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The persistent spin helix in the presence of hyperfine interactions

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Summary

We study the effect of hyperfine interaction in the persistent spin helix (PSH) by investigating the nuclear spin pumping caused by non-equilibrium electron spin dynamics. In the first two chapters we rederive, based on already published literature, the PSH diffusion equation and a formalism for calculating the nuclear spin pumping from non-equilibrium electron spin dynamics. To get some useful insight into the procedure we use, and to show that it actually works, we calculate the nuclear spin pumping in a proof of principle setup in chapter 4. The nuclear spin pumping, and its effects, in the PSH is studied in chapter 5. We find that the lowest-order effect of the nuclear spin polarization stabilizes the electron spin structure. Some higher-order contributions introduce higher-order harmonics, which also rotate the spin structure out of the plane of the helix. The stabilizing effect is considerable when 100 % of the nuclei are polarized. However, for a more realistic nuclear spin polarization, the effect is small.

Sammendrag

Oppgaven tar for seg hvordan interaksjonen mellom kjernespinns og elektronspinn påvirker en spesiell elektronspinnstruktur kalt *persistent spin helix* (PSH), ved å undersøke pumpingen av kjernespinns fra ikke-likevekt elektronspindynamikk. I de to første kapitlene utleides en diffusionsligning for PSH strukturen og en formalisme for å regne ut pumpingen av kjernespinns fra ikke-likevekt elektronspindynamikk, basert på allerede publisert litteratur. For å få en bedre innsikt i prosedyren som brukes, og for å vise at den faktisk virker, regnes kjernespinnpumpingen ut i et konseptbevisoppsett i kapittel 4. Kjernespinnpumpingen, og dens effekter, i PSH strukturen studeres i kapittel 5. Resultatene viser at den laveste-ordens effekten av kjernespinnpolariseringen stabiliserer elektronspinstrukturen. Høyere-ordens bidrag introduserer høyere-ordens harmoniske bølger, som også roterer spinstrukturen ut av helix-planet. Den stabiliserende effekten er betydelig når 100 % av kjernene er polariserte. For en mer realistisk kjernespinnpolarisering er effekten liten.

Preface

This thesis concludes the five year Master's degree programme in Chemical Engineering and Biotechnology at the Norwegian University of Science and Technology. The thesis is written at the Department of Physics, and is a continuation of the project work done during the autumn of 2017 [1].

I would like to thank my supervisor, associate professor Jeroen Danon at the Department of Physics, for all his help and encouragement during the last year. I would also like to thank my co-supervisor, professor Henrik Koch at the Department of Chemistry, for being available for questions regarding practical formalities.

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Chapter 1

Introduction

1.1 Spintronics

Spin is an important property in quantum mechanics. It was originally postulated as an intrinsic "internal" angular momentum of the electron, in an attempt to explain experiments by Stern and Gerlach [2]. A theoretical foundation came in 1928, when Dirac showed how the spin degree of freedom followed naturally in relativistic quantum mechanics [3, 4]. Dirac introduced a relativistic wave equation that fully describes spin $\frac{1}{2}$ particles, i.e. fermions, and the spin-statistics theorem that states that fermions(bosons) have half-integer(integer) spin, and its wave-function is antisymmetric(symmetric) upon particle exchange. The physical properties of fermions, and how their spin degree of freedom couples to fields, can be understood through a nonrelativistic expansion of the Dirac equation in orders of $\frac{1}{c}$, where c is the light speed. To lowest orders in $\frac{1}{c}$, the spin's coupling to magnetic fields is simply given by the Zeeman effect, while the spin's coupling to electric fields is given by spin-orbit coupling through the Hamiltonian

$$H_{\text{so}} = \frac{e\hbar}{4m_0c^2} \boldsymbol{\sigma} \cdot \left(\nabla V \times \frac{\mathbf{p}}{m_0} \right), \quad (1.1)$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices, V is the electrostatic potential, \mathbf{p} is the electron momentum, m_0 is its rest mass, and $\hbar = \frac{h}{2\pi}$ where h is Planck's constant. The \hbar dependence shows that spin-orbit interaction is a pure quantum mechanical effect.

At the end of the 90's, a field called spintronics emerged [5, 6, 7], which studies the spin degrees of freedom in solid state physics, and tries to create devices that take advantage of spin properties. Within this field, information storage and transfer devices are of special interest since spin polarization can be generated and manipulated using external magnetic fields. Such devices offer enhanced operation speed and lower energy consumption when compared to electronic devices. An already successful spintronic device is the giant magnetoresistive (GMR) sandwich structure [8, 9], with alternating ferromagnetic and non-magnetic layers of metal.

GMR structures are used in disk drive read/write heads due to their sensitivity to changing magnetic fields.

During the last decade, semiconductor devices with intrinsic spin-orbit coupling have frequently been studied as platform for spintronic devices. The advantage of such devices is that one could generate and manipulate spin-polarized currents by purely electrical means, without the need for localized magnetic fields or ferromagnetic components. Another advantage of such a platform is the already developed experience and advanced techniques, from electrodynamics, to create micro- and nanometer-scale solid-state devices based on semiconductors.

A common platform for creating low-dimensional semiconducting structures is the 2D electron gas (2DEG) [10], where the energy levels of the conduction band electrons are strongly quantized in one spatial direction. This confines the electrons, on the nanometer scale, in the quantized direction, and makes them free to move in two spatial dimensions. A much used method to create 2DEGs is by using high-electron-mobility transistors (HEMTs) [11], which use the heterojunction between two semiconducting materials, e.g. GaAs and AlGaAs, to confine the electrons in a triangular quantum well, as illustrated in figure 1.1.

In 2DEGs, spin-orbit coupling arises due to space-inversion asymmetry of the potential in which the electrons move. The asymmetry may result from an inversion-asymmetric crystal lattice, so-called bulk inversion-asymmetry, or from an inversion-asymmetric confining potential or structural inversion-asymmetry. The spin-orbit coupling arising from bulk and structure inversion asymmetry are called Dresselhaus [13] and Rashba [14] spin-orbit interaction, respectively [15]. For explicit expressions, we focus on GaAs 2DEGs grown in the $[001] = \hat{z}$ crystallographic direction, where the spin-orbit field behaves as a momentum-dependent in-plane magnetic field. The leading-order Dresselhaus spin-orbit interaction reads $H_D = \beta(\sigma^y k_y - \sigma^x k_x)$, and the Rashba spin-orbit interaction reads $H_R = \alpha(\sigma^x k_y - \sigma^y k_x)$. Here, β and α are the Dresselhaus and Rashba spin-orbit coefficient respectively, and \mathbf{k} is the electron's wave vector.

Spin-orbit coupling in 2DEGs has inspired many spintronic concepts, such as

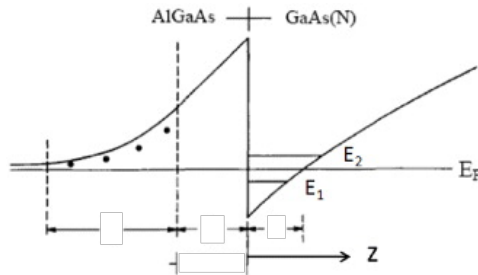


Figure 1.1: Triangular quantum well in a HEMT 2DEG. The heterojunction between AlGaAs and GaAs creates a triangular quantum well that confines the electrons in the z -direction. E_i , $i \in \{1, 2\}$ is the energy of the subband i in the conduction band, and E_F is the Fermi energy. Adapted from Mootabian et al. [12].

the spin Hall effect [16] and the spin field effect transistor (spin-FET) [17]. The spin Hall effect gives rise to a spin current transverse to the direction of an electric current, which leads to accumulation of spin-polarization at the lateral boundaries, which can be controlled by the electric current. The spin-FET is a transistor based on a 2DEG, where strong Rashba spin-orbit coupling is used as a gate to rotate the spin current from a ferromagnetic source to a ferromagnetic detector. These two concepts use the spin-orbit coupling to control the spin polarization, however, there are also negative effects of spin-orbit coupling in 2DEGs, from a spintronics point of view. For semiconductors with diffusive spin transport, spin-orbit coupling gives rise to relaxation of the spin polarization, e.g. through the Elliott-Yafet and D'yakonov-Perel' (DP) mechanisms [18]. The relaxation in the Elliott-Yafet mechanism [19] is caused by spin-orbit coupling that (1) mixes the spin "up" and spin "down" states, which results in relaxation due to spin-independent scattering, and (2) couples to the lattice potential, which results in the creation of phonons that couple to the electron spin directly. The DP mechanism results from the spin precession around the intrinsic spin-orbit field, which depends on the momentum of the electron traveling through the semiconductor. During scattering events the momentum is changed, thus changing the direction of the spin-orbit field, which, in combination with diffusive transport, result in relaxation of the spin polarization. Both of these relaxation mechanisms are dependent on scattering events to cause relaxation.

1.2 The persistent spin helix

The persistent spin helix (PSH) [20] is a spin structure with promising properties for transferring polarized spin currents in diffusive systems. The PSH is a result of the intrinsic spin-orbit coupling in a 2DEG, and arises when the Dresselhaus and Rashba spin-orbit coefficients are of equal magnitude, i.e. $\alpha = \beta$. Experimentally, this situation could be obtained by tuning the ratio of the field coefficients α/β with gating, i.e. changing the confining potential. For $\alpha = \beta$ the spin-orbit Hamiltonian takes the form $H_{so} = \alpha(\sigma^x + \sigma^y)(k_y - k_x) = \alpha\sigma_{x'}k_{y'}$, where x' points in the [110]-direction and y' points in the $[\bar{1}10]$ -direction. In principle this gives rise to an infinite lifetime of the spin polarization, due to the restored SU(2) symmetry of the Hamiltonian. The conservation of spin polarization can be understood by considering the spin-orbit Hamiltonian above. The spin-orbit field points in a fixed direction (which will be perpendicular to the PSH wave vector), and its magnitude is proportional to the momentum in the direction of the wave vector. Thus the spin's rotation angle is only dependent on the distance traveled in the direction of the wave vector, i.e. the y' -direction, and does not depend on the specific path traveled by the electron. An illustration of the PSH is shown in figure 1.2.

The first experimental evidence of the PSH was observed by Koralek et al. [21] in 2009. By using transient spin-grating spectroscopy on GaAs quantum wells, they observed enhanced lifetime of two orders of magnitude, compared to the lifetime of a spin system with diffusive spin transport only, for the spin helices around the symmetry point $\alpha \approx \beta$. The lifetime of the spin polarization was found to be de-

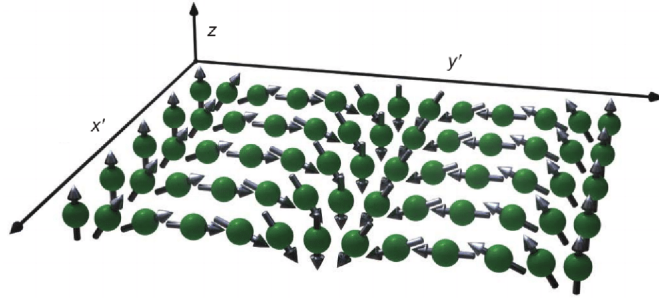


Figure 1.2: Illustration of the persistent spin helix in a 2D electron gas. The PSH is a result of the intrinsic spin-orbit coupling with equal Dresselhaus and Rashba spin-orbit strength. This makes the spin's rotation angle proportional to y' , and independent of momentum. In this figure, z is the growth direction [001], and x' and y' are the [110] and $[\bar{1}10]$ crystal directions respectively. Adapted from Koralek et al. [21].

pendent on temperature, which suggests the effect of electron-electron interactions that relax the spin structure through the spin Coulomb drag effect [22, 23]. By comparing the deviation of the maximum lifetime from $\alpha = \beta$ with theory, they proposed the cubic (higher order in k) Dresselhaus spin-orbit coupling [24] to be the main contribution to the breaking of the SU(2) symmetry. In 2012, the evolution of the persistent spin helix from a local spin excitation in a zincblende semiconductor quantum well was shown by Walser et al. [25] by using temporal and spatial Kerr spectroscopy. They also observed that applying an in-plane magnetic field either destroyed or enhanced the spin structure, depending on its in-plane direction. For the special case of an external magnetic field with magnitude $B = -1$ T in the [110]-direction, the spin structure was rotated into the [100]-direction.

To further improve the lifetime of the PSH, it is necessary to understand its relaxation mechanisms. We know that electron-electron Coulomb interactions reduce the lifetime of the PSH [21], however, these interactions do not break the SU(2) symmetry. The symmetry breaking may result from the cubic Dresselhaus spin-orbit interaction, as suggested by Walser et al., and by extrinsic spin-orbit coupling [26] through the Elliott-Yafet mechanism. To better understand the relaxation mechanisms of the PSH, a diffusion equation was derived by Lüffe et al. (2011) [27] from a semi-classical Boltzmann equation by expanding the spin density in terms of winding numbers and orders of momentum. They found that in the experimentally relevant regime, the lifetime was mainly decided by the cubic Dresselhaus interaction and the electron-electron interactions. Following the same procedure, Lüffe et al. (2013) [28] derived a spin diffusion equation for the PSH now including the Hartree-Fock field set up by electron-electron interactions. They found that for large initial polarizations, the Hartree-Fock field can considerably enhance the PSH lifetime, stabilizing the PSH against the effect of the cubic Dresselhaus interaction. A surprising effect of the Hartree-Fock field was that the

higher order harmonics rotated the spin polarization out of the (S_y, S_z) plane, where S_a is the spin projection in the a -direction. This effect was shown to be enhanced for large cubic Dresselhaus spin-orbit coupling.

1.3 Hyperfine interaction

It is not only the spin-orbit coupling and electron-electron interactions that cause relaxation of the spin polarization in 2DEGs. Another source of relaxation is the hyperfine interaction between the electron spin and the nuclear spins of the host material. This interaction can cause electron-nuclear spin flips and can result in a small random spatially varying effective Zeeman field felt by the electron spins. Under special circumstances, the interaction can also give rise to a feedback cycle, in which a non-equilibrium electron spin polarization affects the nuclear spin polarization, which in turn influences the electron spin dynamics. Such feedback mechanisms are known to be able to lead to dramatic effects, such as very large nuclear polarizations and hysteresis [29, 30].

In PSH experiments, such feedback mechanisms have always been deliberately avoided, usually by modulating the sign of the injected electron spin polarization. One could, however, wonder what would happen if one would not suppress the build-up of nuclear spin polarization, i.e., not modulate the injected spin polarization. This could possibly result in a feedback mechanism where the nuclear spin structure stabilizes the PSH structure. Naïvely one would expect the hyperfine-induced effective field to have a structure similar to the Hartree-Fock field. Based on the work done by Lüffe et al. (2013), we could then expect the nuclear spin density to further enhance the lifetime of the PSH.

We would like to investigate this idea, and see if nuclear spin pumping could indeed lead to a more stable PSH. To that end we should investigate how effective the spin pumping is in the PSH setup, i.e. how fast the nuclear spin polarization is built up, and how this affects the PSH structure. Does the nuclear polarization follow the PSH structure as we expect, and how would this affect the feedback mechanism between the nuclear and electron spin structures? Do we observe an enhanced lifetime of the electron spin polarization as for the Hartree-Fock field?

To answer these questions, we need a diffusion equation for the spin polarization in the PSH setup. This can be obtained by following the derivation of Lüffe et al. (2011). We also need to calculate the nuclear spin pumping caused by the non-equilibrium electron spin dynamics. This has traditionally been calculated using rate equations [31], where by balancing the rates of nuclear spin flip "up" and "down", calculated separately using Fermi's golden rule, one obtains the net nuclear spin pumping. However, for systems with complicated dynamics, such as spin-orbit coupled systems, this approach can fall short due to vanishingly small energy differences between states. Danon and Nazarov (2011) [32] presented an alternative formalism to investigate the nuclear spin pumping due to hyperfine interactions with non-equilibrium electron spins, which does not require separation or basis of electronic states. In this formalism, it is sufficient to investigate the susceptibilities and fluctuations of the electron spins in order to describe the spin

dynamics up to second order in the hyperfine coupling. Using this formalism, we can obtain the nuclear spin pumping from the electron spin diffusion equation. By then including the effect of the nuclear pumping in the electron spin diffusion equation, we should obtain a set of coupled equations that describe the full feedback mechanism between the nuclear and electron spin densities. Investigating these equations should answer the questions above.

1.4 Outline

The first two chapters of this thesis is based on a preparatory study for the calculation of the hyperfine interactions in the PSH [1]. Here, we rederive the components we are going to use to compute the nuclear spin pumping in the persistent spin helix. This includes the diffusion equation for the PSH system, where we follow the derivation by Lüffe et al (2011), in chapter 2, and the formalism for calculating the nuclear spin pumping from electronic fluctuations and susceptibilities, where we follow the derivation by Danon and Nazarov, in chapter 3.

As a proof of principle, in chapter 4, we calculate the nuclear spin pumping due to the susceptibilities of the electron spin density for a simpler, continuous 1D electron spin system. The investigated system has diffusive spin transport, spin relaxation, a constant injected spin polarization, and a magnetic field in the z -direction. We are not evaluating spin-orbit coupling, thus our results is not directly comparable to the PSH. The motivation behind the calculation is to get a better understanding of the procedure, and to show that it actually works, since these two formalisms have never been combined before.

In chapter 5 we scale the procedure used in chapter 4 to 3D in order to calculate the nuclear spin pumping in the PSH. We evaluate the nuclear spin polarization in equilibrium with the electron spin dynamics, and add its effects to the semi-classical Boltzmann equation. By then following a similar approach as Lüffe et al. (2013) we derive a diffusion equation for the electron spin in the presence of hyperfine interaction. Investigating this diffusion equation should give a better understanding of the hyperfine interaction in the PSH.

We indeed find that the nuclear spin pumping is proportional to the PSH, and gives rise to an effective magnetic field experienced by the electron spin. This field mainly stabilizes the PSH, resulting in a longer lifetime of the electron spin polarization. However, a higher-order contribution destabilizes the spin structure by pumping spin polarization into the x -component, as a result of the cubic Dresselhaus scattering coupling to the hyperfine interaction.

Throughout this report we will use natural units, i.e. we set \hbar , c , k_B equal 1.

Chapter 2

Spin diffusion equation

In this chapter we derive an equation of motion for the electron spin in the persistent spin helix. We first describe our model for the PSH, then we define our method for obtaining the equation of motion, and finally, we derive the equation of motion for the PSH.

2.1 Model

We consider electrons near the Fermi-surface in a GaAs 2DEG, described in a basis of Bloch functions. In such a system, the intrinsic spin-orbit coupling behaves as a momentum-dependent in-plane magnetic field. The main contributions to this field come from the linear Dresselhaus and Rashba spin-orbit coupling, but the symmetry-breaking cubic Dresselhaus spin-orbit contributes as well. The system's Hamiltonian can be written as

$$H = H_0 + H_{\text{imp}}, \quad (2.1)$$

where H_0 is the general 2DEG Hamiltonian, and H_{imp} contains the contributions from the electron-impurity interactions. The electron-impurity interactions in the presence of spin-orbit interactions will, in terms of spin dynamics, contribute to an effective relaxation of the electron spin polarization. The mechanisms behind this relaxation are the Elliott-Yafet and DP relaxation, as described in the introduction. We also assume zero temperature, and thus omit the contribution from the Coulomb electron-electron interaction [28].

The 2DEG we are considering has intrinsic spin-orbit coupling and a quadratic dispersion relation $\epsilon_k = \frac{k^2}{2m}$, where m is the electron mass. We write the 2DEG Hamiltonian as

$$H_0 = \sum_{s,s';\mathbf{k}} \psi_{\mathbf{k}s}^\dagger H_{0ss'} \psi_{\mathbf{k}s'} \quad (2.2)$$

where $\psi_{\mathbf{k}s}^\dagger$ ($\psi_{\mathbf{k}s}$) is the creation (annihilation) operator that creates (annihilates) an

electron with momentum \mathbf{k} and spin projection s , and

$$H_{0ss'} = \epsilon_k + \mathbf{b}(\mathbf{k}) \cdot \boldsymbol{\sigma}_{ss'} \quad (2.3)$$

where the spin-orbit field

$$\mathbf{b}(\mathbf{k}) = \mathbf{b}_R(\mathbf{k}) + \mathbf{b}_D(\mathbf{k}) \quad (2.4)$$

contains contributions from the linear Dresselhaus, cubic Dresselhaus, and Rashba spin-orbit interactions. An explicit expression for the spin-orbit field $\mathbf{b}(\mathbf{k})$ in a GaAs 2DEG is shown in section 2.1.1.

The electron-impurity interaction is governed by the single-particle Hamiltonian

$$H_{\text{imp}} = \frac{1}{V} \sum_{s,s';\mathbf{k},\mathbf{k}'} \psi_{\mathbf{k}s}^\dagger U_{\mathbf{k}\mathbf{k}'ss'} \psi_{\mathbf{k}'s'} \quad (2.5)$$

where V is the volume, that we set equal to one, and $U_{\mathbf{k}\mathbf{k}'ss'}$ is the impurity potential in momentum space, which we will obtain from the real-space impurity potential

$$\hat{U}(\mathbf{x}) = V^{\text{imp}}(\mathbf{x}) - \frac{\lambda_0^2}{4} \hat{\boldsymbol{\sigma}} \cdot [\hat{\mathbf{k}} \times \nabla V^{\text{imp}}(\mathbf{x})] \quad (2.6)$$

with λ_0 being a material parameter, and $V^{\text{imp}}(\mathbf{x}) = \sum_i v(\mathbf{x} - \mathbf{R}_i)$, where $v(\mathbf{x})$ is the potential of each impurity, and \mathbf{R}_i is the position of the i -th impurity. The first term contains the contribution from the electron-impurity scattering, while the second term contains the contribution from extrinsic spin-orbit interaction [26] between the electron spin and an electric field it experiences due to the gradients in the impurity potential. We obtain the impurity potential in momentum space, by evaluating the matrix element

$$\begin{aligned} U_{\mathbf{k}\mathbf{k}'} &= \langle \mathbf{k} | V^{\text{imp}}(\mathbf{x}) - \frac{\lambda_0^2}{4} \hat{\boldsymbol{\sigma}} \cdot [\hat{\mathbf{k}} \times \nabla V^{\text{imp}}(\mathbf{x})] | \mathbf{k}' \rangle \\ &= \int d\mathbf{x} \phi_{\mathbf{k}}^*(\mathbf{x}) \left(V^{\text{imp}}(\mathbf{x}) - \frac{\lambda_0^2}{4} \boldsymbol{\sigma} \cdot [\hat{\mathbf{k}} \times \nabla V^{\text{imp}}(\mathbf{x})] \right) \phi_{\mathbf{k}'}(\mathbf{x}) \\ &= V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) - \frac{\lambda_0^2}{4} \boldsymbol{\sigma} \cdot \left[\mathbf{k} \times \int d\mathbf{x} \phi_{\mathbf{k}}^*(\mathbf{x}) \nabla V^{\text{imp}}(\mathbf{x}) \phi_{\mathbf{k}'}(\mathbf{x}) \right] \\ &= V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) - \frac{\lambda_0^2}{4} \boldsymbol{\sigma} \cdot [\mathbf{k} \times i(\mathbf{k} - \mathbf{k}')] V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) \\ &= V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) \left(1 + \frac{i\lambda_0^2}{4} \boldsymbol{\sigma} \cdot [\mathbf{k} \times \mathbf{k}'] \right), \end{aligned} \quad (2.7)$$

where we used the completeness relation for the spatial coordinates $\int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x}| = 1$, defined a complete, orthonormal set of Bloch-functions as $\phi_{\mathbf{k}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{k} \rangle = e^{i\mathbf{k} \cdot \mathbf{x}} u(\mathbf{x})$, where $u(\mathbf{x})$ is a periodic function with the same periodicity as the crystal lattice, and defined $V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) = \sum_j v(\mathbf{k} - \mathbf{k}') e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}_j}$. Since our setup is a 2DEG, \mathbf{k} and \mathbf{k}' are both in-plane, giving

$$U_{\mathbf{k}\mathbf{k}'} = V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) \left(1 + \sigma_z \frac{i\lambda_0^2}{4} [\mathbf{k} \times \mathbf{k}']_z \right). \quad (2.8)$$

Since the only spin-dependent operator in $U_{\mathbf{k}\mathbf{k}'}$ is the Pauli matrix σ_z in the second term, the first term connects states $|k\rangle$ and $|k'\rangle$ with equal spin projections. The Pauli matrix σ_z works as a spin flipping operator for the in-plane spin projections s_x and s_y , and thus, the second term connects states $|k\rangle$ and $|k'\rangle$ where the in-plane spin components are flipped.

2.1.1 Spin-orbit coupling

An important interaction in 2DEGs is the spin-orbit coupling, where the electrons experience a magnetic field due to their movement through an electric field, which couples to the electron's spin. The general spin-orbit Hamiltonian can be derived by expanding the Dirac equation in orders of $\frac{1}{c}$, where the spin-orbit interaction term arises to second order as

$$H_{\text{so}} = \frac{1}{4m_0c^2} \boldsymbol{\sigma} \cdot \left(\nabla V \times \frac{\mathbf{p}}{m_0} \right). \quad (2.9)$$

Spin-orbit coupling in semiconductors arises due to space inversion-asymmetry of the potential in which the electrons move. We here present two types of inversion-asymmetry and their respective spin-orbit coupling. The first is called the Dresselhaus spin-orbit coupling, and is caused by bulk inversion-asymmetry, i.e. the crystal structure's point group lacking an inversion center, which makes the electrons experience an effective electric field. For a GaAs semiconductor with a zinc-blende crystal lattice the Dresselhaus field reads [20]

$$\mathbf{b}_D(\mathbf{k}) = \gamma \begin{pmatrix} k_x(k_y^2 - k_z^2) \\ k_y(k_z^2 - k_x^2) \\ k_z(k_x^2 - k_y^2) \end{pmatrix}, \quad (2.10)$$

where γ is the cubic Dresselhaus coefficient. x , y and z are here simply the [100]-, [010]- and [001]-direction, respectively.

For 2DEGs with sufficiently narrow quantum wells we can approximate the wave vector component along the heterostructure's growth direction by its average within the lowest subband in the 2DEG and integrate out the z -dependence. For a well grown along the [001]-direction we use $\langle k_z \rangle = 0$ and $\beta = \gamma \langle k_z^2 \rangle$, and obtain

$$\begin{aligned} \mathbf{b}_D(\mathbf{k}) = & \cos 2\phi \left[\beta' \begin{pmatrix} -k_x \\ k_y \\ 0 \end{pmatrix} - \gamma \frac{k^3}{4} \begin{pmatrix} \cos 3\theta \\ \sin 3\theta \\ 0 \end{pmatrix} \right] \\ & + \sin 2\phi \left[\beta' \begin{pmatrix} k_y \\ k_x \\ 0 \end{pmatrix} + \gamma \frac{k^3}{4} \begin{pmatrix} \sin 3\theta \\ -\cos 3\theta \\ 0 \end{pmatrix} \right], \end{aligned} \quad (2.11)$$

where v_F is the Fermi velocity, ϕ denotes the angle between the x -axis and the [100] crystal axis, and θ denotes the angle between the x -axis and the \mathbf{k} vector. β is the bare linear Dresselhaus coefficient, and $\beta' = \beta - \gamma k^2/4$ is a momentum-renormalized linear Dresselhaus coefficient, which arises as an effect of the cubic

Dresselhaus coupling. The linear terms typically dominate since $\langle k_z^2 \rangle$ is much larger than any other quadratic contribution because of the strong confinement in the z -direction. In accordance with the PSH structure as we described it in the introduction, we let the x -axis point in the [110]-direction, i.e. we set $\phi = \frac{\pi}{4}$, which gives

$$\mathbf{b}_D(\mathbf{k}) = \beta' \begin{pmatrix} k_y \\ k_x \\ 0 \end{pmatrix} + \gamma \frac{k^3}{4} \begin{pmatrix} \sin 3\theta \\ -\cos 3\theta \\ 0 \end{pmatrix}. \quad (2.12)$$

The second kind of spin-orbit coupling we introduce is the Rashba spin-orbit coupling, which arises due to structure inversion-asymmetry of the confining potential $V(\mathbf{r})$ in semiconductors. When Taylor expanding the confining potential $V(\mathbf{r}) = V_0 + e\mathbf{E} \cdot \mathbf{r} + \dots$, we see that the lowest order inversion-asymmetry of the potential is characterized by the electric field \mathbf{E} . The coupling of \mathbf{E} to the wave vector \mathbf{k} gives rise to the Rashba spin orbit field $\mathbf{b}_R(\mathbf{k}) \propto \mathbf{k} \times \mathbf{E}$. In a 2DEG the confining potential is strictly in the z -direction, i.e. $\mathbf{E} = \hat{e}_z E_z$, as illustrated in figure 1.1 in the introduction. The Rashba spin-orbit coupling in a 2DEG thus reads

$$\mathbf{b}_R(\mathbf{k}) = \alpha \begin{pmatrix} k_y \\ -k_x \end{pmatrix}, \quad (2.13)$$

where α is the Rashba coefficient, which is essentially proportional to the potential gradient over the quantum well, and can experimentally be tuned with gating.

