Application of an average strain energy density criterion to obtain the mixed mode fracture load of granite rock tested with the cracked asymmetric four-point bend specimens

S.M.J. Razavi [a], M.R.M. Aliha [b*], F. Berto [a]

[a] *Department of Mechanical and Industrial Engineering, Norwegian University of Science and Technology (NTNU), Richard Birkelands vei 2b, 7491, Trondheim, Norway.*

[b] *Welding and Joining Research Centre, School of Industrial Engineering, Iran University of Science and Technology (IUST), Narmak, 16846-13114, Tehran, Iran.*

(Corresponding author: mrm_aliha@iust.ac.ir)

Abstract

Mixed mode brittle fracture behaviour of granite rock is studied experimentally and theoretically using Asymmetric Four Point Bend (AFPB) specimens containing pre-cracks subjected to different mixed mode loading conditions, ranging from pure mode I to pure mode II. The main aim of this paper is twofold. First, to present a complete set of experimental results on fracture of pre-cracked granite samples under various in-plane loading mixities, and second, to predict the fracture loads of the tested rock samples under mixed mode I/II conditions using an energy-based criterion, namely the Average Strain Energy Density (ASED) criterion. Good agreement is found between the experimentally obtained fracture loads and the theoretical predictions based on the constancy of the mean strain energy density over the material volume. It is shown that the ASED criterion is able to provide well predictions for the fracture loads of the investigated rock material containing a pre-crack.

Keywords: Average strain energy density criterion; Local approach; Granite rock; Brittle fracture; Mixed mode I/II, Fracture load prediction.

1. Introduction

Cracks and inherent defects are commonly found in rock masses and concrete structures. Crack growth assessment is an important task for various practical applications of the rock and concrete materials such as concrete dams, mining, tunnelling, rock fragmentation and analyses of rock slope stability. The majority of available researches in the field of fracture mechanics of rock and concrete materials are related to opening mode (or mode I loading) which is commonly observed under tensile stresses. However in real world applications, the cracked rock structures are typically subjected to mixed mode loading and hence the fracture of rock and concrete structures may occur under a combination of tensile and shear loading. Therefore, the mixed mode fracture behaviour of rock materials should be analysed and predicted by means of suitable methods to helps researchers in better understanding of the failure mechanism in these brittle or quasi-brittle materials. There are some test specimens available for evaluating the mixed mode tensile and in/out-of-plane shear deformations (i.e. mixed mode I/II or mixed mode I/III fracture behaviour) of rocks [1-15]. The cracked rectangular bar subjected to asymmetric four-point-bend loading (i.e. AFPB specimen) is one of the commonly used test configurations for investigation of in-plane mixed mode I/II fracture phenomenon in rocks and concretes (see for example [1–4]). The simple geometry, compressive applied loading and convenient pre-cracking of the beam are the most important advantages of this specimen for being used in fracture toughness study and ultimate load bearing capacity of brittle materials like rocks.

On the other hand, numerous fracture criteria have been proposed by the scholars in order to estimate the behaviour of brittle and quasi-brittle materials. The maximum tangential stress based criteria [16-20], the maximum tangential strain based criteria [21-24], the minimum strain energy density based criteria [25] and the maximum energy release rate or G criterion [26] are among the mostly used theoretical mixed mode fracture criteria by the researchers. By improving the minimum strain energy density and considering a control volume instead of a critical distance, Lazzarin and Zambardi [27] proposed the Average Stain Energy Density (ASED) criterion. Unlike the SED criterion (which is a point-wise criterion), the ASED criterion as presented in [27-29] states that brittle fracture occurs when the mean value of the strain energy density over a given control volume is equal to a critical value. While successful ability of ASED criterion has been demonstrated in the past for mixed mode fracture behaviour prediction in different engineering materials (such as metals, graphite, polymers and etc. [30-34]), the validity of this criterion has never been examined for rock materials. Therefore, the aim of the present research is to evaluate mixed mode I/II brittle fracture in a typical granite rock both experimentally and theoretically. For this aim, first a series of fracture experiments are undertaken on the AFPB specimens made of granite to obtain the fracture loads under different mixed mode I/II loading conditions. Then the ASED criterion is used to predict the experimental data of fracture loads. It is shown that good agreement exists between the experimental data and the theoretical findings.

