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Exponential Weighted Sums related to the Divisor and Circle Problems

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Foreword

This paper, written in the Spring of 2018, constitutes my master's thesis in mathematics at Norges Teknisk-Naturvitenskapelige Universitet, NTNU, under the supervision of Andrii Bondarenko. I would like to thank Andrii for his enthusiasm, and for the passion he shared both in regards to the material investigated over the course of the thesis, as well as the many other topics he introduced during the process. I would also like to thank my Mother for her support and advice throughout my education. It will forever be appreciated.

Abstract

The classical results of the Dirichlet Divisor Problem and Gauss' Circle Problem are examined, with required information on the Riemann Zeta function presented. In particular, the results derived from the weighted sums of Voronoi and Wigert are considered, along with some slightly altered versions of their approaches.

Contents

Introduction

The Riemann Zeta function has for over one hundred years been an integral part of analytic number theory. The conjectures related to the function, especially the Riemann and Lindelöf Hypotheses, remain some of the most coveted results in modern mathematics.

In this thesis, the Riemann Zeta function is considered with respect to it's role in understanding the Dirichlet divisor problem, and the related Gaussian circle problem.

The first chapter is a brief introduction to the features of the Riemann Zeta function, both as a general number theoretic tool and specifically as it pertains to the Dirichlet divisor and Gaussian circle problems.

With the results of the Riemann Zeta function covered, the second chapter introduces the Dirichlet divisor problem and derives a method of the problem as found by Voronoi [1], and later improved by Kolesnik [2].

The attention of this text then turns to the results found by considering an exponential weighted sum, firstly in the form of Wigert [3], and then in some similar fashions yielding slightly different results, as well as a statement of the conjecture in terms of an integral transform.

Finally, Gauss' circle problem is presented as a close relation to the Dirichlet divisor problem. A similar sum to the previous chapter is considered and an explicit case of Voronoi's formula found, as well as an equivalent statement of the conjectured result for the circle problem, again as an integral transform.

1 The Riemann Zeta Function

The Riemann Zeta function (referred to also as simply the Zeta Function in this text) is one of the most important in the field of analytic number theory. Defined as the Dirichlet series;

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.0.1}$$

The function was initially considered with a real argument, until the function's namesake, Bernhard Riemann, investigated the function with a complex argument in the 1800's. His studies famously led to the proposal of the Riemann Hypothesis in 1859 [4]; that every non-trivial zero of the function lie on the critical line, that is $\Re(s) = 1/2$. The function carries important weight in the study of prime numbers, as we will see later in this chapter, and gives a sharp insight into the Dirichlet Divisor problem. For this reason, we first present some preliminary information on the function.

1.1 Poisson's Summation Formula

The summation formula derived by Poisson is a powerful identity, which holds for all Schwartz functions, as well as certain non-Schwartz functions for which the sums in the formula converge. The formula identifies the sum of the formula over the set of integers to the sum of it's Fourier transform, and gives insight to among other problems, the Riemann Zeta function and later the Voronoi summation formula.

The formula is as follows:

Theorem 1.1 (Poisson Summation Formula). *For $f(x)$ a function in Schwartz space;*

$$\sum_{n \in Z} f(n) = \sum_{k \in Z} \hat{f}(k), \tag{1.1.1}$$

where $\hat{f}$ indicates the Fourier transform of f;

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx. \tag{1.1.2}$$

Proof. To prove Poisson's formula, we consider firstly the finite sum;

$$\sum_{n=M}^{N} f(n) = \frac{1}{2}(f(M)+f(N)) + \int_M^N f(x)dx + \int_M^N (x-[x]-1/2)f'(x)dx, \tag{1.1.3}$$

where square brackets indicate as usual the integer part. This follows from the integration by parts of the following;

$$\sum_{n=M}^{N-1} \int_n^{n+1} (x-n+1/2)f'(x)dx = \sum_{n=M}^{N-1} \frac{1}{2}(f(n+1)+f(n)) - \int_n^{n+1} f(x)dx. \tag{1.1.4}$$

Taking the Fourier series of the periodic argument;

$$x-[x]-1/2 = -\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{\pi n}, \tag{1.1.5}$$

gives

$$\sum_{n=M}^{N} f(n) = \frac{1}{2}(f(M)+f(N)) + \int_M^N f(x)dx - \sum_{n=1}^{\infty} \frac{1}{\pi n} \int_M^N f'(x)\sin(2\pi nx)dx. \tag{1.1.6}$$

Integrating by parts, and taking N and M to infinity and minus infinity respectively, which correspond to f going to zero by the assumption, gives

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x)dx + 2\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(x)\cos(2\pi nx)dx. \tag{1.1.7}$$

From this, the desired identity follows, since cosine is even;

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\cos(2\pi nx)dx = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx. \tag{1.1.8}$$

□

This formula shows that a periodic sum of a "well behaved" function can be exactly defined by a number of discrete values of it's Fourier transform, a remarkable result. The applications of this are far reaching, as shown by the applications in the following chapters.

1.2 The Functional Equation

Whilst the Riemann Zeta function in the form of the Dirichlet series for $\Re s > 1$;

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.2.1}$$

is convergent and examinable in that half of the plane, the true intrigue of the Riemann zeta function lies in it's analytic continuation. This analytic continuation extends the argument of the function to the entire complex plane, minus the point $s = 1$. There are many proofs of this analytic continuation, some of which are outlined in [5]. Here we present in detail a form based on the integral formula of the Gamma function and the Poisson summation derived in the previous section.

Theorem 1.2 (Functional Equation of the Riemann Zeta Function)**.** *For all $s \neq 1, -2k, 2k+1$ where $k \in N$, the following equation holds:*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-1/2+s/2}\Gamma(1/2 - s/2)\zeta(1-s). \tag{1.2.2}$$

Proof. Beginning with the integral

$$\int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx. \tag{1.2.3}$$

Making the change of variables $\pi n^2 x \mapsto z$, the integral becomes in the form of the Gamma function, so one retrieves

$$\int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx = \frac{\Gamma(s/2)}{n^s \pi^{s/2}}. \tag{1.2.4}$$

For the entire complex plane excluding the negative even integers, where the integral diverges. Now to retrieve a form of the Zeta function, we take $\Re s) > 1$ and consider the sum of n over the naturals;

$$\pi^{-s/2}\Gamma(s/2) \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx. \tag{1.2.5}$$

Since the Dirichlet series converges for the given half plane $\Re s > 1$, rearranging the order of the integral and summation gives;

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^{s/2-1} e^{-\pi n^2 x} dx = \int_0^{\infty} x^{s/2-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx. \tag{1.2.6}$$

Now turning to Poisson's formula from the previous section;

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/x} \tag{1.2.7}$$

(as the Gaussian is an eigenvector of the Fourier transform, see appendix). And so, noting that

$$2\sum_{n=1}^{\infty} e^{-\pi n^2 x} + 1 = \frac{2}{\sqrt{x}} \sum_{n=1}^{\infty} e^{-\pi n^2/x} + \frac{1}{\sqrt{x}}, \tag{1.2.8}$$

(Since the exponential is even when evaluated at integers) it follows that

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) = & \int_0^1 x^{s/2-1} \left(\frac{1}{\sqrt{x}} \sum_{n=1}^{\infty} e^{-\pi n^2/x} + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right) dx \\
& + \int_1^{\infty} x^{s/2-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx \\
& = \frac{1}{s-1} + \frac{1}{s} + \int_0^1 x^{s/2-3/2} \sum_{n=1}^{\infty} e^{-\pi n^2/x} dx \\
& + \int_1^{\infty} x^{s/2-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx.
\end{aligned} \tag{1.2.9}$$

Then making the change of variable in the first integral of $x \to 1/x$, we obtain:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} (x^{-s/2-1/2} + x^{s/2-1}) \sum_{n=1}^{\infty} e^{-\pi n^2 x} dx. \tag{1.2.10}$$

This final integral converges over all s. Furthermore, it remains identical under the change of variables $s \to 1-s$, hence, the functional equation is given by

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-1/2+s/2}\Gamma(1/2 - s/2)\zeta(1-s). \tag{1.2.11}$$

$\square$

Here we see the first instance of the applicability of Poisson's formula. This also shows the function having a simple pole at $s = 1$, as expected when considering the Dirichlet series. However, the corresponding point in the functional equation, $s = 0$, has no such pole, as it corresponds to a simple zero in the function reciprocal of the Gamma function.

Since the general divisor function $d_k(n)$ involves higher powers of the zeta function, we will also consider higher powers of zeta, $\zeta^k(s)$. These higher powers are also clearly meromorphic as seen by the definition of the functional equation of Zeta. If we denote

$$\gamma(s) := \pi^{-s/2}\Gamma(s/2). \tag{1.2.12}$$

Then the k-th power of zeta is determined by the functional equation

$$\zeta^k(s) = \frac{\gamma^k(1-s)}{\gamma^k(s)}\zeta^k(1-s). \tag{1.2.13}$$

The notation γ is used also for the Euler Mascheroni constant, so throughout the paper the above function is understood as such only when referred to as $\gamma(s)$ as opposed to merely γ. We also introduce the notation:

$$\chi(s) := \frac{\gamma(1-s)}{\gamma(s)} = \frac{\pi^{-1/2+s/2}\Gamma(1/2-s/2)}{\pi^{-s/2}\Gamma(s/2)}. \tag{1.2.14}$$

Giving:

$$\zeta(s) = \chi(s)\zeta(1-s). \tag{1.2.15}$$

1.3 Euler's Product Formula

The Zeta function is famously connected to the prime numbers, and the most elementary case of this can be seen in the Euler product form of the function. Euler found the formula when he investigated the Dirichlet series for real arguments, before Riemann extended the argument s to the complex plane. The product formula states;

Theorem 1.3 (Euler's Product Formula)**.** *For $\Re s > 1$, the following equation holds:*

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}. \tag{1.3.1}$$

Where the product is taken over all p primes.

This relationship is not so surprising, given the unique representation of the naturals as products of primes. In fact, the proof relies solely on this fact and basic features of geometric series.

Proof. Each of the terms in the product series takes the form;

$$(1-p^{-s})^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ... \tag{1.3.2}$$

Which follows from the expansion of the geometric series in the case $\Re s > 1$. For $\Re s > 1$, since the Zeta function is given by the convergent sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.3.3}$$

one can see how the terms in the sum, the reciprocals of the naturals, will be unique products of the reciprocals of primes from the fundamental theorem of arithmetic. To formalize this argument, the partial product is considered;

$$\left|\zeta(s) - \prod_{p \leq q} (1 - p^{-s})^{-1}\right| < \sum_{n > q} \frac{1}{n^\sigma}. \tag{1.3.4}$$

Since the partial product is equal to the product;

$$\prod_{p \leq q} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...\right), \tag{1.3.5}$$

only integers larger than q will not appear in the partial product. The result then follows by taking $q \to \infty$, since the Dirichlet series converges, so the partial sums must decrease to zero. $\square$

1.4 The Hadamard Product Formula

With the Riemann Zeta function exhibiting a pole at the point $s = 1$, of order 1, it is natural to consider the Xi function;

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s). \tag{1.4.1}$$

It follows from the functional equation of $\zeta(s)$ that $\xi(s)$ satisfies the useful properties that it is entire, that is analytic in the entire plane, and that $\xi(s)$ itself has the simple functional equation:

$$\xi(s) = \xi(1-s). \tag{1.4.2}$$

Furthermore, since the trivial zeroes of the Riemann Zeta function coincide with the poles of the Gamma function, the zeroes of the Xi function are exactly the non-trivial zeroes of the Riemann Zeta function. The Weierstrass factorization theorem then suggests that if $\xi(s)$ is of finite order, one can express it in the form:

$$\xi(s) = s^k e^{g(s)} \prod_{\rho} E_\rho \left(\frac{s}{\rho}\right). \tag{1.4.3}$$

Where E_ρ denote the standard elementary factors, and ρ the zeroes of $\xi(s)$, which coincide with the zeroes of $\zeta(s)$. This formula is indeed true, and $\xi(s)$ in fact has order 1, leading to the following [6]:

Theorem 1.4 (Hadamard's Product Formula)**.** *For all $s \neq 1$, the following product converges:*

$$\zeta(s) = \frac{e^{s(\log(2\pi)-1-\gamma/2)}}{2(s-1)\Gamma(1+s/2)} \prod_\rho (1 - \frac{s}{\rho}) e^{s/\rho} \tag{1.4.4}$$

or equivalently;

$$\zeta(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+s/2)} \prod_\rho (1 - \frac{s}{\rho}) \tag{1.4.5}$$

The convergence of the latter equation is non-trivial, but follows from the convergence of the first and evaluation of $\prod_\rho e^{s/\rho}$, as we will see.