The conduction band electrons in the GaAs 2DEG experience the spin-orbit field as a momentum-dependent magnetic field. Thus, we write the coupling between the spin-orbit field and the electron spin as

$$H_{so,ss'} = \mathbf{b}(\mathbf{k}) \cdot \boldsymbol{\sigma}_{ss'}, \quad (2.14)$$

which is included in the matrix elements for the 2DEG Hamiltonian $H_{0ss'}$ in equation (2.3).

2.2 A semi-classical approach

As a starting point for our derivation of the electron spin diffusion equation we use a semi-classical equation of motion for the spin density $\mathbf{s}_{\mathbf{k}}$, based on a Boltzmann equation. The equation of motion reads

$$\partial_t \mathbf{s}_{\mathbf{k}} + \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) = \mathbf{J}_{\mathbf{k}}^{\text{imp}} \quad (2.15)$$

where $\mathbf{v} = \mathbf{k}/m$ is the electron velocity, and $\mathbf{J}_{\mathbf{k}}^{\text{imp}}$ is the electron-impurity collision integral for the spin density. The spin density $\mathbf{s}_{\mathbf{k}}(\mathbf{r})$ is the average spin polarization of electrons in state \mathbf{k} close to the point \mathbf{r} .¹ The equation of motion (2.15) states how $\mathbf{s}_{\mathbf{k}}$ changes, i.e. $\partial_t \mathbf{s}_{\mathbf{k}}$, in a semi-classical picture where 1) spin polarization flows "in and out" of \mathbf{r} depending on nearby electron's spin polarization and velocity, 2)

¹There is some uncertainty in the position due to Heisenberg's uncertainty principle [33].

spin polarization precesses around the (effective) Zeeman field, 3) scattering events can change the spin polarization.

The electron-impurity collision integral is obtained by using Fermi's golden rule, which for a relevant perturbation H' gives the transition rate from an initial state $|i\rangle$ to a final state $|f\rangle$,

$$\Gamma_{i \rightarrow f} = 2\pi |\langle f | H' | i \rangle|^2 \delta(\epsilon_f - \epsilon_i) \rho_i (1 - \rho_f) \quad (2.16)$$

where $\langle f | H' | i \rangle$ is the matrix element between the initial and final state, $\delta(\epsilon_k - \epsilon_{k'})$ ensures energy conservation, where $\delta(x)$ is the Dirac delta function, and $\rho_{f(i)}$ is the occupation probability that encodes the availability of the final(initial) state. To obtain the electron-impurity collision integral, we investigate all transitions into and out of $|k\rangle$, for the relevant matrix element $U_{\mathbf{k}\mathbf{k}'s's'}$. This is done by using Fermi's golden rule to calculate the transition probabilities in and out of $|k\rangle$, summing over all states, and multiplying with the impurity concentration n_i , which gives

$$\begin{aligned} \mathbf{J}_{\mathbf{k}}^{\text{imp}} &= 2\pi n_i \sum_{\mathbf{k}'} \left\{ |U_{\mathbf{k}\mathbf{k}'s's'}|^2 \delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}) \rho_{\mathbf{k}'} (1 - \rho_{\mathbf{k}}) - |U_{\mathbf{k}'\mathbf{k}s's}|^2 \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \rho_{\mathbf{k}} (1 - \rho_{\mathbf{k}'}) \right\} \\ &= 2\pi n_i \sum_{\mathbf{k}'} |U_{\mathbf{k}\mathbf{k}'s's'}|^2 \delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}) (\rho_{\mathbf{k}'} - \rho_{\mathbf{k}}), \end{aligned} \quad (2.17)$$

where we used that $|U_{\mathbf{k}\mathbf{k}'s's'}|^2 = |U_{\mathbf{k}'\mathbf{k}s's}|^2$. Since the spin diffusion equation describes the evolution of a spin density, where the spin vector has a value between $\pm \frac{1}{2}$, we define the occupation probability for $s_{\mathbf{k}}^a(\mathbf{r})$ as $\rho_{\mathbf{k}}^a = \frac{1}{2} + s_{\mathbf{k}}^a$, where $a \in \{x, y, z\}$.

The probability amplitude $|U_{\mathbf{k}\mathbf{k}'s's'}|^2$ can be obtained by evaluating the matrix element $\langle s | U_{\mathbf{k}'\mathbf{k}} | s' \rangle$, where s is the spin projection and $U_{\mathbf{k}'\mathbf{k}}$ is given in equation (2.8) as

$$\begin{aligned} \langle s | U_{\mathbf{k}'\mathbf{k}} | s' \rangle &= V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) \langle s | \left(1 + \sigma_z \frac{i\lambda_0^2}{4} [\mathbf{k} \times \mathbf{k}']_z \right) | s' \rangle \\ &= V_{\mathbf{k}\mathbf{k}'}^{\text{imp}}(\{\mathbf{R}_j\}) \left\{ \delta_{ss'} + \frac{i\lambda_0^2}{4} [\mathbf{k} \times \mathbf{k}']_z \begin{pmatrix} \delta_{s_x, -s'_x} \\ \delta_{s_y, -s'_y} \\ \delta_{s_z, s'_z} \end{pmatrix} \right\}, \end{aligned} \quad (2.18)$$

where $\delta_{ss'}$ is the Kronecker delta function that makes sure that the initial and final spin density have the same spin projection. We take the absolute value squared of the matrix element above and insert it into equation (2.17), with the occupation probabilities of the spin densities for their respective spin polarization, to obtain

$$\begin{aligned} \mathbf{J}_{\mathbf{k}}^{\text{imp}} &= - \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}) \left\{ \Delta \mathbf{s}_{\mathbf{k}} + \frac{\lambda_0^2}{2} [\mathbf{k} \times \mathbf{k}']_z \begin{pmatrix} -\mathbf{s}_{\mathbf{k}'_y} \\ \mathbf{s}_{\mathbf{k}'_x} \\ 0 \end{pmatrix} \right. \\ &\quad \left. + \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} \mathbf{s}_{\mathbf{k}_x} + \mathbf{s}_{\mathbf{k}'_x} \\ \mathbf{s}_{\mathbf{k}_y} + \mathbf{s}_{\mathbf{k}'_y} \\ \mathbf{s}_{\mathbf{k}_z} - \mathbf{s}_{\mathbf{k}'_z} \end{pmatrix} \right\} \end{aligned} \quad (2.19)$$

where $W_{\mathbf{k}\mathbf{k}'} = 2\pi n_i |v(\mathbf{k} - \mathbf{k}')|^2$ is the transition rate, and $\Delta \mathbf{s}_{\mathbf{k}} = (\mathbf{s}_{\mathbf{k}} - \mathbf{s}_{\mathbf{k}'})$.

2.2.1 Expansion of the spin density

So far, we have obtained a semi-classical equation of motion for the spin density $\mathbf{s}_{\mathbf{k}}(\mathbf{r})$. We now want to expand the spin density in terms that capture the dynamics governed by the equation of motion in order to simplify the derivation of the spin diffusion equation. This is obtained by considering the D'yakonov-Perel' regime of strong scattering.

As we explained in the introduction, the D'yakonov-Perel' mechanism results in an effective relaxation of the electron spin polarization in a semiconductor with intrinsic spin-orbit coupling. The relaxation is caused by momentum altering scattering, which also alters the momentum-dependent spin-orbit field $\mathbf{b}(\mathbf{k})$. The DP relaxation is strongest in the weak scattering regime, where the time between scattering events τ is large. This leads to in a large precession angle $\mathbf{b}(\mathbf{k})\tau$, resulting in a random-walk behaviour of the spin polarization. Stronger scattering thus stabilizes the spin polarization. In the D'yakonov-Perel' regime of strong scattering the precession angle is small, $\mathbf{b}(\mathbf{k})\tau \ll 1$, resulting in little DP relaxation. This effect is often referred to as motional narrowing in spectroscopy, e.g. in nuclear magnetic resonance (NMR) spectroscopy. Motional narrowing in NMR spectroscopy is a phenomenon where the time-averaged magnetic field experienced by the protons is smaller than the instantaneous one. This is caused by a spatially varying magnetic field and diffusive proton spin transport, and results in a smaller linewidth for certain resonant frequencies.

In the D'yakonov-Perel' regime of strong scattering, we can separate spin density into isotropic and anisotropic parts due to their different dynamic timescales. This allows us to derive the spin diffusion equation by evaluating a diffusion equation for the isotropic spin components with respect to the anisotropic components in their steady-state solutions.

The anisotropic parts of the electron spin density are further expanded in terms of winding numbers and powers of momentum k to capture the dynamics governed by the equation of motion. The reasoning behind this expansion can easiest be seen from the spin-orbit fields in equation (2.12) and (2.13), where in order to capture the relevant frequencies, one has to expand the spin density in terms of winding number ± 1 with momentum of powers k and k^3 , and winding numbers ± 3 with momentum of power k^3 . Terms with other winding numbers require coupling between spin-orbit fields, which is not included to leading order in $\mathbf{b}(\mathbf{k})\tau$, and will be neglected. This gives the expanded spin density,

$$\mathbf{s}_{\mathbf{k}} = \mathbf{s}_0 + \mathbf{s}_{\mathbf{k},1} + \tilde{\mathbf{s}}_{\mathbf{k},1} + \mathbf{s}_{\mathbf{k},3} \quad (2.20)$$

with

$$\mathbf{s}_0 = -\frac{2\pi}{m} f'(\epsilon_k) \mathbf{S} \quad (2.21)$$

$$\mathbf{s}_{\mathbf{k},1} = f'(\epsilon_k) \frac{k}{m} \sum_{n=\pm 1} \delta \mathbf{k}_n(\mathbf{x}, t) e^{in\theta} \quad (2.22)$$

$$\tilde{\mathbf{s}}_{\mathbf{k},1} = f'(\epsilon_k) \frac{k^3}{k_F^2 m} \sum_{n=\pm 1} \delta \tilde{\mathbf{k}}_n(\mathbf{x}, t) e^{in\theta} \quad (2.23)$$

$$\mathbf{s}_{\mathbf{k},3} = f'(\epsilon_{\mathbf{k}}) \frac{k^3}{k_F^2 m} \sum_{n=\pm 3} \delta \mathbf{k}_n(\mathbf{x}, t) e^{in\theta} \quad (2.24)$$

where k_F is the Fermi momentum. The isotropic part of $\mathbf{s}_{\mathbf{k}}$ is written as a small spin-dependent shift of the chemical potentials for spin up and spin down along the local direction of the spin polarization, and the anisotropic components are expressed in terms of small (spin-dependent) shifts in momentum of the equilibrium distribution function. This notation ensures that one will obtain the local spin density \mathbf{S} when integrating $\mathbf{s}_{\mathbf{k}}$ over \mathbf{k} . Based on the spin-orbit field in equation (2.4), one expects $\mathbf{s}_{\mathbf{k},1}$ to couple to the Rashba and bare linear Dresselhaus fields, $\tilde{\mathbf{s}}_{\mathbf{k},1}$ to couple to the renormalization of the linear Dresselhaus field, and $\mathbf{s}_{\mathbf{k},3}$ to couple to the cubic Dresselhaus field. However, since $\mathbf{s}_{\mathbf{k},1}$ and $\tilde{\mathbf{s}}_{\mathbf{k},1}$ are identical at zero temperature,² we can omit the contribution from the renormalization of the Dresselhaus field, i.e. $\tilde{\mathbf{s}}_{\mathbf{k},1}$. We thus write the expanded spin density as

$$\mathbf{s}_{\mathbf{k}} = \sum_{n=0,\pm 1,\pm 3} \mathbf{s}_{\mathbf{k},n} e^{in\theta}, \quad (2.25)$$

with $\mathbf{s}_0 = -\frac{2\pi}{m} f'(\epsilon_{\mathbf{k}}) \mathbf{S}$ being the isotropic spin component, and $\mathbf{s}_{\mathbf{k},n} = f'(\epsilon_{\mathbf{k}}) \frac{k^n}{k_F^{n-1} m} \delta \mathbf{k}_n$ being the anisotropic spin components.

2.3 Spin diffusion equation

We will now derive the spin diffusion equation for the model presented in section 2.1, from the semi-classical equation of motion (2.15) and the expanded spin density in equation (2.20). Because of the different time scales of the isotropic and anisotropic components, caused by the DP relaxation, the anisotropic components instantaneously relax into their new equilibrium configuration when compared to the isotropic component. This lets us use the following approach: (1) By inserting the spin density into the semi-classical equation of motion and integrating over \mathbf{k} , we obtain a diffusion equation for the isotropic spin components, as functions of the anisotropic spin components. (2) We then find explicit steady-state expressions for the anisotropic components by separating out the relevant winding number components. (3) Inserting the expression for the anisotropic components into the isotropic diffusion equation gives the spin diffusion equation.

We define a new basis for the anisotropic spin components that we will use during the derivation,

$$\delta \mathbf{k}_{c(3)} = (\delta \mathbf{k}_{1(3)} + \delta \mathbf{k}_{-1(-3)}) \quad \delta \mathbf{k}_{s(3)} = i(\delta \mathbf{k}_{1(3)} - \delta \mathbf{k}_{-1(-3)}). \quad (2.26)$$

Our choice of basis is based on the winding number expansion of the spin density in equation (2.20) and the θ integrals we encounter during the derivation. We will in this section present our main results, while the full derivation is given in appendix A.

²At zero temperature, the Fermi-Dirac distribution takes the form of a heaviside function, $f(\epsilon_{\mathbf{k}}) = \theta(k_F - k)$.

2.3.1 Isotropic diffusion equation

To obtain the spin diffusion equation, we first derive a diffusion equation for the isotropic spin density. We do this by inserting the spin density in equation (2.20) into the semi-classical equation of motion (2.15) and integrating over \mathbf{k} , i.e.

$$\int d\mathbf{k} \{ \partial_t \mathbf{s}_{\mathbf{k}} + \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} \mathbf{J}_{\mathbf{k}}^{\text{imp}}. \quad (2.27)$$

This gives us a diffusion equation for the isotropic components of the spin density,

$$\partial_t S_x = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,x} + \partial_y \delta k_{s,x}) + \alpha \delta k_{c,z} - \beta \delta \bar{k}_{c,z} \right\} - \gamma_{ey} S_x \quad (2.28)$$

$$\partial_t S_y = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,y} + \partial_y \delta k_{s,y}) + \alpha \delta k_{s,z} + \beta \delta \bar{k}_{s,z} \right\} - \gamma_{ey} S_y \quad (2.29)$$

$$\partial_t S_z = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,z} + \partial_y \delta k_{s,z}) - \alpha (\delta k_{c,x} + \delta k_{s,y}) + \beta (\delta \bar{k}_{c,x} - \delta \bar{k}_{s,y}) \right\} \quad (2.30)$$

with $\delta \bar{k}_{c(s)} = \delta \mathbf{k}_{c(s)} - \frac{\gamma k_F^2}{4\beta} (\delta \mathbf{k}_{c(s)} + \delta \mathbf{k}_{c3(s3)})$, and $\delta \bar{k}_{c(s)} = \delta \mathbf{k}_{c(s)} - \frac{\gamma k_F^2}{4\beta} (\delta \mathbf{k}_{c(s)} - \delta \mathbf{k}_{c3(s3)})$, and γ_{ey} being the Elliott-Yafet relaxation rate where the electron spin precess a small angle around the extrinsic spin-orbit field during impurity scattering, given as

$$\gamma_{ey} = \left(\frac{\lambda_0 k_F}{2} \right)^4 \tau^{-1} \quad (2.31)$$

where $\tau^{-1} = mn_i |v(0)|^2$ is the inverse lifetime of the electron spin due to isotropic scattering off a weak impurity potential $v(q)$ in the Born-approximation. We will use this isotropic spin diffusion equation to obtain a spin diffusion equation for the spin-orbit system we are evaluating, by finding steady-state expressions for the anisotropic components appearing in it.

2.3.2 Anisotropic components

To make the isotropic diffusion equation self-consistent, we find steady-state expressions for the anisotropic components as functions of the isotropic component. To obtain this, we multiply the semi-classical Boltzmann equation with an angular part that separates out the winding number ± 1 and ± 3 components separately when performing the \mathbf{k} integration. For the winding number ± 1 part we will use $\cos \theta$ and $\sin \theta$, while for the winding number ± 3 part we will use $e^{\pm i3\theta}$. We choose $\cos \theta$ and $\sin \theta$ instead of $e^{\pm i\theta}$ for the winding number ± 1 part because we then get the contributions expressed directly in terms of $\delta \mathbf{k}_{c(s)}$.

We also omit the time-derivative term in the spin diffusion equation (2.15) due to the fast relaxation of the anisotropic components when compared to the isotropic ones. This is a result of the DP relaxation of the anisotropic components, which we mentioned in section 1.1 and described in section 2.2.1. This makes the anisotropic components relax into their new equilibrium configuration almost instantaneously when compared to the isotropic component.

winding number ± 1

To find the anisotropic components of winding number ± 1 , we multiply the semi-classical diffusion equation (2.15) with $\cos \theta$ and $\sin \theta$ and integrate over \mathbf{k} , i.e.

$$\int d\mathbf{k} \gamma(\theta) \{ \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} \gamma(\theta) \mathbf{J}_{\mathbf{k}}^{\text{imp}} \quad (2.32)$$

where $\gamma(\theta) \in \{\cos \theta, \sin \theta\}$. This gives us two coupled equations, one for $\gamma(\theta) = \cos \theta$, and one for $\gamma(\theta) = \sin \theta$. By solving the two coupled equations for the anisotropic components we get

$$\delta k_{c,x} = \frac{2\pi\tau_1}{m} (S_x \partial_x + \gamma_{sw}\tau_1 S_y \partial_y) + 4\pi\tau_1 S_z \{ \alpha (1 + \gamma_{sw}\tau_1) - \beta'' (1 - \gamma_{sw}\tau_1) \} \quad (2.33)$$

$$\delta k_{c,y} = \frac{2\pi\tau_1}{m} (S_y \partial_x - \gamma_{sw}\tau_1 S_x \partial_y) \quad (2.34)$$

$$\delta k_{c,z} = \frac{2\pi\tau_1}{m} S_z \partial_x - 4\pi\tau_1 S_x (\alpha - \beta'') \quad (2.35)$$

$$\delta k_{s,x} = \frac{2\pi\tau_1}{m} (S_x \partial_y - \gamma_{sw}\tau_1 S_y \partial_x) \quad (2.36)$$

$$\delta k_{s,y} = \frac{2\pi\tau_1}{m} (S_y \partial_y + \gamma_{sw}\tau_1 S_x \partial_x) + 4\pi\tau_1 S_z \{ \alpha (1 + \gamma_{sw}\tau_1) + \beta'' (1 - \gamma_{sw}\tau_1) \} \quad (2.37)$$

$$\delta k_{s,z} = \frac{2\pi\tau_1}{m} S_z \partial_y - 4\pi\tau_1 S_y (\alpha + \beta''), \quad (2.38)$$

where $\beta'' = \beta - \frac{\gamma k_F^2}{4}$, $\tau_1 = (\frac{1}{\tau} + \frac{\gamma_{ey}}{2})^{-1}$ is the effective relaxation time of the winding number ± 1 anisotropic spin components, and $\gamma_{sw} = (\frac{\lambda_0 k_F}{2})^2 \frac{1}{\tau}$ is the "swapping" rate of the spin currents, where a gradient in one of the in-plane spin components generate a current in the other in-plane spin component. We also assumed that $(\gamma_{sw}^2 \tau_1^2 - 1) \rightarrow -1$, which is reasonable in our regime of $\tau_1 \ll 1$. These equations express the anisotropic spin components of winding number ± 1 as functions of the isotropic spin density \mathbf{S} , and will later be used to obtain the spin diffusion equation.

winding number ± 3

To obtain expressions for the anisotropic spin components of winding number ± 3 , we multiply the semi-classical diffusion equation (2.15) with $e^{\pm i3\theta}$ and integrate over \mathbf{k} , i.e.

$$\int d\mathbf{k} e^{\pm i3\theta} \{ \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} e^{\pm i3\theta} \mathbf{J}_{\mathbf{k}}^{\text{imp}}. \quad (2.39)$$

Solving for the anisotropic components $\delta \mathbf{k}_{c3} = \delta \mathbf{k}_3 + \delta \mathbf{k}_{-3}$ and $\delta \mathbf{k}_{s3} = i(\delta \mathbf{k}_3 - \delta \mathbf{k}_{-3})$ gives

$$\delta \mathbf{k}_{c3} = \gamma \pi \tau_3 k_F^2 \begin{pmatrix} S_z \\ 0 \\ -S_x \end{pmatrix} \quad (2.40)$$

$$\delta\mathbf{k}_{s3} = \gamma\pi\tau_3k_F^2 \begin{pmatrix} 0 \\ S_z \\ -S_y \end{pmatrix}, \quad (2.41)$$

where $\tau_3 = \left(\frac{1}{\tau} + \frac{\gamma_{ey}}{2}\right)^{-1}$ is the effective relaxation time of the winding number ± 3 spin components, and γ is the cubic Dresselhaus coefficient. We note that $\tau_1 = \tau_3$, which is a consequence of our choice of system. Adding e.g. a temperature dependence or second-order electron-electron interactions would make $\tau_1 \neq \tau_3$.

We have now calculated the isotropic spin diffusion equation, and the anisotropic spin components appearing in it, hence we can now obtain the spin diffusion equation.

2.3.3 Spin diffusion equation

To obtain the spin diffusion equation, we insert the anisotropic spin components in equation (A.47)-(A.52) and (A.59)-(A.60) into the isotropic spin diffusion equation (A.25)-(A.27) and obtain

$$\partial_t \mathbf{S} = \begin{pmatrix} D\nabla^2 - \Gamma_x - \gamma_{cd} - \gamma_{ey} & 0 & K_{xz}\partial_x \\ 0 & D\nabla^2 - \Gamma_y - \gamma_{cd} - \gamma_{ey} & K_{yz}\partial_y \\ -K_{xz}\partial_x & -K_{yz}\partial_y & D\nabla^2 - \Gamma_x - \Gamma_y - \Gamma_{sw} - 2\gamma_{cd} \end{pmatrix} \mathbf{S}, \quad (2.42)$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$, $D = \frac{1}{2}\tau_1v_F^2$ is the effective diffusion constant, and $q_0 = 4\alpha m$ is the PSH wave vector.

$$\gamma_{cd} = \frac{1}{8}\gamma^2\tau_3k_F^6 \quad (2.43)$$

is the D'yakonov-Perel' relaxation rate due to the precession between electron-impurity scattering events for the winding number ± 3 anisotropic spin components, and

$$\Gamma_{x(y)} = \frac{1}{4}Dq_0^2 \left\{ 1 \mp 2\frac{\beta}{\alpha}(1 - \zeta) + \frac{\beta^2}{\alpha^2}(1 - \zeta)^2 \right\}, \quad (2.44)$$

where $\zeta = \frac{\gamma k_F^2}{4\beta}$, is the D'yakonov-Perel' relaxation rate for the winding number ± 1 anisotropic spin components. The DP relaxation of the $S_{x(y)}$ component is a consequence of precession about the $y(x)$ -component spin-orbit field. The S_z component thus experience both of the in-plane spin-orbit fields, but also get a correction due to the spin-flipping of the in-plane components, due to extrinsic spin-orbit coupling, given by

$$\Gamma_{sw} = \frac{1}{2}\gamma_{sw}\tau_1Dq_0^2 \left\{ 1 - \frac{\beta^2}{\alpha^2}(1 - \zeta)^2 \right\}. \quad (2.45)$$

The three quantities in equations (2.43)-(2.45) relax the spin components, and are thus in the diagonal elements of the spin diffusion equation. From these expressions we can see explicitly that stronger scattering, which gives smaller D and $\tau_{1(3)}$, stabilizes the spin polarization. The off-diagonal elements, however, couple the spin components. For our choice of $\phi = \frac{\pi}{4}$ in section 2.1.1, where ϕ is the angle between the x -axis and the $[001]$ crystal axis in our 2DEG, the coupling between

the in-plane components is zero. The coupling between the in-plane components and the z -component, however, is given by

$$K_{xz(yz)} = Dq_0 \left\{ 1 \mp \frac{\beta}{\alpha}(1 - \zeta) \right\} + \frac{1}{2}\gamma_{sw}\tau_1 Dq_0 \left\{ 1 \pm \frac{\beta}{\alpha}(1 - \zeta) \right\}, \quad (2.46)$$

which contains contributions from the intrinsic and extrinsic spin-orbit coupling.

The spin diffusion equation derived above is valid for any GaAs 2DEG with intrinsic spin-orbit interaction and $\phi = \frac{\pi}{4}$, with no assumption about the the spin-orbit coupling strengths. Below, we find the diffusion equation for the PSH, by setting the the Rashba spin-orbit coupling strength equal the renormalized linear Dresselhaus spin-orbit coupling strength, i.e. $\alpha = \beta(1 - \zeta)$.

2.3.4 PSH diffusion equation

We have derived a spin diffusion equation for a GaAs 2DEG with intrinsic spin-orbit coupling and electron-impurity interaction. What we now want to find is the spin diffusion equation for the persistent spin helix structure. To obtain this, we evaluate the spin diffusion equation (2.42) for the symmetric case of equal Rashba and renormalized linear Dresselhaus coupling strengths, i.e. we set $\alpha = \beta(1 - \zeta)$, which gives

$$\partial_t \mathbf{S} = \begin{pmatrix} D\nabla^2 - \gamma_{cd} - \gamma_{ey} & 0 & \gamma_{sw}\tau_1 Dq_0 \partial_x \\ 0 & D\nabla^2 - Dq_0^2 - \gamma_{cd} - \gamma_{ey} & 2Dq_0 \partial_y \\ -\gamma_{sw}\tau_1 Dq_0 \partial_x & -2Dq_0 \partial_y & D\nabla^2 - Dq_0^2 - 2\gamma_{cd} \end{pmatrix} \mathbf{S}. \quad (2.47)$$

To check the validity of the obtained diffusion equation, we investigate the fully SU(2) symmetric case of no relaxation, i.e. $\gamma_{cd(ey)} = 0$. We then substitute an initial spin polarization $\mathbf{S}(\mathbf{x}) = S_0(0, 0, \cos q_0 y)$, similar to the setup of Koralek et al. (2009), into the diffusion equation and solve for equilibrium. This gives

$$\mathbf{S}(y, t) = \frac{S_0}{2} \begin{pmatrix} (e^{-4Dq_0^2 t} - 1) \sin q_0 y \\ (e^{-4Dq_0^2 t} + 1) \cos q_0 y \end{pmatrix}, \quad (2.48)$$

which in the steady-state limit, i.e. $t \rightarrow \infty$, reduces to the persistent spin helix state. The time-evolution of the fully SU(2) symmetric PSH in equation (2.48) is illustrated in figure 2.1.

We will use the diffusion equation (2.47) to calculate the effects of hyperfine interaction in the PSH in chapter 5. To do so, we first need a formalism to evaluate the nuclear spin pumping caused by the non-equilibrium spin dynamics described by equation (2.47). We derive such a formalism in chapter 3.