2. Experiments

Fig. 1 illustrates the geometry of the AFPB specimen and its position within the loading fixture. By considering suitable values for the loading roller distances relative to the crack (i.e. L_1, L_2 ,L_3 and L_4) different mode mixities can be provided in this specimen. While symmetric loading condition (i.e. $L_1 = L_2$ and $L_3 = L_4$ and $L_1 < L_3$) corresponds to pure mode I (i.e. pure bending and crack opening case), anti-symmetric loading of the beam (i.e. $L_2 = L_3$ and $L_1 = L_4$ and $L_1 > L_2$) introduces pure shear or pure mode II deformation in the crack plane [35,36]. Asymmetric loading of this specimen also results in combined bending – shear deformation of crack flanks (or mixed mode I/II).

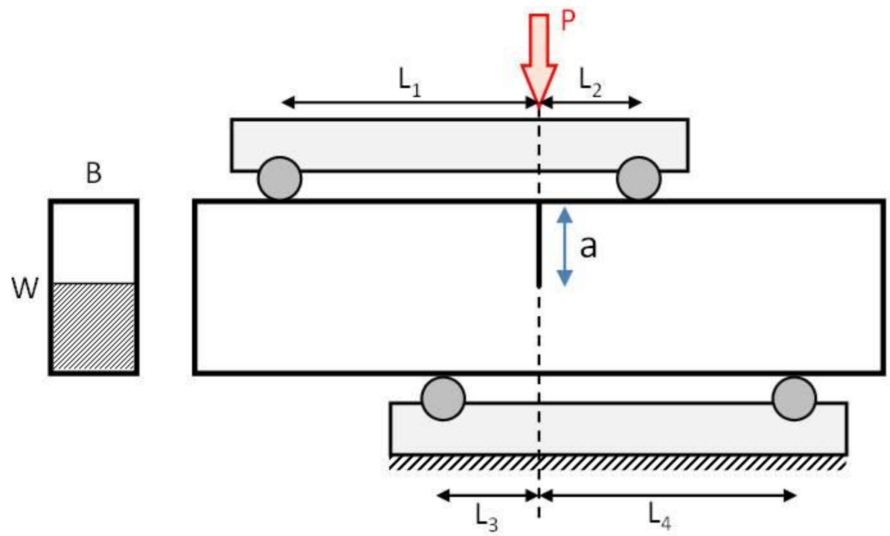


Fig. 1. Schematic view of cracked asymmetric four-point bend (AFPB) specimen

The crack tip stress intensity factors under mixed mode I/II loading conditions (i.e. K_I and K_{II}) in the AFPB specimen are functions of beam geometry, crack length and upper and lower roller distances from the crack. For any desired mixed mode loading conditions of AFPB specimen, K_I and K_{II} can be written as:

$$K_{I} = \frac{P\sqrt{\pi a}\, Y_{I}}{BWL_1} (L_1 - L_3) \quad (1)$$

$$K_{II} = \frac{P\sqrt{\pi a}\, Y_{II}}{BWL_1} (L_1 - L_3) \quad (2)$$

where P is the applied load and Y_I and Y_{II} are the modes I and II geometry factors, respectively that depend on the crack length ratio (a/W) and loading support distances (L_1, L_2 ,L_3 and L_4).

Variations of Y_I and Y_{II} for different geometry and loading conditions of the AFPB specimen (computed from several finite element analyses) have been presented in Fig. 2. The contribution of modes I and II components in crack tip parameters of any cracked specimen is often described and defined using an elastic mode mixity parameter (M_e) defined as:

$$M_e = \frac{2}{\pi} \arctan\left(\frac{K_I}{K_{II}}\right) \tag{3}$$

M_e varies from unity (for symmetric four-point bend specimen) to zero (for anti-symmetric four point bend specimen) which is function of *a/W* and loading support positions (L_1, L_2 ,L_3 and L_4).

