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Classifying Tensor Triangulated Subcategories

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Abstract

We define triangulated and tensor triangulated categories, and classify the thick subcategories of a specific tensor triangulated category using Paul Balmer's theory on the categorical spectrum.

Sammendrag

Vi definerer triangulerte og tensortriangulerte kategorier, og klassifiserer de tjukke underkategoriene til en spesifikk tensortriangulert kategori ved hjelp av Paul Balmers teori om det kategorielle spekteret.

Preface

This thesis marks the end of my time as a student at NTNU. The thesis was written under the supervision of Professor Petter Andreas Bergh.

First and foremost I would like to thank my supervisor Professor Petter Andreas Bergh. Your patience and guidance through my seemingly endless line of questions and struggles has been unparallelled and invaluable, and our many talks and discussions on mathematical and non-mathematical subjects have been very enjoyable and motivating through trying times. For this I am deeply grateful. I hope you won't miss me and my antics too much, as I am almost certain that my absence will leave a little void in your heart that can never truly be filled by anyone but me. I will not say: do not weep; for not all tears are an evil.

Second, a huge thank you to all of my friends at Matteland for the coffee breaks, ping-pong breaks, chess breaks, breaks in general, shared frustations and for the helpful discussions over the years. I would also like to thank Erlend Loe for providing the translations of the quotes in this thesis, and your authorship in general.

Finally, I would like to thank my family, my friends and my partner for the continued encouragement and loving support. Without you this thesis would never have come to exist, and I truly appreciate everything that you have done for me.

Thank you everyone.

Magnus Eggen
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Notation

$\subset$	=	Strict inclusion
$\subseteq$	=	Inclusion
$\mathbb{Z}$	=	The ring of integers
$\mathbb{Z}_n$	=	The ring of integers modulo n
$\mathrm{Mod}(R)$	=	The category of modules over the ring R
$\mathrm{mod}(R)$	=	The category of finitely generated modules over the ring R
$\mathrm{proj}(R)$	=	The category of finitely generated projective modules over the ring R
$\mathbf{D}^b(R)$	=	The bounded derived category of the ring R

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Chapter 1

Introduction

> Fantastic and improbable things happen simultaneously all the time, but mostly we never get to know about it. It takes place hidden away from us, but in this very case we are granted insight, and that should please even the most tenacious cynic.
>
> Erlend Loe, *Volvo Lastvagnar*

Triangulated categories, discovered simultaneously and independently, by Verdier and Puppe in the 60's, are popular structures in today's mathematical disciplines. Finding connections and similarities between the triangulated structures of two categories has proven to be fruitful to better understand the categories in question. In the upcoming chapter we will deal with triangulated categories, define thick subcategories, and present results that will prove useful throughout the thesis. Then we follow up with a presentation of homotopy categories, and the chapter concludes with an example showing that the homotopy category of chain complexes over an additive category is a triangulated category. Even though the triangulated structure is sufficient in many cases, additional structure might be necessary to uncover further details. This is where the tensor product comes into the picture.

Equipping a triangulated category with a tensor product, or a monoidal symmetric structure, gives a *tensor triangulated category*, and opens the door for us to classify *thick subcategories*. Balmer's article on the categorical spectrum of tensor tensor triangulated [2], which the thesis is largely based on, takes on the tensor triangulated category and makes use of the work of Hopkins [5], Neeman [11, 12, 13] to classify its thick subcategories with ideas from commutative algebra.

Balmer introduces the notion of thick tensor ideals, and uses this to define *prime ideals* in a familiar fashion. The spectrum of a tensor triangulated category, $\mathcal{K}$, is named $\mathrm{Spc}(\mathcal{K})$, and consists of these prime ideals. Balmer then continues by defining the Zariski topology and categorical support of an object in the category. Balmer then uses these ideas to generalize and introduce the *support data* on a tensor triangulated category $\mathcal{K}$. With this in place, Balmer presents a bijection between radical thick tensor ideals of $\mathcal{K}$ and subsets of Spc($\mathcal{K}$), where these subsets are unions of support. Although the bijection does not specifically classify the thick subcategories of a tensor triangulated category, it proves to be a useful tool. This theory and some examples are covered in Chapter 3.

Understanding the structure of thick subcategories of mod(R) reveals information about the commutative, noetherian ring R, and these subcategories are related to the thick subcategories of $\mathbf{D}^b(R)$ which is a triangulated category. These sort of connections make thick subcategories interesting to work with, and classifying them has applications across several fields of mathematics. In Chapter 4 we introduce the Hopkins-Neeman bijection [12] and the category we are working with, namely $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$. The ring $\mathbb{Z}_n$ will be further investigated, along with its spectrum and support. We then look at what a product of two tensor triangulated categories is, and how we can use that to our advantage. Finally, we begin to apply the theory we have presented on said tensor triangulated category, and classify its thick subcategories.

In this thesis it is assumed that the reader has a basic knowledge of category theory, as well as homological and commutative algebra.

Chapter 2

Triangulated categories

> Even though my father wasn't called Bongo, I'll name the calf Bongo after him. Sometimes you've got to be open to associations of this kind.
>
> Erlend Loe, *Doppler*

The following chapter will go through the axioms of triangulated categories, some geometric associations related to triangulated categories and the definition of thick subcategories will be presented. Then some useful results will be proved before looking at homotopy categories and why they are triangulated.

2.1 Defining triangulated categories

Definition 2.1.1. A *triangulated category* is an additive category $\mathcal{K}$, together with an autoequivalence $[1] : \mathcal{K} \to \mathcal{K}$, and a class Δ of diagrams of the form
$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ such that

(TR1)
- For any morphism $f : A \longrightarrow B$ in $\mathcal{K}$, there is a diagram $A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$ in Δ.
- For any object A, the diagram $A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow A[1]$ is in Δ.
- The class Δ is closed under isomorphisms. [1]

(TR2) For any diagram $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in Δ, the diagrams

[1] Being closed under isomorphism means that for any triangle T in Δ, if there exists an isomorphism $\phi : T \longrightarrow T'$ where T' is another triangle in $\mathcal{K}$, then T' is also in Δ.

$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$, and
$C[-1] \xrightarrow{-h[-1]} A \xrightarrow{f} B \xrightarrow{g} C$ are in Δ.

(TR3) Given the solid part of a diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

where the leftmost square commutes, and the rows are in Δ, one can always find a morphism *w* as indicated such that the entire diagram becomes commutative.

(TR4) *Octahedral axiom*: Given the solid part of the following diagram, where the two upper rows and the second column are in Δ,

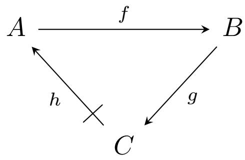
$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C' & \longrightarrow & A[1] \\ \| & & \downarrow{\scriptstyle w} & & \vdots & & \| \\ A & \xrightarrow{w\circ f} & C & \longrightarrow & B' & \longrightarrow & A[1] \\ & & \downarrow & & \vdots & & \downarrow{\scriptstyle f[1]} \\ & & A' & = & A' & \xrightarrow{h} & B[1] \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g[1]\circ h} & & \\ & & B[1] & \xrightarrow{g[1]} & C'[1] & & \end{array}$$

there are morphisms as indicated by the dashed arrows, such that also the third column is in Δ, and the entire diagram commutes.

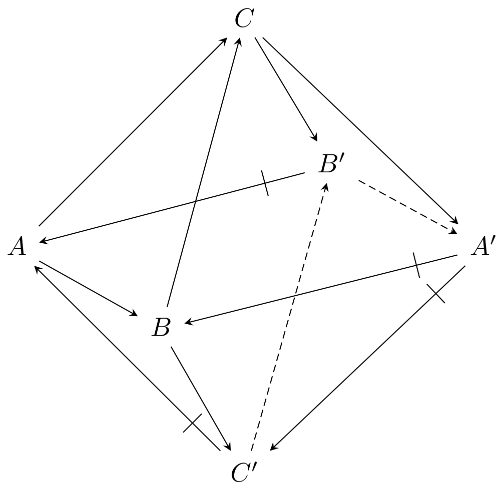
The triangles in Δ are called *distinguished triangles*, and the triangulated structure becomes apparent when remarking that the morphism $C \to A[1]$ can be denoted by the arrow

$$C \mapstochar\longrightarrow A.$$

Then the distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ can be depicted as



The reason that the last axiom is called the octahedral axiom is, not surprisingly, due to the fact that the corresponding diagram can be drawn like an octahedron. With our new geometric understanding of the triangle, the octahedron can be drawn like this:



in which all oriented triangles are in Δ, and all non-oriented triangles and squares commute. If the first three axioms are fulfilled one can swap the Octahedral axiom for another axiom called the *Mapping Cone Axiom*, which was proved by Neeman [11, 13].

Axiom 2.1.2. (Mapping Cone Axiom).
Given a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle u[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1] \end{array}$$

whose rows are distinguished triangles, there exists a map $w\colon C \longrightarrow C'$ such that the diagram commutes, and the mapping cone

$$B \oplus A' \xrightarrow{\begin{pmatrix} -g & 0 \\ v & f' \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} -h & 0 \\ w & g' \end{pmatrix}} A[1] \oplus C' \xrightarrow{\begin{pmatrix} -f[1] & 0 \\ u[1] & h' \end{pmatrix}} B[1] \oplus A'[1]$$

is a distinguished triangle.

Whether the Octahedral axiom is necessary for a category to be triangulated or not, is a widely discussed topic. Now, the Mapping Cone axiom does not resolve the discussion, but rather works as a useful tool if one is having a hard time showing that the traditional Octahedral axiom holds. It is worth mentioning that there are other equivalent versions the axioms, but these will not be covered here[2].

Now, when studying additive categories we often come across additive functors, which are functors between additive categories that preserve finite coproducts, i.e. the additive structure. Analogously we have functors between triangulated categories called triangulated functors, which we will now define.

Definition 2.1.3. A *triangulated functor* is an additive functor

$$F\colon \mathcal{K} \to \mathcal{L}$$

between two triangulated categories, which commutes with the translation, $[1]$, and takes distinguished triangles to distinguished triangles.

With triangulated categories you also get a substructure, namely triangulated subcategories.

Definition 2.1.4. Let $\mathcal{L}$ be an additive subcategory of the triangulated category $\mathcal{K}$. $\mathcal{L}$ is a *triangulated subcategory* when it is closed under isomorphism and translation, and is such that whenever two out of the objects A, B, C in a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ belong to $\mathcal{L}$, then so does the third.

This the latter condition of the definition is often called the "two out of three"-condition. An interesting observation is that this very requirement can be reformulated to demanding that for a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, then, for example, $A, B \in \mathcal{L}$ implies that $C \in \mathcal{L}$. This comes down to the subcategory being closed under translation and (TR2). We can now define one of the structures that will be focused on throughout the thesis, namely the *thick subcategories*.

Definition 2.1.5. (Thick subcategory). A subcategory $\mathcal{L}$ of a triangulated category $\mathcal{K}$ is thick if it is a triangulated subcategory and for any object $A \in \mathcal{L}$ which splits, i.e. $A \cong B \oplus C$, we have that $B, C \in \mathcal{L}$.

2.2 Some useful results

We will now explore some of the traits that the triangulated category structure exhibits. The first result deals with the composition of morphisms in a distinguished triangle.

Lemma 2.2.1. *Let $\mathcal{K}$ be a triangulated category, and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

[2]The interested reader is referred to the work of May[9, 10].

Proof. Let us begin by taking our distinguished triangle, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and pair it up with the following distinguished triangle: $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow A[1]$. This gives us the following diagram:

$$\begin{CD} A @>1>> A @>>> 0 @>>> A[1] \\ @V1VV @VfVV @. @VVf[1]V \\ A @>f>> B @>g>> C @>h>> A[1] \end{CD}$$

Since this is in a triangulated category, and the first square commutes, we have from (TR3) that there exists a map from 0 to C which makes the entire diagram commute. That implies that the composition $g \circ f = 0$. Using the shift axiom, (TR2), and the very same argument for the resulting diagrams, we get that $h \circ g = 0$ and $f[1] \circ h = 0$ as well. This completes the proof. □

Proposition 2.2.2. *Given a triangulated category $\mathcal{K}$ and a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$, if we apply the functor* $\mathrm{Hom}(D, -) := \mathrm{Hom}_{\mathcal{K}}(D, -)$ *for any object D on the distinguished triangle we obtain a long exact sequence of abelian groups:*

$$\cdots \to \mathrm{Hom}(D, A[i]) \to \mathrm{Hom}(D, B[i]) \to \mathrm{Hom}(D, C[i]) \to \mathrm{Hom}(D, A[i+1]) \to \cdots$$

Similarly for the contravariant functor $\mathrm{Hom}(-, D)$.

Proof. Since we are in a triangulated category, it suffices to show that

$$\mathrm{Hom}(D, A) \xrightarrow{f^*} \mathrm{Hom}(D, B) \xrightarrow{g^*} \mathrm{Hom}(D, C)$$

is exact, since we can make use of the rotation axiom. For this sequence to be exact we need $\mathrm{Im}(f^*) = \mathrm{Ker}(g^*)$. Since $g \circ f = 0$, $g^* \circ f^* = 0$ and we have that $\mathrm{Im}(f^*) \subset \mathrm{Ker}(g^*)$. The other inclusion follows from looking at a map $u \in \mathrm{Ker}(g^*)$ and the following diagram

$$\begin{CD} 0 @>>> D @>1>> D @>>> 0 \\ @VVV @. @VuVV @VVV \\ C[-1] @>-h[-1]>> A @>f>> B @>g>> C \end{CD}$$

By (TR1) and (TR2) the two rows are distinguished triangles, also the right square commutes by choice of u. Now, by (TR2) and (TR3) we have a morphism $v\colon D \to A$ making the whole diagram commute. This means that $f \circ v = u$ and hence $u \in \mathrm{Im}(f^*)$. So $\mathrm{Im}(f^*) = \mathrm{Ker}(g^*)$ and the sequence is exact. □

In a triangulated category there are several variants of triangles not necessarily equipped with a classifying adjective and properties. The distinguished triangles are such that when we apply the Hom-functor the result is a long exact sequence. In general, there are triangles that do the exact same thing, but are not distinguished. These are called exact triangles.