Proof. We begin by showing that

$$|\xi(s)| < e^{C|s|\log|s|} \tag{1.4.6}$$

for some constant C. In particular this will show that $\xi(s)$ is an integral function of order 1. Clearly

$$s(s-1)\pi^{-s/2} \ll e^{c_1|s|}, \tag{1.4.7}$$

and for $|\arg(s)| < \frac{\pi}{2}$, Stirling's formula gives;

$$\Gamma(\frac{s}{2}) \ll e^{c_2|s|\log|s|}. \tag{1.4.8}$$

Since $\xi(s)$ is symmetric in the critical line $\sigma = 1/2$, we can assume $\sigma \geq 1/2$, so that the above restriction on $\arg(s)$ holds. Thus it remains only to show that $\zeta(s) \ll e^{C|s|\log|s|}$. In fact using the representation [5];

$$\zeta(s) = \frac{1}{s-1} - s\int_1^\infty (x - [x])x^{-s-1}dx, \tag{1.4.9}$$

since the integral is bounded for $\sigma \geq 1/2$, we have that

$$\zeta(s) \ll s. \tag{1.4.10}$$

Thus by choosing an appropriate C, the entire function $\xi(s)$ is bounded as described above.

To show that the product infinite, that is, that $\zeta(s)$ has an infinite number of zeroes, it must be that the bound on $\xi(s)$ cannot be of the form

$$|\xi(s)| < e^{C|s|}. \tag{1.4.11}$$

To see this one can note that in the case $\sigma \to \infty$:

$$\log(\Gamma(s)) \sim s \log s \tag{1.4.12}$$

By Stirling's Formula, and so the bound $|\xi(s)| < e^{C|s|\log|s|}$ cannot be slackened. It then follows from Jensen's formula [7] that the sum of the reciprocal moduli of the zeroes of $\xi(s)$ diverges, that is, the sequence of partial sums;

$$\sum_{i=1}^{n} \frac{1}{|\rho_i|} \tag{1.4.13}$$

diverges as $n \to \infty$. Thus $\xi(s)$ is an integral product of order 1, and the Weierstrass product

$$\xi(s) = e^{A+Bs} \prod_{\rho} (1 - \frac{s}{\rho}) e^{s/\rho} \tag{1.4.14}$$

has an infinite number of terms [8]. In order to find the constants A and B, the logarithmic derivative is taken:

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \tag{1.4.15}$$

and so by the definition of $\xi(s)$;

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{\log \pi}{2} - \frac{\Gamma'(s/2+1)}{2\Gamma(s/2+1} + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right). \tag{1.4.16}$$

Taking $s \to 0$ in the above equation leaves;

$$\frac{\zeta'(s)}{\zeta(s)} = B + 1 + \frac{\log \pi}{2} - \frac{\Gamma'(1)}{2\Gamma(1)} \tag{1.4.17}$$

which due to the specific values of $\zeta'(s), \zeta(s)$, $\Gamma(1)$ and $\Gamma'(1)$, gives:

$$B = \log 2 + \frac{\log \pi}{2} - 1 - \frac{\gamma}{2} \tag{1.4.18}$$

Finally solving for A, by setting $s = 0$, gives $A = 1/2$. Inserting these constants in the product formula for $\xi(s)$, and then solving for $\zeta(s)$ using the definition yields:

$$\zeta(s) = \frac{e^{s(\log(2\pi)-1-\gamma/2)}}{2(s-1)\Gamma(1+s/2)} \prod_{\rho} (1 - \frac{s}{\rho}) e^{s/\rho} \tag{1.4.19}$$

It now remains to show the equivalence of the two formulations of the infinite product given in the theorem. For this, one can use the function

$$Z(n) := \sum_{\rho} \rho^{-n} \tag{1.4.20}$$

which has the specific value [9]:

$$Z(1) = 1 + \frac{\gamma}{2} - \log(2\pi^{1/2}). \tag{1.4.21}$$

Thus;

$$\begin{aligned} e^{s(\log(2\pi)-1-\gamma/2)} \prod_{\rho} e^{s/\rho} &= e^{s(\log(2\pi)-1-\gamma/2)+\sum s/\rho} \\ &= e^{s(\log(2\pi)-1-\gamma/2+Z(1))} \\ &= e^{(s\log\pi)/2}. \end{aligned} \tag{1.4.22}$$

From which the infinite product;

$$\zeta(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+s/2)} \prod_{\rho}(1 - \frac{s}{\rho}) \tag{1.4.23}$$

follows. □

1.5 The Prime Numbers and the Riemann Zeta Equation

With the previous sections considered, it is possible to show the significance of the zeroes of the Riemann Zeta function, along with the Riemann hypothesis, in relation to the distribution of the primes.

Theorem 1.5. *Let ρ be the non-trivial zeroes of the Riemann Zeta Function. Then the following equation holds:*

$$\psi(x) := \sum_{n\le x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \log(1 - \frac{1}{x^2}). \tag{1.5.1}$$

Where $\Lambda(x)$ indicates the Von Mangoldt function,

$$\Lambda(n) = \begin{cases} \log p & \textit{if } n = p^k \textit{ for some prime } p, k \ge 1 \\ 0 & \textit{otherwise} \end{cases} \tag{1.5.2}$$

Throughout the proof it is taken for granted that the sum over non-trivial zeroes, $\sum_{\rho} x^{\rho}/\rho$, converges, which was shown by Riemann. We start the proof of this formula by comparing the Euler product formula and the Weierstrass-type product expansion of the entire function $(1-s)\zeta(s)$, corresponding to a product similar to that of the previous section, but with slightly different coefficients

$$\prod_{p}(1-p^{-s})^{-1} = \frac{e^{a+bs}}{s-1} \prod_{n\ge 1}(1+s/(2n))e^{-s/2n} \prod_{\rho}(1-s/\rho)s^{s/\rho}. \tag{1.5.3}$$

Convergent for $\Re(s) > 1$. Taking the logarithmic derivative of each side (corresponding to $\zeta'(s)/\zeta(s)$) gives:

$$-\sum_{p}\sum_{m\geq 1}\log(p)p^{-ms} = b - \frac{1}{s-1} - \sum_{n\geq 1}\frac{s}{2n(s+2n)} + \sum_{\rho}\frac{s}{\rho(s-\rho)}. \tag{1.5.4}$$

And since the right hand side is equal to $\zeta'(s)/\zeta(s)$ for all $s \in (C)$, taking $s = 0$ gives $b + 1 = \zeta'(0)/\zeta(0)$, so;

$$\sum_{p}\sum_{m\geq 1}\log(p)p^{-ms} = \frac{s}{s-1} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n\geq 1}\frac{s}{2n(s+2n)} - \sum_{\rho}\frac{s}{\rho(s-\rho)}. \tag{1.5.5}$$

To proceed from here requires the introduction of the formula of Perron. It should be noted that while the version stated is over a finite integral, the classical form over the infinite integral follows [7]:

Lemma 1.6 (Perron's Formula)**.** *For $f(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$, where $a(n)$ is an arithmetic function, if $f(s)$ converges for $Re(s) > \sigma$, $a(n) = O(g(n))$ and $f(\sigma) = O((\sigma-1)^{-\alpha})$, then;*

$$\begin{aligned}\sum_{n\leq x} a(n) =& \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\frac{x^s}{s}ds + O(x^c T^{-1}(\sigma + c - 1)^{-\alpha}) \\ &+ O(g(2x)x^{1-\sigma}\log(x)T^{-1}) + O(g(2x)),\end{aligned} \tag{1.5.6}$$

where $c > \min(0, 1-\sigma)$.

It then follows that:

$$\sum_{n\leq x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}ds. \tag{1.5.7}$$

So taking Perron's formula on the left hand side of the above equation (with $x \neq p^m$ for any prime p), gives;

$$\sum_{p}\sum_{1\leq m}\log p\int_{2-i\infty}^{2+i\infty} p^{-ms}\frac{x^s}{s}ds = \sum_{p}\sum_{1\leq m, p^m\leq x}\log p. \tag{1.5.8}$$

While on the right hand side the residues at simple poles at all $s = -\rho$ and $s = -2n$, are $\frac{x^\rho}{\rho}$ and $\frac{1}{2nx^{2n}}$ respectively, while $s = 1$ and $s = 0$ are simple poles with residues x and $-\frac{\zeta'(0)}{\zeta(0)}$ resp., giving:

$$\sum_{p}\sum_{1\leq m, p^m\leq x}\log p = x - \sum_{\rho}\frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \sum_{1\leq n}\frac{1}{2nx^{2n}}. \tag{1.5.9}$$

Now finally, $\zeta'(0)/\zeta(0) = \log(2\pi)$, and $\sum_{1\leq n} \frac{1}{2nx^{2n}} = \log(1-1/x^2)$, thus the above equation is equivalent to the explicit formula;

$$\sum_{n\leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \log(1 - \frac{1}{x^2}). \tag{1.5.10}$$

One can see from this formula how the distribution of zeroes of the Zeta function will reflect the distribution of primes, since the leading term on the right hand side is indeed x, while the largest order error term is determined by the non-trivial zeroes. Heuristically, one can see that since $|x^\rho| = x^{Re(\rho)}$, and zeroes are symmetric about the critical line, the error term will be minimized if all non-trivial zeroes lie on the critical line, i.e. the Riemann Hypothesis holds.

1.6 Approximate form of the Functional Equation

While the functional equation beautifully expresses the Zeta function, it defines $\zeta(s)$ not explicitly, but rather in terms of $\zeta(1-s)$. For this reason, it proves in many cases useful to understand the approximate behaviour, based not on another value of the zeta function itself. To that end, here is presented the approximate form of the functional equation. This is to be done by considering the integral of a related function and finding the bounds of the moduli of the integrals, which correlate to the error terms.

