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A Fatou-Bieberbach Domain with a Complex Line in the Boundary

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Abstract. The purpose of this thesis is to construct a Fatou-Bieberbach domain in two complex variables with a complex line in the boundary. To achieve this, we will first explore the dynamics of an analytic transformation in one variable. Then we explore a Fatou-Bieberbach domain with a fixed point in its interior, because the procedure to show that this has the wanted properties is easier than for the one we plan to explore later. In the end, the principles for the simpler domain and the one variable dynamics are used to construct the wanted domain, and to show that it is indeed a Fatou Bieberbach domain.

Samandrag. Føremålet med denne oppgåva er å konstruere eit Fatou-Bieberbach-område i to komplekse variablar med ei kompleks line i randa. For å få til dette studerer me dynamikken til ein analytisk transformasjon i ein variabel. Så utforskar me eit Fatou-Bieberbach-område med eit fikspunkt i det indre av området, sidan metoden for å vise at dette har dei ynskte eigenskapane er enklare enn for det me skal undersøke seinare. Til slutt blir prinsippa for det enklare området og dynamikken frå ein variabel tatt i bruk for å konstruere det ynskte området, og å vise at det faktisk er eit Fatou-Bieberbach-område.

Preface

This is the end point of my six years in Trondheim. I am proud to graduate as a teacher and a mathematician from NTNU and for all the experiences I have gained through my time here. At times it has felt like an uphill struggle, but in the end I have climbed and reached social and academic insights I am thankful for and proud of.

My first thank you goes to my supervisors, Berit and Marius, for all your help, patience and critical remarks. I am grateful for our discussions from my first semesters with you as teachers to this final project. You found an intriguing project I have enjoyed working on. A crucial part of working as a math teacher is to find joy in math, and working on this thesis has given me tools and a background I will benefit from when I teach students in the future.

Without support from my family and friends, I would not have been able to work through the obstacles I have met. Thank you for your continued support and encouragement, which has been invaluable to me. In particular I would like to thank my beloved Aldís for helping me become a better man every day.

One of the reasons that I am writing this in 2018 and not a year earlier is the student movement. I was so lucky that I got to spend one year fighting for student's rights, both at NTNU, at national level and in Europe. In the student movement I have made friends for life, who remind me that there is life outside of the reading room and even outside the university campus.

Lastly I want to thank all of you who helped proofread and improve my thesis. You pushed me to do a little bit better the last weeks when I could have considered myself finished. Thank you.

To you who read this: I hope you enjoy my work and that you are able to use this as a resource in your own learning. And I wish you the best of luck when time comes for you to write your own thesis or project!

Stein Olav Romslo
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Introduction

The field of complex dynamics is the study of iterations of analytic transformations in $\mathbb{C}^n$. Pierre Fatou and Gaston Julia worked on this theory in the early 1900s, and the well known Fatou- and Julia sets are named after them.

Iterations of analytic transformations was already studied by, for example, Monique Hakim [**4**] and Jean-Pierre Rosay and Walter Rudin [**7**]. Therefore, this thesis is not as much about discovering something new as about writing out the details. I will use my experiences from the teacher education program to make it accessible for a broader spectrum of readers. My thesis is written with a student reader in mind.

Fatou-Bieberbach domains are proper subsets of $\mathbb{C}^n$ which are biholomorphically equivalent to the whole space. That is, a set which is not the whole space and has a biholomorphic map between itself and $\mathbb{C}^n$. Such domains do not exist in $\mathbb{C}$ because the only domain that is biholomorphically equivalent to $\mathbb{C}$ is $\mathbb{C}$ itself.

The foundation of this thesis is the function $f(z) = z(1 - z)$, which has the property that the fixed point 0 is not an interior point of its basin of attraction. We will use this map to construct a Fatou-Bieberbach domain which is the basin of attraction for the origin under an automorphism which is tangent to the identity at the origin. We also want the origin to lie on the boundary, meaning that it is not an interior point.

Outline of the Text

Chapter 1 gives some preliminary definitions and results which will be a foundation for the later chapters.

In chapter 2 we will study the dynamics of a particular analytic transformation in $\mathbb{C}$, namely $z \mapsto z(1 - z)$, near 0. We will study more general transformations based on this.

Chapter 3 gives an example of a holomorphic transformation in two complex variables. It has a fixed point with a basin of attraction which is a proper subset of $\mathbb{C}^2$. We will then construct a biholomorphic map from the basin of attraction to the whole space, in other words we show that the basin of attraction is a Fatou-Bieberbach domain. This is a simpler variant of the phenomenon we will study in chapters 4 and 5.

In chapters 4 and 5 we construct a map inspired by the map in chapter 2. To ensure that the map has the desired properties, we construct it using shears and overshears. We then show that the map has fixed point with a basin of attraction which is a Fatou-Bieberbach domain. This domain is different from the one in chapter 3 because the fixed point is not attracting, and it is not located in the interior of the basin of attraction.

CHAPTER 1

Background

This thesis builds on basic principles of complex dynamics, some of which will be stated in this chapter. We will consider holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. In particular the aim of this thesis is to study iterations of holomorphic maps in $\mathbb{C}^n$ tangent to the identity at the origin.

Definition 1.1 (Holomorphic map). *A* holomorphic map f *from* $U \subset \mathbb{C}^m$ *to* $V \subset \mathbb{C}^n$ *is a mapping*

$$f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \quad z \in U,$$

where f_j *are holomorphic functions (complex differentiable in a neighbourhood of every* $z \in U$*) for* $1 \leq j \leq n$.

Definition 1.2 (Biholomorphic map). *A* biholomorphic map *is a holomorphic map* $f : U \to V$ *such that* f *is one-to-one and onto.*

If there exists a biholomorphic function between two domains U and V, we say that U and V are *biholomorphically equivalent.*

Lemma 1.3. *The only domain which is biholomorphically equivalent to* $\mathbb{C}$ *is* $\mathbb{C}$ *itself.*

Proof. A biholomorphic function f in $\mathbb{C}$ must be entire. By the Riemann mapping theorem, all proper subsets of $\mathbb{C}$ are biholomorphically equivalent to the unit disc, Δ. So if there is a proper subset of $\mathbb{C}$ that is biholomorphically equivalent to $\mathbb{C}$, there is a bounded entire function $f : \mathbb{C} \to \Delta$. By Liouville's theoerem f is constant. Therefore the only domain that is biholomorphically equivalent to $\mathbb{C}$ is $\mathbb{C}$ itself. □

Definition 1.4 (Automorphism). *The set* $\mathrm{Aut}\,(\mathbb{C}^n)$ *consists of all biholomorphic maps* $f : \mathbb{C}^n \to \mathbb{C}^n$. *If* $f \in \mathrm{Aut}\,(\mathbb{C}^n)$, *we say that* f *is an* automorphism on $\mathbb{C}^n$.

Definition 1.5 (Fatou-Bieberbach domain). *A* Fatou-Bieberbach domain *is a proper subset of* $\mathbb{C}^n$ *that is biholomorphically equivalent to* $\mathbb{C}^n$.

As a consequence of lemma 1.3 there are no Fatou-Bieberbach domains in $\mathbb{C}$.

Lemma 1.6. *A Fatou-Bieberbach domain cannot be bounded.*

Proof. Assume Ω is a Fatou-Bieberbach domain, and that there is an entire map

$$F : \mathbb{C}^n \to \Omega \subset B(0, R)$$

for some large constant R so that

$$F = (f_1, f_2, \ldots, f_n)$$

and $f_j : \mathbb{C}^n \to \mathbb{C}$, $0 \leq j \leq n$. Then each f_j is entire and bounded on $\mathbb{C}^n$. By Liouville's theorem each f_j is constant, hence F is constant. □

Some of our tools in these studies are a functional equation often referred to as the Abel equation and its solutions, called Abel-Fatou functions.

Definition 1.7 (Abel equation). *The* Abel equation *is a functional equation*

$$\alpha(h(z)) = \alpha(z) + k \tag{1.1}$$

for functions α and h, and a constant k.

Definition 1.8 (Abel-Fatou function). *An* Abel-Fatou function *is a function α that satisfies equation* (1.1).

1.1. Preliminaries of Complex Dynamics

In complex dynamics we consider iterations of automorphisms f in $\mathbb{C}^n$, meaning

$$f^k = f \circ f^{k-1} \quad \text{and} \quad f^0 = \mathrm{Id}.$$

For proofs and more discussion on the basic theory of complex dynamics, one can refer to Carleson and Gamelin [**2**] or Fornæss [**3**].

Definition 1.9 (Orbit of z_0). *For a point z_0 we define the* forward orbit *under the map $f \in \mathrm{Aut}\,(\mathbb{C}^n)$ as the sequence $\left\{z_k = f^k(z_0)\right\}_{k=0}^{\infty}$. The* backward orbit *is the forward orbit of z_0 with respect to f^{-1}. We usually say orbit when we mean the forward orbit.*

Definition 1.10 (Fixed point). *Let f be a map in $\mathbb{C}^n$. A point $p \in \mathbb{C}^n$ is said to be a* fixed point of f *if $f(p) = p$.*

When an orbit of a point under an automorphism f converges, the limit point is a fixed point of f.

Definition 1.11 (Basin of attraction). *If p is a fixed point for f, we define*

$$\widehat{\Omega} := \left\{z \in \mathbb{C}^n : f^k(z) \to p \text{ as } k \to \infty\right\}.$$

The basin of attraction *for p is the interior of $\widehat{\Omega}$, denoted Ω.*

A fixed point p of a holomorphic map f is classified by the derivative $f'(p)$. In one complex variable $f'(p) = \lambda$, and p is classified as

- an *attracting fixed point* of f if $|\lambda| < 1$.
- a *repelling fixed point* of f if $|\lambda| > 1$.
- a *neutral fixed point* of f if $|\lambda| = 1$.

In several complex variables $f'(p)$ is a matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}$$

evaluated at p. We classify fixed points according to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $f'(p)$. For the main part we will consider maps for which 0 is a neutral fixed point, meaning that the absolute value of all eigenvalues of $f'(0)$ are 1, and we say that the map is *tangent to the identity* at the origin. For attracting fixed points, $|\lambda_j| < 1$ for all $0 \leq j \leq n$. In chapter 3 we will see an example of a transformation with an attracting fixed point in $\mathbb{C}^2$.

In one complex variable we classify the neutral fixed points into *rationally neutral* and *irrationally neutral* fixed points. The first category has derivative $\lambda = e^{2\pi i\theta}$ satisfying $\lambda^n = 1$ for some $n \in \mathbb{N}$. The second has $\lambda^n \neq 1$ for all $n \in \mathbb{N}$, meaning that $n\theta \in \mathbb{Z}$ or in particular that $\theta \in \mathbb{Q}$. When $p = 0$, the map f is

$$f(z) = e^{2\pi i\theta} z + O\left(z^2\right) \tag{1.2}$$

where $O\left(z^2\right)$ denotes all terms of z of order 2 or higher. For rationally neutral fixed points we can find a basin of attraction. For irrationally neutral fixed points we have the snail lemma, which implies that the basin of attraction does not exist, because $\widehat{\Omega}$ is a union of isolated points.

THEOREM 1.12 (Snail lemma). *Suppose that $f(z)$ is a polynomial of degree $d \geq 2$ with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$, where θ is irrational. Then there is no connected component of the basin of attraction for 0 under f.*

The proof with a slightly different formulation of the theorem can be found in Fornæss [**3**] or more detailed in Romslo [**6**].

1.2. Shears and Overshears

In one complex variable, we can show that the automorphisms are all on the form $z \mapsto az + b$ for a non-zero a. This is not the case in higher dimensions, where the class $\operatorname{Aut}\left(\mathbb{C}^n\right)$ is more complicated. By Andersén and Lempert's theorem (see theorem 1.17), we can generate all automorphisms in $\mathbb{C}^n$ as limits of compositions of less complicated transformations, shears and overshears [**1**].

DEFINITION 1.13 (Shears). *In two complex variables,* shears *are maps of the form*

$$(z, w) \mapsto (z, w + g(z)) \quad \text{or} \quad (z, w) \mapsto (z + h(w), w)$$

where g and h are entire functions.

DEFINITION 1.14 (Overshears). *In two complex variables,* overshears *are maps of the form*

$$(z, w) \mapsto \left(z, we^{u(z)}\right) \quad \text{or} \quad (z, w) \mapsto \left(ze^{v(w)}, w\right)$$

where u and v are entire functions.

LEMMA 1.15. *Shears and overshears are one-to-one.*

We will prove both this and the next lemma for the first given shears and overshears in definitions 1.13 and 1.14. The proofs are similar for the second.