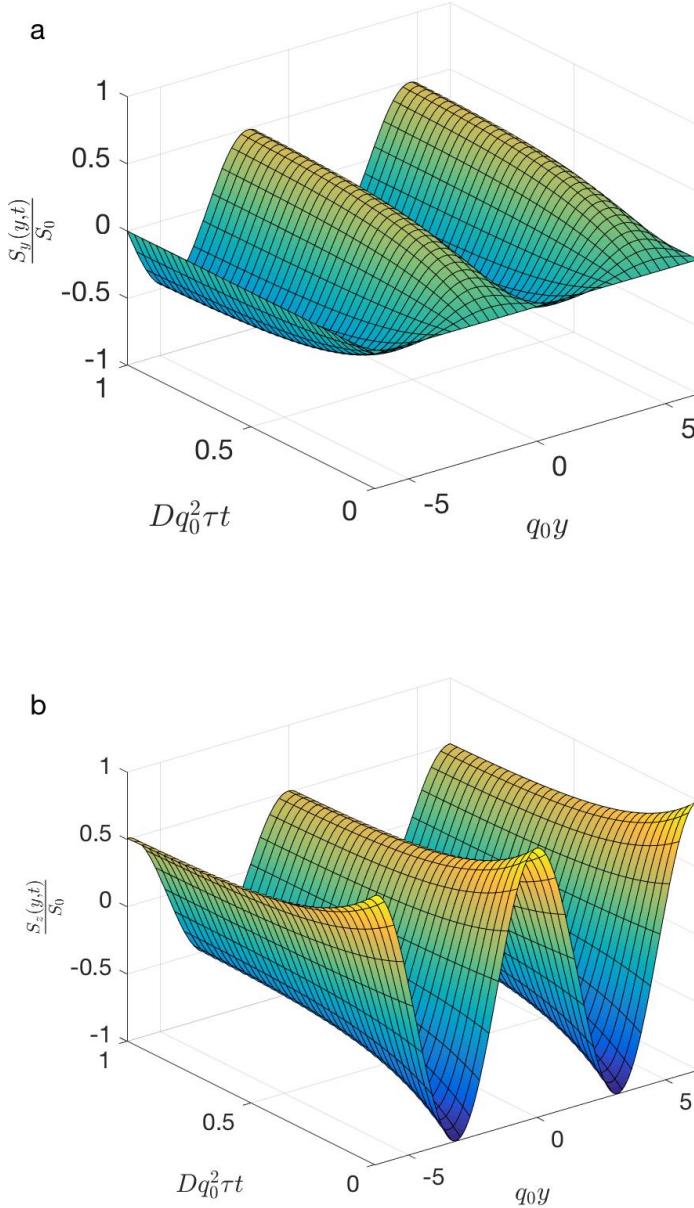


Figure 2.1: **a**, **b**, Illustration of the time-evolution of the initial state $\mathbf{S}(y) = S_0(0, 0, \cos q_0 y)$ into the PSH structure for a GaAg 2DEG with no spin relaxation, and Rashba and linear Dresselhaus intrinsic spin-orbit coupling of equal magnitude, for the y - and z -component respectively.

Chapter 3

Nuclear spin pumping

In our derivation of the spin diffusion equation above we neglected the contribution from the 2DEG's nuclear spins. However, in a real system with non-zero nuclear spins the so-called hyperfine interaction is always present, which couples the electron spins to the magnetic field caused by the nuclear magnetic dipole moments [34]. The effective magnetic field experienced by one electron in the presence of one nucleus is given by the vector potential

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\hat{\boldsymbol{\mu}} \times \mathbf{r}}{r^3} \quad (3.1)$$

where μ_0 is the vacuum permeability, $\hat{\boldsymbol{\mu}}$ is the nuclear magnetic dipole moment, and \mathbf{r} is the vector pointing from the nucleus to the electron. By substituting this vector potential into the Pauli-Schrödinger equation, and averaging over the electron wave function, one obtains the Hamiltonian that governs the hyperfine interaction between the electron and nucleus. This is straightforward given that the electron and nucleus are sufficiently separated, as for electrons in p-, d- and f-orbitals. However, for electrons in s-orbitals, which have a non-zero wave function as $\mathbf{r} \rightarrow 0$, one has to introduce a relativistic correction with a far higher magnitude than the spin-spin interaction between the electron and nucleus. This correction is caused by the large electrostatic potential experienced by the electron when $\mathbf{r} \rightarrow 0$. Calculating this correction and averaging over s-type orbitals yields

$$\hat{H}_{\text{HF}} = \frac{2}{3} \mu_0 g_0 \mu_B \gamma_n \hat{\mathbf{S}} \cdot \hat{\mathbf{K}} |\psi(0)|^2, \quad (3.2)$$

where g_0 is the free electron g-factor, μ_B is the Bohr magneton, γ_n is the nuclear gyromagnetic ratio, $\hat{\mathbf{S}}(\hat{\mathbf{K}})$ is the electron(nuclear) spin operator, and $|\psi(0)|$ is the magnitude of the electron wave function at the nucleus. The electron wave function $\psi(\mathbf{r})$ can, in a semiconductor lattice, be described as a product of a Bloch function $u(\mathbf{r})$ and an envelope function $\Psi(\mathbf{r})$ [35], which gives

$$\hat{H}_{\text{HF}} = \frac{2}{3} \mu_0 g_0 \mu_B \gamma_n \eta \hat{\mathbf{S}} \cdot \hat{\mathbf{K}} |\Psi(0)|^2, \quad (3.3)$$

where $\eta = |u(0)|^2$. The prefactor is collected in a hyperfine coupling energy $A = \frac{2}{3}\mu_0 g_0 \mu_B \gamma_n \eta v_0^{-1}$, where v_0^{-1} is the density of nuclei, and can be assumed to be independent on the nuclear position in a GaAs 2DEG. By writing the product of the electron spin operator and the magnitude of the envelope function as a three dimensional spin density $\hat{\mathbf{S}}(\mathbf{r})$ we obtain

$$\hat{H}_{\text{HF}} = Av_0 \hat{\mathbf{S}}(\mathbf{r}) \cdot \hat{\mathbf{K}}. \quad (3.4)$$

The hyperfine interaction Hamiltonian in a GaAs 2DEG, where the conduction band electrons are in s-type orbitals, can now be obtained by summing over all nuclei,

$$\hat{H}_{\text{HF}} = \sum_n Av_0 \hat{\mathbf{S}}(\mathbf{r}_n) \cdot \hat{\mathbf{K}}_n, \quad (3.5)$$

where \mathbf{r}_n is the position of nucleus n . To get a qualitative understanding of this expression we write out the inner product as

$$\hat{\mathbf{S}} \cdot \hat{\mathbf{K}} = \frac{1}{2} \left(2\hat{S}^z \hat{K}^z + \hat{S}^+ \hat{K}^- + \hat{S}^- \hat{K}^+ \right), \quad (3.6)$$

where $J^\pm = J^x \pm iJ^y$ is the ladder operator that raises(lowers) the angular momentum in the z -direction. One can see here that the hyperfine interaction may cause electron-nuclear spin flips, which for non-equilibrium electron spin dynamics can lead to dynamic nuclear spin polarization [34].

We wish to investigate how the hyperfine interaction, described by the Hamiltonian above, affects a PSH structure in a GaAs 2DEG. To do so we need to investigate how the nuclear spin structure is affected by the non-equilibrium spin dynamics in the PSH structure. We will in this chapter rederive a formalism to calculate the nuclear spin pumping from the local electron spin susceptibilities and fluctuations, as presented by Danon and Nazarov [32]. This will give an equation of motion for the expectation value of the nuclear polarization $\langle \hat{\mathbf{K}} \rangle$ which is dependent on the electron spin dynamics, or more precisely, the local electron spin susceptibilities and fluctuations.

Danon and Nazarov's formalism is based on time-dependent perturbation theory, which will be derived in the first part of this chapter. We then follow the derivation presented by Danon and Nazarov to obtain an equation of motion for $\langle \hat{\mathbf{K}} \rangle$.

3.1 Time-dependent perturbation theory

Our starting point for the derivation of the nuclear spin pumping will be a time-evolution expression of the density matrix which we derive using time-dependent perturbation theory.

3.1.1 Interaction picture

The interaction picture [36] can be interpreted as an intermediate picture between the Schrödinger and Heisenberg picture. The time-dependence in the Schrödinger

picture is governed by the wave-function, $i\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle$, while the time-dependence in the Heisenberg picture is governed by the operators, $\frac{\partial\hat{A}(t)}{\partial t} = i[\hat{H}, \hat{A}(t)]_{-}$, where $[\dots]_{-(+)}$ denotes the commutator(anti-commutator). In the interaction picture, however, the Hamiltonian is split into a simple static part and a small perturbation which may be time-dependent,

$$\hat{H} = \hat{H}_0 + \hat{H}'(t), \quad (3.7)$$

where one assigns the time-dependence of the operators to be governed by the static part \hat{H}_0 , and the dynamics of the wave-function to be governed by the perturbation $\hat{H}'(t)$, i.e.

$$\frac{\partial\hat{A}_I(t)}{\partial t} = i[\hat{H}_0, \hat{A}_I(t)] \quad (3.8)$$

$$i\frac{\partial|\psi_I(t)\rangle}{\partial t} = \hat{H}'_I(t)|\psi_I(t)\rangle, \quad (3.9)$$

where the subscript I indicates that the operator/wave-function is in the interaction picture. Note that the Hamiltonian in equation (3.9) is in the interaction picture and therefore has to satisfy the time-dependence governed by equation (3.8). The advantage with the interaction picture is in time-dependent perturbation theory, where $\hat{H}'(t)$ acts as a small perturbation to a simple Hamiltonian \hat{H}_0 with well known solutions.

Operators and wave-functions can be transformed from the Schrödinger picture to the interaction picture with the following relations

$$\hat{A}_I(t) = e^{i\hat{H}_0 t} \hat{A} e^{-i\hat{H}_0 t}, \quad (3.10)$$

$$|\psi_I(t)\rangle = e^{i\hat{H}_0 t} |\psi(t)\rangle. \quad (3.11)$$

The first relation follows directly from equation (3.8), while the second relation follows from the first relation and the expectation value's independence on choice of picture, i.e. $\langle\hat{A}(t)\rangle = \langle\psi_I(t)|\hat{A}_I(t)|\psi_I(t)\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle$.

3.1.2 Time-evolution operator

The time-evolution of a state in the Schrödinger picture is determined by the time-evolution operator $\hat{U}(t, t')$ [36], that acts on a state $|\psi(t')\rangle$ at time t' and returns a state $|\psi(t)\rangle$ at time t , i.e.

$$|\psi(t)\rangle = \hat{U}(t, t')|\psi(t')\rangle. \quad (3.12)$$

$\hat{U}(t, t')$ is a unitary operator, i.e. it satisfies $\hat{U}^{-1}(t, t') = \hat{U}^\dagger(t, t') = \hat{U}(t', t)$. If the underlying Hamiltonian is time-independent, the time-evolution operator can be written as an exponent operator¹

$$\hat{U}(t, t') = e^{-i\hat{H}(t-t')}. \quad (3.13)$$

¹The exponent operator [36] is defined by the Taylor series of the exponential function, i.e. $e^{\hat{A}} \equiv 1 + \hat{A} + \frac{\hat{A}\hat{A}}{2!} + \dots$

Before we introduce the time-evolution operator for a time-dependent Hamiltonian, we introduce the time-ordering operator \mathcal{T} ,

$$\mathcal{T}\{\hat{A}(t_1)\hat{B}(t_2)\} = \theta(t_1 - t_2)\hat{A}(t_1)\hat{B}(t_2) + \theta(t_2 - t_1)\hat{B}(t_2)\hat{A}(t_1), \quad (3.14)$$

where θ is the heaviside function defined as: $\theta(t_1 - t_2) = \begin{cases} 1, & t_1 \geq t_2 \\ 0, & t_1 < t_2 \end{cases}$. \mathcal{T} makes sure that the operator with the smallest time t acts first on the state. We now introduce the time-evolution operator for a time-dependent Hamiltonian as

$$\hat{U}(t, t') = \mathcal{T} \left\{ \exp \left(-i \int_{t'}^t d\tau \hat{H}(\tau) \right) \right\}. \quad (3.15)$$

We have so far only looked at the time-evolution operator in the Schrödinger picture. However, in time-dependent perturbation theory we need the time-evolution operator in the interaction picture.

To obtain the time-evolution operator in the interaction picture we note that $\hat{H}'_I(t)$ alone governs the time-evolution of the states in the interaction picture. The time-evolution operator in the interaction picture thus reads

$$\hat{U}_I(t, t') = \mathcal{T} \left\{ \exp \left(-i \int_{t'}^t d\tau \hat{H}'_I(\tau) \right) \right\}, \quad (3.16)$$

where $\hat{H}'_I(\tau)$ also has to satisfy the time-dependence of operators in the interaction picture,

$$\hat{H}'_I(t) = e^{i\hat{H}_0 t} \hat{H}'(t) e^{-i\hat{H}_0 t}. \quad (3.17)$$

To find a relation between the time-ordering operator in the interaction picture and the Schrödinger picture we use that the expectation value is independent on choice of picture,

$$\langle \psi(t) | \hat{U}(t, t') | \psi(t') \rangle = \langle \psi_I(t) | \hat{U}_I(t, t') | \psi_I(t') \rangle = \langle \psi(t) | e^{-i\hat{H}_0 t} \hat{U}_I(t, t') e^{i\hat{H}_0 t'} | \psi(t') \rangle, \quad (3.18)$$

which gives

$$\hat{U}(t, t') = e^{-i\hat{H}_0 t} \hat{U}_I(t, t') e^{i\hat{H}_0 t'}. \quad (3.19)$$

We will later use this relation to find a time-dependent expression for the density matrix in the interaction picture with respect to a well known density matrix ρ_0 at $t = t_0$.

3.1.3 The density matrix

In quantum mechanics we encounter two different types of systems. Systems that can be represented by a state vector $|i\rangle$ in an appropriate Hilbert space, called pure states, and systems described by a statistical ensemble of pure states, called mixed states. Mixed states are used to describe systems where the wave-function of the whole system is not known, but where we can obtain a probability of observing

each state in an ensemble. To describe such mixed states we use the density matrix $\hat{\rho}$ [36], where each state in an ensemble is assigned with a probability p_i . For an ensemble of n states, the density matrix reads

$$\hat{\rho} = \sum_{i=1}^n p_i |i\rangle \langle i|. \quad (3.20)$$

The expectation value of an operator \hat{A} in the density matrix formulation is given as

$$\langle \hat{A} \rangle = \sum_{i=1}^n p_i \langle i | \hat{A} | i \rangle = \text{Tr}[\hat{A}\hat{\rho}], \quad (3.21)$$

for a normalized density matrix, i.e. $\text{Tr}[\hat{\rho}] = 1$. The density matrix of a pure state simply reads $\hat{\rho} = |i\rangle \langle i|$, and thus has to satisfy the conditions $\hat{\rho}^2 = \hat{\rho}$ and $\text{Tr}[\hat{\rho}^2] = 1$. A mixed state, however, does not satisfy these conditions. One can thus investigate whether a density matrix describes a pure or mixed state by evaluating $\text{Tr}[\hat{\rho}^2]$.

For a system \mathcal{S} containing two subsystems $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ one can obtain a reduced density matrix operator, $\hat{\rho}^{(i)}$, $i \in \{1, 2\}$, where $\hat{\rho}^{(1)}$ contains all the relevant information about subsystem $\mathcal{S}^{(2)}$, and can be used to calculate the properties of $\mathcal{S}^{(1)}$. The reduced density matrix operator is obtained by tracing out the degrees of freedom of one of the two subsystems, e.g.

$$\hat{\rho}^{(1)} = \text{Tr}_2[\hat{\rho}]. \quad (3.22)$$

If the two subsystems cannot be separated by a tensor product, i.e. $\mathcal{S} = \mathcal{S}^{(1)} \otimes \mathcal{S}^{(2)}$, information about the off-diagonal elements, i.e. the coherence, of the density matrix is lost when tracing out one of the subsystems. We will later use this to obtain a reduced density matrix for the nuclear spins by tracing out the electron spin degrees of freedom.

3.1.4 Time-dependent perturbation theory

We here derive a time-evolution expression for the density matrix in the interaction picture, using the adiabatic approach, and solve it iteratively [36]. The time-evolution of states in the interaction picture is governed by the small perturbation $\hat{H}'(t)$ alone, which ensures that the higher order iterations (in our case, higher than second-order) can be neglected in a perturbative approach.

Our starting point will be a density matrix described by a Hamiltonian as given in equation (3.7), with a simple initial state at $t = t_0 \rightarrow -\infty$. The initial state can e.g. be an eigenstate of \hat{H}_0 or a simple thermal state. This requires that the perturbation $\hat{H}'(t)$ is turned off at $t = t_0$, such that the density matrix $\hat{\rho}_0$ is determined by \hat{H}_0 only. The perturbation is then turned on adiabatically by including an exponential time-dependence, $\hat{H}'(t) \rightarrow e^{\eta t} \hat{H}'$, where η is an infinitesimally small positive number, and the perturbation is assumed to be constant $\hat{H}'(t) = \hat{H}'$. The density matrix, for any time $t > -\infty$, can then be written as

$$\hat{\rho}(t) = \hat{U}(t, t_0) \hat{\rho}_0 \hat{U}(t_0, t), \quad (3.23)$$

where \hat{U} here describes the time-evolution due to $\hat{H}(t) = \hat{H}_0 + e^{\eta t} \hat{H}'$. Formally we only consider $t = 0$, where the perturbation is exactly \hat{H}' . We then use the relation between the time-ordering operator in the Schrödinger and interaction picture in equation (3.19) to obtain

$$\hat{\rho}(t) = e^{-i\hat{H}_0 t} \hat{U}_I(t, t_0) \hat{\rho}_0 \hat{U}_I(t_0, t) e^{i\hat{H}_0 t}, \quad (3.24)$$

where we used that $\hat{\rho}_0$ is stationary under \hat{H}_0 , i.e. $\hat{\rho}_0 = e^{-i\hat{H}_0 t_0} \hat{\rho}_0 e^{i\hat{H}_0 t_0}$. We now introduce the density matrix in the interaction picture, and find a relation between the density matrix in the Schrödinger and interaction picture,

$$\begin{aligned} \hat{\rho}_I(t) &= \sum_n p_n(t) |\psi_{n,I}(t)\rangle \langle \psi_{n,I}(t)| \\ &= \sum_n p_n e^{i\hat{H}_0 t} |\psi_n(t)\rangle \langle \psi_n(t)| e^{-i\hat{H}_0 t} \\ &= e^{i\hat{H}_0 t} \hat{\rho}(t) e^{-i\hat{H}_0 t}. \end{aligned} \quad (3.25)$$

Inserting the expression in equation (3.24) gives the relation between the density matrix in the interaction picture at time t and t_0 ,

$$\hat{\rho}_I(t) = \hat{U}_I(t, t_0) \hat{\rho}_0 \hat{U}_I(t_0, t). \quad (3.26)$$

By substituting $\hat{U}_I(t, t')$ with equation (3.16) and taking the derivative with respect to t , we obtain a time-evolution equation for $\hat{\rho}_I(t)$,

$$\frac{d\hat{\rho}_I}{dt} = -i[\hat{H}'_I(t), \hat{\rho}_I(t)]_-. \quad (3.27)$$

Since we know the initial density matrix at t_0 we can approximate the time-evolution equation iteratively by integrating from $t' = t_0 = -\infty$ to $t' = t$,

$$\hat{\rho}_I(t) - \hat{\rho}_0 = \int_{-\infty}^t -i[\hat{H}'_I(t'), \hat{\rho}_I(t')]_- dt', \quad (3.28)$$

and inserting the expression for $\hat{\rho}_I(t)$ back into the time-evolution equation (3.27). This gives an iterative approximation of the time dependence of the density matrix,

$$\frac{d\hat{\rho}_I(t)}{dt} = -i[\hat{H}'_I(t), \hat{\rho}_0]_- - \int_{-\infty}^t [\hat{H}'_I(t), [\hat{H}'_I(t'), \hat{\rho}_0]_-]_- dt' + \dots \quad (3.29)$$

Note that the leftmost Hamiltonian in the second term on the right hand side is not to be integrated over, and so forth for higher order perturbation terms. This iterative approximation will be our starting point for the derivation of the nuclear spin pumping.

3.2 Nuclear spin pumping

We have so far derived a time-dependent perturbation theory expression for the density matrix in the interaction picture. When we now derive the nuclear spin

pumping we will use the hyperfine Hamiltonian in equation (3.5) as perturbation,

$$\hat{H}'_I(t) = \sum_n Av_0 \hat{S}_I^a(\mathbf{r}_n, t) \hat{K}_{I,n}^a(t), \quad (3.30)$$

where the superscript implies the Einstein summation convention [37]. We can assume that the nuclear spin operators are time-independent on the time-scale of the electron fluctuations due to the different typical time scales of the electron and nuclear spin dynamics, i.e. we set $\hat{K}_n^a(t) = \hat{K}_n^a$. For the rest of this chapter all operators will be in the interaction picture, thus we omit the index I for simplicity.

To derive an expression for the nuclear spin pumping we trace out the electron degrees of freedom from the time-evolution expression of the density matrix in equation (3.29), and insert the hyperfine coupling Hamiltonian in equation (3.30) as perturbation. We keep terms up to second-order in the hyperfine coupling Av_0 because the second-order terms capture the electron-nucleus spin flips, which acts as an effective two-particle operator.

We obtain a time-evolution equation for the reduced density matrix of the nuclear spin system $\hat{\rho}^k$ by tracing out the electron spin degrees of freedom from equation (3.29) to second-order in Av_0 ,

$$\frac{d\hat{\rho}^k(t)}{dt} = \text{Tr}_e \left\{ -i[\hat{H}'(t), \hat{\rho}_0]_- - \int_{-\infty}^t [\hat{H}'(t), [\hat{H}'(t'), \hat{\rho}_0]_-]_- dt' \right\}. \quad (3.31)$$

We assume that the initial density matrix ρ_0 can be separated by a tensor product $\hat{\rho}_0 = \hat{\rho}_0^e \otimes \hat{\rho}_0^k$, i.e. no coherence, such that

$$\frac{d\hat{\rho}^k(t)}{dt} = \text{Tr}_e \left\{ -i[\hat{H}'(t), \hat{\rho}_0^e \otimes \hat{\rho}_0^k]_- - \int_{-\infty}^t [\hat{H}'(t), [\hat{H}'(t'), \hat{\rho}_0^e \otimes \hat{\rho}_0^k]_-]_- dt' \right\}. \quad (3.32)$$

However, tracing out the electron spin degrees of freedom introduces correlation functions of the electron spin coordinates that enter $\hat{H}'_I(t)$. One can see that, by separating the unconnected and connected correlators, the correlators that are not fully-connected² are higher order iteration terms for the first-order perturbation term. We collect the not fully-connected correlators in the first-order perturbation term by substituting $\hat{\rho}_0^k \rightarrow \hat{\rho}^k(t)$. Thus we only keep the fully-connected correlators resulting from the trace, which we indicate by a line over the trace symbol,

$$\frac{d\hat{\rho}^k(t)}{dt} = \overline{\text{Tr}}_e \left\{ -i[\hat{H}'(t), \hat{\rho}_0^e \otimes \hat{\rho}^k(t)]_- - \int_{-\infty}^t [\hat{H}'(t), [\hat{H}'(t'), \hat{\rho}_0^e \otimes \hat{\rho}^k(t)]_-]_- dt' \right\}. \quad (3.33)$$

We now insert the hyperfine Hamiltonian in equation (3.30) into the time-evolution of the reduced density matrix and use that the trace over electron degrees

²For a correlation function $\langle SS \rangle$, the fully-connected correlator $\langle SS \rangle_c$ is given by $\langle SS \rangle = \langle SS \rangle_c + \langle S \rangle \langle S \rangle$.

of freedom is invariant under cyclic permutation to obtain

$$\begin{aligned}
\frac{d\hat{\rho}^k(t)}{dt} &= -iAv_0 \sum_n \langle \hat{S}^a(\mathbf{r}_n, t) \rangle [\hat{K}_n^a, \hat{\rho}^k(t)]_- \\
&\quad - (Av_0)^2 \sum_{n,m} \int_{-\infty}^t dt' \left(\langle \hat{S}^a(\mathbf{r}_n, t) \hat{S}^b(\mathbf{r}_m, t') \rangle_c \hat{K}_n^a \hat{K}_m^b \hat{\rho}^k(t) \right. \\
&\quad - \langle \hat{S}^b(\mathbf{r}_m, t) \hat{S}^a(\mathbf{r}_n, t') \rangle_c \hat{K}_n^a \hat{\rho}^k(t) \hat{K}_m^b - \langle \hat{S}^a(\mathbf{r}_n, t) \hat{S}^b(\mathbf{r}_m, t') \rangle_c \hat{K}_m^b \hat{\rho}^k(t) \hat{K}_n^a \\
&\quad \left. + \langle \hat{S}^b(\mathbf{r}_m, t) \hat{S}^a(\mathbf{r}_n, t') \rangle_c \hat{\rho}^k(t) \hat{K}_m^b \hat{K}_n^a \right) \quad (3.34)
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\rho}^k(t)}{dt} &= -iAv_0 \sum_n \langle \hat{S}^a(\mathbf{r}_n, t) \rangle [\hat{K}_n^a, \hat{\rho}^k(t)]_- \\
&\quad - (Av_0)^2 \sum_{n,m} \int_{-\infty}^t dt' \left(\langle \hat{S}^a(\mathbf{r}_n, t) \hat{S}^b(\mathbf{r}_m, t') \rangle_c [\hat{K}_n^a, \hat{K}_m^b \hat{\rho}^k(t)]_- \right. \\
&\quad \left. - \langle \hat{S}^b(\mathbf{r}_m, t) \hat{S}^a(\mathbf{r}_n, t') \rangle_c [\hat{K}_n^a, \hat{\rho}^k(t) \hat{K}_m^b]_- \right) \quad (3.35)
\end{aligned}$$

with $\langle \dots \rangle = \text{Tr}_e \{ \dots \hat{\rho}_0^e \}$ being the expectation value and $\langle \dots \rangle_c = \overline{\text{Tr}}_e \{ \dots \hat{\rho}_0^e \}$ being the expectation value for the fully connected terms.