Lemma 2.2.3. *Consider the triangulated category $\mathcal{K}$ and the following commutative diagram consisting of two exact triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1]\,. \end{array}$$

If the morphisms u and v are isomorphisms, then so is w.

Proof. We take our diagram and apply the functor $\mathrm{Hom}_{\mathcal{K}}(-, C)$. This results in the following diagram, where $(A, C) = \mathrm{Hom}_{\mathcal{K}}(A, C)$ for the purpose of space saving:

$$\begin{array}{ccccccccc} (A,C) & \longleftarrow & (B,C) & \longleftarrow & (C,C) & \longleftarrow & (A[1],C) & \longleftarrow & (B[1],C) \\ \uparrow{\scriptstyle u^*} & & \uparrow{\scriptstyle v^*} & & \uparrow{\scriptstyle w^*} & & \uparrow{\scriptstyle u^*[1]} & & \uparrow{\scriptstyle v^*[1]} \\ (A',C) & \longleftarrow & (B',C) & \longleftarrow & (C',C) & \longleftarrow & (A'[1],C) & \longleftarrow & (B'[1],C) \end{array}$$

Now this is an exact sequence of abelian groups, so w^* is an isomorphism by the familiar five lemma for abelian categories. Now, since w^* is an isomorphism, and hence an epimorphism, we know there exists a left inverse $p \in \mathrm{Hom}_{\mathcal{K}}(C', C)$ such that $p \circ w = id_C$. Applying the $\mathrm{Hom}_{\mathcal{K}}(C, -)$-functor on the same diagram, and using the same argument, we get that w is an isomorphism. □

Most structures require closedness under various binary operations like addition or multiplication. The following result shows us that adding two distinguished triangles using direct sum gives us another distinguished triangle.

Proposition 2.2.4. *Adding two distinguished triangles through direct sum yields a distinguished triangle.*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} A'[1]$ be distinguished triangles, and consider the following diagram:

$$\begin{array}{ccccccc} A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{(r,r')} & D & \xrightarrow{\left(\begin{smallmatrix} s \\ s' \end{smallmatrix}\right)} & (A \oplus A')[1] \\ \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & & & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)[1]} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \end{array}$$

where $r\colon\ B \to D$, $r'\colon\ B' \to D$, $s\colon\ D \to A[1]$, and $s'\colon D \to A'[1]$ are maps between objects, and $(r, r')\colon B \oplus B' \to D$ and $\left(\begin{smallmatrix} s \\ s' \end{smallmatrix}\right) : D \to A \oplus A'$ are the canonical maps that rises from the previously defined maps. (TR1) assures us that the $D \in \mathcal{K}$ is such that the upper row is a distinguished triangle. Now, (TR3) gives us the morphism $u\colon\ C \to D$ which makes the diagram commute. By the same argument we get a morphism $u'\colon C' \to D$ for the second distinguished triangle. This gives us two morphisms of distinguished triangles, which we add together using the natural isomorphism $\phi\colon A[1] \oplus A'[1] \to (A \oplus A')[1]$. This now results in the following diagram:

$$\begin{array}{ccccccc} A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{(r,r')} & D & \xrightarrow{\left(\begin{smallmatrix} s \\ s' \end{smallmatrix}\right)} & (A \oplus A')[1] \\ \| & & \| & & \uparrow{\scriptstyle (u,u')} & & \| \\ A \oplus A' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & B \oplus B' & \xrightarrow{\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)} & C \oplus C' & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)} & (A \oplus A')[1]\,. \end{array}$$

This diagram commutes. The bottom row is an exact triangle since it is the direct sum of two distinguished (hence exact) triangles. By Proposition 1.2.2, the map (u, u') is an isomorphism, so by (TR1) the bottom row is distinguished. □

2.3 Homotopy categories

Now we move on to homotopy categories and what makes the homotopy category of chain complexes triangulated.

Definition 2.3.1. Let $\mathcal{A}$ be an additive category and $\mathbf{C}(\mathcal{A})$ be the category of chain complexes over this additive category. In $\mathbf{C}(\mathcal{A})$ we construct a translation functor $[1]$ by shifting any complex one degree to the left. More precisely, for an object $A = (A_n, d_n^A)_{n\in\mathbb{Z}}$ in $\mathbf{C}(\mathcal{A})$ we set $A[1] := (A[1]_n, d_n^{A[1]})_{n\in\mathbb{Z}}$ with $A[1]_n = A_{n-1}$ and $d_n^{A[1]} = -d_{n-1}^A$.

The category of chain complexes is not triangulated. In order to construct a triangulated category, we look at the homotopic maps in $\mathbf{C}(\mathcal{A})$ and define the *homotopy category of chain complexes*.

Definition 2.3.2. The homotopy category of chain complexes, $\mathbf{K}(\mathcal{A})$, consists of the same objects as $\mathbf{C}(\mathcal{A})$, and its morphisms are maps of chain complexes modulo homotopy. This means that we define an equivalence relation for homotopic maps, such that $f \sim g$ if f is homotopic to g. Summarized: $\mathrm{Ob}(\mathbf{K}(\mathcal{A})) = \mathrm{Ob}(\mathbf{C}(\mathcal{A}))$, and $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(A, B)/\sim$ for chain complexes A and B.

If we are to have a triangulated structure in a category we need to have a class of triangles. The natural class of triangles in $\mathbf{K}(\mathcal{A})$ need the construction of mapping cones.

Definition 2.3.3. Let $f\colon A \to B$ be a morphism of complexes in $\mathbf{C}(\mathcal{A})$

$$\begin{array}{ccccccccccc} A\colon & & \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_n & \xrightarrow{d_n^A} & A_{n-1} & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ B\colon & & \cdots & \longrightarrow & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} & \longrightarrow & \cdots \end{array}$$

and consider the following diagram

$$\begin{array}{cccccccccc}
B\colon & \cdots \longrightarrow & B_{n+1} & \xrightarrow{d^B_{n+1}} & B_n & \xrightarrow{d^B_n} & B_{n-1} & \longrightarrow \cdots \\
\downarrow{\scriptstyle \alpha(f)} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & \\
M(f)\colon & \cdots \longrightarrow & A_n \oplus B_{n+1} & \xrightarrow{d^{M(f)}_{n+1}} & A_{n-1} \oplus B_n & \xrightarrow{d^{M(f)}_n} & A_{n-2} \oplus B_{n-1} & \longrightarrow \cdots \\
\downarrow{\scriptstyle \beta(f)} & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle (1\ 0)} & \\
A[1]\colon & \cdots \longrightarrow & A_n & \xrightarrow{-d^A_n} & A_{n-1} & \xrightarrow{-d^A_{n-1}} & A_{n-2} & \longrightarrow \cdots
\end{array}$$

The middle row, $M(f)$, is the mapping cone of f where

$$d_n^{M(f)}\colon A_{n-1} \oplus B_n \xrightarrow{\begin{pmatrix} -d^A_{n-1} & 0 \\ f_{n-1} & d^B_n \end{pmatrix}} A_{n-2} \oplus B_{n-1},$$

while

$$\alpha(f)\colon B \to M(f), \ \ \alpha(f)_n := \begin{pmatrix} 0 \\ 1_{B_n} \end{pmatrix}$$

and

$$\beta(f)\colon M(f) \to A[1], \ \ \beta(f)_n := (1_{A_{n-1}}\ 0)$$

are canonical maps.

With this in place we can construct *standard triangles* in $\mathbf{K}(\mathcal{A})$.

Definition 2.3.4. A sequence of objects and morphisms in the homotopy category $\mathbf{K}(\mathcal{A})$ of the form

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1]$$

is called a standard triangle.

We want to show that $\mathbf{K}(\mathcal{A})$ is a triangulated category. First, we define our class Δ of distinguished triangles: if a triangle, T, in $\mathbf{K}(\mathcal{A})$ is isomorphic to a standard triangle in $\mathbf{K}(\mathcal{A})$, then T is a distinguished triangle. This class of distinguished triangles is closed under isomorphism by definition. Now, let us check the axioms.

($\mathbf{TR1}$) For any morphism of complexes $f\colon A \longrightarrow B$ in $\mathbf{K}(\mathcal{A})$ we canonically have a standard triangle

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1]$$

which is in our class of distinguished triangles Δ. Also, for $A \xrightarrow{1_A} A$ we obtain the diagram

$$A \xrightarrow{1_A} A \longrightarrow M(1_A) \longrightarrow A[1].$$

Now, the identity morphism on the mapping cone $M(1_A)$ is actually homotopic to zero via $s = (s_n)_{n\in\mathbb{Z}}$, where $s_n = \begin{pmatrix} 0 & 1_{A_n} \\ 0 & 0 \end{pmatrix}$ in $\mathbf{C}(\mathcal{A})$. This becomes obvious through the following diagram

$$\begin{array}{llcccccccc}
M(1_A)\colon & \cdots \to & A_n \oplus A_{n+1} & \xrightarrow{d} & A_{n-1} \oplus A_n & \xrightarrow{d} & A_{n-2} \oplus A_{n-1} & \to \cdots \\
\downarrow 1_{M(1_A)} & & \downarrow 1 & \swarrow s_n & \downarrow 1 & \swarrow s_{n-1} & \downarrow 1 & \\
M(1_A)\colon & \cdots \to & A_n \oplus A_{n+1} & \xrightarrow{d} & A_{n-1} \oplus A_n & \xrightarrow{d} & A_{n-2} \oplus A_{n-1} & \to \cdots
\end{array}$$

Namely:

$$\begin{aligned}
d_{n+1} \circ s_n + s_{n-1} \circ d_n &= \begin{pmatrix} -d_n^A & 0 \\ 1_{A_n} & d_{n+1}^A \end{pmatrix} \circ \begin{pmatrix} 0 & 1_{A_n} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_{A_{n-1}} \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} -d_{n-1}^A & 0 \\ 1_{A_{n-1}} & d_n^A \end{pmatrix} \\
&= \begin{pmatrix} 1_{A_{n-1}} & 0 \\ 0 & 1_{A_n} \end{pmatrix}
\end{aligned}$$

This means that the identity $1_{M(1_A)}$ equals the zero map in $\mathbf{K}(\mathcal{A})$, hence $M(1_A)$ is isomorphic to the zero complex. So the triangle $A \xrightarrow{1_A} A \to 0 \to A[1]$ is in Δ.

($\mathbf{TR2}$) For this axiom we need to show that for an arbitrary triangle in Δ, the shifted triangles, both left and right, are also in Δ. It suffices to show rotation for a single direction as the proofs are analogous. So we pick a standard triangle

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1]$$

and look at the shifted triangle

$$B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} A[1] \xrightarrow{-f[1]} B[1]$$

to see if it is isomorphic to the following standard triangle

$$B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(\alpha(f))} M(\alpha(f)) \xrightarrow{\beta(\alpha(f))} B[1].$$

This ultimately means that we need to construct an isomorphism between the latter two triangles. Using the identity map between the first, second and fourth entries, the problem boils down to proving that $A[1] \cong M(\alpha(f))$ via an isomorphism that gives a commutative diagram. Recall that $M(\alpha(f))$ is the complex

$$\cdots \to M(\alpha(f))_{n+1} \xrightarrow{d_{n+1}^{M(\alpha(f))}} M(\alpha(f))_n \xrightarrow{d_n^{M(\alpha(f))}} M(\alpha(f))_{n-1} \to \cdots$$

where $M(\alpha(f))_n = B_{n-1} \oplus A_{n-1} \oplus B_n$. So we need to define a pair of morphisms between the two complexes and show that their composition is the identity map for each complex, respectively. We define

$$\phi = (\phi_n)\colon A[1] \to M(\alpha(f))$$

by setting $\phi_n = (-f_{n-1}, 1_{A_{n-1}}, 0)$, and conversely

$$\psi = (\psi_n)\colon M(\alpha(f)) \to A[1]$$

by setting $\psi_n = (0, 1_{A_{n-1}}, 0)$. This gives us the following diagram:

$$\begin{array}{ccccccc}
B & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & A[1] & \xrightarrow{-f[1]} & B[1] \\
\downarrow{\scriptstyle 1_B} & & \downarrow{\scriptstyle 1_{M(f)}} & & {\scriptstyle\psi}\uparrow\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle 1_{B[1]}} \\
B & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\alpha(\alpha(f))} & M(\alpha(f)) & \xrightarrow{\beta(\alpha(f))} & B[1]
\end{array}$$

in which we have that $\beta(\alpha(f)) \circ \phi = -f[1]$, by definition, and $\phi \circ \beta(f) \sim \alpha(\alpha(f))$ via the homotopy given by

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : M(f)_n = A_{n-1} \oplus B_n \to M(\alpha(f))_{n+1} = B_n \oplus A_n \oplus B_{n+1},$$

so $(1_B, 1_{M(f)}, \phi)$ is a morphism of triangles. Looking at ψ we find that $\psi \circ \alpha(\alpha(f)) = \beta(f)$ and $-f[1] \circ \psi \circ \beta(\alpha(f))$ via the homotopy

$$(0, 0, -1)\colon M(\alpha(f))_n \to B[1]_n.$$

Now, for conclusion we show that ψ and ϕ are isomorphisms in our category. We have by definition that $\psi \circ \phi = 1_{A[1]}$ and we have that $\phi \circ \psi \sim 1_{M(\alpha(f))}$ via the homotopy map

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : M(\alpha(f))_n \to M(\alpha(f))_{n+1} = B_{n-1} \oplus A_{n-1} \oplus B_n.$$

This proves that the axiom holds for $\mathbf{K}(\mathcal{A})$.