PROOF (SHEARS ARE ONE-TO-ONE). Assume that

$$(z_1, w_1 + g(z_1)) = (z_2, w_2 + g(z_2)).$$

It follows that $z_1 = z_2$. Hence $g(z_1) = g(z_2)$ and $w_1 = w_2$. In conclusion we get $(z_1, w_1) = (z_2, w_2)$. □

PROOF (OVERSHEARS ARE ONE-TO-ONE). Assume that

$$\left(z_1, w_1 e^{u(z_1)}\right) = \left(z_2, w_2 e^{u(z_2)}\right).$$

It follows that $z_1 = z_2$. Hence $u(z_1) = u(z_2)$ and $w_1 = w_2$. In conclusion we get $(z_1, w_1) = (z_2, w_2)$. □

LEMMA 1.16. *Shears and overshears are onto* $\mathbb{C}^2$*.*

PROOF (SHEARS ARE ONTO $\mathbb{C}^2$). Let $(z_1, w_1) \in \mathbb{C}^2$ and $g : \mathbb{C} \to \mathbb{C}$ be entire. We want to show that there is $(z, w) \in \mathbb{C}^2$ so that $(z, w + g(z)) = (z_1, w_1)$.

Choose $z = z_1$. Then $w = w_1 - g(z_1)$ solves the equation. □

PROOF (OVERSHEARS ARE ONTO $\mathbb{C}^2$). Let $(z_1, w_1) \in \mathbb{C}^2$ and $u : \mathbb{C} \to \mathbb{C}$ be entire. We want to show that there is $(z, w) \in \mathbb{C}^2$ so that $\left(z, we^{u(z)}\right) = (z_1, w_1)$.

Choose $z = z_1$. Then $w = w_1 e^{-u(z_1)}$ solves the equation. □

We note that because shears and overshears are one-to-one and onto $\mathbb{C}^2$, any composition of these will also be one-to-one and onto $\mathbb{C}^2$. Andersén and Lempert [**1**] proves a strong connection between shears and overshears and $\mathrm{Aut}\,(\mathbb{C}^n)$.

THEOREM 1.17 (Andersén-Lempert). *The subgroup of* $\mathrm{Aut}\,(\mathbb{C}^n)$ *generated by (shears and) overshears is dense in* $\mathrm{Aut}\,(\mathbb{C}^n)$*.*

The proof can be found in Andersén and Lempert's paper [**1**]. They give an example to show that not all automorphisms are finite compositions of shears and overshears:

EXAMPLE 1.18. *Not all automorphisms of* $\mathbb{C}^2$ *are finite compositions of shears and overshears. Let* ϕ *be any non-constant holomorphic function. Then*

$$(z, w) \mapsto \left(ze^{\phi(zw)}, we^{-\phi(zw)}\right)$$

is such an automorphism.

CHAPTER 2

Dynamics of an Analytic Transformation in One Variable

In this part we will study the analytic transformation in $\mathbb{C}$

$$f(z) = z - z^2,$$

and variants of it. In particular, we will study maps on the forms

- $z \mapsto z - z^2 + O(z^3) = z(1 - z + O(z^2))$,
- $z \mapsto z(1 - az + O(z^2))$, $\quad a \in \mathbb{C}$,
- $z \mapsto z\left(e^{2\pi i\theta} - z + O(z^2)\right)$.

Clearly 0 is a fixed point for all these transformations. We will show that 0 has a basin of attraction with 0 in the boundary. In chapters 4 and 5 we will use this as inspiration when we construct an automorphism in two dimensions. We will consider some parts of the plane and see how the dynamical behaviour of each transformation is in these domains.

(1) If $|z| \geq 2$.
(2) If $\arg(z) = \pi$.
(3) If $-\pi/4 < \arg(z) < \pi/4$ and $0 < |z| < 1/10$.
(4) If $0 < |z| << 1$, but $\arg(z) \neq \pi$.

2.1. The Origin is a Fixed Point on the Boundary of its Basin of Attraction

We see that $z \mapsto z - z^2$ can be rewritten as

$$f(z) = z(1 - z) \tag{2.1}$$

so we are able to see the effect the transformation has on a point z mainly by looking at the term $1 - z$. Using this term, we see that f will scale and rotate z about the origin, which is the behaviour we need to look into. We will study iterations of f in the four domains mentioned, and then combine the information we get to define the basin of attraction for the fixed point 0.

Lemma 2.1. *If $f(z) = z(1 - z)$, then the basin of attraction for 0 is a subset of the disc $\Delta(0, 2)$.*

Proof. When $|z| \geq 2$, we get that $|1 - z| \geq 1$. Then

$$|z(1 - z)| \geq |z| \cdot 1 = |z| \geq 2,$$

so these values of z will never be attracted to 0, hence the basin of attraction for 0 is contained in the disc $\Delta(0, 2)$. $\square$

LEMMA 2.2. *If $f(z) = z(1-z)$, then the negative real axis is not in the basin of attraction.*

PROOF. If $z \in \mathbb{R}$ and $z < 0$, then $1 - z > 1$. The product $z(1-z) < z < 0$ for all negative real z, so these z are not in the basin of attraction for 0. □

This means that 0 is not in its basin of attraction, which is in accordance with a theorem of Peters, Vivas and Wold [**5**, theorem 2.1], stated here as theorem 2.3. The proof is stated in their paper. Because 0 is a neutral fixed point under f, it cannot be in the interior of the basin of attraction.

THEOREM 2.3 (Peters-Vivas-Wold). *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map such that $f(0) = 0$ and let Ω be basin of attraction. If Ω contains a neighbourhood of the origin, then 0 is an attracting fixed point.*

For the third case we want the opposite conclusion as in lemma 2.1 and 2.2.

THEOREM 2.4. *If $f(z) = z(1-z)$, then the region*

$$V := \left\{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}, 0 < |z| < \frac{1}{10} \right\}$$

is in the basin of attraction for 0.

We will look into the theorem in smaller parts to prove it. First, figure 2.1 gives an illustration of the domain V in question, and shows that when $z \in V$, then $|1 - z| < 1$. The idea of the proof is to show that both the value and argument of z decreases under the map f.

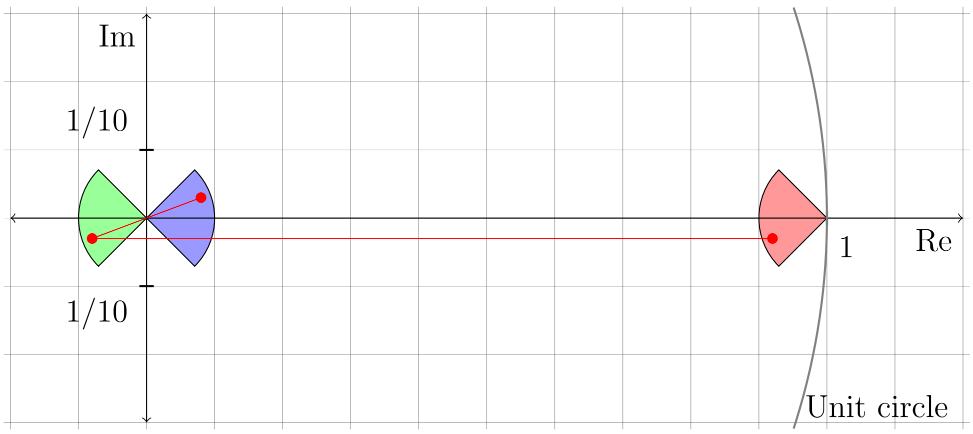

FIGURE 2.1. A figure showing $z \in V$ (blue), $-z$ (green) and $1 - z$ (red), and how the point $z = 0.08 + 0.03i$ goes from z to $-z$ to $1 - z$.

LEMMA 2.5. *If $f(z) = z(1-z)$, then $(0, 1/10)$ is in the basin of attraction for 0.*

PROOF. Let $z \in (0, 1/10)$. Then $1 - z \in (9/10, 1)$, so $0 < z(1-z) < z$. By iteration we have

$$z_1 = z(1-z)$$
$$z_n = z_{n-1}(1 - z_{n-1})$$

and $\{z_n\}$ is a decreasing sequence, bounded below by 0. Hence the limit $\lim_{n\to\infty} z_n = \tilde{z}$ exists. It follows that $\tilde{z} = \tilde{z}\,(1-\tilde{z})$, which implies that $\tilde{z} = 0$. □

LEMMA 2.6. *If $f(z) = z(1-z)$, then the region*

$$V_1 := \left\{ z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{4}, 0 < |z| < \frac{1}{10} \right\}$$

is in the basin of attraction for 0.

PROOF. The idea is to first show that the magnitude and argument of z decreases and stays positive under f, implying that V_1 is invariant under f. In the end we will show that all $z \in V_1$ are taken to 0 under iterations of f.

Let

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$u = s\cos\phi, \quad v = s\sin\phi$$

so

$$x + iy = z = re^{i\theta}$$
$$u + iv = 1 - z = se^{i\phi}.$$

It is clear that $s > r$ follows from the translation by 1. Because of the restrictions placed on θ, we have that $\sin\theta > 0$. We also have that $y + v = 0$. So

$$0 = r\sin\theta + s\sin\phi$$

where all factors except $\sin\phi$ are known to be strictly positive. Because $1-z$ has to be in the fourth quadrant, this implies that $\phi < 0$. We have

$$\begin{aligned} r < s \text{ and } r\sin\theta = -s\sin\phi &\Rightarrow r\sin\theta > -r\sin\phi \\ &\Rightarrow \sin\theta > \sin(-\phi) \\ &\Rightarrow \theta > -\phi. \end{aligned}$$

Because $1-z$ decreases the magnitude of z and rotates z towards the positive real axis, but not across it, we know that $f(V_1) \subset V_1$. We want to show that V_1 under repeated iterations of f tends to 0, and will use the same procedure as in the proof of lemma 2.5. Assume $z \in V_1$. Then

$$|f(z)| = |z(1-z)| \le |z| \left(1 - \frac{1}{2}|z|\right).$$

We write $z_n = f^n(z_0)$ and have that $r_n = |z_n|$. We get

$$r_1 \le r_0 \left(1 - \frac{1}{2} r_0\right)$$
$$r_{n+1} \le r_n \left(1 - \frac{1}{2} r_n\right)$$

by iteration. Let $\rho = r_0$ and

$$\rho_{n+1} = \rho_n \left(1 - \frac{1}{2}\rho_n\right).$$

Then $\{\rho_n\}$ is a decreasing sequence bounded below by 0. This means that the limit $\lim_{n\to\infty} \rho_n = \tilde{\rho}$ exists. Because $r_n < \rho_n$ for all n, it follows that $r_n \to \tilde{r}$ exists and

$$0 \leq \tilde{r} \leq \tilde{\rho}.$$

We use $\tilde{\rho} = \tilde{\rho}\left(1 - \frac{1}{2}\tilde{\rho}\right)$ to show that $\tilde{\rho} = 0$. This means that $\tilde{r} = 0$, so $\lim_{n\to\infty} z_n = 0$. In conclusion V_1 is in the basin of attraction for 0. □

Lemma 2.7. *If $f(z) = z(1 - z)$, then the region*

$$V_2 := \left\{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg(z) < 0, 0 < |z| < \frac{1}{10} \right\}$$

is in the basin of attraction for 0.

Proof. The proof is similar to the one of lemma 2.6. □

Proof (theorem 2.4). The theorem follows from lemma 2.5, 2.6 and 2.7. □

Theorem 2.8. *If $f(z) = z(1 - z)$, then the origin is on the boundary of its basin of attraction.*

Proof. Let Ω be the basin of attraction for 0. By theorem 2.3 we know $0 \notin \Omega$. From theorem 2.4 we know $V \subset \Omega$ and clearly $0 \in \overline{V}$. In conclusion $0 \in \partial\Omega$. □

We also want to show that the negative real axis is the only ray from the origin which does not intersect the basin of attraction for 0.

Lemma 2.9. *If $f(z) = z(1 - z)$ and Ω is the basin of attraction for 0, then $(-\infty, 0]$ is the only ray R originating in 0 such that $\Omega \cap R = \emptyset$.*

The statement in the lemma is equivalent with saying that for any $\theta \neq \pi$, we can find some $r > 0$ depending on θ so that $re^{i\theta} \in \Omega$. We will consider two cases, for which the dynamical behaviour of f is different. These are illustrated in figure 2.2, where we see that the magnitude of the iterates increases in the second quadrant and decreases in the first quadrant. In figure 2.3 we see that the magnitude of the iterates increases, because $|1 - z| > 1$.