We have so far obtained an expression for the time-evolution of the reduced nuclear density matrix with the hyperfine Hamiltonian as perturbation. To obtain the dynamics of the averaged nuclear spins we investigate the following expectation value:

$$\frac{d\langle \hat{K}_n^a \rangle}{dt} = \text{Tr}_k \left\{ \hat{K}_n^a \frac{d\hat{\rho}^k(t)}{dt} \right\}. \quad (3.36)$$

We set $K = \frac{1}{2}$ for simplicity such that we can use the product relation for spin- $\frac{1}{2}$ particles:³ $\hat{K}_n^a \hat{K}_m^b = \delta_{nm} (\frac{i}{2} \epsilon^{abc} \hat{K}_n^c + \frac{1}{4} \delta^{ab})$, where ϵ_{abc} is the Levi-Civita symbol. By inserting equation (3.35) into equation (3.36) we find that the first term reads

$$\begin{aligned}
\frac{d\langle \hat{K}_n^a \rangle}{dt} &\stackrel{(1)}{=} \text{Tr}_k \left\{ \hat{K}_n^a \frac{d\hat{\rho}^k(t)}{dt} \right\}^{(1)} = -iAv_0 \sum_m \langle \hat{S}^b(\mathbf{r}_m, t) \rangle \text{Tr}_k \{ \hat{K}_n^a [\hat{K}_m^b, \hat{\rho}^k(t)]_- \} \\
&= -iAv_0 \sum_n \langle \hat{S}^b(\mathbf{r}_m, t) \rangle \text{Tr}_k \{ [\hat{K}_n^a, \hat{K}_m^b]_- \hat{\rho}^k(t) \} \\
&= \epsilon^{abc} S_n^b \langle \hat{K}_n^c \rangle \quad (3.37)
\end{aligned}$$

where we defined $S_n^b = Av_0 \langle \hat{S}^b(\mathbf{r}_n, t) \rangle$, and used that the trace is invariant under cyclic permutation. This term describes the first-order effects of the hyperfine interactions, i.e. the precession of the nuclear spins around the average electron

³In reality all Ga and As nuclei have spin $K = \frac{3}{2}$, but this usually does not lead to qualitatively different behavior.

spin polarization. The second-order contributions from equation (3.36) read

$$\begin{aligned} \frac{d\langle\hat{K}_n^a\rangle^{(2)}}{dt} = & -(Av_0)^2 \sum_{m,l} \int_{-\infty}^t dt' \text{Tr}_k \left\{ \langle\hat{S}^b(\mathbf{r}_m, t)\hat{S}^c(\mathbf{r}_l, t')\rangle_c \hat{K}_n^a [\hat{K}_m^b, \hat{K}_l^c \hat{\rho}^k(t)]_- \right. \\ & \left. - \langle\hat{S}^c(\mathbf{r}_l, t)\hat{S}^b(\mathbf{r}_m, t')\rangle_c \hat{K}_n^a [\hat{K}_m^b, \hat{\rho}^k(t)\hat{K}_l^c]_- \right\}. \end{aligned} \quad (3.38)$$

We define the correlation functions of the spin density as

$$R_{nm}^{ab} = \int_{-\infty}^t dt' \langle[\hat{S}^a(\mathbf{r}_n, t), \hat{S}^b(\mathbf{r}_m, t')]_+\rangle_c, \quad (3.39)$$

$$\chi_{nm}^{ab} = -i \int_{-\infty}^t dt' \langle[\hat{S}^a(\mathbf{r}_n, t), \hat{S}^b(\mathbf{r}_m, t')]_-\rangle_c, \quad (3.40)$$

and we will later see that only the local correlation functions contribute, where R_{nn}^{ab} are the classical fluctuations of the electron spin density at \mathbf{r}_n , and χ_{nn}^{ab} is the stationary susceptibility of the electron spins, i.e. the linear response of the spin density $S^a(\mathbf{r}_n)$ due to a constant local magnetic field $B^b(\mathbf{r}_n)$. The connection between the correlation function in equation (3.40) and the local susceptibility is shown in Appendix B. We now substitute the expectation values appearing in equation (3.38) with linear combinations of the correlation functions above to obtain

$$\begin{aligned} \frac{d\langle\hat{K}_n^a\rangle^{(2)}}{dt} = & -\frac{(Av_0)^2}{2} \left[(R_{ml}^{bc} + i\chi_{ml}^{bc}) \text{Tr}_k \{ [\hat{K}_n^a, \hat{K}_m^b]_- \hat{K}_l^c \hat{\rho}^k(t) \} \right. \\ & \left. - (R_{ml}^{bc} - i\chi_{ml}^{bc}) \text{Tr}_k \{ \hat{K}_l^c [\hat{K}_n^a, \hat{K}_m^b]_- \hat{\rho}^k(t) \} \right]. \end{aligned} \quad (3.41)$$

By writing out the commutators, using the product relation of spin- $\frac{1}{2}$ particles, and defining the nuclear spin expectation values as $\langle\cdots\rangle = \text{Tr}_k\{\cdots\hat{\rho}^k\}$ we get

$$\begin{aligned} \frac{d\langle\hat{K}_n^a\rangle^{(2)}}{dt} = & -\frac{(Av_0)^2}{2} \left(R_{ml}^{bc} \langle[[\hat{K}_n^a, \hat{K}_m^b]_-, \hat{K}_l^c]_-\rangle + i\chi_{ml}^{bc} \langle[[\hat{K}_n^a, \hat{K}_m^b]_-, \hat{K}_l^c]_+\rangle \right) \\ = & -\frac{(Av_0)^2}{2} \left(R_{nn}^{bc} \langle\delta^{bc}\hat{K}_n^a - \delta^{ac}\hat{K}_n^b\rangle - \chi_{nl}^{bc} \langle\epsilon^{abd}[\hat{K}_n^d, \hat{K}_l^c]_+\rangle \right) \\ = & \frac{(Av_0)^2}{2} \left(R_{nn}^{ba} \langle\hat{K}_n^b\rangle - R_{nn}^{bb} \langle\hat{K}_n^a\rangle + \frac{1}{2}\epsilon^{abc}\chi_{nn}^{bc} \right). \end{aligned} \quad (3.42)$$

These terms contain the second-order effects of the hyperfine interactions and describe the resulting nuclear pumping in terms of the local fluctuations and susceptibilities of the electron spins.

Collecting the first and second-order terms gives an expression for the pumping of the nuclear spins caused by the non-equilibrium spin dynamics,

$$\frac{d\langle\hat{K}_n^a\rangle}{dt} = \epsilon^{abc} S^b \langle\hat{K}_n^c\rangle + \frac{(Av_0)^2}{2} \left(R^{ba} \langle\hat{K}_n^b\rangle - R^{bb} \langle\hat{K}_n^a\rangle + \frac{1}{2}\epsilon^{abc}\chi^{bc} \right), \quad (3.43)$$

where we omitted the subscript n . Recall that this is the pumping of nuclear spin at the point \mathbf{r}_n and thus has the unit $\frac{1}{s}$. The first term contains the effect of the nuclear spins precessing around the average electron spin polarization and is thus linear in the hyperfine coupling Av_0 . The second term contains the contributions from the stationary susceptibilities and fluctuations of the electron density, χ^{ab} and R^{ab} , respectively, and is of second-order in the hyperfine coupling Av_0 . The contribution from the electron spin susceptibilities determine the pumping of the nuclear spin, whereas the fluctuations of the electron spin density contribute to the relaxation of the nuclear spin towards $\langle \hat{\mathbf{K}} \rangle = 0$. We rewrite the expression for the nuclear spin dynamics to better separate the different contributions,

$$\frac{d\langle \hat{K}^a \rangle}{dt} = \epsilon^{abc} S^b \langle \hat{K}_n^c \rangle + \frac{1}{2} Q^{ab} \langle \hat{K}^b \rangle + \frac{1}{4} P^a, \quad (3.44)$$

where

$$Q^{ab} = (Av_0)^2 (\delta^{ab} R^{cc} - R^{ba}), \quad (3.45)$$

$$P^a = (Av_0)^2 \epsilon^{abc} \chi^{bc}, \quad (3.46)$$

is the contribution from the electron spin fluctuations and stationary susceptibilities, respectively. As previously mentioned, the repeated superscripts imply the Einstein summation convention.

This concludes the derivation of the nuclear spin pumping formalism. We will use this formalism to calculate the nuclear spin pumping in a one-dimensional continuous system as a proof of principle calculation in chapter 4, and in the persistent spin helix in chapter 5.

Chapter 4

Proof of principle

In this chapter we will show how one can calculate the nuclear spin pumping by using the formalism from chapter 3 on a system with a diffusion equation similar to what we obtained in chapter 2, which has previously never been done. The motivation behind the proof of principle calculation is to get a better understanding of the formalism, and to show that it actually works, before using it to calculate the nuclear spin pumping in the PSH setup. We will focus on the hyperfine-induced spin pumping by only including the pumping terms (susceptibilities) in our pumping equation, and incorporate the contribution from the electron spin fluctuations to relaxation of the nuclear spin polarization into a phenomenological rate $1/\tau_k$, which can have contributions from many other origins as well.

We investigate a simple one dimensional setup, where the dynamics of the electron spins are described by the diffusion equation

$$\frac{\partial \mathbf{S}(x)}{\partial t} = D \frac{\partial^2 \mathbf{S}(x)}{\partial x^2} - \frac{1}{\tau_e} \mathbf{S}(x) + \mathbf{Q} \delta(x) + B \hat{z} \times \mathbf{S}(x) \quad (4.1)$$

where D is the diffusion constant, τ_e is the electron spin relaxation rate, B is an extrinsic, uniform magnetic field in the z -direction, and \mathbf{Q} is a constant source of injected spin polarization at $x = 0$. The problem is effectively one-dimensional, however, the spin density \mathbf{S} is three-dimensional with units m^{-3} .

There are both similarities and differences when comparing this system to the PSH structure. They are similar in the sense that they are both only dependent on one coordinate, and that they both contain diffusive spin transport and spin relaxation. The PSH, however, relies on having significant spin-orbit interaction, which is absent in equation (4.1). Our motivation behind the spin injection setup is to make it similar to what one would investigate experimentally, e.g. by optical injection of spin or a semiconductor-ferromagnet junction. Depending on choice of spin injection method, one gets a factor 2 difference for systems that only allow diffusive spin transport in one direction compared to systems with diffusive spin transport in both directions. We will use the same spin injection source when calculating the nuclear pumping in the PSH in chapter 5, which is similar to the setup investigated experimentally by Walser et al. [25].

In order to obtain an expression for the nuclear spin pumping in our system we need to calculate the local magnetic susceptibilities $\chi^{ab}(x)$, i.e. the electron spins' linear response to a perturbative, local magnetic field $B'_b(x)$. To find the effect of the perturbative magnetic field we first calculate the steady-state solution of the unperturbed diffusion equation (4.1). We then add a perturbative magnetic field term $\mathbf{B}'(x) \times \mathbf{S}(x)$ to the diffusion equation and calculate the first-order response to the perturbation. The difference caused by the perturbative magnetic field then gives us the magnetic susceptibilities we need.

4.1 Steady-state solution

We start by calculating the steady-state solution to the unperturbed diffusion equation. The diffusion equation (4.1) in steady-state reads

$$D \frac{\partial^2 \mathbf{S}^{(0)}(x)}{\partial x^2} - \frac{1}{\tau_e} \mathbf{S}^{(0)}(x) + \mathbf{Q} \delta(x) + B \hat{z} \times \mathbf{S}^{(0)}(x) = 0, \quad (4.2)$$

where $\mathbf{S}^{(0)}(x)$ is the steady-state solution. Due to the source term, this differential equation is easiest solved in Fourier space. We use the relations $\mathcal{F}\left[\frac{\partial^n f(x)}{\partial x^n}\right](k) = (ik)^n \tilde{f}(k)$ and $\mathcal{F}[\delta(x)](k) = 1$, where $\tilde{f}(k) = \mathcal{F}[f(x)](k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$ is the Fourier transform of $f(x)$, to obtain

$$-Dk^2 \tilde{\mathbf{S}}^{(0)}(k) - \frac{1}{\tau_e} \tilde{\mathbf{S}}^{(0)}(k) + \mathbf{Q} + B \mathcal{F}[\hat{z} \times \mathbf{S}^{(0)}(x)](k) = 0. \quad (4.3)$$

This is a set of three coupled linear equations, one for each spatial component of the spin density, with solution

$$\tilde{\mathbf{S}}^{(0)}(k) = \begin{pmatrix} \frac{(Dk^2 + \frac{1}{\tau_e})Q_x - BQ_y}{(Dk^2 + \frac{1}{\tau_e})^2 + B^2} \\ \frac{(Dk^2 + \frac{1}{\tau_e})Q_y + BQ_x}{(Dk^2 + \frac{1}{\tau_e})^2 + B^2} \\ \frac{Q_z}{Dk^2 + \frac{1}{\tau_e}} \end{pmatrix}. \quad (4.4)$$

The steady-state solution in real space can then be obtained by taking the inverse Fourier transform. This is trivial for the z -component, which reads

$$S_z^{(0)}(x) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} Q_z e^{-\frac{|x|}{\sqrt{D\tau_e}}}. \quad (4.5)$$

This describes the spin density we would get for a system where the injected spin is parallel to the z -axis. In such a system, there is no precession of spin density into the xy -plane since the injected spin polarization is parallel to the external magnetic field B . The spin density in equation (4.5) describes the diffusion and decay of the injected spin, where $\sqrt{D\tau_e}$ is the electron's diffusion length, i.e. the average length an electron travels in time τ_e , and $\sqrt{\frac{D}{\tau_e}}$ is the electron's diffusion

velocity, i.e the electron's average velocity, in the x -direction, in the time period τ_e after injection of spin. The diffusion length decides how fast the spin polarization decays, while the diffusion velocity, together with the injected spin density, decides the spin density's magnitude. The spin density is illustrated in figure 4.1c.

For the coupled x and y components we have to use a different approach. We define a linear combination of the x - and y -component as $\tilde{Z}(x) = \tilde{S}_y^{(0)}(x) + i\tilde{S}_x^{(0)}(x)$, where $S_x^{(0)} = \text{Im}(Z)$ and $S_y^{(0)} = \text{Re}(Z)$, that reads

$$\begin{aligned}\tilde{Z}(k) &= \frac{Dk^2 + \frac{1}{\tau_e} - iB}{(Dk^2 + \frac{1}{\tau_e})^2 + B^2} (Q_y + iQ_x) \\ &= \frac{1}{Dk^2 + \frac{1}{\tau_e} + iB} (Q_y + iQ_x),\end{aligned}\quad (4.6)$$

This linear combination can then easily be inverse Fourier transformed, which gives

$$\begin{aligned}Z(x) &= \frac{(Q_y + iQ_x)}{2D\sqrt{\frac{1}{D\tau_e} + i\frac{B}{D}}} e^{-|x|\sqrt{\frac{1}{D\tau_e} + i\frac{B}{D}}} \\ &= \frac{\sqrt{\frac{1}{D\tau_e} - i\frac{B}{D}}(Q_y + iQ_x)}{2\sqrt{\left(\frac{1}{\tau_e}\right)^2 + B^2}} e^{-|x|\sqrt{\frac{1}{D\tau_e} + i\frac{B}{D}}}.\end{aligned}\quad (4.7)$$

We further simplify by defining $a + ib = \sqrt{\frac{1}{D\tau_e} + i\frac{B}{D}}$ and $\gamma^{-1} = 2\sqrt{\left(\frac{1}{\tau_e}\right)^2 + B^2}$ such that

$$\begin{aligned}Z(x) &= \gamma(a - ib)(Q_y + iQ_x) e^{-|x|(a+ib)} \\ &= \gamma(a - ib)(Q_y + iQ_x) (\cos b|x| - i \sin b|x|) e^{-a|x|}.\end{aligned}\quad (4.8)$$

By then using the relations $S_x^{(0)} = \text{Im}(Z)$ and $S_y^{(0)} = \text{Re}(Z)$, we obtain the x - and y -component of the spin density in real space

$$S_x^{(0)}(x) = \gamma[\cos b|x|(aQ_x - bQ_y) - \sin b|x|(aQ_y + bQ_x)]e^{-a|x|}\quad (4.9)$$

$$S_y^{(0)}(x) = \gamma[\cos b|x|(aQ_y + bQ_x) + \sin b|x|(aQ_x - bQ_y)]e^{-a|x|}.\quad (4.10)$$

We compare these expressions, for the in-plane components of the spin density, to the z -component in equation (4.5) to investigate the effect the external magnetic field has on the electron spin density. We here see that a drives the diffusive spin transport and b drives the precession of spin density in the xy -plane. By investigating the expressions for a and b ,

$$a = \sqrt{\frac{\frac{1}{D\tau_e} + \sqrt{\left(\frac{1}{D\tau_e}\right)^2 + \left(\frac{B}{D}\right)^2}}{2}} \quad b = \text{sgn}(B) \sqrt{\frac{-\frac{1}{D\tau_e} + \sqrt{\left(\frac{1}{D\tau_e}\right)^2 + \left(\frac{B}{D}\right)^2}}{2}},\quad (4.11)$$

one can see that a presents an effective renormalization of the inverse diffusion length $\frac{1}{\sqrt{D\tau_e}}$ due to the magnetic field, which reduces the diffusive spin transport

for the in-plane components when compared to the z -component, and that b has the same sign as B and arises as a consequence of the magnetic field. The magnitude of spin polarization driven by a and b is determined by γa and γb , respectively.

To further investigate the effects of the external magnetic field on the spin density, we look at the special case of $\mathbf{Q} \parallel \hat{x}$. The in-plane components of the spin density then read

$$S_x^{(0)}(x) = \gamma (\cos b|x|Q_x a - \sin b|x|Q_x b) e^{-a|x|} \quad (4.12)$$

$$S_y^{(0)}(x) = \gamma (\cos b|x|Q_x b + \sin b|x|Q_x a) e^{-a|x|}, \quad (4.13)$$

and is illustrated in figure 4.1a and 4.1b, respectively. We here see that in the absence of a magnetic field, i.e. $B = 0$, we get the same solution as for the z -component above, as expected. Hence, for $B = 0$ the three spin components decouple and take the same for as equation (4.5).

To see how the electron spin's lifetime τ_e affects the spin density we investigate the large τ_e limit, i.e. $\tau_e \rightarrow \infty$. In this limit the diffusion length $\sqrt{D\tau_e} \rightarrow \infty$, which effectively removes the spatial damping of the spin density. Also, the diffusion velocity $\sqrt{\frac{\tau_e}{D}} \rightarrow \infty$ which leads to an infinite build up of spin polarization. This is caused by the constant injection of spin polarization \mathbf{Q} in the absence of spin relaxation.

4.2 Steady-state solution of the perturbed system

We now add the perturbative magnetic field term $\mathbf{B}' \times \mathbf{S}(x)$ to the diffusion equation. Adding this term introduces higher powers of \mathbf{B}' in the solution. Since we only need the linear response to \mathbf{B}' to find the susceptibilities we substitute $\mathbf{S}(x)$ with the \mathbf{B}' -independent spin density $\mathbf{S}^{(0)}(x)$. The spin diffusion equation with the perturbative magnetic field then reads

$$\frac{\partial \mathbf{S}(x)}{\partial t} = D \frac{\partial^2 \mathbf{S}(x)}{\partial x^2} + \mathbf{Q} \delta(x) - \frac{1}{\tau_e} \mathbf{S}(x) + B \hat{z} \times \mathbf{S}(x) + \mathbf{B}' \times \mathbf{S}^{(0)}(x). \quad (4.14)$$

We investigate the diffusion equation at steady-state, and define $\mathbf{Q}'(x) = \mathbf{Q} \delta(x) + \mathbf{B}' \times \mathbf{S}^{(0)}(x)$, which gives

$$D \frac{\partial^2 \mathbf{S}^{(1)}(x)}{\partial x^2} - \frac{1}{\tau_e} \mathbf{S}^{(1)}(x) + B \hat{z} \times \mathbf{S}^{(1)}(x) + \mathbf{Q}'(x) = 0, \quad (4.15)$$

where $\mathbf{S}^{(1)}(x)$ is the steady-state solution to the perturbed diffusion equation. The only difference between this differential equation and the one solved in section 5.1 is that this one contains a continuous source instead of a point source. Since we have already solved the differential equation with a point source, we use a Green function to solve the differential equation with a continuous source.

A requirement for using the Green function approach is that $\mathbf{Q}'(x)$ has to be zero on its boundaries, which is satisfied for $\mathbf{Q}'(x)$, since $\lim_{x \rightarrow \pm\infty} \mathbf{S}^{(0)}(x) = 0$. We

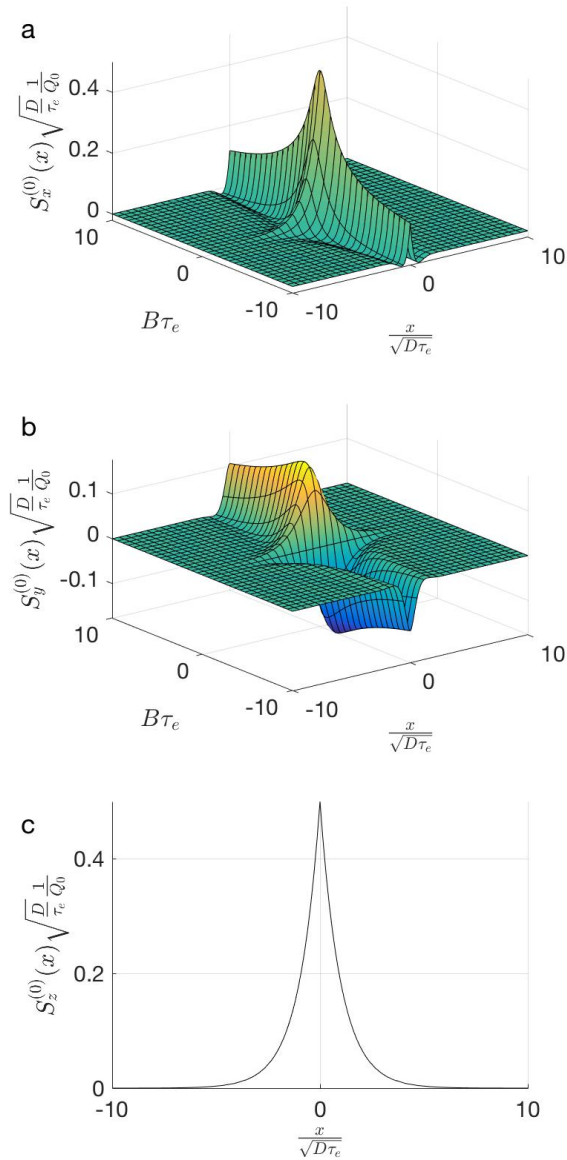


Figure 4.1: **a**, **b**, Illustration of the steady-state solution and its dependence on the magnetic field B for a constant source $Q_x = Q_0$ for the x - and y -component respectively. **c**, Illustration of the steady state solution for a constant source $Q_z = Q_0$. The spin density is independent of B since $\mathbf{S} \parallel \mathbf{B}$.

define a linear differential operator \mathcal{L} such that

$$\mathcal{L}\mathbf{S}^{(1)}(x) = -D\frac{\partial^2\mathbf{S}^{(1)}(x)}{\partial x^2} + \frac{1}{\tau_e}\mathbf{S}^{(1)}(x) - B\hat{z} \times \mathbf{S}^{(1)}(x) = \mathbf{Q}'(x), \quad (4.16)$$

and the Green function $\mathbf{G}(x, x')$ as a function satisfying

$$\mathcal{L}\mathbf{G}(x, x') = \delta(x - x'), \quad (4.17)$$

i.e. the system's response to a point source at $x = x'$ with weight 1. To find the relation between $\mathbf{Q}'(x)$ and the Green function we express the continuous source as an integral over an infinite number of point sources,

$$\begin{aligned} \mathbf{Q}'(x) &= \int_{-\infty}^{\infty} \mathbf{Q}'(x')\delta(x - x')dx' \\ &= \int_{-\infty}^{\infty} \mathbf{Q}'(x')\mathcal{L}\mathbf{G}(x, x')dx' \\ &= \mathcal{L} \int_{-\infty}^{\infty} \mathbf{G}(x, x')\mathbf{Q}'(x')dx', \end{aligned} \quad (4.18)$$

where we used that \mathcal{L} only acts on x in the last step. We insert this relation into equation (4.16) to obtain a Green function expression for the spin density,

$$\mathbf{S}^{(1)}(x) = \int_{-\infty}^{\infty} \mathbf{G}(x, x')\mathbf{Q}'(x')dx', \quad (4.19)$$

where the spin density is described by an infinite number of Green function responses to an infinite amount of point sources that define the continuous source $\mathbf{Q}'(x)$.

We then follow the same approach for the unperturbed spin density to find the point source $\mathbf{Q}\delta(x')$ expressed in terms of $\mathbf{G}(x, x')$:

$$\begin{aligned} \mathbf{Q}\delta(x) &= \int_{-\infty}^{\infty} \mathbf{Q}\delta(x')\delta(x - x')dx' \\ &= \int_{-\infty}^{\infty} \mathbf{Q}\delta(x')\mathcal{L}\mathbf{G}(x, x')dx' \\ &= \mathcal{L}\mathbf{Q}\mathbf{G}(x, 0), \end{aligned} \quad (4.20)$$

which gives us the steady-state solution for the unperturbed system expressed as a Green function response,

$$\mathbf{S}^{(0)}(x) = \mathbf{Q}\mathbf{G}(x, 0). \quad (4.21)$$

Since the perturbed and unperturbed systems have the same differential operator \mathcal{L} , their Green functions have to be equal, as stated by equation (4.17). Equation (4.17) also states that the Green function is only dependent on the distance between x and x' , i.e. $\mathbf{G}(x, x') \equiv \mathbf{G}(x - x')$. By comparing the Green

function expressions for $\mathbf{S}^{(1)}(x)$ and $\mathbf{S}^{(0)}(x)$, we see that one can obtain $\mathbf{S}^{(1)}(x)$ from $\mathbf{S}^{(0)}(x)$ by substituting $\mathbf{Q} \rightarrow \mathbf{Q}'(x')$ and $x \rightarrow x - x'$, and integrate over x' . This gives the steady-state solution to the perturbed diffusion equation,

$$\mathbf{S}^{(1)}(x) = \int_{-\infty}^{\infty} dx' \times \left(\begin{array}{l} \gamma \left[\left\{ Q'_x(x')a - Q'_y(x')b \right\} \cos b\chi - \left\{ Q'_y(x')a + Q'_x(x')b \right\} \sin b\chi \right] e^{-a\chi} \\ \gamma \left[\left\{ Q'_y(x')a + Q'_x(x')b \right\} \cos b\chi + \left\{ Q'_x(x')a - Q'_y(x')b \right\} \sin b\chi \right] e^{-a\chi} \\ \frac{Q'_z(x')\sqrt{\tau_e}}{2\sqrt{D}} e^{-\frac{\chi}{\sqrt{D\tau_e}}} \end{array} \right), \quad (4.22)$$

where $\chi = |x - x'|$. For the special case $\mathbf{Q}'(x') = \mathbf{Q}\delta(x)$ we recover the unperturbed solution $\mathbf{S}^{(0)}(x)$, as expected.

4.3 Electron spin susceptibilities

In order to find the local electron spin susceptibilities we have to investigate the difference in spin density caused by the perturbative magnetic field, i.e. $\delta\mathbf{S}(x) = \mathbf{S}^{(1)}(x) - \mathbf{S}^{(0)}(x)$. By recalling that $\mathbf{Q}'(x) = \mathbf{Q}\delta(x) + \mathbf{B}' \times \mathbf{S}^{(0)}(x)$, we see that we can obtain $\delta\mathbf{S}(x)$ from $\mathbf{S}^{(1)}(x)$ by substituting $\mathbf{Q}'(x') \rightarrow \mathbf{Q}''(x') = \mathbf{B}' \times \mathbf{S}^{(0)}(x')$, giving

$$\delta\mathbf{S}(x) = \int_{-\infty}^{\infty} dx' \times \left(\begin{array}{l} \gamma \left[\left\{ Q''_x(x')a - Q''_y(x')b \right\} \cos b\chi - \left\{ Q''_y(x')a + Q''_x(x')b \right\} \sin b\chi \right] e^{-a\chi} \\ \gamma \left[\left\{ Q''_y(x')a + Q''_x(x')b \right\} \cos b\chi + \left\{ Q''_x(x')a - Q''_y(x')b \right\} \sin b\chi \right] e^{-a\chi} \\ \frac{Q''_z(x')\sqrt{\tau_e}}{2\sqrt{D}} e^{-\frac{\chi}{\sqrt{D\tau_e}}} \end{array} \right). \quad (4.23)$$

From equation (B.3) in Appendix B we get an expression for $\delta\mathbf{S}(x, t)$ in terms of the electron spin correlation functions, now with a uniform magnetic field \mathbf{B} ,

$$\begin{aligned} \delta S^a(x, t) &= -i \int^t dt' \int d\mathbf{r} B^b \langle [\hat{S}^a(x, t), \hat{S}^b(x', t')]_- \rangle \\ &= l_y l_z \int dx' B^b \chi^{ab}(x, x') \end{aligned} \quad (4.24)$$

where l_a is the length of the 2DEG in the a -direction, and $\chi^{ab}(x, x') = -i \int^t dt' \langle [\hat{S}^a(x, t), \hat{S}^b(x', t')] \rangle$ is an electron spin density correlation function. The factor $l_y l_z$ is a result of the 3D spin density only being dependent on x . By comparing the two expressions for $\delta\mathbf{S}$ we see that the local correlation function $\chi^{ab}(x)$, i.e. the local susceptibilities, is obtained from equation (4.23) by omitting the dx' integration, setting $x' = x$, dividing by $l_y l_z$, and finding the terms linear in B^b ,

giving

$$\begin{aligned}
\chi^{xy}(x) &= \frac{\gamma a}{l_y l_z} S_z^{(0)}(x) & \chi^{yx}(x) &= -\frac{\gamma a}{l_y l_z} S_z^{(0)}(x) \\
\chi^{zx}(x) &= \frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_y^{(0)}(x) & \chi^{xz}(x) &= -\frac{\gamma}{l_y l_z} \left(a S_y^{(0)}(x) + \gamma S_x^{(0)}(x) \right) \\
\chi^{yz}(x) &= \frac{\gamma}{l_y l_z} \left(a S_x^{(0)}(x) - b S_y^{(0)}(x) \right) & \chi^{zy}(x) &= -\frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_x^{(0)}(x).
\end{aligned} \tag{4.25}$$

4.4 Nuclear spin pumping

We can now calculate the nuclear spin pumping caused by the local electron spin susceptibilities by using equation (3.44). We incorporate the contribution from the local electron spin fluctuations, Q^{ab} , in a phenomenological relaxation rate $\frac{1}{\tau_k}$ of the nuclear spin polarization, which can have many sources, e.g. the coupling to lattice vibrations. The equation of motion for the expectation value of the nuclear spin then reads

$$\frac{d \langle K_a(x) \rangle}{dt} = \frac{1}{4} (Av_0)^2 \epsilon^{abc} \chi^{bc} - \frac{1}{\tau_k} \langle K_a(x) \rangle, \tag{4.26}$$

where l_0 is the lattice constant in the one-dimensional system. By inserting the local susceptibilities we obtained in section 4.3 we obtain

$$\frac{d \langle \mathbf{K}(x) \rangle}{dt} = \frac{(Av_0)^2}{4l_y l_z} \begin{pmatrix} \gamma a + \frac{\sqrt{\tau_e}}{2\sqrt{D}} & -\gamma b & 0 \\ \gamma b & \gamma a + \frac{\sqrt{\tau_e}}{2\sqrt{D}} & 0 \\ 0 & 0 & 2\gamma a \end{pmatrix} \mathbf{S}^{(0)}(x) - \frac{1}{\tau_k} \langle \mathbf{K}(x) \rangle. \tag{4.27}$$

The pumping term originates from the second-order hyperfine interaction, which can be seen from the prefactor $\frac{(Av_0)^2}{4}$. The matrix in equation (4.27) describes how the nuclear spin polarization responds to the presence of the electron spin polarization. The matrix elements determine the magnitude of the spin pumping, and are dependent on the electron spin's lifetime and diffusion constant, and on the external magnetic field. The off-diagonal elements arise due to the external magnetic field in the z -direction, that couples the in-plane components of the electron and nuclear spin polarization. For the diagonal elements, the z -component is slightly smaller than the x and y components for any $B \neq 0$, as can be seen from

$$\gamma a = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \frac{1}{\sqrt{2}} \sqrt{\frac{1 + \sqrt{1 + (\tau_e B)^2}}{1 + (\tau_e B)^2}}. \tag{4.28}$$

Following the arguments for γa and γb from section 4.1 we find that the matrix becomes diagonal for $B = 0$. This makes the pumping term proportional to the

electron spin density. Thus, in the absence of an external magnetic field, the nuclear spin polarization \mathbf{K} will always be proportional to \mathbf{S} . The electrons experience this nuclear spin polarization as an effective magnetic field $\mathbf{B} \propto \mathbf{K}$, which is also proportional to \mathbf{S} . The relaxation term relaxes the nuclear spin polarization and ensures that there exists a steady-state solution to the pumping equation with a finite nuclear spin polarization.