($\mathbf{TR3}$) Assume that we have a diagram in $\mathbf{K}(\mathcal{A})$

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(u)} & A[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f[1]} \\
A' & \xrightarrow{u'} & B' & \xrightarrow{\alpha(u')} & M(u') & \xrightarrow{\beta(u')} & A'[1]
\end{array}$$

where the left square commutes. This ultimately means that there exist homotopy maps $s_n\colon A_n \to B'_{n+1}$ such that $g_n \circ u_n - u'_n \circ f_n = d^B_{n+1} \circ s_n + s_{n-1} \circ d^A_n$ for all $n \in \mathbb{Z}$. Now, we define a map $h = (h)_n\colon M(u) \to M(u')$, where

$$h_n = \begin{pmatrix} f_{n-1} & 0 \\ s_{n-1} & g_n \end{pmatrix} : M(u)_n \to M(u')_n,$$

and $M(u)_n = A_{n-1} \oplus B_n$ and $M(u')_n = A'_{n-1} \oplus B'_n$, respectively. Since we showed the existence of s, we have that this is a morphism of complexes by the homotopy property of s. This means that the completed diagram gives us a morphism of triangles, and the diagram commutes by the following equalities $h \circ \alpha(u) = \alpha(u') \circ g$ and $\beta(u') \circ h = f[1] \circ \beta(u)$. Note that these are proper equalities, and not only up to homotopy. This proves that the axiom holds for $\mathbf{K}(\mathcal{A})$.

$(\mathbf{TR4})$ For the final axiom we start with two morphisms $u\colon A \to B$ and $v\colon B \to C$, and look at the corresponding standard triangles:

$$A \xrightarrow{u} B \xrightarrow{\alpha(u)} M(u) \xrightarrow{\beta(u)} A[1]$$

and

$$B \xrightarrow{v} C \xrightarrow{\alpha(v)} M(v) \xrightarrow{\beta(v)} B[1].$$

With the composition $v \circ u\colon A \to C$ we get the following standard triangle:

$$A \xrightarrow{vu} C \xrightarrow{\alpha(vu)} M(vu) \xrightarrow{\beta(vu)} A[1].$$

This gives us the following diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(u)} & A[1] \\
\| & & \downarrow{\scriptstyle v} & & & & \| \\
A & \xrightarrow{vu} & C & \xrightarrow{\alpha(vu)} & M(vu) & \xrightarrow{\beta(vu)} & A[1] \\
 & & \downarrow{\scriptstyle \alpha(v)} & & & & \\
 & & M(v) & & & & \\
 & & \downarrow{\scriptstyle \beta(v)} & & & & \\
 & & B[1] & & & &
\end{array}$$

To show that (TR4) is fulfilled, we need to show that there exist dashed arrows that make the following diagram commute:

$$
\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(f)} & A[1] \\
\| & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle f} & & \| \\
A & \xrightarrow{vu} & C & \xrightarrow{\alpha(vu)} & M(vu) & \xrightarrow{\beta(vu)} & A[1] \\
 & & \downarrow{\scriptstyle \alpha(v)} & & \vdots{\scriptstyle g} & & \downarrow{\scriptstyle u[1]} \\
 & & M(v) & = & M(v) & \xrightarrow{\beta(v)} & B[1] \\
 & & \downarrow{\scriptstyle \beta(v)} & & \downarrow{\scriptstyle (\alpha(u)[1])\circ\beta(v)} & & \\
 & & B[1] & \xrightarrow{\alpha(u)[1]} & M(u)[1] & &
\end{array}
$$

Moreover, the triangle $M(u) \to M(vu) \to M(v) \to M(u)[1]$ must be shown to be distinguished. Let $f = (f_n)\colon M(u) \to M(vu)$ be given in degree n by $f_n = \begin{pmatrix} 1_{A_{n-1}} & 0 \\ 0 & v_n \end{pmatrix}$ and set $g = (g_n)\colon M(vu) \to M(u)$ to be given by $g_n = \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1_{C_n} \end{pmatrix}$. Finally, we define $h\colon M(v) \to M(u)[1]$ as the composition $\alpha(u)[1] \circ \beta(v)$, so that it is given by $\begin{pmatrix} 0 & 0 \\ 1_{B_{n-1}} & 0 \end{pmatrix}$. This leaves us with a complete diagram, in which all squares commute by definition of our new maps. Now, we have a situation where we have to show that

$$M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} M(u)[1],$$

is a distinguished triangle in $\mathbf{K}(\mathcal{A})$. This means showing that our triangle is isomorphic to the standard triangle

$$M(u) \xrightarrow{f} M(vu) \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} M(u)[1].$$

We observe that only the triangles' third entries differ from the rest, hence we only need to find morphisms σ and τ such that the diagram

$$
\begin{array}{ccccccc}
M(u) & \xrightarrow{f} & M(vu) & \xrightarrow{g} & M(v) & \xrightarrow{h} & M(u)[1] \\
\| & & \| & & {\scriptstyle\tau}\uparrow\downarrow{\scriptstyle\sigma} & & \| \\
M(u) & \xrightarrow{f} & M(vu) & \xrightarrow{\sigma(f)} & M(f) & \xrightarrow{\beta(f)} & M(u)[1].
\end{array}
$$

commutes in $\mathbf{K}(\mathcal{A})$. Define therefore σ and τ by

$$\sigma_n\colon \begin{pmatrix} 0 & 0 \\ 1_{B_{n-1}} & 0 \\ 0 & 0 \\ 0 & 1_{C_n} \end{pmatrix} \text{ and } \tau_n\colon \begin{pmatrix} 0 & 1_{B_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & 1_{C_n} \end{pmatrix}.$$

Let us first check if σ and τ make the diagram commute in both directions.

By definition we get that $\tau \circ \alpha(f) = g$; where in fact both maps are given in degree n by

$$\begin{pmatrix} u_{n-1} & 0 \\ 0 & 1_{C_n} \end{pmatrix} : A_{n-1} \oplus C_n \to B_{n-1} \oplus C_n.$$

Also, we have that $\beta(f) \circ \sigma = h$ by definition, where both maps are given by

$$\begin{pmatrix} 0 & 0 \\ 1_{B_{n-1}} & 0 \end{pmatrix} : B_{n-1} \oplus C_n \to A_{n-2} \oplus B_{n-1}.$$

All of the remaining commutations will now only hold up to homotopy. The map $\alpha(f) - \sigma \circ g \colon M(vu) \to M(f)$, is in degree n given by

$$\begin{pmatrix} 0 & 0 \\ -u_{n-1} & 0 \\ 1_{A_{n-1}} & 0 \\ 0 & 0 \end{pmatrix} : A_{n-1} \oplus C_n \to A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n.$$

This map is homotopic to zero using the homotopy map $s = (s_n)$, where $s_n \colon M(vu)_n \to M(f)_{n+1}$ is given by

$$\begin{pmatrix} 1_{A_{n-1}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : A_{n-1} \oplus C_n \to A_{n-1} \oplus B_n \oplus A_n \oplus C_{n+1}.$$

Now we consider the map $\beta(f) - h \circ \tau \colon M(f) \to M(u)[1]$ which is given by

$$\begin{pmatrix} 1_{A_{n-2}} & 0 & 0 & 0 \\ 0 & 0 & -u_{n-1} & 0 \end{pmatrix} : A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n \to A_{n-2} \oplus B_{n-1}.$$

Using the homotopy map $s = (s_n)$, where

$$\begin{pmatrix} 0 & 0 & 1_{A_{n-1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n \to A_{n-1} \oplus B_n,$$

all of this can be verified using the differential of the mapping cone $M(f)$ which is given as

$$d_n^{M(f)} = \begin{pmatrix} d_{n-2}^A & 0 & 0 & 0 \\ -u_{n-2} & -d_{n-1}^B & 0 & 0 \\ 1_{A_{n-2}} & 0 & -d_{n-1}^A & 0 \\ 0 & v_{n-1} & (vu)_{n-1} & d_n^C \end{pmatrix}.$$

Now all that remains is showing that τ and σ are isomorphisms in the homotopy category. By definition we have that $\tau \circ \sigma = 1_{M(v)}$. Now we check the composition $\sigma \circ \tau$, and in degree n it is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{B_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{C_n} \end{pmatrix}.$$

We then define the homotopy maps $s_n\colon M(f)_n \to M(f)_{n+1}$ with

$$s_n := \begin{pmatrix} 0 & 0 & -1_{A_{n-1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get that $\sigma \circ \tau - 1_{M(f)} = d_{n+1}^{M(f)} \circ s_n + s_{n-1} \circ d_n^{M(f)}$. Some easy, but lengthy, calculations using the differential of $M(f)$ confirms this. This shows that $\sigma \circ \tau = 1_{M(f)}$ in the homotopy category, and thus $\mathbf{K}(\mathcal{A})$ is a triangulated category.

Chapter 3

Tensor triangulated categories

> The only question that really counts,
> must be this one: are things getting
> better or are they getting worse?
>
> Erlend Loe, *Naïve. Super*

In this chapter we are going to define tensor triangulated categories, present an example, and look at some central definitions in tensor triangular geometry. The concept of tensor triangulated categories may be thought of as a categorical equivalent of a ring, and we can also translate the notions of ideals and prime ideals in a categorical setting. Some of the central results in Balmer's article [2] will also be presented and proved.

3.1 Defining tensor triangulated categories

The triangulated categories in this section will be essentially small, i.e. every category is equivalent to a small category in which the collection of objects form a set.

Definition 3.1.1. A *tensor triangulated category* is a triple $(\mathcal{K}, \otimes, 1)$, where $\mathcal{K}$ is a triangulated category, $\otimes$ is a symmetric monoidal tensor product $\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$ which is a triangulated functor in every variable, while 1 denotes the unit.

We will often denote a tensor triangulated category as $\mathcal{K}$, instead of the triple $(\mathcal{K}, \otimes, 1)$. In this case it will be made explicitly clear that $\mathcal{K}$ is *tensor* triangulated, and not only triangulated.

Definition 3.1.2. A *tensor triangulated functor* $F : \mathcal{K} \to \mathcal{L}$ is a triangulated functor respecting the monoidal structures and mapping the unit to the unit, i.e. $F(1_{\mathcal{K}}) = 1_{\mathcal{L}}$.

Let us now look at an example of a tensor triangulated category.

Example 3.1.3. Let us look at an example of a tensor triangulated category, namely $\mathbf{K}(\mathcal{A})$, where $\mathcal{A} = \mathrm{Mod}(R)$, and R is a commutative noetherian ring. We have already shown that $\mathbf{K}(\mathcal{A})$ is triangulated. So, to see if this actually is a tensor triangulated category, we need to check if there is an identity object, as well as the symmetric, monoidal and functorial triangulated properties of the tensor product. Let A, B be complexes in $\mathbf{K}(\mathcal{A})$. The tensor product in degree n is defined as

$$(A \otimes_R B)_n := \bigoplus_{i \in \mathbb{Z}} (A_i \otimes_R B_{n-i})$$

with the differential, using homogeneous elements $a \in A, b \in B$:

$$d_n^{A \otimes B}(a_i \otimes b_{n-i}) := d_i^A(a_i) \otimes b_{n-i} + (-1)^i a_i \otimes d_{n-i}^B(b_{n-i}).$$

The symmetric property is shown using the definition:

$$\begin{aligned} (A \otimes_R B)_n &= \bigoplus_{i \in \mathbb{Z}} (A_i \otimes_R B_{n-i}) \\ &\cong \bigoplus_{i \in \mathbb{Z}} (B_{n-i} \otimes_R A_i) \\ &= \bigoplus_{j \in \mathbb{Z}} (B_j \otimes_R A_{n-j}) \\ &= (B \otimes_R A)_n. \end{aligned}$$

Note that the isomorphism allowing commutativity is an isomorphism of complexes, which in degree n is given as

$$\begin{aligned} (A \otimes_R B)_n &\xrightarrow{\phi_n} (B \otimes_R A)_n \\ a_i \otimes b_{n-i} &\mapsto (-1)^{i(n-i)} b_{n-i} \otimes a_i \end{aligned}$$

where $a_i \otimes b_{n-i} \in A_i \otimes_R B_{n-i}$. It is straight-forward to show that this is an isomorphism.

