

# Reliability-based measures and prognostic analysis of a $K$ -out-of- $N$ system in random environments

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## Abstract

In this paper, we study reliability based measures and prognostic problems of a  $K$ -out-of- $N$  system in which the failure process of each component depends not only on its intrinsic characteristic but also on its operating environment conditions. The system reliability and the expected remaining useful lifetime are calculated. Under the periodic inspection policy, the system asymptotic availability is derived. We aim at providing explicit expressions for these quantities. **The model allows us to incorporate the observation information of the environment in the evaluation of the system performances. Numerical examples show the efficiency and accuracy of our method by comparing with the Monte-Carlo simulations.** It is pointed out that the environment condition has significant effect on the system reliability based measures and the system prognostic analysis.

*Keywords:* Reliability; remaining useful lifetime (RUL); availability; continuous-time Markov chain;  $K$ -out-of- $N$  systems

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## Notation

|                            |   |
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| $\Phi$                     | $\{1, 2, \dots, N\}$  |
| $S$                        | $\{1, 2, \dots, m\}$  |
| $\emptyset$                | empty set   |
| $I$                        | the $m \times m$ identity matrix  |
| $E_1 \setminus E_2$        | the set-theoretic difference of set $E_1$ and set $E_2$                   |
| $\{c_1, c_2, \dots, c_k\}$ | subset of $\Phi$ with elements $c_1, c_2, \dots, c_k, k = 1, 2, \dots, N$ |
| $\{c_1, c_2, \dots, c_k\}$ | $\Phi \setminus \{c_1, c_2, \dots, c_k\}, k = 1, 2, \dots, N$             |

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| $T_i$                                      | the lifetime of component $i$ , $i \in \Phi$  |
| $T$  | the lifetime of the system  |
| $W(t)$                                     | the environment state at time $t$   |
| $h_i(t, j)$                                | the failure rate of component $i$ at time $t$ when the environment state is $j$ , $i \in \Phi, j \in \mathcal{S}$   |
| $\mathbf{H}_i(t)$                          | the diagonal matrix with the $(k, k)^{th}$ entry $h_i(t, k)$ , $i \in \Phi, k \in \mathcal{S}$ on the primary diagonal  |
| $\mathbf{H}_i$                             | the matrix $\mathbf{H}_i(t)$ when it is time-independent, $i \in \Phi$  |
| $\mathbf{H}_0(t)$                          | the matrix $\mathbf{H}_i(t)$ when it is independent of $i$ , $i \in \Phi$   |
| $\mathbf{H}_0$                             | the matrix $\mathbf{H}_0(t)$ when it is time-dependent  |
| $\mathbf{H}(t)$                            | the sum of $\mathbf{H}_i(t)$ , $i \in \mathcal{S}$ , i.e. $\mathbf{H}(t) = \sum_{i \in \mathcal{S}} \mathbf{H}_i(t)$  |
| $\mathbf{Q}$                               | the transition rate matrix of the continuous-time Markov chain  |
| $q_{ij}$                                   | the element of $\mathbf{Q}$ , $i, j \in \mathcal{S}$  |
| $\text{diag}([a_1, a_2, \dots, a_n])$      | the diagonal matrix with elements $a_1, a_2, \dots, a_n$ in the main diagonal   |
| $B_{ij}^\emptyset(t)$                      | the probability that each component survives by time $t$ when the environment condition is $j$ given its initial value $i$ at time 0, $i, j \in \mathcal{S}$  |
| $\mathbf{B}^\emptyset(t)$                  | the matrix with elements $B_{ij}^\emptyset(t)$ , i.e. $\mathbf{B}^\emptyset(t) = [B_{ij}^\emptyset(t)]$ , $i, j \in \mathcal{S}$  |
| $B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$     | the probability that components $c_1, c_2, \dots, c_k$ fail while the rest are survival by time $t$ when the environment state is $j$ given its initial value $i$ at time 0, $c_i \in \Phi, i = 1, 2, \dots, k$ . |
| $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$ | the matrix with elements $B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$ , $i, j \in \mathcal{S}$  |
| $B_{ij}^{(l)}(t)$                          | the probability that there are $l$ components fail by time $t$ with the environment state $j$ where the initial environment state is $i$ , $l = 0, 1, \dots, N - 1$ when all components are identical             |
| $\mathbf{B}^{(l)}(t)$                      | the matrix with elements $B_{ij}^{(l)}(t)$ , $i, j \in \mathcal{S}$   |
| $B_{ij}(t)$                                | the reliability of the $K$ -out-of- $N$ system at time $t$ when the environment state is $j$ given the initial environment state $i$ , $i, j \in \mathcal{S}$   |
| $\mathbf{B}(t)$                            | the matrix with elements $B_{ij}(t)$ , $i, j \in \mathcal{S}$   |
| $\mathbf{e}$                               | the $m \times 1$ matrix of 1s   |
| $\mathbf{e}_i$                             | the $1 \times m$ matrix whose $i$ th element is 1 and others are 0 respectively   |
| $R(t)$                                     | the reliability function of the system  |
| $R_i(t)$                                   | the conditional reliability function of the system given that   |

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|   | the initial environment state is $i$ , $i \in \mathbf{S}$  |
| $F(t)$  | the lifetime distribution of the system  |
| $F_i(t)$  | the conditional lifetime distribution of the system given that the initial environment state is $i$ , $i \in \mathbf{S}$   |
| $T_{t,i}^{\{c_1, c_2, \dots, c_k\}}$                                    | the remaining useful lifetime of the system given that components $i$ , $\forall i \in \{c_1, c_2, \dots, c_k\}$ fail by time $t$ when the environment state is $i$  |
| $CR_i(u; t, \{c_1, c_2, \dots, c_k\})$                                  | the conditional reliability of the system given that component $l$ fails by time $t$ , $\forall l \in \{c_1, c_2, \dots, c_k\} \subset \Phi$ and the environment state at time $t$ is $i$  |
| $r_i(t; \{c_1, c_2, \dots, c_k\})$                                      | the expected remaining useful lifetime (RUL) of the system at time $t$ given the environment state is $i$ and component $l$ fails by time $t$ , $\forall l \in \{c_1, c_2, \dots, c_k\}$   |
| $L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$     | the probability that component $l$ , $\forall l \in \{d_1, d_2, \dots, d_r\}$ fails by time $t$ where the environment state is $j$ given that component $m$ , $\forall m \in \{c_1, c_2, \dots, c_k\}$ fail at time $x$ with environment state $i$ |
| $\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$ | the matrix with elements $L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$   |
| $L_{ij}^{l_0, l}(\theta, t)$  | the probability that the number of failed components is $l$ at time $t$ with environment state $j$ given that at time $\theta$ the number of failed components is $l_0$ with environment state $i$ for the system with identical components        |
| $\mathbf{L}^{l_0, l}(\theta, t)$  | the matrix with elements $L_{ij}^{l_0, l}(\theta, t)$ , $i, j \in \mathbf{S}$  |
| $X(t)$  | the state of the system, $X(t) = 1$ means it is functional and 0 means it fails at time $t$  |
| $Y$   | the system repair time with distribution function $G(\cdot)$ and density function $g(\cdot)$   |
| $\tau$  | the system inspection period   |
| $A_s$   | the asymptotic availability of the system  |

## 1. Introduction

In Reliability Engineering and Operations research, redundancy technique is widely used to improve the system reliability. For instance, several parts of the control system in hydraulic systems in an aircraft may be triplicated; both mechanical and hydraulic braking are used in a car, redundancy guarantees the regular transmission of power even when some line failures occur in the power grid. There is an extensive literature on the redundant system. Eryilmaz [10, 11] studied the mean residual life of a  $k$ -out-of- $n$  sys-

tem with a standby component. Explicit expressions of the mean residual life functions under various scenarios were provided. Bueno et al. [8] investigated the allocation problem of a redundant system in order to stochastically increase the system reliability. Levitin et al. [18] examined the impact of the modes of redundant elements and the order in which they were initiated on a heterogeneous 1-out-of- $n$  standby systems. Wang [29] considered the reliability estimation problem of weighted  $k$ -out-of- $n$  multi-state systems. Readers are referred to [23, 27, 2, 13, 34, 4] for more information.

It is seen that in the above literature, a 'latent' hypothesis is that the system operates in a static environment which has no effect on system reliability. However, this assumption is not realistic. For example, The space shuttle Challenger accident was related to the low temperature under which the O-rings used to seal the combustion gas didn't work properly [9]; the age of a jet engine consisting hundreds of components is related to the atmospheric flight environment like pressure, temperature, humidity, mechanical vibration; the lifetime of a workstation in a manufacturing system is subjected to its workload [21]; the deterioration of the blade of offshore wind turbine depends on the salt concentration in the air [33]. It is evident that environment has impact on the system lifetime. In particular, for a multi-component system, the impact may cause the dependency among components which complicates the system reliability and prognostic analyses.

To address this need and emphasize the environment impact on the system reliability, in this paper, we develop a model regarding multi-component systems that the lifetime of each component depends on each other through the common impact of their operating environment conditions. The model allows us to evaluate the system reliability based measures by incorporating the observation information of the environment.