To explain the nuclear spin pumping's dependence on the lengths l_y and l_z , we investigate the local electron spin correlation functions where they appear,

$$\chi^{ab}(x) = -i \int dt' \langle [\hat{S}^a(x, t), \hat{S}^b(x, t')] \rangle. \quad (4.29)$$

This expression describes correlations between the electron being at the point x at time t and t' . The larger the system the more space is available for the electrons to diffuse into. This makes it less likely that the electron's path crosses x twice, which reduces this local correlation function. Thus, even if the system is effectively one-dimensional, the availability for electrons to diffuse into the y and z dimensions reduces the local electron spin correlation functions, and hence also the nuclear spin pumping.

We further investigate the case where only the nuclear spin polarization contributes to \mathbf{B} , since this is what we will investigate in chapter 5 for the PSH. We do this by considering a setup where the initial electron spin polarization is injected in the z -direction, with no external magnetic field. The electron spin density is then given by equation (4.5), and the effective magnetic field is determined through the hyperfine coupling $\mathbf{B} = A \langle \mathbf{K} \rangle$. Since the electron spin polarization only has a z -component, the pumping of nuclear spin polarization caused by the electron spin dynamics reads

$$\frac{d \langle K_z(x) \rangle}{dt} = \frac{(Av_0)^2}{2l_y l_z} \gamma a S_z^{(0)}(x) - \frac{1}{\tau_k} \langle K_z(x) \rangle, \quad (4.30)$$

which is, as expected, parallel to \mathbf{S} . When the nuclear spin polarization grows, γa becomes smaller, which leads to a decreasing pumping rate. This can be seen in figure 4.2, where the nuclear spin pumping caused by the electron susceptibilities, i.e. the first term in equation (4.30), and its dependence on the magnetic field B arising from the nuclear spin polarization, is shown for a constant spin source $Q_z = Q_0$. We investigate the steady-state solution of equation (4.30) in the limit $B\tau_e \ll 1$. In this limit, the leading order terms in the Taylor expansion of γa read $\gamma a \approx \frac{\sqrt{\tau_e}}{2\sqrt{D}} \left(1 - \frac{3}{8}(B\tau_e)^2\right)$. Inserting this into the nuclear pumping expression (4.30) gives

$$\frac{d \langle K_z(x) \rangle}{dt} = \frac{(Av_0)^2}{2l_y l_z} \frac{\sqrt{\tau_e}}{2\sqrt{D}} \left(1 - \frac{3}{8}(A \langle K_z(x) \rangle \tau_e)^2\right) S_z^{(0)}(x) - \frac{1}{\tau_k} \langle K_z(x) \rangle, \quad (4.31)$$

with steady-state solution

$$\langle K_z(x) \rangle = \frac{-32e^{\frac{|x|}{\sqrt{D}\tau_e}} + 2\sqrt{16^2 e^{\frac{2|x|}{\sqrt{D}\tau_e}} + 6\zeta^2}}{3A\tau_e\zeta}, \quad (4.32)$$

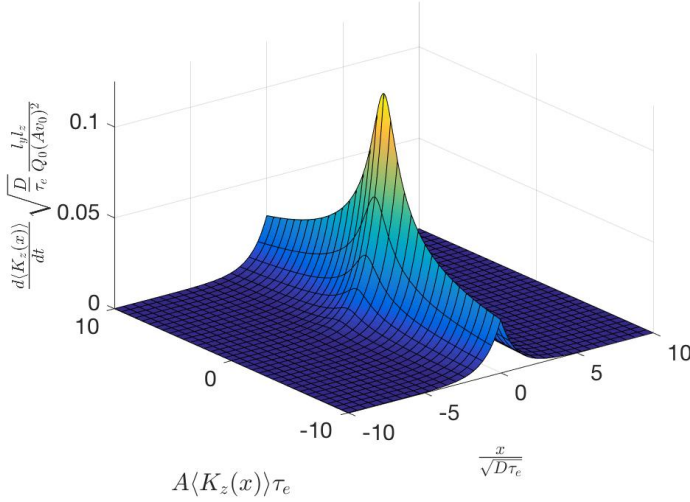


Figure 4.2: Illustration of the nuclear spin pumping caused by the electron spin susceptibilities, and its dependence on the magnetic field B arising from the nuclear spin polarization, for a constant spin source $Q_z = Q_0$.

where $\zeta = \frac{A^3 v_0^2 Q_0 \tau_e^2 \tau_k}{D l_y l_z}$. An illustration of the solution is shown in figure 4.3, where one can see that the magnitude of the spin polarization is dependent on the electron and nuclear spin properties appearing in ζ . The spatial decay is different compared to the one appearing in the electron spin density. This can be explained through the γa expansion in equation (4.31), which increasingly reduces the nuclear spin pumping the bigger the nuclear spin polarization is. This reduces the nuclear spin pumping, and polarization, more the smaller the x , which flattens out the exponential function appearing in the electron spin density $S_z^{(0)}$ in the pumping equation.

We evaluate the nuclear spin polarization at the point of electron spin injection, i.e. $\langle K_z(0) \rangle$, in the limit of large spin injection, $Q_0 \gg \frac{D l_y l_z}{A^3 v_0^2 \tau_e^2 \tau_k}$, to find an estimate of the theoretical maximum nuclear spin polarization that can be obtained from the nuclear spin pumping caused by the non-equilibrium electron spin dynamics. This is realized by setting $\zeta \gg 1$ and $x = 0$ in equation (4.32), and gives a nuclear spin polarization of

$$\langle K_z(0) \rangle^{\max} = \frac{2\sqrt{6}}{3A\tau_e}, \quad (4.33)$$

which has the order of magnitude in a GaAs 2DEG, with $A \sim 100 \mu\text{eV}$ [34] and $\tau_e \sim 100 \text{ ps}$ [38], of ~ 0.10 . An estimate for the order of magnitude of electron spin injection Q_0 needed to obtain $\langle K_z(0) \rangle^{\max}$ can be found by evaluating the scale $\frac{D l_y l_z}{A^3 v_0^2 \tau_e^2 \tau_k}$.

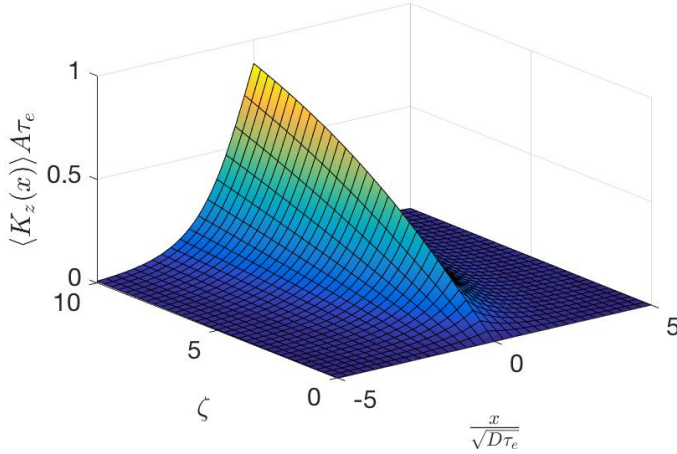


Figure 4.3: Illustration of the steady-state nuclear spin polarization in a system with electron spin injection along the z -axis.

Since only the nuclear spin polarization contributes to \mathbf{B} in our PSH setup, which makes \mathbf{B} parallel to \mathbf{S} , we are not interested to further investigate equation (4.27) for \mathbf{B} non-parallel to \mathbf{S} , i.e. the in-plane components of equation (4.27).

For the PSH we predicted that the nuclear spin polarization would affect the electron spin polarization through a feedback cycle. We see that this is not the case for our proof of principle setup with parallel electron and nuclear spin polarization since S_z is independent of B . This is because only the anisotropic components of the spin density couple to a parallel magnetic field, as we will see in chapter 5, which is not present in our proof of principle setup. We hence have to evaluate a changing \mathbf{S} for the PSH in chapter 5, due to the feedback cycle. This is done by always assuming that the electron spin density is in equilibrium, while the nuclear spin polarization slowly changes, which is a consequence of the vastly different time scales of the electron and nuclear spin dynamics.

We have now shown that we can use the formalism derived in chapter 3 to calculate the nuclear spin pumping in a system described by an equation of motion similar to what we derived for the PSH in chapter 2. In the next chapter we will calculate the nuclear spin pumping in the PSH, and investigate how this affects the electron spin structure through a feedback cycle.

Chapter 5

Hyperfine interaction in the PSH

In chapter 4 we calculated the nuclear spin pumping in a simple spin-injection setup. By following the same approach, we will in this chapter calculate the nuclear spin pumping in the PSH. As in chapter 4, we only consider the build up of nuclear spin polarization caused by the electron spin's magnetic susceptibilities. By then adding the contribution from the nuclear spin polarization to the semi-classical equation of motion from chapter 2, we calculate the feedback effects of the hyperfine interaction in the PSH.

5.1 Nuclear spin pumping in the PSH

In chapter 2 we derived the following equation of motion for the electron spin density in the persistent spin helix,

$$\partial_t \mathbf{S} = \begin{pmatrix} D\nabla^2 - \gamma_{cd} - \gamma_{ey} & 0 & \gamma_{sw}\tau_1 Dq_0 \partial_x \\ 0 & D\nabla^2 - Dq_0^2 - \gamma_{cd} - \gamma_{ey} & 2Dq_0 \partial_y \\ -\gamma_{sw}\tau_1 Dq_0 \partial_x & -2Dq_0 \partial_y & D\nabla^2 - Dq_0^2 - 2\gamma_{cd} \end{pmatrix} \mathbf{S}. \quad (5.1)$$

In order to calculate the nuclear spin pumping caused by the electron spin dynamics we make some simplifications to the equation of motion above. Firstly we assume that the spin density does not have an x -dependence, which is the case that has been investigated experimentally [21][25], such that $\partial_x \mathbf{S} = 0$. This makes the x -component of the spin density decouple from the y and z components. We also assume that the relaxation of the electron spin components is governed by a phenomenological isotropic relaxation rate $1/\tau_e$, which includes the Elliot-Yafet mechanism, cubic Dresselhaus coupling, as well as other sources of relaxation. This is done to make the problem simpler and this part of the description more general, and because it would be naïve to assume that the Elliot-Yafet relaxation would be the only effect that relax the spin density. To calculate the steady-state solution

of the equation of motion we add a spin injection source, similar to the one we investigated in chapter 4. For a three-dimensional PSH, the injected spin is in the xz -plane. In a real system this source could be realized by a semiconductor-ferromagnet junction, assumed that the sample width is small enough to ensure uniform spin injection along the x -axis and no boundary effects. The electron spin equation of motion then reads

$$\partial_t \mathbf{S}(y) = \begin{pmatrix} D\partial_y^2 - \frac{1}{\tau_e} & 0 & 0 \\ 0 & D\partial_y^2 - Dq_0^2 - \frac{1}{\tau_e} & 2Dq_0\partial_y \\ 0 & -2Dq_0\partial_y & D\partial_y^2 - Dq_0^2 - \frac{1}{\tau_e} \end{pmatrix} \mathbf{S}(y) + \mathbf{Q}\delta(y). \quad (5.2)$$

To obtain an expression for the nuclear spin pumping in the PSH we need to calculate the local magnetic susceptibilities of the electron spin. This is obtained through a similar approach as we followed in chapter 4, i.e. we 1) calculate the steady-state solution of the electron spin equation of motion, 2) add a perturbative magnetic field and calculate the new steady-state solution, 3) obtain the susceptibilities from the difference in spin polarization caused by the perturbative magnetic field.

5.1.1 Steady-state solution

As in chapter 4, we calculate the steady-state solution of equation (5.2) in Fourier space. By evaluating equation (5.2) at steady-state, and taking the Fourier transform, we obtain

$$\begin{pmatrix} -Dk_y^2 - \frac{1}{\tau_e} & 0 & 0 \\ 0 & -Dk_y^2 - Dq_0^2 - \frac{1}{\tau_e} & 2iDq_0k_y \\ 0 & -2iDq_0k_y & -Dk_y^2 - Dq_0^2 - \frac{1}{\tau_e} \end{pmatrix} \tilde{\mathbf{S}}^{(0)}(k_y) + \mathbf{Q} = 0, \quad (5.3)$$

where k_y is the y -component of the momentum vector, and $\tilde{\mathbf{S}}^{(0)}$ is the Fourier transformed steady-state spin density. Since the x -component of the spin density is decoupled from the y and z components, we first solve for the x -component, before solving for the coupled y and z components.

The x -component is obtained by simply taking the inverse Fourier transform of $\tilde{S}_x^{(0)}$, which gives

$$S_x^{(0)}(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} Q_x e^{-\frac{|y|}{\sqrt{D\tau_e}}}. \quad (5.4)$$

To solve for the coupled y and z components, we first solve for the Fourier transformed y and z components in equation (5.3), giving

$$\tilde{S}_y^{(0)}(k_y) = \frac{D(2iq_0k_yQ_z + (k_y^2 + q_0^2)Q_y) + \frac{1}{\tau_e}Q_y}{\left(\frac{1}{\tau_e} + D(k_y - q_0)^2\right)\left(\frac{1}{\tau_e} + D(k_y + q_0)^2\right)} \quad (5.5)$$

$$\tilde{S}_z^{(0)}(k_y) = \frac{D(-2iq_0k_yQ_y + (k_y^2 + q_0^2)Q_z) + \frac{1}{\tau_e}Q_z}{\left(\frac{1}{\tau_e} + D(k_y - q_0)^2\right)\left(\frac{1}{\tau_e} + D(k_y + q_0)^2\right)}. \quad (5.6)$$

Since we can not easily calculate the inverse Fourier transform of these expressions, we define linear combinations of the two components as $\tilde{S}_+^{(0)} = \tilde{S}_y^{(0)} + i\tilde{S}_z^{(0)}$ and $\tilde{S}_-^{(0)} = \tilde{S}_y^{(0)} - i\tilde{S}_z^{(0)}$, such that $S_y^{(0)} = \frac{1}{2} (S_-^{(0)} + S_+^{(0)})$ and $S_z^{(0)} = \frac{i}{2} (S_-^{(0)} - S_+^{(0)})$.

Inserting the expressions for $\tilde{S}_y^{(0)}$ and $\tilde{S}_z^{(0)}$ above gives

$$\tilde{S}_+^{(0)}(k_y) = \frac{Q_y + iQ_z}{\frac{1}{\tau_e} + D(k_y + q_0)^2} \quad (5.7)$$

$$\tilde{S}_-^{(0)}(k_y) = \frac{Q_y - iQ_z}{\frac{1}{\tau_e} + D(k_y - q_0)^2}. \quad (5.8)$$

These expressions can easily be Fourier transformed by shifting the transformation variable $k_y \rightarrow k_y \pm q_0$, giving

$$S_+^{(0)}(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} e^{-\frac{|y|}{\sqrt{D}\tau_e}} (\cos q_0 y + i \sin q_0 y) (Q_y + iQ_z) \quad (5.9)$$

$$S_-^{(0)}(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} e^{-\frac{|y|}{\sqrt{D}\tau_e}} (\cos q_0 y - i \sin q_0 y) (Q_y - iQ_z). \quad (5.10)$$

The y and z components of the spin density in real space are then obtained by using the relations $S_y^{(0)} = \frac{1}{2} (S_-^{(0)} + S_+^{(0)})$ and $S_z^{(0)} = \frac{i}{2} (S_-^{(0)} - S_+^{(0)})$,

$$S_y^{(0)}(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} (Q_y \cos q_0 y - Q_z \sin q_0 y) e^{-\frac{|y|}{\sqrt{D}\tau_e}} \quad (5.11)$$

$$S_z^{(0)}(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} (Q_z \cos q_0 y + Q_y \sin q_0 y) e^{-\frac{|y|}{\sqrt{D}\tau_e}}, \quad (5.12)$$

which describe a damped PSH structure. This can be seen by evaluating the electron spin density with a spin injection strictly in the z -direction, i.e. $Q_z = Q_0$, which gives the PSH we obtained in chapter 2, and that was investigated by Walser et al. [25].

The x -component of the spin density in equation (5.4) describes the damping of the x -component of the injected spin. The y and z components of the spin density is given in equation (5.11) and (5.12), respectively. One can here see the precession of the injected spin as a function of y caused by the intrinsic spin-orbit field aligned with the x -axis.

In the next section we add a perturbative magnetic field to the PSH setup and calculate the new steady-state solution. We can then, by evaluating the difference in the spin density caused by the perturbation, obtain the local magnetic susceptibilities we need to calculate the nuclear spin pumping in the PSH.

5.1.2 Perturbed steady-state solution

We now add a perturbative magnetic field term $\mathbf{B} \times \mathbf{S}^{(0)}(y)$ to the equation of motion (5.2),

$$\partial_t \mathbf{S}(y) = \hat{D} \mathbf{S}(y) + \mathbf{Q} \delta(y) + \mathbf{B} \times \mathbf{S}^{(0)}(y) \quad (5.13)$$

where \hat{D} is the diffusion matrix as given in equation (5.2). As in chapter 4, we define a new source $\mathbf{Q}'(y) = \mathbf{Q}\delta(y) + \mathbf{B} \times \mathbf{S}^{(0)}(y)$ such that we can obtain the perturbed steady-state solution from the unperturbed solution through our Green function approach. By using our result from section 4.2 we find that by substituting $\mathbf{Q}\delta(y) \rightarrow \mathbf{Q}'(y)$ and $y \rightarrow y - y'$, and integrating over dy' , we obtain the perturbed spin density

$$S_x(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' Q'_x(y') e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}} \quad (5.14)$$

$$S_y(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' \{Q'_y(y') \cos q_0(y - y') - Q'_z(y') \sin q_0(y - y')\} e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}} \quad (5.15)$$

$$S_z(y) = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' \{Q'_z(y') \cos q_0(y - y') + Q'_y(y') \sin q_0(y - y')\} e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}}. \quad (5.16)$$

We can now find the electron spin's susceptibilities by investigating the spin density difference caused by the perturbative magnetic field.

5.1.3 Susceptibilities

The difference in spin density $\delta\mathbf{S}(y)$, caused by the perturbative magnetic field, is obtained by substituting $\mathbf{Q}'(y) \rightarrow \mathbf{B} \times \mathbf{S}^{(0)}(y)$ in equation (5.14)-(5.16), giving

$$\delta S_x = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}} (B_y S_z^{(0)} - B_z S_y^{(0)}) \quad (5.17)$$

$$\delta S_y = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}} \left\{ (B_z S_x^{(0)} - B_x S_z^{(0)}) \cos q_0(y - y') \right. \\ \left. - (B_x S_y^{(0)} - B_y S_x^{(0)}) \sin q_0(y - y') \right\} \quad (5.18)$$

$$\delta S_z = \frac{\sqrt{\tau_e}}{2\sqrt{D}} \int dy' e^{-\frac{|y-y'|}{\sqrt{D\tau_e}}} \left\{ (B_x S_y^{(0)} - B_y S_x^{(0)}) \cos q_0(y - y') \right. \\ \left. + (B_z S_x^{(0)} - B_x S_z^{(0)}) \sin q_0(y - y') \right\}. \quad (5.19)$$

Following the argumentation from section 4.3, we obtain the local electron spin susceptibility $\chi^{ab}(y)$ by omitting the dy' integration, setting $y' = y$, dividing by $l_y l_z$, and finding the terms linear in B^b , giving

$$\begin{aligned} \chi^{xy}(y) &= \frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_z^{(0)} & \chi^{yx}(y) &= -\frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_z^{(0)} \\ \chi^{zx}(y) &= \frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_y^{(0)} & \chi^{xz}(y) &= -\frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_y^{(0)} \\ \chi^{yz}(y) &= \frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_x^{(0)} & \chi^{zy}(y) &= -\frac{\sqrt{\tau_e}}{2l_y l_z \sqrt{D}} S_x^{(0)}. \end{aligned} \quad (5.20)$$

5.1.4 Nuclear spin pumping

We can now obtain the nuclear spin pumping in the PSH by using equation (3.44). Since we are only interested in the pumping that arises from the local magnetic susceptibilities of the electron spins χ^{ab} , we include the contributions from the relaxation matrices R^{ab} in a phenomenological relaxation rate of the nuclear spin components τ_k , which may also contain other sources of relaxation. The hyperfine-induced pumping of nuclear spin polarization in the PSH then reads

$$\frac{d\langle K_a(y) \rangle}{dt} = \frac{1}{4}(Av_0)^2 \epsilon^{abc} \chi^{bc}(y) - \frac{1}{\tau_k} \langle K_a(y) \rangle. \quad (5.21)$$

By inserting the susceptibilities above we find that the nuclear spin pumping in the PSH reads

$$\frac{d\langle \mathbf{K}(y) \rangle}{dt} = \frac{(Av_0)^2}{4l_y l_z} \sqrt{\frac{\tau_e}{D}} \mathbf{S}^{(0)}(y) - \frac{1}{\tau_k} \langle \mathbf{K}(y) \rangle. \quad (5.22)$$

We see that, as we expected from our calculation in chapter 4, the build up of nuclear spin polarization is proportional to the electron spin density in the absence of an external magnetic field. The magnitude of the nuclear spin pumping is determined by hyperfine coupling, the electron's diffusion velocity $\sqrt{\frac{D}{\tau_e}}$, the size of the 2DEG, and the electron spin density

In the next section we will solve equation (5.22) in steady-state to obtain an expression for the expectation value of the nuclear spin polarization in the PSH. We will use this most general result to investigate the effect of $\langle \mathbf{K} \rangle$ on a specific contribution to spin relaxation, the cubic Dresselhaus interaction. This is done by adding the effective magnetic field caused by $\langle \mathbf{K} \rangle$ to the semi-classical equation of motion from chapter 2, and derive a new diffusion equation for the electron spin density. We then investigate the feedback effect between the electron and nuclear spin polarization. Since the nuclear spin polarization described by equation (5.22) has a similar structure to the Hartree-Fock field, we expect a similar result to what Lüffe et al. [28] obtained for the PSH in the presence of Hartree-Fock interactions.

5.2 Electron spin diffusion equation

In the previous section we obtained an expression for the pumping of nuclear spin polarization in the PSH. We will now evaluate the nuclear spin pumping at steady-state to get an expression for the effective magnetic field experienced by the electron spin. We then add the contribution from that effective magnetic field to the semi-classical equation of motion from chapter 2, and derive a new diffusion equation for the electron spin, which describes the feedback effect between the electron and nuclear spin density. The setup we will use is slightly simplified when comparing to chapter 2, in the sense that we neglect the contribution from the extrinsic spin-orbit coupling with the impurity potential.

Because of the different dynamic time-scales of the electron and nuclear spin, the electron spin density instantaneously reaches a new equilibrium configuration

for an infinitesimal change in the nuclear spin polarization. However, because we anticipate that the change in spin density caused by the hyperfine interaction is small, i.e. $\delta\mathbf{S}_{\text{HF}} \ll \mathbf{S}$, we use the nuclear spin pumping expression we get from solving equation (5.22) in steady-state,

$$\langle \mathbf{K}(y) \rangle = \frac{(Av_0)^2}{4l_y l_z} \sqrt{\frac{\tau_e}{D}} \tau_k \mathbf{S}(y). \quad (5.23)$$

The effective magnetic field caused by the nuclear spin polarization is then obtained by multiplying with the hyperfine coupling energy A , as can be seen from the hyperfine Hamiltonian (3.4) in chapter 3, to get

$$\mathbf{B}(y) = A \langle \mathbf{K}(y) \rangle = \frac{A^3 v_0^2}{4l_y l_z} \sqrt{\frac{\tau_e}{D}} \tau_k \mathbf{S}(y) \equiv \eta \mathbf{S}(y). \quad (5.24)$$

From chapter 4 we know that an isotropic spin density does not couple to a parallel external magnetic field. Hence, we have to investigate the electron spin dynamics in detail to understand how the electron spin density is affected by the nuclear spin polarization. This is done by deriving a diffusion equation for the electron spin density, like we did in chapter 2, which this time includes the effects of the nuclear spin polarization in equation (5.23). When we calculated the nuclear spin pumping in equation (5.23) we assumed isotropic relaxation of the spin density. However, for the electron spin dynamics we will focus on the relaxation by the cubic Dresselhaus terms to investigate how these are affected by the nuclear spin polarization. We hence neglect other sources of relaxation, e.g. the Elliot-Yafet relaxation.

5.2.1 Additional terms in the Boltzmann equation

To investigate the feedback effect between the electron and nuclear spin polarization, i.e. how the nuclear spin polarization, caused by the non-equilibrium electron spin polarization, affects the electron spin polarization, we add a term

$$2\mathbf{s}_{\mathbf{k}} \times \mathbf{B}(\mathbf{r}) \quad (5.25)$$

to the semi-classical equation of motion from chapter 2, and derive a new diffusion equation for the electron spin. The electron spin equation of motion then reads

$$\partial_t \mathbf{s}_{\mathbf{k}} + \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{B} = \mathbf{J}_{\mathbf{k}}^{\text{imp}}, \quad (5.26)$$

where we use a simplified expression for the electron-impurity interaction,

$$\mathbf{J}_{\mathbf{k}}^{\text{imp}} = - \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}) \Delta \mathbf{s}_{\mathbf{k}}, \quad (5.27)$$

where we have set λ_0 to zero because we want to focus the cubic Dresselhaus interaction's effect on the spin relaxation.

As in chapter 2, we expand the electron spin density in terms of winding numbers,

$$\mathbf{s}_{\mathbf{k}} = \sum_{n=0,\pm 1,\pm 3} \mathbf{s}_n(\mathbf{k}) e^{in\theta}, \quad (5.28)$$

with $\mathbf{s}_0 = -\frac{2\pi}{m} f'(\epsilon_k) \mathbf{S}$ being the isotropic component, and $\mathbf{s}_n = f'(\epsilon_k) \frac{k^n}{k_F^{n-1} m} \delta \mathbf{k}_n$ being the anisotropic component with winding number n . We will use a similar approach as in chapter 2. By substituting the expanded spin density into the semi-classical equation of motion (5.26) and integrating over θ we obtain a diffusion equation for \mathbf{s}_0 . We neglect the time-dependence of the anisotropic components due to the different time-scales of the isotropic and anisotropic components, and multiply the semi-classical equation of motion with $\gamma(\theta) \in \{\cos n\theta, \sin n\theta\}$, $n \in \{1, 3\}$. Integrating over θ then gives the expressions for the anisotropic components of winding number $\pm n$ that appear in the isotropic diffusion equation.