The monoidal property is shown similarly:

$$\begin{aligned} ((A \otimes_R B) \otimes_R C)_n &= \bigoplus_{i \in \mathbb{Z}} (A \otimes_R B)_i \otimes_R C_{n-i} \\ &= \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{j \in \mathbb{Z}} A_j \otimes_R B_{i-j} \right) \otimes_R C_{n-i} \\ &= \bigoplus_{i,j \in \mathbb{Z}} A_j \otimes_R B_{i-j} \otimes_R C_{n-i} \\ &= \bigoplus_{i \in \mathbb{Z}} A_i \otimes_R \left(\bigoplus_{j \in \mathbb{Z}} B_{i-j} \otimes_R C_{n-i} \right) \\ &= \bigoplus_{i \in \mathbb{Z}} A_i \otimes_R (B \otimes_R C)_{n-i} \\ &= (A \otimes_R (B \otimes_R C))_n. \end{aligned}$$

The identity element is the stalk complex R, as $A \otimes_R R \cong A$. Now to show that $- \otimes_R D$ is triangulated for an arbitrary object $D \in \mathbf{K}(\mathcal{A})$. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ be a distinguished triangle in $\mathbf{K}(\mathcal{A})$ on which we apply the functor $- \otimes_R D$. We then obtain the following triangle

$$A \otimes_R D \xrightarrow{f \otimes_R 1} B \otimes_R D \xrightarrow{g \otimes_R 1} C \otimes_R D \xrightarrow{h \otimes_R 1} A[1] \otimes_R D$$

and if this turns out to be a distinguished triangle, we are done. Now, consider the following distinguished triangle

$$A \otimes_R D \xrightarrow{f \otimes_R 1} B \otimes_R D \xrightarrow{\alpha(f \otimes_R 1)} M(f \otimes_R 1) \xrightarrow{\beta(f \otimes_R 1)} (A \otimes_R D)[1]$$

which we will pair up with the former triangle to get the following diagram

$$\begin{array}{ccccccc}
A \otimes_R D & \xrightarrow{f \otimes_R 1} & B \otimes_R D & \xrightarrow{g \otimes_R 1} & C \otimes_R D & \xrightarrow{h \otimes_R 1} & (A \otimes_R D)[1] \\
\| & & \| & & & & \| \\
A \otimes_R D & \xrightarrow{f \otimes_R 1} & B \otimes_R D & \xrightarrow{\alpha(f \otimes_R 1)} & M(f \otimes_R 1) & \xrightarrow{\beta(f \otimes_R 1)} & (A \otimes_R D)[1].
\end{array}$$

So if we have that $C \otimes_R D \cong M(f \otimes_R 1)$ we are good. Also, note that for every $X, Y \in \mathbf{K}(\mathcal{A})$ we have $(X \otimes_R Y)[1] = X[1] \otimes_R Y = X \otimes_R Y[1]$. The cone of $f \otimes_R 1$ is defined as

$$(M(f \otimes_R 1))_n := (A_n \oplus B_{n+1}) \otimes_R D_n,$$

also, from $\mathbf{K}(\mathcal{A})$ being triangulated, we have that

$$\phi_n \colon C_n \xrightarrow{\sim} (M(f))_n = A_n \oplus B_{n+1}$$

implies that

$$\phi_n \otimes_R 1 \colon C_n \otimes_R D_n \xrightarrow{\sim} (M(f \otimes_R 1))_n = (A_n \oplus B_{n+1}) \otimes D_n.$$

where the map ϕ consist of $(\phi)_n$ in each degree n, and analogously for the map $\phi \otimes_R 1$. Now what remains is checking that the following diagram commutes:

$$\begin{array}{ccccccc}
A \otimes_R D & \xrightarrow{f \otimes_R 1} & B \otimes_R D & \xrightarrow{g \otimes_R 1} & C \otimes_R D & \xrightarrow{h \otimes_R 1} & (A \otimes_R D)[1] \\
\| & & \| & & \downarrow{\scriptstyle \phi \otimes_R 1} & & \| \\
A \otimes_R D & \xrightarrow{f \otimes_R 1} & B \otimes_R D & \xrightarrow{\alpha(f \otimes_R 1)} & M(f \otimes_R 1) & \xrightarrow{\beta(f \otimes_R 1)} & (A \otimes_R D)[1].
\end{array}$$

We check the maps:

$$(\phi \otimes_R 1) \circ (g \otimes_R 1) = (\phi \circ g) \otimes_R 1 = \alpha(f) \otimes_R 1 = \alpha(f \otimes_R 1)$$

and

$$\beta(f \otimes_R 1) \circ (\phi \otimes_R 1) = \beta(f) \otimes_R 1 \circ (\phi \otimes_R 1) = (\beta(f) \circ \phi) \otimes_R 1 = h \otimes_R 1$$

so the diagram commutes. Hence the functor is triangulated, and the corresponding functor, $D \otimes_R -$, is also triangulated by the symmetry property. So $\mathbf{K}(\mathrm{Mod}(R))$ is a tensor triangulated category.

Definition 3.1.4. Let $\mathcal{K}$ be a tensor triangulated category. A *thick tensor ideal* $\mathcal{A}$ of $\mathcal{K}$ is a thick subcategory such that if $A \in \mathcal{A}$ and $B \in \mathcal{K}$ then $A \otimes B$ also belongs to $\mathcal{A}$.

We see that this is very much like the definition of an ideal for rings, and follow up this definition with an example.

Example 3.1.5. We will show that if $F\colon \mathcal{K} \to \mathcal{L}$ is a tensor triangulated functor between two tensor triangulated categories, then the kernel of F, $\mathrm{Ker}(F)$, is a thick tensor ideal of $\mathcal{K}$. We know that $\mathrm{Ker}(F) := \{A \in \mathcal{K} \mid F(A) \cong 0_{\mathcal{L}}\}$, so all that remains is checking the axioms. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

be a distinguished triangle in $\mathcal{K}$ and $A, B \in \mathrm{Ker}(F)$. We apply the triangulated functor F and get a new distinguished triangle in $\mathcal{L}$:

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(A)[1]$$

which we know is the distinguished triangle

$$0_{\mathcal{L}} \to 0_{\mathcal{L}} \to F(C) \to 0_{\mathcal{L}},$$

so $C \in \mathrm{Ker}(F)$. Let $A \in \mathrm{Ker}(F)$ such that $A \cong B \oplus C$. Then $F(A) \cong F(B) \oplus F(C) \cong 0$ which clearly implies that $B, C \in \mathrm{Ker}(F)$. Let $A \in \mathrm{Ker}(F)$ and $B \in \mathcal{K}$ and consider the tensor product $A \otimes B$. Then $F(A \otimes B) \cong F(A) \otimes F(B)$ since F is a tensor triangulated functor. This in turn means that $F(A) \otimes F(B) \cong 0_{\mathcal{L}} \otimes F(B) = 0_{\mathcal{L}}$, so $A \otimes B \in \mathrm{Ker}(F)$. So $\mathrm{Ker}(F)$ is a thick tensor ideal of $\mathcal{K}$.

3.2 Prime ideals and the spectrum of a category

From the realm of commutative algebra one might already be familiar with the notions of prime ideals and spectrums of commutative rings. Balmer took these ideas and reissued them in the world of categories.

Definition 3.2.1. Let $\mathcal{P} \subset \mathcal{K}$ be a proper thick tensor ideal. $\mathcal{P}$ is called *prime* if

$$A \otimes B \in \mathcal{P} \implies A \in \mathcal{P} \text{ or } B \in \mathcal{P}.$$

The set of all primes of $\mathcal{K}$ is called the *spectrum* of $\mathcal{K}$, and is denoted by $\mathrm{Spc}(\mathcal{K})$. This is analogous to the spectrum for rings that we know from commutative algebra. Among other things, the fact that $1 \notin \mathfrak{p}, \forall\, \mathfrak{p} \in \mathrm{Spec}(R)$ and $0 \in \mathfrak{p}, \forall\, \mathfrak{p} \in \mathrm{Spec}(R)$ for a commutative ring R analogously hold in $\mathrm{Spc}(\mathcal{K})$ for a tensor triangulated category $\mathcal{K}$. This means that any prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ contain the zero object, but not the identity object, of $\mathcal{K}$.

Another important analogy are the subsets of $\text{Spc}(\mathcal{K})$ which we will look at now. Let $\mathcal{S} \subset \mathcal{K}$ be any family of objects, and consider the following subset of $\text{Spc}(\mathcal{K})$, denoted as $\text{Z}(\mathcal{S})$:

$$\text{Z}(\mathcal{S}) = \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid \mathcal{S} \cap \mathcal{P} = \varnothing \}.$$

These subsets have the following main properties:

(1) $\bigcap_{j \in J} \text{Z}(\mathcal{S}_j) = \text{Z}(\bigcup_{j \in J} \mathcal{S}_j)$ for any index set J

(2) $\text{Z}(\mathcal{S}_1) \cup \text{Z}(\mathcal{S}_2) = \text{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$, where $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{a_1 \oplus a_2 \mid a_i \in S_i,\ i \in \{1,2\}\}$

(3) $\text{Z}(\mathcal{K}) = \varnothing$ and $\text{Z}(\varnothing) = \text{Spc}(\mathcal{K})$.

This shows that the collection $\{\text{Z}(\mathcal{S}) \subset \text{Spc}(\mathcal{K}) \mid \mathcal{S} \subseteq \mathcal{K} \}$ defines the closed subsets of a topology on $\text{Spc}(\mathcal{K})$ called the *Zariski topology*. The open complement of $\text{Z}(\mathcal{S})$ is

$$\text{U}(\mathcal{S}) := \text{Spc}(\mathcal{K})/\text{Z}(\mathcal{S}) = \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid \mathcal{P} \cap \mathcal{S} \neq \varnothing\}.$$

With this in place, we can define the support of an object.

Definition 3.2.2. For any object $A \in \mathcal{K}$ we define the *support* of the object to be

$$\text{supp}(A) := \text{Z}(\{A\}) = \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid A \notin \mathcal{P}\}.$$

Viewing tensor triangulated categories as rings is useful for one's intuition, where we think of the coproduct and tensor product of two objects of the category as adding and multiplying two elements of a ring. Note that this is not a precise analogy, as rings have additive inverses and we do not have inverses with respect to coproduct. Regardless, we need to define and understand tensor multiplicativity.

Definition 3.2.3. A collection of objects $\mathcal{S} \subset \mathcal{K}$ is called tensor multiplicative if $1 \in \mathcal{S}$ and if $A_1, A_2 \in \mathcal{S} \implies A_1 \otimes A_2 \in \mathcal{S}$.

We have now defined and presented important notions in the world of tensor triangulated categories. The first result we prove concerning these notions shows how we can locate prime ideals using thick tensor ideals and tensor multiplicative families of objects.

Lemma 3.2.4. *Let $\mathcal{K}$ be a non-zero tensor triangulated category. Let $\mathcal{J} \subset \mathcal{K}$ be a thick tensor ideal and $\mathcal{S} \subset \mathcal{K}$ a tensor multiplicative family of objects such that $\mathcal{S} \cap \mathcal{J} = \varnothing$. Then there exists a prime ideal $\mathcal{P} \in \text{Spc}(\mathcal{K})$ such that $\mathcal{J} \subset \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \varnothing$.*

Proof. We begin constructing a set of objects, $\mathcal{F}$, where the elements are thick tensor ideals $A \subset \mathcal{K}$ such that $\mathcal{A} \cap \mathcal{S} = \varnothing$, $\mathcal{J} \subseteq \mathcal{A}$ and for $C \in \mathcal{S}$, $A \in \mathcal{K}$ with $A \otimes C \in \mathcal{A}$ then $A \in \mathcal{A}$. To show that $\mathcal{F} \neq \varnothing$, consider the following subcategory of $\mathcal{K}$:

$$\mathcal{A}_0 := \{A \in \mathcal{K} \mid \exists\, S \in \mathcal{S} \text{ with } A \otimes C \in \mathcal{J}\}.$$

We now show that this is a thick tensor ideal. Consider a triangle $A \to B \to C \to A[1]$ in $\mathcal{K}$ where $A, B \in \mathcal{A}_0$ and $A[1] \in \mathcal{A}_0$ trivially. For the entries A and B, we know that there are elements $S_A, S_B \in \mathcal{S}$ such that $A \otimes S_A \in \mathcal{J}$ and $B \otimes S_B \in \mathcal{J}$. Let $S := S_A \otimes S_B$ and apply the functor $- \otimes S$ on the mentioned triangle. This yields the triangle

$$A \otimes S \to B \otimes S \to C \otimes S \to A[1] \otimes S$$

in which we find that $A \otimes S = A \otimes (S_A \otimes S_B) = (A \otimes S_A) \otimes S_B$ which implies that $A \otimes S \in \mathcal{J}$, $A[1] \otimes S \in \mathcal{J}$ and analogously that $B \otimes S \in \mathcal{J}$. Since $\mathcal{J}$ is a thick tensor ideal we have that $C \otimes S \in \mathcal{J}$, which implies that $C \in \mathcal{A}_0$.

The ideal is thick since for a direct sum $A \oplus B \in A_0$ there exists a $C \in \mathcal{S}$ such that $(A \oplus B) \otimes C \in \mathcal{J}$. We have that $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$, meaning that $(A \otimes C)$ and $(B \otimes C)$ sit in $\mathcal{J}$ since $\mathcal{J}$ is thick, which in turn means that $A, B \in \mathcal{A}_0$. Finally, assume $A \in \mathcal{A}_0$ and $B \in \mathcal{K}$. We now want to show that $A \otimes B \in \mathcal{A}_0$. A is such that there is a $C \in \mathcal{S}$ such that $A \otimes C \in \mathcal{J}$. Now, $(A \otimes B) \otimes C = B \otimes (A \otimes C) \in \mathcal{J}$ which implies $A \otimes B \in \mathcal{A}_0$. This shows that $\mathcal{A}_0$ is a thick tensor ideal, and the properties for belonging to the family $\mathcal{F}$ are trivial by construction of A_0. This means that the family $\mathcal{F}$ is non-empty, and by Zorn's Lemma there exists a maximal element, $\mathcal{P}$ of this family. We want to show that this maximal element is prime. We assume that $A \otimes B \in \mathcal{P}$ and $B \notin \mathcal{P}$, so we want to show that this leads to $A \in \mathcal{P}$. Consider the following thick tensor ideal

$$\mathcal{A}_1 := \{D \in \mathcal{K} \mid A \otimes D \in \mathcal{P}\}.$$

We see that $\mathcal{P} \subsetneq \mathcal{A}_1$, since $B \in A_1$ but not in $\mathcal{P}$. Since $\mathcal{P}$ is the maximal element of $\mathcal{F}$, $\mathcal{A}_1$ cannot be a part of this family. Now, $\mathcal{J} \subset A_1$ and any tensor product $A \otimes C \in \mathcal{A}_1$ with $A \in \mathcal{K}, C \in \mathcal{S}$ implies that $A \in \mathcal{A}_1$, so two of our three conditions are fulfilled. This means that $\mathcal{A}_1 \cap \mathcal{S} \neq \varnothing$, so there exists a $D \in \mathcal{S}$ such that $A \otimes D \in \mathcal{P}$. Since $\mathcal{P} \subset \mathcal{F}$ we have that $A \in \mathcal{P}$ by the properties of $\mathcal{F}$. This completes the proof. □

This result is essential for further developing the theory of prime ideals and the categorical spectrum.

