximum likelihood estimator.	No theoretical or empirical background was used as evidences about the random depletion policies and its distributions.
Lowalekar et al. (2016)	Aimed at optimum ordering policy parameters considering the order-up-to-level policy for the perishable item with random shelf-life considering of consumer-realized product expiration.	Uniform distribution	Each unit had a xed lifetime; all the units which were not used during their xed useful life were discarded at the end of its shelf-life; a xed cost for every unit that was discarded, a xed cost for every review, a xed ordering cost for every unsatisfied demand unit and a xed cost of shortage for every receivable, holding cost for each unit were charged; no back-order was considered.	Real data from an blood bank was subjected to Wald Wolfowitz runs test for randomness to provide empirical evidences of Randomness. However, no empirical evidence supported the uniform distribution for depletion.
Vaughan (1994)	Worked out an inventory model for determination of inventory ordering and out-date policy for a perishable item with random shelf-life considering of consumer-realized product expiration.	Proportional distribution	Demand was considered to be Poisson distributed and no back-ordering was considered and a order-up-to-level replenishment policy was used to satisfy it. The depletion policy was considered to be random following the proportion of the quantity on stock respective to each shelf-life category. Revenue per unit sold, purchase cost per unit, holding cost per unit, cost per unit shortage, disposal cost per unit outdated and salvage value for outdated units were considered in the model.	It was stated that it is not clear that either the strict FIFO or strict LIFO depletion policy is dominant in an analysis which recognizes both the costs of out-dating and consumer-realized product expiration. The random depletion policy was viewed as a compromise on this issue. However, no theoretical or empirical background was used as evidences about it.
Ferguson and Ketzenberg (2006)	Studied the value of information of the age of the products to be delivery prior to ordering them.	Proportional distribution	Stochastic shelf life of items to be delivered; Constant item utility until zero shelf life; Demand was stochastic and stationary over time; Ordering quantities constrained to multiples of a fixed batch quantity; Lead time for each order was one period; Both FIFO, LIFO and SIRO depletion policy; Lost sales for non-satisfied demand and infinite horizon were considered.	No theoretical or empirical background was used as evidences about the random depletion policies and its distributions and the value of information for the depletion policy was in between than results from LIFO and FIFO, but closer to the LIFO.

2.3   Estimation of censored demand

In the literature, there are several works that address estimation of demand using point of sales data that is usually censored due to limited availability of products on stock. Early studies have focused on the research of Bayesian inventory models assuming that non realized demand was back-ordered. Such problems consider the whole demand observable and therefore uncensored as in Scarf (1959) which applies maximum likelihood estimator considering a Bayes estimate to study the problem conserning demand distributions unknown and known, restricting his attention to exponential family of demand distribution. In an extension of the same study, Iglehart (1964) considers a dynamic inventory problem in which demand distribution possesses a density belonging to exponential or range family of densities. Azoury (1985) worked with the same problem but with the prior demand distribution chosen from the natural conjugate family.

Considering demand uncensored simplifies the problem. However when actual demand is unknown because of censoring form the limited quantity in stock, the problem becomes difficult to solve. Conrad (1976) which studied the estimation of censored sales applying maximum likelihood estimates (MLE) using historical sales data under assumptions of a Poisson distribution shed light on this issue and highlighted that when observations are censored, inventory levels in the current period affects the demand estimate for the next period as well. Nahmias (1994) also applied the MLE method on the same problem under the assumption of normal distribution.

TAN and Karabati (2004) indicated that the MLE procedures work well only when a small fraction of lost sales is present. Hence, there are other methods for demand estimation from observable sales information which comprehends censored demand. For example, some papers proposed data-driven approaches with non parametric modeling to solve the problem of inventory optimization with demand estimation. Burnetas and Smith (2000); Godfrey and Powell (2001); Huh and Rusmevichientong (2009) and Huh et al. (2011) developed adaptive inventory policies based on historical observations and Besbes and Muharremoglu (2013) analyzed the effect–measured in terms of decision makers "regret"–of demand censoring through a non-parametric exploration-exploitation method that used over-ordering to explore the demand distribution, and find exploration to be especially important for integer demand.

Other papers addresses the correction of censoring-induced errors on demand estimation via procedures to uncensor or unconstrain demand data. Wecker (1978), for example, illustrated effects of stock-outs on accuracy of demand forecast in the context of inventory management. Queenan et al. (2007) developed an unconstraining method that employed double exponential smoothing to estimate lost sales. Lau and Lau (1996) worked out a procedure for obtaining the demand distribution from censored data that combines a non-parametric product limit method with extrapolation of hourly sales. Agrawal and Smith (1996) analyze the problem of uncensoring normal and negative binomial demand, respectively.

However, this master thesis sticks to the MLE method in order to also evaluate the statement from TAN and Karabati (2004). Still on the MLE method as pointed out by Vulcano et al. (2010), the data incompleteness derived from the censored demand makes that the estimation of demand distribution parameter(s) becomes too complex to be solved through the standard form of the MLE procedure. The standard form of MLE procedure

requires the calculation of the argument of the maxima of a log-likelihood function. Often when demand is censored by different boundaries as it will be shown throughout this master thesis, this log-likelihood function becomes too complex to be solved analytically or numerically. Moreover, ignoring this censoring can cause a severe bias in estimation.

Thence, several papers have considered the problem of estimating demand or a consumer choice model using the expectation-maximization (EM) algorithm, which is outlined in Dempster et al. (1977) at various levels of generality, to find the maximum likelihood estimator. Anupindi et al. (1998) applied the EM algorithm with POS data to estimate demand rates and substitution probabilities considering demands censored by stock-outs. Talluri and Van Ryzin (2004) developed an estimation procedure based on EM algorithm on the estimation of general choice models from POS data when no-purchase outcomes are unobservable. Kök and Fisher (2007) used the EM method to estimate substitution probabilities along with demands of products in each store within a retail context. Vulcano et al. (2012) combined a multinomial logit choice, nonhomogeneous Poisson and multiperiods model to the EM algorithm to estimate demand and general substitution also in a retail context. Conlon and Mortimer (2013) also did use the EM algorithm to estimate substitution probabilities in a retail context. van Ryzin and Vulcano (2017) used the EM algorithm to estimate demand and substitution considering a rank-based choice model of demand. In a quite original work, Stefanescu (2009) developed an approach for estimating the parameters of the demand models from censored sales data using the EM algorithm considering inter-demand correlation that comprise not just product substitution in an airline industry context.

No research focused on the attempt to estimate the customer behavior regarding preferences of remaining shelf-life of the items was found, either without or with upwards and downwards substitution. That is one among many reasons that make this topic of interest of this master thesis.

Chapter 3

Method and basic theory

This chapter focuses on the methods and theories and why they were considered important and relevant to this thesis. In order to achieve an acceptable level of academic rigour and to facilitate the comprehension of readers and assessors with common and trusted research methods, theories and structures; the guidelines from the supporting portal provided by NTNU for academic writing `https://innsida.ntnu.no/oppgaveskriving` and the book Bryman (2016) were used in the formulation of this document and reported in this chapter.

At first, some theoretical considerations are presented in section 3.1. A brief explanation about the theoretical approach of the literature review is outlined in section 3.2. At last, a general overview of specific theories and methods used throughout this master thesis to address the problem statement and answer the research questions are presented in the further sections of this chapter. Section 3.4 contains an overview of the main concepts within probability theory; section 3.5 reviews and presents the maximum likelihood method; section 3.6 also reviews and presents the Expectation-Maximization algorithm, section 3.7 presents the Monte Carlo method and section 3.8 provides fundamental theory about replenishment policies in inventory management theory and the replenishment policies utilized in this thesis.

3.1 Theoretical considerations and concepts

Theory is a malleable term with a complex definition (Bryman, 2016, p. 18). However, two forms of theory stand out as a good attempt to characterize it: Grand theory, term coined in Mills (1959), which refers to the form of highly abstract and general theorizing in which concepts and the formulation of these concepts are prioritized over understanding the social world and the events subjected to it; and the middle range theory from Merton (1968), which operates in a more limited domain more integrated to empirical research (Bryman, 2016, p. 19). Therefore, the middle range theory was naturally the theory form utilized in the research reported in this document due to its empirical nature of modeling and simulate realistic events of inventory management.

Once the type of theory is established, another concern that deserves careful attention in order to fulfill the desired level of academic rigor is how theory is connected to the research in question. Following the same concepts from middle range theory, Bryman (2016) points out two forms of this theory-research-link: Deductive and inductive strategies. In the first, the research is drawn based on what is known about a particular domain and on relevant proven theoretical concepts. In the last, the implication of the findings is used to support the theory formulation.

The relevance of this matter is supported not by whether one chooses the deductive or inductive approach for his or her research. These two approaches are often complementary. A research is usually planned and grounded to a theory, following a deductive approach, and consecutively its findings follow an inductive approach to fulfill a constructive contribution to the respective theory domain. Indeed, the importance of bringing it to light lies on how the research is structured to effectively achieve the inductive goal of providing a reliable contribution to the relevant theory.

In the deductive approach, theory relates to research adopting usually the following common path: 1. Theory, 2. Hypothesis, 3. Data collection, 4. Findings, 5. Hypothesis confirmed or rejected and 6. Revision of theory (Bryman, 2016, p. 21). Three highlights about this path should be mentioned.

Although, the inductive approach can be identified in this path (in the link from point 5 to 6), this approach is considered by Bryman (2016) as purely deductive. Hereby, this path is also considered as deductive approach to facilitate the comprehension and maintain the methodology in line with the main reference.

In addition, when this approach, which is usually associated to quantitative research, is applied, often it does not follow the sequence presented above. For instance, data may need to be collected in order to support the hypothesis which are validated by the findings resulted from more data collection. The deductive approach does not follow a linear pattern although it looks like that it does. Often, previous steps must be revisited during the performance of the successive steps in order to adapt the research to the unforeseen instances later observed.

The research documented in this master thesis follows a similar path as the common one presented above. A pre-established theory in inventory management and a set of statistical methods which are the foundation of the research comprise the theory step. The coupling of this methods to the inventory management context and its pre-established theoretical background in an attempt to answer the elected problem statement and research questions are used for data collection. The findings from an analysis of the data collection are then used to support the conclusion that derives from the analysis.

As also argued by Crowther and Lancaster (2008) deductive approach is intrinsically related to the positivism. In an epidemiological reach i.e. what is regarded as acceptable knowledge in a discipline (Bryman, 2016, p. 24) opposing the interpretivist, the positivist approach advocates the application of the methods of the natural science to the study of social reality and beyond. Frequently in the literature, this term stretches beyond this description. However, positivism can entail the following principles: Only knowledge confirmed by specific set of phenomena serve as basis to theory used in the research; the purpose of the theory is to develop hypothesis that can be tested and used to complement theory; science is value-free and research is focused on facts instead of meaning. Therefore

from an epidemiological perspective, this master thesis has adopted a positivist approach.

From the ontological point of view, i.e. the set of assumptions one holds about the nature of reality (Blaikie, 1993, p. 6), this theses is embraced by the objectivist position. There are two main ontological positions in social science (Bryman, 2016, p. 29). In the construcionism, social entities such as organization and culture are constructions built up from the perceptions and actions of social actors. In contrast, the objectvist position sees the social entities as objective entities that have reality external to social actors. The social actors have no influence on the social entities in the objectivism.

Another theoretical aspect that was considered in the planning and strategizing of this project, although it can be considered conspicuous, is whether quantitative or qualitative methods should be used. The point of departure was the mathematical modelling of an inventory system which consists of uncertainty variables that require statistical theory and simulations to generate random sampling and analyze the generated data. Thus naturally, the work for the formulation of this master thesis was carried out utilizing a quantitative approach. Moreover, the deductive, positivism and objectivism approaches are fundamentally related to the quantitative strategy as displayed in the table 3.1.

Table 3.1: Fundamental differences between quantitative and qualitative research strategies (Bryman, 2016, pp. 32)

	Quantitative	Qualitative
Principal orientation to the role of theory in relation to research	Deductive, testing of theory	Inductive, generation of theory
Epistemological orientation	Natural science model, in particular positivism	Interpretivism
Ontological orientation	Objectivism	Construcionism

The choice of research strategy is not arbitrary, as it depends on the problem being researched. In this dissertation, the problem statement is related to how customer preferences with regards to the freshness of demanded products available on the shelves can be estimated in an inventory model with complete upwards and downwards substitution in a retail context. Pre-existent theory of inventory modelling and numerical methods to estimate parameters of a censored distribution within a stochastic system is adopted in an original setting. Therefore, a blending between quantitative and qualitative approach is adopted aiming at the best outcome through an efficient process. However the choice for the quantitative, positivist and objectivist approach as the basis for the research presented in this dissertation is considered to be a natural choice.

3.2 Theoretical approach of literature review

The review of literature was planned to be initially extended to a thorough reach of inventory management for perishable products in supply chain and operations management research, the elected theme for this specialization project. This was done with the intent to

gather enough information about how inventory management of perishable items had been researched until the development of this work and what were the possible approaches to the theme that would provide academic relevance to the results. Thence, a reliable foundation for the formulation of the problem statement and research question could be established and used as supporting tool for the complete research. This review is included in this document since it provides a comprehensive overview of the theme to the reader and it shows why the problem statement was considered relevant from an academic perspective.

Once the gathered information was considered to be enough for the formulation of the problem statement and research questions, a more focused literature review was performed, now aiming at specific theories and methods considered relevant to answer the problem statement and research questions. In addition, part of the literature studied was used as a literature base that was used in the design of experiment developed hereby.

The searching for relevant literature was performed primarily using web based search engines and data basis. At first, the searches for relevant literature were performed in Scopus, Web of Knowledge and Google Scholar using combination of more general keywords related to the theme for the first broad literature review. The relevant literature was classified following a systematic review as described in Bryman (2016), and structured using the fundamental mathematical formulation commonly used in inventory control modeling of perishable items as a starting point. The considered main ramifications from the fundamental mathematical formulation was covered in the literature review. This formed a basis for the problem statement and the model. The same was done for the more specific literature review about depletion policies adopted in relevant literature associated with inventory theory and the review about estimation of censored data.

3.3 Mathematical formulation

The mathematical model used in this project followed a similar formulation as the one presented in the equations introduced in the literature review, section 2. In this type of formulation, a state vector $\mathbf{x_t}$ describes the state of the system with respect to t. The future state of this system is a function of the state $\mathbf{x_t}$ and other state variables.

A function is a relation from a set of variables, the domain, to another set of variables, the co-domain, that satisfies two properties: (1) Every element in the domain is related to some element in the co-domain , and (2) No element in the domain is related to more than one element in the co-domain. This formulation can be mathematically generalized by $f : X \rightarrow Y$ where f is the function, X the domain and Y the co-domain. In other words, $f(x)$ is the output of f for the input x where $f = \{y \in Y \mid y = f(x)$, for some $x \in X$. (Epp, 2010, p. 384)

Depending on the model, the state can change continuously in t where $t \in \mathbb{R}$ or, as in the current case, in discrete time increments, t where $t \in \mathbb{Z}$. An example of a function which describes the evolution of the system in a discrete time is

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, q_t, s_t, w_t), \tag{3.1}$$

which is the representation of the transfer equation represented by the equation 2.1.

3.4 Probability theory

It is a challenge to explain concepts of a vast field such as probability theory succinctly. There are numerous different approaches to similar concepts and the definition of some concepts may need multiple other definitions. Therefore in the further lines, only the main concepts utilized in this master thesis are defined.

Maybe the most extensive cornerstone of probability theory necessary for this master thesis is the definition of the probability space which is a triplet $(\Omega, \mathcal{H}, \mathbb{P})$ where Ω is a set, $\mathcal{H}$ is the a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{H})$. A probability space, $(\Omega, \mathcal{H}, \mathcal{P})$, is basically a mathematical model of a random experiment from which an exact outcome cannot be told in advance. The set Ω also called sample space, stands for the collection of all possible outcomes of the experiment. A subset or event H is said to occur if some outcome or outcomes of the experiment happens to belong to H. The σ-algebra $\mathcal{H}$ is the collection of all such subsets. The elements of $\mathcal{H}$ are called event. Finally for each possible event H contained in all subsets of $\mathcal{H}$ contained in Ω, the chance that H occurs is defined by $\mathbb{P}(H)$ which is the probability that H occurs. (Cinlar, 2011, p. 48)

The inventory model used hereby is subjected to two independent and identically distributed (iid.) random variables, representing the demand for the oldest items and the demand for the newest items. This type of variables represents a set of values which have the same probability distribution and are mutually independent from each other. For example, following the formulation used by Bulinskaya (1964) described in the literature review, in section 2.1.1, considering two random variables d_1 and d_2, they are identically distributed iff $P[D \geq d_1] = P[D \geq d_2]$, $\forall D \in \mathbb{I}$; and they are independent iff $P[D \geq d_2] = P[D \geq d_2 | D \geq d_1]$ and $P[D \geq d_1] = P[D \geq d_1 | D \geq d_2]$, $\forall D \in \mathbb{I}$. (Ross, 2010, p. 48)

The two iid. variables followed a Poisson distribution, which is a discrete distribution, and were therefore subjected to the same probability mass functions and cumulative distribution functions. In general, a probability mass function (PMF) gives the probability that a discrete random variable is exactly equal to a specific value. Different from a probability density function (PDF) which is related to a continuous representation of a system, the PMF is the probability distribution of a discrete random variable (Biggs, 2009, p. 382). For instance, considering that $D : \Omega \to A(A \subseteq \mathbb{R})$ is a discrete iid. random variable defined on a sample space Ω. Then the PMF function $\varphi : A \to [0, 1]$ for D can be generally defined as

$$\varphi(d) = P(D = d) = P(\{\omega \in \Omega : D(s) = d\}) \tag{3.2}$$

where

$$\sum_{d \in A} \varphi(d_1) = 1. \tag{3.3}$$

A cumulative distribution function (CDF) represents the probability of the value of a random variable D being less than or equal to a value d. The CDF of a discrete distribution can be represented by

$$\Phi(d) = P(D \leq d) = \sum_{d_i \leq d} \varphi(d_i) \tag{3.4}$$

3.4.1 Expected values

Due to the random iid. variables, the outcomes of the model studied hereby are subjected to uncertainties and it is not possible to determine the exact output of the system. Therefore, it is necessary to base the analysis with respect to expected values given the statistical inputs of the system. In addition, the method utilized in the relevant estimations uses the expectation of certain values in some of its steps. The expected value of a random variable D for example, E(D), can be generally defined by

$$E(D) = \sum_{d_1 \in \Omega} d_1 f(d_1),$$ (3.5)

which is the average value of a series of realizations of this random variable, provided this sum converges absolutely. If the sum does not converge absolutely, then D does not have an expected value (Grinstead and Snell, 1997, p. 226).

3.4.2 Conditional probability

Another concept that is used in the method adopted in the relevant estimations throughout this master thesis is the conditional probability and conditional distributions. Conditional probabilities can be used to describe dependencies between two events A, $B \in \mathcal{H}$ with respect to a probability measure $\mathbb{P}$ on $\mathcal{H}$. For such case considering $P(B) > 0$, the conditional probability can be defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$ (3.6)

which is called the conditional probability of A given B with respect to $\mathbb{P}$. (Steyer and Werner, 2017, p. 138)

The conditional probability can also be defined by the Bayes' Theorem, or Bayes-Price rule, as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)},$$ (3.7)

which was also used throughout this master thesis. The Bayes' theorem relates the probability of an event to prior knowledge of conditions that might be related to the event. It is derived from Bayes and Price (1763), and it is efficiently revised and explained by Stigler (1982). It also provides the basis for the Bayesian inference methodology which is one approach of statistical inference methodology.

3.4.3 Statistical inference

Statistical inference is one subdivision within the branch statistics which uses sample of data to draw inferences about some aspect of the population (real or hypothetical) from which the data were taken (Garthwaite et al., 2002, p. 1). For example, the use of probability distribution to model an inventory system seeking estimates of the behavior of this system from a sample data set, as carried out in this master thesis, constitutes a statistical

inference analysis. There are two main schools within statistical inference, the Bayesian inference and the Frequentist inference. As already explained, the Bayesian inference methodology uses prior knowledge of conditions that may be related to an event to make predictive inferences about the respective event. The Frequentist inference on the other hand is based on hypothetical repetitions of the underlying sampling experiment (Held and Bov, 2014, p. 51).

The most relevant difference between the Bayesian and Frequentist inferences, at least for this master thesis, is that the parameters to be estimated have fixed values and the predictive inference are performed in terms of frequency of these values over real or hypothetical repetitions of an experiment in the Frequentist approach. In the Bayesian approach, the same parameters are associated to probabilities that compose the prior knowledge to be used in the predictive inferences.

3.5 Maximum likelihood estimation method

The estimates performed hereby used concepts derived from the maximum likelihood estimation (MLE) method which is basically a method that aims at the estimation of the parameters of a statistical model from observed data. With its modern version created by Fisher (1922b,a), see Hald et al. (1999) for a revision of the historical routes of the MLE method, and proven by Wilks (1938), the maximum likelihood method fits to the Frequentist inference paradigms since it does not use prior distributions in its estimates.

In order to estimate the distribution parameter(s) of a statistical model, the MLE method departs from a likelihood function that depends on the distribution parameters value and the set of observations that compose the statistical model. A likelihood function is basically a function that provides the probability that certain distribution generates a determined sequence of observations $\mathbf{S} = (s_1, s_2, ..., s_N)$. It is the product of the PMFs, considering discrete distributions and evaluated for this sequence of observations which is given by

$$\mathcal{L}(\mathbf{S}|\theta) = P(\mathbf{S}) = \prod_{t=1}^{N} \varphi_c(s_t|\theta), \tag{3.8}$$

where φ_c is the PMF which hereby denotes a distribution which represent a set of censored observations.

For convenience, the likelihood function is in general substituted by a log-likelihood function. Applying this measure in equation (3.10), its respective log-likelihood is given by

$$l(\mathbf{S}|\theta) = \log \mathcal{L}(\mathbf{S}|\theta) = \log \prod_{t=1}^{N} \varphi_c(s_t|\theta) = \sum_{t=1}^{N} \log \varphi_c(s_t|\theta). \tag{3.9}$$

Therefore, instead of having the product of a series of probabilities, the likelihood is depicted by the summation of a series of probabilities.

Once the censored distribution is established, the estimation of the parameter θ can be performed by finding the parameter θ which maximizes the likelihood. To maximize the

likelihood, the likelihood function shall be differentiated with regards to θ, set to zero, and solved for θ. Hence when the maximum exists, the maximum likelihood estimate is given by

$$\hat{\theta} = \arg \max_{\theta} \sum_{t=1}^{N} \varphi_c(s_t|\theta). \tag{3.10}$$

Several papers have analyzed the consistence and the asymptotic behavior of the MLE such as Wald (1949); Huber et al. (1967); Banker (1993) and Wang and Flournoy (2015); and there are many properties that can be reviewed and analyzed. The most important is the consistency in relation to the size of the sample analyzed since the ML estimator only converges to the its true value only when the number of observations of the sample analyzed goes to infinity. Basically, the more the numbers of observations in the sample analyzed the more consistent and accurate the estimated parameters are. Another consistency property of the MLE is related to the shape of the likelihood function. In multimodal curves for example, there may be more than one maximum. Each maximum is called local maximum which is limited locally within the extension of the domain of the likelihood function, and it may or may not be the global maximum of the function.

Often the calculation the set of parameter(s) that maximizes the likelihood function through a purely analytic approach is not feasible. Hence in this case, one must use numerical methods to estimate the set of parameter(s).

3.6 Expectation-Maximization algorithm

Although not formally entitled and expounded as in its modern version, the expectation-maximization algorithm was explicitly introduced by Hartley (1958) as a procedure for calculating maximum likelihood estimates of discrete and incomplete data sets. In a similar application to the method as the one adopted hereby, the data sets from Hartley's work were considered to follow a discrete statistical model and the observable information was also censored. The censoring applied was rather simpler in contrast to how the data is censored in this master thesis and some examples with different probability distributions such as Poisson and Binomial distributions were studied. Later, Dempster et al. (1977) outlined the EM-algorithm at various levels of generality. Among other outcomes, this work presented a theoretical background for the relation between the monotone behavior of the log likelihood function and the convergence of the algorithm.

As the name suggests, the Expectation-Maximization algorithm (EM-Algorithm) is an iterative method which alternates between an expectation step, the E-step, and a maximization step, the M-step, to find the maximum likelihood estimate of a specific statistical model of a system with censored data. The EM-algorithim starts by the E-step. In the E-step at first, the statistical model with its probability distribution for the censored data is used with a pre-estimated parameter or a set of parameters to calculate the expected value or values for the non-censored data given the conditions in which the data is censored.