In literature, most researchers take the model of Esary et al. [12] as the first environment-related work where the successive damages caused by random shocks were time-dependent relating to environment conditions. Several properties about the system survival time were obtained. Various failure models concerning systems in dynamic environment were established since then [24, 28, 30, 20, 16]. Zhao et al. [32] discussed the optimal maintenance strategies of the degradation system where the impact of the environment to the degradation process was modeled by covariates via the Cox proportional hazards model. Lawless et al. [17] considered a gamma-process model to describe the crack growth by incorporating random effects. Çinlar and ÖZekici [5] presented the intrinsic ageing model where the concept of intrinsic age was proposed to represent the cumulative hazard accumulated in time with varying environment during its operation period. A number of

intrinsic ageing models can be found in [4, 3, 22] for instance. ‘While these models encompass quite general laws for deterioration and are theoretically appealing, they do not readily lend themselves to computational analysis’, as Kiessler et al. [15] pointed out. They investigated the single-component system whose deterioration was driven by its operating environment which was described by a Markov chain. The system asymptotic availability was derived [15, 14].

The main contributions of this work are

- We extend or partially extend some existing models in different aspects in the literature.
  - In terms of the system reliability and remaining lifetime, our model can be degenerate to the problem of the single component system operating in dynamic environment proposed by Banjevic and Jardine [1];
  - In terms of the system availability, when the system is periodically inspected and perfectly repaired if system failures are diagnosed, the single component system presented in [7] and [31] respectively can be seen as special cases of our model ( $K = 1, N = 1$ );
- Comparing to the assumption of the independence between the system reliability and its operating environment, our model is more realistic;
- The methodology is general enough which permits the heterogeneity of components possessing different failure rates;
- **The methodology is also applicable for more complex systems with  $K$ -out-of- $N$  system as sub-systems;**
- Important system indicators such as the system reliability and the system remaining useful lifetime, which are also interested in the PHM analysis are presented;
- Exact numerical calculation method is presented which can enhance the efficiency and accuracy in the evaluation of the system performance measures.

The rest of the paper is organized as follows. In section 2, the system hypotheses and the system reliability function are presented. Section 3 is devoted to the calculation of the system expected remaining useful lifetime. The asymptotic availability of the system under the periodic inspection policy is derived in section 4. Numerical examples are presented to illustrate

the effectiveness of the proposed model in section 5. Finally, we make our conclusions in section 6.

## 2. Model descriptions and the system reliability function

In this section, the mathematical model is described followed by the calculation of the system reliability.

### 2.1. Model descriptions

- Consider a  $K$ -out-of- $N$  system which is put into service at time 0 with as good as new state. Suppose that the components are labeled as component 1, component 2,  $\dots$ , component  $N$ . The system lifetime is  $T$ . The lifetime of component  $i$  is denoted by  $T_i$ ,  $i \in \Phi$ ,  $\Phi = \{1, 2, \dots, N\}$ . Components are independent under the fixed environment.
- The operation of components are impacted by an external environment which is described by a continuous-time Markov chain  $\mathcal{W} = \{W(t), t \geq 0\}$  with a finite state space  $\mathcal{S} = \{1, 2, \dots, m\}$ , infinitesimal generator  $\mathbf{Q}$  and transition probability  $\pi_{ij}(t)$ ,  $i, j \in \mathcal{S}$ . In effect, the environment can be regarded as the working condition which affects the system state (its failure rate). For example, the environment may be mild, normal, and dangerous to the system.
- Component  $k$ ,  $k \in \Phi$  has failure rate  $\lambda_k(t) = h_k(t, W(t))$  where  $h_k(t, j)$  is the hazard rate of component  $k$  at age  $t$  when the environment state is  $j$ ,  $j \in \mathcal{S}$ . It is assumed through this paper that  $\int_0^\infty h_k(t, j) dt = \infty$ ,  $\forall k \in \Phi$ ,  $\forall j \in \mathcal{S}$  which indicates that the mean time to failure of each component under each environment state is finite. Without loss of generality, we further assume that  $h_k(t, 1) < h_k(t, 2) < \dots < h_k(t, m)$  for any  $t \geq 0$  and  $k \in \Phi$ ,  $j \in \mathcal{S}$ .

### 2.2. The system reliability function and lifetime distribution

In order to obtain the system reliability as well as the system lifetime distribution, first denote by

$$B_{ij}^{\mathcal{Q}}(t) = \mathbb{P}(T_k > t, W(t) = j, \forall k \in \Phi, | W(0) = i) \quad (1)$$

the probability that no component failures occur by time  $t$  when the environment state is  $j$  given the initial environment state  $i$ ,  $i, j \in \mathcal{S}$ . Then we have the following lemma.

**Lemma 2.1.** Denoted the  $m \times m$  matrixes  $\mathbf{B}^\varnothing(t) = [B_{ij}^\varnothing(t)]_{m \times m}$  and  $\mathbf{H}(t) = \text{diag}[\sum_{l \in \Phi} h_l(t, j)]_{m \times m}$  which represents the diagonal matrix with elements  $H_{jj}(t) = \sum_{l \in \Phi} h_l(t, j)$  in the main diagonal, the following equation is valid.

$$\frac{d\mathbf{B}^\varnothing(t)}{dt} = \mathbf{B}^\varnothing(t)(\mathbf{Q} - \mathbf{H}(t)) \quad (2)$$

In particular, when the hazard rate is time-independent, i.e.  $\mathbf{H}(t) = \mathbf{H}$ , then it is easily seen that

$$\mathbf{B}^\varnothing(t) = \exp((\mathbf{Q} - \mathbf{H})t)$$

See Appendix A.1 for the proof.

From Lemma 2.1, we can obtain the reliability of the  $N$ -component in series system by  $\mathbf{B}^\varnothing(t)$  and the probability vector of the initial environment state. However, for the general  $K$ -out-of- $N$  system, it is also necessary to evaluate the state of each component (failed or working) at given time epochs.

To do this, let  $\{c_1, c_2, \dots, c_k\}$  be a subset of  $\Phi$  recording the failed components. Correspondingly denote  $\{c_1, c_2, \dots, c_k\}$  the absolute complement of  $\{c_1, c_2, \dots, c_k\}$  in  $\Phi$  which records the survival components. For instance, suppose that  $\Phi = \{1, 2, 3, 4, 5\}$ , then  $\{c_1, c_2\} = \{2, 5\}$  ( $\{c_1, c_2\} = \{1, 3, 4\}$ ) if components 2 and 5 fail, components 1, 3, 4 are functioning. Let

$$B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) = \mathbb{P}\left(T_l > t, T_p < t, W(t) = j, \forall p \in \{c_1, c_2, \dots, c_k\}, \quad (3)\right. \\ \left. \forall l \in \{\underline{c_1, c_2, \dots, c_k}\} \mid W(0) = i\right)$$

be the probability that component  $p$ ,  $\forall p \in \{c_1, c_2, \dots, c_k\}$  fails while the rest are functioning by time  $t$  with environment state  $j$  when the initial environment state is  $i$ ,  $\{\underline{c_1, c_2, \dots, c_k}\} = \Phi \setminus \{c_1, c_2, \dots, c_k\}$ . We have the following theorem.

**Theorem 2.1.** Denoted by  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = [B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)]_{m \times m}$ . For any  $k < N$ , the probability matrix  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$  satisfies

$$\frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} = \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)(\mathbf{Q} - \sum_{j \in \{\underline{c_1, c_2, \dots, c_k}\}} \mathbf{H}_j(t)) \quad (4) \\ + \sum_{l=1}^k \mathbf{B}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l}(t)$$

where  $\mathbf{B}^{\{i_m\} \setminus \{i_m\}}(t) = \mathbf{B}^\varnothing(t)$ , and  $\mathbf{H}_i(t) = \text{diag}[h_i(t, j)]_{m \times m}$  is  $m \times m$  matrix,  $i \in \Phi$ ,  $j \in \mathcal{S}$ .

See Appendix A.2 for the proof.

It is seen that the probability matrix  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$  can be derived by Theorem 2.1 for the general case that components are heterogeneous. However, the closed-form solution of equation (9) in Theorem 2.1 is almost impossible to obtain except the case that the lifetime of each component under each environment state is exponentially distributed. Hence in the following in Corollary 1, the exact expression of  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$  is provided under this special case. Besides, components in some redundant system can be seen as ones possessing identical failure rates. For example, the four engines in aircraft, three blades in wind turbine, etc. In such special case the matrix utilized for the reliability evaluation is provided in Corollary 2.

**Corollary 1.** *When the hazard rates of components are time-independent, i.e.  $\mathbf{H}_i(t) = \mathbf{H}_i$ ,  $\forall i \in \Phi$ , the matrix  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)$  in Theorem 2.1 can be represented as*

$$\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = \sum_{m=0}^k (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_k\}, \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left( \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_k\}} \mathbf{H}_j - \sum_{i=1}^m \mathbf{H}_{l_i} \right) t \right) \quad (5)$$

where  $\sum_{i=1}^0 = 0$ .

See Appendix A.3 for the proof.