Proposition 3.2.5. *The following claims hold for a non-zero tensor triangulated category $\mathcal{K}$.*

$i)$ *If $\mathcal{S}$ is a tensor multiplicative collection of objects not containing zero, then there exists a prime ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ such that $\mathcal{P} \cap \mathcal{S} = \varnothing$.*

$ii)$ *If $\mathcal{J} \subset \mathcal{K}$ is a proper thick tensor ideal, then there exists a maximal proper thick tensor ideal such that $\mathcal{J} \subseteq \mathcal{M} \subset \mathcal{K}$.*

$iii)$ *Maximal proper thick tensor ideals are prime.*

$iv)$ *The spectrum of $\mathcal{K}$ is non-empty.*

Proof. $i)$ follows directly from looking at the thick tensor ideal $\mathcal{J} = 0$ and Lemma 3.2.4. $ii)$ also is a direct consequence of 3.2.4, namely the case of $\mathcal{S} = \{1\}$ where we obtain a proper maximal ideal $\mathcal{P}$ which contains $\mathcal{J}$. For $iii)$ we assume $\mathcal{S} = \{1\}$ and that our proper thick tensor ideal $\mathcal{J}$ is maximal. From Lemma 3.2.4 we get that there exists a prime $\mathcal{P}$ containing $\mathcal{J}$, but since $\mathcal{J}$ is maximal they have to be equal. $iv)$ trivially follows from $i)$. □

Proposition 3.2.6. *The following hold for $A, B, C \in \mathcal{K}$:*

$i)$ supp$(0) = \varnothing$ *and* supp$(1) =$ Spc$(\mathcal{K})$.

$ii)$ supp$(A \oplus B) =$ supp$(A) \cup$ supp(B)

$iii)$ supp$(A \otimes B) =$ supp$(A) \cap$ supp(B)

$iv)$ supp$(A[1]) =$ supp(A)

$v)$ supp$(A) \subseteq$ supp$(B) \cup$ supp(C) *for a distinguished triangle* $A \to B \to C \to A[1]$.

Proof. For $i)$ we use the fact that every additive subcategory includes the 0-element, and hence every prime has 0 in it and supp$(0) = \varnothing$. If any proper thick tensor ideals contains 1 it must generate the whole category and hence cannot be proper. So supp$(0) =$ Spc$(\mathcal{K})$.

For $ii)$ we take a $\mathcal{P} \in$ supp$(A \oplus B)$ and use that $A \oplus B \notin \mathcal{P}$ which means that A and B cannot both be in $\mathcal{P}$. This means that either $\mathcal{P} \in$ supp(A) or $\mathcal{P} \in$ supp(B) which implies $\mathcal{P} \in$ supp$(A) \cup$ supp(B). Now, if $\mathcal{P} \notin$ supp$(A \oplus B)$ we have that $A \oplus B \in \mathcal{P}$ which by thickness of $\mathcal{P}$ implies that $\mathcal{P} \notin$ supp$(A) \cup$ supp(B).

For $iii)$ we look at a $\mathcal{P} \in$ supp$(A \otimes B)$. This means $A \otimes B \notin \mathcal{P}$ so neither of A or B could be in $\mathcal{P}$ by its tensor property. So $\mathcal{P} \in$ supp$(A) \cap$ supp(B). Now, for $\mathcal{P} \notin$ supp$(A \otimes B)$ we know that $A \otimes B \in \mathcal{P}$ so either $A \in \mathcal{P}$ or $B \in \mathcal{P}$ since $\mathcal{P}$ is prime. Either case concludes with $\mathcal{P} \in$ supp$(A) \cap$ supp(B).

For $iv)$ we use that triangulated categories are closed under translation. This includes primes.

For $v)$ we use the triangulated property of $\mathcal{P}$. This means that for a distinguished triangle $A \to B \to C \to A[1]$, if $A, B \in \mathcal{P}$ then $C \in \mathcal{P}$. From this the deduce that if $\mathcal{P} \in$ supp(A) then $\mathcal{P} \in$ supp$(B) \cup$ supp(C). This completes the proof. □

Remark 3.2.7. Since $\{$supp(A) $\mid A \in \mathcal{K}\}$ forms a basis of closed subsets, we can understand what the closure of a subset $\mathcal{S} \subseteq$ Spc$(\mathcal{K})$ is. From [2, Proposition 2.8] we learn that if $\mathcal{S} \subseteq$ Spc$(\mathcal{K})$, then its closure, $\overline{\mathcal{S}}$, is the smallest closed subset of Spc$(\mathcal{K})$ which includes $\mathcal{S}$. This means that

$$\overline{\mathcal{S}} := \bigcap_{\substack{A \in \mathcal{K} \ s.t. \\ \mathcal{S} \subset \text{supp}(A)}} \text{supp}(A)$$

and we can understand the closure of a prime ideal, or point, $\mathcal{P}$ in the categorical spectrum.

Proposition 3.2.8. *The closure of a point* $\mathcal{P} \in$ Spc$(\mathcal{K})$ *is* $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \text{Spc}(\mathcal{K}) \mid \mathcal{Q} \subseteq \mathcal{P}\}$.

Proof. Let $\mathcal{S}_0 := \mathcal{K} \setminus \mathcal{P}$, with Z$(\mathcal{S}_0) = \{\mathcal{P} \in \text{Spc}(\mathcal{K}) \mid \mathcal{P} \cap \mathcal{S}_0\}$. Then $\mathcal{P} \in \mathcal{S}$ implies $\mathcal{S} \subseteq \mathcal{S}_0$, which in turn implies Z$(\mathcal{S}_0) \subseteq$ Z$(\mathcal{S})$, so Z$(\mathcal{S}_0)$ is the smallest closed subset containing $\mathcal{P}$. Hence $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \text{Spc}(\mathcal{K}) \mid \mathcal{Q} \subseteq \mathcal{P}\}$. □

The closure of a point also admits another nice property. If $\mathcal{P}_1, \mathcal{P}_2 \in$ Spc$(\mathcal{K})$, and $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$, then $\mathcal{P}_1 = \mathcal{P}_2$. Now, the next definition deals with the relationship between a topological space and a function associating objects of a tensor triangulated category to a closed subset of said topological space, called the assignment. The pairing of the topological space and the assignment is called a support data.

Definition 3.2.9. [2, Definition 3.1] A *support data* on a tensor triangulated category $(\mathcal{K}, \otimes, 1)$ is a pair (X, σ) where X is a topological space and σ is an assignment which associates to any object $A \in \mathcal{K}$ a *closed* subset $\sigma(A) \subset X$ subject to the following rules:

$i)$ $\sigma(0) = \varnothing$ and $\sigma(1) = X$.

$ii)$ $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.

$iii)$ $\sigma(A[1]) = \sigma(A)$.

$iv)$ Given any distinguished triangle $A \to B \to C \to A[1]$, $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$.

$v)$ $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$.

Example 3.2.10. Let $\mathcal{K}$ be a tensor triangulated category. Then $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is a support data on $\mathcal{K}$. This falls on $\mathrm{Spc}(\mathcal{K})$ being a topological space and Proposition 3.2.6.

With this in place it might prove useful to define what a morphism between two support data is.

Definition 3.2.11. Let $\mathcal{K}$ be a tensor triangulated category, and (X, σ) and (Y, τ) be two support data on $\mathcal{K}$. Then a morphism of support data, $f\colon (X, \sigma) \to (Y, \tau)$, is a continuous map $f\colon X \to Y$ such that $\sigma(A) = f^{-1}(\tau(A))$ for all objects $A \in \mathcal{K}$.

Note that this morphism is an isomorphism if and only if f is a homeomorphism.

Lemma 3.2.12. *For a set X, if we have two maps $f_1, f_2\colon X \to \mathrm{Spc}(\mathcal{K})$ such that $f_1^{-1}(\mathrm{supp}(A)) = f_2^{-1}(\mathrm{supp}(A))$ for all $A \in \mathcal{K}$, then $f_1 = f_2$.*

Proof. The relation $f_1^{-1}(\mathrm{supp}(A)) = f_2^{-1}(\mathrm{supp}(A))$ tells us that $f_1(x) \in \mathrm{supp}(A) \iff f_2(x) \in \mathrm{supp}(A)$. Then, look at the closure of $f_1(x)$:

$$\overline{\{f_1(x)\}} = \bigcap_{\substack{A \in \mathcal{K}\ s.t. \\ f_1(x) \in \mathrm{supp}(A)}} \mathrm{supp}(A) = \bigcap_{\substack{A \in \mathcal{K}\ s.t. \\ f_2(x) \in \mathrm{supp}(A)}} \mathrm{supp}(A) = \overline{\{f_2(x)\}}$$

which means that $f_1(x) = f_2(x)$ by Proposition 3.2.8. Hence $f_1 = f_2$. □

Lemma 3.2.13. *Let (X, σ) be a support data on $\mathcal{K}$, and $Y \subseteq X$ any subset. Then the full subcategory $\{A \in \mathcal{K} \mid \sigma(A) \subseteq Y\}$ of $\mathcal{K}$ is a thick tensor ideal.*

Proof. Let $\mathcal{I} := \{A \in \mathcal{K} \mid \sigma(A) \subseteq Y\}$. For objects $A \in \mathcal{I}, B \in \mathcal{K}$, we obtain $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B) \subseteq Y$, hence $A \otimes B \in \mathcal{I}$. For thickness, consider $A \oplus B \in \mathcal{I}$. This means that $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B) \subseteq Y$, and so $\sigma(A) \subseteq Y$ and $\sigma(B) \subseteq Y$, i.e. $A, B \in \mathcal{I}$. For the triangulated part, consider a distinguished triangle $A \to B \to C \to A[1]$, where $A, B, A[1] \in \mathcal{I}$. Now, shifting the distinguished triangle to $C \to A[1] \to B[1] \to C[1]$ we have $\sigma(C) \subseteq \sigma(A) \cup \sigma(B)$ and so $C \in \mathcal{I}$ which completes the proof. □

We now arrive at the universal property of the spectrum, which explains the relation between a support data (X, σ) on a tensor triangulated category $\mathcal{K}$, and the spectrum of $\mathcal{K}$.

Theorem 3.2.14. (Universal property of the spectrum)[2, Theorem 3.2]
Let $\mathcal{K}$ be a tensor triangulated category. The spectrum $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is the final support data on $\mathcal{K}$. This means that* $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is a support data, and for any support data (X, σ) on $\mathcal{K}$ there exists a unique continuous map $f\colon X \to \mathrm{Spc}(\mathcal{K})$ such that $\sigma(A) = f^{-1}(\mathrm{supp}(A))$ for any object $A \in \mathcal{K}$. Explicitly, the map f is defined, for all $x \in X$, by*

$$f(x) = \{A \in \mathcal{K} \mid x \notin \sigma(A)\}.$$

Proof. First, we note that $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is a support data by Proposition 3.2.6. Now, let (X, σ) be a support data on $\mathcal{K}$ and $f\colon X \to \mathrm{Spc}(\mathcal{K})$ be a morphism of support data such that $f(x) := \{A \in \mathcal{K} \mid x \notin \sigma(A)\}$. By Lemma 3.2.12 we have that f is unique. For continuity, we have to show that $f^{-1}(\mathrm{supp}(A)) = \sigma(A)$ by the definition of continuity of topological maps. The definition of $\mathrm{supp}(A)$ gives that $f(x) \in \mathrm{supp}(A) \iff A \notin f(x) \iff x \in \sigma(A)$, which means that $f^{-1}(\mathrm{supp}(A) = \sigma(A)$ and we have continuity. Moreover, by Lemma 3.2.13 with $Y \setminus \{x\}$ we have that $f(x)$ is a thick tensor ideal. For primality, consider $A \otimes B \in f(x)$. This means that $x \notin \sigma(A \otimes B)$ which implies that $x \notin \sigma(A)$ or $x \notin \sigma(B)$, so $A \in f(x)$ or $B \in f(x)$. □

Remark 3.2.15. Shortly, we will talk about the spectrum being functorial, and in that regard we need to specify our notation to avoid confusion. We denote the support of an object $A \in \mathcal{K}$ by $\mathrm{supp}_{\mathcal{K}}(A) := \mathrm{supp}(A) \subseteq \mathrm{Spc}(\mathcal{K})$ to emphasize the support's dependency on $\mathcal{K}$.