Theorem 1.7 (Approximate Functional Equation)**.** *In the case $\sigma > 0, |t| < \frac{2\pi x}{C}$, for some constant $C > 1$;*

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}). \tag{1.6.1}$$

Proof. Suppose initially that $\sigma > 1$, and x is not an integer. Then clearly;

$$\zeta(s) - \sum_{n<x} \frac{1}{n^s} = \sum_{n>x} \frac{1}{n^s}. \tag{1.6.2}$$

So we now turn our attention an integral form of the right hand side of the above equation;

$$\sum_{x<n} \frac{1}{n^s} = -\frac{1}{2i} \int_{x-i\infty}^{x+i\infty} z^s \cot\pi z \mathrm{d}z. \tag{1.6.3}$$

This equality follows from the sum identity for the Cotangent (presented in the Appendix);

$$\cot\pi z = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{z+n}. \tag{1.6.4}$$

Where if the integral above is taken, and without loss of generality, it is assumed that x is a half-integer, the sum identity of Cotangent shows that the poles of the function will be at integers in the half plane to the right of x. It then follows that the pole at $z = k$ will have residue of k^{-s}, and so by residue theory the desired equation follows.
Manipulation of the previous integral yields;

$$\begin{aligned}&-\frac{1}{2i}\int_{x-i\infty}^{x} z^s(\cot\pi z - i) + iz^s \mathrm{d}z - \frac{1}{2i}\int_{x}^{x+i\infty} z^s(\cot\pi z + i) - iz^s \mathrm{d}z \\ &= -\frac{1}{2i}\int_{x-i\infty}^{x} z^s(\cot\pi z - i)\mathrm{d}z - \frac{1}{2i}\int_{x}^{x+i\infty} z^s(\cot\pi z + i)\mathrm{d}z + \frac{x^{1-s}}{1-s}.\end{aligned} \tag{1.6.5}$$

It now remains to show that the integrals are of order $O(x^{-\sigma})$. By first considering cotangent in it's exponential form;

$$\begin{aligned}\int_{x-i\infty}^{x} z^s(\cot\pi z - i)\mathrm{d}z &= \int_{x-i\infty}^{x} iz^s\left(\frac{\mathrm{e}^{i\pi z}+\mathrm{e}^{-i\pi z}}{\mathrm{e}^{i\pi z}-\mathrm{e}^{-i\pi z}} - 1\right)\mathrm{d}z \\ &= \int_{x-i\infty}^{x} iz^s\left(\frac{\mathrm{e}^{i\pi z}+\mathrm{e}^{-i\pi z}-\mathrm{e}^{i\pi z}+\mathrm{e}^{-i\pi z}}{\mathrm{e}^{i\pi z}-\mathrm{e}^{-i\pi z}}\right)\mathrm{d}z \\ &= \int_{x-i\infty}^{x} iz^s\left(\frac{2\mathrm{e}^{-i\pi z}}{\mathrm{e}^{i\pi z}-\mathrm{e}^{-i\pi z}}\right)\mathrm{d}z.\end{aligned} \tag{1.6.6}$$

.

Letting $z = x + i\beta$, $s = \sigma + it$, and considering the modulus of the argument of the integral gives;

$$\left|iz^{\sigma+it}\cdot\frac{2\mathrm{e}^{-i\pi x}\mathrm{e}^{\pi\beta}}{\mathrm{e}^{i\pi x}\mathrm{e}^{-\pi\beta}-\mathrm{e}^{-i\pi x}\mathrm{e}^{\pi\beta}}\right| \leq |z^{\sigma+it}|\cdot\left|\frac{2}{\mathrm{e}^{2i\pi x}\mathrm{e}^{-2\pi\beta}-1}\right|. \tag{1.6.7}$$

Using that

$$|z^{\sigma+it}| \leq |z|^{\sigma}\mathrm{e}^{t\cdot\arctan(\beta/x)} \leq |z|^{\sigma}\mathrm{e}^{|t|\beta/x}. \tag{1.6.8}$$

and (remembering x is half-integer and $\beta \leq 0$)

$$\left|\frac{2}{\mathrm{e}^{2i\pi x}\mathrm{e}^{-2\pi\beta}-1}\right| = \left|\frac{2}{-\mathrm{e}^{-2\pi\beta}-1}\right| = \frac{2}{\mathrm{e}^{-2\pi\beta}+1}. \tag{1.6.9}$$

It then follows that:

$$\left|\int_{x-i\infty}^{x} z^s(\cot\pi z - i)\mathrm{d}z\right| \leq x^{-\sigma}\int_{-\infty}^{0} \mathrm{e}^{2\pi\beta+|t|\beta/x}\mathrm{d}\beta$$
$$= \frac{x^{-\sigma}}{2\pi - |t|/x}$$
$$= O(x^{-\sigma}). \tag{1.6.10}$$

The same process yields the same result for the integral in the upper half of the plane, giving the same O term. Hence the Approximate functional equation follows. $\square$

2 The Divisor problem

The Divisor problem has been a feature of the field of analytic number theory since Dirichlet found the leading term of the divisor summation formula [10];

$$D(n) := \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x). \tag{2.0.1}$$

Where $d(n)$ is the multiplicative arithmetic function given by the number of divisors of n. The divisor problem involves finding the lowest possible order of $\Delta(x)$, that is, the smallest α such that $\Delta x = O(x^{\alpha+\epsilon})$ for all ϵ greater than zero. While conjectured to be $1/4$, the exact order remains undetermined, the bound is being lowered, most recently by Huxley in 2003 [11].

The relation between the divisor problem and the Riemann Zeta function is clear, as the square of the Riemann Zeta function is given by

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}. \tag{2.0.2}$$

Which follows from a simple multiplication of sums. Furthermore, letting $d_k(n)$ denote the function which counts the number of ways n can be represented as a product of k factors, gives the more general

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}. \tag{2.0.3}$$

And the corresponding

$$D_k(n) := \sum_{n \leq x} d_k(n) = x P_k(\log x) + \Delta_k(x). \tag{2.0.4}$$

Where P_k is a polynomial of order $k-1$, which will be seen to correspond to the residue of a certain function. This relationship allows the use of a range of techniques to analyze the behaviour of the divisor function. To examine the divisor function, one can use certain tools of complex analysis to give asymptotic formulas for $D_k(x)$, as shown in this chapter.

2.1 Voronoi's Formula for arbitrary order d_k

With the analytic nature of all powers of the zeta function described, one can derive Voronoi's summation formula, developed by Georgy Voronoi in 1904 [1], which in particular can be used to derive a good error term of $d(n)$.

Initially we derive a sum over all naturals, then later aim to restrict this to finite sums, allowing us to consider the nature of D_k. Initially, the Mellin transform and it's inverse are presented, as found in [12].

Definition 2.0.1. *The Mellin transform of a function f is;*

$$Mf(s) = \int_0^\infty f(x)x^{s-1}dx. \tag{2.1.1}$$

And the Mellin Inversion Theorem states that if $g(s)$, a function of complex variable, is analytic in a strip $a < Re(s) < b$, along with the condition that g approaches zero uniformly in the limit $\lim_{t\to\infty} g(c+it)$ for every $a < c < b$, then

$$M^{-1}g(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)x^{-s}ds = f(x) \tag{2.1.2}$$

implies that

$$Mf(s) = g(s). \tag{2.1.3}$$

This restriction on g leads to the equivalent statement that

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Mf(s)x^{-s}ds. \tag{2.1.4}$$

As long as $\frac{f(x)}{x^k} \to 0$ in the limit $x \to \infty$ for all k, and $f(x) = O(x^{-\kappa})$ for some $\kappa > 0$ in the limit $x \to 0$. Another feature of the Mellin transform is shown by integrating the expression for $Mf(s)$, to yield the identity:

$$Mf(s) = f(x)\left.\frac{x^s}{s}\right|_0^\infty + \int_0^\infty f'(x)\frac{x^s}{s}dx = \frac{1}{s}Mf'(s+1). \tag{2.1.5}$$

Showing that there exists an analytic continuation of Mf to the complex plane, with possible poles at zero and the negative reals, depending on f.

Now considering the sum;

$$\sum_{n=1}^\infty d_k(n)f(n) \tag{2.1.6}$$

with the above restriction on f allows the manipulation;

$$\begin{aligned}\sum_{n=1}^\infty d_k(n)f(n) &= \sum_{n=1}^\infty \frac{d_k(n)}{2\pi i}\int_{Re(s)=c>1} n^{-s}Mf(s)ds \\ &= \frac{1}{2\pi i}\int_{Re(s)=2} Mf(s)\sum_{n=1}^\infty \frac{d_k(n)}{n^s}ds \\ &= \frac{1}{2\pi i}\int_{Re(s)=2} Mf(s)\zeta^k(s)ds. \end{aligned} \tag{2.1.7}$$

From the identity above, it remains to find the residue at the point $s = 0$:

$$\lim_{s\to 0} Mf(s)\zeta^k(s) = -Mf(1)\zeta^k(0) = f(0)\zeta^k(0). \tag{2.1.8}$$

Due to the definition of the transform, and remembering the restriction on f in the limit $x \to \infty$. There is also the pole of $\zeta^k(1)$, so moving the path of integration gives:

$$\sum_{n=1}^{\infty} d_k(n)f(n) = f(0)\zeta^k(0) + Res_{s=1}(Mf(s)\zeta^k(s)) \\ + \frac{1}{2\pi i}\int_{Re(s)=-1/2} Mf(s)\zeta^k(s)ds. \tag{2.1.9}$$

Recalling, from the previous chapter, the functional equation of the zeta function gives the formula for arbitrary k;

$$\sum_{n=1}^{\infty} d_k(n)f(n) = \frac{f(0)}{2^k} + Res_{s=1}(Mf(s)\zeta^k(s)) \\ + \frac{1}{2\pi i}\int_{Re(s)=3/2} Mf(1-s)\frac{\pi^{-ks/2}\Gamma(s/2)^k}{\pi^{-k/2+ks/2}\Gamma(1/2-s/2)^k}\zeta^k(s)ds. \tag{2.1.10}$$

2.2 Voronoi Summation Formula for $k = 2$

The traditional case of Voronoi's formula results in the specific case of $k = 2$, giving the famous result based on the Bessel functions:

Theorem 2.1 (Voronoi Summation Formula)**.** *For a Schwartz function $f(x)$, the following summation formula holds:*

$$\sum_{n=1}^{\infty} d(n)f(n) = \frac{f(0)}{4} + \int_0^{\infty} f(x)(\log x + 2\gamma)dx \\ + \sum_{n=1}^{\infty} d(n)\int_0^{\infty} f(x)(4K_0(4\pi(nx)^{1/2}) - 2\pi Y_0(4\pi(nx)^{1/2}))dx. \tag{2.2.1}$$

Proof. Setting $k = 2$ in the general equation;

$$\sum_{n=1}^{\infty} d(n)f(n) = \frac{f(0)}{4} + Res_{s=1}\int_0^{\infty} f(x)x^{s-1}\zeta^2(s)dx \\ + \frac{1}{2\pi i}\int_{Re(s)=3/2}\int_0^{\infty} f(x)x^{-s}dx\frac{\pi^{-s}\Gamma(s/2)^2}{\pi^{-1+s}\Gamma(1/2-s/2)^2}\zeta^2(s)ds. \tag{2.2.2}$$

Then the various properties of the gamma function (see Appendix), give the identity

$$\frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-1/2-s/2}\Gamma(1/2-s/2)} = 2\cos(\frac{\pi s}{2})(2\pi)^{-s}\Gamma(s). \tag{2.2.3}$$

Thus the final double integral gives

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{Re(s)=3/2}\int_0^\infty f(x)x^{-s}4\cos^2(\frac{\pi s}{2})(2\pi)^{-2s}\Gamma^2(s)\zeta^2(s)dxds \\
&= \frac{1}{2\pi i}\int_{Re(s)=3/2}\int_0^\infty f(x)x^{-s}(2+2\cos(\pi s))(2\pi)^{-2s}\Gamma^2(s)\zeta^2(s)dxds \\
&= \frac{1}{2\pi i}\int_0^\infty\int_{Re(s)=3/2} f(x)x^{-s}(2+2\cos(\pi s))(2\pi)^{-2s}\Gamma^2(s)\zeta^2(s)dsdx.
\end{aligned} \tag{2.2.4}$$

Since the complex integral is over a real part greater than 1, the Dirichlet series converges so we are permitted to make the substitution

$$= \frac{1}{2\pi i}\sum_{n=1}^\infty d(n)\int_0^\infty f(x)\int_{Re(s)=3/2}(nx)^{-s}(2+2\cos(\pi s)(2\pi)^{-2s}\Gamma^2(s)dsdx. \tag{2.2.5}$$