Lemma 2.10. *Let $f(z) = z(1 - z)$ and Ω be the basin of attraction for 0. For every $\theta \in [\pi/4, \pi/2)$, there is an $r > 0$ so that $z = re^{i\theta} \in \Omega$.*

Proof. Let $f(z) = z(1 - z)$ and Ω be the basin of attraction for 0. Let $\theta \in [\pi/4, \pi/2)$. Then there is an $r > 0$ so that when $z = re^{i\theta}$, $|(1 - z)| < 1$. We can use the same procedure as in the proof of lemma 2.6 to show that the argument and the magnitude decreases, and that $f^n(z) \to 0$ as $n \to \infty$, so $z \in \Omega$. □

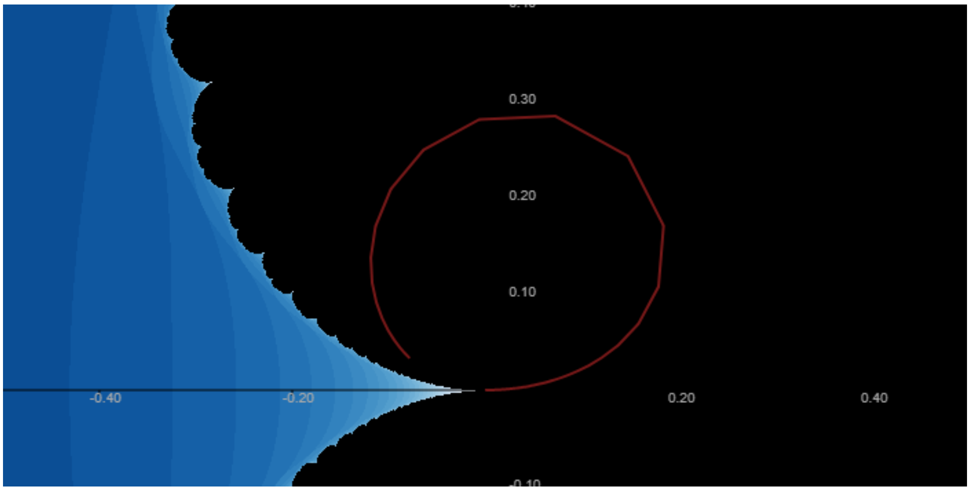

FIGURE 2.2. A detail view of the basin of attraction for 0 under the map $f(z) = z(1-z)$ and the orbit of a point z_0 in the second quadrant of the complex plane. The magnitude of z_n increases in the second quadrant, and decreases in the first quadrant. The figure is generated online, at `www.marksmath.org/visualization/polynomial_julia_sets/`.

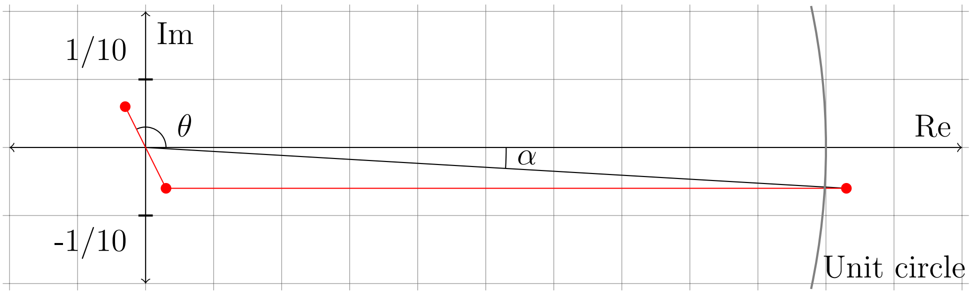

FIGURE 2.3. The point $z = -0.03 + 0.06i$, $-z$ and $1-z$ with arguments marked relative to the positive real axis.

LEMMA 2.11. *Let $f(z) = z(1-z)$ and Ω be the basin of attraction for 0. For every $\theta \in [\pi/2, \pi)$, there is an $r > 0$ so that $z = re^{i\theta} \in \Omega$.*

OUTLINE OF PROOF. Let $f(z) = z(1-z)$ and let Ω be the basin of attraction for 0. Let $\theta \in [\pi/2, \pi)$. We want to find an $r > 0$ so that when $z = re^{i\theta}$, there is a k so that

$$z_k = f^k(z) \in V_1 = \left\{ z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{4}, 0 < |z| < \frac{1}{10} \right\}.$$

Because the magnitude increases for each iteration when z lies in the second quadrant, we must have $r < 1/10$, and we know that $|\arg(1 - z_n)|$ increases for each iteration. Combining these facts, we can define a lower bound

$$\alpha = \left| \arg\left(1 - \frac{1}{10}e^{i\theta}\right) \right| \leq |\arg(1 - z_n)|, \quad n \geq 0.$$

The number of iterations k needed to ensure that $\left|\arg\left(f^k(z)\right)\right| < \pi/4$ is determined by solving $\theta + k\alpha < \pi/4$. We will determine the magnitude of z by solving the iteration problem

$$r_{n+1} = r_n(1 + r_n), \quad r_k < \frac{1}{10}, \quad n < k$$

for $r_0 > 0$. In conclusion, $z = r_0 e^{i\theta}$ gives $f^k(z) \in V_1$. □

Proof (lemma 2.9). The lemma follows from lemma 2.10 and 2.11, and the fact that these lemmas and their proofs can be restated for the same domains reflected over the real axis. □

We have shown that any point in a neighbourhood of the origin that is taken to 0 under repeated iterations of $f(z) = z(1 - z)$ first is taken to (or is already inside) V as described in theorem 2.4. This means that we can describe the basin of attraction for 0 as

$$\Omega := \bigcup_{n=0}^{\infty} f^{-n}(V),$$

and Ω is shown in figure 2.4.

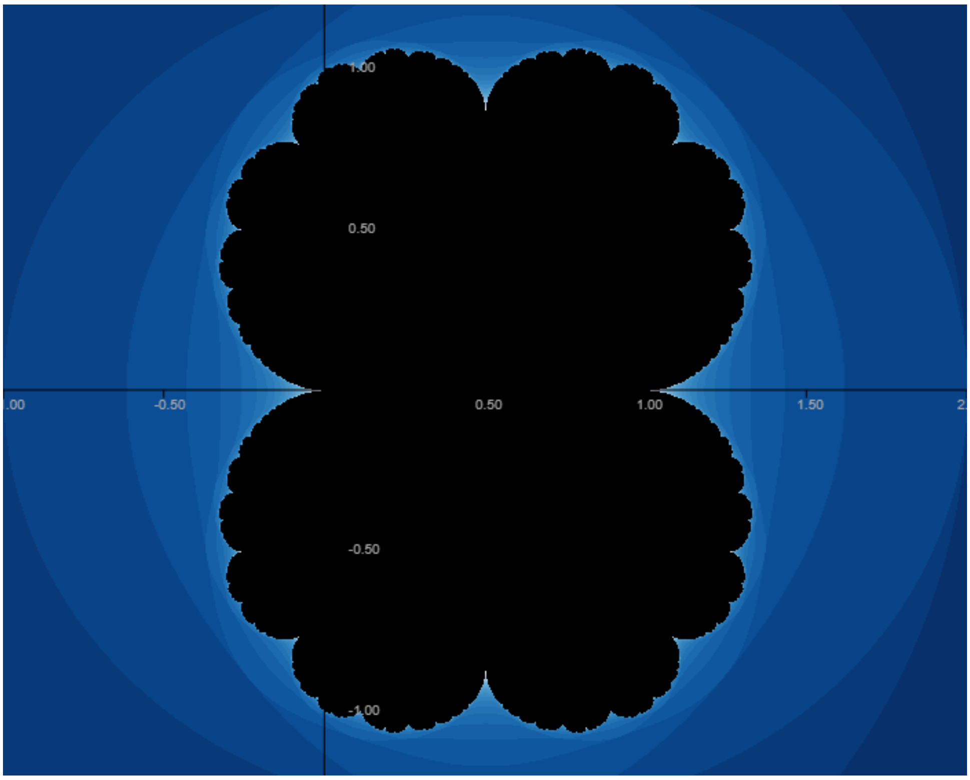

Figure 2.4. The basin of attraction for 0 under the transformation $f(z) = z(1 - z)$. This is known as the cauliflower because of its shape. The figure is generated online, at `www.marksmath.org/visualization/polynomial_julia_sets/`.

2.2. Higher Order Terms

If we add terms of higher order to our transformation $z \mapsto z(1 - z)$, we can write

$$f(z) = z - z^2 + O\left(z^3\right) = z\left(1 - z + O\left(z^2\right)\right).$$

Since we are interested in the dynamical behaviour of f in a neighbourhood of 0, the higher order terms are small enough to not change the dynamics too much from what we saw in chapter 2.1. We will use this to give some results that are similar to the ones we saw there.

LEMMA 2.12. *If f is holomorphic, 0 is a fixed point and ∞ is an attracting fixed point, then the basin of attraction for 0 is bounded.*

PROOF. Let f be a holomorphic function, let the basin of attraction for 0 be Ω_0 and let the basin of attraction for ∞ be Ω_∞. Because ∞ is an attracting fixed point there is a large disc $D(0,R)$ so that any $z \notin D(0,R)$ is in Ω_∞. Because the basin of attraction for two different fixed points must be disjoint, we know that $\Omega_0 \subset D(0,R)$, hence Ω_0 is bounded. □

Because ∞ is an attracting fixed point for all polynomials, we know that the basin of attraction for all finite fixed points of a polynomial is bounded. In example 2.13 we see an example of a holomorphic function with fixed point 0, but where ∞ is not an attracting fixed point.

EXAMPLE 2.13. *Let $f(z) = ze^{-z}$. There is a small $\delta > 0$ such that when $z \in (0,\delta)$, $f^n(z) \to 0$. Hence $(0,\delta)$ is in the basin of attraction for 0 under f. There is also an $M > 0$ so that when $z \in [M,\infty)$, $f(z) \in (0,\delta)$, implying that there are z in a neighbourhood of ∞ that are attracted to 0 under f.*

THEOREM 2.14. *If $f(z) = z\left(1 - z + O\left(z^2\right)\right)$, then there is an $\varepsilon > 0$ so that the region*

$$V_\varepsilon := \left\{z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}, 0 < |z| < \varepsilon\right\}$$

is in the basin of attraction for 0 under f.

PROOF. There is an $\varepsilon > 0$ so that when $|z| < \varepsilon$, then

$$\left|O\left(z^2\right)\right| \leq C\,|z|^2 \leq C\varepsilon\,|z|,$$

which lets us ensure that $|1 - z + O\left(z^2\right)| < 1$ as long as $|1 - z| < 1$. Set

$$z = re^{i\theta}$$
$$1 - z + O\left(z^2\right) = se^{i\alpha}$$

Because of the translation by 1, $1 - z + O\left(z^2\right)$ will have a smaller argument α than θ, the argument of z. So $|\alpha| < |\theta|$. Now $s < 1$ and $|\alpha| < |\theta|$ so $f(V_\varepsilon) \subset V_\varepsilon$. We use the same procedure as in the proof of lemma 2.6 and conclude that V_ε is in the basin of attraction for 0. □

Because 0 is a neutral fixed point we can use theorem 2.3 by Peters, Vivas and Wold [**5**] to conclude that 0 is on the boundary of its basin of attraction. The results proven in this chapter are corresponding to the results in chapter 2.1. In conclusion we have shown that $f(z) = z\left(1 - z + O\left(z^2\right)\right)$ has the same dynamical behaviour near 0 as $z \mapsto z(1-z)$.

2.3. Scaling the Second Term

Here we will look at iterations of

$$f(z) = z\left(1 - az + O\left(z^2\right)\right), \quad a \in \mathbb{C}.$$

We will conjugate f to a function for which we already know the dynamical behaviour.

THEOREM 2.15. *The dynamical behaviour of $f(z) = z(1 - az + O(z^2))$ near 0 is the same as for a function $g(w) = w(1 - w + O(w^2))$.*

PROOF. Let $f(z) = z(1 - az + O(z^2))$. We will define the conjugation $w = az$ so $z = w/a$. Then

$$f(z) = f(w/a) = \frac{w}{a}\left(1 - w + O\left(w^2\right)\right)$$

so $g(w) = af(w/a)$ is on the form

$$g(w) = w\left(1 - w + O\left(w^2\right)\right).$$

By iteration

$$f^n(z) = \frac{1}{a}g^n(az)$$

so the dynamical behaviour is the same for f and its conjugate g near 0. □

The iterations of maps of the same form as g was studied in chapter 2.2, so we conclude that f has the same dynamical behaviour near 0 as $z \mapsto z(1 - z)$.

2.4. Rotation of the First Term

We are now prepared to study maps of the form

$$f(z) = z\left(e^{2\pi i\theta} - z + O\left(z^2\right)\right)$$

and we will consider f for rational and irrational values of θ separately.