The functoriality of the spectrum follows from looking at a tensor triangulated functor $F\colon \mathcal{K} \to \mathcal{L}$ with a map

$$\begin{aligned} \mathrm{Spc}F\colon \mathrm{Spc}(\mathcal{L}) &\to \mathrm{Spc}(\mathcal{K}) \\ \mathcal{S} &\mapsto F^{-1}(\mathcal{S}). \end{aligned}$$

which is well-defined, continuous and for all objects $A \in \mathcal{K}$

$$(\mathrm{Spc}F)^{-1}(\mathrm{supp}_{\mathcal{K}}(A)) = \mathrm{supp}_{\mathcal{L}}(F(A))$$

in $\mathrm{Spc}(\mathcal{L})$. The latter part means that we can tie knots between the support of objects in different categories using the functor F. To get a better look at how the inverse Spc-functor actually works, consider the following:

$$\begin{aligned} & F\colon \mathcal{K} \to \mathcal{L} \\ \implies & \mathrm{Spc}F\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K}) \\ & \qquad \mathcal{Q} \mapsto F^{-1}\mathcal{Q} = \{x \in \mathcal{K} \mid F(x) \in \mathcal{Q}\} \end{aligned}$$

Nothing new here, and we remind ourselves that for $A \in \mathcal{K}$ the support of A is such that $\mathrm{supp}_{\mathcal{K}}(A) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid A \notin \mathcal{P}\} \subseteq \mathrm{Spc}(\mathcal{K})$. This gives us the following when we apply $(\mathrm{Spc}F)^{-1}$:

$$\begin{aligned}(\mathrm{Spc}F)^{-1}(\mathrm{supp}_{\mathcal{K}}(A)) &:= \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{L}) \mid (\mathrm{Spc}F)(\mathcal{Q}) \in \mathrm{supp}_{\mathcal{K}}(A)\} \\ &= \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{L}) \mid F^{-1}(\mathcal{Q}) \in \mathrm{supp}_{\mathcal{K}}(A)\} \\ &= \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{L}) \mid A \notin F^{-1}(\mathcal{Q})\} \\ &= \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{L}) \mid F(A) \notin \mathcal{Q}\} \\ &= \mathrm{supp}_{\mathcal{K}}(F(A))\end{aligned}$$

Hence the inverse Spc-functor occur naturally. Note that $\mathrm{Spc}(-)$ is a contravariant functor between the 2-category of tensor triangulated categories and the category of topological spaces. It is also worth mentioning that if the functor $F\colon \mathcal{K} \to \mathcal{L}$ is dense[3], then we have that the smallest thick tensor triangulated subcategory generated by $F(F^{-1}(\mathcal{P}))$ is $\mathcal{P}$. Hence $F^{-1}(\mathcal{P}_1) = F^{-1}(\mathcal{P}_2)$ implies that $\mathcal{P}_1 = \mathcal{P}_2$, and $\mathrm{Spc}(F)$ is injective.

The following definitions will help us understand the radical in a categorical sense, as well as the support in various manners.

Definition 3.2.16. The radical $\sqrt{\mathcal{J}}$ of a thick tensor ideal $\mathcal{J} \subset \mathcal{K}$ is defined to be

$$\sqrt{\mathcal{J}} := \ \{A \in \mathcal{K} \mid \exists\, n \geq 1 \text{ such that } A^{\otimes n} \in \mathcal{J}\}.$$

Moreover, a thick subcategory $\mathcal{J}$ is called *radical* if $\sqrt{\mathcal{J}} = \mathcal{J}$.

We note that the radical of $\mathcal{J}$, $\sqrt{\mathcal{J}}$, can be written as an intersection of the prime ideals containing $\mathcal{J}$.

Lemma 3.2.17. $\sqrt{\mathcal{J}} = \bigcap\limits_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \\ \mathcal{J} \subseteq \mathcal{P}}} \mathcal{P}.$

Proof. Indeed, $\sqrt{\mathcal{J}} \subseteq \mathcal{P}$ for any $\mathcal{P}$ containing $\mathcal{J}$ since $\mathcal{P}$ is prime, so $\sqrt{\mathcal{J}} \subset \bigcap\limits_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \\ \mathcal{J} \subseteq \mathcal{P}}} \mathcal{P}.$ Also, for reverse inclusion, consider an element $A \in \mathcal{K}$ such that $A \in \mathcal{P}$ for all $\mathcal{P} \supseteq \mathcal{J}$. The multiplicative set $\mathcal{S} := \{A^{\otimes n} \mid n \geq 1\}$ is such that $\mathcal{S} \cap \mathcal{J} \neq \varnothing$ since Lemma 3.2.4 says that if $\mathcal{S} \cap \mathcal{J} = \varnothing$ then there is a prime ideal $\mathcal{P}$ such that $\mathcal{J} \subseteq \mathcal{P}$, but $A \notin \mathcal{P}$, which is a contradiction. Hence the equality holds. □

Definition 3.2.18. Let $\mathcal{E} \subseteq \mathcal{K}$ be a collection of objects. The support of $\mathcal{E}$ is the union of the support of its elements:

$$\mathrm{supp}(\mathcal{E}) = \bigcup_{A \in \mathcal{E}} \mathrm{supp}(A) \subseteq \mathrm{Spc}(\mathcal{K}).$$

This means, by definition of the support, that $\mathrm{supp}(\mathcal{E}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{E} \nsubseteq \mathcal{P}\}$. Also, let $Y \subseteq \mathrm{Spc}(\mathcal{K})$ be a subset. The subcategory *supported on* Y, $\mathcal{K}_Y$ is defined to be

$$\mathcal{K}_Y = \{A \in \mathcal{K} \mid \mathrm{supp}(A) \subseteq Y\} \subseteq \mathcal{K}.$$

[3]A functor $F\colon \mathcal{K} \to \mathcal{L}$ is dense if every $B \in \mathcal{L}$ is isomorphic to $F(A)$ for an object $A \in \mathcal{K}$.

The subcategory $\mathcal{K}_Y$ is a thick tensor ideal by Lemma 3.2.13. Also, by the nature of the support, it is clear that $\mathcal{K}_Y = \bigcap_{\mathcal{P}\in \mathrm{Spc}(\mathcal{K})\setminus Y} \mathcal{P}$.

Proposition 3.2.19. *For any thick tensor ideal $\mathcal{J} \subseteq \mathcal{K}$ we have that $\mathcal{K}_{\mathrm{supp}(\mathcal{J})} = \sqrt{\mathcal{J}}$.*

Proof. By Definition 3.2.18 we have that $P \notin \mathrm{supp}(\mathcal{J})$ is equivalent to $\mathcal{J} \subset \mathcal{P}$. Hence $\mathcal{K}_{\mathrm{supp}(\mathcal{J})} = \bigcap_{\mathcal{P}\in \mathrm{Spc}(\mathcal{K})\setminus \mathrm{supp}(\mathcal{J})} \mathcal{P}) = \bigcap_{\mathcal{P}\notin \mathrm{supp}(\mathcal{J})} \mathcal{P} = \bigcap_{\substack{\mathcal{P}\in \mathrm{Spc}(\mathcal{K})\\ \mathcal{J}\subseteq\mathcal{P}}} \mathcal{P} = \sqrt{\mathcal{J}}$. □

With all this in place, we are ready to present Balmer's bijection. The bijection lets us establish a connection between the thick tensor ideals of a tensor triangulated category $\mathcal{K}$ and the subsets of $\mathrm{Spc}(\mathcal{K})$. These subsets are unions of supports on objects.

Theorem 3.2.20. [2, Theorem 4.10] *Let $\mathcal{S}$ be the set of those subsets $Y \subseteq \mathrm{Spc}(\mathcal{K})$ of the form $Y = \bigcup_{i\in I} Y_i$ for closed subsets Y_i of $\mathrm{Spc}(\mathcal{K})$ with $\mathrm{Spc}(\mathcal{K}) \setminus Y_i$ quasi-compact*[4] *for all $i \in I$. Let $\mathcal{R}$ be the set of radical thick tensor ideals of $\mathcal{K}$. Then there is an order-preserving bijection $\mathcal{S} \xrightarrow{\sim} \mathcal{R}$ given by*

$$Y \mapsto \mathcal{K}_Y := \{A \in \mathcal{K} \mid \mathrm{supp}(A) \subseteq Y\}$$

whose inverse is

$$\mathcal{J} \mapsto \mathrm{supp}(\mathcal{J}) := \bigcup_{A\in\mathcal{J}} \mathrm{supp}(A).$$

Proof. Our maps need to be well-defined. $\mathcal{K}_Y$ is a thick tensor ideal and is in fact radical as $\mathrm{supp}(A^{\otimes n}) = \mathrm{supp}(A)\cap\cdots\cap\mathrm{supp}(A) = \mathrm{supp}(A)$ which we learn from Proposition 3.2.6. Now, [2, Proposition 2.14] tells us that $\mathrm{supp}(\mathcal{J})$ is a union of closed subsets with quasi-compact complements $\mathrm{Spc}(\mathcal{K}) \setminus \mathrm{supp}(A) = \mathrm{U}(A)$. Both maps also preserve inclusions, so the maps are well-defined.

To complete the proof we need to show that the two compositions of the maps equal the identity. From Proposition 3.2.19 we see that $\mathcal{J} \mapsto \mathrm{supp}(\mathcal{J}) \mapsto \mathcal{K}_{\mathrm{supp}(\mathcal{J})} = \sqrt{\mathcal{J}} = \mathcal{J}$ for $\mathcal{J}$ radical, so the first composition is okay. The second composition $Y \mapsto \mathcal{K}_Y \mapsto \mathrm{supp}(\mathcal{K}_Y)$ is such that $\mathrm{supp}(\mathcal{K}_Y) \subseteq Y$ for any subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$ by definition, so we need to show the reverse inclusion to finish up the proof.

We pick a $\mathcal{P} \in Y$, and by the nature of Y there exists a $Y_i \subset Y$ such that $\mathcal{P} \in Y_i$ and $\mathrm{Spc}(\mathcal{K}) \setminus Y_i$ is quasi-compact. We know from [2, Proposition 2.14 b)] that there exists an object $A \in \mathcal{K}$ such that $\mathrm{Spc}(\mathcal{K}) \setminus Y_i = \mathrm{U}(A)$ from which we conclude that $Y_i = \mathrm{supp}(A)$, so $\mathcal{P} \in \mathrm{supp}(A)$. Then $\mathcal{P} \in \bigcup_{A\in\mathcal{K}_Y} = \mathrm{supp}(\mathcal{K}_Y)$, which completes the proof.

□

[4]For us this means that for a topological space X, for each open covering of X, there is a finite subcover.

Chapter 4

Application

One can't know what one doesn't know. One can just hope that lucky associations over time bring us forward. Personally my only hope is that I'm prone to get such associations. So far it doesn't look too promising.

Erlend Loe, *L*

Here we will look at a specific derivation of the main result of the previous chapter, called the Hopkins-Neeman-bijection. This result was first presented in [12] and will be used in classifying the subcategories of our chosen category $\mathbf{K}^b(\text{proj}(R))$, with R being a commutative noetherian ring. We will show an isomorphism of support data in this very category, before we choose a specific commutative noetherian ring to work with.

4.1 Classifying subcategories

To obtain the Hopkins-Neeman-bijection, we need to redefine the notions of spectrum and support in a ring theoretical manner. If not stated otherwise, $\mathcal{K}$ will be a tensor triangulated category and R will be a commutative, noetherian ring.

Definition 4.1.1. A subset $Y \subseteq \text{Spec}(R)$, is called *specialization closed* if $\mathfrak{p} \in Y$ and $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mathfrak{q} \in Y$.

The spectrum of R is as known from commutative algebra, while the support needs a clearer definition.

Definition 4.1.2. The support of a finitely generated module M over the commutative, noetherian ring R, is given as

$$\text{Supp}_R M := \{\mathfrak{p} \in \text{Spec}(R) \mid M_\mathfrak{p} \neq 0\}.$$

First, with our category being $\mathbf{K}^b(\text{proj}(R))$, we want to show that every thick subcategory $\mathcal{J} \subset \mathbf{K}^b(\text{proj}(R))$ is actually a radical tensor ideal. To show this we will use the bijection of Hopkins-Neeman [12]:

Theorem 4.1.3. *There is a bijection of sets*

$$\begin{Bmatrix} Thick\ subcategories \\ of\ \mathbf{K}^b(\text{proj}(R)) \end{Bmatrix} \underset{T}{\overset{S}{\rightleftarrows}} \begin{Bmatrix} Specialization\ closed \\ subsets\ of\ \text{Spec}(R) \end{Bmatrix}$$

where the maps are defined as

$$S\colon \mathcal{T} \mapsto \bigcup_{M \in \mathcal{T}} \text{Supp}_R M$$

$$T\colon \mathcal{V} \mapsto \{M \in \mathbf{K}^b(\text{proj}(R)) \mid \text{Supp}_R M \subset \mathcal{V}\},$$

where $\mathcal{T}$ is a thick subcategory of $\mathbf{K}^b(\text{proj}(R))$ and $\mathcal{V}$ is a specialization closed subset of $\text{Spec}(R)$.

This has proved to be a useful tool in developing new theory in several branches of algebra. We will now also make use of this beautiful theorem, showing that every thick subcategory of $\mathbf{K}^b(\text{proj}(R))$ is a radical thick tensor ideal.

Proposition 4.1.4. *Every thick subcategory of $\mathbf{K}^b(\text{proj}(R))$ is a radical thick tensor ideal.*

Proof. For the sake of keeping things simple, we let our category $\mathbf{K}^b(\text{proj}(R)) = \mathcal{K}$. First, we take a thick subcategory $\mathcal{J} \subset \mathcal{K}$ and run it through the Hopkins-Neeman-bijection:

$$\mathcal{J} \overset{S}{\longmapsto} \bigcup_{M \in \mathcal{J}} \text{Supp}_R M \overset{T}{\longmapsto} \{N \in \mathcal{K} \mid \text{Supp}_R N \subseteq \bigcup_{M \in \mathcal{J}} \text{Supp}_R M\} = \mathcal{J}.$$

Since $\mathcal{J}$ is assumed to be thick, we go directly to the tensor property. We show this by taking an $A \in \mathcal{J}$ and a $B \in \mathcal{K}$ and look at $\text{Supp}_R(A \otimes_R B)$. We know that $\text{Supp}_R(A \otimes_R B) = \text{Supp}_R A \cap \text{Supp}_R B$ from [3, Chapter 2.1.2 - Property (5)], in which $\text{Supp}_R A \subseteq \bigcup_{M \in \mathcal{J}} \text{Supp}_R M$, hence $\text{Supp}_R(A \otimes_R B) \subset \bigcup_{M \in \mathcal{J}} \text{Supp}_R M$ so $A \otimes_R B \in \mathcal{J}$.