The classical result of Voronoi then follows from the Inverse Mellin theorem applied to Bessel functions (see Appendix), firstly;

$$\begin{aligned}
\sum_{n=1}^\infty d(n)f(n) &= \frac{f(0)}{4} + Res_{s=1}\int_0^\infty f(x)x^{s-1}\zeta^2(s)dx \\
&\quad + \sum_{n=1}^\infty d(n)\int_0^\infty f(x)(4K_0(4\pi(nx)^{1/2}) - 2\pi Y_0(4\pi(nx)^{1/2}))dx.
\end{aligned} \tag{2.2.6}$$

Then considering the following asymptotic expansions around $s=1$;

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \implies \zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + O(1) \tag{2.2.7}$$

where here γ represents the Euler-Mascheroni constant, not the function of the zeta functional equation, and

$$x^{s-1} = 1 + (s-1)\log x + O(|s-1|^2) \tag{2.2.8}$$

gives that

$$Res_{s=1}\big(\int_0^\infty f(x)x^{s-1}\zeta^2(s)dx\big) = \int_0^\infty f(x)(\log x + 2\gamma)dx. \tag{2.2.9}$$

Giving the final result;

$$\begin{aligned}\sum_{n=1}^{\infty} d(n)f(n) =& \frac{f(0)}{4} + \int_0^{\infty} f(x)(\log x + 2\gamma)dx \\ &+ \sum_{n=1}^{\infty} d(n) \int_0^{\infty} f(x)(4K_0(4\pi(nx)^{1/2}) - 2\pi Y_0(4\pi(nx)^{1/2}))dx.\end{aligned} \tag{2.2.10}$$

□

2.3 The Truncated Voronoi Summation Formula

We now consider the bounded sum $D_k(x)$, ie

$$D_k(x) = \sum_{n=1}^{x} d_k(n). \tag{2.3.1}$$

The correspondence between this formula and the Voronoi formula derived in the previous section is clear. Indeed, by restricting the $f(x)$ weighting function in the previous chapter to a bounded support, we get the formula;

$$\begin{aligned}{\sum_{n=a}^{b}}' d(n)f(n) =& \int_a^{b} f(x)(\log x + 2\gamma)dx \\ &+ \sum_{n=1}^{\infty} d(n) \int_a^{b} f(x)(4K_0(4\pi(nx)^{1/2}) - 2\pi Y_0(4\pi(nx)^{1/2}))dx.\end{aligned} \tag{2.3.2}$$

Where the prime on the sum indicates that if a or b is integer, the sum term is $\frac{1}{2}d(a)f(a)$ (resp. b). Taking $a = 1$ and $b = x$ and $f(x)$ to be the constant function on compact support $[0, x]$, using the Bessel function identities in the appendix gives that:

$$\begin{aligned}{\sum_{n=1}^{x}}' d(n) =& \frac{1}{4} + x(\log x + 2\gamma - 1) \\ &- \frac{2\sqrt{x}}{\pi} \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} (4K_1(4\pi(nx)^{1/2}) - 2\pi Y_1(4\pi(nx)^{1/2})).\end{aligned} \tag{2.3.3}$$

Without knowing the convergence of the infinite sum the formula is barely useful, so a truncated version of the Voronoi summation formula is of far more use, presented here is an elaboration of the proof outlined by Ivic in [7]:

Theorem 2.2 (Truncated Voronoi Summation Formula)**.**

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{\epsilon}) + O(x^{1/2+\epsilon} N^{-1/2}). \tag{2.3.4}$$

Proof. To start, by using the version of Perron's formula over a finite contour of integration;

$$\sum_{n=1}^{x}{}' d(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta^2(s) \frac{x^s}{s} ds + O(x^c T (c-1)^{-2}) + O(x^{1+\epsilon} T^{-1}). \tag{2.3.5}$$

Letting $c = 1+\epsilon$, $T = 2\pi\sqrt{x(N+1/2)}$ in the previous equation. Now taking the integral around the rectangle with vertices $c+iT, -a+iT, -a-iT, c-iT$ for some $a > 0$, gives the following, by utilizing the functional equation for the Zeta function;

$$\Delta(x) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} d(n) \int_{-a-iT}^{-a+iT} \chi^2(s) n^{s-1} \frac{x^s}{s} ds + O(x^{-2a} T^{2a}) + O(x^{\epsilon}) + O(x^{1+\epsilon} T^{-1}). \tag{2.3.6}$$

(Recalling that $\Delta(x) := \sum_{n=1}^{x}{}' d(n) - x(\log x + 2\gamma - 1) - 1/4$). The Horizontal segments from the rectangle are contained in the error terms, since on the contour $\zeta(s) = O(t^{(a+1/2)(c-\sigma)/(a+c)})$, by the Phragmem-Lindelöf principle, and bounds presented by Titchmarsh in [5]. Thus the integral on the horizontal sections are of order;

$$\int_{-a+iT}^{c+iT} \zeta^2(s) \frac{x^s}{s} ds = O(T^{2a} x^{-2a}) + O(T^{-1} x^c). \tag{2.3.7}$$

Which is contained in the error terms for the expression of $\Delta(x)$. The same result holds for the integral in the $\Im(s) < 0$ half of the plane. Furthermore, the choices of c and N give the rest of the error terms for $\Delta(x)$. Now to avoid the possible problems arising from the infinite sum, the order of the contribution of the terms corresponding to $n > N$:

$$\begin{aligned} &\frac{1}{2\pi i} \sum_{n=N}^{\infty} d(n) \int_{-a-iT}^{-a+iT} \chi^2(s) n^{s-1} \frac{x^s}{s} ds \\ &= \frac{1}{2\pi i} \sum_{n=N}^{\infty} d(n) \Big(\int_{\sigma=-a, 1 \le |t| \le T} \chi^2(s) n^{s-1} \frac{x^s}{s} ds + \int_{\sigma=-a, |t|<1}^{-a+iT} \chi^2(s) n^{s-1} \frac{x^s}{s} ds \Big). \end{aligned} \tag{2.3.8}$$

Then using the asymptotic formula for $\chi(s)$ based on Stirling's formula $(\chi(s) = (\frac{2\pi}{t})^{\sigma+it-1/2}e^{i(t+\pi/4)}(1+O(t^{-1}))$ gives for half of the first integral;

$$\int_{\sigma=-a,1\leq T} \chi^2(s)n^{s-1}\frac{x^s}{s}ds$$
$$= \frac{1}{(2\pi)^{2a-1}x^a n^{1+a}} \int_1^T e^{2ti(-\log t+\log(2\pi)+1+(1/2)\log(nx))}(t^{2a}+O(t^{2a-1})dt. \tag{2.3.9}$$

Writing the integrand in exponential form shows that is appropriate to use the following lemma:

Lemma 2.3. *For $F(x)$ a real differentiable function, such that $F'(x) \geq m > 0$ and monotonic;*

$$|\int_a^b e^{iF(x)}dx| \leq \frac{4}{m}. \tag{2.3.10}$$

Essentially an application of the second mean value theorem applied to the real and imaginary parts of the exponential [5].

Thus since $F'(x) = \log(4\pi^2 nx/t^2) \geq \log(n/(N+1/2)$, the previous integral becomes;

$$\frac{1}{x^a n^{1+a}}\big(O(\frac{T^{2a}}{\log(n/(N+1/2))}) + O(T^{2a})\big). \tag{2.3.11}$$

Splitting the range of the sum into two parts, $[N, 2N]$ and $[2N, \infty)$ gives for the first part:

$$\frac{1}{2\pi i}\sum_{n=2N}^{\infty} d(n)\int_{\sigma=-a,1\leq t\leq T} \chi^2(s)n^{s-1}\frac{x^s}{s}ds = O\big(\sum_{n=2N}^{\infty}\frac{d(n)T^{2a}}{x^a n^{1+a}}\big) = O(N^{\epsilon}). \tag{2.3.12}$$

And for $N \leq n < 2N$;

$$\frac{1}{2\pi i}\sum_{n=N}^{2N} d(n)\int_{\sigma=-a,1\leq t\leq T} \chi^2(s)n^{s-1}\frac{x^s}{s}ds = O\big(\sum_{n=N}^{2N}\frac{d(n)T^{2a}}{x^a n^{1+a}\log(n/(N+1/2)}\big)$$
$$= O(N^{\epsilon}\sum_{k=1}^{N}\frac{1}{k}) = O(N^{\epsilon}). \tag{2.3.13}$$

And again recalling our choice of N, we are left with both integrals being $O(T^{2\epsilon}x^{-\epsilon})$ which are contained in the error terms of $\Delta(x)$ above. The same follows for the integral corresponding to $-T \leq t < 1$. For the middle part of the integral corresponding to $|t| < 1$;

$$\frac{1}{2\pi i}\sum_{n=N}^{\infty} d(n)\int_{\sigma=-a,-1\leq t\leq 1}\chi^2(s)n^{s-1}\frac{x^s}{s}ds = O(\sum_{n=N}^{\infty} d(n)\frac{1}{x^a n^{a+1}}) = O(x^{-a}N^{\epsilon}). \tag{2.3.14}$$

Hence by limiting our consideration to the partial sum $\sum_{n=1}^{N} d(n)$, the error terms for the expression of $\Delta(x)$ remain unchanged, that is;

$$\Delta(x) = \frac{1}{2\pi i}\sum_{n=1}^{N} d(n)\int_{-a-iT}^{-a+iT}\chi^2(s)n^{s-1}\frac{x^s}{s}ds + O(x^{-2a}T^{2a}) + O(x^{\epsilon}) + O(x^{1+\epsilon}T^{-1}). \tag{2.3.15}$$

The manipulations thus far have depended on the contour of integration being finite, however, to derive the Bessel functions appearing in the truncated Voronoi formula, we desire an integral ranging over an infinite line. Thus, we must check by how much such an integral differs from the finite one considered so far. In fact, we can split the integral up in the following way;

$$\int_{-a-iT}^{-a+iT} = \int_{-i\infty}^{i\infty} -(\int_{iT}^{i\infty} + \int_{-i\infty}^{-iT} + \int_{-iT}^{-a-iT} + \int_{-a+iT}^{iT}) \tag{2.3.16}$$

The two integrals on the imaginary axis are of the form;

$$\int_{T}^{\infty} e^{iF(t)}dt. \tag{2.3.17}$$

(or $(-\infty, -T)$ resp.) so as per the above lemma these are given by:

$$\frac{1}{n}\int_{T}^{i\infty} e^{2ti(-\log t + \log(2\pi) + 1 + (1/2)\log(nx)} \cdot O(t^{-1})dt$$
$$= O(\frac{1}{\log((N+1/2)/n)}). \tag{2.3.18}$$

The change in sign of the logarithm in the error term corresponds heuristically to the change of taking the integral "to" T to "from" T. Thus for the sum of the integral over n;

$$\sum_{n=1}^{N}\frac{d(n)}{n\log((N+1/2)/n)} = O(N^{\epsilon}). \tag{2.3.19}$$

And similarly for the negative part of the integral on the imaginary axis. For the horizontal segments, the integrals are of the form:

$$O(\int_{-a}^{0}\frac{1}{n}(\frac{nx}{T^2})^{\sigma}d\sigma). \tag{2.3.20}$$

The maximum of which (in the range of integration) is achieved at $\sigma = -a$, so over the sum;

$$O(\sum_{n=1}^{N} \frac{d(n)}{n} (\frac{T^2}{nx})^a) = O(T^{2a} x^{-a}). \tag{2.3.21}$$