Let's consider the sequence of observations presented in section 3.5, the observed data set $\mathbf{S} = (s_1, s_2, ..., s_N)$, and that the observed data lacks information that was censored by any reason. Data can be censored by limitations on the acquaintance of the data, due to

contamination or simply due to loss of data. In addition, let's also consider that the unob-served uncensored data, which are often called latent data, is given by $\mathbf{D} = (d_1, d_2, ..., d_N)$ and that $\mathbf{D}$ is a random variable which is generated by a parameterized probability distri-bution dependent on the unknown set of parameters θ. In the E-step, the expected value of the log-likelihood function with respect to the conditional distribution of the uncensored demand given $\mathbf{S}$ and under the pre-estimate of θ is calculated and given by

$$E_{\mathbf{D}|\mathbf{S},\hat{\theta}}\left[l\left(\mathbf{Z},\mathbf{X},\hat{\theta}\right)\right] = E\left[\log P(\mathbf{D}|\mathbf{S})\right], \qquad (3.11)$$

where $\log P(\mathbf{S}) = \sum_{t=1}^{N} \varphi_c(s_t|\theta)$ and $\log P(\mathbf{D}) = \varphi(d_t|\theta)$. Note that both $P(\mathbf{S})$ and $P(\mathbf{D})$ depend on the set of parameters θ. Consequently, $\log P(\mathbf{D}|\mathbf{S})$ also depends on θ.

Basically and in a rough explanation, the expectation of the uncensored data calculated under the conditions of the censored data in the E-step fills the gaps of the information that was not captured by the observed data. Then once the expectation, or expectations as in this master thesis, are calculated in the E-step, a new parameter that uses the information of the observed uncensored data and the expectation of the non-observed uncensored data that were hidden by the censoring can be estimated. This estimation is conveniently elected as the parameter that maximizes the expectation from equation (3.11), given by

$$\hat{\theta}_{\mathbf{i+1}} = arg\ max\ E_{\mathbf{Z}|\mathbf{X},\hat{\theta}_{\mathbf{i}}}\left[l(\mathbf{Z},\mathbf{X},\hat{\theta}_{\mathbf{i}})\right], \qquad (3.12)$$

where $\hat{\theta}_{\mathbf{i+1}}$ is the estimated set of parameters at iteration $i+1$ and $\hat{\theta}_{\mathbf{i}}$ is its previous estimation.

Both the E-steps and M-steps are iterated until satisfactory convergence is reached. The estimations for the set θ of the M-step from the previous iteration is always used in the E-step of the current iteration. If the algorithm converges, the algorithm should monotonically approach a local maximum of the respective $l(\cdot)$ function.

There are several properties related to the convergence of the EM-algorithm and they are reviewed in detail by McLachlan and Krishnan (2008). Therefore in the following lines, the general convergence of the algorithm is briefly discussed. Since for each iteration of the EM-algorithm the estimated set of parameters θ is the estimation which maximizes the expectation generalized in equation (3.11) which is established under the estimated set of parameters from the previous iteration, for each iteration the respective expectation of function $l(\cdot)$ increases monotonically to a local maximum when the estimation of θ reaches a stationary point. However, this stationary point may not be a local maximum. It is possible for the algorithm to converge to local minima or saddle points in unusual cases as illustrated in McLachlan and Krishnan (2008).

The term local maximum is used above because although an EM iteration does increase the likelihood function of the observed data, it is not assured that the algorithm converges to a global maximum likelihood estimator. To which local maximum the EM-algorithm converges depends basically on the initial pre-estimated values of θ.

3.7 Monte Carlo method

A Monte Carlo simulation was used to determine the expected values of the system studied in this master thesis. A Monte Carlo simulation method estimates values by building discrete models and substitutes a range of values probability distribution for any factor that has inherent uncertainty. It then calculates the output of the model over and over, each time using a different set of respective random inputs from the probability functions. Depending on the number of uncertainties and the ranges specified for the model, a Monte Carlo Simulation could involve thousands or tens of thousands of recalculations before it is complete. Monte Carlo simulation produces distributions of possible outcome values.

In the Monte Carlo method, values are sampled randomly from the input probability distributions. Each set of samples is called an iteration, and the resulting outcome from that sample is recorded. Monte Carlo simulation does it iteratively, and the result is a probability distribution of possible outcomes. In this way, Monte Carlo simulation provides a much more comprehensive view of what may happen. It tells you not only what could happen, but how likely it is to happen. By using probability distributions, variables can have different probabilities of different outcomes occurring (Kalos and Whitlock, 2008, pp.77–101).

3.8 Replenishment policies

The considered most relevant concepts related to inventory modeling for this project were introduced in the literature review chapter, section 2.1. Anyhow, some specific concepts about replenishment decision are formally presented in this section since they play an important role in this master thesis.

The replenishment policies utilized in the inventory modeling of this project are based on periodic reviewing. It means that instead of monitoring the inventory continuously to evaluate if the inventory is low or not, such as in the continuous review, the inventory is modeled considering the position only at certain given points in time (Axster, 2006, p.47). A typical periodic review largely used in the industry and studied by the academia is the (R, Q) policy. In this policy, when the inventory state declines to or below the reorder point R, a batch quantity of order Q is ordered.

Haijema and Minner (2018) which investigated the value of stock-age, or freshness as referred in this master thesis, in different replenishment policies, has presented a comprehensive review of replenishment policies which suits well to this master thesis. The replenishment policies reviewed were divided into three groups: Stock level dependent ordering, stock age dependent ordering and policies that track the time elapsed since the last order. The last group of policy has its ordering control on a determined period of non-purchase and it does not play an important role in this master thesis. However,the other two group of policies do play an important role in this master thesis.

As the name suggests, stock level dependent ordering are policies which set the replenishment decision exclusively on the level of stock at the ordering period. The most fundamental policy of this category is the base stock policy (BSP). In this policy which was extensively investigated by Cohen (1976), a base stock level usually denoted by the term S is used as basis for the ordering quantity. At the ordering period, the ordering quantity

is defined as the quantity necessary to increase the inventory level up to S. If the inventory level is equal or higher than S no new items are ordered. Cohen (1976) analytically studied this policy for a two-period shelf-life with FIFO depletion policy. The optimal value of S requires numerical calculations to be acquired through a pre-established search range considering all possibilities.

An extension of the BSP policy is the SQmax which adds a parameter to the BSP policy. In addition to the base stock level, a maximum order quantity is used to limit the quantity to be ordered even though it may not be sufficient to reach the base stock level. This policy aims at reducing waste resulted from large orders. The optimal values of S and the maximum ordering quantity can be calculated similarly to the calculation of the optimal S in the BSP. Two search domains for each parameter are established and their optimal value are the outcome of the application of a numerical search method as reported in Haijema and Minner (2016).

There are other stock level dependent ordering policies also based on the BSP policy such as the BSP-low which uses three parameter, two base stock levels and one break-point, and aims for example at waste reduction and increase service level. In this case if the stock level is lower than the break-point, a base stock level is used for ordering. If the stock level is higher than the break-point, the other base stock level is used for ordering. However, this goes beyond the scope of this master thesis since for the data generation, as it is going to be explained further, a simple base stock policy is used in an inventory simulation.

The stock-age dependent ordering policies use not just the quantity of inventory at the ordering period but also the remaining shelf-life of each item in stock to determine the ordering quantity. An example of this policy is a base stock policy referred by Karaesmen et al. (2011) as the TIS-NIS policy. In this policy, the inventory items are divided in between old products and new products. Three parameters are used, one to define until which remaining shelf-life the items are considered new, one base stock level for all items and one base stock level for the new items. The ordering quantity is defined as the maximum between zero, the difference between total stock level and base stock level for all items and difference between the quantity of new items in stock and the base stock level for the new items. Again, the search of optimal parameters are performed by using established search range for each parameter and utilizing numerical methods considering all possibilities.

In addition, a stochastic dynamic programming (SDP) methodology may be used as in Fries (1975), Nahmias (1975) and Haijema et al. (2007) to calculate the optimal ordering quantity. In this case, for each possible state, an optimal ordering quantity is determined using value iteration through the adoption of a recursive equation which aims at minimizing costs or maximize profits considering the expected costs or profits for a determined ordering quantity and the future optimal expected cost or profit considering the state at the current period. As mentioned in Haijema and Minner (2018), the practical application of such policy is hampered by the calculation complexity and computational cost of the SDP policy considering that the expected values mentioned must be calculated for all possible states at each ordering period. For problems with many states, approximately heuristics may be utilized.

In this master thesis for the calculations of the impact of the estimated demand in relation to the remaining-shelf life of the items, an heuristic policy using a similar approach to

the SDP methodology is adopted by using the inventory level to assess how much should be ordered to fulfill the expected demand until the delivery of the next potential order in order to maximize profitability. This was performed by adopting a look-ahead sparse sampling tree search policy. As described in Powell (2011, pp. 200-202), the look-ahead policies make a decision at ordering period by solving an approximation of the problem over some horizon. In the sparse sampling approach, statistical sample of outcomes calculated for limited horizon using for example the Monte Carlo online simulation is adopted to stipulate the best policy decision.

Chapter 4

Model development

This chapter presents the modeling adopted in this master thesis and how the methods outlined in chapter 3 were applied to address the problem statements and answer the research questions. The chapter is systematically organized to answer each research question and it is therefore divided in three sections: Inventory model, demand estimation and design of experiment.

Section 4.1 describes how the inventory was mathematically modeled. The most relevant conceptual traits of the inventory model are presented with emphasis on the explanation of how the preference of customers in relation to the remaining shelf-life of available products were modeled. Thereafter in section 4.2, it is described how the estimation methods reviewed from literature, presented at chapter 3, are applied to the specific case and context studied hereby. This section aims at the second research question. Finally, section 4.3 provides a design of a factorial experiment built up to answer the third research question. This design of experiment is divided in three phases.

The first phase from the DOE is designed to generate relevant data from which the estimations were performed. These data were generated through the simulation of the inventory model that is already presented in section 4.1. Therefore, section 4.3.1 presents specifically the replenishment policy and the parameters that were used in the inventory simulations for the data generation. In addition, the fixed and factorized parameters of the simulations that were used in the factorial experiment are presented.

The second phase focuses on the estimate of the demand parameters, which models the customers preferences related to the remaining shelf-life, from the data generated in the first phase considering the demand distributions from the inventory model. Section 4.3.2, presents a convergence test for the EM-Algorithm used to estimate these demand parameters as described in section 4.2. The results and conclusions of the convergence test are presented in addition to the parameters of the EM-Algorithm that are used in the third phase of the design of experiment.

In the third phase, inventory was simulated also adopting the same inventory model, but a stock-age dependent policy was used in this case. The stock-age dependent policy was used to capture the influence of the estimation performed in the second phase. In

this phase, different set of parameters for the distribution of the shelf-life of items on stock were used in addition to the estimated parameters in different experiments and an extra experiment was performed with the replenishment of the first phase for the sake of comparison. It was done for each experiment from the first phase. Therefore, the variants of the first phase jointly to the variants of phase three comprehend the full factorial design of experiment. At the end of section 4.3, the fixed and factorized parameters of the simulations from the third phase are presented. These factorized parameters represent the full factorial experiment designed hereby.

4.1   Inventory Model

This thesis focuses on a realistic setting of a retailer that sells perishable products and receives replenishment from a large supplier. For this, an inventory model which can realistically simulate the inventory of perishables in a retail context was developed. This model was intended to be used to generate similar observed data that can be captured by a modern POS system at grocery stores, for instance. It is assumed a periodic review of a discrete inventory state since this is the most common system used in the grocery industry (Ferguson and Ketzenberg, 2006).

The inventory system from the model developed hereby covers the control of one perishable product stored at a single location that is provided by a single supplier (Single-echelon). It was considered that the available supply was sufficient to cover all orders quantities at the same unit cost.

The items' maximum remaining shelf life is an arbitrary fixed integer M in periods. All ordered items are delivered with the same remaining shelf life M. Once in stock, the remaining shelf life of each item is reduced by one unit each period and once the item reaches a remaining shelf life of 0, the item perishes and cannot be sold anymore.

The demand is discrete, stochastic and stationary following a Poisson distribution with a probability mass function (PMF)

$$\varphi_P(d_t|\Lambda) = \frac{e^{\Lambda}\Lambda^{d_t}}{d_t!},\tag{4.1}$$

and cumulative distribution function (CDF)

$$\Phi_P(d_t|\Lambda) = \frac{\Gamma(d_t+1,\Lambda)}{d_t!},\tag{4.2}$$

where Λ is the mean demand, D_t is an idd. random variable denoting total demand at period t, d_t is its realization at period t and $\Gamma(\cdot)$ is the gamma function.

Each customer has a probability p of picking up the oldest item on the shelf (through a Bernoulli trial) following a FIFO depletion policy. Therefore for a known total demand, the demand for the oldest items follows a binomial distribution $D_{a,t}|D_t \sim Binomial(d_t, p)$. Then, the demand for the oldest products follows a Poisson distribution with mean demand $\lambda_a = \Lambda p$,

$$D_{a,t} \sim \varphi_a(d_{a,t}|\Lambda, p) = \sum_{n=0}^{\infty} \varphi_P(n|\Lambda)\varphi_B(1|n,p) = \frac{e^{\Lambda p}(\Lambda p)^{d_t}}{d_t!} = \frac{e^{\lambda_a}\lambda_a^{d_t}}{d_t!}, \qquad (4.3)$$

where φ_a is the PMF of the demand for the oldest items and φ_B is the Binomial PMF. Analogously considering that each customer has a probability $(1-p)$ of selecting the newest item on the shelf (through a Bernoulli trial) following a LIFO depletion policy, the demand for the newest products follows a Poisson distribution with parameter $\lambda_b = \Lambda(1-p)$,

$$D_{b,t} \sim \varphi_b(d_{b,t}|\Lambda, 1-p) = \sum_{n=0}^{\infty} \varphi_P(n|\Lambda)\varphi_B(1|n,p) = \frac{e^{\Lambda(1-p)}[\Lambda(1-p)]^{d_t}}{d_t!} = \frac{e^{\lambda_b}\lambda_a^{d_t}}{d_t!},$$

$$(4.4)$$

where φ_b is the PMF of the demand for the newest items. Hence, $D_{a,t}$ and $D_{b,t}$ are two independent identically distributed (iid.) random variables denoting demand at period t for the oldest and youngest items respectively which follow each a Poisson distribution with mean demands λ_a and λ_b also respectively. Their realization in period t are denoted by $d_{t,a}$ and $d_{t,b}$. Since the demand for the inventory system modeled in this dissertation is comparatively low, the choice for Poisson distribution in the modeling of demand is based on the recommendations from Syntetos et al. (2009) i.e. modeling of demand with Poisson distribution for low demand cases, as it was explained in the literature review, section 2.1.1.

Each order is periodically placed with an ordering frequency of τ periods. The delivery happens L periods after ordering date. It means that the replenishment quantity, q_t, is ordered at period $t - L$ and delivered at period t.

Items depleted are not returned and there is no backlogging of excess demand.

The unit cost, c_u is constant and does not vary over the periods by any circumstances. The sales price is denoted by c_s and the utility of each product remains constant over its whole life time. Hence, sales price is also constant, and it is not altered by the perishable nature of the product. No fixed ordering cost is accounted in the profits calculation due to transport costs are usually distributed over many stock keeping units (SKUs) in a re-tail context. A linear holding cost, c_h, per unit of inventory stored is charged. The holding cost is also constant, and it is not altered by the perishable nature of the product. No salvage value and no disposal costs are accounted in the profits calculations. The loss due to perishability is accounted in the items purchased and perished that do not have their sales income added to the profits.

For the inventory balance and profits calculations, the order of events in each period follows the sequence: (1) Review inventory and place replenishment order (2) receive delivery, (3) satisfy demand, (4) out-date inventory (5) calculate profits, waste and stock-outs. Since the orders must always be placed τ periods after the previous ordering period, let establish the following relation:

$$q_t = \begin{cases} 0 & t \neq \alpha\tau \\ \gamma & t = \alpha\tau \end{cases}, \quad \text{where } \alpha \in \mathbb{Z}^+, \ \gamma \in \mathbb{Z}^* \ . \qquad (4.5)$$

Considering $(y)^+ = max(y, 0)$, the inventory balance equations for FIFO depletion policy and LIFO depletion policy are respectively:

FIFO

$$x_{m,t+1} = \begin{cases} q_{t+1-L} & m = M \\ \\ \left(\left(x_{m+1,t} - \left(d_{t+1} - \sum_{z=1}^{m} x_{z,t} \right)^+ \right)^+ \right)^+ & m < M \end{cases}$$ (4.6)

LIFO

$$x_{m,t+1} = \begin{cases} q_{t+1-L} & m = M \\ \\ = \left(\left(x_{m+1,t} \left(d_{t+1} - \sum_{z=m+2}^{M} x_{z,t} \right)^+ \right)^+ \right)^+ & m < M \end{cases}$$ (4.7)

where the vector of ending inventory in the period t, $\mathbf{x_t} = \{x_{t,1}, x_{t,2}, ...x_{t,M}\}$ at period t, represents the inventory state i.e. the vector which contains the inventory held for each remaining shelf life until the maximum shelf life, M. The demand in period t is d_t, $q_t + 1 - L$ denotes the quantity of products ordered at period $t+1-L$. Since in the inventory model used in this thesis it is considered demand for both FIFO and LIFO depletion policies, a combined version of the balance equation, $\xi(\mathbf{x}_t, d_{a,t+1}, d_{b,t+1}, q_{t+1-L})$ where $\xi(\mathbf{x}_t, d_{a,t+1}, d_{b,t+1}, q_{t+1-L}) = \mathbf{x}_{t+1}$, shows necessary. This balance equation is given by

FIFO and LIFO combined

$$x_{m,t+1} =$$

$$\begin{cases} q_{t+1-L} & m = M \\ \\ \left(\left(x_{m+1,t} - \left(d_{b,t+1} - \sum_{z=m+2}^{M} x_{z,t} \right)^+ - \left(d_{a,t+1} - \sum_{z=1}^{m} x_{z,t} \right)^+ \right)^+ \right)^+ & m < M \end{cases}$$ (4.8)

where $d_{t+1,a}$ represents the demand for the oldest item,FIFO depletion policy, and $d_{t+1,a}$ represents the demand for the youngest item,LIFO depletion policy; both at period $t+1$. The total demand for each period t is the sum of both a and b demands, $d_t = d_{t,a} + d_{t,b}$ respectively.

The equation for the expected profit incurred in each period is

$$\pi_t(X_t, q_t) = c_s \sum_{d_t=0}^{X_t} d_t \, \varphi_P(d_t|\Lambda) - c_h \sum_{d_t=0}^{X_t} (X_t - d_t) \, \varphi_P(d_t|\Lambda) - c_u q_t,$$ (4.9)

where $X_t = \sum_{i=0}^{M} x_{t,i}$ is the total quantity of items in inventory considering all remaining shelf-life categories. The first term of equation (4.9) represents the income due to sold items predicated only on $\varphi_P(\cdot)$. The second term represents the holding cost of current inventory also predicated on $\varphi_P(\cdot)$. At last, the third term represents the ordering cost which depends on the unit cost and the quantity ordered in each period.

The setting analyzed in this master thesis comprehends the inventory system of a product with maximum remaining shelf life of two periods in a finite horizon of T periods. Both delivery time and ordering frequency are equal to one period. Therefore, all equations presented in the further sections are modeled for the specific scenario in which $L = 1$, $\tau = 1$, $M = 2$ and with a finite value of T.

4.2 Demand estimation

The demand parameters are estimated from POS observed data available. Thence considering that the retail store has the control over the quantities of items and their remaining shelf-life in stock, the quantity of sold products per remaining shelf-life is the main observed data that can be used for estimation. This realized sales depicts the demand which is censored by the available quantities of items in stock which are also observed.

The quantity of items with remaining shelf-life equals to one period and remaining shelf-life equals to two periods are given by equations (4.8) and can be simplified to

$$x_{1,t+1} = \left(x_{2,t} - d_{b,t} - (d_{a,t} - x_{1,t})^+ \right)^+ \tag{4.10}$$

and

$$x_{2,t+1} = q_t \tag{4.11}$$

when M=2 and L=1.

The realized sales or censored demand for the product a, the oldest product available correspondent to the items with remaining shelf-life equals to 1 period and for the product b, the newest product available correspondent to items with remaining shelf-life equals to 2 periods are respectively censored and given by

$$r_{1,t} = x_{1,t} - \left((x_{1,t} - d_{a,t})^+ - (d_{b,t} - x_{2,t})^+ \right)^+ \tag{4.12}$$

and

$$r_{2,t} = x_{2,t} - \left((x_{2,t} - d_{b,t})^+ - (d_{a,t} - x_{1,t})^+ \right)^+. \tag{4.13}$$

which can be rewritten as

$$r_{1,t} = \begin{cases} x_{1,t} & d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t} \\ x_{1,t} & d_{a,t} \geq x_{1,t} \text{ and } d_{b,t} < x_{2,t} \text{ and } d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \\ d_{a,t} + d_{b,t} - x_{2,t} & d_{a,t} < x_{1,t} \text{ and } d_{b,t} \geq x_{2,t} \text{ and } d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \\ d_{a,t} & d_{a,t} < x_{a,t} \text{ and } d_{b,t} < x_{2,t} \end{cases}$$

$$\tag{4.14}$$

and

$$
r_{2,2} = \begin{cases}
x_{2,t} & d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t} \\
x_{2,t} & d_{a,t} < x_{1,t} \text{ and } d_{b,t} \geq x_{2,t} \text{ and } d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \\
d_{a,t} + d_{b,t} - x_{1,t} & d_{a,t} \geq x_{1,t} \text{ and } d_{b,t} < x_{2,t} \text{ and } d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \\
d_{b,t} & d_{a,t} < x_{1,t} \text{ and } d_{b,t} < x_{2,t}
\end{cases}
$$

$$(4.15)$$

Do note that there is upwards and downwards substitution with relation to the remaining shelf-life. It means that when there is stock-out for the oldest item, any remaining stock of the newest item, after the complete demand for the newest item is satisfied, is used to satisfy the demand for the oldest item and vice versa.

In the current setting, there are two censored distributions which are the object of study: distribution of censored demand for the product a and distribution of censored demand for the product b which are respectively the distributions for the realized sales presented by equations (4.14) and (4.15). The parameters Λ and p are unknown and the objects of interest i.e. the parameters to be estimated. In addition as it was already indicated, these parameters are correlated with λ_a and λ_b as follows:

$$\Lambda = \lambda_a + \lambda_b, \tag{4.16}$$

$$\lambda_a = \Lambda p, \tag{4.17}$$

$$\lambda_b = \Lambda p(1-p). \tag{4.18}$$

Therefore by estimating λ_a and λ_b, Λ and p are also estimated.

The estimation of λ_a and λ_b from the observed data can be done using the maximum likelihood method as outlined in section 3.5, which is a method that applies a log-likelihood function related to the distribution of censored observations to find the parameters of the uncensored distributions. The likelihood functions for the observed data of the model presented in this section are given by

$$\mathcal{L}(\mathbf{r}|\boldsymbol{\lambda_m}) = \sum_{t=1}^{T} \log \boldsymbol{\varphi_c}(\mathbf{r_t}|\boldsymbol{\lambda_m}), \tag{4.19}$$

where $\mathbf{r}$ denotes the set $\{\mathbf{r_1}, \mathbf{r_2}, ...\mathbf{r_t}, \mathbf{r_t}+\mathbf{1}, ...\mathbf{r_T}\}$, $\boldsymbol{\varphi_c}$ is the vector of the PMFs of the censored observations which depend on the censored observations denoted by $\mathbf{r_t} = \{r_{1,t}, r_{2,t}\}$ and the mean demand vector $\boldsymbol{\lambda_m}$.

Once the censored distribution is established, the estimation of the set of parameters $\boldsymbol{\lambda_m}$ can be done by finding the set of parameter $\boldsymbol{\lambda_m}$ which maximizes the likelihoods from equation (4.19). To maximize these likelihoods, the log-likelihood functions shall be differentiated with regards to $\boldsymbol{\lambda_m}$, set to zero, and solved for $\boldsymbol{\lambda_m}$. Therefore when the maximum exists, the maximum likelihood estimate is given by

$$\hat{\lambda}_m = \arg\max_\lambda \sum_{t=1}^{T} \log \boldsymbol{\varphi_c}(\mathbf{r_t}|\boldsymbol{\lambda_m}), \tag{4.20}$$

where $\boldsymbol{\lambda_m} = \{\lambda_a, \lambda_b\}$.

Hereby, the demands $d_{a,t}$ and $d_{b,t}$ are censored by $x_{1,t}$ and $x_{2,t}$, and the censored observations are represented by the realized sales $r_{1,t}$ and $r_{2,t}$ respectively. In order to facilitate the formulation and understanding of the problem, the observed censored demands are divided over four events that are classified with regard to how the demand is censored by the inventory quantities. These events follow the conditions displayed in equations (4.14) and (4.15). For each event, one censored distribution for product a, φ_{ca}, and one censored distribution for product b, φ_{cb}, is given. Both of them constitute the set φ_c which is then equal to $\{\varphi_{ca}, \varphi_{cb}\}$ as outlined in equation (4.19). The events, its conditions and the respective censored distributions are presented and explained below.