Recall that a thick tensor ideal is called radical if $\sqrt{\mathcal{J}} = \mathcal{J}$. So we take an $A \in \sqrt{\mathcal{J}}$, and since $\sqrt{\mathcal{J}} = \{A \in \mathcal{K} \mid \exists\, n \geq 1 \text{ s.t. } A^{\otimes n} \in \mathcal{J}\}$ we know that $A^{\otimes n} \in \mathcal{J}$ for some n. Now, since $\text{Supp}_R A^{\otimes n} = \text{Supp}_R A \cap \text{Supp}_R A \cap \cdots \cap \text{Supp}_R A = \text{Supp}_R A$, it is evident that $\sqrt{\mathcal{J}} = \mathcal{J}$ and so $\mathcal{J}$ is a radical thick tensor ideal. □

This will ultimately lead us to a useful result where we get an isomorphism of support data, namely $(\text{Spc}(\mathbf{K}^b(\text{proj}(R)), \text{supp}) \simeq (\text{Spec}(R), \text{Supp}_R)$, but before we get there we need to define what a classifying support data is, and present a theorem from Balmer stating the relationship between a noetherian topological space X and the spectrum of $\mathcal{K}$. We begin with the classifying support data.

Definition 4.1.5. (Classifying support data) [2, Definition 5.1]
A support data (X, σ) on a tensor triangulated category $\mathcal{K}$ is a *classifying support data* if the following two conditions hold:

(a) The topological space X is noetherian and any non-empty irreducible closed subset $Z \subseteq X$ has a unique generic point: $\exists!\, x \in Z$ with $\overline{\{x\}} = Z$.

(b) We have a bijection

$$S\colon \{Y \subseteq X \mid Y \text{ specialization closed}\} \xrightarrow{\sim} \{\mathcal{J} \subseteq \mathcal{K} \mid \mathcal{J} \text{ radical thick tensor ideal}\}$$

defined by $Y \mapsto \{A \in \mathcal{K} \mid \sigma(A) \subseteq Y\}$, with inverse $Y \mapsto \sigma(\mathcal{J}) := \bigcup\limits_{A \in \mathcal{J}} \sigma(A)$.

Now, the theorem of Balmer.

Theorem 4.1.6. *Let (X, σ) be a classifying support data on $\mathcal{K}$. Then the universal property of the spectrum grants a canonical map $f\colon X \to \mathrm{Spc}(\mathcal{K})$ which is a homeomorphism.*

Proof. We have from Theorem 3.2.14 that the map f is continuous and is such that $f^{-1}(\mathrm{supp}(A)) = \sigma(A)$ for all objects $A \in \mathcal{K}$. Now, before we check bijectivity and closedness, we need to prove that any closed subset $Z \subseteq X$ is of the form $Z = \sigma(A)$ for some object $A \in \mathcal{K}$. In lack of a better name, this will be referred to as the Z-σ-property. We use the bijection from Definition 4.1.5, and choose an irreducible[5] $Z = \overline{\{x\}}$ for some $x \in X$. Now, making use of Definition 3.2.9 $ii)$ and the fact that X is noetherian, we have that $\overline{\{x\}} = Z = S^{-1}(S(Z)) = \bigcup\limits_{A \in S(Z)} \sigma(A)$. This means that there exists an object $A \in \mathcal{K}$ such that $x \in \sigma(A) \subseteq Z$. So $\overline{\{x\}} \subseteq \sigma(A) \subseteq Z = \overline{\{x\}}$, so $Z = \sigma(A)$, and the Z-σ-property is proved.

For injectivity, we define for an $x \in X$ that $Y(x) := \{y \in X \mid x \notin \overline{\{y\}}\ \}$ which is easily verified as specialization closed. If $\sigma(A) \subset Y(x)$ then $x \notin \sigma(A)$, so $\sigma(A) \subset Y(x) \implies x \notin \sigma(A)$. Now assume that $x \notin \sigma(A)$. We know that $\sigma(A)$ is specialization closed, so for all $y \in \sigma(A)$, $x \notin \overline{\{y\}}$ and we have $\sigma(A) \subseteq Y(x)$, so $\sigma(A) \subset Y(x) \iff x \notin \sigma(A)$. Let us now look at $S(Y(x))$:

$$S(Y(x)) := \{a \in \mathcal{K} \mid \sigma(A) \subseteq Y(x)\} = \{A \in \mathcal{K} \mid x \notin \sigma(A)\} = f(x).$$

Ultimately this means that if $f(x_1) = f(x_2)$, then $Y(x_1) = Y(x_2)$, which means that $\overline{\{x_1\}} = \overline{\{x_2\}}$ and so $x_1 = x_2$, which implies that f is injective.

For surjectivity, let $\mathcal{P}$ be a prime ideal in $\mathcal{K}$ such that $\mathcal{P} = S(Y)$ for a specialization closed subset $Y \subset X$. Since $\mathcal{P} \neq \mathcal{K}$ we have a non-empty complement $X \setminus Y$. Now let $x, y \in X \setminus Y$, we have by the Z-σ-property the existence of $A, B \in \mathcal{K}$ such that $\overline{\{x\}} = \sigma(A)$ and $\overline{\{y\}} = \sigma(B)$. This means that $A, B \notin \mathcal{P} = S(Y)$ since $x, y \notin Y$, so $A \otimes B \notin \mathcal{P}$ and $\sigma(A \otimes B) \nsubseteq Y$. We then have that there is a point $z \in X \setminus Y$ such that $z \in \sigma(A \otimes B) = \sigma(A) \cap \sigma(B) = \overline{\{x\}} \cap \overline{\{y\}}$, so $\overline{\{z\}}$ sits in the closure of both x and y. Hence the non-empty family of closed subsets $\mathcal{F} = \{\overline{\{x\}} \subset X \mid x \in X \setminus Y\}$ is such that any two elements admit a lower bound for inclusion. This lower bound is the minimal element in $\mathcal{F}$, given by X being noetherian. We now know that there exists a point $x \in X \setminus Y$ such that $X \setminus Y \subset \{y \in X \mid x \in \overline{\{y\}}\ \}$, and since $x \notin Y$ the reverse inclusion also holds. This yields $Y = \{y \in X \mid x \notin \overline{\{y\}}\ \}$, which gives us that $\mathcal{P} = S(Y) = S(Y(x)) = f(x)$, and surjectivity is proved.

[5] For a subset $Z \subset \mathrm{Spc}(\mathcal{K})$, being irreducible means that for any open subsets U_1, U_2 in $\mathrm{Spc}(\mathcal{K})$ if $Z \cap U_1 \cap U_2 = \varnothing$ then $Z \cap U_1 = \varnothing$ or $Z \cap U_2 = \varnothing$ [2, Proposition 2.18].

Now, $f^{-1}(\text{supp}(A)) = \sigma(A)$ which in turn gives $f(\sigma(A)) = \text{supp}(A)$, so f is a closed map by the Z-σ-property. Hence f is a homeomorphism. □

Corollary 4.1.7. $(\text{Spc}(\mathbf{K}^b(\text{proj}(R)), \text{supp})$ *and* $(\text{Spec}(R), \text{Supp}_R)$ *are isomorphic as support data.*

Proof. It is easily verified that $(\text{Spec}(R), \text{Supp}_R)$ is a classifying support data on $\mathbf{K}^b(\text{proj}(R))$, using Proposition 4.1.4 and Theorem 3.2.20. So by Theorem 3.2.14 there exists a unique map $f\colon \text{Spec}(R) \to \text{Spc}(\mathbf{K}^b(\text{proj}(R))$, which by Theorem 4.1.6 is a homeomorphism. Hence $\text{Spec}(R) \simeq \text{Spc}(\mathbf{K}^b(\text{proj}(R))$, and $(\text{Spec}(R), \text{Supp}_R) \cong (\text{Spc}(\mathbf{K}^b(\text{proj}(R)), \text{supp})$ as support data. □

4.2 Picking a specific ring

We will now look at a specific commutative noetherian ring, namely $\mathbb{Z}_n = \mathbb{Z}/\langle n\rangle$, and we will consider different variations of n. The varations of n will of course influence the structure of $\mathbb{Z}_n$. So let $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$ where the p_i are distinct primes for $i \in \{1, 2, \ldots, m\}$. It is known that when a ring R is semi-simple, all modules over R are semi-simple and hence every module is automatically projective and injective. What makes this interesting is that with the ring R being semi-simple, the category $\text{mod}(R)$ also becomes semi-simple, i.e. global dimension 0. This means that every module in $\text{mod}(R)$ is projective, and hence $\text{mod}(R) = \text{proj}(R)$. The choice of n influences $\mathbb{Z}_n$ which has consequences for our category, so when is $\mathbb{Z}_n$ semi-simple?

We can factor $\mathbb{Z}_n$ depending on what n is. So, let $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, such that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_m^{k_m}}$. $\mathbb{Z}_n$ is semi-simple when $k_1 = k_2 = \cdots = k_m = 1$, i.e. n is a product of distinct primes. We have that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_j}$, which means that $\mathbb{Z}_n$ is isomorphic to a product of fields. This is in accordance with the Wedderburn-Artin theorem, which states that a ring is semi-simple if and only if it is isomorphic to a product of matrix rings over division rings. Now, if we let $\mathbb{Z}_n$ be our ring of choice, what does the spectrum look like?

Proposition 4.2.1. *Let* $\mathbb{Z}_n$ *be such that* $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$*, where the* p_i *are distinct primes, and* $k_i \geq 1$*, for* $i \in \{1, 2, \ldots, m\}$*. Then* $\text{Spec}(\mathbb{Z}_n) = \{\langle p_1\rangle, \langle p_2\rangle, \ldots, \langle p_m\rangle\}$*, i.e. the ideals generated by the residue classes of the distinct primes in the factorization of* n*.*

Proof. We know that since $\mathbb{Z}_n = \mathbb{Z}/\langle n\rangle$, every prime ideal in $\mathbb{Z}_n$ is of the form $\mathfrak{p}/\langle n\rangle$, where $\mathfrak{p}$ is a prime ideal in $\mathbb{Z}$ and $n \in \mathfrak{p}$. Then $\mathfrak{p}$ is of the form $\langle p\rangle$ for a prime $p \in \mathbb{Z}$, which implies that $\mathfrak{p}/\langle n\rangle = \langle p\rangle/\langle n\rangle$. Now, $n \in \langle p\rangle$, so we have that $p \mid n$. Since $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$, $p = p_i$, $i \in \{1, 2, \ldots, m\}$. So every prime ideal in $\mathbb{Z}/\langle n\rangle$ is generated by the residue classes of one of the primes in n, which is what we wanted to show. □

If we are to classify the thick subcategories of $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$, it might prove useful to decompose the category into a product of "smaller" tensor triangulated categories, analogously to what we are doing when we decompose $\mathbb{Z}_n$ into products of smaller rings. Firstly, the equivalence $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n)) \simeq \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1}^{k_1} \times \mathbb{Z}_{p_2}^{k_2} \times \cdots \times \mathbb{Z}_{p_m}^{k_m}))$ holds trivially from the fact that $Z_n \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_m^{k_m}}$. Secondly, we need to show that

$\mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1}^{k_1} \times \mathbb{Z}_{p_2}^{k_2} \times \cdots \times \mathbb{Z}_{p_m}^{k_m}))$ decomposes to $\mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}})) \times \cdots \times \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_m^{k_m}}))$, as tensor triangulated categories. For this to be true we need to present some general results.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two tensor triangulated categories. We define the product category $\mathcal{K}_1 \times \mathcal{K}_2$ as follows: The objects of $\mathcal{K}_1 \times \mathcal{K}_2$ are of the form (A_1, A_2) where $A_1 \in \mathcal{K}_1$ and $A_2 \in \mathcal{K}_2$. The morphisms between two objects of the product category are inherited componentwise from the factor categories. This means that for a morphism $f\colon (A_1, A_2) \to (A_1', A_2')$, $f = (f_1, f_2)$ where $f_1\colon A_1 \to A_2$ and $f_2\colon A_2 \to A_2'$.

Lemma 4.2.2. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be tensor triangulated categories. Then $\mathcal{K}_1 \times \mathcal{K}_2$ is tensor triangulated.*

Proof. We take a triangle $T_1\colon A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \xrightarrow{h_1} A_1[1]$ in $\mathcal{K}_1$, and a triangle $T_2\colon A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2 \xrightarrow{h_2} A_2[1]$ in $\mathcal{K}_2$ and compose them to get the following diagram:

$$\begin{array}{cccccc} T_1: & A_1 \xrightarrow{f_1} & B_1 \xrightarrow{g_1} & C_1 \xrightarrow{h_1} & A_1[1] \\ \times & \times & \times & \times & \times \\ T_2: & A_2 \xrightarrow{f_2} & B_2 \xrightarrow{g_2} & C_2 \xrightarrow{h_2} & A_2[1]. \end{array}$$

This is a distinguished triangle in the product category. Now the axioms for triangulated categories follows trivially, likewise the monoidal property required for the category to be tensor triangulated. □

Note that by induction this holds for any finite composition of tensor triangulated categories. By this result we know that for two noetherian commutative rings R and S, $\mathbf{K}^b(\text{proj}(R)) \times \mathbf{K}^b(\text{proj}(S))$ is a tensor triangulated category. The only thing that is missing at this point is showing that $\mathbf{K}^b(\text{proj}(R \times S))$ is equivalent to $\mathbf{K}^b(\text{proj}(R)) \times \mathbf{K}^b(\text{proj}(S))$ as tensor triangulated categories.