THEOREM 2.16. *The dynamical behaviour of $f(z) = z\left(e^{2\pi i\theta} - z + O(z^2)\right)$ near 0, for $\theta \in \mathbb{Q}$, is the same as for a function $g(w) = w(1 - w + O(w^2))$.*

PROOF. Suppose that θ is rational, and choose N to be the smallest natural number so that $N\theta \in \mathbb{Z}$. Because 0 is a fixed point, we know that $f^k(0) = 0$ for all $k \geq 0$. Then by the chain rule

$$f^{N\prime}(0) = f'\left(f^{N-1}(0)\right)\left(f^{N-1}\right)'(0) = \ldots = \underbrace{f'(0) \cdot \ldots \cdot f'(0)}_{N \text{ times}} = f'(0)^N,$$

which shows that the first coefficient of f^N is 1. We use the Taylor series expansion around 0 and the chain rule to obtain

$$\begin{aligned}
f^N(z) &= \left(f^N\right)(0) + \left(f^N\right)'(0)z + \left(f^N\right)''(0)z^2 + O\left(z^3\right) \\
&= 0 + 1z + bz^2 + O\left(z^3\right) \\
&= z\left(1 - (-b)z + O\left(z^2\right)\right)
\end{aligned}$$

which, for $a = -b$, has the same dynamical behaviour near 0 as a function

$$g(w) = w\left(1 - w + O\left(w^2\right)\right)$$

by theorem 2.15. But iterations are not always done in multiples of N. For any $k \in \mathbb{N}$ there are $m, j \in \mathbb{N}_0$ so $k = mN + j$ and $0 \leq j < N$. Then

$$f^k(z) = f^{mN+j}(z) = f^j \circ f^{mN}(z) = f^j\left(\frac{1}{a} g^m(az)\right).$$

We already know that there exists a region V_ε so that $f^N(V_\varepsilon) \subset V_\varepsilon$ and when $z \in V$ $f^{mN}(z) \to 0$ as $m \to \infty$ by theorem 2.14. It follows that $f^j\left(f^{mN}(z)\right) \to f^j(0) = 0$, because f is continuous and 0 is a fixed point under f. □

For an irrational θ we must recall the snail lemma (theorem 1.12).

THEOREM (Snail lemma). *Suppose that $f(z)$ is a polynomial of degree $d \geq 2$ with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$, where θ is irrational. Then there is no connected component of the basin of attraction for 0 under f.*

The snail lemma tells us that there is no connected Fatou component for transformations where the first coefficient is an irrational rotation. Hence we will regard them as exceptions. On the other hand, we have shown that for a rational θ the dynamical behaviour of $f(z) = z\left(e^{2\pi i\theta} - z + O\left(z^2\right)\right)$ near 0 is the same as for $z \mapsto z(1-z)$.

2.5. Rate of Convergence

In addition to saying that we have a basin of attraction, we will say at which rate the iterations converge towards the fixed point. A high rate of convergence means that the sequence which is the orbit of a point converges faster than with a lower rate of convergence. We will introduce Fatou coordinates, meaning conjugates by the conjugation $\phi(z) = 1/z$, to the transformation

$$f(z) = z\left(1 - z + O\left(z^2\right)\right).$$

Let $F = \phi \circ f \circ \phi^{-1}$. Then F is defined where $X = 1/z$ is defined, and

$$\begin{aligned}
F(X) &= \frac{1}{f\left(\frac{1}{X}\right)} \\
&= \frac{1}{\frac{1}{X}\left(1 - \frac{1}{X} + O\left(\frac{1}{X^2}\right)\right)} \\
&= X\left(1 + \left(\frac{1}{X} + O\left(\frac{1}{X^2}\right)\right) + \left(\frac{1}{X} + O\left(\frac{1}{X^2}\right)\right)^2 + \dots\right) \\
&= X\left(1 + \frac{1}{X} + \frac{a}{X^2} + O\left(\frac{1}{X^3}\right)\right) \\
&= X + 1 + \frac{a}{X} + O\left(\frac{1}{X^2}\right).
\end{aligned}$$

By iteration we have that

$$F^n(X) = X + n + \frac{an}{X} + O\left(\frac{1}{X^2}\right) \tag{2.2}$$

so

$$\begin{aligned} f^n\left(\frac{1}{X}\right) &= \frac{1}{F^n(X)} = \frac{1}{X + n + \frac{an}{X} + O\left(\frac{1}{X^2}\right)} \\ &= \frac{1}{n} \cdot \frac{1}{1 + \frac{X}{n} + O\left(\frac{1}{X}\right)}. \end{aligned}$$

Because $X = 1/z$ is fixed, we see that when n is large,

$$f^n\left(\frac{1}{X}\right) \sim \frac{1}{n}.$$

This means that the rate of convergence for f is $\frac{1}{n}$. The rate of convergence is illustrated in figure 2.5, where the iterates under f go towards 0.

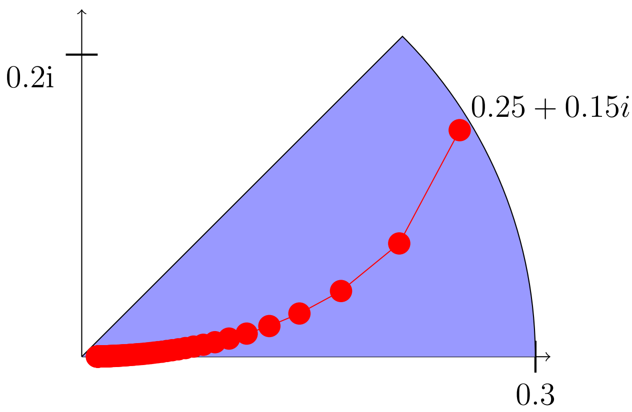

FIGURE 2.5. Iterations of $f(z) = z(1-z)$. This illustrates that the rate of convergence decreases as the iterates approaches the fixed point.

We will do a similar argument for

$$g(z) = z\left(1 - bz + O\left(z^2\right)\right),$$

again by the conjugation $X = \phi(z) = 1/z$. Let $G = \phi \circ g \circ \phi^{-1}$, and

$$\begin{aligned} G(X) &= \frac{1}{\frac{1}{X}\left(1 - b\frac{1}{X} + O\left(\frac{1}{X^2}\right)\right)} \\ &= X\left(1 + \frac{b}{X} + \frac{c}{X^2} + O\left(\frac{1}{X^3}\right)\right) \\ &= X + b + \frac{c}{X} + O\left(\frac{1}{X^2}\right). \end{aligned}$$

By iteration, the same way as for f and F, we get

$$G^n(X) = X + bn + \frac{cn}{X} + O\left(\frac{1}{X^2}\right)$$

so, by the same argument as for (2.2), the rate of convergence for g is $\frac{1}{|b|n}$.

CHAPTER 3

A Fatou-Bieberbach Domain in Two Variables

In this chapter we will study a holomorphic map in two variables and its basin of attraction. We will then construct a biholomorphic map from the basin of attraction to $\mathbb{C}^2$ to prove that the basin of attraction is a Fatou-Bieberbach domain.

LEMMA 3.1. *The map*

$$f(z,w) = \left(\frac{1}{2}w + z^2, \frac{1}{2}z\right) \tag{3.1}$$

is holomorphic, one-to-one and onto $\mathbb{C}^2$.

PROOF. The map f is clearly holomorphic, because it is complex differentiable everywhere. Assume $f(z_1, w_1) = f(z_2, w_2)$.

$$\begin{aligned} f(z_1, w_1) &= f(z_2, w_2) \\ \left(\frac{1}{2}w_1 + z_1^2, \frac{1}{2}z_1\right) &= \left(\frac{1}{2}w_2 + z_2^2, \frac{1}{2}z_2\right) \\ &\Rightarrow z_1 = z_2 \\ &\Rightarrow \frac{1}{2}w_1 + z_1^2 = \frac{1}{2}w_2 + z_1^2 \\ &\Rightarrow w_1 = w_2 \\ &\Rightarrow (z_1, w_1) = (z_2, w_2) \end{aligned}$$

so f is one-to-one. We can now find the inverse

$$f^{-1}(z,w) = \left(2w, 2\left(z - 4w^2\right)\right)$$

which is also complex differentiable everywhere, hence holomorphic. $\square$

We could also have said that it is a finite composition of a linear map, a shear and a flip, which are all biholomorphic maps. The next step is to verify that the origin is a fixed point under f, and that it has a basin of attraction.

LEMMA 3.2. *The origin is an attracting fixed point under f, and its basin of attraction contains a ball centred at the origin.*

PROOF. The proof will follow the proof given by Rosay and Rudin [**7**] in a more general case. Clearly $f(0,0) = (0,0)$. Define

$$A := f'(0,0) = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

so $\|A\| = \frac{1}{2}$. The origin is an attracting fixed point under f because the eigenvalues for A are $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$. Then choose constants α, β_1, β_2 and β so that

$$\alpha < \frac{1}{2} < \beta_1 < \beta_2 < \beta \quad \text{and} \quad \beta^2 < \alpha. \tag{3.2}$$

The spectral radius formula, see Rudin's book [**8**, p. 360], tells us that there is an m so that $\|A^{-N}\| < \alpha^{-N}$ and $\|A^N\| < \beta_1^N$ for all $N > m$. Also, there is an $r > 0$ so that

$$\|f^m(z, w)\| \leq \beta_2^m \|(z, w)\|$$

when $(z, w) \in B_r = B(0, r)$, the ball of radius r centred at the origin. Then choose C such that $|f^j(z, w)| \leq C\|(z, w)\|$ for $(z, w) \in B_r$ and $0 \leq j < m$. We can write $N = j + km$ for $0 \leq j < m$, $k = 1, 2, 3, \ldots$ for all $N \in \mathbb{N}$. If also $(z, w) \in B_r$, then

$$\|f^N(z, w)\| = \|f^j\left(f^{km}(z, w)\right)\| \leq C\|f^{km}(z, w)\| \leq C\beta_2^{km}\|(z, w)\|.$$

Thus there is an N_0 dependent of m and r so that for all $N \geq N_0$ and $(z, w) \in B_r$

$$\|f^N(z, w)\| < \beta^N, \tag{3.3}$$

because $\beta > \beta_2$ so $\beta^N/\beta_2^N \to \infty$ as $N \to \infty$. The restrictions given on β ensures that $\beta < 1$, so

$$\Omega := \bigcup_{n=0}^{\infty} f^{-n}(B_r). \tag{3.4}$$

is the basin of attraction for $(0, 0)$, clearly containing the ball B_r. $\square$

Now that we have defined Ω, we will show that it actually is a Fatou-Bieberbach domain in $\mathbb{C}^2$. We need to show two things:

(1) That Ω is a proper subset of $\mathbb{C}^2$.
(2) That there is a one-to-one holomorphic map $\Psi : \Omega \to \mathbb{C}^2$ which is onto.

Lemma 3.3. *The basin of attraction for* $(0, 0)$ *under* f *is a proper subset of* $\mathbb{C}^2$.

Proof. Let $(u, v) = f(z, w)$, so

$$\begin{cases} u = \frac{1}{2}w + z^2 \\ v = \frac{1}{2}z, \end{cases}$$

and let E be the region where $|z| > 2 + |w|$. Then for $(z, w) \in E$

$$\begin{aligned} |u| &\geq |z|^2 - \frac{1}{2}|w| \\ &> |z| \cdot |z| - \frac{1}{2}|z| \\ &> 2|z| - \frac{1}{2}|z| \\ &= |z| + \frac{1}{2}|z| \\ &> 2 + |v| \end{aligned}$$

which proves that $f(E) \subset E$. Because $(0,0)$ is an attracting fixed point, $(0,0) \in \Omega$. Also, $(0,0)$ is not in E because $|z| > 2$. This shows that $E \cap \Omega = \emptyset$, so Ω is a proper subset of $\mathbb{C}^2$. $\square$

We still have left to show that there is a biholomorphic map from Ω which is onto $\mathbb{C}^2$. The matrix $A = f'(0,0)$ has the inverse

$$A^{-1} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

which we will use to construct the map we want.