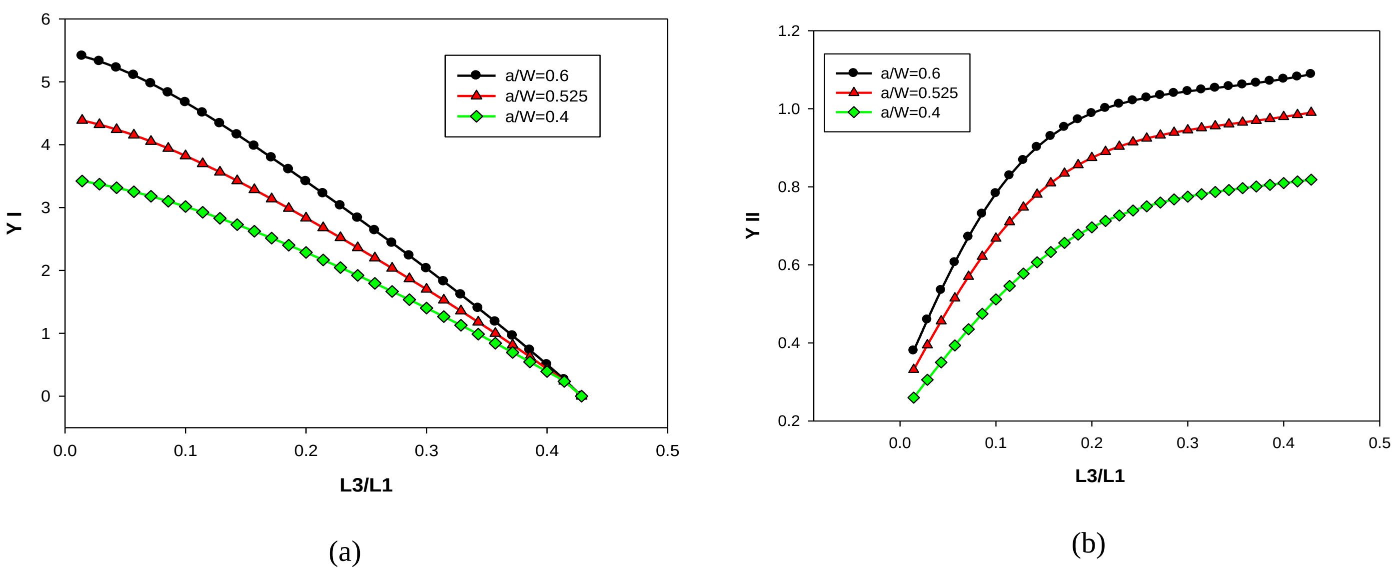


Fig. 2. Variations of (a) mode I geometry factor (Y_I) and (b) mode II geometry factor (Y_{II}) in the AFPB specimen for different mode I/II combinations (L_2 = 30 mm and L_4 = 70 mm).

For conducting the fracture toughness experiments, a number of rectangular beam specimens with dimensions of 220 mm (length), 40 mm (width) and 20 mm (thickness) were cut from a sheet of Takaab granite excavated from north western of Iran. An edge crack of length 20 mm was introduced at the middle of each beam by using a high speed rotary diamond saw machine. The radius and thickness of rotary saw blade was 55 mm and 0.3 mm, respectively. Meanwhile, in order to cover full mode mixities from pure mode I to pure mode II in fracture toughness experiments, the locations of upper and lower roller supports were considered at suitable distances relative to the crack. Accordingly, by considering constant values of $L_1 = L_4$ =70 mm, L_2 = 30 mm and varying the L_3 from 30 mm toward the crack location, different mode mixities were produced in the entire range of modes I and II mixities. As stated earlier, pure mode I case was produced by symmetric loading of beam (i.e. $L_1 = L_2$ = 70 mm and $L_3 = L_4$ = 30 mm) and pure mode II tests were performed under anti-symmetric four point bend loading conditions (i.e. $L_1 = L_4$ = 70 mm and $L_2 = L_3$ = 30 mm). For introducing the intermediate mode I/II mixities, different support length ratios of L_3/L_1 = 0.285, 0.357, 0.393, 0.405, 0.407, 0.429 were considered. In order to conduct the fracture toughness experiments under different mode mixities, the manufactured rectangular shape granite specimens were located inside a four-point bend fixture according to the loading span ratios given in Table 1. Fig. 3 shows a typical loading set-up for the AFPB specimen made of Takaab granite. The loading rate for tests was constant and equal to 0.25 mm/min. For each mode mixity at least three fracture experiments were conducted and the corresponding values of fracture loads (P_{cr}) were obtained from the load-displacement curves of tested AFPB specimens. In the next section, the obtained fracture loads are predicted theoretically using the averaged strain energy density (ASED) criterion.

Table 1. Loading roller distances for introducing different mode mixities in the investigated four-point bend specimen.