Hence all parts of the integral in the bracket in the decomposition above are contained within the error terms for $\Delta(x)$, and only the integral over $(-i\infty, i\infty)$ remains:

$$\begin{aligned}
\sum_{n=1}^{N} \frac{d(n)}{n} \int_{-i\infty}^{i\infty} \chi^2(s) \frac{(nx)^s}{s} ds &= \sum_{n=1}^{N} \frac{d(n)}{n} \int_{1-i\infty}^{1+i\infty} \chi^2(1-s)(nx)^{1-s}(1-s)^{-1} ds \\
&= \sum_{n=1}^{N} \frac{d(n)}{n} \int_{1-i\infty}^{1+i\infty} 4\cos^2(\pi s/2)(2\pi)^{-2s}\Gamma^2(s) \frac{(nx)^{1-s}}{1-s} ds \\
&= \sum_{n=1}^{N} \frac{d(n)}{n\pi^2} \int_{1-i\infty}^{1+i\infty} \cos^2(\pi s/2)(2\pi\sqrt{nx})^{2-2s}\Gamma(s) \frac{(s-1)\Gamma(s-1)}{1-s} ds \\
&= -\sum_{n=1}^{N} \frac{d(n)}{n\pi^2} \int_{1-i\infty}^{1+i\infty} \cos^2(\pi s/2)(2\pi\sqrt{nx})^{2-2s}\Gamma(s)\Gamma(s-1) ds.
\end{aligned} \tag{2.3.22}$$

Giving for $\Delta(x)$, in terms of the Bessel functions:

$$\begin{aligned}
\Delta(x) = -\sum_{n=1}^{N} \frac{2\sqrt{x}d(n)}{\pi\sqrt{n}} &\big(K_1(4\pi\sqrt{nx}) + (\pi/2)Y_1(4\pi\sqrt{nx})\big) \\
&+ O(x^{\epsilon}) + O(T^{2a}x^{-a}) + O(x^{1+\epsilon}T^{-1}).
\end{aligned} \tag{2.3.23}$$

And finally, expressing the error terms in terms of N (so that $O(T^{2a}x^{-2a} = O(N^a x^{-a})$ and $O(x^{1+\epsilon}T^{-1}) = O(x^{1/2+\epsilon}N^{-1/2})$), and taking $a = \epsilon$, gives the desired truncated Voronoi formula:

$$\begin{aligned}
\Delta(x) = -\sum_{n=1}^{N} \frac{2\sqrt{x}d(n)}{\pi\sqrt{n}} &\big(K_1(4\pi\sqrt{nx}) + (\pi/2)Y_1(4\pi\sqrt{nx})\big) \\
&+ O(x^{\epsilon}) + O(x^{1/2+\epsilon}N^{-1/2}).
\end{aligned} \tag{2.3.24}$$

Replacing the Bessel functions by their asymptotic approximations [13] gives the simpler expression;

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{\epsilon}) + O(x^{1/2+\epsilon}N^{-1/2}). \tag{2.3.25}$$

□

2.4 An Estimate for $\Delta(x)$ from Voronoi's Formula

From the truncated version of Voronoi's summation formula found above, it is natural to go on to develop an estimate of the error term $\Delta(x)$ based on the result. This will give the result:

Theorem 2.4.

$$\Delta(x) = O(x^{35/108+\epsilon}) \tag{2.4.1}$$

for all $\epsilon > 0$.

Proof. By considering the truncated Voronoi formula, it is clear that deriving a bound on $\Delta(x)$ will involve finding a bound on the sum

$$\sum_{n=1}^{N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx}) \ll \sum_{n=1}^{N} d(n) n^{-3/4} e^{4\pi i\sqrt{nx}}. \tag{2.4.2}$$

Such a bound was found by Kolesnik [2] in 1982 by first rearranging the above sum into the form;

$$\sum_{n=1}^{\sqrt{N}} \sum_{m=1}^{N/n} (mn)^{-3/4} e^{4\pi i\sqrt{mnx}}. \tag{2.4.3}$$

Kolesnik then proved in particular that:

Lemma 2.5 (Kolesnik)**.**

$$\begin{aligned} \sum_{n=1}^{\sqrt{N}} \sum_{m=1}^{N/n} (mn)^{-3/4} e^{4\pi i\sqrt{mnx}} \ll & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.4.4) \\ & \ll \log^2 x \max_{K\le N} \left(x^{-1/16} K^{173/152-3/4} + x^{1/16} K^{119/152-3/4} \right). \end{aligned}$$

Kolesnik, in his paper, found the above form as a bound for a class of functions satisfying a series of equations and restrictions, as outlined in [2], however the above is all that is required to proceed.

With this lemma, it follows that:

$$\begin{aligned} \Delta(x) \ll & \log^2 x \left(x^{3/16} N^{59/152} + x^{5/16} N^{5/152} \right) \\ & + x^{\epsilon} + x^{1/2+\epsilon} N^{-1/2}. \end{aligned} \tag{2.4.5}$$

By taking $N = x^{19/54}$, the above then reduces to:

$$\begin{aligned} \Delta(x) &\ll \log^2 x \left(x^{35/108} + x^{35/108} \right) \\ &+ x^{\epsilon} + x^{35/108+\epsilon} \\ &\ll x^{35/108+\epsilon} \end{aligned} \tag{2.4.6}$$

as stated. $\square$

3 The Exponential Weighted Divisor Sum

3.1 A Note on the Riemann Zeta Functional Equation

In 1.2, the functional equation of the Riemann Zeta function was derived, which served well in deriving a generalization of Voronoi's formula for the divisor function. However, in the following argument it will be convenient to consider a different (although clearly equivalent) version of the analytic continuation of the Dirichlet sum. The reason for this is that it will present us an easy method of finding certain values of the Riemann Zeta function which are required in the following section.

We begin with the integral (assuming for a moment that $\sigma > 1$);

$$\int_0^\infty x^{s-1}e^{-nx}dx \tag{3.1.1}$$

Which after a change of variable yields

$$\int_0^\infty \frac{u^{s-1}e^{-u}}{n^s}du = \frac{\Gamma(s)}{n^s}, \tag{3.1.2}$$

so

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^\infty \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-nx}dx \\ &= \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}dx \\ &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx. \end{aligned} \tag{3.1.3}$$

Here we are permitted to change the order of summation and integration due to our assumption that $\sigma > 1$, so the integral is absolutely convergent. In order to extend the argument to the rest of the complex plane, we consider the related contour integral;

$$\int_C \frac{z^{s-1}}{e^z-1}dz. \tag{3.1.4}$$

Where the contour C is a standard keyhole integral from positive infinity on the real axis to a circle of circumference ϵ then returning to positive infinity. The interval then falls into three parts,

$$\int_C \frac{z^{s-1}}{e^z-1}dz = -\int_0^\infty \frac{x^{s-1}}{e^x-1}dx + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^x-1}dx + \int_{|\epsilon|=1} \frac{z^{s-1}}{e^z-1}dz \tag{3.1.5}$$

Turning our attention to the final integral above, on this path we find the bound on the z^{s-1} term;

$$|z^{s-1}| = |e^{(s-1)\log(z)}| = |e^{(\sigma-1+it)\log(z)}| = |z|^{(\sigma-1)}. \tag{3.1.6}$$

Hence by taking our path on a radius approaching zero our integral will disappear for $\sigma > 1$ (since $|e^z - 1|$ is asymptotically larger than $|z|$,) and our initial integral reduces to the integral

$$(e^{2\pi i(s-1)} - 1)\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx = (e^{2\pi is} - 1)\Gamma(s)\zeta(s) = \frac{2i\pi e^{\pi is}}{\Gamma(1-s)}\zeta(s), \tag{3.1.7}$$

where the last equality follows from Eulers reflection formula of the Gamma function.

Hence we have a form of the functional equation;

$$\zeta(s) = \frac{\Gamma(1-s)}{2i\pi e^{\pi is}}\int_C \frac{z^{s-1}}{e^z - 1}dz. \tag{3.1.8}$$

With C the keyhole contour from above. From this form of the equation, the explicit values of $\zeta(s)$ are found using the equation;

$$\frac{z}{e^z - 1} = B_0 - B_1\frac{z}{1!} + B_2\frac{z^2}{2!} - B_4\frac{z^4}{4!} + ..., \tag{3.1.9}$$

where B_n denotes the n-th Bernoulli number. Thus for negative odd integer values of s, the theorem of residues gives:

$$\begin{aligned}\zeta(-2n-1) &= (-1)^{2n+1}\frac{(2n+1)!}{2\pi i}\cdot\frac{2\pi iB_{2n+2}}{(2n+2)!}\\ &= -\frac{B_{2n+2}}{2n+2}.\end{aligned} \tag{3.1.10}$$

3.2 On an Exponential Weighted Divisor Sum

The exponential weighted divisor sum has been considered by many authors, famously by Wigert in [3], who's findings we present here, through to the more recent investigations of Jutila [14], who considered a weight in the form $e^{nh/k}$, as a generalization of the Voronoi formula derived above. Equipped with the particular values for $\zeta(s)$ from the previous section, we look at the weighted divisor sum considered first by Wigert:

Theorem 3.1 (Wigert's Summation theorem). *For* $\arg(z) < \pi/2$, *and* z *sufficiently close to zero;*

$$\sum_{n=1}^{\infty} d_k(n)e^{-nz} = \frac{1}{(-2)^k} + \frac{P_k(\log z)}{z} + \frac{-B_{2n+2}^2}{(2n+2)(2n+2)!}x^{2n+1} + O(|z|^{2K}). \tag{3.2.1}$$

Proof. Making the observation;

$$\begin{aligned}\sum_{n=1}^{\infty} d_k(n)e^{-nz} &= \sum_{n=1}^{\infty} d_k(n)\frac{1}{2\pi i}\int_{Re(s)=2}\Gamma(s)(nz)^{-s}ds \\ &= \frac{1}{2\pi i}\int_{Re(s)=2}\Gamma(s)\zeta^k(s)z^{-s}ds \\ &:= \frac{1}{2\pi i}\int_{Re(s)=2}\varphi_k(s)ds. \end{aligned} \tag{3.2.2}$$

Which follows using that $Mf(s) = \Gamma(s)$ is the inverse Mellin transform of $f(x) = e^x$, and so the equation follows by the change of variables $x \to nz$. It is noted that the choice of integral path is arbitrary so long as $s > 0$ when $nz > 0$, so for convenience the explicit case of $\Re(s) = 2$ is used for the rest of the proof.

Turning the attention to the poles of φ_k, it is seen immediately that there is a pole of order k at $s = 1$. Furthermore, there are poles of order one at each of $0, -1, -2, -3, ...$ in the Gamma function. However, due to zeroes of order k at the negative even integers in the Zeta function, the poles of φ are only at $1, 0$, and the negative odd integers.