One step further, when the  $N$  components have identical lifetime distributions under given environments, i.e.  $\mathbf{H}_i(t) = \mathbf{H}_0(t)$  for any  $i \in \Phi$ . Let  $B_{ij}^{(l)}(t)$  be the probability that there are  $l$  components fail by time  $t$  with the environment state  $j$  where the initial environment state is  $i$ ,  $l = 1, 2, \dots, N - 1$ . The matrix expression is  $\mathbf{B}^{(l)}(t) = [B_{ij}^{(l)}(t)]_{m \times m}$ . The following corollary can be derived.

**Corollary 2.** *For any  $l = 1, 2, \dots, N - 1$ ,  $\mathbf{B}^{(l)}(t)$  satisfies*

$$\frac{d\mathbf{B}^{(l)}(t)}{dt} = \mathbf{B}^{(l)}(t)(\mathbf{Q} - (N - l)\mathbf{H}_0(t)) + \mathbf{B}^{(l-1)}(t)(N - l + 1)\mathbf{H}_0(t) \quad (6)$$

where  $\mathbf{B}^{(l)}(t) = \mathbf{B}^\varnothing(t)$  when  $l = 0$ .

In particular, when the hazard rates are time-independent, i.e.  $\mathbf{H}_0(t) = \mathbf{H}_0$  we have



$$\mathbf{B}^{(l)}(t) = \binom{N}{l} \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \exp((\mathbf{Q} - (N-i)\mathbf{H}_0)t) \quad (7)$$

where  $\mathbf{H}_0 = \text{diag}[h(j)]_{m \times m}$ .

Corollary 2 can be easily verified by Theorem 2.1 and Corollary 1.

In the following, the reliability function  $R(t)$  and the lifetime distribution function  $F(t)$  of the  $K$ -out-of- $N$  system are derived respectively. Denoted by

$$B_{ij}(t) = B_{ij}^{\emptyset}(t) + \sum_{k=1}^{N-K} \sum_{1 \leq c_1 < c_2 < \dots < c_k \leq N} B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) \quad (8)$$

and the matrix form

$$\mathbf{B}(t) = \mathbf{B}^{\emptyset}(t) + \sum_{k=1}^{N-K} \sum_{1 \leq c_1 < c_2 < \dots < c_k \leq N} \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) \quad (9)$$

where  $\sum_{l=1}^0 = 0$ . Equation (9) can also be rewritten as follows.

$$\mathbf{B}(t) = \mathbf{B}^{\emptyset}(t) + \sum_{l=1}^{N-K} \mathbf{B}^{(l)}(t)$$

when the  $N$  components have identical lifetime distributions under given environment conditions.

Assume that the initial probability row vector of the environment process is given by  $\boldsymbol{\alpha} = [\alpha_i]$ , where  $\alpha_i = \mathbb{P}(W(0) = i)$ ,  $i \in \mathcal{S}$ . Let  $\mathbf{e}$  be a column vector of 1s,  $\mathbf{e}_i$  be a  $1 \times m$  matrix whose  $i$ th element is 1 and others are 0 respectively. The reliability function  $R(t)$ , the lifetime distribution function  $F(t)$  of the  $K$ -out-of- $N$  system are given as

$$R(t) = \boldsymbol{\alpha} \mathbf{B}(t) \mathbf{e} \quad (10)$$

$$F(t) = 1 - \boldsymbol{\alpha} \mathbf{B}(t) \mathbf{e} \quad (11)$$

The conditional reliability  $R_i(t)$  and the conditional distribution of the system  $F_i(t)$  given the initial environment state  $i$  are

$$R_i(t) = \mathbf{e}_i \mathbf{B}(t) \mathbf{e} \quad (12)$$

$$F_i(t) = 1 - \mathbf{e}_i \mathbf{B}(t) \mathbf{e} \quad (13)$$

### 3. The system remaining useful lifetime

Besides the reliability, the remaining useful lifetime (RUL) is also an important criterion considered extensively in the reliability analysis [19, 25]. Accurate estimation of RUL permits us to predict the system failure process by taking advantage of the system monitoring information. It is beneficial to the formulation of maintenance policies. There are many definitions of the RUL, and here we apply the definition of RUL as in [1] which means

$$\begin{aligned} & \mathbb{P}(T_{t,i}^{\{c_1, c_2, \dots, c_k\}} > h) \\ &= \mathbb{P}\left(T - t > h \mid T_u < t, T_v > t, \forall u \in \{c_1, c_2, \dots, c_k\}, \forall v \in \underline{\{c_1, c_2, \dots, c_k\}}, W(t) = i\right) \end{aligned} \quad (14)$$

where  $T$  is the system lifetime,  $\{c_1, c_2, \dots, c_k\} = \Phi \setminus \{c_1, c_2, \dots, c_k\}$ . To calculate system expected RUL, it is necessary to record the current condition of each component (failed or work). Let

$$\begin{aligned} L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\}) &= \mathbb{P}\left(T_s > t, T_p < t, \forall s \in \underline{\{c_1, c_2, \dots, c_k\}} \setminus \{d_1, d_2, \dots, d_r\}, \right. \\ & \quad \left. \forall p \in \{d_1, d_2, \dots, d_r\}, W(t) = j \mid T_u > x, T_v < x, \forall v \in \{c_1, c_2, \dots, c_k\}, \forall u \in \underline{\{c_1, c_2, \dots, c_k\}}, \right. \\ & \quad \left. W(x) = i\right) \end{aligned}$$

be the probability that component  $s, \forall s \in \{d_1, d_2, \dots, d_r\}$  fails in the time interval  $(x, t]$  and the environment state is  $j$  at time  $t$  given that component  $v, \forall v \in \{c_1, c_2, \dots, c_k\}$  fails by time  $x$  when the environment state is  $i$  where  $\{d_1, d_2, \dots, d_r\} \subseteq \{c_1, c_2, \dots, c_k\}$ . The following lemma is given before the calculation of the system remaining useful lifetime.

**Lemma 3.1.** *Denoted by*

$$\mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\}) = [L_{ij}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})]_{m \times m}.$$

*It can be derived that*

$$\begin{aligned} \frac{\partial \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})}{\partial t} &= \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})(\mathbf{Q} - \sum_{u \in \sigma(p, r)} \mathbf{H}_u(t)) \\ &+ \sum_{l=1}^r \mathbf{L}^{\{d_1, d_2, \dots, d_r\} \setminus \{d_l\}}(x, t; \{c_1, c_2, \dots, c_k\}) \mathbf{H}_{d_l}(t) \end{aligned} \quad (15)$$

where  $\sigma(p, r) = \Phi \setminus \{c_1, \dots, c_p, d_1, \dots, d_r\}$ .

The proof is omitted as it is similar to the proof of Theorem 2.1.

It is pointed out that both  $\{c_1, c_2, \dots, c_k\}$  and  $\{d_1, d_2, \dots, d_r\}$  can be  $\emptyset$ . In particular,

- $\{c_1, c_2, \dots, c_k\} = \emptyset$  means that there is no components failures by time  $x$  and  $\mathbf{L}^\emptyset(x, x; \emptyset) = \mathbf{I}$ ;
- $\{d_1, d_2, \dots, d_r\} = \emptyset$  means that there is no components failures in the interval  $(x, t]$ , in this case equation (15) can be represented as

$$\frac{\partial \mathbf{L}^\emptyset(x, t; \{c_1, c_2, \dots, c_k\})}{\partial t} = \mathbf{L}^\emptyset(x, t; \{c_1, c_2, \dots, c_k\})(\mathbf{Q} - \sum_{u \in \sigma(p)} \mathbf{H}_u(t))$$

where  $\sigma(p) = \Phi \setminus \{c_1, c_2, \dots, c_p\}$

- $\{c_1, c_2, \dots, c_k\} = \{d_1, d_2, \dots, d_r\} = \emptyset$  means no components failures occur by time  $t$  and  $\mathbf{L}^\emptyset(x, x; \emptyset) = \mathbf{I}$ .

Therefore we can obtain the system conditional reliability  $CR_i(u; t, W(t), \{c_1, c_2, \dots, c_k\})$  and the expected RUL  $r_i(t; \{c_1, c_2, \dots, c_k\})$  in the following.

$$CR_i(u; t, W(t), \{c_1, c_2, \dots, c_k\}) = \mathbf{e}_i \mathbf{L}(t, u; \{c_1, c_2, \dots, c_k\}) \mathbf{e} \quad (16)$$

$$r_i(t, W(t); \{c_1, c_2, \dots, c_k\}) = \int_t^\infty \mathbf{e}_i \mathbf{L}(t, u; \{c_1, c_2, \dots, c_k\}) \mathbf{e} du \quad (17)$$

where  $\mathbf{L}(x, t; \{c_1, c_2, \dots, c_k\}) = \sum_{r=0}^{N-K-k} \sum_{\substack{d_1 < d_2 < \dots < d_r \\ d_i \in \{c_1, c_2, \dots, c_k\} \\ i=1,2,\dots,r}} \mathbf{L}^{\{d_1, d_2, \dots, d_r\}}(x, t; \{c_1, c_2, \dots, c_k\})$ ,

$\mathbf{e}$  is a column vector of 1s,  $\mathbf{e}_i$  is a  $1 \times m$  matrix whose  $i$ th element is 1 and others are 0 respectively.