M_e	1	0.7	0.51	0.31	0.24	0.21	0
L1 (mm)	70	70	70	70	70	70	70
L2 (mm)	70	30	30	30	30	30	30
L3 (mm)	30	19.95	23.45	27.51	28.35	28.5	70
L4 (mm)	30	30	30	30	30	30	30

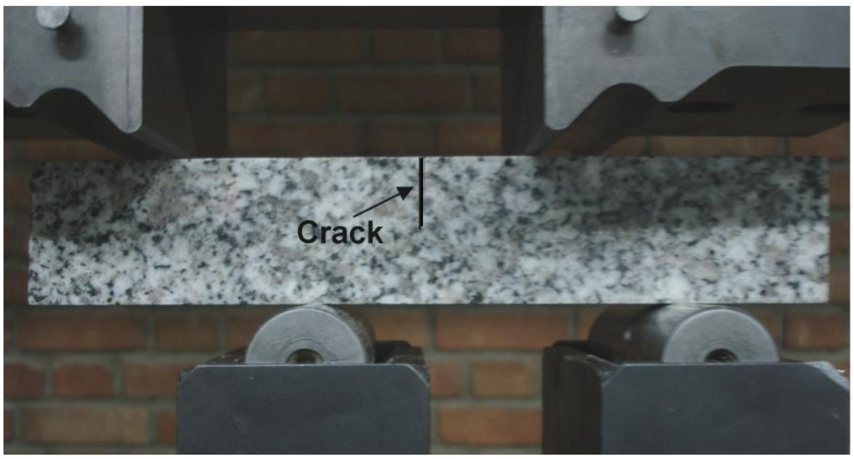


Fig. 3. A typical cracked rectangular granite rock specimen subjected to asymmetric four-point bend loading.

3. Application of ASED criterion for cracked specimens

In order to predict the fracture load of cracked components, an appropriate fracture criterion is required which is based on the mechanical behaviour of material around the crack tip. In this paper, a strain energy based criterion namely the Averaged Strain Energy Density (ASED) criterion [27-29] is described and used for estimating the fracture loads of the tested specimens under mixed mode I/II loading. According to the ASED criterion, brittle fracture occurs when the average value of strain energy density (SED) over a given control volume, $\overline{W}$ is equal to a critical value W_c which is a material property and is independent from the geometry of cracked specimen. The control volume dependents on the ultimate tensile strength (σ_t) and the fracture toughness (K_{Ic}) of brittle or quasi-brittle materials under static loads.

Dealing with the cracked specimens, the control volume is a circle of radius R_c centred at the crack tip (see Fig. 4). Considering a plane-strain condition, the critical length, R_c, can be obtained using the following expression [37]:

$$R_c = \frac{(1+\nu)(5-8\nu)}{4\pi}\left(\frac{K_{Ic}}{\sigma_t}\right)^2 \tag{4}$$

where ν is the Poisson's ratio of the material. It is worth mentioning that the radius of control volume was considered to be constant for different mode mixities.

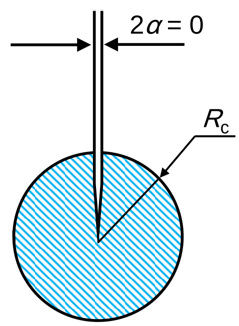


Fig. 4. Schematic view of the control volume around the crack tip.

The elastic deformation energy, averaged on the control volume turns out to be [27-29]:

$$\bar{W} = \frac{e_1}{E} \times \frac{K_I^2}{R_c^{2(1-\lambda_1)}} + \frac{e_2}{E} \times \frac{K_{II}^2}{R_c^{2(1-\lambda_2)}} \tag{5}$$

where e_1 and e_2 are geometric constants which depend on the geometry of notch and the Poisson's ratio, λ_1 and λ_2 are Williams' eigenvalues, E is the elastic modulus and K_I and K_{II} are the mode I and mode II stress intensity factors. For the specific case of a cracked specimen (i.e. $2\alpha = 0$), the geometric constants and eigenvalues are equal to $e_1 = 0.133$, $e_2 = 0.34$ and $\lambda_1 = \lambda_2 =$

0.5. Therefore, the average strain energy density in a control volume around the crack tip can be calculated using Eq. (6).

$$\bar{W} = (0.133)\frac{K_I^2}{ER_c} + (0.34)\times\frac{K_{II}^2}{ER_c} \tag{6}$$