Beginning with the easiest residue to deal with, at $s = 0$;

$$\begin{aligned}Res_{s=0}\varphi_k(s) &= \zeta^k(0)Res_{s=0}\Gamma(s) \\ &= \frac{1}{(-2)^k}.\end{aligned} \tag{3.2.3}$$

Since $Res_{s=1}\Gamma(s) = 1$. Furthermore, since $\Gamma(s+1) = s\Gamma(s)$, induction gives that for negative integers;

$$\begin{aligned}Res_{s=-n}\Gamma(s) &= \lim_{s\to -n}(s+n)\Gamma(s) \\ &= \lim_{s\to -n}\frac{\Gamma(s+n+1)}{(s(s+1)...(s+n-1)} \\ &= \frac{(-1)^n}{n!}.\end{aligned} \tag{3.2.4}$$

Combining this with the result of the previous section, it is found that the general formula for the residue at the negative odd integers is given by:

$$Res_{s=-2n-1}\varphi_k(s) = \frac{-B_{2n+2}^k}{(2n+2)(2n+2)!}z^{2n+1}. \tag{3.2.5}$$

Now finally it remains to find the residue at $s = 1$. This require the following expansions around $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n \frac{(-1)^n}{n!}(s-1)^n \tag{3.2.6}$$

$$\Gamma(s) = \sum_{n=0}^{\infty} \Gamma_n (s-1)^n \tag{3.2.7}$$

$$z^{-s} = \sum_{n=0}^{\infty} \frac{(s-1)^n(-\log z)^n}{n!z}. \tag{3.2.8}$$

Where γ_n are the Stieltjes constants (for convenience, γ refers hereafter to γ_0), and Γ_n are the coefficients of the Gamma function, usually given in terms of γ, and the digamma function. Expanding the $\zeta^k(s)$ thus gives;

$$\begin{aligned}\zeta^k(s) = \sum_{\kappa=1}^{k} (&\frac{1}{(s-1)^{k-\kappa}} \sum_{n_1+\ldots+n_j \leq \kappa} \frac{(1-s)^{n_1+\ldots+n_\kappa}}{\prod_{l \leq j} n_l!} \prod_{l \leq j} \gamma_{n_l}) \\ &+ \frac{1}{(s-1)^k} + O(|s-1|). \end{aligned} \tag{3.2.9}$$

Using these expansions, the residue is found by considering as usual the coefficient of their products corresponding to $(s-1)^{-1}$. As k grows large, this becomes especially cumbersome. In the example of the simplest case of $k = 2$, this gives the expression;

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma_0}{(s-1)} + O(1). \tag{3.2.10}$$

Thus combining all three of the expansions, for φ_2 around $s = 1$;

$$\begin{aligned}\varphi_2(s) &= \frac{1}{z(s-1)^2} + \frac{2\gamma}{z(s-1)} - \frac{\gamma}{z(s-1)} - \frac{\log z}{z(s-1)} + O(1) \\ &= \frac{1}{z(s-1)^2} + \frac{\gamma}{z(s-1)} - \frac{\log z}{z(s-1)} + O(1). \end{aligned} \tag{3.2.11}$$

(Since $\Gamma_0 = 1$ and $\Gamma_1 = -\gamma$.) And hence;

$$Res_{s=1}\varphi_2(s) = \frac{\gamma - \log z}{z}. \tag{3.2.12}$$

For the slightly more complicated case $k = 4$;

$$\begin{aligned}\zeta^4(s) &= \frac{1}{(s-1)^4} + \frac{4\gamma_0}{(s-1)s} + \frac{6\gamma_0^2 - 4\gamma_1}{(s-1)^2} + \frac{-12\gamma_1\gamma_0 + 2\gamma_2 + 4\gamma_0^3}{s-1} + O(1) \\ &:= \frac{1}{(s-1)^4} + \frac{(\zeta^4)_3}{(s-1)^3} + \frac{(\zeta^4)_2}{(s-1)^2} + \frac{(\zeta^4)_1}{s-1} + O(1). \end{aligned} \tag{3.2.13}$$

Gives;

$$\begin{aligned}Res_{s=1}\varphi_4(s) =& \frac{1}{z}\Big(\frac{\Gamma_1 \log^2 z}{2} + \Gamma_2 \log z + \Gamma_3 + \frac{\log^3 z}{6} \\ &+ (\zeta^4)_1\Gamma_0 - (\zeta^4)_2(\Gamma_1 - \log z) \\ &+ (\zeta^4)_3(\Gamma_2 - \Gamma_1 \log z + \frac{\log^2 z}{2})\Big).\end{aligned} \tag{3.2.14}$$

And in fact for the general k, we have

$$Res_{s=1}\varphi_k(s) = \frac{1}{z}P_k(\log z). \tag{3.2.15}$$

Where $P_k(x)$ is a polynomial of order $k-1$. This is seen from the series expansions given above, by calculating the form of the $(s-1)^{-1}$ coefficient. Thus with all the residues of $\varphi_k(s)$ calculated, or at least calculable, the contour of integration is constructed by taking the contour C around the rectangle with vertices at $2+iR, -2K+iR, -2K-iR$ and $2-iR$. Then the residue theorem gives us;

$$\frac{1}{2\pi i}\int_C \varphi_k(s)ds = \frac{1}{(-2)^k} + Res_{s=1}\varphi_k(s) + \sum_{n=1}^{K-1} \frac{-B_{2n+2}^k}{(2n+2)^{k-1}(2n+2)!}z^{2n+1}. \tag{3.2.16}$$

Firstly, it is shown that the horizontal sections of the contour tend to zero as we take R to infinity. From [15], it is known that for $-2K \le \sigma \le 2$, as $|t| \to \infty$;

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}e^{-\pi|t|/2}|t|^{\sigma-1/2}. \tag{3.2.17}$$

And from [5];

$$\zeta(s) = O(|t|^{\epsilon}). \tag{3.2.18}$$

For some $\epsilon > 0$ in the half plane $-2K \le \sigma$. Finally the control on z^{-s} is given by;

$$|z^{-s}| = e^{-\sigma \log|z| + t\arg(z)} \le |z|^{-\sigma}e^{\lambda|t|}. \tag{3.2.19}$$

For any $\arg(z) \le \lambda$. Thus the bound on the integral is;

$$\begin{aligned}\Big|\int_{2+iR}^{-2K+iM} \varphi_k(s)ds\Big| &\le (2+2K) \sup_{-2K\le\sigma\le 2} |\varphi_k(\sigma+iR)| \\ &= O(e^{-\pi R/2}R^{\sigma-1/2}R^{k\epsilon}|z|^{-\sigma}e^{\lambda|R|}).\end{aligned} \tag{3.2.20}$$

Here we see the necessity of the restriction on the argument of z, keeping the exponent decreasing. Hence by taking $R \to \infty$ out integral goes to zero.

The same reasoning gives that the integral on the lower horizontal side of the contour also goes to zero. Now considering the integral

$$-\int_{-2K-iR}^{-2K+iR} \varphi_k(s)ds = \int_{\sigma=-2K,|t|\leq 1} \varphi_k(s)ds + \int_{\sigma=-2K,|t|\geq 1} \varphi_k(s)ds. \tag{3.2.21}$$

For the first integral, the Gamma and Zeta functions are continuous on the bounded path, so;

$$|\int_{\sigma=-2K,|t|\leq 1} \varphi_k(s)ds| \leq \int_{\sigma=-2K,|t|\leq 1} |\Gamma(s)\zeta^k(s)||z|^{-\sigma}e^{\lambda|t|}ds = O(|z|^{2K}). \tag{3.2.22}$$

For the remaining part of the integral, the order of φ_k from the previous argument can be used to give;

$$\begin{aligned} |\int_{\sigma=-2K,|t|\geq 1} \varphi_k(s)ds| &\leq \int_{\sigma=-2K,|t|\geq 1} e^{-\pi|t|/2}|t|^{\sigma-1/2}|t|^{k\epsilon}|z|^{-\sigma}e^{\lambda|t|}ds \\ &= |z|^{-2K} \int_{\sigma=-2K,|t|\geq 1} e^{-\pi|t|/2}|t|^{\sigma-1/2}|t|^{k\epsilon}e^{\lambda|t|}ds \\ &= O(|z|^{-2K}). \end{aligned} \tag{3.2.23}$$

Now all components of the contour integral are understood, apart from the vertical section corresponding to $\sigma = 2,$. Thus by taking $R \to \infty$ and applying the residue theorem, we are left with the formula:

$$\sum_{n=1}^{\infty} d_k(n)e^{-nz} = \frac{1}{(-2)^k} + Res_{s=1}\varphi_k(s) + \sum_{n=1}^{K-1} \frac{-B_{2n+2}^k}{(2n+2)^{k-1}(2n+2)!} z^{2n+1} + O(|z|^{2K}). \tag{3.2.24}$$

Taking the case $k = 2$, with the residue as found above thus gives:

$$\sum_{n=1}^{\infty} d(n)e^{-nz} = \frac{1}{4} + \frac{\gamma - \log z}{z} + \sum_{n=1}^{K-1} \frac{-B_{2n+2}^2}{(2n+2)(2n+2)!} z^{2n+1} + O(|z|^{2K}). \tag{3.2.25}$$

Discovered by Wigert in his paper. □

Similarly, we find the more complicated case of $k = 4$:

$$\begin{aligned} \sum_{n=1}^{\infty} d_4(n)e^{-nz} = \frac{1}{16} &+ \frac{1}{z}\big(3\pi^2\gamma_0 - 2\gamma_0^3 - 8\gamma_0\gamma_1 + 2\gamma_2 + 2\psi''(1) \\ &+ \log z(4\gamma_1 - \frac{5}{2}\gamma_0^2 - \pi^2) + \frac{3}{2}\gamma_0 \log^2 z - \frac{1}{6}\log^3 z\big) \\ &+ \sum_{n=1}^{K-1} \frac{-B_{2n+2}^4}{(2n+2)^3(2n+2)!} x^{2n+1} + O(|z|^{2K}). \end{aligned} \tag{3.2.26}$$

Where $\psi''(s)$ is the second derivative of the digamma function:

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}. \tag{3.2.27}$$

For the general case:

$$\sum_{n=1}^{\infty} d_k(n)e^{-nz} = \frac{1}{(-2)^k} + \frac{P_k(\log z)}{z} + \frac{-B_{2n+2}^2}{(2n+2)(2n+2)!}x^{2n+1} + O(|z|^{2K}). \tag{3.2.28}$$

3.3 A New Exponential Sum

With the aim of reducing the error term of the previous section, we introduce a new exponential weight, with the property that the negative integers will no longer be a consideration. To that end, we turn to the sum;

$$\sum_{n=1}^{\infty} d_k(n)e^{-n^2z^2}. \tag{3.3.1}$$

With the argument of z now restricted by $\arg(z) < \pi/4$, since it will follow that the gamma function now decays at half the rate. Following the same process as previously gives;

$$\begin{aligned}\sum_{n=1}^{\infty} d_k(n)e^{-n^2z^2} &= \sum_{n=1}^{\infty} \frac{d_k(n)}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)(nz)^{-2s}ds \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\zeta^k(2s)z^{-2s}ds.\end{aligned} \tag{3.3.2}$$

Or alternatively

$$= \frac{1}{4\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s/2)\zeta^k(s)z^{-s}ds. \tag{3.3.3}$$

Comparing with the previous part, it is seen that this will clearly behave similarly to before, however, now both $\Gamma(s/2)$ and $\zeta^k(s)$ will have poles and zeroes, respectively, at the negative even integers (and of course the zeroes will have higher order) so it is only required to consider the two poles at $s = 0$ and $s = 1$.

At $s = 0$ the new case has the residue $\frac{1}{(-2)^k}$. For the pole at $s = 1$, the same process as before is followed, considering the series expansions at this point. In the case of $k = 2$, this is;

$$Res_{s=1}\Gamma(s/2)\zeta^2(s)z^{-s} = \frac{\sqrt{\pi}}{2z}(4\gamma_0 + \psi(1/2) - 2\log z). \tag{3.3.4}$$

For ψ as defined above. In the general k case the residue is given by;

$$Res_{s=1}\Gamma(s/2)\zeta^k(s)z^{-s} = \frac{Q_k(\log z)}{z}. \tag{3.3.5}$$

It now remains to construct the integral contour. The integrand has only the two poles, so the contour is constructed around the rectangle with points $2 - iR, 2 + iR, -\delta + iR, -\delta - iR$. By the same argument as above (with a new restriction on z), the horizontal parts of the rectangle tend to zero as R increases, it remains to find the integral from $-\delta + iR$ to $-\delta - iR$:

$$\begin{aligned}\int_{-\delta-iR}^{-\delta+iR}\Gamma(s/2)\zeta^k(s)z^{-s}ds &= \int_{\sigma=-\delta,|t|\leq 1}\Gamma(s/2)\zeta^k(s)z^{-s}ds \\ &+ \int_{\sigma=-\delta,1\leq|t|\leq R}\Gamma(s/2)\zeta^k(s)z^{-s}ds. \end{aligned} \tag{3.3.6}$$

The first integral on the RHS gives;

$$\begin{aligned}|\int_{\sigma=-\delta,|t|\leq 1}\Gamma(s/2)\zeta^k(s)z^{-s}ds| &\leq \int_{\sigma=-\delta,|t|\leq 1}|\Gamma(s/2)\zeta^k(s)||z|^{-\sigma}e^{\lambda|t|}ds \\ &= O(|z|^{\delta}). \end{aligned} \tag{3.3.7}$$

Since the Gamma and Zeta functions are analytic and so bounded over the range of integration. While the second integral gives;

$$\begin{aligned}|\int_{\sigma=-\delta,|t|\geq 1}\Gamma(s/2)\zeta^k(s)z^{-s}ds| &\leq \int_{\sigma=-\delta,|t|1} e^{-\pi|t|/4}|t/2|^{\sigma-1/2}|t|^{k\epsilon}|z|^{-\sigma}e^{\lambda|t|}ds \\ &= |z|^{\delta}\int_{\sigma=-\delta,|t|\geq 1} e^{-\pi|t|/4}|t/2|^{\sigma-1/2}|t|^{k\epsilon}e^{\lambda|t|}ds \\ &= O(|z|^{\delta}). \end{aligned} \tag{3.3.8}$$

Where again the restriction $\arg(z) < \pi/4$ gives an exponentially decaying integral in t. Thus the expression for the sum in the case $k = 2$ is given by;

$$\sum_{n=1}^{\infty} d(n)e^{-n^2z^2} = \frac{1}{4} + \frac{\sqrt{\pi}}{2z}(4\gamma_0 + \psi(1/2) - 2\log z) + O(|z|^{\delta}). \tag{3.3.9}$$

And the general case:

$$\sum_{n=1}^{\infty} d(n)e^{-n^2z^2} = \frac{1}{(-2)^k} + \frac{Q_k(\log z)}{z} + O(|z|^{\delta}) \tag{3.3.10}$$

for any $\delta > 0$.