For the special case when each component of the  $K$ -out-of- $N$  system possesses identical, constant failure rate under each environment state, denoted by  $r_i^{l_0}(\theta)$  the expected remaining useful lifetime of the  $K$ -out-of- $N$  system given that the system environment is  $i$  at time  $\theta$  and the number of component failures is  $l_0$ . The following corollary is obtained.

**Corollary 3.** *For the  $K$ -out-of- $N$  system with identical components possessing constant hazard rates under given environment conditions,  $r_i^{l_0}(\theta)$  satisfies*

$$r_i^{l_0}(\theta) = \sum_{l=l_0}^{N-K} \int_\theta^\infty \mathbf{e}_i (\mathbf{L}^{l_0, l}(\theta, t)) \mathbf{e} dt$$

where

$$\mathbf{L}^{l_0, l}(\theta, t) = \binom{N-l_0}{l-l_0} \sum_{i=0}^{l-l_0} (-1)^{l-l_0-i} \binom{l-l_0}{i} \exp\left(\left(\mathbf{Q} - (N-l_0-i)\mathbf{H}_0\right)(t-\theta)\right)$$

$0 < \theta < t$ ,  $l_0 \leq l < N$ ,  $\mathbf{L}^{l_0, l_0}(\theta, \theta) = \mathbf{I}$ ,  $\mathbf{L}^{l_0, l}(\theta, \theta) = \mathbf{0}$ ,  $l_0 < l$ .

See Appendix A.4 for the proof.

#### 4. The system asymptotic availability

In the following, we intend to derive the asymptotic availability (the limiting proportion of time that the system is functional [7]) of the system undergoing periodic inspections. Let  $\mathcal{X} := X(t)$  be the state of the system where  $X(t) = 1$  if the system is in the up-state and  $X(t) = 0$  if it is in the down-state. Other assumptions and notations are presented as follows.

- The system is new at time  $t = 0$  and the duration between two consecutive inspections length is  $\tau$ ;
- The system failure is not self-announcing and can be revealed only by system inspections;
- Upon inspection, it is perfectly repaired with a random time of length  $Y$  with distribution function  $G(y)$  (density function  $g(y)$ ) if the system failure is diagnosed; however, nothing is done if the system is in the up-state.

Thus the system asymptotic availability can be defined as [31] follows.

$$A_s = \lim_{t \rightarrow \infty} \frac{\int_0^t \mathbb{E}[X(s)] ds}{t}$$

Denoted by  $U_i$  and  $R_i$  the  $i$ th system failure epoch and the  $i$ th system renewal epoch respectively. According to [15],  $\{(W_{R_n}, R_n), n = 1, 2, \dots\}$  is a Markov renewal process. We further define  $W_n = W_{R_n}$  the environment state at the  $n$ th system replacement epoch. It is seen that  $\{W_n, n = 1, 2, \dots\}$  is an irreducible, discrete-time Markov chain with one-step transition probability matrix  $\mathbf{P}$  and stationary distribution  $\mathbf{p} = [p_i], i \in \mathcal{S}$  which satisfy

$$\begin{aligned} p_j &= \sum_{i \in \mathcal{S}} p_i P_{ij}, j \in \mathcal{S} \\ \sum_{i \in \mathcal{S}} p_i &= 1 \end{aligned} \tag{18}$$

Therefore according to [6], the system asymptotic availability  $A_s$  can be obtained:

$$A_s = \frac{\sum_{k \in \mathcal{S}} p_k \mathbb{E}_k(U_1)}{\sum_{k \in \mathcal{S}} p_k \mathbb{E}_k(R_1)} \tag{19}$$

The following theorem shows the system asymptotic availability.

**Theorem 4.1.** *Under the periodic inspection policy, the asymptotic availability of the  $K$ -out-of- $N$  system can be given as*

$$A_s = \frac{\sum_{k \in \mathcal{S}} p_k \int_0^\infty (1 - F_k(t)) dt}{\sum_{k \in \mathcal{S}} p_k (\sum_{i=0}^\infty \tau (1 - F_k(i\tau)) + \int_0^\infty (1 - G(y)) dy)}$$

where  $F_k(t)$  is the system lifetime distribution with initial environment state  $k$  given in equation (13),  $G(y)$  is the distribution function of the repair time,  $p_k$  is the stationary probability derived from the transition probability with

$$P_{ij} = \sum_{m \in \mathcal{S}} \sum_{l \in \mathcal{S}} \sum_{k=1}^\infty \int_0^\infty \pi_{lj}(y) dG(y) \sum_{p=0}^{N-K} \sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} \left( B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \pi_{ml}(\tau) - B_{il}^{\{c_1, c_2, \dots, c_p\}}(k\tau) \right)$$

where  $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t)$  is defined in equations (3),  $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t) = B_{im}^\emptyset(t)$  when  $p = 0$ ,  $\pi_{ij}(t)$  is the transition probability of the continuous time Markov environment.

See Appendix A.5 for the proof.

*Remark:* In effect, the methods we proposed can also be implemented on more complex systems with  $K$ -out-of- $N$  system as subsystems. For instance, consider a series system consisting two subsystems  $s_1$  and  $s_2$  and  $s_i$  is a 1-out-of-2 system with two identical components  $(X_1, X_2; Z_1, Z_2)$  respectively,  $i = 1, 2$ . The failure rate matrix of each component in  $s_i$  is  $R_i$ ,  $i = 1, 2$ . The infinitesimal generator of the Markov environment is defined as  $Q$ . Therefore the system lifetime  $T_s$  satisfies

$$\begin{aligned} R^s(t) &= \mathbb{P}(T_s > t \mid W(0) = i) \\ &= \mathbb{P}(X_j > t, Z_j > t, \forall j = 1, 2 \mid W(0) = i) + 2\mathbb{P}(X_1 > t, X_2 < t, Z_j > t, \forall j = 1, 2 \mid W(0) = i) \\ &\quad + 2\mathbb{P}(X_j < t, Z_1 > t, \forall j = 1, 2 \mid W(0) = i) + 4\mathbb{P}(X_1 > t, X_2 < t, Z_1 > t, Z_2 < t, \forall j = 1, 2 \mid W(0) = i) \end{aligned} \quad (20)$$

By utilizing equation (5), equation (20) can be solved:

$$\begin{aligned} R^s(t) &= \exp\left((Q - 2R_1 - 2R_2)t\right) - 2\left(\exp((Q - R_1 - 2R_2)t)\right) \\ &\quad - 2\exp\left((Q - 2R_1 - R_2)t\right) + 4\exp\left((Q - R_1 - R_2)t\right). \end{aligned}$$

Therefore, given the initial environment state vector, the system reliability can be calculated. Similarly, the system expected remaining lifetime and the asymptotic availability can be derived.

## 5. Numerical examples

### 5.1. Reliability illustration by two methods

To show the advantage of our method, the system reliability is illustrated in the following example. Here we suppose that the environment state changes fast from one to another. For instance, the wind speed may be classified into 4 levels (A, B, C and D) regarding its impact on the wind turbine. In level A, its sojourn time is exponentially-distributed with mean  $\frac{1}{4}$  day and when it leaves level A, the probability that it goes to level B is  $\frac{1}{2}$ , to level C is  $\frac{1}{4}$  and to level D is  $\frac{1}{4}$  too respectively. Similarly wind level transformations can be defined when the weed speed is in level B, C, and D. The continuous time Markov process with infinitesimal generator  $Q$  is defined in the following:

$$Q = \begin{pmatrix} -4 & 2 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -2.5 & 0.5 \\ 2 & 1 & 1 & -4 \end{pmatrix}$$

Suppose that a small area consists 5 identical wind turbines They are called component  $i$  in the following,  $i \in \{1, 2, 3, 4, 5\}$ . The failure rate Matrix of component  $i$  is given as

$$H_i(t) = \text{diag}([0.002, 0.01, 0.005, 0.007])$$

for  $i \in \{1, 2, 3, 4, 5\}$ . Let be the initial environment probability vector  $\alpha = [0, 1, 0, 0]$ . Table 1 presents the run-times of MATLAB2010b in calculating the system reliability of  $K$ -out-of-5 system by time  $t = 40$  through our proposed method and the Monte-Carlo simulation (the number of simulation times is 10000) respectively under different values of  $K$ . It is seen that our

|                   | K=4     |            | K=3    |            |
|-------------------|---------|------------|--------|------------|
|                   | $R(40)$ | run-time   | $R(t)$ | run-time   |
| our method        | 0.6963  | 0.000479s  | 0.9255 | 0.00089s   |
| simulation method | 0.6909  | 54.923931s | 0.9230 | 57.749855s |

Table 1: The system reliability and the calculation run-time of  $K$ -out-of-5 system by time  $t = 40$  through our method and the Monte-carlo simulation

calculation is more efficient and practical especially in dealing with large complex systems consisting of numerous components.