Some other closed-form expressions for obtaining the SED from the stress field around the crack tip are available in the literature [38-41], however, to avoid any simplifications, the SED values were directly obtained from the finite element model and the results were compared with those of obtained from Eq. (6). At the onset of fracture, $\bar{W}$ reaches its critical value W_c. The critical SED value can be determined in terms of σ_t and E [27,29]:

$$W_c = \frac{\sigma_t^2}{2E} \tag{7}$$

The critical SED value was considered to be constant for different mixed mode loading conditions. The same assumption was verified in numerous research papers in the same field [28,29,42,43]. By imposing the average SED value $\bar{W}$ equal to the critical value W_c, the fracture load of the cracked specimens can be predicted. Therefore, the theoretical fracture loads (P_{th}) of the pre-cracked specimens were obtained using a simple proportion between the applied load P and the square root values of averaged SED as given below

$$P_{th}/P = \sqrt{W_c/\bar{W}} \tag{8}$$

4. ASED approach in fracture analysis of the tested granite specimens

Finite element models of pre-cracked AFPB specimens with dimensions of 220 mm$\times$40mm$\times$20 mm and initial crack length a = 20 mm (i.e. exactly the same geometry utilized for the manufactured and tested granite rock samples) were considered for fracture analysis with linear elastic behaviour assumption. Mechanical properties of the tested granite rock (i.e. E = 45 GPa, v = 0.28, and σ_t = 12.2 MPa) were employed in the numerical analysis and SED calculations. According to the experimental data, an average value of fracture toughness equal to 44.06 $MPa.mm^{0.5}$ was used in critical SED calculations. The specimens were assumed to be under plane strain conditions and the 8-node biquadratic plane strain quadrilateral elements were used to mesh the FE model. As illustrated in Fig. 5, higher mesh density was used near the crack tip to improve the accuracy of the results. Besides, the singular elements (quadratic elements with the mid-side nodes placed at the quarter points) were used for the first ring of elements around the crack tip. A mesh convergence study was also performed to ensure that a proper number of elements were used in the finite element modelling.

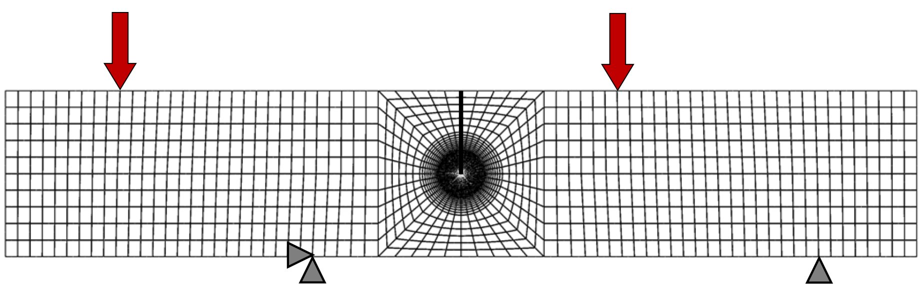
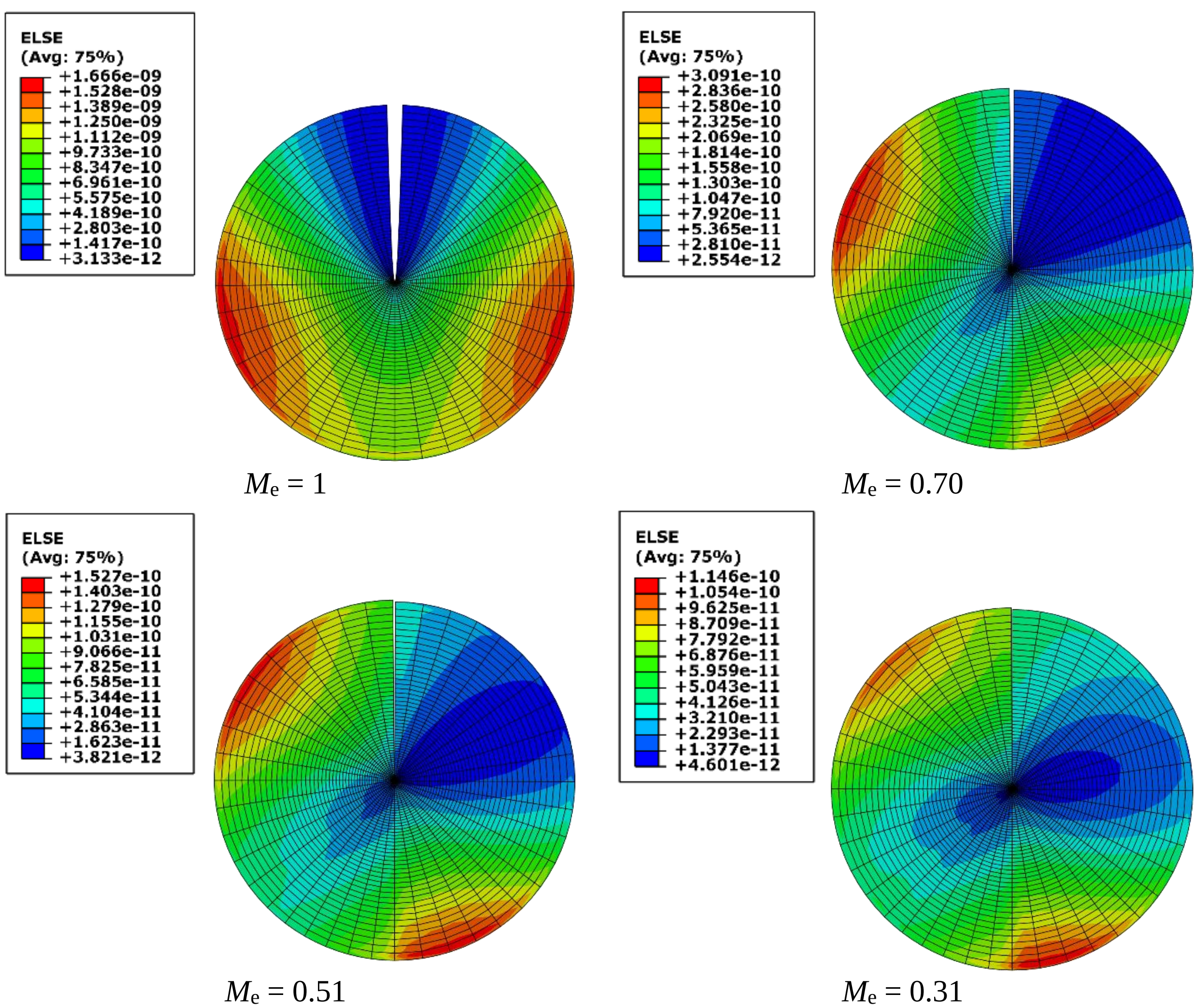