Event $\overline{AB}$ - Fully censored demand:
The condition which characterizes this event is $d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}$. It means that in this event the sum of the demands are equal or higher than the sum of the inventory quantities and both demands are censored at a specific period t. Thence, the realized sales for the oldest and newest items are respectively equal to the quantity in stock of the oldest and newest items, $r_{1,t} = x_{1,t}$ and $r_{2,t} = x_{2,t}$. The only valuable information in terms of probability is that the probability of both censored demands (or realized sales) to achieve this specific state has the same probability as for the sum of the demands being equal to the sum of order quantities. Therefore, both the censored distribution for the censored demand of the oldest items, φ_{ca}, and the censored distribution of the demand for the newest items, φ_{cb}, are the cumulative distribution function for $P(d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})$ which are given by

$$\varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t}, \lambda_a, \lambda_b) = 1 - \Phi_p(x_{1,t} + x_{2,t} | \lambda_a + \lambda_b) \tag{4.21}$$

$$\varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t} | \lambda_a, \lambda_b) = 1 - \Phi_p(x_{1,t} + x_{2,t} | \lambda_a + \lambda_b). \tag{4.22}$$

Event $\overline{A}B$ - Partially censored demand with contamination from A to B:
The conditions which characterize this event are $d_{a,t} \geq x_{1,t}$ and $d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}$. It means that in this event the demand for the oldest items is equal or higher than the quantity of the oldest items in stock and the sum of both demands is lower than the total quantity of all items in stock. That also means that the demand for the newest items is lower than the quantity of the newest items in stock which shall also be sustained for this event to happen, but this condition is implicit in the condition established for the sum of demands and for the demand of the oldest items. Therefore for a specific period t, the demand for product a is censored and the demand from product b is partially censored since it has contamination from the excess of demand from product a (excess of $d_{a,t}$ in relation to $x_{1,t}$). The realized sales for the oldest items is equal to the quantity in stock of the oldest items and the realized sales for the newest items is less than the quantity in stock of the newest items, $r_{1,t} = x_{1,t}$ and $r_{2,t} < x_{2,t}$.

The demand for the product a is again fully censored and its distribution also consists of a cumulative distribution. In this case, the censored distribution of product a must follow the two conditions established and the probability of $r_{1,t}$ being equal to $x_{1,t}$ is therefore

equals to $P(d_{a,t} \geq x_{1,t} \cap d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$. On the other hand, the demand of product b is partially censored i.e. a portion of its value corresponds to the total demand of product b and the rest corresponds to the excess of demand that was not fulfilled by $x_{1,t}$. In this case, the censored distribution of product b is equal to $P(d_{a,t} \geq x_{1,t} \cap d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \cap d_{a,t} + d_{b,t} < x_{1,t} + r_{2,t})$. The censored demand distributions are given by

$$\varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t}, \lambda_a, \lambda_b) = \sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} \omega(i,j) \tag{4.23}$$

$$\varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t} | \lambda_a, \lambda_b) = \sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} \kappa_b(i,j) \omega(i,j) \tag{4.24}$$

where $\kappa_a(i,j) = \boldsymbol{\delta}(i + j - r_{1,t} - x_{2,t})$, $\kappa_b(i,j) = \boldsymbol{\delta}(i + j - r_{2,t} - x_{1,t})$, $\omega(i,j) = \varphi_p(i | \lambda_a) \varphi_b(j | \lambda_b)$ and $\boldsymbol{\delta}(\cdot)$ is the Kronecker delta.

Event $A\bar{B}$ - Partially censored demand with contamination from B to A:

This event is symmetric to the event $\mathbf{A}\bar{\mathbf{B}}$. The conditions which characterize it are $d_{b,t} \geq x_{2,t}$ and $d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}$. It means that in this event the demand for the newest items is equal or higher than the quantity of the newest items in stock and the sum of both demands is lower than the total quantity of all items in stock. That also means that the demand for the oldest items is lower than the quantity of the oldest items in stock which shall also be sustained for this event to happen, but this condition is implicit in the condition established for the sum of demands and for the demand of the oldest items. Therefore for a specific period t, the demand for product b is censored and the demand from product a is partially censored since it has contamination from the excess of demand from product b (excess of $d_{b,t}$ in relation to $x_{2,t}$). The realized sales for the oldest items is less than the quantity in stock of the oldest items and the realized sales for the newest items is equal to the quantity in stock of the newest items, $r_{1,t} < x_{1,t}$ and $r_{2,t} = x_{2,t}$.

The demand for the product b is fully censored and its distribution also consists of a cumulative distribution. In this case, the censored distribution of product b must follow the two conditions established and the probability of $r_{2,t}$ being equal to $x_{2,t}$ is therefore equals to $P(d_{b,t} \geq x_{2,t} \cap d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$. On the other hand, the demand of product a is partially censored i.e. a portion of its value corresponds to the total demand of product a and the rest corresponds to the excess of demand that was not fulfilled by $x_{2,t}$. The censored distribution of product a is equal to $P(d_{b,t} \geq x_{2,t} \cap d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t} \cap d_{a,t} + d_{b,t} < r_{1,t} + x_{2,t})$. The censored demand distributions are given by

$$\varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t}, \lambda_a, \lambda_b) = \sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} \kappa_a(i,j) \omega(i,j) \tag{4.25}$$

$$\varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t} | \lambda_a, \lambda_b) = \sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} \omega(i,j). \tag{4.26}$$

Event AB - Fully uncensored demand:
The conditions which characterize this event are $d_{a,t} < x_{1,t}$ and $d_{a,t} < x_{2,t}$. It means that in this event the demand for the oldest items is lower than the quantity of the oldest items in stock and that the demand for the newest items is less than the quantity of the newest items in stock and both demands are fully uncensored at a specific period t. Thence, the realized sales for the oldest and newest items are respectively less than the quantity of the oldest and newest items. With both demands lower than their respective inventory quantities, both censored observations, the realized sales, for the oldest and newest items are respectively equal to the demands for the oldest and newest items, $r_{1,t} = d_{a,t}$ and $r_{2,t} = d_{b,t}$. The censored distribution for product a in this case is the probability of the demand for product a being equal to the realized sales of product a at the same time that the demand for product b is less than its quantity in stock, $P(d_{a,t} = r_{1,t} \cap d_{b,t} < x_{2,t})$. Analogously, the censored distribution for product b in this case is the probability of the demand for product b being equal to the realized sales of product b at the same time that the demand for product a is less than its quantity in stock, $P(d_{b,t} = r_{2,t} \cap d_{a,t} < x_{1,t})$. The censored demand distributions are given by

$$\varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t}, \lambda_a, \lambda_b) = \varphi_p(r_{1,t}|\lambda_a)\Phi_p(x_{2,t} - 1|\lambda_b) \tag{4.27}$$

$$\varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t}|\lambda_a, \lambda_b) = \Phi_p(x_{1,t} - 1|\lambda_a)\varphi_p(r_{2,t}|\lambda_b) \tag{4.28}$$

The distribution of each censored demand for each product is given as the union of the probability distributions of their respective censored demand for each event as outlined below:

$$\varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t}, \lambda_a, \lambda_b) =$$

$$\begin{cases} 1 - \Phi(x_{1,t} + x_{2,t}|\lambda_a + \lambda_b) & r_{1,t} = x_{1,t} \text{ and } r_{2,t} = x_{2,t} \\ \sum\limits_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum\limits_{j=0}^{x_{1,t}+x_{2,t}-i-1} \varepsilon(i,j) & r_{1,t} = x_{1,t} \text{ and } r_{2,t} < x_{2,t} \\ \sum\limits_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum\limits_{i=0}^{x_{1,t}+x_{2,t}-j-1} \kappa_a(i,j)\varepsilon(i,j) & r_{1,t} < x_{1,t} \text{ and } r_{2,t} = x_{2,t} \\ \varphi_p(r_{1,t}|\lambda_a)\Phi_p(x_{2,t} - 1|\lambda_b) & r_{1,t} < x_{1,t} \text{ and } r_{2,t} < x_{2,t} \end{cases} \tag{4.29}$$

and

$$\varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t} | \lambda_a, \lambda_b) =$$

$$
\begin{cases}
1 - \Phi(x_{1,t} + x_{2,t} | \lambda_a + \lambda_b) & r_{1,t} = x_{1,t} \text{ and } r_{2,t} = x_{2,t} \\[2mm]
\displaystyle\sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} \kappa_b(i,j)\varepsilon(i,j) & r_{1,t} = x_{1,t} \text{ and } r_{2,t} < x_{2,t} \\[2mm]
\displaystyle\sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} \varepsilon(i,j) & r_{1,t} < x_{1,t} \text{ and } r_{2,t} = x_{2,t} \\[2mm]
\varphi_p(r_{1,t} | \lambda_a)\Phi_p(x_{2,t} - 1 | \lambda_b) & r_{1,t} < x_{1,t} \text{ and } r_{2,t} < x_{2,t}
\end{cases}
\tag{4.30}
$$

With the censored distributions available, the estimation of λ_a and λ_b can be performed by finding the set of parameters which maximizes the log-likelihood functions of $l(\mathbf{r_1}, \mathbf{x_1}, \mathbf{x_2} | \hat{\lambda}_a, \hat{\lambda}_b)$ and $l(\mathbf{r_2}, \mathbf{x_1}, \mathbf{x_2} | \hat{\lambda}_a, \hat{\lambda}_b)$ as outlined in section 3.5. For the current setting, these likelihood functions are respectively given by

$$l(\mathbf{r_1}, \mathbf{x_1}, \mathbf{x_2} | \hat{\lambda}_a, \hat{\lambda}_b) = \sum_{t=1}^{T} \log \varphi_{ca}(r_{1,t}, x_{1,t}, x_{2,t} | \hat{\lambda}_a, \hat{\lambda}_b) \tag{4.31}$$

and

$$l(\mathbf{r_2}, \mathbf{x_1}, \mathbf{x_2} | \hat{\lambda}_a, \hat{\lambda}_b) = \sum_{t=1}^{T} \log \varphi_{cb}(r_{2,t}, x_{1,t}, x_{2,t} | \hat{\lambda}_a, \hat{\lambda}_b), \tag{4.32}$$

where $\mathbf{r_1} = (r_{1,1}, r_{1,2}, ..., r_{1,T})$, $\mathbf{r_2} = (r_{2,1}, r_{2,2}, ..., r_{r,T})$, $\mathbf{x_1} = (x_{1,1}, x_{1,2}, ..., x_{1,T})$ and $\mathbf{x_2} = (x_{2,1}, x_{2,2}, ..., x_{2,T})$ are respectively the vectors for the observed demand a, observed demand b, inventory quantity for the product with remaining shelf-life equals to one and inventory quantity for the product with remaining shelf-life equals to two. In addition, $\hat{\lambda}_a$ and $\hat{\lambda}_b$ are the maximum likelihood estimators for the mean demands a and b. These maximum likelihood estimators are then given by the values which satisfy the multivariate system formed by equations 4.31 and 4.32.

Due to the presence of some peculiarities in both censored distributions such as the Gamma functions for example, but mainly because the system formed by equations (4.31) and (4.32) have Markovian property i.e. each observation from vectors $\mathbf{x_1}$ and $\mathbf{x_1}$ depend on the state of the previous observation, it shows necessary the use of numerical methods to find the solution of this system. Hereby, the maximum likelihood estimators are pursued by using the expectation-maximization algorithm (EM-Algorithm) which was outlined in section 3.6.

To estimate the λ_a and λ_b through the EM-algorithm, the following four steps were applied:

1. **Initializing** - All non-censored observations are used to estimate λ_a and λ_b by calculating MLE of non-censored data points available for Poisson which is basically the average of non-censored observations of the censored demand. The non-censored observations are all set of observable data points of the censored demand $\{r_{1,t}, r_{2,t}\}$

correspondent to the event AB. The first estimated mean demands are then given by $\hat{\lambda}_a = \{\bar{r}_{1,t}|r_{1,t} \in \Theta_{AB}\}$ and $\hat{\lambda}_b = \{\bar{r}_{2,t}|r_{2,t} \in \Theta_{AB}\}$ where Θ_{AB} is the state space of the set $\{r_{1,t}, r_{2,t}\}$ correspondent to the event AB.

2. **E-Step** - The conditional expectations of non-censored demand given each event where the demand is censored under the respective lambda estimates, $\hat{\lambda}_a$ and $\hat{\lambda}_b$, are calculated. Among the events that depict how the demand is censored by the inventory quantities, there are three in which demand is censored differently (events $\overline{AB}$, $\bar{A}B$ and $A\bar{B}$). New conditional expectations of non-censored demand for both demands, $d_{a,t}$ and $d_{b,t}$, are calculated for each set of censored observation $\{r_{1,t}, r_{2,t}\}$ correspondent to these events.

3. **M-Step** - Each set of censored observation $\{r_{1,t}, r_{2,t}\}$ correspondent to their respective events are replaced by their respective conditional expectations i.e. following the conditions that characterize each event. The new $\hat{\lambda}_a$ and $\hat{\lambda}_b$ are obtained by using the MLE for Poisson without censoring on the whole data-set (non-censored values, combined with conditional expectations).

4. **Convergence evaluation** - Steps 2 and 3 are iteratively repeated until satisfactory convergence is achieved.

Figure 4.1 displays a basic schematics showing how the EM-algorithm was implemented. At first, a fixed number of inventory simulations, I, were run and their output were stored in different data-sets, one for each simulation. As it was already indicated throughout this master thesis, the main data-sets that are used from the simulation on the studied estimations are the inventory quantity for the items for each shelf-life available in stock per period, $\mathbf{x} = \{\{x_{1,1}, x_{2,1}\}, \{x_{1,2}, x_{2,2}\}, ..., \{x_{1,T}, x_{2,T}\}\}$; which of these items that were sold per period, $\mathbf{r} = \{\{r_{1,1}, r_{2,1}\}, \{r_{1,2}, r_{2,2}\}, ..., \{r_{1,T}, r_{2,T}\}\}$; and the first estimate of λ_a and λ_b. Each of these data-sets, of each inventory simulation, are then applied to the EM-Algorithm.

For each period t of each inventory simulation, the censored observations and the conditions which censure these observations have specific values. Therefore, the observed censored data for each period is evaluated whether it constitutes censored demand or not. If it constitutes censored demand, the E-step to calculate the expected values for the non-censored demand given the respective conditions for the censored data according to each event among the events $\overline{AB}$, $\bar{A}B$ and $A\bar{B}$ is run. Thereafter, part of the M-step is run replacing the censored observation for the respective period with the respective expected values calculated in the E-step. Once all periods have passed through this process, the MLEs for the λ_a and λ_b are calculated considering the complete set of censored data for each shelf-life, now replaced by the expected values, is not censored anymore. Since the non-censored data in the current case (the demands a and b) are Poisson distributed, the MLE for The estimations is the average of the considered non-censored data. The estimates for λ_a and λ_b are then used again in a new loop for the EM-Algorithm until satisfactory convergence is reached.

The EM-Algorithm implemented achieved satisfactory converge when the differences between both estimates of one loop and the respective estimations of the previous loop was equal or less than 0.005, or if the EM-algorithm had run for 100 loops. The final estimates
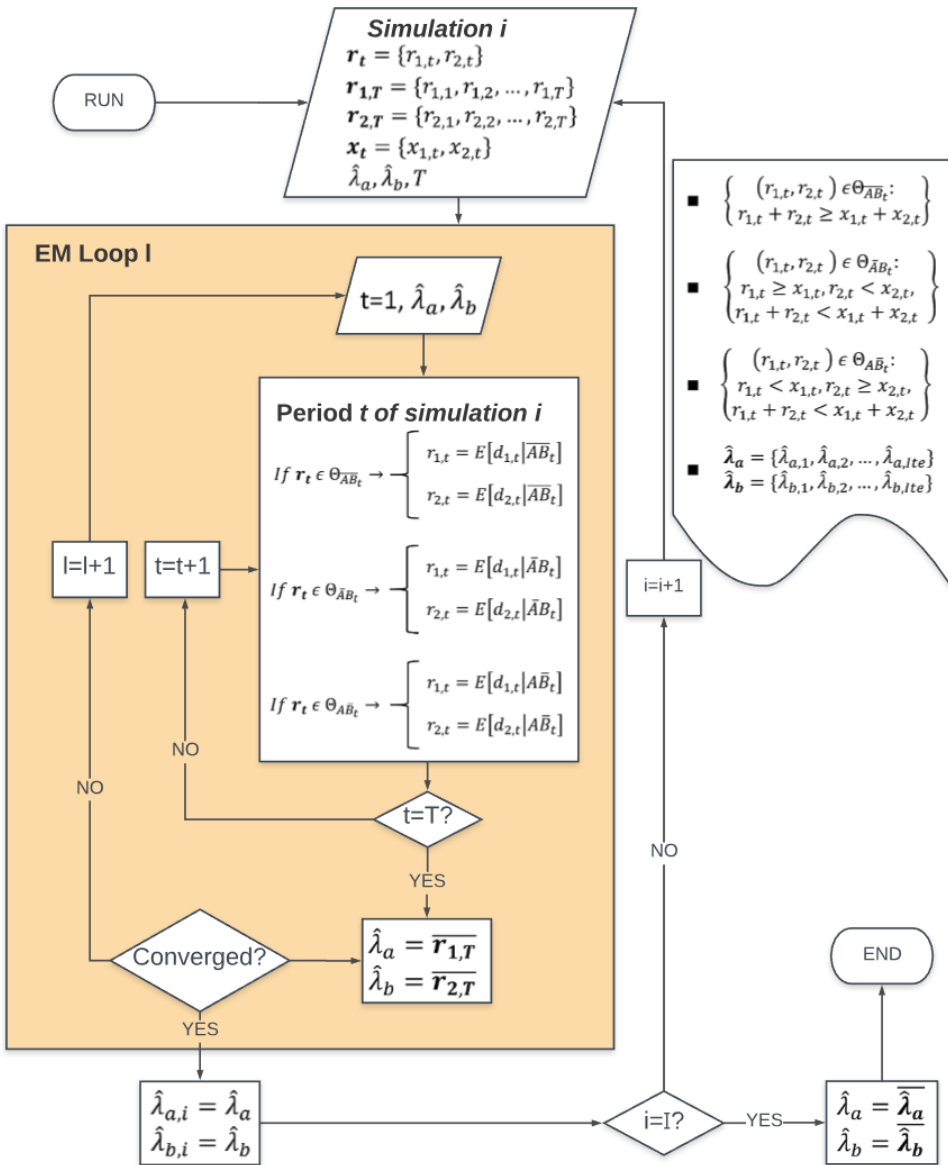
Figure 4.1: Flow chart for the implementation of EM-Algorithm.

are then given by the expected value of the estimate over all inventory simulations run. It means that the final estimation of λ_a is the average of all $\hat{\lambda}_a$ over all simulations and the final estimation of λ_b is the average of all $\hat{\lambda}_b$ over all simulations.

As indicated before; the conditions for each event depend on both demands and both inventory states correspondent to the items categorized as a and b, the oldest and newest

respectively. Therefore, the conditional expectations for $d_{a,t}$ and $d_{b,t}$ in each event are calculated using conditional joint probability distribution for both $d_{a,t}$ and $d_{b,t}$ correspondent to each event as presented below.

Event $\overline{AB}$:

In event $\overline{AB}$, the conditional expectations of non-censored demand are $E(d_{a,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})$ and $E(d_{b,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})$. The conditional joint probability distribution in this case can be analytically calculated, since $d_{a,t}$ and $d_{b,t}$ are two independent variables and by using the Bayers' theorem $P(A|B) = P(A)\frac{P(B|A)}{P(B)}$, as follows:

$$\varphi_{\overline{AB}}(d_{1,t}, d_{2,t}) = P(d_{1,t}, d_{2,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}) =$$

$$P(d_{a,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})P(d_{b,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}) =$$

$$\frac{P(d_{b,t} \geq x_{1,t} + x_{2,t} - d_{a,t})P(d_{a,t})}{d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}} \frac{P(d_{a,t} \geq x_{1,t} + x_{2,t} - d_{b,t})P(d_{b,t})}{P(d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})} = \quad (4.33)$$

$$\frac{P(d_{b,t} \geq x_{1,t} + x_{2,t} - d_{a,t})P(d_{a,t})P(d_{a,t} \geq x_{1,t} + x_{2,t} - d_{b,t})P(d_{b,t})}{[P(d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t})]^2} =$$
$$\frac{\Psi(d_{a,t}, \lambda_b)\varphi_P(d_{a,t}|\lambda_a)\Psi(d_{b,t}, \lambda_a)\varphi_P(d_{b,t}|\lambda_b)}{\Psi(0, \lambda_a + \lambda_b)^2}$$

where $\Psi(z, \alpha) = 1 - \Phi_P(x_{1,t} + x_{2,t} - z - 1|\alpha)$.

Hence, the conditional expectations for the non-censored demands from the data-set which falls into the conditions from the event $\overline{AB}$ are given by

$$E(d_{a,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}) = \sum_{d_{a,t}=0}^{\infty} \sum_{d_{b,t}=0}^{\infty} [\varphi_{\overline{AB}}(d_{a,t}, d_{b,t})d_{a,t}] \quad (4.34)$$

and

$$E(d_{b,t}|d_{a,t} + d_{b,t} \geq x_{1,t} + x_{2,t}) = \sum_{d_{a,t}=0}^{\infty} \sum_{d_{b,t}=0}^{\infty} [\varphi_{\overline{AB}}(d_{a,t}, d_{b,t})d_{b,t}]. \quad (4.35)$$

Event $\bar{A}B$:

In event $\bar{A}B$ the conditional expectations of the non-censored demands are given by $E(d_{a,t}|d_{a,t} \geq x_{1,t}, d_{b,t} < x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$ and $E(d_{b,t}|d_{a,t} \geq x_{1,t}, d_{b,t} < x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$. The conditional joint probability distribution in this case is calculated as follows:

$$\varphi_{\bar{A}B}(d_{1,t}, d_{2,t}) =$$
$$P(d_{a,t}, d_{b,t} | d_{a,t} \geq x_{1,t}, d_{b,t} < x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$
$$\frac{E(d_{a,t}, d_{b,t})}{\sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} \varepsilon(i,j)}. \tag{4.36}$$

Hence, the conditional expectations for the non-censored demands from the data-set which falls into the conditions from the event $\bar{A}B$ are given by

$$E(d_{a,t} | d_{a,t} \geq x_{1,t}, d_{b,t} < x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$
$$\sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} [\varphi_{\bar{A}B}(d_{a,t}, d_{b,t}) d_{a,t}] \tag{4.37}$$

and

$$E(d_{b,t} | d_{a,t} \geq x_{1,t}, d_{b,t} < x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$
$$\sum_{i=x_{1,t}}^{x_{1,t}+x_{2,t}-1} \sum_{j=0}^{x_{1,t}+x_{2,t}-i-1} [\varphi_{\bar{A}B}(d_{a,t}, d_{b,t}) d_{b,t}]. \tag{4.38}$$

Event A$\bar{B}$:

Symmetrically to event $\bar{A}B$ in event $A\bar{B}$, the conditional expectations of the non-censored demands are given by $E(d_{a,t} | d_{a,t} < x_{1,t}, d_{b,t} \geq x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$ and $E(d_{b,t} | d_{a,t} < x_{1,t}, d_{b,t} \geq x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t})$. The conditional joint probability distribution in this case is calculated as follows:

$$\varphi_{A\bar{B}}(d_{1,t}, d_{2,t}) =$$
$$P(d_{a,t}, d_{b,t} | d_{a,t} < x_{1,t}, d_{b,t} \geq x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$
$$\frac{E(d_{a,t}, d_{b,t})}{\sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} \varepsilon(i,j)}. \tag{4.39}$$

Finally, the conditional expectations for the non-censored demands from the data-set which falls into the conditions from the event $\bar{A}B$ are given by

$$E(d_{a,t} | d_{a,t} < x_{1,t}, d_{b,t} \geq x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$
$$\sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} [\varphi_{A\bar{B}}(d_{a,t}, d_{b,t}) d_{a,t}] \tag{4.40}$$

and

$$E(d_{b,t}|d_{a,t} < x_{1,t}, d_{b,t} \geq x_{2,t}, d_{a,t} + d_{b,t} < x_{1,t} + x_{2,t}) =$$

$$\sum_{j=x_{2,t}}^{x_{1,t}+x_{2,t}-1} \sum_{i=0}^{x_{1,t}+x_{2,t}-j-1} \left[\varphi_{A\bar{B}}(d_{a,t}, d_{b,t})d_{b,t}\right]. \tag{4.41}$$

4.3 Design of experiment

In order to answer the research question number three and evaluate the impact of the model presented hereby a full factorized design of experiment (DOE) is applied. The DOE is divided in three phases as presented below:

1. Inventory is simulated following a standard stock level dependent heuristic replenishment policy to generate the data for demand estimation. Two main cases are simulated:

   - Demand following a Poisson distribution with determined mean demand λ, and a probability p of customers picking up the oldest item which leads to the two different demands, λ_a, demand for the oldest items, and λ_b, demand for the newest items.

   - Inventory following a Poisson distribution with determined mean demand λ and demand distribution over remaining-shelf life is proportional to the quantity of items correspondent to each remaining-shelf life available in stock as implemented in Vaughan (1994) and Ferguson and Ketzenberg (2006).

2. Demand is estimated for each inventory simulation from both cases described in the first phase.

3. The estimated demand is used in new inventory simulations with a stock-age dependent replenishment policy to evaluate the estimations considering the following depletion policies:

   - Full LIFO
   - Full FIFO
   - $p = 50\%$
   - Estimated parameters for demand a and demand b in the second phase.
   - Proportional depletion policy from Vaughan (1994) and Ferguson and Ketzenberg (2006).