5.2.2 Spin diffusion equation

We insert the expanded spin density into the semi-classical Boltzmann equation and integrate over θ to obtain a time-evolution equation for the isotropic spin component,

$$\partial_t \mathbf{s}_0 = -\frac{v}{2} (\partial_x \mathbf{s}_c + \partial_y \mathbf{s}_s) - \mathbf{s}_c \times \mathbf{b}_c - \mathbf{s}_s \times \mathbf{b}_s - \mathbf{s}_{c3} \times \mathbf{b}_{c3} - \mathbf{s}_{s3} \times \mathbf{b}_{s3}, \quad (5.29)$$

where we defined a new basis for the anisotropic spin components, $\mathbf{s}_{cn} = \mathbf{s}_n + \mathbf{s}_{-n}$, $\mathbf{s}_{sn} = i(\mathbf{s}_n - \mathbf{s}_{-n})$, and the spin-orbit field that couples to the respective anisotropic components

$$\begin{aligned} \mathbf{b}_c &= k(-\alpha + \beta'') \hat{\mathbf{e}}_y & \mathbf{b}_s &= k(\alpha + \beta'') \hat{\mathbf{e}}_x \\ \mathbf{b}_{c3} &= \gamma \frac{k^3}{4} \hat{\mathbf{e}}_y & \mathbf{b}_{s3} &= \gamma \frac{k^3}{4} \hat{\mathbf{e}}_x. \end{aligned} \quad (5.30)$$

This is identical to the isotropic diffusion equation obtained in chapter 2, except from the missing Elliot-Yafet relaxation terms.

To make the isotropic diffusion equation self-consistent, we find steady-state expressions for the anisotropic components appearing in it. We neglect the time-evolution term due difference in dynamics between the isotropic and anisotropic components. The anisotropic components are then obtained by multiplying the semi-classical Boltzmann equation with $\gamma(\theta) \in \{\cos n\theta, \sin n\theta\}$, $n \in \{1, 3\}$ before integrating over θ . For the winding number ± 1 components we multiply with $\cos \theta$ and $\sin \theta$ before integrating over θ , and find that

$$\frac{\mathbf{s}_c}{\tau_1} = -v \partial_x \mathbf{s}_0 + 2\mathbf{b}_c \times \mathbf{s}_0 + 2\mathbf{B} \times \mathbf{s}_c \quad (5.31)$$

$$\frac{\mathbf{s}_s}{\tau_1} = -v \partial_y \mathbf{s}_0 + 2\mathbf{b}_s \times \mathbf{s}_0 + 2\mathbf{B} \times \mathbf{s}_s, \quad (5.32)$$

where $\tau_1 = mn_i |v(0)|^2$ is the effective relaxation time of the winding number ± 1 components of the spin density. We solve for \mathbf{s}_c and \mathbf{s}_s iteratively, and use that $\mathbf{B} \parallel \mathbf{s}_0$, which gives

$$\begin{aligned} \mathbf{s}_c = \tilde{\tau}_1 \{ & -v\partial_x \mathbf{s}_0 + 2\mathbf{b}_c \times \mathbf{s}_0 - 2v\tau_1(\mathbf{B} \times \partial_x \mathbf{s}_0) \\ & + 4\tau_1 \mathbf{B}_1 \times (\mathbf{b}_c \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B}(\mathbf{B} \cdot \partial_x \mathbf{s}_0) \} \end{aligned} \quad (5.33)$$

$$\begin{aligned} \mathbf{s}_s = \tilde{\tau}_1 \{ & -v\partial_y \mathbf{s}_0 + 2\mathbf{b}_s \times \mathbf{s}_0 - 2v\tau_1(\mathbf{B} \times \partial_y \mathbf{s}_0) \\ & + 4\tau_1 \mathbf{B}_1 \times (\mathbf{b}_s \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B}(\mathbf{B} \cdot \partial_y \mathbf{s}_0) \}, \end{aligned} \quad (5.34)$$

with $\tilde{\tau}_1 = \frac{\tau_1}{1+(2B\tau_1)^2}$, where $B = |\mathbf{B}|$. For the anisotropic components of winding number ± 3 we multiply with $\cos 3\theta$ and $\sin 3\theta$ before integrating over θ , which gives

$$\frac{\mathbf{s}_{c3}}{\tau_3} = 2\mathbf{b}_{c3} \times \mathbf{s}_0 + 2\mathbf{B} \times \mathbf{s}_{c3} \quad (5.35)$$

$$\frac{\mathbf{s}_{s3}}{\tau_3} = 2\mathbf{b}_{s3} \times \mathbf{s}_0 + 2\mathbf{B} \times \mathbf{s}_{s3}, \quad (5.36)$$

where $\tau_3 = mn_i |v(0)|^2$ is the effective relaxation time of the winding number ± 3 components of the spin density. Solving iteratively for \mathbf{s}_{c3} and \mathbf{s}_{s3} gives

$$\mathbf{s}_{c3} = 2\tilde{\tau}_3 \{ \mathbf{b}_{c3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{c3} \times \mathbf{s}_0) \} \quad (5.37)$$

$$\mathbf{s}_{s3} = 2\tilde{\tau}_3 \{ \mathbf{b}_{s3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{s3} \times \mathbf{s}_0) \}, \quad (5.38)$$

where $\tilde{\tau}_3 = \frac{\tau_3}{1+(2B\tau_3)^2}$.

The electron spin diffusion equation is obtained by substituting the expressions for the anisotropic spin components into the time-evolution expression for the isotropic spin component, and integrating over k . We then evaluate the diffusion equation for the PSH condition $\alpha = \beta''$, and use that $\mathbf{S} \parallel \mathbf{B}$ to obtain

$$\partial_t \mathbf{S} = \hat{D} \mathbf{S} + \mathbf{H}_1 + \mathbf{H}_3 \quad (5.39)$$

with

$$\hat{D} = \begin{pmatrix} \tilde{D}(\partial_x^2 + \partial_y^2) - \tilde{\gamma}_{cd} & 0 & 0 \\ 0 & \tilde{D}(\partial_x^2 + \partial_y^2 - q_0^2) - \tilde{\gamma}_{cd} & 2\tilde{D}q_0\partial_y \\ 0 & -2\tilde{D}q_0\partial_y & \tilde{D}(\partial_x^2 + \partial_y^2 - q_0^2) - 2\tilde{\gamma}_{cd} \end{pmatrix}, \quad (5.40)$$

$$\begin{aligned} \mathbf{H}_1 = & 2\tilde{D} \left[\tau_1 \mathbf{B} \times \partial_x^2 \mathbf{S} + \tau_1 \mathbf{B} \times \partial_y^2 \mathbf{S} - \partial_y \{ \tau_1 \mathbf{B} \times (q_0 \hat{e}_x \times \mathbf{S}) \} \right. \\ & + 2\partial_x \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_x \mathbf{S}) \} + 2\partial_y \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) \} + q_0 \hat{e}_x \times (\partial_y \mathbf{S} \times \tau_1 \mathbf{B}) \\ & \left. + (\tau_1 \mathbf{B} \cdot q_0 \hat{e}_x) (\mathbf{S} \times q_0 \hat{e}_x) + 2(\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) (\tau_1 \mathbf{B} \times q_0 \hat{e}_x) \right], \end{aligned} \quad (5.41)$$

and

$$\mathbf{H}_3 = -2\tilde{\gamma}_{cd}\tau_3 \mathbf{B} \times \begin{pmatrix} S_x \\ S_y \\ 2S_z \end{pmatrix}, \quad (5.42)$$

where $\tilde{D} = \frac{1}{2}v_F^2\tilde{\tau}_1 = \frac{\frac{1}{2}v_F^2\tau_1}{1+(2B\tau_1)^2}$ is the renormalized diffusion constant, $\tilde{\gamma}_{\text{cd}} = \frac{\frac{1}{8}\gamma^2k_F^6\tau_3}{1+(2B\tau_3)^2}$ is the renormalized cubic Dresselhaus scattering rate, and $q_0 = 4m\alpha$ is the PSH wave vector. The full calculation, from the expressions for the isotropic and anisotropic components, is given in appendix C. The obtained diffusion equation is similar to the expression obtained by Lüffe et al. [28] for a static Hartree-Fock field parallel to the spin polarization profile, as we expected.

The first term in the diffusion equation (5.39) is identical to the one we derived in chapter 2, except from the neglected spin-orbit coupling to the impurity potential, and that the diffusion constant \tilde{D} and relaxation rate $\tilde{\gamma}_{\text{cd}}$ contain an additional factor $\frac{1}{1+(2B\tau)^2}$, where $\tau = \tau_1 = \tau_3$, as a result of the nuclear spin polarization. Since \mathbf{B} is proportional to \mathbf{S} , $\mathbf{H}_{1(3)}$ contains terms that are non-linear in \mathbf{S} , which have their origin in the coupling of the nuclear spin polarization to the anisotropic components of winding number $\pm 1(\pm 3)$ of the electron spin density. $\mathbf{H}_{1(3)}$ contains additional factors of τB compared to $\hat{D}\mathbf{S}$ which assure the effect of $\hat{H}_{1(3)}$ to be small, and its contribution can be seen as a small deviation from $\partial_t\mathbf{S} = \hat{D}\mathbf{S}$.

5.3 Feedback effects in the PSH

We here investigate how the diffusion equation (5.39) affects two different PSH setups. Firstly, we look at how an injected PSH changes with time to lowest order in $\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{\Gamma}}$, where $\tilde{\Gamma}$ is the DP relaxation rate. This is inspired by work done by [28], and gives an idea of how the nuclear spin polarization is affecting the PSH structure. Secondly, we investigate how the PSH is affected by equation (5.39) in a system with spin injection along the x -axis. We choose the x -component of the PSH due to its simpler spin profile (decay of injected spin polarization) compared to the helix components, i.e. the y and z components, while still having the same prolonged lifetime due to the intrinsic spin-orbit field.

5.3.1 Injection of a PSH

To investigate how the PSH is affected by the nuclear spin polarization we look at how the diffusion equation (5.39) affects the PSH. To get a first estimate of this effect we inject a PSH, similar to the one we discussed in chapter 2, for then to turn on the effects of the hyperfine interaction. As mentioned above, $\mathbf{H}_{1(3)}$ contains additional factors of τB compared to the first term, which means that they can be seen as a small deviation from $\partial_t\mathbf{S} = \hat{D}\mathbf{S}$. From the diagonal relaxation terms in \hat{D} one can see that the PSH spin density is elliptical. Based on this, we inject the following PSH,

$$\mathbf{S} = S_0 \begin{pmatrix} 0 \\ -\sin q_0 y \\ (1 - \delta) \cos q_0 y \end{pmatrix}, \quad (5.43)$$

where δ is some offset from a circular spin polarization caused by the cubic Dresselhaus interaction. By writing the effective magnetic field arising from the nuclear

spin polarization as $\mathbf{B} = B \frac{\mathbf{S}}{|\mathbf{S}|}$, and inserting this and the injected PSH (5.43) into the diffusion equation (5.39), we find that with

$$\delta = \frac{\tilde{\gamma}_{\text{cd}}}{\tilde{\Gamma}} \frac{1}{1 + \frac{1}{2}(\tau B)^2} \quad (5.44)$$

where $\tilde{\Gamma} = 4q_0^2 \tilde{D}$ is the DP relaxation rate, the diffusion equation to lowest order in $\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{\Gamma}}$ can be written as

$$\partial_t \mathbf{S} = -\frac{\mathbf{S}}{\tilde{\tau}_{\text{PSH}}} + \frac{\mathbf{S}_{2x}}{\tau_{2x}} + \frac{\mathbf{S}_{3yz}}{\tau_{3yz}} \quad (5.45)$$

with $\tilde{\tau}_{\text{PSH}}^{-1} = \frac{3}{2} \tilde{\gamma}_{\text{cd}}$ being the PSH relaxation rate,

$$\frac{\mathbf{S}_{2x}}{\tau_{2x}} = \frac{S_0}{3\tilde{\tau}_{\text{PSH}}} \tau B \left(1 - \frac{1}{1 + \frac{1}{2}(\tau B)^2} \right) \begin{pmatrix} \sin 2q_0 y \\ 0 \\ 0 \end{pmatrix} \quad (5.46)$$

being the contribution from \mathbf{H}_3 , and

$$\frac{\mathbf{S}_{3yz}}{\tau_{3yz}} = \frac{2_0}{3\tilde{\tau}_{\text{PSH}}} \frac{(\tau B)^2}{1 + \frac{1}{2}(\tau B)^2} \begin{pmatrix} 0 \\ -\sin 3q_0 y \\ \cos 3q_0 y \end{pmatrix} \quad (5.47)$$

being the contribution from \mathbf{H}_1 . The first term in equation (5.45) describes the temporal decay of the spin helix. The two rightmost terms describe higher-order harmonics, which is an additional effect of the isotropic spin components coupling to the nuclear spin polarization.

From this we can see that the hyperfine interaction affects the PSH by increasing the PSH lifetime, introducing higher-order harmonics, and giving rise to a non-zero x -component. The increased lifetime is a result of the reduction in the cubic Dresselhaus scattering rate $\tilde{\gamma}_{\text{cd}} = \frac{\gamma_{\text{cd}}}{1 + (2B\tau)^2} < \gamma_{\text{cd}}$. The higher-order harmonics are a result of the anisotropic spin components coupling to the nuclear spin polarization, where the non-zero x -component arise from the winding number ± 3 anisotropic components. We have shown that the ± 3 components give rise to the cubic Dresselhaus relaxation rate. Hence, the driving of electron spin polarization out of the yz -plane can be seen as an effect of the nuclear spin polarization coupling to the cubic Dresselhaus spin-orbit coupling.

The solutions for the higher-order harmonics terms are smaller by factors $\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{\Gamma}}$ [28]. Hence, we write the diffusion equation (5.45) in the $\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{\Gamma}} \ll 1$ limit as $\partial_t \mathbf{S} = -\frac{\mathbf{S}}{\tilde{\tau}_{\text{PSH}}}$, which has the time-dependent solution

$$\mathbf{S}(t) = \mathbf{S} e^{-\frac{t}{\tilde{\tau}_{\text{PSH}}}}. \quad (5.48)$$

This gives the primary effect of the hyperfine interaction in the PSH, which increases the lifetime of the spin helix by reducing the cubic Dresselhaus relaxation

rate. This effect is dependent on the magnitude of electron spin polarization, since the electron spin polarization affects the magnetic field appearing in $\tilde{\gamma}_{\text{cd}}$ through the nuclear spin pumping.

We calculate the theoretical maximum feedback effect we can get from equation (5.48) by assuming that all nuclear spins are fully polarized. A 100 % nuclear spin polarization corresponds to a magnetic field of ~ 5 T in GaAs [34], which has a g-factor of 0.44 [39]. The momentum relaxation time τ is approximately 1 ps in the PSH [21]. This gives a reduction in the effective cubic Dresselhaus scattering rate of $\tilde{\gamma}_{\text{cd}}^{\text{max}} \sim 0.87\gamma_{\text{cd}}$. Thus, the nuclear spin polarization can, in theory, maximum increase the PSH lifetime by 15 %, when all nuclear spins are fully polarized. Figure 5.1 shows the time-dependent relaxation of $\mathcal{S}(t)$ for a nuclear spin polarization of 0, 50 and 100 %.

5.3.2 Spin injection along the x -axis

In the last subsection we looked at how the nuclear spin polarization affects an injected PSH. Here, we investigate how the diffusion equation (5.39) affects the PSH in a spin injection setup, similar to the setup we looked at in section 5.1. We focus on the x -component of the PSH because it has a simpler structure than the helix components, while still having the prolonged PSH lifetime. Due to its simpler structure, of spatial decay of injected spin polarization, the x -component can be investigated in more detail analytically, and may even be more useful for spintronic applications. We assume here a spin injection setup with diffusive transport in only one direction, which makes $y \geq 0$, and doubles the magnitude of the spin

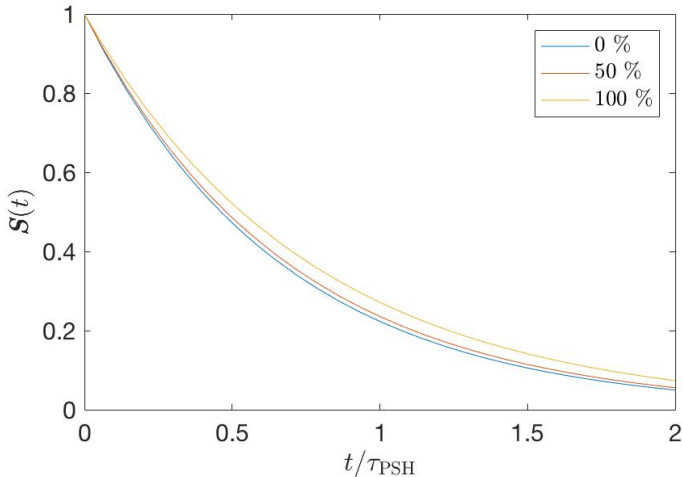


Figure 5.1: Time-dependence of an injected PSH, $\mathcal{S}(t)$, in a system with a nuclear spin polarization of 0, 50 and 100 %. The prolonged lifetime for higher nuclear polarizations shows how the feedback effect affects the PSH.

polarizations compared to diffusive transport in both directions.

We investigate the feedback effect governed by equation (5.39) for a spin polarization strictly along the x -axis by inserting $\mathbf{S} \parallel \mathbf{B} \parallel \hat{x}$ into equation (5.39), which yields

$$\partial_t S_x = \tilde{D} \partial_y^2 S_x - \tilde{\gamma}_{\text{cd}} S_x + 4\tilde{D} \partial_y \{ \tau^2 B_x^2 \partial_y S_x \}, \quad (5.49)$$

where the last term is the only non-zero contribution from \mathbf{H}_1 . To get an estimate for the effective magnetic field B_x appearing in equation (5.49) we first solve for S_x in steady-state with the last term set equal zero, which gives

$$S_x^{(0)}(y) = S_0 e^{-y \sqrt{\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{D}}}}, \quad (5.50)$$

where we used the boundary condition $S_x(0) = S_0$. This solution is used to get an estimate for the effective magnetic field B_x experienced by the electrons in the presence of nuclear spin polarization. As we showed in section 5.2, this field is given by $\mathbf{B}(y) = \eta \mathbf{S}(y)$. Inserting this field into the last term in equation (5.49) gives us a lowest-order correction,

$$\partial_t S_x = \tilde{D} \partial_y^2 S_x - \tilde{\gamma}_{\text{cd}} S_x + 4\tilde{D} \partial_y \left\{ (\tau \eta S_0)^2 e^{-y \sqrt{\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{D}}}} \partial_y S_x \right\}. \quad (5.51)$$

This differential equation can, in steady-state, be written as a hypergeometric differential equation,

$$z(1-z) \partial_y^2 S_x + \{c - (a+b+1)\} \partial_y S_x - ab S_x = 0, \quad (5.52)$$

with parameters $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$ and $z = -\frac{1}{(2\tau\eta S_0)^2} e^{2y \sqrt{\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{D}}}}$. The hypergeometric differential equation has six solutions. One of these solutions is finite and physical, i.e., it decays to zero for $y \rightarrow \infty$. Since this solution also is negative and imaginary, we multiply with i . This gives

$$S_x(y) = i z^{-b} {}_2F_1(b+1, b, b+1-a, z^{-1}), \quad (5.53)$$

where ${}_2F_1$ is the Gauss hypergeometric function. We normalize the spin density by requiring that $S_x(0) = S_0$, which yields

$$\begin{aligned} S_x(y) &= \frac{z^{-b} {}_2F_1(b+1, b, b+1-a, z^{-1})}{2\tau\eta {}_2F_1(b+1, b, b+1-a, -(2\tau\eta S_0)^2)} \\ &= S_0 e^{-y \sqrt{\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{D}}}} \frac{{}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 2, -(2\tau\eta S_0)^2 e^{-2y \sqrt{\frac{\tilde{\gamma}_{\text{cd}}}{\tilde{D}}}}\right)}{{}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 2, -(2\tau\eta S_0)^2\right)}. \end{aligned} \quad (5.54)$$

This gives the electron spin density for a PSH with hyperfine interactions and spin injection along the x -axis.

The spin density in equation (5.54) is plotted in figure 5.2 for three magnitudes of nuclear spin polarization $\tau\eta S_0 \in \{0, 0.19, 0.50\}$. The figure shows how \mathbf{H}_1 affects the electron spin density by reducing the spatial decay of spin polarization. We

see that this effect is largest for small y , which is where the magnitude of nuclear spin polarization is considerable. For large y the nuclear spin polarization vanishes, and the spin density converges towards $S_x^{(0)}$. From equation (5.54) we can see that increasing the spin polarization S_0 should give the same effect as increasing the hyperfine coupling η . Increasing the lifetime τ , however, does not have the same effect since this also alters the the cubic Dresselhaus interaction that is $\propto \tau$.

Another advantage with the x -component of the PSH, in addition to the simpler structure and the stabilizing effect of \mathbf{H}_1 , is the absence of coupling to the y and z components since $\mathbf{H}_3 = 0$. When we investigated the helix components of the PSH in section 5.3.1 we observed that the cubic Dresselhaus interaction combined with the nuclear spin polarization, i.e. \mathbf{H}_3 , rotated the spin polarization out of the yz -plane, into the x -component.

We investigate how \mathbf{H}_1 affects the spin density close to $y = 0$, since this is where the feedback effect is largest. Because of our boundary condition, $S_x \rightarrow S_x^{(0)}$ as $y \rightarrow 0$. Thus, to estimate an effect of \mathbf{H}_1 close to $y = 0$ we calculate the

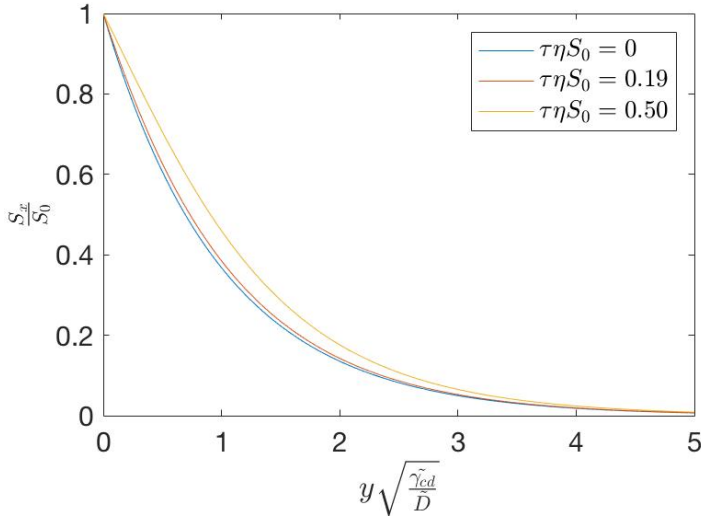


Figure 5.2: Feedback effect for the electron spin density in a setup with spin injection along the y -axis. The spin density is plotted for three different magnitudes of nuclear spin polarization, $\tau\eta s_0 \in \{0, 0.19, 0.50\}$. The plots show that the nuclear spin polarization slows the decay of electron spin polarization for small x , which is where the nuclear spin polarization is largest. The red graph shows the feedback effect's theoretical maximum, which is obtained with a 100 % polarization of nuclei at $y = 0$.

derivatives of S_x and $S_x^{(0)}$. The derivative of S_x at $y = 0$ reads

$$\partial_y S_x(0) = -S_0 \sqrt{\frac{\tilde{\gamma}_{cd}}{\tilde{D}}} \frac{{}_2F_1\left(\frac{3}{2}, \frac{3}{2}, 2, -(2\tau\eta S_0)^2\right)}{{}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, -(2\tau\eta S_0)^2\right)}, \quad (5.55)$$

which we expand in a Taylor series to second order in $2\tau\eta S_0$,

$$\partial_y S_x(0) \approx -S_0 \sqrt{\frac{\tilde{\gamma}_{cd}}{\tilde{D}}} \left(1 - \frac{3}{4}(2\tau\eta S_0)^2\right). \quad (5.56)$$

By comparing this to the derivative of $S_x^{(0)}$ at $y = 0$, reading $\partial_y S_x^{(0)}(0) = -S_0 \sqrt{\frac{\tilde{\gamma}_{cd}}{\tilde{D}}}$, we see that \mathbf{H}_1 effectively changes the electron spin density by increasing the diffusion length $\sqrt{\frac{\tilde{D}}{\tilde{\gamma}_{cd}}} \rightarrow \sqrt{\frac{\tilde{D}}{\tilde{\gamma}_{cd}}} (1 - 3(\tau\eta S_0)^2)^{-1}$. This has a stabilizing effect on the spin density, as can be seen from the expression for $\mathbf{S}^{(0)}$ in equation (5.50). To find an estimate for the theoretical maximum feedback effect, we assume a 100 % polarizaiton of nuclear spin at $y = 0$, i.e. $\eta S_0 = B_0 = 5$ T. This gives a diffusion length of $\sim 1.13 \sqrt{\frac{\tilde{D}}{\tilde{\gamma}_{cd}}}$, and $\tau\eta S_0 \sim 0.19$, which we plotted in figure 5.2. We can here see that the change in spin density caused by the hyperfine interaction is indeed small, which confirms our assumption from section 5.2

Chapter 6

Conclusion

Lüffe et al. [28] derived a diffusion equation for the persistent spin helix in the presence of an effective Hartree-Fock field, which arises from the mean polarization of the PSH itself, through electron-electron interactions. They predicted enhanced PSH lifetimes due to the Hartree-Fock field, but also an altering of the spin structure due to higher-order harmonics. Naïvely, one expects the Hartree-Fock field to have a similar structure to the effective field set up by a nuclear spin polarization that is created by the PSH, thus one could expect an enhanced PSH lifetime due to the hyperfine interactions. This is what we want to investigate. In PSH experiments, one has deliberately avoided the effects of hyperfine interaction by modulating the sign of the spin injection. Thus, one could easily investigate the hyperfine effects experimentally by simply allowing for nuclear spin build-up.

Following work done by Lüffe et al. [27] we rederived a semi-classical diffusion equation for a GaAs 2DEG with Rashba and Dresselhaus intrinsic spin-orbit coupling and electron-impurity scattering in chapter 2. Our starting point was a Boltzmann equation for the electron spin polarization in a 2DEG. We then expanded the momentum-dependent spin density in terms of winding numbers and orders of momentum to capture the relevant frequencies, and derived an effective spin diffusion equation. From the 2DEG diffusion equation, we obtained the PSH diffusion equation by investigating the limit of equal Rashba and Dresselhaus coupling strengths. We further investigated the fully $SU(2)$ symmetric PSH case, i.e. no spin relaxation, where we obtained the persistent spin helix state in the steady-state limit, as expected.

Nuclear spin pumping rates in semiconductor-based nanostructures has traditionally been calculated by balancing rate equations obtained from Fermi's golden rule. This procedure falls short for systems with complicated dynamics, such as spin-orbit coupling. We thus rederived a formalism presented by Danon et al. to calculate the nuclear spin pumping from non-equilibrium electron spin dynamics in chapter 3. Using this formalism, one only needs to calculate the electron spin susceptibilities and fluctuations in order to obtain the nuclear spin pumping up to second order in the hyperfine interactions.

In chapter 4 we calculated the nuclear spin pumping in a simpler, effectively

one-dimensional spin-diffusion system as a warm up to the calculation of the nuclear pumping in the PSH, and to show that the procedure actually works. We calculated the nuclear spin pumping caused by the non-equilibrium electron spin dynamics, including spin diffusion, relaxation, a constant point source of spin injection, and an external magnetic field in the z -direction. We found that in the absence of an external magnetic field, the nuclear spin pumping was proportional to the steady-state electron spin distribution $\mathbf{S}^{(0)}(x)$. In the presence of a magnetic field, the z -component remained the same, while for the in-plane components, the nuclear spin pumping precessed away from the steady-state solutions $S_x^{(0)}(x)$ and $S_y^{(0)}(x)$. Finally, we investigated the case of parallel electron and nuclear spin polarization, where we obtained a theoretical maximum order of magnitude for the nuclear spin polarization of $\sim 10\%$ in the limit of large electron spin injection.