## 5.2. Method implementation in Engineer

In the following, an example is presented to demonstrate how to use the methods we proposed to calculate the system reliability-based measures as well as the system expected remaining useful lifetime. Here we consider a 10-out-of-15 system which operates in a dynamic environment. The following procedures need to be implemented for the evaluation of the quantities.

### 1. Identification of the environment states

The first step is to identify all the environment states. For instance, the failure process of the system is related to 3 environmental elements—the weather condition which can be classified into: "sunny", "cloudy" and "rain"; the visibility condition: "good" and "bad"; and the temperature which can be categorized as " $< 0$  °C" and " $\geq 0$  °C". Therefore the number of the environment states is 12. We use number 1 to 12 to represent the environment states which are listed in the following:

$$\begin{aligned}
 1 &= \{\text{"sunny", "good", "< 0 °C"}\}; & 2 &= \{\text{"sunny", "good", "> 0 °C"}\}; \\
 3 &= \{\text{"sunny", "bad", "< 0 °C"}\}; & 4 &= \{\text{"sunny", "bad", "\ge 0 °C"}\}; \\
 5 &= \{\text{"cloudy", "good", "< 0 °C"}\}; & 6 &= \{\text{"cloudy", "good", "\ge 0 °C"}\}; \\
 7 &= \{\text{"cloudy", "bad", "< 0 °C"}\}; & 8 &= \{\text{"cloudy", "bad", "\ge 0 °C"}\}; \\
 9 &= \{\text{"rain", "good", "< 0 °C"}\}; & 10 &= \{\text{"rain", "good", "\ge 0 °C"}\}; \\
 11 &= \{\text{"rain", "bad", "< 0 °C"}\}; & 12 &= \{\text{"rain", "bad", "\ge 0 °C"}\};
 \end{aligned}$$

### 2. Identification of the transition matrix $\mathbf{Q}$

Following by the identification of the environment state, the transition matrix  $\mathbf{Q}$  of the environment states should also be recognized. It is clear that in this example, the environment states can be described by a  $12 \times 12$  matrix with  $\mathbf{S} = \{1, 2, \dots, 12\}$  where the system holding time at state  $i$  is exponentially distributed with expectation  $-\frac{1}{q_{ii}}$ ,  $\frac{q_{ij}}{-q_{ii}}$  represents the probability that the system goes to state  $j$  once it leaves state  $i$ . For instance, to complete the first row of the matrix  $\mathbf{Q}$ , we assume that when the system is in state 1 which means the weather is fine, the visibility is good and the temperature is lower than 0 °C, its holding time is exponentially distributed with expectation  $\frac{1}{2 \times 10^{-3}}$  time units. Assume that once leaving state 1, it may transfer only to states 2, 3 and 5 with probabilities  $\frac{1}{2}$ ,  $\frac{1}{6}$  and  $\frac{1}{3}$ . Under these assumptions, the first row of the transition matrix is

$$10^{-3} \times [-2, 1, \frac{1}{3}, 0, \frac{2}{3}, 0, 0, 0, 0, 0, 0, 0].$$

Similarly, we may define the transition matrix  $\mathbf{Q}$ . Assume that it is given

as in the following.

$$Q = 10^{-3} \times \begin{pmatrix} -2 & 1 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{2} & -5 & \frac{3}{4} & \frac{1}{2} & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{2} & -4 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -2 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & -\frac{5}{2} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{5}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -3 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & -2 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -2 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & -\frac{3}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 1 & -2 \end{pmatrix}$$

### 3. Recognition of the failure rate of each component under each environment state

In order to evaluate the system reliability based measures, the failure rate of each component under each environment state is also required. Considering that in the area of Engineering, Exponential distribution is the most commonly used distribution to describe the lifetime of a component/system because of its tractability and effectiveness. In this example, we assume that given the environment state, all the components possess constant failure rate and the corresponding failure rate matrices are in the following.

$$H_i = \begin{cases} \text{diag}(10^{-3} \times [2, 1, 2, 3, 4, 5, 4.5, 6, 4, 4, 5, 7]), & i = 1, 2, 3, 4, 5; \\ \text{diag}(10^{-3} \times [4, 7, 6.5, 8, 7.5, 9, 8, 10, 9, 7, 8, 9]), & i = 6, 7, 8, 9, 10; \\ \text{diag}(10^{-2} \times [6, 7, 8, 6, 10, 9, 8.5, 9, 11, 8, 15, 9]), & i = 11, 12, 13, 14, 15. \end{cases}$$

where  $\text{diag}([a_i], i = 1, 2, \dots, 12)$  is a  $12 \times 12$  diagonal matrix with the  $(k, k)^{th}$  entry  $a_k, k = 1, 2, \dots, 12$ .

### 4. Calculation of the quantities by the proposed formulae

Given all the information in procedures 1-3, the system reliability can be derived by equations (5) and (9). Table 2 shows the variation of the system reliability with respect to different values of  $t$  and the initial environment state. As expected, it is seen that under each scenario, the system reliability decreases with time  $t$ . Environment state '1' (sunny day with good visibility and the temperature is lower than  $0^\circ\text{C}$ ) is very friendly to the system lifetime comparing to environment state '4'. System exposed under environment '7' and '12' are very dangerous which implies that the visibility is very important to the system reliability.



Further more, Table 3 shows the system reliability under static environment (state '1' and '12' respectively). It can be observed that the system reliability is either overestimated or underestimated under the two situations. Therefore it is necessary to incorporate the external environment effect on the system lifetime distribution obtained under ideal, controlled, static laboratory condition.

| $t$         | $t = 10$ | $t = 20$ | $t = 30$ | $t = 50$ | $t = 70$ | $t = 80$ | $t = 100$ |
|-------------|----------|----------|----------|----------|----------|----------|-----------|
| $W(0) = 1$  | 0.9904   | 0.8762   | 0.6604   | 0.3014   | 0.1365   | 0.0945   | 0.0474    |
| $W(0) = 4$  | 0.9772   | 0.7647   | 0.4595   | 0.1197   | 0.0312   | 0.0169   | 0.0055    |
| $W(0) = 7$  | 0.9368   | 0.5845   | 0.2750   | 0.0560   | 0.0138   | 0.0072   | 0.0021    |
| $W(0) = 12$ | 0.9011   | 0.4684   | 0.1772   | 0.0249   | 0.0046   | 0.0022   | 0.0005    |

Table 2: The system reliability  $R(t)$  with different values of  $t$  and initial environment state

| $t$        | $t = 10$ | $t = 20$ | $t = 30$ | $t = 50$ | $t = 70$ | $t = 80$ | $t = 100$ |
|------------|----------|----------|----------|----------|----------|----------|-----------|
| state '1'  | 0.9906   | 0.8796   | 0.6682   | 0.3113   | 0.1442   | 0.1010   | 0.0520    |
| state '12' | 0.9014   | 0.4680   | 0.1756   | 0.0235   | 0.0040   | 0.0017   | 0.0003    |

Table 3: The system reliability  $R(t)$  with different  $t$  under static environment state '1' and '12' respectively

Table 4 illustrates the expected remaining useful lifetime of the system given different environment conditions and the component states (functioning or failed). It is obvious that the more component failure occurs, the less is the system expected remaining useful lifetime. Besides, environment state '12' is very furious comparing to environment state '1'. The expected system RUL under environment '1' is around 75.6% larger than the quantity under environment '12' given that the component 1 fails by now. This proportional number is more larger (near 76.8% ) when the failure component set is  $\{1, 15\}$ , which implies the importance of the redundancy to the system lifetime especially when the environment state is fierce. Furthermore, component 1 is more important to the system lifetime comparing to component 15, which is very logical as component 1 possesses a lower failure rate under each environment state comparing to component 15.

| Failure component set | $\{1\}$ | $\{1\}$ | $\{15\}$ | $\{1, 15\}$ | $\{1, 15\}$ | $\{1, 7, 15\}$ | $\{1, 4, 7, 15\}$ |
|-----------------------|---------|---------|----------|-------------|-------------|----------------|-------------------|
| Environment state     | '1'     | '12'    | '1'      | '1'         | '12'        | '1'            | '1'               |
| Expected system RUL   | 26.6496 | 15.1764 | 42.9584  | 24.1890     | 13.6792     | 15.0620        | 8.6472            |

Table 4: The expected system RUL given different initial states of the components and environment states

In the following, by assuming that the system repair time is exponentially distributed  $G(y) = 1 - \exp(-\lambda t)$  with mean value 1. Table 5 presents the asymptotic availability respect to various inspection period  $\tau$ . Obviously it decreases with the inspection period as the larger  $\tau$  is, the later the system failure is detected.

| $\tau$ | 1      | 2      | 5      | 6      | 8      | 9      | 10     | 15     | 20     |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $A_s$  | 0.9523 | 0.9362 | 0.9174 | 0.9025 | 0.8741 | 0.8560 | 0.8475 | 0.7899 | 0.7428 |

Table 5: The asymptotic availabilities of the 10-out-of-15 system with different values of the inspection period  $\tau$