Fig. 5. Typical mesh pattern and boundary conditions for finite element model of AFPB specimen.

A unit external load was applied to the FE models (i.e. P = 1 N) in order to obtain the mean value of strain energy density in a control volume around the crack tip with a radius of R_c = 3.67 mm and also stress intensity factor which are used in Eq. (6). The critical value of SED was equal to 1.65×10^{-3} mJ/mm^3. Considering Eq. (8), the theoretical values of fracture load were obtained. Fig. 6 illustrates strain energy contours in the control volumes around the crack tip for the analysed models. According to Fig. 6, distributions of strain energy in control volume for various loading conditions are quite different. Considering the pure mode I loading condition, the strain energy has a symmetric variation on both sides of the crack plane with minimum value in from of the crack tip. This variations are changed under different mode mixities showing kinking of the crack from its initial direction. However, for all mode mixities, the average SED value in the control volumes are the same under fracture loads.



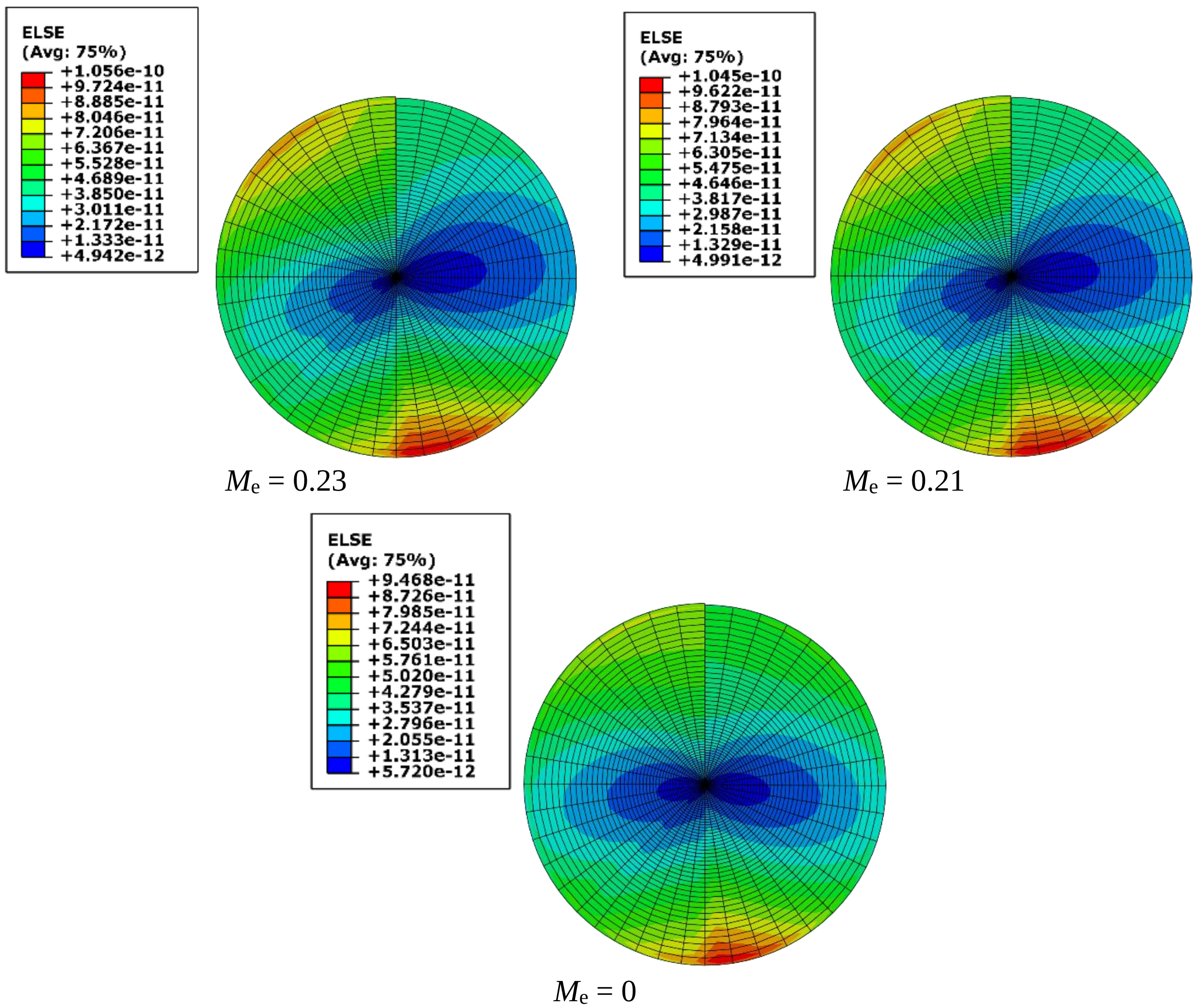


Fig. 6. The strain energy contours in the control volumes of AFPB.

A comparison between the experimental data and ASED predictions is shown in Fig. 7. Table 2 summarizes the stress intensity factors (K_I and K_{II}) corresponding to the mean value of the experimental fracture loads as well as the mean values of SED for the same loads. The average SED value was also obtained directly from the FE model and it was considered for fracture load prediction (see Table 3). According to the theoretical results, the ASED predictions based on the constant value W_c = 1.65×10^{-3} mJ/mm^3 of the critical local SED for different mode mixities is acceptable having the maximum deviation of 10.7% which is lower than the scatter reported in a review paper by Berto and Lazzarin [29] for a number of different materials.

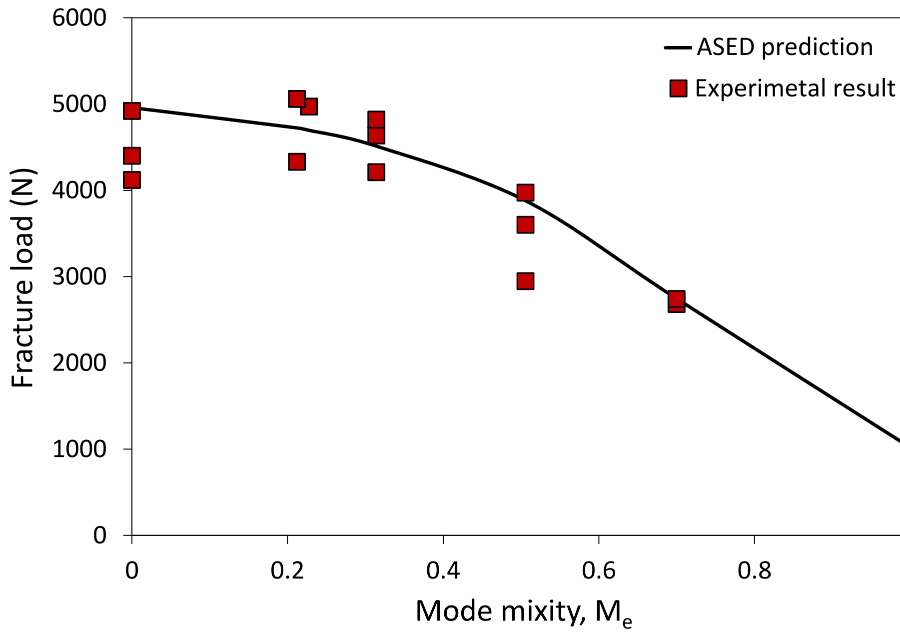


Fig. 7. Comparative results of the experimental fracture loads and ASED predictions.