The nuclear spin pumping in the PSH was calculated in chapter 5 and was, as expected, proportional to the electron spin polarization. By evaluating the nuclear spin pumping in equilibrium with the nuclear spin relaxation rate we obtained an expression for the effective magnetic field caused by the nuclear spin polarization. To then investigate the feedback effect between the electron and nuclear spin polarization, we added an additional precession term to the semi-classical Boltzmann equation from chapter 2. From this equation of motion, which included hyperfine interaction, we derived a new diffusion equation for the electron spin density by using a similar approach as in chapter 2.

We then investigated the effects of hyperfine interaction in the PSH by investigating two different PSH setups. We first looked at how an injected PSH was affected by the hyperfine interaction in the $\frac{\tilde{\gamma}_{\text{hd}}}{\Gamma} \ll 1$ limit. We found that the hyperfine interaction mainly increased the PSH lifetime, but also introduced higher-order harmonics and rotated the spin density out of the yz -plane. The finite x -component was a result of the coupling between the cubic Dresselhaus interaction and the nuclear spin polarization. The stabilizing effect of the PSH lifetime had a theoretical maximum at 15 %, for a nuclear polarization of 100 %. Secondly, we looked at the PSH in a setup with injection of electron spin along the x -axis. We found that the hyperfine interaction stabilized the PSH structure by increasing the effective electron diffusion length. The effect was largest close to the injection point, where the nuclear spin polarization was considerable. As an upper limit, we found that for a nuclear polarization of 100 % the diffusion length was increased by approximately 13 %.

The main contribution from the hyperfine interaction in the PSH results in an effective stabilization of the electron spin polarization, just as expected, and as observed by Lüffe et al. [28] for the Hartree-Fock field. This effect is considerable when 100 % of the nuclei are polarized. However, from our results in chapter 4 we know that the nuclear spin polarization arising due to the non-equilibrium electron spin dynamics is smaller than than 100 %. For a nuclear spin polarization of $\sim 10\%$, approximately what we found in chapter 4, the stabilizing effect is less than 0.15 %.

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Appendix A

Derivation of the spin diffusion equation

This appendix contains the full derivation of the electron spin diffusion equation derived in chapter 2, section 2.3. The derivation follows closely the derivation presented in my project report [1].

Throughout the derivation we encounter a lot of angular integrals. We state the most frequent angular integrals here:

$$\int_0^{2\pi} d\theta e^{\pm in\theta} = 0 \tag{A.1}$$

$$\int_0^{2\pi} d\theta \cos n\theta e^{\pm im\theta} = \pi \delta_{n,m} \tag{A.2}$$

$$\int_0^{2\pi} d\theta \sin n\theta e^{\pm im\theta} = \pm i\pi \delta_{n,m} \tag{A.3}$$

where $n, m \in \{1, 3\}$. This shows that only the terms in the spin-orbit field and spin density with equal or opposite winding number give a non-zero contribution.

To solve the integration over k , we change the integration variable to ϵ_k , due to the Fermi-Dirac distribution derivatives. To do so, we need the surface elements $d\epsilon_k$, which is obtained from the system's quadratic dispersion relation, giving $dk = \frac{m}{k} d\epsilon_k$. Since we are assuming zero temperature, the Fermi-Dirac distribution takes the form of a Heaviside function. Thus its derivative is a delta function, giving $f'(\epsilon_k) = -\frac{m}{k} \delta(k_F - k)$. This gives us a relation for the k integration,

$$\int_0^\infty d\epsilon_k k^n f'(\epsilon_k) = -k_F^n, \tag{A.4}$$

that we will use without explicit calculation, together with the surface element $d\epsilon_k$, during our derivation.

Finally, we repeat the expansion of the spin density

$$\mathbf{s}_{\mathbf{k}} = \sum_{n=0,\pm 1,\pm 3} \mathbf{s}_{\mathbf{k},n} e^{in\theta} \quad (\text{A.5})$$

with $\mathbf{s}_0 = -\frac{2\pi}{m} f'(\epsilon_k) \mathbf{S}$ being the isotropic spin component, and $\mathbf{s}_{\mathbf{k},n} = f'(\epsilon_k) \frac{k^n}{k_F^{n-1} m} \delta \mathbf{k}_n$ being the anisotropic spin components, and the new basis for the anisotropic components that we defined in section 2.3,

$$\delta \mathbf{k}_{c(c3)} = (\delta \mathbf{k}_{1(3)} + \delta \mathbf{k}_{-1(-3)}) \quad \delta \mathbf{k}_{s(s3)} = i(\delta \mathbf{k}_{1(3)} - \delta \mathbf{k}_{-1(-3)}). \quad (\text{A.6})$$

This choice of basis is based on the winding number expansion of the spin density in equation (2.20) and the θ integrals we encounter during the derivation.

A.1 Isotropic diffusion equation

To obtain the spin diffusion equation, we first derive a diffusion equation for the isotropic spin density. We do this by inserting the spin density in equation (2.20) into the semi-classical equation of motion (2.15) and integrating over \mathbf{k} , i.e.

$$\int d\mathbf{k} \{ \partial_t \mathbf{s}_{\mathbf{k}} + \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} \mathbf{J}_{\mathbf{k}}^{\text{imp}}. \quad (\text{A.7})$$

We will go through the derivation termwise, starting with the leftmost term.

Time-evolution For the leftmost term, including the time derivative of the spin density, only the isotropic spin component contributes due to the θ integration. This gives

$$\begin{aligned} \int d\mathbf{k} \partial_t \mathbf{s}_{\mathbf{k}} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k \partial_t \left(-\frac{2\pi}{m} f'(\epsilon_k) \mathbf{S} + \sum_{n=\pm 1, \pm 3} \mathbf{s}_{\mathbf{k},n} e^{in\theta} \right) \\ &= -\frac{1}{m} \int_0^\infty dk k f'(\epsilon_k) \partial_t \mathbf{S} \\ &= -\int_0^\infty d\epsilon_k f'(\epsilon_k) \partial_t \mathbf{S} \\ &= \partial_t \mathbf{S}. \end{aligned} \quad (\text{A.8})$$

This is the only term with time-evolution, thus we can easily obtain an isotropic diffusion equation by calculating the contributions from rest of the terms.

Spin transport The second term, responsible for the spin polarization currents, has an angular dependence through the velocity \mathbf{v} . Thus only the anisotropic

components with winding number ± 1 contribute, giving

$$\begin{aligned}
& \int d\mathbf{k} \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} \\
&= \frac{1}{(2\pi)^2} \int_0^\infty dk \int_0^{2\pi} d\theta k \frac{k}{m} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \left(-\frac{2\pi}{m} f'(\epsilon_k) \mathbf{S} + \sum_{n=\pm 1, \pm 3} \mathbf{s}_{\mathbf{k}, n} e^{in\theta} \right) \\
&= \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{m} \int_0^{2\pi} d\theta (\partial_x \cos \theta + \partial_y \sin \theta) f'(\epsilon_k) \frac{k}{m} \sum_{n=\pm 1} \delta \mathbf{k}_n e^{in\theta} \\
&= \frac{1}{4\pi} \int_0^\infty dk \frac{k^3}{m^2} f'(\epsilon_k) (\partial_x \delta \mathbf{k}_c + \partial_y \delta \mathbf{k}_s) \\
&= -\frac{k_F^2}{4\pi m} (\partial_x \delta \mathbf{k}_c + \partial_y \delta \mathbf{k}_s). \tag{A.9}
\end{aligned}$$

Spin-orbit coupling The contribution from spin-orbit interaction has an angular dependence through the spin-orbit fields, thus we only expect contributions from the anisotropic components. The spin-orbit field couples the in-plane components to the anisotropic z -component, and vice versa, as can be seen by writing out the cross product $\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k})$ for an in-plane \mathbf{k} ,

$$2 \int d\mathbf{k} \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) = \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k \begin{pmatrix} -s_{\mathbf{k}, z} b_y(\mathbf{k}) \\ s_{\mathbf{k}, z} b_x(\mathbf{k}) \\ s_{\mathbf{k}, x} b_y(\mathbf{k}) - s_{\mathbf{k}, y} b_x(\mathbf{k}) \end{pmatrix}, \tag{A.10}$$

where $\mathbf{b}(\mathbf{k})$ is the intrinsic spin-orbit field given in equation (2.4). The only terms in the spin-orbit field and spin density that couple are the ones with equal winding number. We will calculate the winding number ± 1 contributions first, then continue with the winding number ± 3 contributions.

The linear Dresselhaus and Rashba spin-orbit field couple to the winding number ± 1 terms of the spin density. By writing out the expression in (\dots) for the winding number ± 1 components we get

$$\frac{k^2}{m} f'(\epsilon_k) \sum_{n=\pm 1} \begin{pmatrix} -\delta k_{n, z} e^{in\theta} b_y(\mathbf{k}) \\ \delta k_{n, z} e^{in\theta} b_x(\mathbf{k}) \\ \delta k_{n, x} e^{in\theta} b_y(\mathbf{k}) - \delta k_{n, y} e^{in\theta} b_x(\mathbf{k}) \end{pmatrix}, \tag{A.11}$$

where $b_x(\mathbf{k}) = \sin \theta(\beta' + \alpha)$, and $b_y(\mathbf{k}) = \cos \theta(\beta' - \alpha)$. By performing the θ integration we obtain

$$\frac{\pi k^2}{m} f'(\epsilon_k) \begin{pmatrix} -\delta k_{c, z}(\beta' - \alpha) \\ \delta k_{s, z}(\beta' + \alpha) \\ \delta k_{c, x}(\beta' - \alpha) - \delta k_{s, y}(\beta' + \alpha) \end{pmatrix}. \tag{A.12}$$

Recall that the renormalized linear Dresselhaus coefficient $\beta' = \beta - \frac{\gamma k^2}{4}$ has a k -dependence. Thus when including the prefactor $\frac{2}{(2\pi)^2}$ and performing the integ-

ration $\int_0^\infty dk k$ (equation (A.12)), we get

$$-\frac{k_F^2}{2\pi} \begin{pmatrix} -\delta k_{c,z}(\beta'' - \alpha) \\ \delta k_{s,z}(\beta'' + \alpha) \\ \delta k_{c,x}(\beta'' - \alpha) - \delta k_{s,y}(\beta'' + \alpha) \end{pmatrix}, \quad (\text{A.13})$$

where $\beta'' = \beta - \frac{\gamma k_F^2}{4}$. This gives the contribution to the spin diffusion equation from the linear Dresselhaus and Rashba spin-orbit interaction

$$-\frac{\alpha k_F^2}{2\pi} \begin{pmatrix} \delta k_{c,z} \\ \delta k_{s,z} \\ -\delta k_{c,x} - \delta k_{s,y} \end{pmatrix}, \quad (\text{A.14})$$

$$-\frac{\beta'' k_F^2}{2\pi} \begin{pmatrix} -\delta k_{c,z} \\ \delta k_{s,z} \\ \delta k_{c,x} - \delta k_{s,y} \end{pmatrix}, \quad (\text{A.15})$$

respectively.

The only contributions from the spin-orbit field that couple to the winding number ± 3 term of the spin density are the cubic Dresselhaus terms. We follow the same approach as for the winding number ± 1 contributions, and obtain an expression for the expression in (\dots) in equation (A.10) for the cubic Dresselhaus term,

$$\frac{\gamma k^6}{4k_F^2 m} f'(\epsilon_k) \sum_{n=\pm 3} \begin{pmatrix} \delta k_{n,z} e^{in\theta} \cos 3\theta \\ \delta k_{n,z} e^{in\theta} \sin 3\theta \\ -\delta k_{n,x} e^{in\theta} \cos 3\theta - \delta k_{n,y} e^{in\theta} \sin 3\theta \end{pmatrix}. \quad (\text{A.16})$$

By integrating over θ , we get

$$\frac{\gamma \pi k^6}{4k_F^2 m} f'(\epsilon_k) \begin{pmatrix} \delta k_{c3,z} \\ \delta k_{s3,z} \\ -\delta k_{c3,x} - \delta k_{s3,y} \end{pmatrix}. \quad (\text{A.17})$$

We now include the prefactor $\frac{2}{(2\pi)^2}$ and compute the integration $\int_0^\infty dk k$ (equation (A.17)), giving the cubic Dresselhaus terms' contribution to the spin diffusion equation,

$$-\frac{\gamma k_F^4}{8\pi} \begin{pmatrix} \delta k_{c3,z} \\ \delta k_{s3,z} \\ -\delta k_{c3,x} - \delta k_{s3,y} \end{pmatrix}. \quad (\text{A.18})$$

Collecting the terms for the Rashba and the linear and cubic Dresselhaus contribution we get the total contribution from the intrinsic spin-orbit coupling, reading

$$2 \int d\mathbf{k} \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) = -\frac{\alpha k_F^2}{2\pi} \begin{pmatrix} \delta k_{c,z} \\ \delta k_{s,z} \\ -\delta k_{c,x} - \delta k_{s,y} \end{pmatrix} - \frac{\beta'' k_F^2}{2\pi} \begin{pmatrix} -\delta k_{c,z} \\ \delta k_{s,z} \\ \delta k_{c,x} - \delta k_{s,y} \end{pmatrix} \\ - \frac{\gamma k_F^4}{8\pi} \begin{pmatrix} \delta k_{c3,z} \\ \delta k_{s3,z} \\ -\delta k_{c3,x} - \delta k_{s3,y} \end{pmatrix}. \quad (\text{A.19})$$

Electron-impurity interaction For the electron-impurity interactions we start from equation (2.19), and perform the \mathbf{k}' sum in the continuous limit, i.e. $\sum_{\mathbf{k}'} \rightarrow \frac{1}{(2\pi)^2} \int d\mathbf{k}' = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta' \int_0^\infty dk' k'$. We start with the the left-most term, containing the $\mathbf{s}_{\mathbf{k}} - \mathbf{s}_{\mathbf{k}'}$ spin dependence. Following the same argument as for the electron-electron interactions, i.e. no angular dependence, and $\mathbf{s}_{\mathbf{k}}$ and $\mathbf{s}_{\mathbf{k}'}$ having the same isotropic density \mathcal{S} , results in zero contribution.

The second term contains the cross product $[\mathbf{k} \times \mathbf{k}']_z = k_x k'_y - k_y k'_x = k k' (\cos \theta \sin \theta' - \sin \theta \cos \theta')$. This cross product is the term's only contribution to θ , which, when integrated over a period, is zero.

The cross product in the third term reads $[\mathbf{k} \times \mathbf{k}']_z^2 = k^2 k'^2 (\cos^2 \theta \sin^2 \theta' + \sin^2 \theta \cos^2 \theta' - 2 \cos \theta \sin \theta \cos \theta' \sin \theta')$. The anisotropic contribution is zero due to θ integration of $\cos^2 \theta e^{in\theta}$, $\sin^2 \theta e^{in\theta}$, and $\cos \theta \sin \theta e^{in\theta}$, where $n \in \{1, 3\}$ and $\theta \in \{\theta, \theta'\}$. Thus, the electron-impurity interaction's only contribution is

$$\mathbf{J}_{\mathbf{k}}^{\text{imp}} = \frac{\lambda_0^4}{2(4\pi)^3 m} \int d\mathbf{k} d\mathbf{k}' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} S_x(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_y(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_z(f'(\epsilon_{\mathbf{k}}) - f'(\epsilon_{\mathbf{k}'})) \end{pmatrix}, \quad (\text{A.20})$$

where $W_{\mathbf{k}\mathbf{k}'} = 2\pi n_i |v(\mathbf{k}' - \mathbf{k})|^2$, with $v(\mathbf{k}' - \mathbf{k})$ and n_i being the impurity potential and impurity concentration respectively. The only θ and θ' dependence is in the cross product $[\mathbf{k} \times \mathbf{k}']_z^2$ and the impurity potential $|v(\mathbf{k}' - \mathbf{k})|^2$. The integrals average out the contribution from the impurity potential to $|v(0)|^2$, and the integrals over the cross product read

$$\int_0^{2\pi} d\theta d\theta' (\cos^2 \theta \sin^2 \theta' + \sin^2 \theta \cos^2 \theta' + 2 \cos \theta \sin \theta \cos \theta' \sin \theta') = 2\pi^2. \quad (\text{A.21})$$

We then perform the integration over k and k' :

$$\begin{aligned} \mathbf{J}_{\mathbf{k}}^{\text{imp}} &= \frac{\lambda_0^4 n_i}{32m} |v(0)|^2 \int_0^\infty dk dk' k^3 k'^3 \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \begin{pmatrix} S_x(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_y(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_z(f'(\epsilon_{\mathbf{k}}) - f'(\epsilon_{\mathbf{k}'})) \end{pmatrix} \\ &= \frac{\lambda_0^4 m n_i}{32} |v(0)|^2 \int_0^\infty d\epsilon_{\mathbf{k}} d\epsilon_{\mathbf{k}'} k^2 k'^2 \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \begin{pmatrix} S_x(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_y(f'(\epsilon_{\mathbf{k}}) + f'(\epsilon_{\mathbf{k}'})) \\ S_z(f'(\epsilon_{\mathbf{k}}) - f'(\epsilon_{\mathbf{k}'})) \end{pmatrix} \\ &= \frac{\lambda_0^4 m n_i}{16} |v(0)|^2 \int_0^\infty d\epsilon_{\mathbf{k}} k^4 f'(\epsilon_{\mathbf{k}}) \begin{pmatrix} S_x \\ S_y \\ 0 \end{pmatrix} \\ &= - \left(\frac{\lambda_0 k_F}{2} \right)^4 \tau^{-1} \begin{pmatrix} S_x \\ S_y \\ 0 \end{pmatrix} \\ &= -\gamma_{ey} \begin{pmatrix} S_x \\ S_y \\ 0 \end{pmatrix} \end{aligned} \quad (\text{A.22})$$

where γ_{ey} is the Elliott-Yafet relaxation rate where the electron spin precess a small angle around the extrinsic spin-orbit field during impurity scattering, and is given as

$$\gamma_{ey} = \left(\frac{\lambda_0 k_F}{2} \right)^4 \tau^{-1} \quad (\text{A.23})$$

where $\tau^{-1} = mn_i |v(0)|^2$ is the inverse lifetime of the electron spin due to isotropic scattering off a weak impurity potential $v(q)$ in the Born-approximation.

A.1.1 Isotropic diffusion equation

By collecting the terms for the isotropic diffusion equation, i.e. equations (A.8), (A.9), (A.19) and (A.22), we obtain

$$\begin{aligned} \partial_t \mathbf{S} = & \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta \mathbf{k}_c + \partial_y \delta \mathbf{k}_s) + \alpha \begin{pmatrix} \delta k_{c,z} \\ \delta k_{s,z} \\ -\delta k_{c,x} - \delta k_{s,y} \end{pmatrix} \right. \\ & + \left(\beta - \frac{\gamma k_F^2}{4} \right) \begin{pmatrix} -\delta k_{c,z} \\ \delta k_{s,z} \\ \delta k_{c,x} - \delta k_{s,y} \end{pmatrix} \\ & \left. + \frac{\gamma k_F^2}{4} \begin{pmatrix} \delta k_{c3,z} \\ \delta k_{s3,z} \\ -\delta k_{c3,x} - \delta k_{s3,y} \end{pmatrix} \right\} - \gamma_{ey} \begin{pmatrix} S_x \\ S_y \\ 0 \end{pmatrix}. \quad (\text{A.24}) \end{aligned}$$

We collect the isotropic components in a more compact notation to get

$$\partial_t S_x = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,x} + \partial_y \delta k_{s,x}) + \alpha \delta k_{c,z} - \beta \delta \bar{k}_{c,z} \right\} - \gamma_{ey} S_x \quad (\text{A.25})$$

$$\partial_t S_y = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,y} + \partial_y \delta k_{s,y}) + \alpha \delta k_{s,z} + \beta \delta \bar{k}_{s,z} \right\} - \gamma_{ey} S_y \quad (\text{A.26})$$

$$\partial_t S_z = \frac{k_F^2}{2\pi} \left\{ \frac{1}{2m} (\partial_x \delta k_{c,z} + \partial_y \delta k_{s,z}) - \alpha (\delta k_{c,x} + \delta k_{s,y}) + \beta (\delta \bar{k}_{c,x} - \delta \bar{k}_{s,y}) \right\} \quad (\text{A.27})$$

with $\delta \bar{k}_{c(s)} = \delta \mathbf{k}_{c(s)} - \frac{\gamma k_F^2}{4\beta} (\delta \mathbf{k}_{c(s)} + \delta \mathbf{k}_{c3(s3)})$ and $\delta \bar{k}_{c(s)} = \delta \mathbf{k}_{c(s)} - \frac{\gamma k_F^2}{4\beta} (\delta \mathbf{k}_{c(s)} - \delta \mathbf{k}_{c3(s3)})$. We use this isotropic spin diffusion equation to obtain a spin diffusion equation for the spin-orbit system we are evaluating, by finding steady-state expressions for the anisotropic components appearing in it.

A.2 Winding number ± 1 components

To find the anisotropic components of winding number ± 1 , we multiply the semi-classical diffusion equation (2.15) with $\cos \theta$ and $\sin \theta$ and integrate over \mathbf{k} , i.e.

$$\int d\mathbf{k} \gamma(\theta) \{ \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2 \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} \xi(\theta) \mathbf{J}_{\mathbf{k}}^{\text{imp}} \quad (\text{A.28})$$

were $\xi(\theta) \in \{\cos \theta, \sin \theta\}$. We go through the derivation using the same procedure as with the isotropic diffusion equation, i.e. we start with the leftmost term.

Spin transport The first term we investigate is responsible for the spin polarization currents. Here, only the isotropic component of the spin density contributes due to the integration over θ . This gives

$$\begin{aligned} \int d\mathbf{k} \cos\theta \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} &= -\frac{2\pi}{(2\pi)^2 m} \int_0^{2\pi} d\theta \int_0^\infty dk k^2 (\partial_x \cos^2 \theta + \partial_y \cos \theta \sin \theta) f'(\epsilon_k) \mathbf{S} \\ &= -\frac{1}{2m} \int_0^\infty dk k^2 f'(\epsilon_k) \partial_x \mathbf{S} \\ &= \frac{k_F}{2m} \partial_x \mathbf{S} \end{aligned} \quad (\text{A.29})$$

for $\gamma(\theta) = \cos \theta$ and, as can be seen from the expression above,

$$\int d\mathbf{k} \sin\theta \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} = \frac{k_F}{2m} \partial_y \mathbf{S} \quad (\text{A.30})$$

for $\gamma(\theta) = \sin \theta$.

Spin-orbit coupling The second term is responsible for the precession due to the spin-orbit coupling. We here expect, as for the isotropic diffusion equation, that the in-plane components couple to the z -component of the spin density, and vice versa. When calculating the contribution for $\gamma(\theta) = \cos \theta$, we get

$$\begin{aligned} &2 \int d\mathbf{k} \cos\theta \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \\ &= \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k \cos\theta \begin{pmatrix} -s_{\mathbf{k},z} b_y(\mathbf{k}) \\ s_{\mathbf{k},z} b_x(\mathbf{k}) \\ s_{\mathbf{k},x} b_y(\mathbf{k}) - s_{\mathbf{k},y} b_x(\mathbf{k}) \end{pmatrix} \\ &= \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k^2 \cos^2 \theta \begin{pmatrix} s_{\mathbf{k},z} (\alpha - \beta') \\ 0 \\ -s_{\mathbf{k},x} (\alpha - \beta') \end{pmatrix} \\ &= -\frac{1}{m} \int_0^\infty dk k^2 f'(\epsilon_k) (\alpha - \beta') \begin{pmatrix} S_z \\ 0 \\ -S_x \end{pmatrix} \\ &= k_F (\alpha - \beta'') \begin{pmatrix} S_z \\ 0 \\ -S_x \end{pmatrix}, \end{aligned} \quad (\text{A.31})$$

and for $\gamma(\theta) = \sin \theta$ we get

$$\begin{aligned}
& 2 \int d\mathbf{k} \sin \theta \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \\
&= \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k \sin \theta \begin{pmatrix} -s_{\mathbf{k},z} b_y(\mathbf{k}) \\ s_{\mathbf{k},z} b_x(\mathbf{k}) \\ s_{\mathbf{k},x} b_y(\mathbf{k}) - s_{\mathbf{k},y} b_x(\mathbf{k}) \end{pmatrix} \\
&= \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k^2 \sin^2 \theta \begin{pmatrix} 0 \\ s_{\mathbf{k},z}(\alpha + \beta') \\ -s_{\mathbf{k},y}(\alpha + \beta') \end{pmatrix} \\
&= -\frac{1}{m} \int_0^\infty dk k^2 f'(\epsilon_k)(\alpha + \beta') \begin{pmatrix} 0 \\ S_z \\ -S_y \end{pmatrix} \\
&= k_F(\alpha + \beta') \begin{pmatrix} 0 \\ S_z \\ -S_y \end{pmatrix}. \tag{A.32}
\end{aligned}$$

Electron-impurity interaction The electron-impurity interaction gives three contributions to the winding number ± 1 anisotropic components. We calculate them termwise, starting with the leftmost term. This term contains the first-order interaction between the electrons and the impurity potential, and gives a contribution

$$\begin{aligned}
& - \int d\mathbf{k} \int d\mathbf{k}' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) (\mathbf{s}_{\mathbf{k}} - \mathbf{s}_{\mathbf{k}'}) \\
&= -\frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' k k' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) (\mathbf{s}_{\mathbf{k}} - \mathbf{s}_{\mathbf{k}'}) \\
&= -\frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) f'(\epsilon_k) \cos \theta \frac{k^2 k'}{m} \sum_{n=\pm 1} \delta \mathbf{k}_n e^{in\theta} \\
&= -\frac{\pi}{(2\pi)^3} \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) f'(\epsilon_k) \frac{k^2 k'}{m} \delta \mathbf{k}_c \\
&= \frac{1}{4\pi} m n_i |v(0)|^2 k_F \delta \mathbf{k}_c \\
&= \frac{k_F}{4\pi\tau} \delta \mathbf{k}_c, \tag{A.33}
\end{aligned}$$

for $\xi(\theta) = \cos \theta$. For $\xi(\theta) = \sin \theta$ we get

$$- \int d\mathbf{k} \int d\mathbf{k}' \sin \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) (\mathbf{s}_{\mathbf{k}} - \mathbf{s}_{\mathbf{k}'}) = \frac{k_F}{4\pi\tau} \delta \mathbf{k}_s, \tag{A.34}$$

as can be seen by replacing $\cos \theta$ with $\sin \theta$ above.