   In addition, inventory simulations using the same parameters, but instead of using any estimated value for the demand in a stock age dependent replenishment policy, it uses the BSP with the same base stock as the one adopted in the first phase run.

From the base literature, only the proportional depletion policy from Vaughan (1994) and Ferguson and Ketzenberg (2006). Due to schedule constraints for the completion

of the master thesis, the other two approaches from base literature, considering uniform distribution and hypothetical probability from Pegels and Jelmert (1970), were not applied. The proportional depletion policy was considered the most relevant in terms of efficiency and realistic traits. However, it is acknowledged in this thesis that the use of two policies in further research may improve the scientific quality of the research and the reliability of the results.

All inventory models and estimation algorithms adopted to execute the design of experiments were implemented in Mathematica 11.1.1.0. The results were analyzed with the support of Microsoft Excel 2016.

4.3.1 Observed data generation

At the first phase, the observed data that are used for the estimation of the demand and customer preferences parameters are generated from the standard inventory model described in section 4.1 applied jointly with a standard base stock replenishment policy, the BSP, with a base stock. In this policy the order quantity, q_t, depends on the total quantity of items available in stock, $X_t = \sum_{i=1}^{M} x_i$, as follows:

$$q_t = (S - X_t)^+ \tag{4.42}$$

where S is the order up-to level or the base stock. This is the target level of inventory. In this replenishment policy, items are ordered at each reviewing period to fulfill the gap between current inventory and the base stock. Cohen (1976) analytically studied this policy for a two-period shelf-life with FIFO depletion policy. The optimal value of S requires numerical calculations to be acquired. Since the generation of the observed data is performed for the sake of estimation of some parameters, an arbitrary value of S based on the newsvendor model is then used in the simulations adopted for the estimation of targeted parameters. The newsvendor model identifies a profit-maximizing order quantity considering that the probability of ending the day with positive stock should equal the profit margin (Churchman et al., 1957). Since the items purchased arrive in stock with a remaining shelf-life of 2 periods in the model for data generation differing from the newsvendor model, the calculation of S is adapted as follows:

$$Pr(d_t + d_{t+1} \leq S^*) = 2m_u \to S^* \approx \Phi_P^{-1}(2u_m|2\lambda) \tag{4.43}$$

where d_t and d_{t+1} represent the demand for two consecutive periods and are two independent variables following a Poisson distribution with the same total mean demand λ. In addition, m_u, which is equal to $\frac{c_s - c_u - c_h}{c_s}$, is the unit sales margin accounted from the unit sales price. In equation (4.43), the targeted S^* is the inventory quantity which results in the probability of ending the next day with positive stock approximately equals to twice the profit margin.

The parameters used in the inventory simulation from the first phase are presented in table 4.1. At the second phase for each set of simulations considering the factorized parameters presented in the table, the demand of the oldest items and the demand of the newest items are estimated by applying the EM-Algorithm as outlined in section 3.6.

Table 4.1: Parameters used in the data generation for the simulations from the first phase of the DOE.

Parameter	Value	Unit
Fixed parameters		
Maximum shelf-life (M)	2	days
Ordering frequency (τ)	1	days
Delivery time (L)	1	days
Inventory horizon (T)	500	days
Number of simulations (I)	20	Simulations
Factorized parameters		
Mean demand (λ)	$\{3, 20, 50\}$	units
FIFO probability (p)	$\{0.3, 0.7, Prop^*\}$	units

*Demand distribution over remaining-shelf life is proportional to the quantity of items correspondent is proportional to the quantity of items correspondent to each remaining-shelf life available in stock.

4.3.2 Convergence test for EM-Algorithm

The goal of this convergence test is to evaluate the behavior of the model and to heuristically stipulate the most recommended parameters from the EM-Algorithm that should be used in the second phase. Hence, a convergence and estimation accuracy evaluation was performed for the implementation of the EM-Algorithm to estimate λ_a and λ_b. A detailed evaluation of the properties of the model developed hereby was not performed since this goes beyond the scope of this master thesis.

The consistency of the estimations provided by the EM-Algorithm depends on basically four key factors as explained in sections 3.5 and 3.6: How much of the observed data-set is censored, the number of observations which are analyzed, the initializing estimated parameters and the number of loops in which the EM-Algorithm is run.

The first one is especially important because the errors from the estimates derive from the expected values that substitute the censored data among the observations i.e. the more censored data among the observations, the less accurate the results from the EM-Algorithm are. The second one is related to the MLE method. The MLE consistency depends on the number of observations used in the estimation. Since the EM-Algorithm converges to the MLE estimator, the number of samples has to be high enough to provide valid results i.e. the higher the number observations, the closest to the actual value the maximum likelihood estimator becomes. The importance of the initializing estimator is related to the shape of the curve of the likelihood function. If the curve has more than one maximum, the estimator converges to the one which is closer to the initializing estimator this may not be the global maximum of the likelihood function.

This convergence evaluation was performed through a design of experiment which consisted in varying some key parameters. The inventory model used in this convergence analysis followed the description from section 4.1 and the replenishment policy of the inventory simulations used was the base stock policy explained in section 4.3.1. The main parameters of the inventory model used in the experiments convergence analysis are dis-

played at table 4.2

Table 4.2: Parameters used in the inventory model for the convergence test.

Parameter	Value	Unit
Fixed parameters		
Mean demand (λ)	20	units
FIFO probability (p)	0.7	——
Maximum shelf-life (M)	2	days
Ordering frequency (τ)	1	days
Delivery time (L)	1	days
Factorized parameters		
Base stock (S)	$\{35, 40, S^* = 45, 50, 60, 70\}$	units
Inventory Horizon (T)	$\{50, 100, 500\}$	units

The two parameter from the inventory model that varied in the experiments from the convergence testing was the base stock S and the inventory horizon T that denoted the highest period which the inventory simulation reached. The base stock was elected to be factorized because it is related to the proportion of the data-set which denotes the demand that was censored. The less the quantity of the base stock, the less expected available stock the inventory simulations are expected to generate. This should lead to more censored demand making the estimates less accurate. The inventory horizon was elected to be factorized because this value is related to the quantity of observations. Each single data-set comprised of the realized sales for each period are the observable data-points that are used in the EM-Algorithm to the estimation of λ_a and λ_b. Therefore, the more periods are used in the inventory simulation, the more observable data-points are available for the estimation improving the accuracy of the estimations. In addition, one parameter to be set directly at the EM-Algorithm was elected to vary in the convergence test experiments. The number of simulations performed in each experiment, I.

The algorithm implemented in the experiments followed the flowchart presented in figure 4.1 and the results are presented in table 4.3. Due to limitations on time, a full factorized design of experiment was not performed. The EM-Algorithm for the experiments from the convergence analysis took between approximately 7 and 78 hours of running time. Therefore, eight fully registered experiments were considered enough to reach a conclusion about the EM-Algorithm behavior for the current model and to heuristically estimate the satisfactory parameters of the EM-Algorithm that should be applied to the analysis.

The first column of the table presents the value of the three factorized parameters respective for each experiment in a series. The series correspond to the variables S-T-I which are respectively the base stock, the inventory horizon and the number of simulations. Then in the second column, the table displays the estimated value of $\hat{\lambda}_a$ in units followed by the percentage error in relation to the original value of demand a, $\lambda_a = 14$

units, in the third column. The fourth and fifth columns, contain the same results as the second and third respectively but for the demand b, $\lambda_b = 6$. The sixth, seventh and eighth columns contain respectively the quantity of censored data-sets, the quantity of partially censored data-sets and fully uncensored data-sets. These terms are related to the events $\overline{AB}$, $\bar{A}B$ and $A\bar{B}$ as outlined in section 4.2. These values depict the quantity of data-sets correspondent to each event normalized by the total number of periods and the total number of simulation. Therefore, they can be considered to be the proportion of quantity of data-sets correspondent to each event over the total number EM-Iterations or periods within all inventory simulation of each experiment. The last three columns are respectively the expected lost sales of the oldest items (item a), expected lost sales of the newest items (item b) and the expected waste over all periods and over all simulations in units.

Table 4.3: Results of convergence analysis.

S-T-I	$\hat{\lambda}_a$	PE $\hat{\lambda}_a$	$\hat{\lambda}_b$	PE $\hat{\lambda}_b$	Cen.	Part. Cens.	Uncen.	LS a	LS b	Waste
70-500-20	13.99	0.15	6.01	0.06	0.00	0.28	0.72	0.01	0.04	5.72
60-500-20	13.99	0.08	5.98	0.19	0.01	0.48	0.51	0.01	0.04	2.78
50-500-20	13.98	0.25	6.03	0.16	0.08	0.74	0.19	0.01	0.19	0.74
45-500-20	14.13	0.93	5.94	1.00	0.21	0.72	0.07	0.01	0.62	0.24
45-50-100	14.01	0.07	5.89	1.83	0.22	0.71	0.07	0.13	0.88	0.35
40-500-20	13.92	0.57	6.07	1.17	0.42	0.56	0.02	0.02	1.52	0.05
40-50-100	13.89	0.79	6.14	2.33	0.43	0.55	0.02	0.15	1.76	0.10
35-100-100	14.31	2.21	5.76	4.00	0.67	0.32	0.00	0.10	3.17	0.01

At first, two experiments were run considering the base stock equal to 45 units. In one of them 20 inventory simulations were run over an inventory horizon of 500 periods and in the other 100 inventory simulations were run over an inventory horizon of 50 periods. Then, two experiments were run considering base stock equals to 40 units. In one of them 20 inventory simulations were also run over an inventory horizon of 500 periods and the other were run with 100 inventory simulations were run over an inventory horizon of 50 periods in the other one. Thereafter, 100 inventory simulations were run over an inventory horizon of 100 periods an experiment considering base stock equals to 35 units. At this point the experiment took too long time to run, the results were considered satisfactory enough for a heuristic evaluation and as an outcome of the evaluation the values of T and I considered the most appropriate for the data generation were elected. At last, three extra experiments with higher base stock than the S^* were run in order to evaluate effect of increasing the base stock and potentially increasing the uncensored data over the observations. The experiments adopted $I = 20$ inventory simulations and $T = 500$ periods. The base stock values evaluated in these three last experiments were 50, 60 and 70 units.

With regards to the running time, two factors are expected to be pivotal. How many times the EM-Algorithm is iterated and what are the operations run for each EM-iteration. The EM-algorithm is iterated for each set of observations respective to each period in our current problem. These iterations are called hereby EM-iterations. These EM-iterations over all periods simulated are run for all EM-Loops necessary to achieve the convergence criteria elected, denoted by the variable l, and this is done for all simulations. Therefore, the total number of EM-iterations run over one single experiment is equal to $T \times l \times I$. The number of necessary loops to achieve the convergence criteria depends on the outcome of the simulations. The necessary EM-loops and its influence in the running time and

consistency of results are also a consequence of available data generated by the inventory simulations. Thence, the controllable parameters of the algorithm presented hereby that has influence in the running time and possibly in the consistency of the outcomes are T and I.

In addition, each EM-Iteration can correspond to one of four different operations as illustrated by figure 4.1. The operations depend on the status of the censored demand and the quantity of stock available for each period. As it was already outlined in the case AB, when the censored demand for the item with remaining shelf-life equals to one and two are less than the quantity of items in stock with the same remaining shelf-life, one and two, no operation is run in the EM-Algorithm i.e. no operation is run in the E-step and M-step. Therefore, the AB is expected to be the fastest case. The E-step and M-step are applied only when the data-set for the observed data $\mathbf{r_t}$ is censored. Moreover, the calculation for the conditional expectation of non-censored demand for censored data from event $\overline{AB}$ takes longer time than the same calculations for censored data from event $\bar{A}B$ and $A\bar{B}$. This happens because the conditional expectations for elements from $\bar{A}B$ and $A\bar{B}$ result from a double summation with limited indexes, both up to $x_{1,t} + x_{x_2,t} - 1$, and the conditional expectations for elements from $\overline{AB}$ result from a double summation with unlimited indexes, both up to ∞. Consequently, the $\overline{AB}$ is expected to be the slowest and the cases $A\bar{B}$ and $\bar{A}B$ are expected to be equally the second fastest cases.

The running time for the experiments that had $T \times I$ equals to 1000 EM-Iterations were indeed higher than the running time for the experiments with $T \times I$ equal to 500 EM-Iterations. In the experiments which used $S = 45$ units, the one with 1000 EM-Iterations took about 52 hours to be completed and the one with 500 EM-iterations took approximately 25 hours of running time. In the experiments with $S = 40$ units, the one with 1000 EM-Iterations took approximately 64 hours to run and the one with 500 EM-Iterations took about 29 hours of running time. The last one with $S = 35$ units and 1000 EM-Iterations almost reached 78 hours, over than three days, of running time.

These numbers also indicates that indeed the running speed for the EM-Algorithm is related to quantity of censored, partially censored and uncensored data-sets among the observed data-sets. The quantity of fully censored data for the experiments with $S = 45$ units was about 21 % of the total data-sets for each experiment. The quantity of partially censored data-sets was about 72 % and the quantity for the uncensored data-sets was about 7 %. With an increase of about 100 %, the fully censored data-sets for the experiments with $S = 40$ units was equals to approximately 42 %, that led to a decrease on the quantity of partially censored and uncensored data-sets, about 56 and 2 % respectively. The experiment with $S = 35$ units had an increase of 300 % on the quantity of fully censored data-sets in relation to the experiments with $S = 45$ units. It had a total of about 67 % of censored data-sets. The quantity of partially censored and uncensored data-sets were approximately 32 % and 0 % in this experiment.

The three last experiments simulated with base stock equals to 50, 60 and 70 units resulted in between 7 and 12 hours of running time. Much less than the other experiments. The proportion of non-censored demand were significantly higher as well. In the experiment with $S = 50$ units, the non-censored observations corresponded to 19 % of the total of observations. In the experiment with $S = 60$ units, the non-censored observations corresponded to 51 % of the total of observations. In the experiment with $S = 70$ units, the

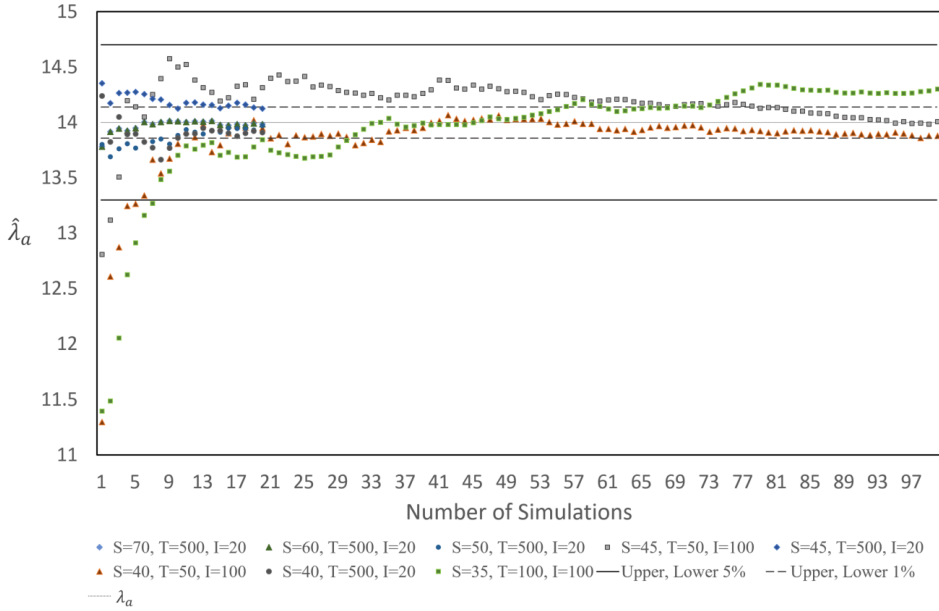non-censored observations corresponded to 72 % of total observations.

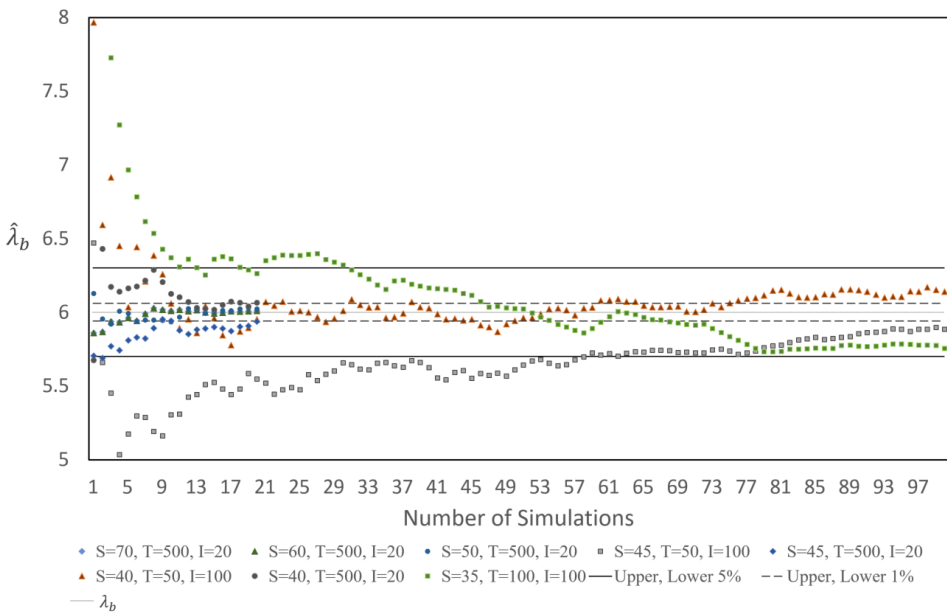**Figure 4.2:** Convergence evaluation for estimation of mean demand a.

Figure 4.3: Convergence evaluation for estimation of mean demand b.

As expected these results show that the more censored data-sets over the periods of the inventory simulations the longer the EM-Algorithm takes to run. An increase from about 21 % of fully censored data-sets to approximately 42 % between the experiments with $S = 45$ and $S = 40$ units led to an increase of approximately 20 % on the running time. Then the increase to 67 % of the fully censored data-sets on the experiment with $S = 35$ units led to an increase of approximately 50 % on the running time in relation to the running time of the experiment with $S = 45$ hours. The experiments with base stock higher than 45 units with significantly higher number of non-censored observations ran much faster than the others.

With regards to the consistency of the estimations, all simulations provided satisfactory results with a small exception of the experiment with $S = 35$ units. The percentage error for λ_a was less than 1 % for all simulations but the one with $S = 35$ units which had a percentage error for λ_a equals to 2.21 % considered substantially high. The percentage error for λ_b presented values between 1 and 2.5 % in all experiments with the exception of the experiment with $S = 35$ units which had a percentage error of 4 %. Considering that the experiment with $S = 35$ units had a substantial high percentage error in relation to the others experiments and that the running time was also much higher, it was decided that any extra simulation with $S = 35$ units or less were unnecessary regarding the objective of this convergence analysis. In addition, this led to an indication that indeed the more censored data-set the inventory simulations present the poorer results for the estimations the EM-Algorithm provides.

A more carefully comparison with regards to the base stock for the experiments with $S = 40$ and $S = 45$ units, led to the same conclusion. Here, it was more confusing. Because the experiments with $S = 45$ units and 1000 EM-iterations presented a poorer result for $\hat{\lambda}_a$, 0.93 % of percentage error, than both experiments with $S = 40$ units, 0.57 % and 0.79 %, and the opposite happened in relation to the experiment with $S = 45$ units and 500 EM-iterations and the experiments with $S = 40$ units. The experiment with $S = 45$ units and 500 EM-iterations presented better result for $\hat{\lambda}_a$, 0.07 % of percentage error, than the experiments with $S = 40$ units. In contrast, the experiment with $S = 45$ units and 1000 EM-iterations presented better result for $\hat{\lambda}_b$ than both the experiments with $S = 40$ units and the experiment with $S = 45$ units and 500 EM-iterations. This may have happened due to the randomness present in the experiments and due to the number of simulations run. These results derived from expectations and although they converge to the expected values which correspond to demand a and demand b, there are still variations present over the simulations. However, the results of both experiments with $S = 45$ units combined provided better results for both $\hat{\lambda}_a$ $\hat{\lambda}_b$ than the results of both experiments with $S = 40$ units. The expected percentage error combined was lower for both $\hat{\lambda}_a$ $\hat{\lambda}_b$ and this gives a more solid indication that indeed by reducing S, the algorithm results in poorer convergence.

The impact of the randomness mentioned above can be clearly seen in figures 4.2 and 4.3. These figures show the expected estimations of λ_a and λ_b over the simulations, or number of simulations, with 5 lines of reference: The center line for the actual value of the mean demand, two lines for the 1 % boundary (upper and lower) in relation to the actual mean demand and two lines for the 5 % boundary in relation to the actual mean demand. For example in figure 4.2, the value of $\hat{\lambda}_a$ for the experiment with $S = 45$ units and 500

EM-iterations is the closest to the center line and it has a percentage error of 0.07 % as shown in table 4.3. Considering 100 simulations, this experiment gives the most accurate result for $\hat{\lambda}_a$. However considering the values of $\hat{\lambda}_a$ correspondent to 73 simulations, the experiment with $S = 40$ units and 500 EM-iterations presented the lowest percent error. This could also happen if more simulations were used for each simulation changing the perception that one may have analyzing the results presented in table 4.3. The results are too close from each other and the randomness may impact the perceptions for the results of each isolated experiment. However by taking the expected value of $\hat{\lambda}_a$ and $\hat{\lambda}_b$ considering all experiments when $S = 45$ units and $S = 40$ units, the results confirm the conjecture that reducing S the algorithm results in poorer convergence.

Figures 4.2 and 4.3 also show the general behavior of the algorithm in relation to the settings of the inventory horizon and the number of iteration considered. For higher T, the outcomes of the EM-Algorithm is closer to the actual values of the mean demand a and mean demand b in the cases where T is lower. Both figures 4.2 and 4.3 show that its respective estimated mean demand is closest to the actual values when only few first simulations are considered in the experiments with $T = 500$. In addition, the curves of these experiments vary much less over the simulations. The experiments with lower T needs more simulations to reach more trustworthy values. This happens because the EM-Algorithm generates the estimation of the expected values of the mean demands. Therefore if the inventory horizon is short, the expected values are less accurate. This can be compensated by running the EM-Algorithm to a higher number of simulations. This indicates that high variances on the outcomes are not expected when instead of using long inventory horizon and few simulations in the experiment, it is used short inventory horizon and high number of simulations.

The results of the experiments with base stock higher than S^*, presented general more consistent results than the one with S^*. All percentage errors in these cases were less than 0.25. However, the waste rate was significantly higher what makes them not feasible from the perspective of a realistic retail context which aims at high profitability and low waste.

With regards to the initializing estimated parameters, a more detailed analysis of the likelihood function and its shape is necessary for a more precise conclusion. But the choice of calculating the expected value of mean demand considering the available observations that were not censored, the realized sales, showed to be adequate.

Table 4.4: Elected parameters for the estimations of the experiments from phase 2 of the DOE.

Parameter	Value	Unit
Periods (t)	500	days
Number of simulations (I)	20	Simulations

As a conclusion, running the simulations used in the estimation of the parameters with the replenishment policy outlined in section 4.3.1, with base stock equals to S^*, was considered enough to provide adequate estimations. The expected percentage error is expected to be less than 1 % in this case considering all experiments analyzed hereby. Therefore due to the long running time, the simulations to generate the data for the analysis

are run considering S correspondent to the base stock replenishment policy, $t = 500$ periods and $I = 20$ simulations as presented in table 4.4.

4.3.3 Estimations evaluation

The replenishment heuristic policy adopted in the third phase of the design of experiment is the same as the one used in Montojos (2017), a stock-age dependent look-ahead sparse sampling tree policy which uses online simulations at each reviewing period to estimate the most profitable replenishment quantity considering a determined demand distribution. This replenishment policy is applied in this thesis because it is well balanced considering its computational efficiency over its near optimal performance. This replenishment policy was based on the ones validated in Ferguson and Ketzenberg (2006). Their heuristics represented myopic policies modeled for an inventory problem which a retailer could place an order each period and the lead-time was one day. The order decisions were based on whether sufficient stock existed each period that would carry over and minimize expected cost in the next period only. If sufficient stock existed, then the decision would be postponed to the next day.