According to the experimental results, for the tested granite material the mode I fracture toughness (K_{Ic}) is less than the mode II fracture toughness (K_{IIc}). However, there are some published researches which suggest that unlike the mode I fracture toughness K_{Ic}, the value of measured K_{IIc} is not purely a material constant and it is also dependent on the geometry of test specimen and loading configuration [44-47]. The typical ratio of K_{IIc}/K_{Ic} for various pre-cracked rock specimens varies in a wide range from 0.45 to 2.3 [48-52]. The experimental results in the current paper show that the tested granite rock is weaker under shear loading compared to the tensile loading since K_{IIc}/K_{Ic} is about 0.58.

Table. 2. Outline of numerical results for the AFPB specimens under mixed mode loading condition

(The numerical values of $\bar{W}$ are obtained using Eq. (4) based on stress intensity factor values)

L_3/L_1	K_I [MPa.mm$^{0.5}$]	K_{II} [MPa.mm$^{0.5}$]	M_e	$\bar{W}$ [mJ/mm^3]	Failure load P_{exp} [N] (Experiments.)	Failure load P_{th} [N] (ASED prediction)	Discrepancy (%)
Mode I	4.71e-2	0	1	1.78e-9	936	963	2.88
0.285	1.33e-2	6.79e-3	0.70	2.38e-10	2711	2638	-2.70
0.357	6.40e-3	6.29e-3	0.51	1.14e-10	3506	3803	8.46
0.393	3.23e-3	6.01e-3	0.31	8.28e-11	4555	4469	-1.90
0.405	2.21e-3	5.92e-3	0.23	7.60e-11	4970	4663	-6.17
0.407	2.04e-3	5.90e-3	0.21	7.51e-11	4695	4693	-0.04
Mode II	0	5.74e-3	0	6.78e-11	4480	4939	10.24

Table. 3. Outline of numerical results for the AFPB specimens under mixed mode loading condition.

(The numerical values of $\bar{W}$ are obtained directly from the finite element analysis)

L_3/L_1	M_e	$\bar{W}_{FEM}$ [mJ/mm^3]	Failure load (Expriment.) P_{exp} [N]	Failure load (ASED) $P_{th,FEM}$ [N]	Discrepancy (%)
Mode I	1	1.59E-09	936	1020	8.95
0.285	0.70	2.20E-10	2711	2745	1.25
0.357	0.51	1.10E-10	3506	3880	10.67
0.393	0.31	8.12E-11	4555	4512	-0.96
0.405	0.23	7.50E-11	4970	4695	-5.53
0.407	0.21	7.41E-11	4695	4723	0.59
Mode II	0	6.73E-11	4480	4958	10.66

It is finally reminded that cracks make the rock and concrete structures vulnerable to brittle failure as a result of high stress concentration around the crack tip. Because conducting fracture tests on the full-scale structure is not practical, it is preferred to predict and determine the fracture strength of cracked rock masses by a suitable fracture criterion. Generally, the validated

ASED criterion can be used by engineers and scientists to estimate the onset of fracture in complicated pre-cracked components without requiring costly and time-consuming mixed mode fracture experiments. The results presented in this paper for granite rock, demonstrated the accuracy and practical ability of ASED criterion for evaluating the mixed mode I/II fracture toughness behaviour of cracked granite beams subjected to asymmetric four point bend loading. However, the same methodology can also be employed in order to estimate the fracture loads of other cracked components with various geometries made of different rock materials.

Conclusion

- Brittle fracture in cracked specimens made of granite rock was investigated both experimentally and theoretically under in-plane mixed mode loading conditions ranging from pure mode I to pure mode II.
- The ASED criterion was employed for crack domains in order to predict the fracture load of pre-cracked granite samples subjected to symmetric, anti-symmetric and asymmetric four-point bend loading.
- It was shown that the mentioned method is suitable for fracture load prediction of rock materials loaded under mixed mode I/II loading conditions since the theoretical predictions of ASED criterion were in good agreement with the experimental data.
- According to the fracture test results on granite, this material is weaker in mode II fracture than in mode I fracture.

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