THEOREM 3.4. *The map* $\Psi := \lim_{n\to\infty} A^{-n} f^n$ *is holomorphic in* Ω*, one-to-one and onto* $\mathbb{C}^2$*.*

PROOF. In this proof we will use the notation $p = (z,w) \in \mathbb{C}^2$. We will use constants defined in the proof of lemma 3.2, see (3.2). Let K be a compact, proper subset of Ω. For any fixed $r > 0$ there is a j depending only on K so that $p \in K$ implies $f^j(p) \in B(0,r)$. Recall the proof for lemma 3.2, and use (3.3) to deduce that for $p \in K$ and $N > N_0 + j$

$$\|f^N(p)\| \le \beta^{N-j} = a\beta^N, \quad \text{where } a = \frac{1}{\beta^j}.$$

We use that $A^{-1}f'(0) = I$ to deduce that for $q \in \mathbb{C}^2$, $|q| < a$, there is a constant b so

$$\left|q - A^{-1}f(q)\right| \le b\,|q|^2.$$

We set $q_N = f^N(p)$ so

$$\begin{aligned} \left|A^{-N}f^N(p) - A^{-N-1}f^{N+1}(p)\right| &\le \|A^{-N}\|\,\left|q_N - A^{-1}f(q_N)\right| \\ &\le \alpha^{-N} b\,|q_N|^2 \\ &\le ba^2\left(\frac{\beta^2}{\alpha}\right)^N \end{aligned}$$

for all $N \ge N_0 + j$. By the triangle inequality

$$\left|A^{-N}f^N(p) - A^{-N-M}f^{N+M}(p)\right| \le \sum_{k=N}^{N+M-1} ba^2\left(\frac{\beta^2}{\alpha}\right)^k.$$

Recall that $\beta^2/\alpha < 1$ so we have a convergent geometric series. For any $\varepsilon > 0$ there is an M_ε so that $\left|A^{-N}f^N(p) - A^{-N-M}f^{N+M}(p)\right| \le \varepsilon$ whenever $N \ge \max\{M_\varepsilon, N_0\}$ and $M \ge 0$. This is the definition of a Cauchy sequence, hence

$$\lim_{N\to\infty} A^{-N}f^N = \Psi$$

exists. Because A and f are biholomorphic, so is Ψ. Finally, notice that

$$\Psi = A^{-1} \circ \Psi \circ f$$

and $f(\Omega) = \Omega$ by the construction in (3.4). Then $A^{-1}\Psi$ and Ψ has the same range. Since A^{-1} is an expansion, the range must be all of $\mathbb{C}^2$, as claimed in the theorem. $\square$

CHAPTER 4

Construction of an Automorphism Tangent to the Identity at the Origin

We will construct an example of the transformations described in Hakim's paper [**4**], section 5. We want our transformation in $\mathbb{C}^2$ to have the same dynamical behaviour near the origin as the previously studied transformation $z\left(1-z+O\left(z^2\right)\right)$ in the first variable, and a contracting behaviour in the second variable. In fact we want to produce an automorphism

$$h(z,w)=\begin{cases} z(1-z)+O\left(z^3, wz^5\right) \\ w(1-z)+O\left(z^3, z^2w\right). \end{cases}$$

The procedure is a composition of shears and overshears, which ensures that the transformation is holomorphic, one-to-one and onto $\mathbb{C}^2$:

$$\begin{aligned}
(z_0,w_0)\mapsto (z_1,w_1) &= \left(z_0, w_0e^{-z_0}\right) &&= h_1(z_0,w_0)\\
\mapsto (z_2,w_2) &= (z_1e^{w_1}, w_1) &&= h_2(z_1,w_1)\\
\mapsto (z_3,w_3) &= (z_2, w_2+z_2) &&= h_3(z_2,w_2)\\
\mapsto (z_4,w_4) &= \left(z_3e^{-w_3}, w_3\right) &&= h_4(z_3,w_3)\\
\mapsto (z_5,w_5) &= (z_4, w_4-z_4) &&= h_5(z_4,w_4)\\
\mapsto (z_6,w_6) &= \left(z_5, w_5-z_5^2\right) &&= h_6(z_5,w_5)\\
\mapsto (z_7,w_7) &= \left(z_6, w_6-az_6^3\right) &&= h_7(z_6,w_6)\\
\mapsto (z_8,w_8) &= \left(z_7, w_7-bz_7^4\right) &&= h_8(z_7,w_7)\\
\mapsto (z_9,w_9) &= \left(z_8, w_8-dz_8^5\right) &&= h_9(z_8,w_8)
\end{aligned}$$

The constants a, b and d will be determined through the calculations later, but the idea is stated above. We will denote

$$\tilde{h}=h_9\circ h_8\circ h_7\circ h_6\circ h_5\circ h_4\circ h_3\circ h_2\circ h_1$$

so

$$\tilde{h}(z,w)=(z_9,w_9).$$

We now do the calculations to show explicitly how the composition of all these functions works. Some of the calculations will look messy, but we will do them step by step to be as clear as possible and point at the main ideas.

$$\begin{aligned}
(z_2, w_2) &= \left(ze^{we^{-z}}, we^{-z}\right) \\
(z_3, w_3) &= \left(ze^{we^{-z}}, we^{-z} + ze^{we^{-z}}\right) \\
(z_4, w_4) &= \left(ze^{we^{-z}}e^{-we^{-z}-ze^{we^{-z}}}, we^{-z} + ze^{we^{-z}}\right) \\
&= \left(ze^{-ze^{we^{-z}}}, we^{-z} + ze^{we^{-z}}\right) \\
(z_5, w_5) &= \left(ze^{-ze^{we^{-z}}}, we^{-z} + ze^{we^{-z}} - ze^{-ze^{we^{-z}}}\right)
\end{aligned}$$

We will take a closer look at w_5 by expanding each term into power series at $(0, 0)$, expanding into as many terms as we will need later:

$$\begin{aligned}
we^{-z} &= w\left(1 - z + O\left(z^2\right)\right) \\
ze^{we^{-z}} &= z\left(1 + w\left(1 - z + O\left(z^2\right)\right) + O\left(w^2, z^2w^2\right)\right) \\
&= z\left(1 + w + O\left(w^2, zw\right)\right) \\
ze^{-ze^{we^{-z}}} &= z\sum_{n=0}^{\infty}\frac{1}{n!}\left(-z\left(1 + w + O\left(w^2, zw\right)\right)\right)^n \\
&= z\left(1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{24}z^4 + O\left(z^5, zw\right)\right) \\
&= z - z^2 + \frac{1}{2}z^3 - \frac{1}{6}z^4 + \frac{1}{24}z^5 + O\left(z^6, z^2w\right).
\end{aligned}$$

Notice that the last calculation is the same as z_5, because the previous step was $(z_4, w_4) \mapsto (z_4, w_4 - z_4) = (z_5, w_5)$. The first coordinate remains unchanged through steps 5 to 9, hence

$$z_9 = z_8 = z_7 = z_6 = z_5 = z - z^2 + \frac{1}{2}z^3 - \frac{1}{6}z^4 + \frac{1}{24}z^5 + O\left(z^6, z^2w\right) \tag{4.1}$$

which we will use later. Combining the expressions above we get

$$\begin{aligned}
w_5 = w&\left(1 - z + O\left(z^2\right)\right) + z\left(1 + w + O\left(zw, w^2\right)\right) \\
&- \left(z - z^2 + \frac{1}{2}z^3 - \frac{1}{6}z^4 + \frac{1}{24}z^5 + O\left(z^6, z^2w\right)\right) \\
= w& - wz + z + wz - z \\
&+ z^2 - \frac{1}{2}z^3 + \frac{1}{6}z^4 - \frac{1}{24}z^5 + O\left(z^6, z^2w, zw^2\right) \\
= w& + z^2 - \frac{1}{2}z^3 + \frac{1}{6}z^4 - \frac{1}{24}z^5 + O\left(z^6, z^2w, zw^2\right).
\end{aligned}$$

We will use the expression for z_5 from equation (4.1) in the following calculations. We want to get rid of the pure terms of z up to and including order 5. Using this we obtain

$$\begin{aligned}
w_6 &= w + z^2 - \frac{1}{2}z^3 + \frac{1}{6}z^4 - \frac{1}{24}z^5 + O\left(z^6, z^2w, zw^2\right) - z_5^2 \\
&= w + \frac{3}{2}z^3 - \frac{11}{6}z^4 + \frac{31}{24}z^5 + O\left(z^6, z^2w, zw^2\right) \\
w_7 &= w_6 - \frac{3}{2}z_6^3 = w_6 - \frac{3}{2}z_5^3 & a = \frac{3}{2} \\
&= w + \frac{8}{3}z^4 - \frac{131}{24}z^5 + O\left(z^6, z^2w, zw^2\right) \\
w_8 &= w_7 - \frac{8}{3}z_7^4 = w_7 - \frac{8}{3}z_5^4 & b = \frac{131}{24} \\
&= w + \frac{125}{24}z^5 + O\left(z^6, z^2w, zw^2\right) \\
w_9 &= w_8 - \frac{125}{24}z_8^5 = w_8 - \frac{125}{24}z_5^5 & d = \frac{125}{24} \\
&= w + O\left(z^6, z^2w, zw^2\right)
\end{aligned}$$

where a, b and d were chosen to fit in the calculation of w_7, w_8 and w_9, respectively. The need to remove the lower order terms of z will become apparent later, first in lemma 4.1, and then in the proof of lemma 5.1, see (5.5) on page 26. We may also run into a problem with the higher order term of z^2w in z_5 later. This becomes evident when we want to show that the sum of these higher order terms converges in lemma 5.5 on page 29. To solve this problem before it occurs we will use a conjugation trick. Let us take a step back first and notice that $\tilde{h}(0, w) = (0, w)$.

LEMMA 4.1. *Let $\phi(z, w) = (z, wz^3)$. The map $\phi^{-1} \circ \tilde{h} \circ \phi$ defined on $\mathbb{C}^* \times \mathbb{C}$ extends as an automorphism on $\mathbb{C}^2$.*

PROOF. The map $\phi(z, w) = (z, wz^3)$ is holomorphic and one-to-one as long as $z \neq 0$. Then

$$\left(\tilde{h} \circ \phi\right)(z, w) = \begin{cases} z\left(1 - z + O\left(z^2, z^4w\right)\right) \\ \left(wz^3 + O\left(z^6, z^5w, z^7w^2\right)\right) \end{cases}$$

is holomorphic and one-to-one. The inverse of ϕ is $\phi^{-1}(z, w) = \left(z, \frac{w}{z^3}\right)$, which is also holomorphic and one-to-one as long as $z \neq 0$. We get the conjugation

$$\left(\phi^{-1} \circ \tilde{h} \circ \phi\right)(z, w) = \begin{cases} z_{10} &= z\left(1 - z + O\left(z^2, z^4w\right)\right) \\ w_{10} &= \left(w + O\left(z^3, z^2w\right)\right)e^{3ze^{we^{-z}}}. \end{cases} \tag{4.2}$$

The composition in equation (4.2) is defined on

$$\left\{(z, w) \in \mathbb{C}^2 : z \neq 0\right\},$$

but we see that

$$\lim_{z \to 0}\left(\phi^{-1} \circ \tilde{h} \circ \phi\right)(z, w) = (0, w)$$

so the map can be extended to $\mathbb{C}^2$. Because the map is a composition of biholomorphic maps, the map itself is biholomorphic. $\square$

The last step is an overshear

$$(z_{10}, w_{10}) \mapsto \left(z_{10}, w_{10}e^{-4z_{10}}\right) = (z_{11}, w_{11}),$$

which we call h_{11} (notice that the 10th step was the conjugation with ϕ), and obtain

$$\begin{cases} z_{11} & = z\left(1 - z + O\left(z^2, z^4 w\right)\right) \\ w_{11} & = \left(w + O\left(z^3, z^2 w\right)\right)\left(1 - z + O\left(z^2\right)\right) \end{cases}$$

which we can recognise as very similar to the one variable functions that were studied in chapter 2.

We will denote the whole composition of maps by h, so

$$h = h_{11} \circ \phi^{-1} \circ h_9 \circ h_8 \circ h_7 \circ h_6 \circ h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1 \circ \phi$$

and

$$h(z, w) = \begin{cases} z\left(1 - z + O\left(z^2, wz^4\right)\right) \\ \left(w + O\left(z^3, z^2 w\right)\right)\left(1 - z + O\left(z^2\right)\right). \end{cases}$$

We can rewrite h as

$$h(z, w) = \begin{cases} z(1 - z) + O\left(z^3, wz^5\right) \\ w(1 - z) + O\left(z^3, z^2 w\right) \end{cases}$$

which we recognise as the map in the introduction to this chapter. The final remark is that h is composed by shears and overshears, and a trick that was commented earlier, so h is an automorphism. This will be used in the next chapter, when we will show that the basin of attraction for $(0, 0)$ under h is a Fatou-Bieberbach domain.

CHAPTER 5

Construction of the Fatou-Bieberbach Domain with a Complex Line in the Boundary

We want to show that the basin of attraction for h constructed in chapter 4 is a Fatou-Bieberbach domain. Further this will be a domain with the w-axis in the boundary. The difference between this and the example in chapter 3 is that the automorphism h is now tangent to the identity at the origin, which is a fixed point and lies in the boundary of its basin of attraction.