3.4 The Divisor Problem and the Exponential Weighted Sum

The sum in the previous section,

$$\sum_{n=1}^{\infty} d_k(n) e^{-n^2 z^2} \tag{3.4.1}$$

has a close correspondence to Dirichlet's divisor problem. Indeed by writing the sum, as above, in the form:

$$\sum_{n=1}^{\infty} d_k(n) e^{-n^2 z^2} = \frac{1}{(-2)^k} + \frac{Q_k(\log z)}{z} + E_k(z). \tag{3.4.2}$$

Where E_k is the error term in the previous section, a certain parallel can be seen in the formula;

$$\sum_{n=1}^{x} d_k(n) = x P_k(\log x) + \Delta_k(x). \tag{3.4.3}$$

Where P_k is the polynomial in Chapter 2. It is shown in this section that in fact;

Theorem 3.2. *The equation*

$$z \int_0^{\infty} \Delta_k(\sqrt{x}) e^{-zx} = \frac{1}{(-2)^k} + E_k(\sqrt{z}) \tag{3.4.4}$$

holds for the polynomial E_k described above.

Proof.

Lemma 3.3 (Abel's Summation Formula)**.** *If $A(x) = \sum_{0 \le n \le x} a_n$, and the derivative of $f(x)$ exists, then:*

$$\sum_{n=0}^{x} a_n f(n) = A(x) f(x) - \int_0^{x} A(y) f'(y) dy. \tag{3.4.5}$$

Which follows by integrating by parts. Using Abel's summation formula,

$$\begin{aligned} \sum_{n=1}^{\infty} d_k(n) e^{-n^2 z^2} &= z^2 \int_0^{\infty} 2x D_k(x) e^{-x^2 z^2} dx \\ &= z^2 \int_0^{\infty} 2x (x P_k(\log x) + \Delta_k(x)) e^{-x^2 z^2} dx. \end{aligned} \tag{3.4.6}$$

Now considering the polynomial part of the integrand;

$$z^2 \int_0^\infty 2x^2 P_k(\log x) e^{-x^2z^2} dx = \sum_{n<k} c_n z^2 \int_0^\infty 2x^2 \log^n x e^{-x^2z^2} dx. \quad (3.4.7)$$

Considering the separate Log powers, making the change of variable $u = xz$ shows;

$$\begin{aligned} z^2 \int_0^\infty 2x^2 \log^n x e^{x^2z^2} dx &= \frac{2}{z} \int_0^\infty u^2 e^{-u^2} \log^n(u/z) du \\ &= \frac{2}{z} \Big(\int_0^\infty u^2 e^{-u^2} \log^n u du - \log^n z \int_0^\infty u^2 e^{-u^2} du \Big). \end{aligned} \quad (3.4.8)$$

Thus, since both integrals on the RHS above converge for $n < \infty$;

$$z^2 \int_0^\infty 2x^2 P_k(\log x) e^{-x^2z^2} dx = \frac{1}{z} \tilde{Q}_k(\log z). \quad (3.4.9)$$

For some polynomial $\tilde{Q}_k$ of order $k-1$. It remains to show that $Q_k = \tilde{Q}_k$. To this end, note that since;

$$\frac{1}{z} Q_k(\log z) + \frac{1}{(-2)^k} + E_k(z) = z^2 \int_0^\infty 2x(xP_k(\log x) + \Delta_k(x)) e^{-x^2z^2} dx. \quad (3.4.10)$$

Then if

$$z^3 \int_0^\infty 2x \Delta_k(x) e^{-x^2z^2} dx = O(z^{\epsilon_k}) \quad (3.4.11)$$

in the limit $z \to 0$, the two polynomials Q_k and $\tilde{Q}_k$ must be equal, since the above integral will not have any terms of order large enough to contribute to the polynomial $Q_k(\log z)$ in the equation (3.4.10). To show this is the case;

$$\begin{aligned} z^3 \int_0^\infty 2x \Delta_k(x) e^{-x^2z^2} dx &= z^3 \int_1^\infty 2x \Delta_k(x) e^{-x^2z^2} dx + O(z^3) \\ &= z^3 \int_1^\infty 2x^{2-\epsilon_k} e^{-x^2z^2} dx + O(z^3) \\ &= z^3(O(z^{\epsilon_k - 3})) = O(z^{\epsilon_k}). \end{aligned} \quad (3.4.12)$$

And thus $Q_k = \tilde{Q}_k$. Hence by the above, it follows that:

$$\begin{aligned} \frac{1}{(-2)^k} + E_k(z) &= z^2 \int_0^\infty 2x \Delta_k(x) e^{-x^2z^2} dx \\ &= z^2 \int_0^\infty \Delta_k(\sqrt{u}) e^{-uz^2} du. \end{aligned} \quad (3.4.13)$$

Which gives the desired result under the substitution $z \mapsto \sqrt{z}$. $\square$

The immediate consequence of this lemma is of course that the divisor problem is equivalent to finding the growth of an in inverse Laplace transform. In particular, the conjectured result on the Dirichlet divisor problem, that $\Delta(x) = O(x^{1/4+\epsilon})$, is equivalent to:

$$\mathcal{L}^{-1}\left[\frac{1}{z}E_2(\sqrt{z})\right](x) = O(x^{1/8+\epsilon}) \tag{3.4.14}$$

For the error term $E_2(z)$ as defined previously.

4 The Circle Problem of Gauss

4.1 Introduction to Gauss' Circle Problem

The circle problem of Gauss is the problem of finding the number of integer lattice points contained in circles of increasing radius in the 2 dimensional plane. Specifically, for the function;

$$r_2(n) := \#\{(x,y) \in Z^2 | x^2 + y^2 = n\}, \tag{4.1.1}$$

The Gauss Circle Problem corresponds to examining the behaviour of;

$$R_2(x) := \sum_{n=0}^{x} r_2(n). \tag{4.1.2}$$

Heuristically, one can see that the leading term of $r_2(n)$ will be πn, corresponding to the area of the circle increasing in n. The problem then becomes a matter of finding the order of the error term;

$$T_2(x) = R_2(x) - \pi x. \tag{4.1.3}$$

(It should be noted that in the literature, $T_k(x)$ is referred to often as $P_k(x)$, which will be avoided here to avoid confusion with the polynomial from the previous chapter.) As in the the case of $\Delta_2(x)$, the lowest order of $T_2(x)$ (that is, the smallest α such that $T_2(x) = O(x^{\alpha+\epsilon})$ for all $\epsilon > 0$,) is conjectured to be $\frac{1}{4}$.

The similarity between the circle and divisor problems is clear, one involves the lattice points under the curve $xy = n$, while the other within the circle $x^2 + y^2 = n$. As a result of this geometric similarity, the formulas of the Dirichlet series with coefficients $d_2(n)$ and $r_2(n)$ are similar, the latter taking the form;

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{n^s} = \zeta(s)\beta(s), \tag{4.1.4}$$

where $\beta(s)$, the Dirichlet Beta function, is defined by;

$$\beta(s) := L(s, \chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s}, \tag{4.1.5}$$

for the dirichlet character

$$\chi_4(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod n \\ -1 & \text{if } n \equiv 3 \mod n \\ 0 & \text{otherwise} \end{cases} \tag{4.1.6}$$

The theory covered on the Riemann Zeta function in some ways carries well across to the Circle problem, as a result of some basic properties of Dirichlet L-functions [12]:

Lemma 4.1. *For all Dirichlet L-functions $L(\chi, s)$, such that the Dirichlet character $\chi \neq \chi_0$, the associated Dirichlet series;*

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{4.1.7}$$

Converges for $\sigma > 0$, and $L(\chi, s)$ admits an analytic continuation on this region.

And in the particular case of the Dirichlet Beta function:

Lemma 4.2. *The Dirichlet Beta function $\beta(s)$ satisfies the functional equation:*

$$\beta(1-s) = \left(\frac{2}{\pi}\right)^s \sin\left(\frac{\pi s}{2}\right) \Gamma(s)\beta(s). \tag{4.1.8}$$

Which follows from the general L-function functional equation.

4.2 Exponential Weighted Sum of the Circle Problem

As with the divisor problem, the circle problem has been examined with an exponential weighted sum, for example by Šleževičienė and Steuding in [16] using a method parallel to that of Jutila in [14] to generalize the Voronoi formula for $r_2(n)$. These cases examine the weighting of $e^{\pi ink/2l}$ for coprime naturals k and l, over finite intervals. Instead here, we examine the general exponential weight $e^{\pi nz}$ over the naturals. With the features of $r_2(n)$ found in above, a similar process to the previous chapter could be used on the Circle problem as well. However, the function $r_2(n)$ has an interesting geometric feature which lends itself to the following application of the Poisson summation formula. We begin with the claim:

$$\mathcal{L}[T_2(x/\pi)](z) = O(z^{-1}e^{-\pi/z}). \tag{4.2.1}$$

To that end, we introduce the following lemma:

Lemma 4.3. *For the function $r_2(n)$ as defined above and $\Re(z) > 0$;*

$$\sum_{n=0}^{\infty} r_2(n)e^{-\pi nz} = \frac{1}{z}\sum_{n=0}^{\infty} r_2(n)e^{-\pi n/z}. \tag{4.2.2}$$

An immediate consequence then follows that

$$\sum_{n=0}^{\infty} r_2(n)e^{-\pi nz} = \frac{1}{z} + O(z^{-1}e^{-\pi/z}). \tag{4.2.3}$$

Proof. We begin by taking the Poisson summation formula in the case of $f(x)$ being the Gaussian, which is it's own Fourier transform up to a scaling factor (see appendix):

$$\sum_{x=-\infty}^{\infty} e^{-\pi x^2 z} = \frac{1}{\sqrt{z}} \sum_{x=-\infty}^{\infty} e^{-\pi x^2/z}. \tag{4.2.4}$$

Taking the square of both sides:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} e^{-\pi(x^2+y^2)z} = \frac{1}{z} \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} e^{-\pi(x^2+y^2)/z}. \tag{4.2.5}$$

And so by the definition of the function $r_2(n)$;

$$\sum_{n=0}^{\infty} r_2(n)e^{-\pi nz} = \frac{1}{z} \sum_{n=0}^{\infty} r_2(n)e^{-\pi n/z}. \tag{4.2.6}$$

As required. $\square$

Applying Abel's summation formula yields;

$$\int_0^{\infty} \pi z(\pi x + T_2(x))e^{-\pi xz}dx = \frac{1}{z} \sum_{n=0}^{\infty} r_2(n)e^{-\pi n/z}. \tag{4.2.7}$$

And since $\int_0^{\infty} \pi^2 zxe^{-\pi xz}dx = \frac{1}{z}$;

$$\int_0^{\infty} \pi z T_2(x)e^{-\pi zx}dx = \frac{1}{z} \sum_{n=1}^{\infty} r_2(n)e^{-\pi n/z} = O(z^{-1}e^{-\pi/z}). \tag{4.2.8}$$

Hence, under the change of variables $\pi x \mapsto x$, it becomes clear that:

$$\mathcal{L}[T_2(x/\pi)](z) = O(z^{-1}e^{-\pi/z}). \tag{4.2.9}$$

As claimed.