### 5.3. Method implementation to time dependent failure rates

In the previous paragraph, we have explained how to apply the proposed formulas in the calculation of the system quantities from an engineering point of view. We have supposed that the failure rate of each component was time-independent under each environment state. It was a logical and tractable assumption which may effectively facilitate the quantity calculation especially when the system is complex and large. In this paragraph, the applicability of our method to a more general case-systems with time dependent failure rates is presented. Under this situation, the first three procedures in section 5.2 are also valid. With respect to procedure 4, it is noticed that we always confront the equation of the following form.

$$\begin{aligned} \frac{d\mathbf{A}(t)}{dt} &= \mathbf{A}(t)\mathbf{M}(t) + \mathbf{N}(t), \\ \mathbf{A}(a) &= \mathbf{A}_0, a < t, \end{aligned} \quad (21)$$

where  $\mathbf{A}(t)$ ,  $\mathbf{M}(t)$  and  $\mathbf{N}(t)$  are  $n \times n$  matrixes. The closed-form solution of it is nearly impossible to obtain. Here we adopt the product-integration method proposed by Vito Volterra for the numerical approximated solution. Details about the method can be found in [26]. According to which, for the homogeneous case when  $\mathbf{N}(t) = 0$ , the solution of equation (21) is

$$\mathbf{A}(t) = \mathbf{A}_0 \prod_{i=\frac{a}{\delta}}^{\frac{t}{\delta}} \left( I + \mathbf{M}(i\delta)\delta \right),$$

where  $\delta$  is the step size in the calculation. More details can be found in [26].

For the non-homogeneous case, the solution is

$$\mathbf{A}(t) = \mathbf{A}_0 \mathbf{Z}(t) + \int_a^t \mathbf{N}(x) \prod_{i=\frac{x}{\delta}}^{\frac{t}{\delta}} (I + \mathbf{M}(i\delta)\delta) dx,$$

where  $\mathbf{Z}(t) = \Pi_{i=\frac{a}{\delta}}^{\frac{t}{\delta}} (I + \mathbf{M}(i\delta)\delta)$ .

In the next, several examples are given to demonstrate the corresponding quantities of the system with time-dependent failure rate. Let

$$\mathbf{Q} = 10^{-4} \times \begin{pmatrix} -2 & 2 & 0 \\ 3 & -7 & 4 \\ 0 & 4 & -4 \end{pmatrix}$$

Inspired by the numerical example used by [1] where they analyzed the transmissions oil data. In their case, they assumed that the hazard rate of a transmission system were determined by a baseline hazard function (Weibull function) and also the iron values which was taken as the main diagnosed indicator (covariate). Here we assume that the system consists of three components with failure rates

$$h_i(t, j) = \frac{\beta_i}{\lambda_i} \left(\frac{t}{\lambda_i}\right)^{\beta_i-1} \exp(ENV(i, j)), i = 1, 2, 3; j = 1, 2, 3.$$

where  $ENV(i, j)$  is the environment related factor with respect to component  $i$  under environment  $j$  and

$$\begin{aligned} [\beta_1, \beta_2, \beta_3] &= [1.5, 2, 2.5] \\ [\lambda_1, \lambda_2, \lambda_3] &= [10^4, 0.5 \times 10^4, 0.3 \times 10^4] \\ [\exp(ENV(i, 1)), \exp(ENV(i, 2)), \exp(ENV(i, 3))] &= [1, 5, 10], i = 1, 2, 3. \end{aligned}$$

Table 6 shows the conditional radiabilities  $CR_i(t; x, W(x), \{c_2\})$ , ( $x = 1000$ ) of the 2-out-3 system with different initial environment conditions respectively.

Similarly, Figure 1 illustrates the RUL of the series system and the 2-out-of-3 system with different components states (fail or work) and environment conditions ( $W(x) = 1, 2, 3$ ) respectively. In each case, it can be observed that the system operating environment has significant impact on the system reliability as well as the system remaining useful lifetime.

| $W(x)/t$   | $t = 1100$ | $t = 1200$ | $t = 1300$ | $t = 1400$ | $t = 1500$ | $t = 1600$ | $t = 1700$ |
|------------|------------|------------|------------|------------|------------|------------|------------|
| $W(x) = 1$ | 0.9783     | 0.9533     | 0.9254     | 0.8951     | 0.8627     | 0.8285     | 0.7931     |
| $W(x) = 2$ | 0.8990     | 0.7982     | 0.7021     | 0.6126     | 0.5307     | 0.4573     | 0.3924     |
| $W(x) = 3$ | 0.8108     | 0.6458     | 0.5082     | 0.3958     | 0.3055     | 0.2342     | 0.1786     |

Table 6: The conditional reliability of the 2-out-of-3 system given that component 2 fails by time  $x = 1000$  and different initial environment state

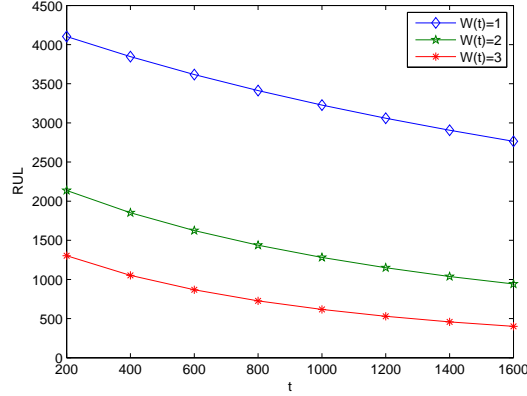


Figure 1: The remaining useful lifetimes  $r_i(t, W(t); \{c_2\})$  of the 2-out-of-3 system with different initial survival time  $t$  and environment state  $W(t) \in \{1, 2, 3\}$

Let

$$Q = \begin{pmatrix} -0.1 & 0.1 \\ 0.2 & -0.2 \end{pmatrix}$$

and

$$h_i(j, t) = 0.05 \times j \times i^2, i = 1, 2, 3; j = 1, 2.$$

Assume that the repair time is exponentially distributed with mean value  $\frac{1}{2}$ . Table 7 illustrates the asymptotic availabilities of the  $k$ -out-of-3 system under the periodic inspection policy with respect to different values of  $\tau$ . As expected, the system limiting availability decreases with respect to the inspection period  $\tau$ .

| $\tau$     | 0.2    | 0.4    | 0.8    | 1      | 1.2    | 1.5    | 1.8    | 2      | 2.5    |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $A_s(K=1)$ | 0.9636 | 0.9579 | 0.9465 | 0.9410 | 0.9355 | 0.9274 | 0.9194 | 0.9142 | 0.9015 |
| $A_s(K=2)$ | 0.8560 | 0.8363 | 0.7994 | 0.7822 | 0.7656 | 0.7418 | 0.7192 | 0.7047 | 0.6701 |
| $A_s(K=3)$ | 0.6448 | 0.6062 | 0.5366 | 0.5054 | 0.4765 | 0.4371 | 0.4021 | 0.3810 | 0.3348 |

Table 7: The asymptotic availabilities of the  $K$ -out-of-3 system under the equidistant inspection policy with different values of the inspection period  $\tau$

## 6. Conclusions

In this paper, we have presented a framework of the  $K$ -out-of- $N$  system operating in random environment, where the objectives are to evaluate the

system reliability-based measures such as the system reliability, the asymptotic availability and to carry out the prognostic analysis. It is shown that the environment state has significant influence to the system reliability and the remaining useful lifetime. The results obtained here are very useful which permit the heterogeneity of components possessing different failure rates as well as the flexibility of the system configuration such as series systems or parallel systems. Besides, the methods we have proposed can also be implemented to more complex systems with  $K$ -out-of- $N$  system as sub-systems. Even still, there still some problems which may be interesting to be investigated in our future work. First, we have considered a system with redundancy which on the one hand, may increase the system reliability, on the other hand, may lead to a rise of the system maintenance cost. Hence it is important to find an equilibrium between the two perspectives. Secondly, it may be interesting to consider the system with cold standby redundancy and to evaluate the corresponding system performance measures. Also, the spare allocation problem remains open. Furthermore, the system asymptotic availability under other maintenance policies could be investigated.