The second contribution from the electron-impurity term gives rise to precession due to extrinsic spin-orbit coupling with the impurity potential. This effect is

similar to the Elliott-Yafet relaxation, but in lower orders of λ_0 . The $\xi(\theta) = \cos \theta$ contribution reads

$$\begin{aligned}
& - \int d\mathbf{k} \int d\mathbf{k}' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^2}{2} [\mathbf{k} \times \mathbf{k}']_z \begin{pmatrix} -s_{\mathbf{k}',y} \\ s_{\mathbf{k}',x} \\ 0 \end{pmatrix} \\
&= - \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' k k' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^2}{2} [\mathbf{k} \times \mathbf{k}']_z \begin{pmatrix} -s_{\mathbf{k}',y} \\ s_{\mathbf{k}',x} \\ 0 \end{pmatrix} \\
&= - \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^2}{2} \cos^2 \theta \sin \theta' \frac{k^2 k'^3}{m} \sum_{n=\pm 1} \begin{pmatrix} -\delta k_{n,y} \\ \delta k_{n,x} \\ 0 \end{pmatrix} e^{in\theta'} \\
&= - \frac{\pi^2}{(2\pi)^4} \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^2}{2} \frac{k^2 k'^3}{m} \begin{pmatrix} -\delta k_{s,y} \\ \delta k_{s,x} \\ 0 \end{pmatrix} \\
&= \frac{\lambda_0^2}{16\pi} m n_i |v(0)|^2 k_F^3 \begin{pmatrix} -\delta k_{s,y} \\ \delta k_{s,x} \\ 0 \end{pmatrix} \\
&= \frac{\gamma_{sw} k_F}{4\pi} \begin{pmatrix} -\delta k_{s,y} \\ \delta k_{s,x} \\ 0 \end{pmatrix}, \tag{A.35}
\end{aligned}$$

where $\gamma_{sw} = \left(\frac{\lambda_0 k_F}{2}\right)^2 \frac{1}{\tau}$ is the "swapping" rate of the spin currents, where a gradient in one of the in-plane spin components generates a current in the other in-plane spin component. Performing the same calculation for $\xi(\theta) = \sin \theta$, we get

$$- \int d\mathbf{k} \int d\mathbf{k}' \sin \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^2}{2} [\mathbf{k} \times \mathbf{k}']_z \begin{pmatrix} -s_{\mathbf{k}',y} \\ s_{\mathbf{k}',x} \\ 0 \end{pmatrix} = - \frac{\gamma_{sw} k_F}{4\pi} \begin{pmatrix} -\delta k_{c,y} \\ \delta k_{c,x} \\ 0 \end{pmatrix}. \tag{A.36}$$

The third part of the electron-impurity term gives rise to the Elliott-Yafet relaxation, as observed for the isotropic diffusion equation. For $\xi(\theta) = \cos \theta$ we get

$$\begin{aligned}
& - \int d\mathbf{k} \int d\mathbf{k}' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} s_{\mathbf{k},x} + s_{\mathbf{k}',x} \\ s_{\mathbf{k},y} + s_{\mathbf{k}',y} \\ s_{\mathbf{k},z} - s_{\mathbf{k}',z} \end{pmatrix} \\
&= - \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' k k' \cos \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} s_{\mathbf{k},x} + s_{\mathbf{k}',x} \\ s_{\mathbf{k},y} + s_{\mathbf{k}',y} \\ s_{\mathbf{k},z} - s_{\mathbf{k}',z} \end{pmatrix}, \tag{A.37}
\end{aligned}$$

where the angular integral yields

$$\begin{aligned} & \int_0^{2\pi} d\theta d\theta' (\cos^3 \theta \sin^2 \theta' + \cos \theta \sin^2 \theta \cos^2 \theta' + 2 \cos^2 \theta \sin \theta \cos \theta' \sin \theta') (s_{\mathbf{k},a} \pm s_{\mathbf{k}',a}) \\ &= f'(\epsilon_k) \frac{\pi^2}{m} k \delta k_{c,a}, \end{aligned} \quad (\text{A.38})$$

where $a \in \{x, y, z\}$. By inserting the angular integral into equation (A.37) we obtain

$$\begin{aligned} & -\frac{\pi^2}{(2\pi)^4 m} \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} f'(\epsilon_k) k^4 k'^3 \delta \mathbf{k}_c \\ &= \frac{1}{8\pi} m n_i |v(0)|^2 k_F^5 \frac{\lambda_0^4}{16} \delta \mathbf{k}_c \\ &= \frac{\gamma_{ey} k_F}{8\pi} \delta \mathbf{k}_c. \end{aligned} \quad (\text{A.39})$$

Similar, for $\xi(\theta) = \sin \theta$ we get

$$-\int d\mathbf{k} \int d\mathbf{k}' \sin \theta W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} s_{\mathbf{k},x} + s_{\mathbf{k}',x} \\ s_{\mathbf{k},y} + s_{\mathbf{k}',y} \\ s_{\mathbf{k},z} - s_{\mathbf{k}',z} \end{pmatrix} = \frac{\gamma_{ey} k_F}{8\pi} \delta \mathbf{k}_s. \quad (\text{A.40})$$

We point out that this is, by a factor $\frac{1}{2}$, different from Lüffe's derivation [27]. However, this is not a big deal, as we will later see, since the Elliott-Yafet relaxation only contributes to the effective lifetime of the anisotropic components.

A.2.1 Anisotropic components of winding number ± 1

We now collect the terms calculated for the winding number ± 1 anisotropic components of the spin density. This gives us two coupled equations, one for $\xi(\theta) = \cos \theta$, and one for $\xi(\theta) = \sin \theta$. The coupling arise in the electron-impurity interaction term responsible for the spin "swapping". By solving the two coupled equations for the anisotropic components we get

$$\delta k_{c,x} = -\frac{2\pi\tau_1 (S_x \partial_x + \gamma_{sw} \tau_1 S_y \partial_y) + 4\pi\tau_1 m S_z \{\alpha(1 + \gamma_{sw} \tau_1) - \beta''(1 - \gamma_{sw} \tau_1)\}}{m(\gamma_{sw}^2 \tau_1^2 - 1)} \quad (\text{A.41})$$

$$\delta k_{c,y} = -\frac{2\pi\tau_1 (S_y \partial_x - \gamma_{sw} \tau_1 S_x \partial_y)}{m(\gamma_{sw}^2 \tau_1^2 - 1)} \quad (\text{A.42})$$

$$\delta k_{c,z} = \frac{2\pi\tau_1}{m} S_z \partial_x - 4\pi\tau_1 S_x (\alpha - \beta'') \quad (\text{A.43})$$

$$\delta k_{s,x} = -\frac{2\pi\tau_1 (S_x \partial_y - \gamma_{sw} \tau_1 S_y \partial_x)}{m(\gamma_{sw}^2 \tau_1^2 - 1)} \quad (\text{A.44})$$

$$\delta k_{s,y} = -\frac{2\pi\tau_1 (S_y \partial_y + \gamma_{sw} \tau_1 S_x \partial_x) + 4\pi\tau_1 m S_z \{\alpha(1 + \gamma_{sw} \tau_1) + \beta''(1 - \gamma_{sw} \tau_1)\}}{m(\gamma_{sw}^2 \tau_1^2 - 1)} \quad (\text{A.45})$$

$$\delta k_{s,z} = \frac{2\pi\tau_1}{m} S_z \partial_y - 4\pi\tau_1 S_y (\alpha + \beta''), \quad (\text{A.46})$$

where $\tau_1 = \left(\frac{1}{\tau} + \frac{\gamma_{ey}}{2} + \frac{1}{\tau_{e-e,1}} \right)^{-1}$ is the effective relaxation time of the winding number ± 1 anisotropic spin components. Because of the short relaxation time τ_1 , we assume that $(\gamma_{sw}\tau_1)^2 \ll 1$. Thus we substitute the denominator $(\gamma_{sw}^2\tau_1^2 - 1) \rightarrow -1$, and obtain

$$\delta k_{c,x} = \frac{2\pi\tau_1}{m} (S_x \partial_x + \gamma_{sw}\tau_1 S_y \partial_y) + 4\pi\tau_1 S_z \{ \alpha (1 + \gamma_{sw}\tau_1) - \beta'' (1 - \gamma_{sw}\tau_1) \} \quad (\text{A.47})$$

$$\delta k_{c,y} = \frac{2\pi\tau_1}{m} (S_y \partial_x - \gamma_{sw}\tau_1 S_x \partial_y) \quad (\text{A.48})$$

$$\delta k_{c,z} = \frac{2\pi\tau_1}{m} S_z \partial_x - 4\pi\tau_1 S_x (\alpha - \beta'') \quad (\text{A.49})$$

$$\delta k_{s,x} = \frac{2\pi\tau_1}{m} (S_x \partial_y - \gamma_{sw}\tau_1 S_y \partial_x) \quad (\text{A.50})$$

$$\delta k_{s,y} = \frac{2\pi\tau_1}{m} (S_y \partial_y + \gamma_{sw}\tau_1 S_x \partial_x) + 4\pi\tau_1 S_z \{ \alpha (1 + \gamma_{sw}\tau_1) + \beta'' (1 - \gamma_{sw}\tau_1) \} \quad (\text{A.51})$$

$$\delta k_{s,z} = \frac{2\pi\tau_1}{m} S_z \partial_y - 4\pi\tau_1 S_y (\alpha + \beta''). \quad (\text{A.52})$$

These equations show the anisotropic spin components of winding number ± 1 as functions of the isotropic spin density \mathbf{S} , and will be used to obtain the spin diffusion equation.

A.3 Winding number ± 3 components

To obtain expressions for the anisotropic spin components of winding number ± 3 , we multiply the semi-classical diffusion equation (2.15) with $e^{\pm i3\theta}$ and integrate over \mathbf{k} , i.e.

$$\int d\mathbf{k} e^{\pm i3\theta} \{ \mathbf{v} \cdot \partial \mathbf{s}_{\mathbf{k}} + 2\mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \} = \int d\mathbf{k} e^{\pm i3\theta} \mathbf{J}_{\mathbf{k}}^{\text{imp}}. \quad (\text{A.53})$$

Spin transport The leftmost term is zero because $\mathbf{v} \cdot \partial$ only couples to the winding number ± 1 components of the spin density. This can be seen from the θ integration, which reads $\int d\theta \cos \theta e^{\pm i3\theta} = 0$, $\int d\theta \sin \theta e^{\pm i3\theta} = 0$.

Spin-orbit coupling The spin-orbit interaction term reads

$$\begin{aligned}
& 2 \int d\mathbf{k} e^{\pm i3\theta} \mathbf{s}_{\mathbf{k}} \times \mathbf{b}(\mathbf{k}) \\
&= \frac{2}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^\infty dk k e^{\pm i3\theta} \begin{pmatrix} -s_{\mathbf{k},z} b_y(\mathbf{k}) \\ s_{\mathbf{k},z} b_x(\mathbf{k}) \\ s_{\mathbf{k},x} b_y(\mathbf{k}) - s_{\mathbf{k},y} b_x(\mathbf{k}) \end{pmatrix} \\
&= \int_0^{2\pi} d\theta \int_0^\infty dk \frac{\gamma k^4}{4\pi m} f'(\epsilon_k) e^{\pm i3\theta} \begin{pmatrix} -S_z \cos 3\theta \\ -S_z \sin 3\theta \\ S_x \cos 3\theta + S_y \sin 3\theta \end{pmatrix} \\
&= \int_0^\infty dk \frac{\gamma k^4}{4m} f'(\epsilon_k) \begin{pmatrix} -S_z \\ \mp i S_z \\ S_x \pm i S_y \end{pmatrix} \\
&= \frac{\gamma k_F^3}{4} \begin{pmatrix} S_z \\ \pm i S_z \\ -S_x \mp i S_y \end{pmatrix}. \tag{A.54}
\end{aligned}$$

This gives, as for the isotropic and winding number ± 1 anisotropic components, a coupling between the in-plane components and the z -component of the spin density.

Electron-impurity interaction For the electron-impurity interaction, the second term does not contribute because of the θ integration of $e^{\pm i3\theta}$. For the first and third term, only the winding number ± 3 component of the spin density contributes, which gives

$$\begin{aligned}
& \int d\mathbf{k} \int d\mathbf{k}' e^{\pm i3\theta} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) (\mathbf{s}_{\mathbf{k}'} - \mathbf{s}_{\mathbf{k}}) \\
&= \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' k k' e^{\pm i3\theta} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) (\mathbf{s}_{\mathbf{k}'} - \mathbf{s}_{\mathbf{k}}) \\
&= -\frac{1}{(2\pi)^2} \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) f'(\epsilon_k) \frac{k^4 k'}{k_F^2 m} \delta\mathbf{k}_{\mp 3} \\
&= \frac{1}{2\pi} m n_i |v(0)|^2 k_F \delta\mathbf{k}_{\mp 3} \\
&= \frac{k_F}{2\pi\tau} \delta\mathbf{k}_{\mp 3}, \tag{A.55}
\end{aligned}$$

for the first term, and

$$\begin{aligned}
& - \int d\mathbf{k}' e^{\pm i3\theta} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} s_{\mathbf{k},x} + s_{\mathbf{k}',x} \\ s_{\mathbf{k},y} + s_{\mathbf{k}',y} \\ s_{\mathbf{k},z} - s_{\mathbf{k}',z} \end{pmatrix} \\
& = - \frac{1}{(2\pi)^4} \int_0^{2\pi} d\theta d\theta' \int_0^\infty dk dk' k k' e^{\pm i3\theta} W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} [\mathbf{k} \times \mathbf{k}']_z^2 \begin{pmatrix} s_{\mathbf{k},x} + s_{\mathbf{k}',x} \\ s_{\mathbf{k},y} + s_{\mathbf{k}',y} \\ s_{\mathbf{k},z} - s_{\mathbf{k}',z} \end{pmatrix} \\
& = - \frac{2\pi^2}{(2\pi)^4} \int_0^\infty dk dk' W_{\mathbf{k}\mathbf{k}'} \delta(\epsilon_k - \epsilon_{k'}) \frac{\lambda_0^4}{16} f'(\epsilon_k) \frac{k^6 k'^3}{k_F^2 m} \delta\mathbf{k}_{\mp 3} \\
& = \frac{1}{4\pi} m n_i |v(0)|^2 \frac{\lambda_0^4}{16} k_F^5 \delta\mathbf{k}_{\mp 3} \\
& = \frac{\gamma e y k_F}{4\pi} \delta\mathbf{k}_{\mp 3}. \tag{A.56}
\end{aligned}$$

for the third term. We point out, as for the winding number ± 1 anisotropic components, that this is different by a factor $\frac{1}{2}$ from Ref. [27].

A.3.1 Anisotropic components of winding number ± 3

By collecting the terms for the winding number ± 3 anisotropic spin components in equation (A.54)-(A.56) we obtain

$$\delta\mathbf{k}_{\mp 3} = \frac{\gamma\pi\tau_3 k_F^2}{2} \begin{pmatrix} S_z \\ \pm i S_z \\ -S_x \mp i S_y \end{pmatrix}, \tag{A.57}$$

where $\tau_3 = \left(\frac{1}{\tau} + \frac{\gamma e y}{2} + \frac{1}{\tau_{e-e,3}}\right)^{-1}$ is the effective relaxation time of the winding number ± 3 spin components. We want to express the winding number ± 3 components as they appear in the diffusion equation for the isotropic component of the spin density. This can be obtained by using the basis we defined above:

$$\delta\mathbf{k}_{c3} = (\delta\mathbf{k}_3 + \delta\mathbf{k}_{-3}) \quad \delta\mathbf{k}_{s3} = i(\delta\mathbf{k}_3 - \delta\mathbf{k}_{-3}), \tag{A.58}$$

which gives

$$\delta\mathbf{k}_{c3} = \gamma\pi\tau_3 k_F^2 \begin{pmatrix} S_z \\ 0 \\ -S_x \end{pmatrix} \tag{A.59}$$

$$\delta\mathbf{k}_{s3} = \gamma\pi\tau_3 k_F^2 \begin{pmatrix} 0 \\ S_z \\ -S_y \end{pmatrix}. \tag{A.60}$$

Appendix B

Susceptibilities

We here show that the local correlation functions χ_{nn}^{ab} are identical to the local magnetic susceptibilities to first order in the coupling Hamiltonian $\hat{H}^{(1)} = \int d\mathbf{r} B^a(\mathbf{r}) \hat{S}^a(\mathbf{r}, t)$ that couples the electron spin to an extrinsic magnetic field. Our starting point will be the time-evolution equation for the density matrix,

$$\frac{d\hat{\rho}_I(t)}{dt} = -i[\hat{H}^{(1)}(\mathbf{r}, t), \hat{\rho}_I(t)]_- \quad (\text{B.1})$$

By integrating from t_0 to t we find that the difference between the perturbed density matrix $\hat{\rho}^{(1)}(t)$ and the unperturbed density matrix $\hat{\rho}^{(0)}$ is given by

$$\delta\hat{\rho}(t) = -i \int^{t_0} dt' [\hat{H}^{(1)}(\mathbf{r}, t'), \hat{\rho}^{(0)}]_- \quad (\text{B.2})$$

The electron spin density is defined as $\hat{S}^a(\mathbf{r}, t) = \sum_n \hat{S}_n^a \delta(\mathbf{r} - \mathbf{r}_n)$. The change in the expectation value of the spin density caused by the perturbation is thus determined by the change in the density matrix alone, such that

$$\begin{aligned} \delta \langle \hat{S}^a(\mathbf{r}, t) \rangle &= \text{Tr} \left\{ \hat{S}^a(\mathbf{r}, t) \delta\hat{\rho}(t) \right\} \\ &= -i \int^{t_0} dt' \text{Tr} \left\{ \hat{S}^a(\mathbf{r}, t) [\hat{H}^{(1)}(\mathbf{r}, t'), \hat{\rho}^{(0)}]_- \right\} \\ &= -i \int^{t_0} dt' \int d\mathbf{r}' B^b(\mathbf{r}') \text{Tr} \left\{ \hat{S}^a(\mathbf{r}, t) [\hat{S}^b(\mathbf{r}', t'), \hat{\rho}^{(0)}]_- \right\} \\ &= -i \int^{t_0} dt' \int d\mathbf{r}' B^b(\mathbf{r}') \langle \hat{S}^a(\mathbf{r}, t) \hat{S}^b(\mathbf{r}', t') - \hat{S}^b(\mathbf{r}', t') \hat{S}^a(\mathbf{r}, t) \rangle \\ &= -i \int^{t_0} dt' \int d\mathbf{r}' B^b(\mathbf{r}') \langle [\hat{S}^a(\mathbf{r}, t), \hat{S}^b(\mathbf{r}', t')]_- \rangle \end{aligned} \quad (\text{B.3})$$

where $\langle \hat{A} \rangle = \text{Tr} \{ \hat{A} \hat{\rho}^{(0)} \}$ is the expectation value of operator \hat{A} in a system described by $\hat{\rho}^{(0)}$. We write the local magnetic field as $B^b(\mathbf{r}) = \beta^b \delta(\mathbf{r} - \mathbf{r}')$, in order to obtain

the local susceptibilities, which gives

$$\delta \langle \hat{S}^a(\mathbf{r}, t) \rangle = -i \int^t dt' \beta^b \langle [\hat{S}^a(\mathbf{r}, t), \hat{S}^b(\mathbf{r}, t')]_- \rangle, \quad (\text{B.4})$$

where $\chi^{ab} = -i \int dt' \langle [\hat{S}^a(\mathbf{r}, t), \hat{S}^b(\mathbf{r}, t')]_- \rangle$ is the local electron spin correlation function we used in our derivation of the nuclear spin pumping formalism. We have thus shown that the local magnetic susceptibility is identical to the correlation function χ^{ab} .

Appendix C

Calculation of spin diffusion equation with hyperfine interaction

This appendix includes the calculation of the electron spin diffusion equation in section 5.2.2. We start from the already obtained expressions for the isotropic and anisotropic spin components:

$$\partial_t \mathbf{s}_0 = -\frac{v}{2} (\partial_x \mathbf{s}_c + \partial_y \mathbf{s}_s) - \mathbf{s}_c \times \mathbf{b}_c - \mathbf{s}_s \times \mathbf{b}_s - \mathbf{s}_{c3} \times \mathbf{b}_{c3} - \mathbf{s}_{s3} \times \mathbf{b}_{s3} \quad (\text{C.1})$$

$$\mathbf{s}_c = \tilde{\tau}_1 \left\{ -v \partial_x \mathbf{s}_0 + 2\mathbf{b}_c \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_x \mathbf{s}_0) + 4\tau_1 \mathbf{B}_1 \times (\mathbf{b}_c \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_x \mathbf{s}_0) \right\} \quad (\text{C.2})$$

$$\mathbf{s}_s = \tilde{\tau}_1 \left\{ -v \partial_y \mathbf{s}_0 + 2\mathbf{b}_s \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_y \mathbf{s}_0) + 4\tau_1 \mathbf{B}_1 \times (\mathbf{b}_s \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_y \mathbf{s}_0) \right\} \quad (\text{C.3})$$

$$\mathbf{s}_{c3} = 2\tilde{\tau}_3 \left\{ \mathbf{b}_{c3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{c3} \times \mathbf{s}_0) \right\} \quad (\text{C.4})$$

$$\mathbf{s}_{s3} = 2\tilde{\tau}_3 \left\{ \mathbf{b}_{s3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{s3} \times \mathbf{s}_0) \right\}. \quad (\text{C.5})$$

Substituting the expressions for the anisotropic spin components into the isotropic diffusion equation gives

$$\begin{aligned} \partial_t \overline{\mathbf{s}_0} = & -\frac{v}{2} \left(\partial_x \tilde{\tau}_1 \left\{ -v \partial_x \mathbf{s}_0 + 2\mathbf{b}_c \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_x \mathbf{s}_0) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_c \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_x \mathbf{s}_0) \right\} \right. \\ & + \partial_y \tilde{\tau}_1 \left\{ -v \partial_y \mathbf{s}_0 + 2\mathbf{b}_s \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_y \mathbf{s}_0) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_s \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_y \mathbf{s}_0) \right\} \\ & - \tilde{\tau}_1 \left\{ -v \partial_x \mathbf{s}_0 + 2\mathbf{b}_c \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_x \mathbf{s}_0) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_c \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_x \mathbf{s}_0) \right\} \times \mathbf{b}_c \\ & - \tilde{\tau}_1 \left\{ -v \partial_y \mathbf{s}_0 + 2\mathbf{b}_s \times \mathbf{s}_0 - 2v\tau_1 (\mathbf{B} \times \partial_y \mathbf{s}_0) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_s \times \mathbf{s}_0) - 4v\tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_y \mathbf{s}_0) \right\} \times \mathbf{b}_s \\ & \left. - 2\tilde{\tau}_3 \left\{ \mathbf{b}_{c3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{c3} \times \mathbf{s}_0) \right\} \times \mathbf{b}_{c3} - 2\tilde{\tau}_3 \left\{ \mathbf{b}_{s3} \times \mathbf{s}_0 + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{s3} \times \mathbf{s}_0) \right\} \times \mathbf{b}_{s3} \right) \end{aligned} \quad (\text{C.6})$$

By now performing the integration over k , dividing by $(2\pi)^2$, and by using that $\int \frac{d\mathbf{k}}{(2\pi)^2} = \mathbf{S}$ we get an expression where \mathbf{s}_0 is substituted with \mathbf{S} and all properties

are evaluated by their Fermi values, i.e. $k \rightarrow k_F$, $v \rightarrow v_F$. This simplification occurs due to our zero temperature assumption, and can easily be seen from the derivation of the spin diffusion equation in chapter two. The diffusion equation then reads

$$\begin{aligned} \partial_t \mathbf{S} = & -\frac{v_F}{2} \left(\partial_x \tilde{\tau}_1 \{ -v_F \partial_x \mathbf{S} + 2\mathbf{b}_c \times \mathbf{S} - 2v_F \tau_1 (\mathbf{B} \times \partial_x \mathbf{S}) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_c \times \mathbf{S}) - 4v_F \tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_x \mathbf{S}) \} \right. \\ & + \partial_y \tilde{\tau}_1 \{ -v_F \partial_y \mathbf{S} + 2\mathbf{b}_s \times \mathbf{S} - 2v_F \tau_1 (\mathbf{B} \times \partial_y \mathbf{S}) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_s \times \mathbf{S}) - 4v_F \tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_y \mathbf{S}) \} \\ & - \tilde{\tau}_1 \{ -v_F \partial_x \mathbf{S} + 2\mathbf{b}_c \times \mathbf{S} - 2v_F \tau_1 (\mathbf{B} \times \partial_x \mathbf{S}) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_c \times \mathbf{S}) - 4v_F \tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_x \mathbf{S}) \} \times \mathbf{b}_c \\ & - \tilde{\tau}_1 \{ -v_F \partial_y \mathbf{S} + 2\mathbf{b}_s \times \mathbf{S} - 2v_F \tau_1 (\mathbf{B} \times \partial_y \mathbf{S}) + 4\tau_1 \mathbf{B} \times (\mathbf{b}_s \times \mathbf{S}) - 4v_F \tau_1^2 \mathbf{B} (\mathbf{B} \cdot \partial_y \mathbf{S}) \} \times \mathbf{b}_s \\ & \left. - 2\tilde{\tau}_3 \{ \mathbf{b}_{c3} \times \mathbf{S} + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{c3} \times \mathbf{S}) \} \times \mathbf{b}_{c3} - 2\tilde{\tau}_3 \{ \mathbf{b}_{s3} \times \mathbf{S} + 2\tau_3 \mathbf{B} \times (\mathbf{b}_{s3} \times \mathbf{S}) \} \times \mathbf{b}_{s3}. \right. \end{aligned} \quad (\text{C.7})$$

We evaluate the spin-orbit fields at the PSH condition $\alpha = \beta''$, which gives $\mathbf{b}_c = 0$ and $\mathbf{b}_s = 2v_F k_F \alpha \hat{e}_x$, and use that $\mathbf{S} \parallel \mathbf{B}$, to obtain

$$\begin{aligned} \partial_t \mathbf{S} = & \tilde{D} \partial_x^2 \mathbf{S} + \tilde{D} \partial_y^2 \mathbf{S} + 2\tilde{D} q_0 \partial_y \begin{pmatrix} 0 \\ S_z \\ -S_y \end{pmatrix} - \tilde{D} q_0^2 \begin{pmatrix} 0 \\ S_y \\ S_z \end{pmatrix} - \tilde{\gamma}_{cd} \begin{pmatrix} S_x \\ S_y \\ 2S_z \end{pmatrix} \\ & 2\tilde{D} \left[\tau_1 \mathbf{B} \times \partial_x^2 \mathbf{S} + \tau_1 \mathbf{B} \times \partial_y^2 \mathbf{S} - \partial_y \{ \tau_1 \mathbf{B} \times (q_0 \hat{e}_x \times \mathbf{S}) \} \right. \\ & + 2\partial_x \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_x \mathbf{S}) \} + 2\partial_y \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) \} \\ & \left. + q_0 \hat{e}_x \times (\partial_y \mathbf{S} \times \tau_1 \mathbf{B}) + (\tau_1 \mathbf{B} \cdot q_0 \hat{e}_x) (\mathbf{S} \times q_0 \hat{e}_x) + 2(\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) (\tau_1 \mathbf{B} \times q_0 \hat{e}_x) \right] \\ & - 2\tilde{\gamma}_{cd} \tau_3 \mathbf{B} \times \begin{pmatrix} S_x \\ S_y \\ 2S_z \end{pmatrix}, \end{aligned} \quad (\text{C.8})$$

where $\tilde{D} = \frac{1}{2} v_F^2 \tilde{\tau}_1 = \frac{\frac{1}{2} v_F^2 \tau_1}{1 + (2B\tau_1)^2}$ is the renormalized diffusion constant, $\tilde{\gamma}_{cd} = \frac{\frac{1}{8} v_F^2 \gamma^2 k_F^6 \tau_3}{1 + (2B\tau_3)^2}$ is the renormalized cubic Dresselhaus scattering rate, and $q_0 = 4k_F \alpha$ is the PSH wave vector. We collect the terms that are independent on \mathbf{B} in the diffusion matrix \hat{D} , and the terms arising due to the winding number ± 1 and ± 3 components in \mathbf{H}_1 and \mathbf{H}_3 , respectively, such that

$$\partial_t \mathbf{S} = \hat{D} \mathbf{S} + \mathbf{H}_1 + \mathbf{H}_3 \quad (\text{C.9})$$

with

$$\hat{D} = \begin{pmatrix} \tilde{D}(\partial_x^2 + \partial_y^2) - \tilde{\gamma}_{cd} & 0 & 0 \\ 0 & \tilde{D}(\partial_x^2 + \partial_y^2 - q_0^2) - \tilde{\gamma}_{cd} & 2\tilde{D}q_0 \partial_y \\ 0 & -2\tilde{D}q_0 \partial_y & \tilde{D}(\partial_x^2 + \partial_y^2 - q_0^2) - 2\tilde{\gamma}_{cd} \end{pmatrix}, \quad (\text{C.10})$$

$$\begin{aligned} \mathbf{H}_1 = & 2\tilde{D} \left[\tau_1 \mathbf{B} \times \partial_x^2 \mathbf{S} + \tau_1 \mathbf{B} \times \partial_y^2 \mathbf{S} - \partial_y \{ \tau_1 \mathbf{B} \times (q_0 \hat{e}_x \times \mathbf{S}) \} \right. \\ & + 2\partial_x \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_x \mathbf{S}) \} + 2\partial_y \{ \tau_1 \mathbf{B} (\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) \} \\ & \left. + q_0 \hat{e}_x \times (\partial_y \mathbf{S} \times \tau_1 \mathbf{B}) + (\tau_1 \mathbf{B} \cdot q_0 \hat{e}_x) (\mathbf{S} \times q_0 \hat{e}_x) + 2(\tau_1 \mathbf{B} \cdot \partial_y \mathbf{S}) (\tau_1 \mathbf{B} \times q_0 \hat{e}_x) \right], \end{aligned} \quad (\text{C.11})$$

and

$$\mathbf{H}_3 = -2\tilde{\gamma}_{cd}\tau_3\mathbf{B} \times \begin{pmatrix} S_x \\ S_y \\ 2S_z \end{pmatrix}. \quad (\text{C.12})$$