First, recall that we constructed an automorphism

$$h(z,w) = \begin{cases} z\left(1 - z + O\left(z^2, z^4 w\right)\right) \\ \left(w + O\left(z^3, z^2 w\right)\right)\left(1 - z + O\left(z^2\right)\right) \end{cases} \tag{5.1}$$

in chapter 4. We will continue to use this here. Then define the sets

$$V_\varepsilon := \left\{ z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}, 0 < |z| < \varepsilon \right\} \text{ and } \Delta_\varepsilon := \left\{ w \in \mathbb{C}^2 : |w| < \varepsilon \right\}$$

and their cross-product

$$B_\varepsilon := V_\varepsilon \times \Delta_\varepsilon. \tag{5.2}$$

Lemma 5.1. *Given h as in (5.1). Then there is an $\varepsilon > 0$ such that $h(B_\varepsilon) \subset B_\varepsilon$.*

We will prove the lemma soon. First let $\Omega := \bigcup_{n=0}^{\infty} h^{-n}(B_\varepsilon)$. In this chapter we will show that Ω is a domain with $(0,w)$ in the boundary, and construct a one-to-one holomorphic map G satisfying

$$G : \Omega \to \mathbb{C}^2, \qquad G(\Omega) = \mathbb{C}^2.$$

Proof (lemma 5.1). Let $(z_1, w_1) = h(z,w)$. We have

$$z_1 = z\left(1 - z + O\left(z^2, z^4 w\right)\right)$$

$$w_1 = w\left(1 + O\left(\frac{z^3}{w}, z^2\right)\right)\left(1 - z + O\left(z^2\right)\right) \tag{5.3}$$

$$= w\left(1 - z + O\left(z^2\right)\right) + O\left(z^3, z^2 w\right). \tag{5.4}$$

The proof for the second coordinate will be split in two, where w_1 will be expressed as in equations (5.3) and (5.4). The cases are

(1) The case when $|w| < \frac{1}{2}\varepsilon$.
(2) The case when $\frac{1}{2}\varepsilon \leq |w| < \varepsilon$.

We start with the first coordinate. Note that for $(z,w)\in B_\varepsilon$, $|z|$, $|w|<\varepsilon$. Then

$$\begin{aligned} z_1 &= z\left(1-z+O\left(z^2,\varepsilon z^4\right)\right)\\ &= z\left(1-z+O\left(z^2\right)\right)\end{aligned}$$

which we recognise from chapter 2. Therefore $(z,w)\in B_\varepsilon \Rightarrow z_1\in V_\varepsilon$. Here we made use of the conjugation we did to obtain (z_{10},w_{10}) in chapter 4, see (4.2) on page 23. We use the fact that we can choose a small enough ε so when $z\in V_\varepsilon$ is fixed we get

$$\left|1-z+O\left(z^2\right)\right| < 1-\frac{1}{10}|z|.$$

In case 1 we have

$$\begin{aligned} |w_1| = &\left|w\left(1-z+O\left(z^2\right)\right)+O\left(z^3,z^2w\right)\right|\\ &\le |w|\left(1-\frac{1}{10}|z|\right)+C\varepsilon^2\\ &< \frac{1}{2}\varepsilon\left(1-\frac{1}{10}|z|+2C\varepsilon\right)\\ &<\varepsilon\end{aligned}$$

as long as $\left|1-\frac{1}{10}|z|+2C\varepsilon\right|<2$.

In case 2 we have that w is away from the origin, so $|w|>\kappa|z|$ for some $\kappa>0$. Then

$$\begin{aligned} |w_1| = &\ |w|\cdot\left|1+O\left(\frac{z^3}{w},z^2\right)\right|\left|1-z+O\left(z^2\right)\right|\\ &\le |w|\cdot\left(1+C_1\left|\frac{z^3}{w}\right|+C_2\left|z^2\right|\right)\left|1-\frac{1}{10}|z|\right| \qquad (5.5)\\ &\le |w|\cdot\left(1+C_1\frac{\varepsilon^2}{\kappa}+C_2\varepsilon^2\right)\left(1-\frac{1}{10}|z|\right)\\ &\le \varepsilon\left(1-\frac{1}{10}|z|+C_3\varepsilon^2-C_4\varepsilon^3\right)\\ &<\varepsilon \qquad (5.6)\end{aligned}$$

when $\left(1-\frac{1}{10}|z|+C_3\varepsilon^2-C_4\varepsilon^3\right)<1$.

The calculations above show that $|w_1|\le|w|$, hence $(z,w)\in B_\varepsilon\Rightarrow w_1\in\Delta_\varepsilon$, and the lemma is proved. □

Lemma 5.2. *If h is given by* (5.1) *and B_ε by* (5.2), *then B_ε is in the basin of attraction for the origin under h.*

Proof. Let $(z,w)\in B_\varepsilon$. We want to show that $h^n(z,w)\to(0,0)$ when $n\to\infty$.

We know that $|z|$, $|w|<\varepsilon$. Let $(z_n,w_n)=h^n(z_0,w_0)$. From chapter 2 we know that $(z_0,w_0)\in B_\varepsilon\Rightarrow z_n\to 0$, because $|w_n|<\varepsilon$ from lemma 5.2. In particular from chapter 2.5 we can deduce that $z_n\sim 1/n$. We use (5.1) and by induction

$$w_{n+1}=\left(w_n+O\left(z_n^3,z_n^2w_n\right)\right)\left(1-z_n+O\left(z_n^2\right)\right).$$

We are in B_ε, so we can choose a small enough ε to get

$$\left|1 - z_n + O\left(z_n^2\right)\right| < 1 - \frac{1}{10}\left|z_n\right|,$$

and write

$$\left|z_{n+1}\right| \leq \left|z_n\right|\left(1 - \frac{1}{10}\left|z_n\right|\right).$$

We use this to calculate

$$\begin{aligned}
\left|w_{n+1}\right| &\leq \Big(\left|w_n\right| + \left|O\left(z_n^3, z_n^2 w_n\right)\right|\Big)\left|1 - z_n + O\left(z_n^2\right)\right| \\
&\leq \Big(\left|w_n\right| + C_1\left|z_n\right|^3 + C_2\left|z_n\right|^2\left|w_n\right|\Big)\left(1 - \frac{1}{10}\left|z_n\right|\right) \\
&= \left|w_n\right|\left(1 + C_2\left|z_n\right|^2\right)\left(1 - \frac{1}{10}\left|z_n\right|\right) + C_1\left|z_n\right|^3\left(1 - \frac{1}{10}\left|z_n\right|\right) \\
&\leq \left|w_0\right|\prod_{j=0}^{n}\left(1 + C_2\left|z_j\right|^2\right)\prod_{j=0}^{n}\left(1 - \frac{1}{10}\left|z_j\right|\right) \\
&\quad + C_1\sum_{j=0}^{n}\left(\left|z_j\right|^3\prod_{k=j}^{n}\left(1 - \frac{1}{10}\left|z_k\right|\right)\right) \\
&\leq \left|w_0\right|\prod_{j=1}^{n}\left(1 + C_2\frac{1}{j^2}\right)\prod_{j=1}^{n}\left(1 - \frac{1}{10j}\right) + C_1\sum_{j=1}^{n}\frac{1}{j^3}\exp\left(\sum_{k=j}^{n}\ln\left(1 - \frac{1}{10k}\right)\right)
\end{aligned}$$

for constants C_1 and C_2. We will use that $\ln(1-u) = -u + O\left(u^2\right)$ several times and rewrite the expression by using the following method:

$$\begin{aligned}
\exp\sum_{k=j}^{n}\ln\left(1 - \frac{1}{10k}\right) &= \exp\sum_{k=j}^{n}\left(-\frac{1}{10k} + O\left(\frac{1}{k^2}\right)\right) \\
&= \exp\left(-\frac{1}{10}\sum_{k=j}^{n}\left(\frac{1}{k} + O\left(\frac{1}{k^2}\right)\right)\right) \\
&\leq C_3\exp\left(-\frac{1}{10}\big(\ln(n) - \ln(j-1)\big)\right) \\
&\leq C_3\left(\frac{j-1}{n}\right)^{1/10}.
\end{aligned}$$

Then, for new constants C_4, C_5 and C,

$$\begin{aligned}
\left|w_{n+1}\right| &\leq C_4\left|w_0\right|\frac{1}{n^{1/10}} + C_5\sum_{j=1}^{n}\frac{1}{j^3}\left(\frac{j-1}{n}\right)^{1/10} \\
&\leq \left(C_4\left|w_0\right| + C_5\sum_{j=1}^{n}\frac{(j-1)^{1/10}}{j^3}\right)\frac{1}{n^{1/10}} \\
&\leq \left(C_4\left|w_0\right| + C_5\sum_{j=1}^{n}\frac{1}{j^{3-1/10}}\right)\frac{1}{n^{1/10}} \leq C\frac{1}{n^{1/10}}
\end{aligned}$$

which goes to 0 as $n \to \infty$. $\square$

LEMMA 5.3. *The basin of attraction for* $(0,0)$ *under* h *is a proper subset of* $\mathbb{C}^2$.

PROOF. The lemma follows from the fact that

$$\{(0,w) : w \in \mathbb{C}\} \cap \Omega = \emptyset,$$

because h preserves $(0,w)$. □

LEMMA 5.4. *The basin of attraction for* $(0,0)$ *under* h *has the line* $(0,w)$ *in the boundary.*

PROOF. The lemma follows from two facts. First the fact that $h(0,w) = (0,w)$. Secondly the fact that for any $w \in \mathbb{C}$ and any $\delta > 0$, we can find an $\varepsilon \in (0,\delta)$ so that whenever $z \in V_\varepsilon$, it follows that $h^n(z,w) \to (0,0)$ as $n \to \infty$. This means that in a neighborhood of all $(0,w)$ there are both points in and not in Ω, hence $(0,w) \in \partial\Omega$. □

To construct G as indicated before, we will in the next chapters construct two one-to-one holomorphic maps

$$g_1, g_2 : \Omega \to \mathbb{C} \text{ so that } (g_1, g_2)(\Omega) = \mathbb{C}^2.$$

5.1. Constructing a Map in the First Coordinate

We will construct an Abel-Fatou function Ψ satisfying the Abel equation

$$\Psi \circ H = \Psi + 1$$

where H is the Fatou coordinate for h, given as

$$H \circ \phi = \phi \circ h \tag{5.7}$$

for $\phi(z,w) = \left(\frac{1}{z}, w\right)$. As we did in chapter 2.5, we will denote $X = \frac{1}{z}$ and see that the first coordinate of H is

$$\pi_1 H(X,w) = X + 1 + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right).$$

First we define

$$\Psi_1(X,w) = X - a \log X.$$

Then, composed with H, we get

$$\Psi_1 \circ H(X,w) = \pi_1 H(X,w) - a \log(\pi_1 H(X,w)),$$

which written out is

$$\begin{aligned}
&= X + 1 + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) - a\log\left(X + 1 + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right)\right) \\
&= X + 1 + \frac{a}{X} - a\log\left(X\left(1 + \frac{1}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right)\right)\right) + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) \\
&= X + 1 + \frac{a}{X} - a\log(X) - a\log\left(1 + \frac{1}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right)\right) + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) \\
&= X + 1 + \frac{a}{X} - a\log(X) - \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) \\
&= X - a\log(X) + 1 + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) \\
&= \Psi_1(X, w) + 1 + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right),
\end{aligned}$$

where we used the Taylor series for $\log(1+u)$ around 0. If we denote the higher order terms by $v(X, w)$, so

$$\Psi_1(H(X, w)) = \Psi_1(X, w) + 1 + v(X, w),$$

then proceed inductively with

$$\begin{aligned}
\Psi_1\left(H^N(X, w)\right) &= \Psi_1\left(H\left(H^{N-1}(X, w)\right)\right) \\
&= \Psi_1(X, w) + N + \sum_{k=0}^{N-1} v\left(H^k(X, w)\right).
\end{aligned}$$

We define

$$\Psi(X, w) = X - a\log X + \sum_{k=0}^{\infty} v\left(H^k(X, w)\right). \tag{5.8}$$

where we have included the higher order terms from Ψ_1. Now we need to answer two questions:

(1) Does the series $\sum_{k=0}^{\infty} v\left(H^k(X, w)\right)$ converge?
(2) Does Ψ satisfy the Abel equation?

LEMMA 5.5. *If H is given by* (5.7) *and Ψ is defined as in* (5.8)*, then the series*

$$\sum_{k=0}^{\infty} v\left(H^k(X, w)\right)$$

converges.