These policies were validated by comparing their results with optimal results for various scenarios. Although they were not designed considering the same context utilized in the inventory model of this project, as for example addressing the fixed ordering costs, it was assumed that this validation was sufficient for the purpose of this project. In addition, a convergence evaluation for the same heuristics evaluated hereby were performed in Montojos (2017) which enabled its utilization without further convergence evaluations.
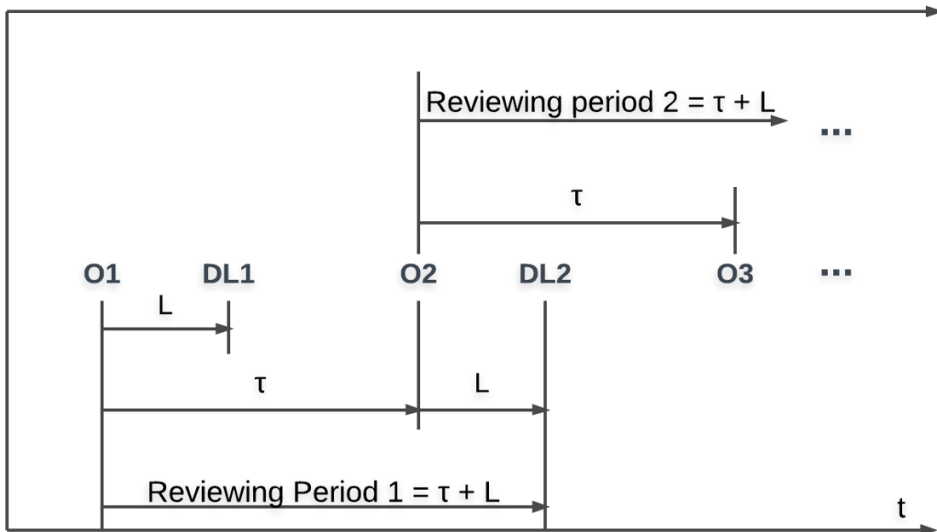

Figure 4.4: Overview of replenishment decision review in the look-ahead replenishment policy.

Another difference of the context in which the replenishment policies were used in Ferguson and Ketzenberg (2006) is the fact that in this thesis look-ahead sparse sampling

tree search policies were used adopting arbitrary delivery time and ordering frequencies. The order decisions were based whether sufficient stock existed in each ordering period to be carried over and maximize expected profit until the delivery for the next ordering period ($\tau + L$ periods after the period which ordering decision had to be made). If sufficient stock existed, the ordering would be postponed and a reevaluation would be performed in the next ordering period.

Figure 4.4 outlines the reviewing period used in the replenishment decision of the heuristic used in this project. The ordering periods are represented by the letter O followed by its respective number inside the series of orders ($O1$ represents the first ordering period, $O2$ the second ordering period and so forth). The delivery periods are represented by DL also followed by its respective number inside the series of deliveries and linked to its respective order ($DL1$ represents the delivery from order $O1$, $DL2$ represents the delivery for order $O2$ and so forth). Considering the figure, if the stock level at $O1$ is sufficient to be carried over and maximize the profits until $DL2$, the decision is set to no ordering at $O1$ and the next review happens at $O2$. Otherwise, the order is placed with a n_t that maximizes the profits until delivery period of next ordering period.

Let $g(\mathbf{x_t})$ denote the total estimated future profit associated with the inventory $\mathbf{x_t}$, from period t until $t + \tau + L$, exclusive. The total estimated cost from period $t + \tau + L$ is excluded from $g(\mathbf{x_t})$ since the stock received at $t + \tau + L$ is used to realize the sales and is accounted in the profit calculation at this period. Thence, the recursion for the maximum total profit of this heuristics is given by

$$
g(\mathbf{x_t}) = max_{q_t \geq 0} \left\{ \pi(\mathbf{x}_t, q_t) + \right.
$$
$$
\left. \sum_{d_{a,t}=0}^{\mathbf{x_t}} \sum_{d_{b,t}=0}^{\mathbf{x_t}-d_{a,t}} g(\xi(\mathbf{x}_t, d_{a,t+1}, d_{a,t+1}, q_{t+1-L})) \varphi_P(d_{a,t+1}|\lambda_a) \varphi_P(d_{b,t+1}|\lambda_b) \right\},
$$
(4.44)

where in the right hand side the total expected profit that comprises the expected profit in period t and the future expected profit is calculated. Both terms are predicated on $\varphi_P(\cdot)$ and depend on the replenishment decision depicted by the q_t among the other variables.

When the decision space for q_t is a set of positive integer values, the state and decision spaces are discrete and finite, and the cost is bounded; there is an optimal policy that does not randomize (pp. 102-111, Puterman, 1994, cited by Ferguson and Ketzenberg, 2006). However, the implementation of this optimal policy is impractical for many realistically sized problems given that the size of the state space expands exponentially with the age dependent vector of inventory (Nahmias, 1982; Ferguson and Ketzenberg, 2006). The complexity of such optimal policy leads to high computational costs. It is why that a more conceivable heuristic look-ahead replenishment policy is used hereby.

Still on the third phase, the data collection to answer the research question 3 was performed through the application of a Monte Carlo simulation implemented. A basic schematics of the algorithm flow chart is outlined in figure 4.5. This schematic does not represent the complete flow chart of the algorithm, but instead it does intend to show the most relevant features of it in a sketch. Each main simulation was run for 1100 periods.

But the calculation of the expected outputs was performed setting aside the first 100 periods. These 100 periods were set aside as warm up interval so the results could capture only the steady state of the system in each iteration. Hundred periods are more than enough to capture the steady state behavior. Furthermore for each ordering period, another simulation, called hereby online simulation, was run for the look-ahead replenishment policy in the ordering decision as represented by equation (4.44).

The simulations are marked in the orange process boxes in figure 4.5. The number of iterations set for the main simulations were determined as the same number of iterations adopted in the online simulations. Thence, the number of iterations for the simulations had a great impact on the computational cost. This is better understood by calculating the number of iterations of the online simulation for each main simulation, I_O, considering the total number of iterations of main simulation, I_M, the ordering frequency and the total number of periods, which is given by approximately

$$I_O = I_M \, T \, \frac{I_M}{\tau} = T \, \frac{I_M^2}{\tau} \, . \tag{4.45}$$

Equation 4.45 shows that the computational cost increases quadractically with the number of iterations elected for the simulations. The number of iterations for each simulations was assigned as 1100 in figure 4.5 just for illustration purpose. As it was already mentioned, a convergence evaluation for convergence evaluation for the same heuristics evaluated hereby was performed in Montojos (2017). This evaluation showed quite scattered results until the iteration number 50, approximately. Close to this point, the results were inside the interval of confidence at 50% considering a simulation performed over 1500 iterations. This interval of confidence depicted deviations of approximately $\pm 0.1\%$ in relation to the total expected profit over all iterations.

Other similar analysis was performed in Montojos (2017) taking into consideration other outputs and other experiments. All of them presented similar outcomes considering the intervals of confidence and their respective results. Therefore, 50 iterations were considered enough for the simulations performed hereby and considering the intent of this master thesis. Furthermore, any difference between results of two different experiments below $\pm 0.1\%$ is either disregarded in the analysis or presented with proper notification about the uncertainty.

The simulations are run considering the demand parameters from the first phase in the main simulation and the estimated parameters from the second phase in the online simulations. Five different set of parameters are used in the online simulations from the third phase: Considering full FIFO demand ($p = 100\%$) as usually applied by the industry, full LIFO ($p = 0$), half demand FIFO and half LIFO ($p = 50\%$), parameters estimated in the second phase and considering the proportional depletion policy from Vaughan (1994) and Ferguson and Ketzenberg (2006).

It is the third phase that provides the results that are used in the comparison of the performance of the model which uses the estimated parameters and the other models proposed in the same phase. In addition, a sixth set of results in the factorized design of experiment is used to serve as a basis for the comparison. The sixth set considers the base stock policy used in the generation of data from the first phase.

The parameters used in the simulations from the third phase are presented in table 4.5. In table 4.5 at the replenishment policy line, the letters LA stand for look-ahead and
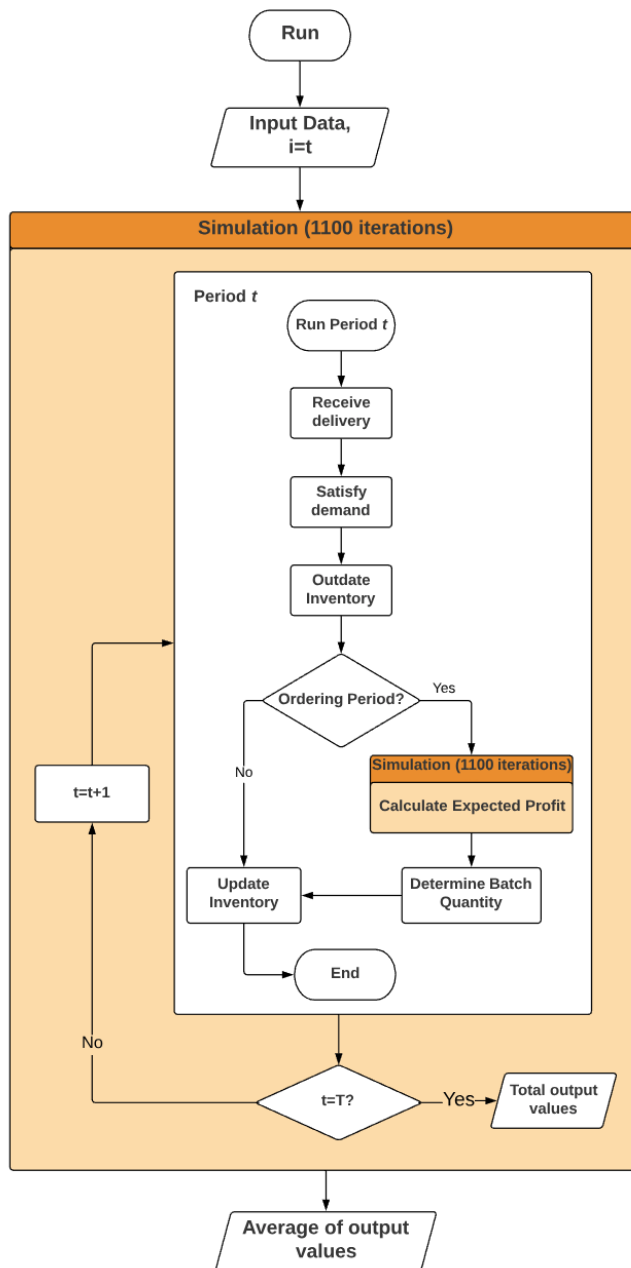
Figure 4.5: Sketch of the flow chart for the inventory simulations with the look-ahead replenishment policy.

refer to the policies which uses the look-ahead sparse sampling tree policy and the prefix which follows these two letters indicates the demand distribution adopted. The prefix LIFO indicates that the demand was satisfied with a full LIFO depletion policy in the look-ahead replenishment policy; the prefix FIFO indicates that the demand was satisfied with a full FIFO depletion policy; the number 50 indicates that p was equal to 50 %, the prefix EST stands for estimated and refers to the look-ahead policy which considers the estimated value of p from the data generation; the prefix $PROP$ stands for proportional and refers to the look-ahead policy which considers the proportional demand from Vaughan (1994) and Ferguson and Ketzenberg (2006). At last, the acronym BSP stands for base stock policy and refers to the simulations which used this replenishment policy.

Hence, the set of experiments performed for the analysis comprise all combinations of the factorized parameters presented in table 4.1 and 4.5. The experiments are differentiated by 3 variants of the mean demand of the main simulation, 3 variants distributions of the FIFO probability and a total of 6 variants of the replenishment policy. This leads to 54 experiments.

Three main outcomes of each experiment are used in the analysis as performance measurements for comparison: Waste, Fill-rate and Profit.

The profit per period is given by equation (4.9) and the overall profit used in the analysis for each experiment is given by its expected value over all periods excluded the first 100 and over all iterations. The waste is given by

$$w_t = \left(x_{1,t} - d_{a,t} - (d_{b,t} - x_{2,t})^+ \right)^+, \tag{4.46}$$

and its overall value used in the analysis for each experiment is given by its expected value over all periods excluded the first 100 and over all iterations. At last, the fill-rate is given by

$$\zeta = \begin{cases} \dfrac{(d_{a,t} + d_{b,t} - x_{1,t} - x_{2,t})^+}{d_{a,t} + d_{b,t}} & d_{a,t} + d_{b,t} > 0 \\ 1 & d_{a,t} + d_{b,t} = 0 \end{cases}, \tag{4.47}$$

and its overall value used in the analysis for each experiment is also given by its expected value over all periods excluded the first 100 and over all iterations.

Table 4.5: Parameters used in the estimations evaluations simulations from the third phase of the DOE.

Parameter	Value	Unit
Fixed parameters		
Maximum shelf-life (M)	2	days
Ordering frequency (τ)	1	days
Delivery time (L)	1	days
Inventory horizon (T)	1100	days
Number of simulations (I)	100	Simulations
Sales price (c_s)	25	*NOK*
Unit cost (c_u)	15	*NOK*
Holding cost (c_h)	0.15	*NOK*
Factorized parameters		
Mean demand of main simulation (λ)	$\{3, 20, 50\}$	units
FIFO probability of main simulation (p)	$\{0.3, 0.7, Prop^*.\}$	units
Replenishment policy	{LA_LIFO, LA_FIFO, LA_50, LA_EST, LA_PROP, BSP}	——

*Demand distribution over remaining-shelf life is proportional to the quantity of items correspondent is proportional to the quantity of items correspondent to each remaining-shelf life available in stock.

Chapter 5

Analysis

The third research question about the impact of the estimations derived of the model provided hereby on profit, fill-rate and waste is addressed in this chapter. This is performed by using the estimations in a stock-age dependent replenishment policy, called hereby look-ahead policy, which uses a look-ahead spare sampling tree method to calculate in each ordering period the expected non-satisfied demand until the next ordering period. The ordering quantity is stipulated to satisfy this non-satisfied expected demand aiming at profit maximization. Therefore, this chapter basically presents the analysis of the outcomes of the experiments performed following the factorized DOE outlined in section 4.3.

Primarily in section 5.1, the estimations of the factorized experiments from the first phase of the DOE are displayed and evaluated. Thereafter in section 5.2, the values for the performance measurements resulted from the experiments from the third phase which used these estimations as input are analyzed. At first, general considerations about the results are presented at section 5.2.1. Then, a more detailed analysis only considering the variations of the demand distributions applied to the main simulation of each experiment is performed in section 5.2.2. At last, a more detailed analysis only considering the variations of the replenishment policies applied to the online simulation of each experiment is performed in section 5.2.3.

5.1 Estimations

Table 5.1 displays the estimated values of the demands ($\hat{\lambda}_a$, $\hat{\lambda}_b$ and $\hat{\lambda}$) and their percentage errors in relation to their respective real values (λ_a, λ_b and λ) of all experiments performed in the first phase from the DOE as indicated in section 4.3. Each experiment can be identified by a number followed by a trace and a distribution description displayed in the first column of the table. The number depicts the mean demand which was used in the experiment and the distribution description indicates either the value of p adopted in the simulations or if the simulations of the experiment followed the proportional distribution from Vaughan (1994) and Ferguson and Ketzenberg (2006). The estimated value of the total mean demand, $\hat{\lambda}$ is the sum of $\hat{\lambda}_a$ and $\hat{\lambda}_b$. The experiments which adopted the

proportional distribution of demands in relation to the remaining shelf-life, called hereby proportional-distributed, did not present values of the percentage error of $\hat{\lambda}_a$ and $\hat{\lambda}_b$ since the simulations were not modeled with Poisson distributed demand for the FIFO and LIFO detached.

Table 5.1: Results of estimations of the experiments from the first phase of the DOE.

λ-**Dist.**	$\hat{\lambda}_{\mathbf{a}}$	**PE** $\hat{\lambda}_{\mathbf{a}}$	$\hat{\lambda}_{\mathbf{b}}$	**PE** $\hat{\lambda}_{\mathbf{b}}$	λ	**PE** $\hat{\lambda}$
3-p=0.3	0.98	2.00	2.00	0.00	2.98	0.67
3-p=0.7	1.99	0.50	1.00	0.00	2.99	0.33
3-Prop.	1.30	——	1.62	——	2.92	2.67
20-p=0.3	5.99	0.17	14.00	0.00	19.99	0.05
20-p=0.7	14.13	0.91	5.94	1.00	20.07	0.35
20-Prop.	5.31	——	14.27	——	19.58	2.10
50-p=0.3	14.92	0.53	34.84	0.46	49.76	0.48
50-p=0.7	34.75	0.71	14.85	1.00	49.60	0.80
50-Prop.	9.41	——	39.99	——	49.41	1.18

The estimates of demand a and demand b of the experiments which used the factor p in the distribution of demand over the remaining shelf-lives of the items available, called hereby p-distributed, presented small percentage errors as expected considering the results from the convergence analysis from section 4.3.2. All percentage errors of $\hat{\lambda}_a$ and $\hat{\lambda}_b$ were equal or lower than 1 % with the exception of $\hat{\lambda}_a$ of experiment 3-p=0.3 which presented a percentage error equals to 2 %. The value of $\hat{\lambda}_a$ in this case was 0.98 and the value of λ_a was 1.00.

The estimates of λ for the experiments which were p-distributed also showed accurate values. All values of $\hat{\lambda}$ presented percentage error lower than 1 %. In contrast, the experiments which were proportional-distributed presented values of $\hat{\lambda}$ with higher percentage error. All percentage errors were over 1 % in this case. The experiment with mean demand equals to 3 units presented a percentage error of 2.67 %, the one with mean demand equals to 20 units presented a percentage error equals to 2.10 % and the one with mean demand equals to 50 units presented a percentage error of 1.18 %.

When the demand was distributed over the remaining-shelf life available following a Binomial distribution i.e. each demand a and demand b followed a Poisson-Binomial distribution as a consequence of the use of the adoption of the probability p, the EM-Algorithm provided, in general, accurate estimations for both λ_a and λ_b, the parameters for demand a and demand b, and for λ, the parameter for the total demand. However when demand was not distributed over the remaining-shelf life available, the use of the same EM-Algorithm in the application adopted hereby, estimating demand a and demand b separately, provided less accurate estimations of λ. It is expected that applying the EM-Algorithm to estimate only the value of λ would provide more precise estimations. In addition, this indicates that the adoption of the Poisson distribution for the demand distribution when demand is not Poisson distributed does not model properly the behavior of the demand in an inventory system.

5.2 Comparison

Three main factors varied over the experiments in the third phase of the DOE which identify the experiments from which the results analyzed hereby derived. Two of them varied in the experiments from the first phase from the DOE which were the mean demand and how the demand was distributed in relation to the remaining shelf-life of the items. In addition to these two factors, the replenishment policy adopted in the experiment varied only in the third phase of the DOE.

As it was already explained, three mean demands were adopted. One with lower magnitude, equals to 3 units, one with medium magnitude, equals to 20 units, and the last with higher magnitude, equals to 50 units. Each of these demands were used with three different distributions considering the remaining shelf-life of the items in stock. Two distributions had a fixed probability p of customers picking up the oldest item and a probability $p-1$ of customers picking up the newest item considering perfect substitution between these two different demands. The two values of p elected to be part of experiments were 0.3 and 0.7. The third distribution implemented the proportional depletion policies also applied to Vaughan (1994) and Ferguson and Ketzenberg (2006), with the inventory following a Poisson distribution with determined mean demand λ and with a demand distribution over remaining-shelf life proportional to the quantity of items correspondent to each remaining-shelf life available in stock.

Six different replenishment policies were used in the experiments. Five of them consisted of a stock-age-dependent policy: The look-ahead policy considering demand distribution in relation to the remaining shelf-life correspondent to the FIFO depletion policy; correspondent to the LIFO depletion policy; correspondent to a depletion policy with 50 % of LIFO demand and the other 50 % of FIFO demand; considering p equivalent to the estimated values outlined in section 5.1; and the proportional distribution from Vaughan (1994) and Ferguson and Ketzenberg (2006). All input parameters from the look-ahead policies, i.e. the mean demands, utilized in the third phase of the DOE corresponded the estimated values displayed in table 5.1. The last replenishment policy applied was the BSP utilizing the same base stock as the one adopted in the inventory simulations for the data generation from the first phase of the DOE.

5.2.1 General results

Figure 5.1 presents the expected profit in NOK over all periods and all simulations from each experiment. Each chart of the figure contains the results of the experiments correspondent to one of the three mean demands adopted as identified above each of them. The units of the mean demands presented above each chart are units of items. Each curve in each chart indicated by a color and a symbol corresponds to the experiments which adopted one specific replenishment policy identified in the legend of the figure. At last, the horizontal axis of each chart contains three variables which identify how the actual demand was distributed over the items for each remaining shelf-life group available in the inventory simulations. The values of the profits for all experiments used to build up the charts from figure 5.1 are presented in table 5.2. The decimals of these values were not considered hereby due to the uncertainties derived from the simulations.
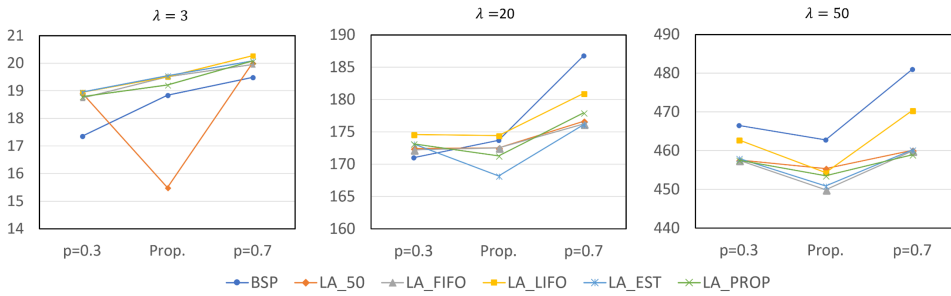
Figure 5.1: Expected profits vs. actual demand distribution for each mean demand applied.

The general results presented in each of the three charts from figure 5.1, provide clear evidences that the higher the demand, the higher is the expected profit per period. This result is expected since the higher the demand, the more items may be sold. Because each unit sold generates profits; each unit purchased, not sold and wasted generates loss; and all replenishment policies adopted hereby aimed at the profit maximization; the more items available to be sold, the higher is consequently be the profit generation. This also gives an indication that indeed the policies adopted succeed in maximizing the profitability in terms of the demand available although none of them were optimal policies.

In relation to the fill-rate results, higher demands led to higher fill-rates in all cases as it can be seen in figure 5.2 and the table 5.6. The differences between the values of the cases for $\lambda = 3$ units and $\lambda = 20$ units were higher than the difference between the values of the cases for $\lambda = 20$ units and $\lambda = 50$ units. The maximum value for the fill-rate within the experiments that considered $\lambda = 3$ units was about 0.86. The maximum value for the fill-rate within the experiments that considered $\lambda = 3$ units was virtually under 0.98 and the same maximum value for the experiments that considered $\lambda = 50$ units was virtually above 0.98 although both of them can be considered 0.98 due to the uncertainties derived from the simulations.

The general results of the waste expectations, displayed in figure 5.3 and table 5.7, showed similar behavior to the values for the profit and the fill-rate in relation to performance. The values for the experiments that considered $\lambda = 3$ units were lower than the values of the equivalent experiments considering $\lambda = 20$ and $\lambda = 50$ units. However the values for equivalent experiments considering $\lambda = 20$ and $\lambda = 50$ units are very proximate and therefore were in general considered the same.

The relation between profit and fill-rate tends to be direct i.e. when profits increase the fill-rate increases in an approximate proportion. This is true when the waste is not simultaneously high. That because by increasing the fill-rate, the profits also increase until the point that the loss on the profit due to waste becomes significant. Low fill-rate means that a high portion of the demand was not satisfied and potential profit was not incurred. In this case, the waste expectation tend to be low because a major part of the stock was used to satisfy the demand. Also when the fill-rates are high, the level of stock tend to exceed the demand and there is a higher occurrence of waste. The relation between profit and waste tend then to be indirect until a certain limit which is when there is mostly excess of stock in relation to the demand. When it happens, the higher the stock levels, the lower

the profit expectation becomes. This relation brings the necessity to use a term that in this master thesis is called hypothetical interval close to optima.

The hypothetical interval close to optima is considered to be the virtual interval close to the turning point of the expected profit; when by increasing the expected ordering quantity the expected waste increases, expected fill-rate increases and the expected profit starts to decrease. Following this dynamic relation between the expected ordering quantity, the expected profit, the expected fill-rate and the expected waste; five hypothetical points are identified:

1. Low ordering quantity, lower profit, lower fill-rate, lower waste - At this point the expected ordering quantity, the expected profit, the expected fill-rate and the expected waste are relatively low. The lower the expected ordering quantity becomes; the lower the expected profit, the expected fill-rate and the expected waste become.

2. High profit, high fill-rate and low waste - At this point the expected ordering quantity is higher than at the point number 1, the expected profit and expected fill-rate are considered relatively high and the expected waste is considered relatively low.

3. Highest profit, high fill-rate, waste high - That is the turning point aforementioned. The expected ordering quantity is higher than at the point number 2, the expected profit is close to the highest that it can be, the optimal value, the expected fill-rate is high and the expected waste becomes relatively high. By increasing the expected ordering quantity after this point, the expected profits decreases.

4. High profit, high fill-rate and high waste - The expected ordering quantity is higher than at the point 3, the expected profit, the expected fill-rate and the expected waste are still high. But by increasing the expected ordering quantity, the expected profit decreases.

5. Lower profit, higher fill-rate and higher waste - At this point the expected ordering quantity, the the expected fill-rate and the expected waste are relatively high. The expected profit is relatively low. The higher the expected ordering quantity becomes, the lower the expected profit becomes and the higher the expected fill-rate and the expected waste become.

The hypothetical interval close to optima is than the interval between the points 2 and 4. The designation hypothetical is used in this term because no deeper mathematical and statistical analysis proving the existence of these points were provided hereby. This hypothetical interval is defined by the set of results that are analyzed and by a subjective definition. Therefore, no concrete values are presented as the limits of this interval. However, the results analyzed hereby as much as results from literature available provides indications of the existence of these points.