## Appendix A. 1

### Appendix A.1. Proof of Lemma 2.1

From the definition of  $B_{ij}^{\varnothing}(t)$  it is easily seen that

$$\begin{aligned}
B_{ij}^{\varnothing}(t + \Delta t) &= \mathbb{P}(T_l > t + \Delta t, W(t + \Delta t) = j, \forall l \in \Phi \mid W(0) = i) \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}(T_l > t + \Delta t, \forall l \in \Phi, W(t + \Delta t) = j, \mid T_l > t, W(t) = k, \forall l \in \Phi) \\
&\quad \times \mathbb{P}(T_l > t, W(t) = k, \forall l \in \Phi \mid W(0) = i) \\
&= \sum_{k \in \mathcal{S}} \exp\left(-\int_t^{t+\Delta t} \sum_{l \in \Phi} h_l(\theta, k) d\theta\right) \pi_{kj}(\Delta t) B_{ik}^{\varnothing}(t)
\end{aligned}$$

As we know that when  $\Delta t$  is very small,

$$\pi_{kj}(\Delta t) \approx \begin{cases} 1 + q_{jj}\Delta t & k = j \\ q_{kj}\Delta t & k \neq j \end{cases}$$

where  $q_{kj}$  are the entries of  $\mathbf{Q}$ . Therefore

$$\frac{dB_{ij}^{\varnothing}(t)}{dt} = -B_{ij}^{\varnothing}(t) \left( \sum_{l \in \Phi} h_l(t, j) \right) + \sum_{k \in \mathcal{S}} B_{ik}^{\varnothing}(t) q_{kj} \quad (\text{A.1})$$

which can also be represented in the matrix form as

$$\frac{d\mathbf{B}^\varnothing(t)}{dt} = \mathbf{B}^\varnothing(t)(\mathbf{Q} - \mathbf{H}(t)) \quad (\text{A.2})$$

where  $\mathbf{B}^\varnothing(t) = [B_{ij}^\varnothing(t)]_{m \times m}$  and  $\mathbf{H}(t) = \text{diag}[\sum_{l \in \Phi} h_l(t, j)]$  which represents the diagonal matrix with elements  $H_{jj}(t) = \sum_{l \in \Phi} h_l(t, j)$  in the main diagonal. In particular, when the hazard rate is time-independent, i.e.  $\mathbf{H}(t) = \mathbf{H}$ , then it is easily seen that

$$\mathbf{B}^\varnothing(t) = \exp((\mathbf{Q} - \mathbf{H})t)$$

◇

#### Appendix A.2. Proof of Theorem 2.1

*Proof:* The similar method as utilized in the calculation of  $\mathbf{B}^\varnothing(t)$  in Lemma 2.1 is implemented here. From the definition of  $B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)$ , we have

$$\begin{aligned} & B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t + \Delta t) \\ = & \mathbb{P}\left(T_l > t + \Delta t, T_p < t + \Delta t, W(t + \Delta t) = j, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}} \right. \\ & \left. | W(0) = i\right) \\ = & \sum_{m \in \mathcal{S}} \mathbb{P}\left(T_l > t + \Delta t, T_p < t + \Delta t, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}}, \right. \\ & W(t + \Delta t) = j, | T_l > t, T_p < t, \forall p \in \{c_1, c_2, \dots, c_k\}, \forall l \in \underline{\{c_1, c_2, \dots, c_k\}}, \\ & \left. W(t) = m\right) B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) \end{aligned}$$

$$\begin{aligned}
& + \sum_{m \in \mathbf{S}} \sum_{v=1}^k \mathbb{P} \left( T_l > t + \Delta t, \forall l \in \{c_1, c_2, \dots, c_k\}, T_p < t + \Delta t, \forall p \in \{c_1, c_2, \dots, c_k\}, \right. \\
& \quad W(t + \Delta t) = j, | T_l > t, \forall l \in \{c_1, c_2, \dots, c_k\} \bigcup \{c_v\}, T_p < t, \forall p \in \{c_1, c_2, \dots, c_k\} \setminus c_v, \\
& \quad \left. W(t) = m \right) B_{im}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_v\}}(t) + o((\Delta t)) \\
& = \sum_{m \in \mathbf{S}} \pi_{mj}(\Delta t) \exp \left( - \sum_{u \in \{c_1, c_2, \dots, c_k\}} \int_t^{t+\Delta t} h_u(\theta, m) d\theta \right) B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) \\
& + \sum_{m \in \mathbf{S}} \sum_{v=1}^k \pi_{mj}(\Delta t) \exp \left( - \sum_{u \in \{c_1, c_2, \dots, c_k\}} \int_t^{t+\Delta t} h_u(\theta, m) d\theta \right) \\
& \quad \times \left( 1 - \exp \left( - \int_t^{t+\Delta t} h_{c_v}(\theta, m) d\theta \right) \right) B_{im}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_v\}}(t) + o((\Delta t))
\end{aligned}$$

where  $\lim_{t \rightarrow \infty} \frac{o(\Delta t)}{\Delta t} = 0$ . It is known that when  $\Delta t$  is very small,

$$\pi_{kj}(\Delta t) \approx \begin{cases} 1 + q_{jj}\Delta t & k = j \\ q_{kj}\Delta t & k \neq j \end{cases}$$

where  $q_{kj}$  are the entries of  $\mathbf{Q}$ . By implementing the similar method as in the calculation of  $B_{ij}^{\emptyset}(t)$  in equation (A.1), we have

$$\begin{aligned}
\frac{dB_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} & = \sum_{m \in \mathbf{S}} B_{im}^{\{c_1, c_2, \dots, c_k\}}(t) q_{mj} + \sum_{l=1}^k B_{ij}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) h_{c_l}(t, j) \\
& \quad - B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t) \sum_{l \in \{c_1, c_2, \dots, c_k\}} h_l(t, j)
\end{aligned}$$

which can also be represented by the matrix form

$$\frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t)}{dt} = \mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) (\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_k\}} \mathbf{H}_j(t)) + \sum_{l=1}^k \mathbf{B}^{\{c_1, c_2, \dots, c_k\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l}(t)$$

where  $\mathbf{B}^{\{c_1, c_2, \dots, c_k\}}(t) = [B_{ij}^{\{c_1, c_2, \dots, c_k\}}(t)]_{m \times m}$  and  $\mathbf{H}_i(t) = \text{diag}[h_i(t, j)]_{m \times m}$ ,  $i \in \Phi, j \in \mathbf{S} \diamond$

### Appendix A.3. The proof of Corollary 1

*Proof:* The mathematical induction method is carried out in the following proof.

When  $k = 1$ , from Theorem 2.1, we have

$$\frac{d\mathbf{B}^{\{c_1\}}(t)}{dt} = \mathbf{B}^{\{c_1\}}(t)(\mathbf{Q} - \sum_{j \in \Phi \setminus \{c_1\}} \mathbf{H}_j) + \mathbf{B}^\emptyset(t)\mathbf{H}_{c_1}(t)$$

where  $\mathbf{B}^\emptyset(t) = \exp((\mathbf{Q} - \mathbf{H})t)$ . As  $\mathbf{B}^{\{c_1\}}(0) = \mathbf{0}$ , On can deduce that

$$\mathbf{B}^{\{c_1\}}(t) = \exp\left(\left(\mathbf{Q} - \sum_{j \in \Phi \setminus \{c_1\}} \mathbf{H}_j\right)t\right) - \exp((\mathbf{Q} - \mathbf{H})t)$$

Assume that when  $k = p$ , equation (5) works which means

$$\mathbf{B}^{\{c_1, c_2, \dots, c_p\}}(t) = \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_p\} \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp\left(\left(\mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_p\}} \mathbf{H}_j - \sum_{i=1}^m \mathbf{H}_{l_i}\right)t\right) \quad (\text{A.3})$$

When  $k = p + 1$ , as

$$\begin{aligned} \frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t)}{dt} &= \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j \right) \\ &\quad + \sum_{l=1}^{p+1} \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\}}(t) \mathbf{H}_{c_l} \end{aligned} \quad (\text{A.4})$$

where  $\{c_1, c_2, \dots, c_{p+1}\} = \Phi \setminus \{c_1, c_2, \dots, c_{p+1}\}$ . It can be seen that the expression of  $\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\}}(t)$ ,  $l \in \{c_1, c_2, \dots, c_{p+1}\}$  can be derived in equation (A.3) as one can notice that it is the case when the number of failed components is  $p$ . Therefore by substituting the expression of  $\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\}}(t)$



into equation (A.4) we have

$$\begin{aligned}
& \frac{d\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t)}{dt} \\
&= \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j \right) \\
&+ \sum_{l=1}^{p+1} \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_{p+1}\} \setminus \{c_l\} \\ u=1, 2, \dots, m \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left( \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j - \mathbf{H}_{c_l} - \sum_{i=1}^m \mathbf{H}_{l_i} \right) t \right) \mathbf{H}_{c_l} \\
&= \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j \right) \\
&+ \sum_{m=0}^p (-1)^m \sum_{\substack{l_u \in \{c_1, c_2, \dots, c_{p+1}\} \\ u=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left( \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j - \sum_{i=1}^{m+1} \mathbf{H}_{l_i} \right) t \right) \sum_{i=1}^{m+1} \mathbf{H}_{l_i}
\end{aligned} \tag{A.5}$$