Proposition 4.2.3. *Let R and S be two noetherian commutative rings. Then $\mathbf{K}^b(\text{proj}(R \times S))$ and $\mathbf{K}^b(\text{proj}(R)) \times \mathbf{K}^b(\text{proj}(S))$ are equivalent as tensor triangulated categories.*

Proof. First, let us rename our categories to keep things a little cleaner. Let $\mathbf{K}^b(\text{proj}(R \times S)) = \mathcal{K}(R \times S)$ and $\mathbf{K}^b(\text{proj}(R)) \times \mathbf{K}^b(\text{proj}(S)) = \mathcal{K}(R) \times \mathcal{K}(S)$. Also, let

$$F\colon \mathcal{K}(R \times S) \to \mathcal{K}(R) \times \mathcal{K}(S)$$

be the functor such that

$$0 \to (P_R^n, P_R^n) \to (P_R^{n-1}, P_S^{n-1}) \to \cdots \to (P_R^0, P_S^0) \to 0 \mapsto (P_R^\bullet, P_S^\bullet).$$

We observe that every complex of $\mathcal{K}(R) \times \mathcal{K}(S)$ is naturally derived componentwise from the complexes of $\mathcal{K}(R \times S)$. The functor is triangulated by construction, and $1_{\mathcal{K}(R\times S)} \mapsto (1_{\mathcal{K}(R)}, 1_{\mathcal{K}(S)})$. Let $A, B \in \mathcal{K}(R \times S)$ where

$$A = 0 \to (A_R^n, A_S^n) \to (A_R^{n-1}, A_S^{n-1}) \to \cdots \to (A_R^0, A_S^0) \to 0$$

and

$$B = 0 \to (B_R^n, B_S^n) \to (B_R^{n-1}, B_S^{n-1}) \to \cdots \to (B_R^0, B_S^0) \to 0$$

and take the tensor product $A \otimes_{R\times S} B$. If we examine this complex in degree n, we have $(A_R^n, A_S^n) \otimes_{R\times S} (B_R^n, B_S^n)$. Since $(A_R \times A_S) \otimes_{R\times S} (B_R \times B_S) = (A_R \otimes_R B_R) \times (A_S \otimes_S B_S)$, we have that $(A_R^n, A_S^n) \otimes_{R\times S} (B_R^n, B_S^n) = (A_R^n \otimes_R B_R^n, A_S^n \otimes_S B_S^n)$. This makes the complex

$$0 \to (A_R^n, A_S^n) \otimes_{R\times S} (B_R^n, B_S^n) \to \cdots \to (A_R^0, A_S^0) \otimes_{R\times S} (B_R^0, B_S^0) \to 0$$

equal to

$$0 \to (A_R^n \otimes_R B_R^n, A_S^n \otimes_S B_S^n) \to \cdots \to (A_R^0 \otimes_R B_R^0, A_S^0 \otimes_S B_S^0) \to 0.$$

So $F(A \otimes_{R\times S} B) = (A_R \otimes_R B_R, A_S \otimes_S B_S)$, and our functor is tensor triangulated. We construct a functor

$$G : \mathcal{K}(R) \times \mathcal{K}(S) \to \mathcal{K}(R \times S)$$

which maps

$$(P_R^\bullet, P_S^\bullet) \mapsto 0 \to (P_R^n, P_R^n) \to (P_R^{n-1}, P_S^{n-1}) \to \cdots \to (P_R^0, P_S^0) \to 0.$$

This is tensor triangulated by the same arguments as for F. Now we look at the compositions $F \circ G$ and $G \circ F$, which by construction give us the identity functor for each category respectively. This means that $\mathbf{K}^b(\text{proj}(R \times S))$ and $\mathbf{K}^b(\text{proj}(R)) \times \mathbf{K}^b(\text{proj}(S))$ are equivalent as tensor triangulated categories. □

Remark 4.2.4. We observe that if $\mathbb{Z}_n$ is a semi-simple ring such that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_m}$, then $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n)) = \mathbf{K}^b(\text{mod}(\mathbb{Z}_n))$ since every module over a semi-simple ring is projective. This also decomposes and we get that $\mathbf{K}^b(\text{mod}(\mathbb{Z}_n)) \cong \mathbf{K}^b(\text{mod}(\mathbb{Z}_{p_1})) \times \mathbf{K}^b(\text{mod}(\mathbb{Z}_{p_2})) \times \cdots \times \mathbf{K}^b(\text{mod}(\mathbb{Z}_{p_m}))$.

4.3 Consequences

Now that we have picked a specific ring, $\mathbb{Z}_n$, and analyzed its spectrum, we can examine what consequences this has for the categorical spectrum of $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$. Let us give our diagram from Theorem 4.1.3 a slight update with the knowledge we have gathered so far:

$$\left\{\begin{matrix}\text{Thick subcategories of}\\ \mathbf{K}^b(\text{proj}(\mathbb{Z}_n))\end{matrix}\right\} \underset{T}{\overset{S}{\rightleftarrows}} \left\{\begin{matrix}\text{Subsets of}\\ \text{Spec}(\mathbb{Z}_n)\end{matrix}\right\}$$

and recall the map $T\colon \mathcal{V} \mapsto \{M \in \mathcal{K} \mid \text{Supp}_R M \subseteq \mathcal{V}\}$. Here, we can specify that $\mathcal{V}$ is some subset of $\text{Spec}(\mathbb{Z}_n)$ of the form $\{\langle p_1\rangle, \langle p_2\rangle, \ldots, \langle p_i\rangle\}$, $i \in \{1, 2, \ldots, m\}$. Also, $\text{Supp}_{\mathbb{Z}_n} M = \{p \in \text{Spec}(\mathbb{Z}_n) \mid M_p \neq 0\}$. Since we did a lot of ground work in the previous section, we can simplify things to make them more understandable. We know that $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$ can be decomposed into factor categories, so let us pick one specific category, $\mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_i}^{k_i}))$ and look at the bijection diagram:

$$\left\{\begin{matrix}\text{Thick subcategories of}\\ \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_i^{k_i}}))\end{matrix}\right\} \underset{T}{\overset{S}{\rightleftarrows}} \left\{\begin{matrix}\text{Subsets of}\\ \text{Spec}(\mathbb{Z}_{p_i^{k_i}})\end{matrix}\right\}$$

We have already understood the subsets of Spec($\mathbb{Z}_{p_i^{k_i}}$), and so to generalize it:

$$\text{Spec}(\mathbb{Z}_{p_i^{k_i}}) = \begin{cases} \{0\}, & k_i = 1 \\ \{\langle p_i \rangle\}, & k_i > 1 \end{cases}$$

which means that in each case ($k_i = 1$ or $k_i > 1$) the ring spectrum has two subsets, namely the aforementioned subsets and $\varnothing$. Hence $\mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) has two thick subcategories. We actually know what these subcategories are, since $\{0\} \subset \mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) and $\mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) itself are thick subcategories and we only have room for exactly two by our bijection, these are exactly the thick subcategories of $\mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)).

Theorem 4.3.1. *Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Then $\mathbf{K}^b$(proj($\mathbb{Z}_n$)) has 2^m thick subcategories, which correspond to the thick subcategories of the equivalent product category*

$$\mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}})) \times \cdots \times \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_m^{k_m}})),$$

namely $C_1 \times C_2 \times \cdots \times C_m$, where $C_i \subseteq \mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) and $C_i = \mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) or $\{0\}$.

Proof. The fact that $\mathbf{K}^b$(proj($\mathbb{Z}_n$)) has 2^m thick subcategories comes down to checking the amount of subsets of Spec($\mathbb{Z}_n$). There are two subsets of Spec($\mathbb{Z}_{p_i^{k_i}}$) for each $i \in \{1, 2, \ldots, m\}$. Since Spec($\mathbb{Z}_n$) $\cong$ Spec($\mathbb{Z}_{p_1^{k_1}}$) $\times$ Spec($\mathbb{Z}_{p_2^{k_2}}$) $\times \cdots \times$ Spec($\mathbb{Z}_{p_m^{k_m}}$), and since we have two subsets for each of the m different spectrums, we get 2^m different subsets in Spec($\mathbb{Z}_n$), so by the bijection of Theorem 4.1.3 we have 2^m thick subcategories in $\mathbf{K}^b$(proj($\mathbb{Z}_n$)).

We know that a thick subcategory $C_i \subseteq \mathbf{K}^b$(proj($\mathbb{Z}_{p_i^{k_i}}$)) is either the category itself or $\{0\}$. This means that the composition of thick subcategories from each of the m factor categories, $C_1 \times C_2 \times \cdots \times C_m$, yields a total of 2^m thick subcategories, hence these correspond to the thick subcategories of $\mathbf{K}^b$(proj($\mathbb{Z}_n$)). ☐

We have now learned what the thick subcategories of $\mathbf{K}^b$(proj($\mathbb{Z}_n$)) look like, but it is also interesting to see how the Hopkins-Neeman bijection works and what connections it uncovers.

We can pick a subset of Spec($\mathbb{Z}_n$) and see what it maps to, using

$$T\colon \mathcal{V} \mapsto \{M \in \mathcal{K} \mid \text{Supp}_R M \subseteq \mathcal{V}\}.$$

Let Spec($\mathbb{Z}_n$) $= \{\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_m \rangle\}$, choose $\{\langle p_1 \rangle\} \subset$ Spec($\mathbb{Z}_n$), and map this using T:

$$T(\{\langle p_1 \rangle\}) = \{M \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_n)) \mid \text{Supp}_{\mathbb{Z}_n} M \subseteq \{\langle p_1 \rangle\}\ \}$$

This means that we have two options for what $\text{Supp}_{\mathbb{Z}_n} M$ can be for an $M \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$, namely $\varnothing$ or $\{\langle p_1 \rangle\}$. To clarify, this means that $M_{\langle p_1 \rangle} \neq 0$ if and only if $p_1 \in \text{Supp}_{\mathbb{Z}_n} M$ which is if and only if $\{\langle p_1 \rangle\} \subset \text{Supp}_{\mathbb{Z}_n} M$. Also, $M_{\langle p_1 \rangle} = 0$ if and only if $\text{Supp}_{\mathbb{Z}_n} M = \varnothing$, which in turn implies $M \in T(\{\langle p_1 \rangle\})$. So we are looking for $M \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$ such that $M_{\langle p_1 \rangle} \neq 0$ and at the same time $M_{\langle p_2 \rangle} = M_{\langle p_3 \rangle} = \cdots = M_{\langle p_m \rangle} = 0$.

We know that a non-acyclic complex $M_1 \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}}))$ is still non-acyclic after localizing it with the prime ideal $\langle p_1 \rangle$, i.e. $M_{1_{\langle p_1 \rangle}} \neq 0$. However, this does not hold when we localize using a prime ideal not equal to $\langle p_1 \rangle$, so $M_{1_{\langle p \rangle}} \neq 0$ if and only if $p = p_1$. This means that if we localize M_1 with $\langle p_2 \rangle$, then $M_{1_{\langle p_2 \rangle}} = 0$ since we cannot have that $\{\langle p_2 \rangle\} \subseteq \{\langle p_1 \rangle\}$.

Now, if we choose a complex $M \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$, we know by Proposition 4.2.3 that this corresponds to a complex $M_1 \times M_2 \times \cdots \times M_m$ in $\mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}})) \times \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_2^{k_2}})) \times \cdots \times \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_m^{k_m}}))$ where $M_i \in \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_i^{k_i}}))$, $i \in \{1, 2, \ldots, m\}$. This means that $M_{\langle p_1 \rangle} \cong M_{1_{\langle p_1 \rangle}} \times \{0\} \times \{0\} \times \cdots \times \{0\}$ which implies that $M \cong M_1 \times \{0\} \times \{0\} \times \cdots \{0\}$. Hence $T(\{\langle p_1 \rangle\}) = \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}})) \times \{0\} \times \{0\} \times \cdots \{0\}$. Now, let $\{0\} \subset \{\langle p_1 \rangle\} \subset \{\langle p_1 \rangle, \langle p_2 \rangle\} \subset \cdots \subset \{\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_{m-1} \rangle\} \subset \{\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_m \rangle\} = \text{Spec}(R)$ be a chain of subsets in $\text{Spec}(R)$. The following diagram illustrates the connection between said chain of subsets of $\text{Spec}(R)$ and the associated thick subcategories of $\mathbf{K}^b(\text{proj}(\mathbb{Z}_n))$:

$$
\begin{array}{ccc}
\{0\} & \longleftrightarrow & \{0\} \\
\cap & & \cap \\
\{\langle p_1 \rangle\} & \longleftrightarrow & \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}})) \\
\cap & & \cap \\
\{\langle p_1 \rangle, \langle p_2 \rangle\} & \longleftrightarrow & \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}})) \\
\cap & & \cap \\
\vdots & & \vdots \\
\cap & & \cap \\
\{\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_{m-1} \rangle\} & \longleftrightarrow & \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_{m-1}^{k_{m-1}}})) \\
\cap & & \cap \\
\{\langle p_1 \rangle, \langle p_2 \rangle, \ldots, \langle p_m \rangle\} & \longleftrightarrow & \mathbf{K}^b(\text{proj}(\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_m^{k_m}})) \\
\| & & \| \\
\text{Spec}(R) & \longleftrightarrow & \mathbf{K}^b(\text{proj}(\mathbb{Z}_n))
\end{array}
$$

Remark 4.3.2. The Erdös-Kac-theorem [4] states that for large n, the number of distinct prime divisors of n has a normal distribution with mean and variance $\log(\log(n))$. This means that we can expect the amount of distinct prime divisors to be approximately $\log(\log(n))$ for large n, and hence we can expect the amount of thick tensor ideals of the category $\mathbf{K}^b(\mathrm{proj}(\mathbb{Z}_n))$ to be $2^{\log(\log(n))}$ for large n.

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Appendix: Norwegian translations

Here is a list of suggested translations for terminology used in the thesis.

English	Norwegian (bokmål)
categorical spectrum	kategorielt spektrum
distinguished triangle	distingvert triangel
Hopkins-Neeman theorem	Hopkins-Neeman-teoremet
perfect complex	perfekt kompleks
semi-simple	semisimpel
specialization closed subset	spesialiseringslukket delmengde
spectrum	spektrum
support	støtte
support data	støttedatum
tensor triangulated category	tensortriangulert kategori
thick subcategory	tykk underkategori
triangulated category	triangulert kategori
triangulated subcategory	triangulert underkategori
Zariski topology	zariskitopologien