5.2.2 Results considering variations on demand distribution

Considering the actual demand distribution over the items for each remaining shelf-life group available, the results showed as one can expect that the profitability is higher when $p = 0.7$ for all mean demands and replenishment policies adopted over the experiments.

This result is expected because in the experiments with $p = 0.7$ the oldest items are depleted in a higher proportion than the newest items what leads to less waste and therefore more of the money spent on the ordered items are turned into profit. Indeed, the waste results were lower for the experiments which considered $p = 0.7$ than for the other experiments with the exception of one experiment that considered proportional distribution of actual demand, $\lambda = 3$ units and look-ahead replenishment policy with $p = 50$ %. The fill-rate values have indicated tendency to be higher when $p = 0.7$ however there were some exceptions and have shown less variation over the actual demand distribution variations.

With regards to the proportional demand distribution, the results were less consistent in relation to the general results of the experiments for each mean demand adopted. In general, the experiments that adopted proportional distribution of demand presented poorer results considering profitability. This is expected because all ordered items arrive with the highest remaining shelf-life possible in the experiments analyzed hereby. Since all replenishment policies aimed at maximization of profit meaning that they also aimed at fulfilling the expected demand avoiding excess of stock and waste in this case, at each period the quantity in stock of items with remaining shelf-life equals to 2 periods that corresponds to the quantity of items ordered at the previous period should approximate the quantity of total demand. Then once these items fulfill the demand, a little quantity or none of them remain in stock and are carried over to the next period. Therefore in each period, the quantity of items with remaining shelf-life equals to 2 periods in stock is much higher than the quantity of items with remaining shelf-life equals to 1 period. Since the demand for the oldest products and the demand for the newest products are proportional to their respective quantity in stock, the demand for the newest products i.e. the products with the highest remaining-shelf life tends to be much higher driving virtually the probability p to lower levels.

This effect can be grasped in table 5.1 which has shown that the experiment that considered $\lambda = 3$ and proportional distribution of demand resulted in $\hat{\lambda_a} = 1.30$ and $\hat{\lambda_b} = 1.62$. In this experiment the virtual estimation of p was approximately 0.45. This value was designated as a virtual value because in the cases that demand was considered proportionally distributed there was no value of p attributed to the model. The experiment that considered $\lambda = 20$ and proportional distribution of demand resulted in $\hat{\lambda_a} = 5.31$ and $\hat{\lambda_b} = 14.27$. In this experiment the virtual estimation of p was approximately 0.27. The experiment that considered $\lambda = 50$ and proportional distribution of demand resulted in $\hat{\lambda_a} = 9.41$ and $\hat{\lambda_b} = 39.99$. In this experiment the virtual estimation of p was approximately 0.19. The virtual value of p clearly decreased once the value of λ increased when the proportional distribution was adopted in the inventory model.

The reason that these results were considered less consistent is that not all curves of the three charts from figure 5.1 presented the expected behavior, poorer results for the proportional distribution. This effect occurred in a more accentuated scale for higher demands. For instance in the chart for $\lambda = 50$ units from figure 5.1, all curves presented clearly the poorest results for the experiments that adopted proportional distribution demand. In the chart for $\lambda = 20$ units, the curves which presented this effect presented it in a much less accentuated scale of difference. In addition, four curves didn't present this effect such as the one which used the look-ahead replenishment policy considering full LIFO, full FIFO and 50 % demand distribution; and the one which used the BSP replenishment policy. In

Table 5.2: Expected profits results per actual demand distribution for each mean demand applied.

Mean Demand	Replenishment Policy	Demand Distribution		
		p=0.3	Prop.	p=0.7
3	BSP	17	19	19
	LA_50	19	15	20
	LA_EST	19	20	20
	LA_FIFO	19	20	20
	LA_LIFO	19	20	20
	LA_PROP	19	19	20
20	BSP	171	174	187
	LA_50	172	173	177
	LA_EST	173	168	176
	LA_FIFO	172	173	176
	LA_LIFO	175	174	181
	LA_PROP	173	171	178
50	BSP	467	463	481
	LA_50	458	455	460
	LA_EST	458	451	460
	LA_FIFO	458	450	460
	LA_LIFO	463	454	470
	LA_PROP	458	454	459

the chart for $\lambda = 3$ units, none of the curves presented this effect with the exception of one experiment. In all experiments in this case the profits for when the actual demand was proportional distributed were higher than when p was equal to 0.3 but the experiment which considered look-ahead replenishment policy with $p = 50 \%$.

The occurrence or non-occurrence of this effect can be clearly grasped through examination of table 5.2. The accentuated difference between the values for the experiments with proportional demand distribution and the other two for the experiments with mean demand equals to 50 units can be clearly observed. However, when the mean demand is equal to 20 and 3 units the difference between the values for the profits for the experiments with proportional demand distribution and demand distribution with $p = 0.3$ becomes less accentuated and can be disregarded due to the uncertainties derived from the simulations.

Tables 5.3, 5.4 and 5.5 present the full set of outcomes besides profits, fill-rates and waste data for all experiments that adopted respectively mean demand equals to 3, 20 and 50 units. All values presented in these tables present expected outcomes over all periods and all simulations analyzed. The first column of these tables outline what were the demand distribution and replenishment policy used in each experiment displayed. The second and third column present the values correspondent to stock quantity for the items with remaining shelf-life equals to 2 and 1 period respectively. The fourth column presents the values correspondent to the order quantity. The fifth and sixth columns present the values correspondent to the demand of the newest items and the oldest items respectively.

The seventh and eighth columns present the values correspondent to satisfied demand for the newest and oldest items available respectively. The ninth and tenth columns present the correspondent values of sold items for the items with remaining shelf-life equals to 2 and 1 period respectively. Finally, the eleventh and twelfth columns present the correspondent values of lost sales related to the demand for the newest and oldest items respectively.

The effect of the proportionality on the demand for the experiments that used the proportional demand distribution can be seen in these tables. As mentioned, the objective of maximizing profit of all replenishment policies adopted leads to little carry-over stock from one period to the other. Indeed, the quantity of items with remaining shelf-life equals to 1 period in stock is not significant if compared to the quantity of items with remaining shelf-life equals to 2 periods in stock. Consequently since the demand in relation to the remaining shelf-life of the items is proportional to the quantity of items in stock correspondent to each remaining-shelf life, the values of the demand for the oldest items were much lower than the demand for the newest items for the experiments that considered proportional distribution of demand. In fact in all three tables, tables 5.3, 5.4 and 5.5, the demand a for all experiments that used proportional distribution of demand were the lowest ones for each replenishment policy considered.

When demand is high, the difference between the stock quantities is also high. This difference is indeed higher than the same difference when demand is lower. For example considering mean demand equals to 50 units, the ordering level will be about 48 units and therefore the expected quantity of items with remaining shelf-life equals to 2 periods in stock per period will be also equals to about 48 units. Considering the replenishment policies used hereby aim at maximizing the profitability and therefore reducing excess on the purchases, quantity of items that are carried over and remain in stock with a remaining shelf-life equals to 1 period should be much lower than the stock of items with remaining shelf-life equals to 2 periods. But they are not zero. The stochastic nature of the inventory system which leads to demand uncertainty naturally generates positive stock of items with remaining shelf-life equals to 1 period.
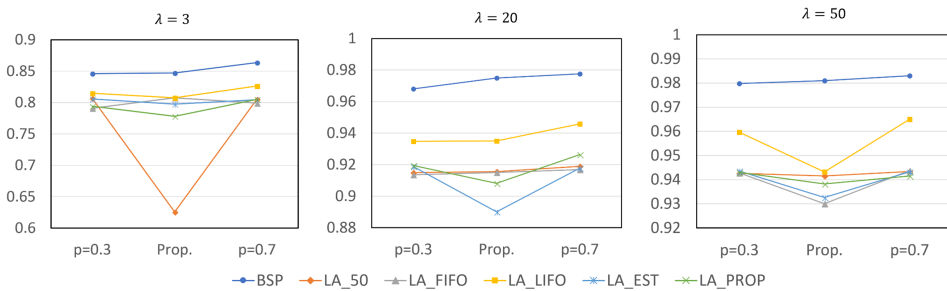
Figure 5.2: Expected fill-rate vs. actual demand distribution for each mean demand applied.

In the experiments that adopted mean demand equals to 50 units, the expected quantity of items with remaining shelf-life equals to 1 period is equal to values between 2 and 5 units depending on the replenishment policy adopted with some exceptions that did not surpass 10 units. These values are less than 20 % of the total demand. The experiments with mean demand equals to 20 units, presented an expected quantity of items with re-

Table 5.3: Experiments outcome when mean demand was equal to 3 units.

Dist.-Rep.	Stock 2	Stock 1	Ord.	Dem. b	Dem. a	Sat. Dem. b	Sat. Dem. a	Sal. 2	Sal. 1	L. Sal. b	L. sal. a
p=0.3-LA_LIFO	2.57	0.61	2.57	2.00	1.00	1.70	0.61	1.96	0.36	0.30	0.38
p=0.3-LA_FIFO	2.42	0.53	2.42	1.99	1.00	1.65	0.57	1.89	0.33	0.35	0.42
p=0.3-LA_50	2.50	0.57	2.50	1.99	1.00	1.68	0.60	1.94	0.34	0.31	0.40
p=0.3-LA_EST	2.51	0.57	2.51	2.00	1.00	1.68	0.60	1.94	0.34	0.32	0.40
p=0.3-LA_PROP	2.45	0.54	2.45	2.00	1.00	1.66	0.58	1.91	0.33	0.34	0.42
p=0.3-BSP	2.99	1.03	2.99	2.00	1.00	1.75	0.76	1.96	0.55	0.26	0.24
p=0.7-LA_LIFO	2.57	0.78	2.57	1.00	1.99	0.95	1.42	1.79	0.58	0.05	0.57
p=0.7-LA_FIFO	2.40	0.62	2.40	1.00	2.00	0.93	1.32	1.78	0.48	0.07	0.68
p=0.7-LA_50	2.43	0.64	2.43	1.00	1.99	0.94	1.34	1.78	0.49	0.06	0.65
p=0.7-LA_EST	2.43	0.64	2.43	1.00	2.00	0.94	1.34	1.78	0.49	0.06	0.66
p=0.7-LA_PROP	2.45	0.66	2.45	1.00	2.00	0.94	1.35	1.79	0.50	0.06	0.65
p=0.7-BSP	2.92	1.16	2.92	1.00	1.99	0.94	1.62	1.76	0.80	0.06	0.37
Prop.-LA_LIFO	2.57	0.64	2.57	2.54	0.46	1.93	0.41	1.93	0.41	0.61	0.05
Prop.-LA_FIFO	2.57	0.64	2.57	2.54	0.46	1.93	0.41	1.93	0.41	0.61	0.05
Prop.-LA_50	1.71	0.23	1.71	2.74	0.26	1.48	0.18	1.48	0.18	1.26	0.07
Prop.-LA_EST	2.50	0.59	2.50	2.55	0.45	1.91	0.39	1.91	0.39	0.64	0.05
Prop.-LA_PROP	2.39	0.53	2.39	2.57	0.42	1.86	0.36	1.86	0.36	0.72	0.05
Prop.-BSP	2.97	1.05	2.98	2.26	0.74	1.92	0.64	1.92	0.64	0.34	0.10

Table **5.4**: Experiments outcome when mean demand was equal to 20 units.

Dist.-Rep.	Stock 2	Stock 1	Ord.	Dem. b	Dem. a	Sat. Dem. b	Sat. Dem. a	Sal. 2	Sal. 1	L. Sal. b	L. sal. a
p=0.3-LA_LIFO	18.77	2.28	18.77	14.01	6.01	13.81	4.56	16.50	1.88	0.20	1.45
p=0.3-LA_FIFO	18.08	1.70	18.08	14.00	6.00	13.66	4.20	16.39	1.47	0.34	1.79
p=0.3-LA_50	18.12	1.71	18.12	14.00	6.00	13.68	4.21	16.41	1.48	0.33	1.78
p=0.3-LA_EST	18.24	1.76	18.24	13.99	6.02	13.71	4.28	16.48	1.51	0.28	1.74
p=0.3-LA_PROP	18.24	1.79	18.24	14.00	5.99	13.71	4.28	16.46	1.53	0.29	1.72
p=0.3-BSP	20.34	4.31	20.34	14.02	5.99	13.80	5.40	16.03	3.17	0.22	0.59
p=0.7-LA_LIFO	18.77	3.76	18.77	6.00	14.00	6.00	12.63	15.01	3.63	0.00	1.37
p=0.7-LA_FIFO	17.98	2.00	17.98	6.01	14.00	6.01	11.94	15.98	1.98	0.00	2.06
p=0.7-LA_50	18.02	2.06	18.02	6.00	14.00	6.00	12.00	15.96	2.04	0.00	2.01
p=0.7-LA_EST	17.97	2.05	17.97	6.00	13.99	6.00	11.95	15.92	2.02	0.00	2.04
p=0.7-LA_PROP	18.18	2.32	18.18	6.00	13.99	6.00	12.15	15.86	2.29	0.00	1.84
p=0.7-BSP	19.65	5.69	19.65	6.01	14.00	6.01	13.41	13.96	5.45	0.00	0.59
Prop.-LA_LIFO	18.77	2.16	18.77	18.20	1.80	16.61	1.76	16.61	1.76	1.60	0.05
Prop.-LA_FIFO	18.13	1.63	18.13	18.52	1.48	16.49	1.40	16.49	1.40	2.03	0.07
Prop.-LA_50	18.15	1.65	18.15	18.52	1.48	16.50	1.41	16.50	1.41	2.02	0.07
Prop.-LA_EST	17.46	1.19	17.46	18.87	1.11	16.27	1.04	16.27	1.04	2.60	0.06
Prop.-LA_PROP	17.93	1.48	17.93	18.65	1.34	16.45	1.28	16.45	1.28	2.20	0.06
Prop.-BSP	20.34	4.33	20.34	16.59	3.37	16.01	3.29	16.01	3.29	0.58	0.07

Table 5.5: Experiments outcome when mean demand was equal to 50 units.

Dist.-Rep.	Stock 2	Stock 1	Ord.	Dem. b	Dem. a	Sat. Dem. b	Sat. Dem. a	Sal. 2	Sal. 1	L. Sal. b	L. sal. a
p=0.3-LA_LIFO	48.05	4.84	48.05	35.02	14.99	34.97	12.69	43.21	4.45	0.04	2.31
p=0.3-LA_FIFO	46.83	3.04	46.83	34.99	15.00	34.87	11.82	43.79	2.91	0.12	3.17
p=0.3-LA_50	46.84	3.04	46.84	35.00	15.01	34.88	11.82	43.80	2.90	0.12	3.18
p=0.3-LA_EST	46.90	3.10	46.90	35.01	15.01	34.91	11.85	43.80	2.96	0.10	3.16
p=0.3-LA_PROP	46.84	3.05	46.84	34.99	14.99	34.90	11.80	43.79	2.91	0.09	3.19
p=0.3-BSP	49.68	7.64	49.68	35.00	15.01	34.88	13.93	42.04	6.77	0.11	1.08
p=0.7-LA_LIFO	48.05	7.41	48.05	15.00	35.02	15.00	32.98	40.64	7.34	0.00	2.04
p=0.7-LA_FIFO	46.74	3.27	46.74	14.97	35.01	14.97	31.77	43.47	3.27	0.00	3.24
p=0.7-LA_50	46.76	3.23	46.76	15.02	35.00	15.02	31.74	43.53	3.23	0.00	3.26
p=0.7-LA_EST	46.77	3.20	46.77	15.02	35.01	15.02	31.75	43.57	3.20	0.00	3.27
p=0.7-LA_PROP	46.64	3.15	46.64	15.00	35.01	15.00	31.64	43.50	3.15	0.00	3.37
p=0.7-BSP	49.02	8.96	49.02	15.00	35.01	15.00	34.00	40.05	8.95	0.00	1.01
Prop.-LA_LIFO	47.07	2.93	47.07	47.36	2.62	44.14	2.57	44.14	2.57	3.22	0.05
Prop.-LA_FIFO	46.19	2.20	46.19	47.90	2.09	43.99	2.00	43.99	2.01	3.91	0.08
Prop.-LA_50	46.96	2.74	46.96	47.52	2.55	44.22	2.47	44.22	2.47	3.29	0.08
Prop.-LA_EST	46.35	2.29	46.35	47.85	2.14	44.06	2.08	44.06	2.08	3.79	0.07
Prop.-LA_PROP	46.71	2.54	46.71	47.65	2.36	44.17	2.29	44.17	2.29	3.48	0.07
Prop.-BSP	50.01	6.97	50.01	44.10	5.90	43.04	5.82	43.04	5.82	1.06	0.08

maining shelf-life equals to 1 period equals to values between 1 and 3 units depending on the replenishment policies with some exceptions that did not surpass 5 units. Although these values were lower than the values for the experiments with mean demand equals to 50 units, they correspond to a proportion of up to 25 % of total demand, higher than the 20 % proportion of the experiments with mean demand equals to 50 units.

Following this tendency, the experiments with mean demand equals to 3 units, presented an expected quantity of items with remaining shelf-life equals to 1 period equals to values between 0 and 1 units depending on the replenishment policies with some exceptions that did not surpass 2 units. These values correspond to a proportion of up to 66 % of the expected quantity of the items with remaining shelf-life equals to 2 periods. Much higher compared to the experiments that adopted mean demand equals to 20 and 50 units. That indicates that the lower the mean demand, the more balanced the stock becomes in terms of quantity per remaining shelf-life category. This leads to a more balanced demand in terms of remaining shelf-life of items when the replenishment stock applied is a stock-age dependent considering the proportional probability distribution.

The waste and fill-rate results follow this effect as well. For the experiments with $\lambda = 50$ units, the waste values are higher and the fill-rate values are lower when the demand is proportionally distributed in the major of the cases. For the experiments with $\lambda = 20$ units, this effect is less apparent. For the experiments with $\lambda = 3$ units, the waste values are lower and fill-rate values are higher in the major of the cases.

5.2.3 Results considering variations on replenishment policies

The general results considering the different replenishment policies adopted weren't the same over the experiments with different mean demands either. In the experiments with mean demand equals to 3 units, the look-ahead policy considering FIFO, LIFO and the estimated distributions presented best performances. The results for these policies in this case are considered equivalent because the difference between them is virtually non-existent due to the uncertainties derived from the simulations as one can see in table 5.3. The experiments which adopted the look-ahead replenishment policy considering proportional distribution and $p = 50$ % presented equivalent best performance results when the actual demand was distributed considering $p = 0.3$ and $p = 0.7$ but an accentuated poorer performance when the actual demand was proportional distributed. The experiment with look-ahead policy considering proportional distribution presented a slightly poorer result, considered equivalent to the result from the experiment which adopted BSP policy. But the experiment with look-ahead policy considering $p = 50$ % presented an accentuated poorer result. In this case one experiment presented lower profits than the BSP which presented the poorest performances with the exception of this case. This accentuated poorer result was perceived as a deviation since it does not follow the variations over the actual demand distribution of the other experiments that considered other replenishment policies.

The deviation mentioned above is assumed to be originated from the estimation inaccuracy. In this case, the expected ordering quantity, of $1,71$ units, was much lower than the expected ordering quantity from the other experiments that adopted the look-ahead replenishment policy with mean demand equals to 3 units, between 2.39 and 2.57 units, as shown in table 5.3. This led to a much lower profit. Interestingly, the second worse performance among the experiments that considered proportional distribution of actual demand

was the BSP which presented an expected ordering quantity of 2.92 units, much higher than the other ones. These results indicate that the BSP policy overestimated the necessary ordering quantities and the look-ahead policy considering $p = 50\%$ underestimated the necessary ordering quantities in relation to close to optima ordering quantities, most probably in between 2.39 and 2.57 units, or close to it.

In the experiments that used the look-ahead policies, the ordering quantities were stipulated to fulfill expected demand that would not be fulfilled by the items in stock on the ordering period until the next ordering period. The ordering quantities depended on the estimations used in the look-ahead policy model. If the estimated parameters had adopted in the look-ahead replenishment policies underestimated the demand in relation to the actual demand of the inventory system, the ordering quantities could also be underestimated. Indeed, the estimation performed in the experiment with mean demand equals to 3 units and proportional demand distribution from the first phase of the DOE underestimated the actual demand. The actual mean demand was equal to 3 units and the estimation was equal to 2.92 units as one can see in table 5.1. Although, this estimation was used in all look-ahead policies for the experiments with $\lambda = 3$ units and demand proportionally distributed, it had a significant impact only on the experiment which considered the look-ahead replenishment policy with $p = 50\%$.

The fill-rate and the waste values for the experiments that considered $\lambda = 3$ units supported all indications mentioned above. In the experiments that considered the base stock replenishment policy, the fill-rate and the waste values were the highest ones among the other experiments that considered $\lambda = 3$ units. This indicates that the ordering quantities for the experiments with the base stock policy were indeed overestimated in this case. On the other hand in the experiment that adopted the look-ahead replenishment policy considering $p = 50\%$, both the fill-rate and waste values were much lower than the values of the other experiments that considered $\lambda = 3$ units. This indicates that the ordering quantities for the experiment with look-ahead replenishment policy considering $p = 50\%$ were indeed overestimated in this case. As expected, the fill-rates and waste values for the other experiments lied in between these two boundaries showing no abnormality.

Further investigation is required for more explanatory conclusions. Anyway, these indications show that the estimations may impact the inventory system even with a virtual low deviation. Furthermore, this supports the indication mentioned in section 5.1 that the adoption of the Poisson distribution for the demand distribution when demand is not Poisson does not model properly the behavior of the demand in an inventory system and this may impact the performance of replenishment policy if applied to a stock-age dependent replenishment policy.

In addition, the fact that all three look-ahead policies considering FIFO, LIFO and the estimated distributions presented equivalently the best performances indicated that knowing the actual distribution of the demand over the remaining shelf-life does not provide great advantages over standard LIFO and FIFO policies usually applied by industry, as mentioned in chapter 1, at least for low demands. Even the look-ahead policies considering proportional demand distribution were in general also approximately equivalent to the estimated case, with one exception that demands further investigation as it was already mentioned.

In the experiments with mean demand equals to 20 units, the base stock policy pre-

Table 5.6: Expected fill-rate results per actual demand distribution for each mean demand applied.

Mean Demand	Replenishment Policy	Demand Distribution		
		p=0.3	Prop.	p=0.7
3	BSP	0.85	0.85	0.86
	LA_50	0.81	0.63	0.81
	LA_EST	0.81	0.8	0.8
	LA_FIFO	0.79	0.81	0.8
	LA_LIFO	0.81	0.81	0.83
	LA_PROP	0.79	0.78	0.81
20	BSP	0.97	0.97	0.98
	LA_50	0.91	0.92	0.92
	LA_EST	0.92	0.89	0.92
	LA_FIFO	0.91	0.92	0.92
	LA_LIFO	0.93	0.94	0.95
	LA_PROP	0.92	0.91	0.93
50	BSP	0.98	0.98	0.98
	LA_50	0.94	0.94	0.94
	LA_EST	0.94	0.93	0.94
	LA_FIFO	0.94	0.93	0.94
	LA_LIFO	0.96	0.94	0.96
	LA_PROP	0.94	0.94	0.94

sented great variation on profit over the actual demand distribution considered. Regarding the experiments with $p = 0.3$, the BSP policy presented the poorest profit. It presented intermediate result considering the experiments with proportional demand distribution and presented the best performance considering the experiments with $p = 0.7$ in terms of profit. The experiments with look-ahead replenishment policy considering full LIFO and FIFO demand distributions and the case considering $p = 50\%$ presented virtually the same level of profits for the experiments with $p = 0.3$ and proportional demand distribution and higher profits when $p = 0.7$. The experiments that considered look-ahead policy with proportional and estimated demand distribution presented poorer results for the proportional demand distributed case, intermediate results for $p = 0.3$ and the best results for $p = 0.7$. The experiment with look-ahead replenishment policy considering LIFO demand distribution presented the best results among the experiments with look-ahead replenishment policy. The experiments that considered in their replenishment policies FIFO and proportional distributions and $p = 50\%$ presented virtually the same performance although the one which considered proportional distribution presented slightly poorer performance when the actual demand was proportionally distributed. The experiment with look-ahead replenishment policy which considered the estimated demand distribution presented virtually the same performance as the one which considered FIFO demand distribution when p was equal to 0.3 and 0.7. However, it presented a more accentuated poorer performance when the actual demand was proportionally distributed.
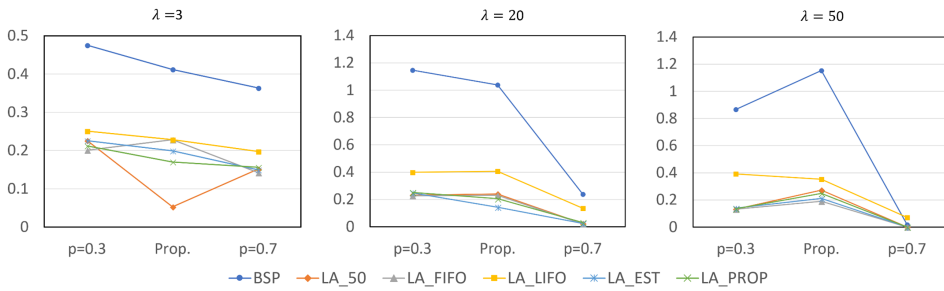
Figure 5.3: Expected waste vs. actual demand distribution for each mean demand applied.