Like in the case of $d_k(n)$, $r_2(n)$ has an associated Voronoi summation formula. This is given by the formula [1]:

$$\sum_{a \le n \le b} r_2(n) = \sum_{n=0}^{\infty} r_2(n) \int_a^b f(x)\pi J_0(2\pi\sqrt{nx})dx, \tag{4.2.10}$$

Or taking the characteristic function on the interval $[0, x]$, and the Bessel function identity in the appendix;

$$\sum_{n \leq x} r_2(n) = \sum_{n=0}^{\infty} r_2(n) \sqrt{\frac{x}{n}} J_1((2\pi\sqrt{nx}) \tag{4.2.11}$$

Using this in the Abel summation from above gives:

$$\sum_{n=0}^{\infty} r_2(n) e^{-\pi n z} = \int_0^{\infty} \sum_{n=0}^{\infty} r_2(n) \pi z \sqrt{\frac{x}{n}} J_1((2\pi\sqrt{nx}) e^{-\pi x z} dx \tag{4.2.12}$$

Then integrating each term in the sum individually, using the fact that:

$$\int_0^{\infty} \sqrt{\frac{x}{n}} J_1(n\pi\sqrt{nx}) e^{-\pi x z} dx = \frac{e^{-\pi n/z}}{\pi z^2} \tag{4.2.13}$$

Gives exactly the same result as the previous claim. In this instance, the method of Voronoi's formula is distilled into the underlying essence, which is an application of Poisson's formula.

Now with the goal of formulating the Circle problem conjecture in terms of an integral transform, we start with the sum (again with $\Re(z) > 0$;

$$\begin{aligned} \sum_{n=0}^{\infty} r_2(n) e^{-n^2 z^2} &= \sum_{n=0}^{\infty} \frac{r_2(n)}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s)(zn)^{-2s} ds \\ &= \frac{1}{4\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)\beta(s)\Gamma(s/2) z^{-s} ds. \end{aligned} \tag{4.2.14}$$

We claim that as above, this is equivalent to:

$$\begin{aligned} &\frac{1}{4\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \zeta(s)\beta(s)\Gamma(s/2) z^{-s} ds + Res_{s=0} + Res_{s=1} \\ &= \frac{1}{4\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \zeta(s)\beta(s)\Gamma(s/2) z^{-s} ds - \frac{1}{4} + \frac{\sqrt{\pi}(\psi(3/4) - \psi(1/4))}{8z} \end{aligned} \tag{4.2.15}$$

To verify this claim, it must be the case that the integrand disappears as the contour from the previous chapter is taken to infinity. To see this it is enough to see that any asymptotic bound on the Zeta function will also bound the Dirichlet Beta function, so the same bounds follow. Hence, by writing

$$\frac{1}{4\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \zeta(s)\beta(s)\Gamma(s/2) z^{-s} ds = \tilde{E}(z), \tag{4.2.16}$$

It follows that

$$\sum_{n=0}^{\infty} r_2(n)e^{-n^2z^2} = \tilde{E}(z) - \frac{\sqrt{\pi}(\psi(3/4) - \psi(1/4))}{8z}. \tag{4.2.17}$$

Abel's formula then tells us that:

$$\int_0^{\infty} T_2(\sqrt{x})z^2e^{-xz^2}dx = \tilde{E}(z) - \frac{1}{4} + \frac{\sqrt{\pi}(\psi(3/4) - \psi(1/4)) - 4\pi^{3/2}}{8z}. \tag{4.2.18}$$

under the substitution $x^2 \mapsto x$. Denoting the RHS as $F(z)$, the inverse Laplace transform then follows:

$$\mathcal{L}^{-1}[\frac{1}{z}F(\sqrt{z})](x) = T_2(\sqrt{x}). \tag{4.2.19}$$

And as in the case of the divisor problem, the conjectures result is equivalent to:

$$\mathcal{L}^{-1}[\frac{1}{z}F(\sqrt{z})](x) = O(x^{1/8+\epsilon}). \tag{4.2.20}$$

5 Discussion

The mysteries surrounding the Riemann Zeta function are vast, and in many cases remain elusive, none more so than the Riemann Hypothesis. In the case of the divisor problem, while the bound has been lowered repeatedly over the last century, it remains quite removed from the conjectured value.

The Summation techniques of Voronoi, and on a more fundamental level Poisson, have been instrumental in developing the understanding of the divisor problem, and have found application in a range of similar problems. We hope that by understanding better the implications of exponential weights similar to those presented in the preceding chapters, there remains an insight to be found in these lattice point type problems.

Further work on the topic would most likely involve a better understanding of the cancellation in such exponent sums. Furthermore, an analogue of standard A and B procedures for corresponding exponent sums on Fourier side for different z could be expected to produce a deal more cancellation.

A Appendix

A.1 Cotangent Summation Identity

In deriving the approximation of the Riemann Zeta Function, the following identity was used:

$$\pi\cot(\pi x) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{x+n}. \tag{A.1.1}$$

This can be deduced by considering the two functions;

$$f(x) = \pi\cot(\pi x), \tag{A.1.2}$$

and

$$g(x) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{x+n} = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}, \tag{A.1.3}$$

Where the rearrangement of the sum shows convergence in the limit $N\to\infty$. Both functions are clearly periodic in x, with a period of 1, and both odd functions. Furthermore, they both satisfy the property;

$$f(\frac{x}{2}) + f(\frac{x+1}{2}) = 2f(x) \tag{A.1.4}$$

(resp. $g(x)$). While perhaps not entirely obvious in the case of $\cot(\pi x)$, it follows from considering the trigonometric identities;

$$\begin{aligned}
f(\frac{x}{2}) + f(\frac{x+1}{2}) &= \pi(\frac{\cos(\pi x/2)}{\sin(\pi x/2)} + \frac{\cos(\pi(x+1)/2)}{\sin(\pi(x+1)/2)}) \\
&= \pi(\frac{\cos(\pi x/2)}{\sin(\pi x/2)} - \frac{\sin(\pi x/2)}{\cos(\pi x/2)}) \\
&= 2\pi\frac{\cos(\pi x/2+\pi x/2)}{\sin(\pi x/2+\pi x/2)} \\
&= 2f(x).
\end{aligned} \tag{A.1.5}$$

Then defining the new function;

$$h(x) := f(x) - g(x), \tag{A.1.6}$$

it remains to show that $h(x)\equiv 0$. Firstly;

$$\lim_{x\to 0}(\pi\cot(\pi x) - \frac{1}{x}) = \lim_{x\to 0}\frac{\pi x\cos(\pi x) - \sin(\pi x)}{x\sin(\pi x)} = 0 \tag{A.1.7}$$

and

$$\lim_{x\to 0} \frac{2x}{n^2 - x^2} = 0 \tag{A.1.8}$$

so we define $h(x) := 0$ for $x \in Z$. Now $h(x)$ is periodic and continuous, so is bounded by a maximum we can denote M. Define x_0 such that $h(x_0) = M$. Since $h(x)$ inherits the properties of $f(x)$ and $g(x)$ discussed above, it follows that:

$$h(\frac{x_0}{2}) + h(\frac{x_0 + 1}{2}) = 2h(x_0) = 2M. \tag{A.1.9}$$

Since $h(x)$, is 1-periodic, it follows that $h(\frac{x_0}{2}) = M$, and reiterating the above process, $h(\frac{x_0}{2^k}) = M$. Taking the limit as $k \to \infty$, it is seen that $h(0) = M$. But $h(0) = 0$, so $h(x) \leq 0$. Finally, since $h(x)$ is odd, it must be the case that $h(x) \equiv 0$, and hence the identity holds.

A.2 Gamma Function identity

We have the following features of the Gamma function;

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \tag{A.2.1}$$

and

$$\Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\pi^{1/2}\Gamma(2s). \tag{A.2.2}$$

Thus we get

$$\begin{aligned}\Gamma(\frac{1-s}{2}) &= \Gamma(1 - \frac{s+1}{2}) \\ &= \frac{\pi}{\sin(\frac{\pi(s+1)}{2})\Gamma(\frac{s}{2} + \frac{1}{2})} \\ &= \frac{\pi\Gamma(\frac{s}{2})}{2^{1-s}\sin(\frac{\pi(s+1)}{2})\pi^{1/2}\Gamma(s)}.\end{aligned} \tag{A.2.3}$$

So substituting, one gets

$$\begin{aligned}\frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1-s)/2}\Gamma((1-s)/2)} &= 2\sin\left(\frac{\pi(s+1)}{2}\right)(2\pi)^{-s}\Gamma(s) \\ &= 2\cos(\frac{\pi s}{2})(2\pi)^{-s}\Gamma(s).\end{aligned} \tag{A.2.4}$$

A.3 The Fourier Transform of the Gaussian

For the function

$$f(x) = e^{-\pi x^2}, \tag{A.3.1}$$

We have that $\hat{f}(x) = f(y)$ (where $\hat{f}$ indicates that usual Fourier transform of f). To show this;

$$\begin{aligned} \hat{f}(x) &= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi ixy} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi xy) dx. \end{aligned} \tag{A.3.2}$$

Where the imaginary component of the exponential vanishes, as it is an even function over a symmetric interval. Then

$$\begin{aligned} \frac{d}{dy} \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi xy) dx &= -\int_{-\infty}^{\infty} (2\pi x) e^{-\pi x^2} \sin(2\pi xy) dx \\ &= -\int_{-\infty}^{\infty} (2\pi y) e^{-\pi x^2} \cos(2\pi xy) dx. \end{aligned} \tag{A.3.3}$$

By integrating by parts, and using that fact that sin is an odd function over a symmetric range. Hence $\frac{d}{dy}\hat{f}(y) = -(2\pi y)\hat{f}(y)$, so $\hat{f}(y) = Ce^{-\pi y^2}$. Finally, since the Gaussian is normalized, $C = 1$ and the identity follows.

A.4 Bessel Function identities

In finding the Voronoi equation we used the following identities of the Second kind Bessel functions;

$$\int_0^{\infty} x^{s-1} K_0(4\pi x^{1/2}) dx = \frac{(2\pi)^{-2s}\Gamma(s)^2}{2}. \tag{A.4.1}$$

And

$$\int_0^{\infty} x^{s-1} Y_0(4\pi x^{1/2}) dx = -\frac{(2\pi)^{-2s}\cos(\pi s)\Gamma(s)^2}{\pi}. \tag{A.4.2}$$

The integrals are clearly Mellin transforms on the Bessel functions, so the inverse Mellin transform, as appears in our Voronoi sum, will return the Bessel functions as desired.

The Voronoi formula can also be written in terms of K_1 and Y_1 via the identities [13];

$$\frac{d}{dx}(xK_1(x)) = -xK_0(x) \tag{A.4.3}$$

$$\frac{d}{dx}(xY_1(x)) = xY_0(x) \tag{A.4.4}$$

And in the case of the circle problem, the identity:

$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x) \tag{A.4.5}$$

is used.

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