PROOF. Recall that $v(X, w) = O\left(\frac{1}{X^2}, \frac{w}{X^4}\right)$, so

$$\sum_{k=0}^{\infty} v\left(H^k(X, w)\right) = \sum_{k=0}^{\infty} O\left(\frac{1}{\left(H^k(X, w)\right)^2}, \frac{w}{\left(H^k(X, w)\right)^4}\right).$$

Recall that $|w| < \varepsilon$. Set $X_k = \pi_1 H^k(X, w)$ and use that

$$\frac{1}{X_k} = \frac{1}{X+k} + O\left(\frac{1}{(X+k)^2}\right)$$

which is proven in lemma 5.14. Then there are C_k so that

$$\left| O\left(\frac{1}{(H^k(X,w))^2}, \frac{w}{(H^k(X,w))^4}\right) \right| \leq \left| O\left(\frac{1}{(X+k)^2}\right) \right| \leq C_k \left| \frac{1}{(X+k)^2} \right|.$$

Let $C := \max C_k$. Then

$$\sum_{k=0}^{\infty} \left| v\left(H^k(X,w)\right) \right| = \sum_{k=0}^{\infty} \left| O\left(\frac{1}{(H^k(X,w))^2}\right) \right| \leq \sum_{k=0}^{\infty} C_k \left| \frac{1}{(X+k)^2} \right| \leq C \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which we know converges. $\square$

LEMMA 5.6. *If H is given by (5.7) and Ψ is defined as in (5.8), then Ψ satisfies the Abel equation with H.*

PROOF. The proof is a calculation.

$$\begin{aligned}
(\Psi \circ H)(X, w) &= \Psi_1(H(X,w)) + \sum_{j=0}^{\infty} v\left(H^j(H(X,w))\right) \\
&= \Psi_1(X,w) + 1 + v(X) + \sum_{j=0}^{\infty} v\left(H^{j+1}(X,w)\right) \\
&= \Psi_1(X,w) + 1 + \sum_{j=0}^{\infty} v\left(H^j(X,w)\right) \\
&= \Psi(X,w) + 1
\end{aligned}$$

so Ψ satisfies the Abel equation with $k = 1$. $\square$

We will use Ψ to construct a one-to-one holomorphic map from Ω which is onto $\mathbb{C}^2$. Define W so that

$$W = \left\{ z \in \mathbb{C} : \frac{1}{z} \in V_\varepsilon \right\} \text{ and } \tilde{B}_\varepsilon = W \times \Delta_\varepsilon. \tag{5.9}$$

DEFINITION 5.7. *Let $p \in \Omega$. Define*

$$g_1(p) = \Psi\left(H^N(\phi(p))\right) - N$$

where $H^N(\phi(p)) \in W$.

LEMMA 5.8. *If $H^{N_0}(p) \in \tilde{B}_\varepsilon$, then*

$$\left(\Psi \circ H^N \circ \phi\right)(p) - N = (\Psi \circ H \circ \phi)(p)$$

for all $N \geq N_0$ and g_1 is well defined in Ω.

PROOF. Assume that $H^N\left(\frac{1}{z}, w\right) \in \tilde{B}_\varepsilon$. Then

$$\begin{aligned}\Psi\left(H^{N+1}\left(\frac{1}{z}, w\right)\right) - (N-1) &= \Psi\left(H^{N}\left(\frac{1}{z}, w\right)\right) + 1 - (N-1)\\ &= \Psi\left(H^{N}\left(\frac{1}{z}, w\right)\right) - N.\end{aligned}$$

We already know that $\Psi \circ H^N = \Psi \circ H + N$ in $\tilde{B}_\varepsilon$ so the statement is proven. $\square$

LEMMA 5.9. *For any large compact $Q \subset B_\varepsilon$ and any $N \in \mathbb{N}$, there is a compact set $K_N \subset \Omega$ so that*

$$Q \subset h^N(K_N) \subset B_\varepsilon.$$

PROOF. Pick a large compact $Q \subset B_\varepsilon$. Given any N, we can choose $K_N = h^{-N}(Q)$. By definition Ω is completely invariant under h, so $K_N \subset \Omega$. $\square$

THEOREM 5.10. *The map $g_1 : \Omega \to \mathbb{C}$ is onto.*

First, we need another piece of information about Ψ. The following lemma is illustrated in figure 5.1.

LEMMA 5.11. *If $\tilde{W} = \Psi\left(\tilde{B}_\varepsilon\right)$, then there is a $W_1 \subset \tilde{W}$ so that*

$$\bigcup_{n=0}^{\infty} (W_1 - n) = \mathbb{C}.$$

PROOF. From the construction, Ψ projects $\tilde{B}_\varepsilon$ to its first coordinate W and distorts it. For large X (small z) the distortion is small, because the first term dominates the expression. $\square$

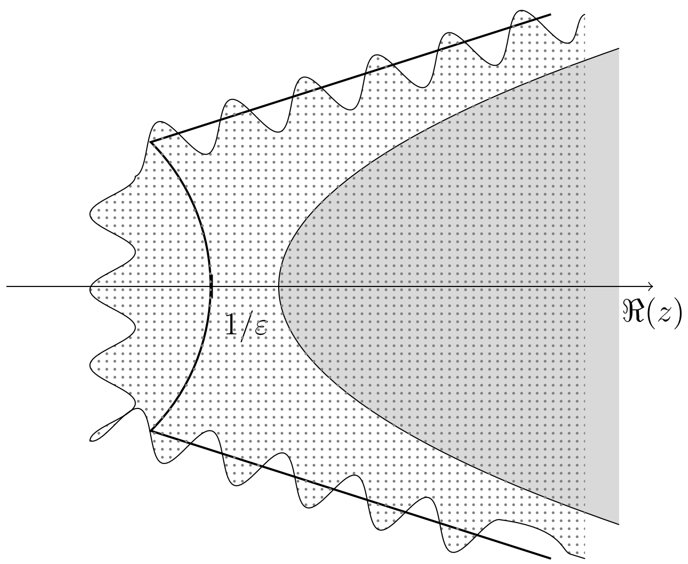

FIGURE 5.1. An illustration showing how the domains W, $\tilde{W}$ (dotted) and W_1 (shaded) as described in lemma 5.11 can be.

PROOF (g_1 IS ONTO $\mathbb{C}$). Given $\tilde{B}_\varepsilon$ as in (5.9). There is a family of compacts C_j so that

(1) $C_j \subset C_{j+1}$
(2) $\bigcup_{j=1}^{\infty} C_j = \tilde{B}_\varepsilon$

For all n and j there is a compact $K_{n,j}$ so that

$$H^n(K_{n,j}) = C_j.$$

We then use lemma 5.11, and the fact that Ψ is continuous to show

$$\begin{aligned} g_1(\Omega) &= \bigcup_{n=0}^{\infty} \left(\Psi\left(H^n(\phi(\Omega)\right) - n\right) \\ &\supset \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{\infty} \left(\Psi(C_j) - n\right) \\ &= \bigcup_{n=0}^{\infty} \left(\Psi\left(\tilde{B}_\varepsilon\right) - n\right) \\ &\supset \bigcup_{n=0}^{\infty} (W_1 - n) = \mathbb{C} \end{aligned}$$ □

5.2. Constructing a Map in the Second Coordinate and Conclusion

The next step is to construct a map that can fill out the whole complex plane from a small disc, Δ_ε. To achieve that we want to find a σ which satisfies

$$\sigma \circ h = \exp\left(\frac{-1}{\Psi}\right)\sigma. \tag{5.10}$$

We start with

$$\sigma_n(z, w) = h_2^n(z, w) \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(\frac{1}{z}, w\right) + j}\right) \tag{5.11}$$

We need to answer two questions about this map:

(1) Does it converge?
(2) Does the relation in (5.10) hold?

We will start with the second question, and come back to the first later, in theorem 5.16.

LEMMA 5.12. *If σ_n converges, then* (5.10) *holds.*

PROOF. Assume that $\sigma_n \to \sigma$ as $n \to \infty$. Then

$$\begin{aligned}
\sigma_n(h(z,w)) &= h_2^n(h(z,w)) \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(H\left(\frac{1}{z}\right)\right)+j}\right)\\
&= h_2^{n+1}(z,w) \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(\frac{1}{z},w\right)+1+j}\right)\\
&= h_2^{n+1}(z,w) \exp\left(\sum_{j=1}^{n} \frac{1}{\Psi\left(\frac{1}{z},w\right)+j} + \frac{1}{\Psi\left(\frac{1}{z},w\right)} - \frac{1}{\Psi\left(\frac{1}{z},w\right)}\right)\\
&= h_2^{n+1}(z,w) \exp\left(\sum_{j=0}^{n} \frac{1}{\Psi\left(\frac{1}{z},w\right)+j}\right) \exp\left(-\frac{1}{\Psi\left(\frac{1}{z},w\right)}\right)\\
&= \exp\left(\frac{-1}{\Psi\left(\frac{1}{z},w\right)}\right)\sigma_{n+1}(z,w)
\end{aligned}$$

Let $n \to \infty$ on both sides, and obtain $\sigma \circ h = \exp\left(\frac{-1}{\Psi}\right)\sigma$. ☐

To show that σ_n converges, we will do the calculation in smaller steps. Each part is stated as a lemma, and we will combine them in the end. Note that we set

$$(z_n, w_n) = h^n(z_0, w_0),$$

so $w_n = h_2^n(z_0, w_0)$ and (z_0, w_0) is fixed. It follows that $X_0 = 1/z_0$ is fixed.

LEMMA 5.13. *We can write*

$$w_n = w_0 \exp\left(\sum_{k=0}^{n-1} \left(-z_k + O\left(z_k^2\right)\right)\right)$$

PROOF. Because w_0 is fixed,

$$\frac{1}{w_0}\sum_{k=0}^{n-1} O\left(z_k^2\right) = \sum_{k=0}^{n-1} O\left(z_k^2\right).$$

Therefore

$$\begin{aligned}
&\ln\left(w_0 \prod_{k=0}^{n-1}\left(1 - z_k + O\left(z_k^2\right)\right) + \sum_{k=0}^{n-1} O\left(z_k^2\right)\right)\\
&\qquad = \ln\left(w_0\left(1 + \sum_{k=0}^{n-1} O\left(z_k^2\right)\right)\prod_{k=0}^{n-1}\left(1 - z_k + O\left(z_k^2\right)\right)\right).
\end{aligned}$$

We then use that $\ln(1-u) = -u + O\left(u^2\right)$ to calculate

$$\ln \prod_{k=0}^{n-1}\left(1 - z_k + O\left(z_k^2\right)\right) = \sum_{k=0}^{n-1} \ln\left(1 - z_k + O\left(z_k^2\right)\right) = \sum_{k=0}^{n-1}\left(-z_k + O\left(z_k^2\right)\right)$$

and get

$$\begin{aligned}
w_n &= w_0 \prod_{k=0}^{n-1} \left(1 - z_k + O\left(z_k^2\right)\right) + \sum_{k=0}^{n-1} O\left(z_k^2\right) \\
&= \exp\left(\ln\left(w_0\left(1 + \sum_{k=0}^{n-1} O\left(z_k^2\right)\right) \prod_{k=0}^{n-1} \left(1 - z_k + O\left(z_k^2\right)\right)\right)\right) \\
&= w_0 \exp\left(\ln\left(1 + \sum_{k=0}^{n-1} O\left(z_k^2\right)\right) + \sum_{k=0}^{n-1} \ln\left(1 - z_k + O\left(z_k^2\right)\right)\right) \\
&= w_0 \exp\left(\sum_{k=0}^{n-1} O\left(z_k^2\right) + \sum_{k=0}^{n-1} \left(-z_k + O\left(z_k^2\right)\right)\right) \\
&= w_0 \exp\left(\sum_{k=0}^{n-1} \left(-z_k + O\left(z_k^2\right)\right)\right).
\end{aligned}$$

□

LEMMA 5.14. *For a fixed X_0 and when N grows larger,*

$$\frac{1}{X_N} = \frac{1}{X_0 + N} + O\left(\frac{1}{(X_0 + N)^2}\right).$$

PROOF. The proof is a calculation. Fix X_0, so

$$\begin{aligned}
\frac{1}{X_N} &= \frac{1}{X_0 + N + \sum_{k=0}^{N-1} \left(\frac{a}{X_k} + O\left(\frac{1}{X_k^2}, \frac{w_k}{X_k}\right)\right)} \\
&= \frac{1}{X_0 + N} \cdot \frac{1}{1 + \frac{1}{X_0+N} \sum_{j=0}^{N-1} \left(\frac{a}{X_k} + O\left(\frac{1}{X_k^2}, \frac{w_k}{X_k}\right)\right)} \\
&= \frac{1}{X_0 + N} \cdot \frac{1}{1 + O\left(\frac{1}{X_0+N}\right)} \\
&= \frac{1}{X_0 + N} + O\left(\frac{1}{(X_0 + N)^2}\right).
\end{aligned}$$