Again, the look-ahead policy using the estimated distribution of demand didn't present any advantage in comparison to the other look-ahead policies. In fact, its results showed to be the poorest for when the demand was proportionally distributed. I general, the look-ahead policy using the estimated data impacted the inventory system differently in comparison to the impact of the base-stock replenishment policy which didn't use any replenishment data. The policies that presented the best performance in terms of profitability were the ones that presented the highest expected ordering quantities within the hypothetical interval close to optima, as mentioned before, that limits the underestimated ordering quantities, when the loss of potential sales impacts significantly the profitability, and overestimated ordering quantities, when the loss due to excess of stock also impacts significantly the profitability.

This conclusion is confirmed by the fill-rate and waste values as well. For example, the base stock replenishment policy provided the worst results regarding profitability among the experiments that considered $p = 0.3$ in the actual demand distribution. However, the same experiment provided the highest fill-rate and waste values. This confirms that the necessary ordering quantities in this case were overestimated. The waste values decreased in the BSP experiment that considered actual demand proportionally distributed. The fill-rate values increased but not so sharply as the profit values.

The difference between the waste and fill-rate values and the respective values from the other experiments also decreased. That means that the ordering quantities were close to the hypothetical close to optima interval and the profitability increased in relation to the other experiments considering the actual demand proportionally distributed. But its ordering quantities were still too high to overcome all the experiments with look-ahead policy. At last, fill-rate increased, again less sharply than the profit, and the waste values decreased when considering the actual demand distributed with $p = 0.7$. The difference between these waste and fill-rate values and the respective values from the other experiments decreased sharply. That led the profitability of the BSP experiment to become the highest one and its ordering quantity was most probably and also the highest one within the hypothetical close to optima interval.

In the experiments with mean demand equals to 50 units the results were more consistent. The experiments with base-stock replenishment policy presented the best performance. All experiments in this case presented the trend of the best performance when p is equal to 0.7, intermediate performance when p is equal to 0.3 and the poorest performance when the demand is proportionally distributed. This was expected considering the rela-

tively high total demand as it was already explained. The experiment that used look-ahead replenishment policy considering LIFO policy showed the best performance for all three demand distributions followed by the experiments that adopted look-ahead replenishment policies considering $p = 50$ % and the experiment that adopted look-ahead replenishment policy with proportional demand distribution. These three experiments had equivalent performance when actual demand was proportionally distributed. The experiments which presented the worst performance in terms of profit were the ones which adopted the look-ahead replenishment policy with FIFO demand distribution and estimated demand distributions. These two cases presented equivalent results as the experiments that adopted look-ahead replenishment policy with $p = 50$ % and proportional demand distribution.

The fill-rate and waste values were coherent with the profit results considering all of them were within the hypothetical close to optima interval as explained before. Most probably the BSP experiment which considered actual demand proportionally distributed resulted in a high peak for the waste value what shows that it probably overestimated slightly the necessary ordering quantities. But the profit for this experiment is still the highest among the equivalent experiments considering other replenishment policies. This leads to the conclusion that the overestimation was not too far from the hypothetical close to optima interval.

Once again, the look-ahead policy using the estimated demand distribution showed the poorest results in terms of profit for when the demand was proportionally distributed. In this case, the fill-rate and waste values showed that the look-ahead policy using the estimated demand distribution underestimated the necessary ordering quantities to provide the highest profits within the hypothetical close to optima interval.

The performance of the experiments which considered look-ahead replenishment policy with LIFO demand distribution also confirmed the effect of the higher ordering quantities in the hypothetical close to optima interval. This policy over all variations of the factorized parameters from the DOE in comparison to experiments adopting other replenishment policies but the same λ and actual demand distribution always provided either the best or the second best performance in terms of profit. A reason for this occurrence is considered to be the fact that considering that the demand distribution in the replenishment policy was LIFO. In the LIFO distribution, the newest items are picked first and the oldest items remain in stock. When this occurs with the estimations from the look-ahead replenishment policy, the estimated remaining demand until the next ordering period for each ordering period tend to increase, this leads to a stipulation of higher ordering quantities than the ones from the look-ahead policies considering other demand distributions. This is confirmed by the ordering quantity values from tables 5.3, 5.4 and 5.5. The ordering quantities from all experiments with look-ahead replenishment policy considering LIFO demand distribution were always the second highest in comparison to experiments adopting other replenishment policies but the same λ and actual demand distribution always provided either the best or the second best performance in terms of profit.

In general, the look-ahead policy using the estimated data impacted the inventory system differently in comparison to the impact of the base-stock replenishment policy which didn't use any estimated data. Although the base-stock policy experiments presented better performance in terms of profitability for higher actual demands, the look-ahead replenishment policy experiments provided more stable performance over the different parameters

and distributions of actual demand with the exception of one abnormality, the underestimation of ordering quantities of the experiment with look-ahead replenishment policy with $p = 50\,\%$, $\lambda = 3$ units and proportional demand distribution, as it was already mentioned. The look-ahead replenishment policy showed to have a higher adaptability over different demand factors since it didn't underestimate or overestimate the necessary ordering quantities within the hypothetical close to optima interval. But this conclusion may be changed if the heuristics base-stock quantities are calibrated properly for each demand factor. At the same time, the performance of the look-ahead policy may be improved by using a correction factor applied to each ordering quantity decision to increase slightly the ordering quantities achieving higher values within the hypothetical close to optima interval.

Table 5.7: Expected waste results per actual demand distribution for each mean demand applied.

Mean Demand	Replenishment Policy	Demand Distribution		
		p=0.3	Prop.	p=0.7
3	BSP	0.48	0.41	0.36
	LA_50	0.23	0.05	0.15
	LA_EST	0.23	0.2	0.15
	LA_FIFO	0.2	0.23	0.14
	LA_LIFO	0.25	0.23	0.2
	LA_PROP	0.21	0.17	0.16
20	BSP	1.15	1.04	0.24
	LA_50	0.23	0.24	0.02
	LA_EST	0.25	0.14	0.02
	LA_FIFO	0.23	0.23	0.02
	LA_LIFO	0.4	0.41	0.14
	LA_PROP	0.25	0.21	0.03
50	BSP	0.87	1.15	0.02
	LA_50	0.13	0.27	0
	LA_EST	0.14	0.21	0
	LA_FIFO	0.13	0.19	0
	LA_LIFO	0.39	0.35	0.07
	LA_PROP	0.13	0.25	0

Chapter 6

Discussion

This chapter presents a review of the outcomes of this master thesis relating them to the problem statement and research questions with a critical discussion investigating whether they were answered or not and how they were answered in section 6.1. Thereafter in section 6.2, the features of this project, potential improvements and suggestions for future research are discussed.

6.1 Review of outcomes

This master thesis was performed focusing on customers preferences with regards to the freshness of demanded same products available on the shelves from a retail store. The problem statement and main question is how customer preferences in relation to the remaining shelf-lives of demanded products can be estimated in an inventory model with complete upwards and downwards substitution. To answer this problem statement the planning to develop this thesis was dived in three research questions. These research questions did not just drive the actual work carried out but also the structure of this document. Therefore, the structure of this chapter also follows this organization. The following lines present the research questions and the main findings from the analysis relevant for each of them.

1. How can customer preferences of products' freshness be modeled within a stochastic inventory system?

This question was answered in section 2.1. In the inventory system presented hereby, the demand was modeled following a Poisson distribution and the preference of customers with relation to the remaining shelf-life of items available was modeled through the adoption of a Binomial distribution. Each unit of demand had a probability p of being satisfied with the oldest items i.e. FIFO depletion. This led to the conception of two demands. One demand for the oldest items and one demand for the newest items which were also

Poisson distributed. Furthermore, another demand distribution with relation to the remaining shelf-life of items found in the relevant literature available, called proportionally distributed, was also adopted in the developed inventory model in order to evaluate the behavior of the inventory system in a scenario which the demand is not necessarily constrained to the Poisson-Binomial distribution as initially modeled. The model allowed perfect substitution when the inventory system adopted both demand distributions.

The main difference between these two distributions is that the depletion of item in relation to the remaining-shelf life depends on the quantity of items for each remaining shelf-life category when the demand is proportionally distributed, and the demand does not depend on stock level when demand follows the Poisson-Binomial distribution. It means that no matter the replenishment policy, the depletion of items follows the pre-established distribution according to the arbitrated parameters when the demand follows the Poisson-Binomial distribution. On the other hand, how the demand is realized in relation to the remaining shelf-life of items may change over the periods depending on the replenishment policies adopted when the proportional distribution of demand is adopted.

Since all replenishment policies aimed successfully at profitability maximization, few items were carried over to the next period in the experiments carried out hereby. Therefore, the expected quantity of items with remaining-shelf life equals to 2 units were always as low as possible considering the stochastic nature of the model. Since the demand for the oldest products and the demand for the newest products are proportional to their respective quantity in stock when the proportional distribution of demand is adopted, the demand for the newest products i.e. the products with highest remaining-shelf life tend to be much higher driving virtually the probability p to lower levels as identified in the analysis, chapter 5.

The assessment whether either the proportional distribution or the Poisson-Binomial distribution represents the customer preferences with relation to the remaining shelf-life of the items more realistically require further research using a real case scenario. Even more because the use of both of them seems plausible from a speculative point of view.

The retail stores adopt usually merchandising and displaying strategies that provide incentives for customers to pick up the oldest items in order to avoid the losses incurred from waste. On the other hand, the customers tend to prefer the newest items to avoid the loss of waste at their premises. In this non-explicit tug-of-war between customers and retail stores, there is most probably a portion of customers who are eager to pursue the newest items in spite of the strategies from the stores and there is a portion which is usually satisfied with the oldest items in light of the displaying strategies. At the same time that these portions or probabilities may exist in an approximate distribution as the Binomial one, these probabilities may be affected by different factors such as the day of the week or, as adopted hereby, the quantity of items correspondent to each remaining-shelf life.

2. How can the parameters of the demand of such a model be estimated?

This research question was answered in section 4.2. Demand was estimated through the application of the maximum likelihood estimator with the support of the Expectation-Maximization algorithm. Among the methods available for such estimations, the setting of methods applied hereby was considered the most effective in terms of the quality of results in view of the complexity of the problem.

Indeed, the estimations performed in the convergence evaluation, section 4.3.2, and in the experiments from the design of experiment analyzed in section 5.1 presented satisfactory results. Four factors in the convergence analysis were considered fundamental for the consistency of the estimations and the running time of the algorithm: How much of the observed data-set is censored, the number of observations which are analyzed, the initializing estimated parameters and the number of loops in which the EM-Algorithm is run.

All experiments from the convergence analysis provided satisfactory results, with one exception, the experiment with $S = 35$ units which was the experiment that adopted the lowest base stock quantity. This led to an indication that indeed the more censored data-set in the inventory simulations the poorer results for the estimations the EM-Algorithm is expected to provide. A confirmation on this indication requires a deeper and structured analysis of the statistical and mathematical properties of the model.

With regards to the initializing estimated parameters, a more detailed analysis of the likelihood function and its shape is considered necessary for a more precise conclusion. But the choice of calculating the expected value of mean demand considering the available observations that were not censored, the realized sales, showed to be adequate. Running the simulations used in the estimation of the parameters with the heuristic replenishment policy adopted in the data generation was considered enough to provide adequate estimations. This includes the number of observations which were analyzed, about 500 periods and the number of simulations, 20 simulations.

The running time was also considered practicable. Although the results showed that the more censored data-sets over the periods of the inventory simulations the longer the EM-Algorithm took to run. The simulations which used the heuristic policies for data generation were taking in between 8 and 24 hours to run.

The experiments from the design of experiment which had their results analyzed in section 5.1 showed that when the demand was distributed over the remaining-shelf life available following a Binomial distribution i.e. each demand a and demand b followed a Poisson-Binomial distribution as a consequence of the use of the adoption of the probability p, the EM-Algorithm provided, in general, accurate estimations for both λ_a and λ_b, the parameters for demand a and demand b, and for λ, the parameter for the total demand. However when demand was not distributed over the remaining-shelf life available, the use of the same EM-Algorithm in the application adopted hereby, estimating demand a and demand b separately, provided less accurate estimations of λ. It is expected that applying the EM-Algorithm to estimate only the value of λ would provide more precise estimations. This indicated that the adoption of the Poisson distribution for the demand distribution when demand is not Poisson does not model properly the behavior of the demand in an inventory system.

3. What is the impact on profit, fill-rate, availability and waste of this model in relation to similar models which consider differently the preferences of products' freshness?

The starting point to answer this research question was the design of experiment presented in section 4.3. The design of experiment was built up to provide results that could be used to evaluate the impact on profit, fill-rate and waste of the model presented hereby, with the

Poisson-Binomial demand distribution, and the same impact of the estimations comparing it to different scenarios, including the proportional demand distribution.

Therefore in the design of experiment, a preliminary set of experiments were performed in order to generate data so the demand parameters could be estimated. For this, simulations with three demand distributions with a heuristic standard base stock replenishment policy were used in this preliminary set of experiments: Two demand distributions followed the Poisson-Binomial model proposed hereby and one of them followed the proportionally distributed model from Vaughan (1994) and Ferguson and Ketzenberg (2006). Afterwards, for each experiments the demand parameters were estimated considering that all experiments were modeled with the Poisson-Binomial distribution, even the ones modeled with the proportionally distributed demand. These estimations were used in a last set of experiments which was designed to test these estimations. For this, two different replenishment policies were used: One stock-age dependent policy, which depends on the level of stock for each shelf-life category and could use the estimated data from the first set of experiments executed, and the base stock policy used in the first set of experiments, which does not depend on the stock level for each age category and didn't use the estimated data. In addition, the experiments which considered the stock-age dependent replenishment policy were performed with five different variations of this replenishment policy: One considering full FIFO depletion policy, one considering full LIFO depletion policy, one considering $p = 50$ %, one considering the proportional distribution from Vaughan (1994) and Ferguson and Ketzenberg (2006), and the last one considering the estimated distribution.

The heuristic stock-age dependent policy used in the design of experiment called look-ahead sparse sampling tree policy, was adopted with the intention to investigate how the estimation would impact the inventory system. The investigation was performed through a sensitivity analysis of the factors that varied in the design of experiment over the experiments comparing the expected profits, fill-rate and waste resulted from the experiments.

The results presented clear evidences that the higher the demand, the higher the expected profit per period as expected is since the higher the demand the more items may be sold. This also gave an indication that indeed the policies adopted succeed in maximizing the profitability in terms of the demand available although none of them were optimal policies.

The term hypothetical interval close to optima was often used in the analysis chapter. This term is associated to the relation between the profit, fill-rate, waste and the ordering quantity. The results presented in the analysis indicated that that the relation between expected profit and expected fill-rate tends to be direct i.e. when profits increase the fill-rate increases in an approximate proportion. This was clearly true when the waste was not simultaneously high. When the expected fill-rate increases, the expected profit also increases until the point that the loss on the profit due to waste becomes significant. Low fill-rate means that a high portion of the demand was not satisfied and potential profit was not incurred. In this case, the waste expectation tend to be low because a major part of the stock was used to satisfy the demand. Also when the fill-rates are high, the level of stock tend to exceed the demand and there is a higher occurrence of waste. The relation between profit and waste also follows then a direct relation until a certain limit which is when there is mostly excess of stock in relation to the demand. When that happen, the higher the stock

levels, the lower the profit expectation becomes due to the loss because of waste. These levels of profit, fill-rate and waste depend on the ordering quantity which is derived from the replenishment policy and drives the stock-level.

The designation hypothetical is used in this term because no deeper mathematical and statistical analysis proving the existence of these points were provided hereby. This hypothetical interval is defined by the set of results that are analyzed and by a subjective definition. Therefore, no concrete values are presented as the limits of this interval. However, the analyzed results hereby as much as the results from literature available provide indications of the existence of these points.

Considering the actual demand distribution over the items for each remaining shelf-life group available, the results showed as one can expect that the profitability is higher when $p = 0.7$ for all mean demands and replenishment policies adopted over the experiments. This result is expected because in the experiments with $p = 0.7$ the oldest items are depleted in a higher proportion than the newest items what leads to less waste and therefore more of the money spent on the ordered items are turned into profit. The experiments which adopted the proportional demand distribution, presented less consistent results considering the general results of the experiments for each mean demand adopted.

In general, the experiments that adopted proportional distribution of demand presented poorer results considering profitability. This is expected because all items ordered arrive with the highest remaining shelf-life possible in the experiments analyzed hereby. Since all replenishment policies aimed at maximization of profit meaning that they also aimed at fulfilling the expected demand avoiding excess of stock and waste in this case, at each period the quantity in stock of items with remaining shelf-life equals to 2 periods that corresponds the quantity of items ordered at the previous period should approximate the quantity of total demand. Then once these items fulfilled the demand, a little quantity or none of them remained in stock and were carried over to the next period. Therefore in each period, the quantity of items with remaining shelf-life equals to 2 periods in stock is much higher than the quantity of items with remaining shelf-life equals to 1 period. Since the demand for the oldest products and the demand for the newest products are proportional to their respective quantity in stock, the demand for the newest products i.e. the products with highest remaining-shelf life tend to be much higher driving virtually the probability p to lower levels. This leads to the conclusion that the adoption of proportion demand distribution tend to increase the impact of the replenishment policy on the inventory system.

With regards to the replenishment policy, the look-ahead policy using the estimated data impacted the inventory system differently in comparison to the impact of the base-stock replenishment policy which didn't use any estimated data. Although the base-stock policy experiments presented better performance in terms of profitability for higher actual demands, the look-ahead replenishment policy experiments provided more stable performance over the different parameters and distributions of actual demand in general. Therefore, the look-ahead replenishment policy showed to have a higher adaptability over different demand factors since it didn't underestimated or overestimated the necessary ordering quantities within the hypothetical close to optima interval. But this conclusion may be changed if the heuristics base-stock quantities are calibrated properly for each demand factor most probably placing the outcome into the hypothetical close to optima interval. At the same time, the performance of the look-ahead policy may be improved by using

a correction factor applied in each ordering quantity decision to increase slightly the ordering quantities achieving higher values within the hypothetical close to optima interval. The benefit of it is that a general correction factor that places the outcome of the inventory system into the hypothetical close to optima interval over different demand parameters without the need for calibration for each demand scenario. Any concrete conclusion about it requires further research on this issue.

The look-ahead policy using the estimated distribution of demand didn't present any general advantage in comparison to the other look-ahead policies. Furthermore, the policies that presented best performance in terms of profitability were the ones that presented the highest expected ordering quantities within an hypothetical interval close to optima, for example the one adopting the LIFO distribution. This was related to how the replenishment policy was estimating the potential non-satisfied demand until the next ordering period for each ordering period. When LIFO distribution was considered in the replenishment policy, more non-satisfied demand was estimated and the ordering quantity was slightly higher than other demand distributions were considered, including the estimated one. The efficacy of the adoption of the estimated distribution in the replenishment policy in the inventory system hereby was considered low since it was considered that by using the correction factor mentioned above in the look-ahead policy used with any of the distribution considered hereby would improve their outcomes. However, further investigations of this model in a similar system with other variations of the parameter of the inventory system such as the ordering frequency, ordering quantity, incurred costs and delivery time are considered necessary for a better conclusion about the efficacy of the adoption of the estimated distributions as presented hereby in the inventory system.

Although the look-ahead policy using the estimated distribution has shown a higher adaptability over different demand factors, one experiment provided a particularly poor result, the experiment with $\lambda = 3$ units, proportional demand distribution and the look-ahead policy considering $p = 50$ %. This particularly poor result was considered as a deviation since it did not follow the pattern of variations over the actual demand distribution of the other experiments that considered other replenishment policies. This deviation is assumed to be originated from the estimation inaccuracy since the performed estimation in the experiment with mean demand equals to 3 units and proportional demand distribution from the first phase of the DOE underestimated the actual demand. The actual mean demand was equal to 3 units and the estimation was equal to 2.92 units as one can see in table 5.1. Although, this estimation was used in all look-ahead policies for the experiments with $\lambda = 3$ units and demand proportionally distributed, it had a significant impact only on the experiment which considered the look-ahead replenishment policy with $p = 50$ %. Further investigation is required for more explanatory conclusions. Anyway, this gives an indication that the estimations may impact the inventory system even with a virtual low deviation. Furthermore, this also indicates that the adoption of the Poisson-Binomial distribution for the demand distribution when demand is not Poisson-Binomial does not model properly the behavior of the demand in an inventory system and this may impact the performance of replenishment policy if applied in a stock-age dependent replenishment policy.

6.2   Limitations and research opportunities

Most of limitations presented in this section may become ingredients for further research. Therefore at the same time that the limitations are presented as drawbacks, they are often presented as suggestions for future investigation and research opportunities. That is the main reason that both limitation and research opportunities were combined in this section.

The first important limitation, or set of limitations, that should be addressed in this section are the limited range of factors in the design of experiment. These factors are features usually present in realistic inventory systems and that impact the outcome and the dynamics of the system. However, they also make the system more complex and difficult to study. One of the main guideline found in any book that handles mathematical modeling as a subject is, as Ingels (1985) puts, "the overriding principle for developing models is to start simple and develop to more complex forms" (Ingels, 1985, p. 32). Since no similar study as the one presented in this master thesis was found in the literature, the following limitations were conveniently employed in order to simplify the problem and assure the consistency of the model's results.

The most important factor that was limited in the design of experiment was the maximum shelf-life. As indicated in the literature review, chapter 2, most of the problems within inventory management of perishable items that were worked out in the references presented hereby considered first the simplified version of the problem with maximum shelf-life equals to 1 period and 2 periods before they studied the general case with M periods such as Van Zyl (1963) and Nahmias and Pierskalla (1973) for example. This master thesis did the same with the inventory model studied hereby. The maximum shelf-life of the items in the inventory was equal to 2 periods, not more nor less. Although this limitation may be considered realistic to the perishable products that do not have a long shelf-life, there are many products that may have their shelf-life modeled as more than 2 periods. Thence to study the same problem as the one studied hereby for an inventory system that consider a generic limitation of the shelf-life of the items is also relevant.

Modeling an inventory system of perishables considering different demands that are addressed to the depletion of items with different remaining shelf-life when the maximum remaining shelf-life is a generic term M higher than 2 periods is not a trivial task. The inventory model for such problem differs in a lot of aspects the inventory model which considers the maximum remaining shelf-life equals to 2. It is suggested hereby two approaches for such problem. One that considers one specific demand designated by a demand stochastic variable, considering the demand is stochastic, for each group of items with a unique remaining shelf-life. In this approach the demand variable may be split into M variables equivalent to the value of maximum remaining shelf-life. This may increase the complexity of the problem. A simplest version may most probably the most recommended approach for a next step in the research presented hereby. The approach that considers the same two components of demand as in the model adopted in this master thesis; demand a for the oldest items and a demand b for the newest items. Although the modeling of demand is equivalent as the model of demand used hereby, the fact that now the maximum remaining shelf-life is equal a generic variable M assumed to be higher than 2 periods increase the level of complexity of the problem in a lot of aspects.

First of all, opposing to the 2 periods system, in the M periods system it is not possible to quantify the quantity in stock of the items assumed old and the quantity in stock of the

items assumed new. Therefore, the events that may be observed in the inventory system under the consideration of M periods of maximum shelf-life products have a different characterization as the ones presented for the 2 periods case. Figure 6.1 shows a basic schematics of what are the events that are expected to be seen when the inventory considering the maximum shelf-life of the items equal to M periods counts with one demand for the oldest items and one demand for the newest items. In the figure, the demand for the oldest items is represented by the customer a and the demand for the newest items is represented by the customer b. The status for the inventory at period t can be seen at the upper section. Each square with or without the red dots represent the shelf for items with the same shelf-life. The shelves are designated with their respective shelf-life categories which varies from 1 to M. All three possible events considering the status of the inventory at period $t + 1$ that are used in the suggested model are outlined in the lower section. The events, further features and a sketch of the modeling for such inventory system are presented in the Appendix B.
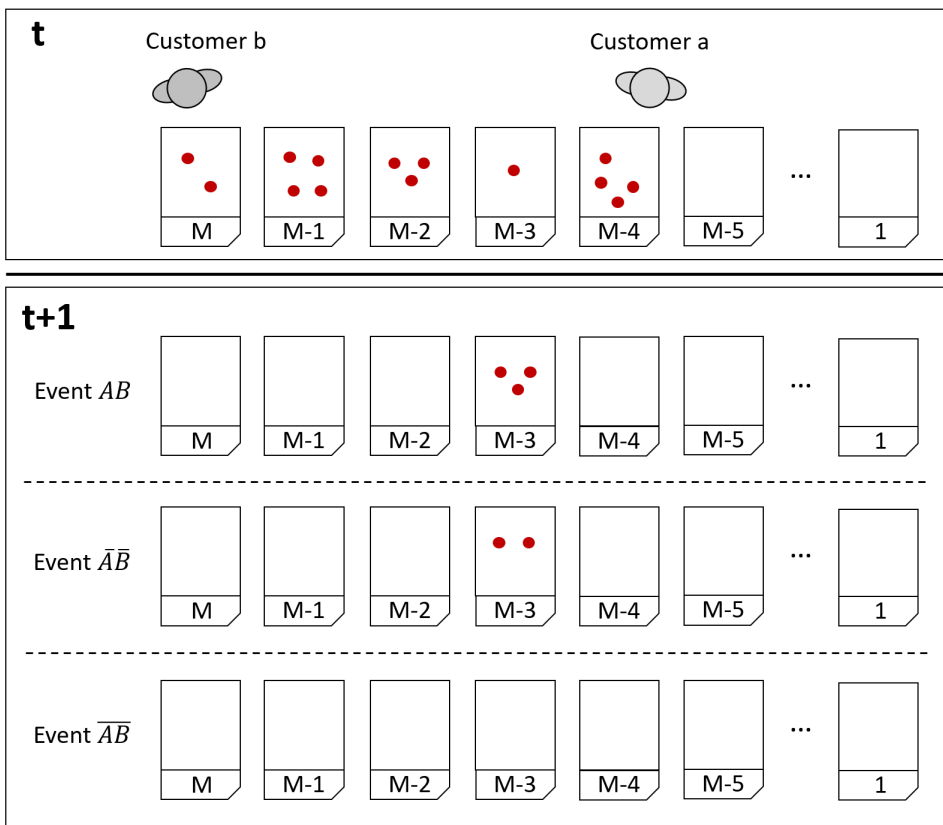

Figure 6.1: Schematic showing the possible events of the M-shelf-lives setting.