Let be  $\mathbf{a} = \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j$ ,  $\mathbf{a}_m(l_1, l_2, \dots, l_{m+1}) = \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j - \sum_{i=1}^{m+1} \mathbf{H}_{l_i}$ ,  $\mathbf{k}_m = (-1)^m \sum_{i=1}^{m+1} \mathbf{H}_{l_i}$ , by solving equation (A.5) we have

$$\begin{aligned}
& \mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t) \\
&= \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \left( \exp(\mathbf{a}t) - \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \right) \mathbf{k}_m (\mathbf{a} - \mathbf{a}_m(l_1, l_2, \dots, l_{m+1}))^{-1} \\
&= \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^m \left( \exp(\mathbf{a}t) - \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \right) \\
&= \sum_{m=0}^p (-1)^m \binom{p+1}{m+1} \exp(\mathbf{a}t) + \sum_{m=0}^p \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^{m+1} \exp(\mathbf{a}_m(l_1, l_2, \dots, l_{m+1})t) \\
&= \exp(\mathbf{a}t) + \sum_{m=1}^{p+1} \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} (-1)^m \exp(\mathbf{a}_{m-1}(l_1, l_2, \dots, l_m)t) \\
&= \sum_{m=0}^{p+1} (-1)^m \sum_{\substack{l_i \in \{c_1, c_2, \dots, c_{p+1}\} \\ i=1, 2, \dots, m+1 \\ l_i \neq l_j \text{ if } i \neq j}} \exp \left( \left( \mathbf{Q} - \sum_{j \in \{c_1, c_2, \dots, c_{p+1}\}} \mathbf{H}_j - \sum_{i=1}^m \mathbf{H}_{l_i} \right) t \right)
\end{aligned}$$

where  $\binom{p+1}{m+1}$  is the  $(m+1)$ -combination of the set with  $p+1$  distinguished elements. It is seen that equation (5) holds for the  $\mathbf{B}^{\{c_1, c_2, \dots, c_{p+1}\}}(t)$  and therefore the proof is complete.  $\diamond$

*Appendix A.4. The proof of Corollary 3*

*Proof:* Denoted

$$L_{ij}^{l_0, l}(\theta, t) = \mathbb{P}(l \text{ components fail at } t, W(t) = j \mid l_0 \text{ components fail at } \theta, W(\theta) = i)$$

with its matrix form  $\mathbf{L}^{l_0, l}(\theta, t) = [L_{ij}^{l_0, l}(\theta, t)]_{m \times m}$  where  $\theta > 0$ ,  $l_0 \leq l < N$ . It is obviously that  $\mathbf{L}^{l_0, l_0}(\theta, \theta) = \mathbf{I}$ ,  $\mathbf{L}^{l_0, l}(\theta, \theta) = \mathbf{0}$  for  $l_0 < l$ . By applying the similar method, we have

$$\frac{d\mathbf{L}^{l_0, l}(\theta, t)}{dt} = \mathbf{L}^{l_0, l-1}(\theta, t)(N-l+1)\mathbf{H}_0 + \mathbf{L}^{l_0, l}(\theta, t)(\mathbf{Q} - (N-l)\mathbf{H}_0)$$

where  $\mathbf{L}^{l_0, l_0-1}(\theta, t) = \mathbf{0}$  which yields

$$\mathbf{L}^{l_0, l}(\theta, t) = \binom{N-l_0}{l-l_0} \sum_{i=0}^{l-l_0} (-1)^{l-l_0-i} \binom{l-l_0}{i} \exp\left((\mathbf{Q} - (N-l_0-i)\mathbf{H}_0)(t-\theta)\right)$$

Therefore  $r_i^{l_0}(\theta)$  can be obtained.

$$r_i^{l_0}(\theta) = \sum_{l=l_0}^{N-M} \int_{\theta}^{\infty} \mathbf{e}_i(\mathbf{L}^{l_0, l}(\theta, t)) \mathbf{e} dt$$

where  $\mathbf{e}$  is a column vector of 1s,  $\mathbf{e}_i$  is a  $1 \times m$  matrix whose  $i$ th element is 1 and others are 0 respectively.  $\diamond$

*Appendix A.5. The proof of Theorem 4.1*

*Proof:* It is easily seen that the expectation of the system lifetime distribution when the initial environment is  $k$ ,  $k \in \mathbf{S}$  can be derived as

$$\mathbb{E}_k(U_1) = \int_0^{\infty} (1 - F_k(t)) dt$$

where  $F_k(t)$  is the conditional system lifetime distribution given is equation (13).

Furthermore, the length of a renewal cycle when the initial environment is  $k$  can be obtained:

$$\begin{aligned}\mathbb{E}_k(R_1) &= \sum_{i=1}^{\infty} i\tau \mathbb{P}((i-1)\tau < U_1 \leq i\tau \mid W_0 = k) + \mathbb{E}Y \\ &= \tau \sum_{i=0}^{\infty} (1 - F_k(i\tau)) + \int_0^{\infty} (1 - G(y))dy\end{aligned}$$

Denoted by  $h(t)$  the system failure rate at time  $t$ . Define

$$H_{\min}(t) = \int_0^t \sum_{i \in \Phi} h_i(t, 1)dt$$

where  $h_i(t, 1)$  is the failure rate of component  $i$  under environment 1. Then

$$1 - F_k(t) \leq \exp(-H_{\min}(t))$$

As  $\int_0^{\infty} h_i(t, j)dt = \infty$  for  $i \in \Phi, j \in \mathcal{S}$  and  $\mathbb{E}_k(U_1) = \int_0^{\infty} (1 - F_k(t))dt$  therefore  $\mathbb{E}_k(U_1)$  is convergent. Also from the expression of  $E_k(R_1)$ , it is easily seen the convergence of  $\mathbb{E}_k(R_1)$ . So  $\frac{\mathbb{E}_k(U_1)}{\mathbb{E}_k(R_1)}$  exists for any  $k \in \mathcal{S}$ .

In the next, we focus on the calculation of the transition kernel  $Q_{ij}(t)$ . Let us define

$$\mathcal{L}\{Q_{ij}\}(s) = \mathbb{E}[e^{-sR_1} \mathbf{1}_{W_{R_1}=j} \mid W_0 = i]$$

where  $\mathbf{1}_A(x)$  is the indicator function which equals 1 when  $x \in A$  and 0 otherwise. By conditioning on the repair time  $Y$  and the first failure diagnosed time  $U_1$  we may write

$$\begin{aligned}&\mathcal{L}\{Q_{ij}\}(s) \\ &= \int_0^{\infty} \mathbb{E}[e^{-s(U_1+y)} \mathbf{1}_{W_{U_1+y}=j} \mid W_0 = i]dG(y) \\ &= \sum_{l \in \mathcal{S}} \sum_{k=1}^{\infty} \mathbb{P}(U_1 = k\tau, W_{k\tau} = l \mid W_0 = i) \int_0^{\infty} \pi_{lj}(y) e^{-s(k\tau+y)} dG(y) \\ &= \sum_{l \in \mathcal{S}} \sum_{m \in \mathcal{S}} \sum_{k=1}^{\infty} \sum_{p=0}^{N-M} \mathbb{P}(U_1 = k\tau, W_{k\tau} = l \mid T_u > (k-1)\tau, T_v < (k-1)\tau,\end{aligned}$$

$$\begin{aligned}
& \forall u \in \{c_1, c_2, \dots, c_p\}, \forall v \in \{c_1, c_2, \dots, c_p\}, W_{(k-1)\tau} = m) \\
& \times \mathbb{P}(T_u > (k-1)\tau, T_v < (k-1)\tau, \forall u \in \{c_1, c_2, \dots, c_p\}, \forall v \in \{c_1, c_2, \dots, c_p\}, \\
& W_{(k-1)\tau} = m, | W_0 = i) \int_0^\infty \pi_{lj}(y) e^{-s(k\tau+y)} dGy \\
& = \sum_{l \in \mathcal{S}} \sum_{m \in \mathcal{S}} \sum_{k=1}^\infty \sum_{p=0}^{N-M} \int_0^\infty \pi_{lj}(y) e^{-s(k\tau+y)} dG(y) B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \\
& \quad \times \mathbb{P}(U_1 = k\tau, W_{k\tau} = l | T_u > (k-1)\tau, T_v < (k-1)\tau, \forall u \in \{c_1, c_2, \dots, c_p\}, \\
& \quad \forall v \in \{c_1, c_2, \dots, c_p\}, W_{(k-1)\tau} = m) \\
& = \sum_{m \in \mathcal{S}} \sum_{l \in \mathcal{S}} \sum_{k=1}^\infty \int_0^\infty \pi_{lj}(u) e^{-s(k\tau+y)} dG(y) \sum_{p=0}^{N-M} \sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} \left( B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \right. \\
& \quad \left. \times \pi_{ml}(\tau) - B_{ml}^1(k\tau) \right)
\end{aligned}$$

Therefore, the  $(i, j)$ th element of the transition probability matrix  $\mathbf{P}$  can be given as:

$$\begin{aligned}
P_{ij} &= \lim_{s \rightarrow 0} \mathcal{L}\{Q_{ij}\}(s) \\
&= \sum_{m \in \mathcal{S}} \sum_{l \in \mathcal{S}} \sum_{k=1}^\infty \int_0^\infty \pi_{lj}(y) dG(y) \sum_{p=0}^{N-M} \sum_{1 \leq c_1 < c_2 < \dots < c_p \leq N} \left( B_{im}^{\{c_1, c_2, \dots, c_p\}}((k-1)\tau) \pi_{ml}(\tau) \right. \\
& \quad \left. - B_{il}^{\{c_1, c_2, \dots, c_p\}}(k\tau) \right)
\end{aligned}$$

where  $B_{im}^{\{c_1, c_2, \dots, c_p\}}(t) = B_{im}^\emptyset(t)$  when  $p = 0$  which is given in Equation (A.1).

◇

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