□

LEMMA 5.15. *For a fixed X and when k grows larger,*

$$\frac{1}{\Psi(X, w) + k} = \frac{1}{X + k} + O\left(\frac{1}{(X + k)^2}\right)$$

PROOF. The proof is a calculation. Fix X, so

$$\begin{aligned}
\frac{1}{\Psi(X, w) + k} &= \frac{1}{X - a\log(X) + O\left(\frac{1}{X^2}, \frac{w}{X}\right) + k} \\
&= \frac{1}{X + k}\left(\frac{1}{1 - \frac{a\log(X)}{X+k} + O\left(\frac{1}{X^2(X+k)}, \frac{w}{X(X+k)}\right)}\right) \\
&= \frac{1}{X + k}\left(\frac{1}{1 + O\left(\frac{a\log X}{X+k}\right)}\right) \\
&= \frac{1}{X + k} + O\left(\frac{1}{(X + k)^2}\right).
\end{aligned}$$

□

THEOREM 5.16. *The sequence of maps σ_n converges.*

PROOF. By combining the three lemmas 5.13, 5.14 and 5.15 we get that

$$\begin{aligned}
\sigma_n(z_0,w_0) &= w_n \exp\left(\sum_{k=0}^{n-1} \frac{1}{\Psi\left(\frac{1}{z_0}, w_0\right)+k}\right)\\
&= w_0 \exp\left(\sum_{k=0}^{n-1}\left(-\frac{1}{X_k}+O\left(\frac{1}{X_k^2}\right)+\frac{1}{X_0+k}+O\left(\frac{1}{(X_0+k)^2}\right)\right)\right)\\
&= w_0 \exp\left(\sum_{k=0}^{n-1}\left(-\frac{1}{X_0+k}+\frac{1}{X_0+k}+O\left(\frac{1}{(X_0+k)^2}\right)\right)\right)\\
&= w_0 \exp\left(\sum_{k=0}^{n-1} O\left(\frac{1}{(X_0+k)^2}\right)\right)\\
&= w_0 \exp\left(\sum_{k=1}^{n} O\left(\frac{1}{k^2}\right)\right)
\end{aligned}$$

which clearly converges for all n. □

We shall define g_2 and prove that $g_2 : \Omega \to \mathbb{C}^2$ is onto in several steps. First observe what we see in lemma 5.17.

LEMMA 5.17. *If $h^{N_0}(p) \in B_\varepsilon$, then*

$$\begin{aligned}
&\exp\left(\sum_{j=0}^{N_0+n-1} \frac{1}{\Psi\left(H^{N_0+n}(\phi(p))\right)+j-N_0-n}\right)\sigma\left(h^{N_0+n}(p)\right)\\
&\quad= \exp\left(\sum_{j=0}^{N_0-1} \frac{1}{\Psi\left(H^{N_0}(\phi(p))\right)+j-N_0}\right)\sigma\left(h^{N_0}(p)\right)
\end{aligned}$$

for all $n \geq 0$.

PROOF. The proof is a calculation. Assume that $q = h^{N_0}(p) \in B_\varepsilon$. Recall that

$$\sigma\left(h^k(q)\right) = \exp\left(\frac{-1}{\Psi\left(H^{k-1}(\phi(q))\right)}\right)\sigma\left(h^{k-1}(q)\right)$$

because of the relation in (5.10) and

$$\Psi\left(H^k(\phi(q))\right) = \Psi\left(H^{k-m}(\phi(q))\right) + m$$

because of the construction of Ψ. Then set $M = N_0 + n$ and get

$$\begin{aligned}
&\exp\left(\sum_{j=0}^{M-1} \frac{1}{\Psi\left(H^M(\phi(p))\right) + j - M}\right)\sigma\left(h^M(p)\right) \\
&\quad= \exp\left(\sum_{j=0}^{M-1} \frac{1}{\Psi\left(H^j(\phi(p))\right)}\right)\sigma\left(h^M(p)\right) \\
&\quad= \exp\left(\sum_{j=0}^{N_0-1} \frac{1}{\Psi\left(H^j(\phi(p))\right)} + \sum_{j=N_0}^{M-1} \frac{1}{\Psi\left(H^j(\phi(p))\right)} - \sum_{j=N_0}^{M-1} \frac{1}{\Psi\left(H^j(\phi(p))\right)}\right)\sigma\left(h^{N_0}(p)\right) \\
&\quad= \exp\left(\sum_{j=0}^{N_0-1} \frac{1}{\Psi\left(H^j(\phi(p))\right)}\right)\sigma\left(h^{N_0}(p)\right) \\
&\quad= \exp\left(\sum_{j=0}^{N_0-1} \frac{1}{\Psi\left(H^{N_0}(\phi(p)) + j - N_0\right)}\right)\sigma\left(h^{N_0}(p)\right) \qquad \square
\end{aligned}$$

Definition 5.18. *Let $S = g_1(B_\varepsilon)$ and find $g_1^{-1}(S)$. Let $p \in g_1^{-1}(S)$ and N so that $h^N(p) \in B_\varepsilon$. Define*

$$g_2(p) = \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(H^n(\phi(p))\right) + j - n}\right)\sigma\left(h^n(p)\right).$$

for $n \geq N$.

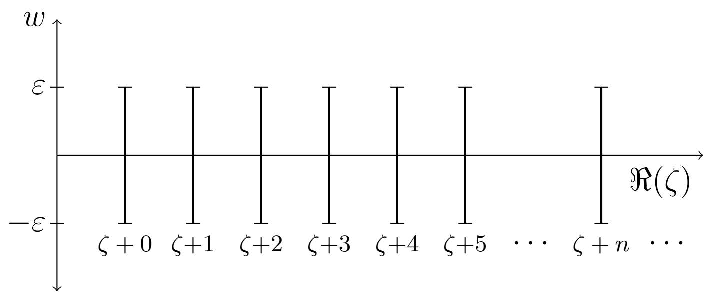

Figure 5.2. This is an illustration of the image $(\Psi \circ \phi, \sigma)\left(h^n\left(\widetilde{\Gamma}\right)\right)$ from (5.12). The horizontal axis represents the real part of a complex number ζ in the z-plane. Each vertical line represents a disc with radius ε in the w-plane, centred at the origin (Δ_ε).

Theorem 5.19. *Let $z \in \mathbb{C}$ and define $\Gamma_{z,n} := \{q \in \Omega : g_1(q) = z\}$. The map $g_2 : \Gamma_{z,n} \to \mathbb{C}$ is onto.*

Proof. The domain B_ε and the maps used are as given before. Define

$$S := g_1(B_\varepsilon)$$

and find the preimage $g_1^{-1}(S)$. We want to show that

$$(g_1, g_2) : g_1^{-1}(S) \to S \times \mathbb{C}$$

is onto.

Let $\zeta \in S$ and define

$$\Gamma_{\zeta,n} := \{q \in \Omega : g_1(q) = \zeta\}.$$

By writing out the expression for g_1

$$g_1(q) = \zeta \quad \Leftrightarrow \quad \Psi\left(H^n(\phi(q))\right) = \zeta + n.$$

We know that σ takes $g_1^{-1}(S)$ to Δ_ε. By using that $\Psi \circ H^n \circ \phi = \Psi \circ \phi \circ h^n$ because h and H are conjugates by the conjugation ϕ, we get

$$(\Psi \circ \phi, \sigma)\left(h^n(\Gamma_{\zeta,n})\right) = \{\zeta + n\} \times \Delta_\varepsilon.$$

Define

$$\widetilde{\Gamma}_\zeta = \bigcup_{n=0}^{\infty} \Gamma_{\zeta,n}$$

and get

$$(\Psi \circ \phi, \sigma)\left(h^n\left(\widetilde{\Gamma}_\zeta\right)\right) = \bigcup_{n=0}^{\infty} \{\zeta + n\} \times \Delta_\varepsilon. \tag{5.12}$$

Recall that

$$\Psi\left(H^n(X,w)\right) = X + n + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right).$$

Then

$$\begin{aligned}
\sum_{j=0}^{n-1} \frac{1}{\Psi\left(H^n(X,w)\right) + j - n} &= \sum_{j=0}^{n-1} \frac{1}{X + n + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) + j - n} \\
&= \sum_{j=0}^{n-1} \frac{1}{X + \frac{a}{X} + O\left(\frac{1}{X^2}, \frac{w}{X^4}\right) + j} \\
&= \sum_{j=0}^{n-1} \frac{1}{X+j} \frac{1}{1 + O\left(\frac{1}{X(X+j)}, \frac{w}{X^4(X+j)}\right)} \\
&= \sum_{j=0}^{n-1} \frac{1}{X+j}\left(1 + O\left(\frac{1}{X+j}\right)\right).
\end{aligned}$$

We know that there is some N_0 so that for $n \geq N_0$,

$$\sum_{j=0}^{n-1} \left|\frac{1}{X+j}\right| > \ln\left(\frac{1}{2}n\right).$$

We use this, and get

$$g_2\left(\widetilde{\Gamma}_\zeta\right) = \bigcup_{n=0}^{\infty} \Delta_\varepsilon \exp\left(\ln\left(\frac{1}{2}n\right)\right) = \bigcup_{n=0}^{\infty} \frac{1}{2} n \Delta_\varepsilon = \mathbb{C}.$$

Because $\mathbb{C} = g_1(\Omega) \supset g_1(B_\varepsilon) = S$, it is clear that $g_1^{-1}(S) \subset g_1^{-1}(\mathbb{C})$. In conclusion, we have shown that for any $z \in \mathbb{C}$ there is a set

$$\Gamma_{z,n} := \{q \in \Omega : \Psi\left(H^n(\phi(q))\right) = z + n\}$$

so the map $g_2 : \Gamma_{z,n} \to \mathbb{C}$ is onto. $\square$

There is still one issue left. The denominator in the exponent of g_2 may become 0. To solve this we may create a covering of Ω given by $\{U_k\}$, so

$$\Omega \subset \bigcup_{k=0}^{\infty} U_k,$$

and $g_2 : U_k \to \mathbb{C}$ is given by

$$g_{2,k}(p) = \sigma\left(h^n(p)\right) \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(H^n(\phi(p))\right) + j + k - n}\right).$$

Notice that the g_2 constructed before, see definition 5.18, will be denoted $g_{2,0}$ here. These are constructed in such a way that

$$\frac{g_{2,k+1}}{g_{2,k}} = \exp\left(-\frac{1}{\Psi + k}\right)$$

and none have 0 in the denominator. We can then use a Cousin problem like Weickert does [**9**], and glue together all $g_{2,k}$, extending the solution to be defined on all of Ω.

We have constructed two maps $g_1, g_2 : \Omega \to \mathbb{C}$

$$g_1(p) = \Psi\left(H^n(\phi(p))\right) - n$$
$$g_2(p) = \exp\left(\sum_{j=0}^{n-1} \frac{1}{\Psi\left(H^n(\phi(p))\right) + j - n}\right) \sigma\left(h^n(p)\right)$$

so that the map $G : \Omega \to \mathbb{C}^2$ given by

$$G = (g_1, g_2)$$

is onto.

We have shown that $G = (g_1, g_2)$ is holomorphic on Ω, one-to-one and onto $\mathbb{C}^2$. In conclusion we have shown that Ω, the basin of attraction for a neutral fixed point, is a Fatou-Bieberbach domain with the w-axis in the boundary.

Bibliography

[1] E. Andersén and L. Lempert. On the group of holomorphic automorphisms of $\mathbb{C}^n$. *Inventiones mathematicae*, 110(1):371–388, 1992.

[2] L. Carleson and T. W. Gamelin. *Complex Dynamics*. Springer-Verlag New York Inc., 1993.

[3] J. E. Fornæss. Introduction to complex dynamics in one variable. Lecture notes, July 2013. Downloaded September 18, 2017 from `https://arxiv.org/pdf/1605.09562.pdf`.

[4] M. Hakim. Analytic transformations of $(\mathbb{C}^p, 0)$ tangent to the identity. *Duke mathematical journal*, 92(2):403–428, 1998.

[5] H. Peters, L. R. Vivas, and E. F. Wold. Attracting basins of volume preserving automorphisms of $\mathbb{C}^k$. *International Journal of Mathematics*, 19(07):801–810, 2008.

[6] S. O. Romslo. BSc Thesis: The snail lemma in one complex variable. Norwegian University of Science and Technology (NTNU), 2017.

[7] J.-P. Rosay and W. Rudin. Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. *Transactions of the American Mathematical Society*, 310(1):47–86, 1988.

[8] W. Rudin. *Real and complex analysis*. McGraw-Hill, Inc., 1987.

[9] B. J. Weickert. Attracting basins for automorphisms of $\mathbb{C}^2$. *Inventiones mathematicae*, 132(3):581–605, 1998.