Another factor that was limited in the design of the experiment was the incurred costs in the profit equation. The ordering cost is a cost that is often incurred in the retail industry

and may result in a relevant impact to the decision making about replenishment and other inventory strategies was disregarded in the modeling presented hereby. Another factor related to the ordering of the items is the minimum ordering quantity or the batch quantity. These two factors, or variables, may impact the system differently depending on their values. Hence, a sensitivity analysis considering a range of values related to these variables also has great relevance to the study presented hereby.

Other two factors that plays a similar role to the inventory system as the ordering quantity and batch quantity concerning their impact to the system, are the ordering frequency and delivery time. Hereby, both of them were considering two invariant variables in the design of experiment. In addition, their value was as small as possible, equals to 1 period. By increasing both the ordering frequency and delivery time, it is expected that the impact of the uncertainties derived from the demand on the inventory also increase due to the stochastic nature of the problem. This may impact directly the performance of the replenishment policies and decisions what can result in different conclusions as the one presented hereby. Any further conclusion about the impact of the ordering frequency and delivery time on the inventory system considering the setting studied hereby requires also a sensitivity analysis considering a range of values designated to these variables.

Still on the factors that could be extended in the design of experiment, different demand distributions over the shelf-lives of the items in stock that was not considered hereby may also be considered in further research. Section 2.2 presented basically what is called hereby the base literature containing different references which used different approaches to model the demand distribution over the shelf-lives of the items in stock. One of the demand distributions was the proportional distribution associated to Vaughan (1994) and Ferguson and Ketzenberg (2006), exhaustively commented in this master thesis, and adopted in the design of experiment. However, two other distributions that were presented in section 2.2 were not adopted in the design of experiment: the uniform distribution associated to Sapountzis (1985) and Lowalekar et al. (2016), and the hypothetical distribution associated to Pegels and Jelmert (1970). These distributions were not adopted due to the resources limitations to perform the actual work for this master thesis, because the adoption of only one extra distribution was considered of extreme relevance for the sake of comparison and because the proportional distribution was considered the most relevant and realistic distribution. However, it is acknowledged that the adoption of one or two extra distributions may improve the scientific quality of the research and the reliability of the results. In addition, the adoption of other distributions could provide opening for other findings.

At last as explained in the literature review, pricing marketing actions such as pricing manipulation can drive consumer demand, a phenomenon called price elasticity of demand, which significantly influences operations management decisions in areas such as capacity planning and inventory control (Maddah et al., 2011). Price elasticity may also impact the choice of customer considering the remaining shelf-life of demanded items available on shelf. In fact, such strategy is often used by retail stores in order to avoid the wastege of items that are about to expire. Therefore, price elasticity is other feature that was not modeled in the inventory system studied hereby and could improve the the research on the preference of customers regarding remaining shelf-life.

Another limitation that also impacted the analysis of the results was the use of only

heuristics policies. No experiments considering optimal ordering on the replenishment policies adopted was used hereby. In an attempt to overcome this drawback, the hypothetical interval close to optima, an interval which hypothetically surrounds the optimal expected ordering quantity results, was identified. However, undoubtedly the use of optimal ordering policies may enrich the analysis and the conclusions.

No deeper mathematical and statistical analysis of the properties of the model adopted hereby was performed. Such analysis may also improve the scientific rigour of the results by improving for example the evaluation of the consistence of the EM-Algorithm considering the setting adopted in thus master thesis. An evaluation of the curve may for instance indicate if there is more than one local maximum, supporting the assessment about the initializing estimator from the EM-Algorithm.

At last, the considered most promising limitation and research opportunity is the fact that no real data was used in this thesis. A real case scenario data could be used to validate the model for specific conditions and enrich the design of experiment extending the factorized analysis. In addition, it may be used to improve the model adapting it to a more realistic context. In the initial discussions about the topic to be covered in this research was to find or develop a realistic model that captures the behavior of customer considering their preferences against the freshness of demanded items in in inventory simulations. The time consuming complications to access such real data within the time frame for submission of this work was the main reason for the non-acquaintance of the data. However, such validations is a far-reaching step regarding the progress of this work.

Chapter 7

Conclusion

This chapter provides a proper closure for this master thesis by linking the introduction to the discussion chapter. It starts by discussing what was the main contribution of the thesis. It then presents a straightforward answer to the problem statement and research questions.

A primary ambition for the thesis was to validate an existent depletion policy found in available literature or develop a new one that could capture the behavior of customer considering their preferences against the freshness of demanded items in in inventory simulations. The relevance of such topic ascends from motivations from the industry inasmuch as the technological development found in the retail environment that enables data collection and their use, and the statistical methods developed in the academia environment in its majority.

Due to the time consuming complications to access the real data that would enable such aspiration within the time frame for submission of this work, a step back was taken and this aspiration was not directly pursued. A more fundamental work towards the development of a model that may be used for the desired estimations and their investigations was carried out. Hence, this thesis provided a considered insightful outcome in spite of the limitations mentioned. The developed model frames a bedrock with its fruitful results for verification and validation with real data, and for further research extensions. In addition, the results and analysis presented a meaningful reasoning about the use of estimated values of demand considering an inventory system in the stochastic context presented hereby. That is why it is considered that most of limitations presented within the conclusion may become ingredients for further research.

The problem statement asked: How can customer preferences be estimated in an inventory model with complete upwards and downwards substitution? Therefore, this master thesis presented one alternative for the considered viable answer describing and evaluating a setting of tools adopted for such estimation. Moreover, it provided the results of a series of analysis to assess its impact in an inventory context.

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Appendix A

This meeting log is an adaptation of all minutes of the meetings that were held to discuss this master thesis. Therefore, a summary of what happened in each meeting with the main decisions are presented. The content and terms used in each minute of meeting were maintained in order to secure the realistic historic overview about what was discussed and decided. Hence, this appendix should be used just to support the comprehension of the development of the thesis and not as part of the scientific content of it.

#	Date	Topic	Participants	Notes
1	15.JAN.2018	Meeting with Relex Solutions AS: Master thesis proposal.	Bruno Montojos Heidi Dreyer Fred Graham Andreas Fischer	Bruno Montojos, the thesis author, Heidi Dreyer, co-supervisor for the master thesis, Fred Graham, country manager of Relex, and Andreas Fischer, project manager at Relex. Carl Philip, the supervisor for the thesis, was in China and wouldn't come back before second week of February. Discussions and decisions: • Formally, NTNU, through Bruno Montojos and Heidi was seeking a problem statement for Bruno's master thesis which has relevance in the industry context. • It was suggested that Relex contributed to the master thesis with suggestions for the problem statement. • Any further involvement of Relex could be decided later. • The scopes discussed in the meeting which most matched both NTNU and Relex interests and expertise were: Value chain, inventory management in retail context (Potentially management of perishables) and information sharing. • Mainly, two potential broad challenges to be worked out were discussed in the meeting: "Value of shared information to manufacturer/supplier perspective" and "What kind of information is worth to be shared and how this information can be shared to attend manufacturer/supplier interests". • Decided that Andreas Fischer would consult the researching steering group the discussed topics and would check if they had any other suggested points for the thesis.

| 2 | 26.JAN.2018 | Meeting with Relex Solutions AS: Master thesis proposal | Bruno Montojos Andreas Fischer | Suggested topics from Relex's steering group brought by Andreas Fischer:

"Relex suggested the following topic for Bruno's M.Sc. thesis:
- *Based on literature (and intuition), define a handful of potential issuing policies that could reflect the true shopping behavior of customers for fresh goods*
  - *E.g. LIFO, FIFO, random, combination of the previous models, something totally different / wild?*
  - *Different kinds of products may naturally require different models or model parameters*
- *Fit and validate the model(s) by simulating spoilage of products based on the model(s) and by comparing the predicted spoilage to realized spoilage. This study would be made using real-life balance, delivery, sales and spoilage data."*

NTNU, through Bruno Montojos and Heidi Dreyer, agreed that the suggested topics had relevance and could be worked out in a master thesis project. Then, the theme of the master thesis theme was settled to be the investigation of realistic depletion policies in a retail context.

Both parts decided that real data acquaintance showed to be necessary for such investigations. Heidi make herself available to contact companies for data acquaintance. |
| 3 | 30.JAN.2018 | Email to Relex: Master Proposal | Bruno Montojos Andreas Fischer Tuomas Viitanen | Aiming at the data collection, Bruno Montojos sent a series of questions about the inventory models adopted by Relex in order to match the data collection to Relex proceedings.

Tuomas Viitanen replied the email answering the questions and making himself available for future approach. |

| 4 | 13.FEB.2018 | Meeting with Supervisor: Master Thesis Kick-off | Bruno Montojos Carl P. Hedenstierna | This meeting was the formally kick-off for the project.

After discussed thoroughly with Heidi Dreyer about the direction which the master thesis could take, Carl Philip suggested that the time required for data acquaintance, quality check of data and the actual work with the data could be too much to be done over the five months available for the master thesis.
Carl Philip suggested that by using inventory simulation different depletion policies could be investigated. The following directions were suggested:
- Compare different depletion policies with regard to remaining shelf-life of items available on shelves in a retail context.
- Apply estimation theory to estimate the parameter(s) of the depletion considering censored demand from a demand modelled with known probability distributions. Methods suggested:
  - Maximum likelihood estimator (MLE),
  - Expectation-Maximazation Algorithm (EM-Algorithm),
  - Method of moments
- Bruno suggested he could research available literature and decide about the direction that the master thesis could take for the next meeting. |
| 5 | 15.FEB.2018 | Meeting with supervisor: Master Thesis follow up | Bruno Montojos Carl P. Hedenstierna | After a literature research and other considerations, Bruno decided that working on the estimation of customer preferences in relation to the remaining shelf-life of items available on shelves from stores in a retail context was the most relevant topic. The impact of different depletion policies considering the remaining shelf-life by using inventory simulation was already investigated in few works found in the literature available and no work adopting estimation methods to estimate such depletion policies were found.

Bruno decided for the application of MLE. Since the EM-algorithm is an extension of the MLE that is usually applied when the model is too complex for estimation performed purely using MLE, working on the MLE |

				at first and evaluating the need for the EM-Algorithm later was considered reasonable. Carl Philip indicated material that could be used to start the modelling of inventory and the application of the estimation methods elected.
6	22.FEB.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Bruno worked out an one period inventory model with maximum shelf life equals to two considering stochastic demand. Two demand distributions were used: one for the items with remaining shelf-life equals to one (old items) and one for the items with remaining shelf-life equals to two (new items). A manuscript with the work developed was shown to Carl Philip. Bruno proposed a sketch of a framework for the master thesis to be presented to Relex. Carl Philip elucidated some doubts and agreed with the sketch. It was decided that Bruno should keep working on the same model since it was not finished.

7	22.FEB.2018	Email to Relex: Master Thesis follow up	Bruno Montojos
			Andreas Fischer
			Tuomas Viitanen
				Bruno explained that the use of real data in the master thesis project could not fit the schedule considering the delivery date and communicate what was the plan.
				
				The following sketch of framework for the master thesis was proposed:
				
				1. Generate demand randomly following a specific distribution.
				2. Specify an issuing conversion rate p for FIFO (p-1 for LIFO policy. Two cases can be analyzed:
				   ○ Fixed p
				   ○ Different values of p.
				3. Consider as unknown information: Censored observations, distribution parameter and issuing conversion rate p.

				○ Known information is sales, inventory, waste, and other inventory parameters.
				4. Estimate demand by estimating the unknown information and apply it to a specified replenishment policy.
				   ○ Using the MLE / Expectation-Maximization algorithm.
				5. Test the performance of the literature issuing policies and the estimation policies using simulations. Following group of cases will be tested:
				   ○ Model that only uses FIFO and one that only uses LIFO.
				   ○ Literature cases,
				   ○ The case using the MLE / Expectation-Maximization algorithm.
				
				Tuomas replied the email, understood the issues regarding real data and agreed that using simulations in the research would be the next best option. He made some suggestions about the scope of the master thesis.

8	01.MAR.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Discussion and decisions: 1. Fix the equation discussed in the meeting. 2. Implement the balance equation from the manuscript (equation 13 from the MLE Problem document), with any fixing if necessary, for the two periods proposed problem in Mathematica. 3. A Mathematica license was provided to Bruno. 4. Debug and check if it works. 5. Simulate and extract the expected values of waste, sales for product a and product b, and any other relevant output. 6. Include the references in the main manuscript. 7. Add the pdf of the main papers with the relevant FIFO+LIFO issuing policies in the shared folder from dropbox. 8. Do further literature research.

9	12.MAR.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Good progress was made, and Bruno communicated that the model was too complex for the application of only the MLE method in the estimations. Therefore, he started working on the EM-Algorithm application for the estimations. Carl Philip approved.
10	23.MAR.2018	Meeting with Jørn Vatn: About data acquaintance	Jørn Vatn Bruno Montojos Swapnil Bhalla Carl P. Hedenstierna	Professor Jørn Vatn suggested this meeting because one of the students who he was supervising, Swapnil Bhalla, also worked with inventory management and was seeking to work with real data. He would attempt to acquire data with one of the partners companies. Bruno sent Jørn a list of demanded data for potential further research on the theme. Nothing was heard after this meeting.
11	23.MAR.2018	Meeting with supervisor: Master Thesis follow up		Bruno communicated that he was having some problems with the conditional distribution and the calculation of the expectations. Carl Philip indicated what could be done. Decided that Bruno should keep working with the EM-Algorithm with the same one period model with two different demand distributions, one for the old and other for the new items.
11	06.APR.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Bruno showed the results from the EM-Algorithm for the one period model. Bruno also communicated that he started working on the model considering a general maximum shelf-life of the items on shelf higher than two periods for a general inventory horizon over than two periods. Carl Philip revised the results and indicated that they looked coherent.

|				Carl Philip warned that the general model could add complications difficult to solve and that maybe Bruno should work on the one period case with maximum remaining shelf-life equals to two periods.
12	24.APR.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	After one week struggling with the modelling, Bruno decided that indeed the general period was too complex, and a simpler model should be worked out before. A model with maximum remaining shelf-life equals to two and a general inventory horizon considering more than two periods was developed and applied with the EM-Algorithm. Carl Philip approved the changes and suggested the what was done for the general case could be added as suggestion for further research in the thesis document.
13	16.05.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Follow up meeting to show the results and the thesis report.
14	30.05.2018	Meeting with supervisor: Master Thesis follow up	Bruno Montojos Carl P. Hedenstierna	Meeting to show the report. Carl Philip made his final suggestions about the report.

Appendix B

This appendix presents a sketch of the modeling that it is suggested for a similar inventory model developed in this master thesis but with a general maximum remaining shelf-life higher than 2 periods. In the M-shelf-lives setting similarly to the two-shelf-lives setting, λ_a and λ_b are also be estimated from their respective observed censored demand which is given by the realized sales. However, two differences between the settings adds two complications to the problem.

The realized sales in the two-shelf-lives setting was censored by $x_{1,t}$ and $x_{2,t}$. There were only two categories of items considering the remaining shelf-lives in the system which were related to these two ordering quantities and they could be compered directly to $d_{a,t}$ and $d_{b,t}$ as shown by the equations (4.29) and (4.30). In the M-shelf-lives setting, the realized sales is censored by the stock available to fulfill both demands. The stock available have potentially more than two shelf-lives, fact that, in this case, adds the first complication: With more than two shelf-lives, which of the items do compose the realized sales a and which of the items do compose the realized sales b for the demand estimation?

Analogously to how was described to the two-shelf-lives setting, the figure presented in this appendix shows all possible events that can happen in the M-shelf-lives setting. In this figure, customer a and customer b represent hypothetically the demand for the items a and items b respectively. The items a are the oldest items on the shelf and items b are the newest items on the shelf. Also hypothetically and in order to facilitate the comprehension, the items are distributed through the shelves which correspond respectively to their remaining shelf-life. Hence, the demand a starts to be gradually satisfied by the oldest items following a FIFO issuing policy and demand b starts to be gradually satisfied by the newest items following a LIFO policy. Customer a starts picking up the items from the shelf with lowest remaining shelf-life (far right). Once the shelf is empty and there is still remaining demand, customer a starts picking up the items from the immediate next available shelf correspondent to the less higher shelf-life items. Similarly, customer b starts picking up the items from the shelf with highest remaining shelf-life (far left). Once the shelf is empty and there is still remaining demand, customer b starts picking up the items from the immediate next available shelf correspondent to the less lower shelf-life items.

Once again, the event $\overline{AB}$ corresponds to the event which the demand is fully censored. That means that in this event the total demand at period $t + 1$, $d_{a,t+1} + d_{b,t+1}$ is equal or higher than the total inventory at period t. In such event, there is no tangible information available about demand a and demand b that can be used to estimate them since the magnitude of the demands were censored by the complete available stock.

The event AB corresponds to the fully non-censored demand similarly to the same event in the one-period setting. In such event, both the demand a and demand b are tangible and can be indicated by the realized sales at period $t+1$. The demand a corresponds to the realized sales of the items with lower shelf-life than the lowest shelf-life of the unaltered
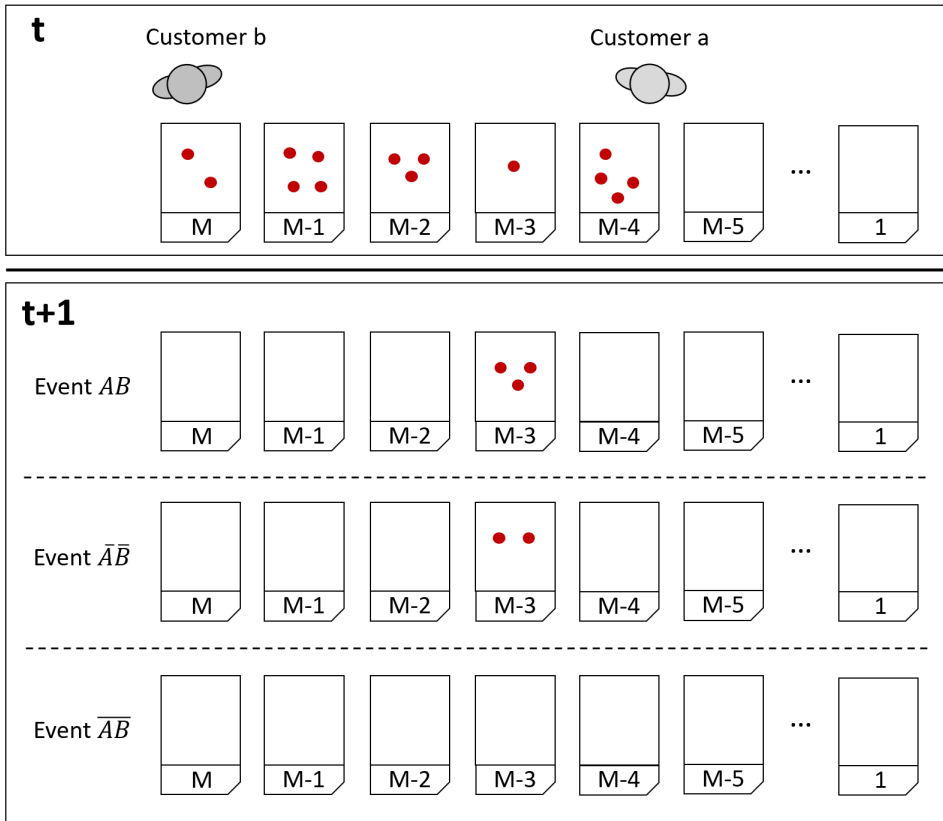
inventory. In the figure, the unaltered stock is represented by the shelf $M-2$ at period t and shelf $M-3$ at period $t+1$. Do note that the quantity of items from shelf $M-2$ didn't change at period $t+1$ at shelf $M-3$ when the stock was updated for the next period and items from shelves m were transferred to the shelves $m-1$. Equivalently, the demand b corresponds to the realized sales of the items with higher shelf-life than the higher shelf-life of the unaltered inventory. The unaltered inventory can consist of more than one shelf-life and it always corresponds to the items within the unaltered inventory boundaries i.e. all shelves that didn't have their quantity altered after and before the shelves that were altered on the inventory update from the period t to the period $t+1$.

At last, the event $\bar{A}\bar{B}$, which is analogous to the events $\bar{A}B$ and $A\bar{B}$, corresponds to a partially censored demand. In this case, there is no unaltered inventory, but there is remaining stock correspondent to one shelf-life. This event gives an idea about what is the demand a and the demand b, but there is no available information which supports the identification of which demand the items correspondent to the shelf which still contains items satisfied at the previous period. For example in the figure, there are still 2 units at period $t+1$, shelf $M-3$ and event $\bar{A}\bar{B}$. These items were transferred from shelf $M-2$ at period t which contained 3 units. The system does not provide any information to identify if this 1 unit of difference satisfied demand a or demand b. In order to overcome

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this lack of information an hypothetical new demand n is considered. This demand n is used similarly to the demand a and demand b to model the demand for the items from the unique remaining stock at event $\bar{A}\bar{B}$

The realized sales for demand a, demand b and demand n are given by the following equations:

$$r_{1,t+1} = \begin{cases} 0 & \text{for} \quad \overline{\mathbf{AB}} \\ \bar{x}_{a,t+1} & \text{for} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \\ d_{a,t+1} & \text{for} \quad \mathbf{AB} \end{cases} \qquad (7.1)$$

and

$$r_{2,t+1} = \begin{cases} 0 & \text{for} \quad \overline{\mathbf{AB}} \\ \bar{x}_{b,t+1} & \text{for} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \\ d_{b,t+1} & \text{for} \quad \mathbf{AB} \end{cases} \qquad (7.2)$$

and

$$r_{n,t+1} = \begin{cases} \bar{x}_t & \text{for} \quad \overline{\mathbf{AB}} \\ d_{n,t+1} & \text{for} \quad \bar{\mathbf{A}}\bar{\mathbf{B}} \\ 0 & \text{for} \quad \mathbf{AB} \end{cases} \qquad (7.3)$$

where the events $\overline{AB}$, $\bar{A}\bar{B}$ and AB correspond to the following conditions:

$$\begin{cases} \overline{AB} \rightarrow \bar{x}_t \leq d_{a,t+1} + d_{b,t+1} \\ \bar{A}\bar{B} \rightarrow \bar{x}_{a,t} + \bar{x}_{b,t} < d_{a,t+1} + d_{b,t+1} < \bar{x}_t \\ AB \rightarrow d_{a,t+1} + d_{b,t+1} < \bar{x}_t \cap d_{a,t+1} \leq \bar{x}_{a,t} \cap d_{b,t+1} \leq \bar{x}_{b,t} \end{cases} \qquad (7.4)$$

Having the realized sales for the items correspondent to the demand a, b and n, and the inventory quantities from each side of the inventory, $\bar{x}_{a,t}$ and $\bar{x}_{b,t}$; the censored demand distributions can be acquired similarly to the ones given for the two-shelf-life setting. The censored demand distributions can be used to estimate the parameters correspondent to the demand distribution of product a, product b and product n, which is also an hypothetical one that corresponds the items that are associated to the unique remaining stock at event $\bar{A}\bar{B}$. At last, the demand n may be proportionally distributed between the demand a and the demand b. For example if the demands are Poisson distributed, the final estimated mean demand a can be set to be equal to the estimated mean demand a plus the quantity of the estimated mean demand n proportional to the quantity of estimated mean demand a in relation to the sum of the estimated mean demand a and the estimated mean demand b. Analogously, the final estimated mean demand b can be set to be equal to the estimated mean demand b plus the quantity of the estimated mean demand n proportional to the quantity of estimated mean demand b in relation to the sum of the estimated mean demand a and the estimated mean demand b.