

SPECTRAL THEORY AND RANDOM OPERATORS

THE ENERGY SPECTRUM OF
THE QUANTUM ELECTRON IN A DISORDERED SOLID

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December 1992

A dissertation presented for partial fulfillment
of the requirements for the degree of
Doktor Ingeniør

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The Norwegian Institute of Technology

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Mathematics Subject Classification (1991): 47-02. 60H25, 35J10, 82B44, 47B39, 47A35
Research exposition.

1. Random operators and equations. 2. Schrödinger operator.
3. Disordered systems. 4. Difference operators. 5. Ergodic theory.

Taraldsen, Gunnar, 1963-
Spectral theory and random operators.
The energy spectrum of the quantum electron in a disordered solid.
Includes figures, bibliographical references, and index.
ISBN 82-7119-444-5
ISSN 0802-3271

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Printed in Norway

Dedicated
to
Peter and Ida

Preface

The quantum mechanics of an electron moving in a solid is the starting point for this work. The Bloch theory for electrons in a perfect crystal is generalized to include random deviations, such as impurities. This extension may give a better theoretical understanding of the electronic properties of solids.

The mathematical models which arise are also of interest for workers in the field of pure mathematics. The physical observables are represented by selfadjoint, linear operators acting in a Hilbert space. The operators in question are differential operators with coefficients which in some sense contain too much information to be tractable. A fruitful approach—in analogy with the moral in statistical mechanics—is to turn to a statistical description of the coefficients. The result is a linear random operator. The general study of linear random operators is still in its infancy.

The main content in this monograph is on foundational questions, on the level of giving an appropriate definition of what we mean by the term “linear random operator”. This is a delicate question when we allow unbounded operators. A minimum demand seems to be a definition such that the spectrum of a random operator is a random set, in a measure theoretic sense. Guided by these questions we are led to a study of topologies in spaces of closed sets. The presented work suggests that both the definition of random spectra and random operators are made in terms of the lower topology in the set of closed sets. In making this choice we have also been guided by one of the main questions about transitive random operators: Which of the quantities derived from a transitive random operator are nonrandom? We have found a seemingly optimal answer to this question—in terms of very general sufficient conditions which ensure nonrandomness. Still, this monograph gives more questions, than answers. A good starting point for further research. The summary gives a more specific description of the contents.

The organization and further content in this monograph are seen from a glance at the table of contents, but we will nonetheless be explicit on a few points. The text is divided into chapters. Each chapter is meant to be reasonably self contained. A separate introduction is given to each chapter. References to the literature and additional remarks have all been collected in a “Notes” section at the end of each chapter. In particular I have acknowledged valuable suggestions from various people at the proper points. The articles and books referred to are collected in a separate “Bibliography” found at the end. This “Bibliography” does not pretend to be complete, but is rather a reflection of the books and articles I have found to be useful, a subjective judgement. The references [79], [31], and [107] are recommended as general introductions to this field of research, and they contain extensive references to the literature.

I have tried to use standard notation whenever possible, modeled after [136], [115], and [43]. These references should also provide (more than) the necessary mathematical prerequisites. The reader is advised to glance through the list of symbols, which is found after the table of contents. The word “iff” is used as a short form of the more cumbersome “if and only if”. I apologize for other possible abuses of the English language.

I like to express my appreciation of the constant support and encouragement from my su-

pervisor Helge Holden. Without his guidance I would not have entered into this important and most interesting field of research. I am grateful for the support from the staff at The Department of Mathematical Sciences during this work. In particular I would like to thank Harald Hanche-Olsen for many helpful comments. Ola Bratteli and Henrik Martens have also been most helpful. The scholarship received from The Norwegian Institute of Technology for the period January 1990 to January 1993 is acknowledged. My sincere thanks go to my wife Svanhild, who supported me during this work, even when I worked at very irregular hours. Finally, I wish to thank my parents for their help and encouragement, and for always believing in what I have been doing.

The Norwegian Institute of Technology

Gunnar Taraldsen

December, 1992

Mechanics is the paradise of mathematicians, where they can see (and harvest)
the fruits of mathematics.

Leonardo da Vinci, 1452-1519.

Summary

This thesis consists of seven chapters. Chapter 1 is more informal than the rest, and is meant partly as an introduction. It demonstrates that models with periodic potentials and simple impurities can be exactly solved. Chapters 2,3,4, and 5 give general theory, which are relevant for the study of the Schrödinger operator defined from a random potential. In chapters 6 and 7 we consider respectively the random multiplication operator and the random Schrödinger operator, both in the discrete case.

Chapter 1 We consider the discrete Schrödinger operator. The Hilbert space $l^2(\mathbb{Z}^d)$ is decomposed into a direct integral of Bloch Hilbert spaces. This construction is used to compute the explicit band structure for the chessboard Schrödinger operator. We present a method which determines all possible new eigenvalues in the gaps of the spectrum of an unperturbed Schrödinger operator, when the potential is perturbed on a finite number of sites.

Chapter 2 We consider abstract valued random variables. This abstract setting is motivated from the study of ergodic random Schrödinger operators. We prove nonrandomness of a random variable, invariant with respect to a metrically transitive system. This is a generalization from real valued variables to variables with a σ -separated range.

Chapter 3 Some properties of topologies and σ -algebras on spaces of closed sets are studied. Upper and lower topologies are introduced and corresponding convergence concepts are discussed. We prove that the lower topology in the set of closed sets in a second countable space gives a second countable T_0 -space. Our discussion is motivated by the spectral theory of linear operators, but the topologies and σ -algebras introduced are most general.

Chapter 4 We consider random measures, motivated by the spectral theory of random Schrödinger operators. The randomness of the Lebesgue-Hammer parts of a random measure is proven. Several topologies in a set of measures are considered, including in particular the pointwise topology and the weak topology. Finally we consider how decompositions behave under perturbations.

Chapter 5 The lower operator topology is introduced. It gives a proper generalization of strong resolvent convergence of selfadjoint operators. We prove that the set of finite matrix operators is dense. The lower topology is applied for the definition of unbounded, closed random operators. We prove nonrandomness of the spectrum of a transitive random operator, given that the spectrum is random in the sense defined by the lower topology. We prove lower semicontinuity of the spectrum of normal operators. A general formula for the spectrum of a normal, transitive random operator is given. We give conditions which imply presence of continuous spectrum or pure point spectrum.

Chapter 6 A random sequence is interpreted as a random multiplication operator. We investigate the properties of this random operator. The motivation for the study is the electron theory of disordered solids, and more generally the theory of linear random operators. We gain insight into general concepts from this elementary model.

Chapter 7 The lower topology is applied to define random Schrödinger operators on a random graph, defined by generalized potentials. We prove that the spectrum is a lower semicontinuous function of the generalized real potential. This gives an explicit formula

for the spectrum, when the potential is a metrically transitive, random process. The significance of the Lebesgue decompositions of a selfadjoint Hamiltonian is discussed in terms of the time dependent wave function. Local spectral measures are defined and applied. Firstly for the proof of randomness of the Lebesgue decomposition of the spectrum, and secondly in an explicit formula for the integrated density of states measure. We determine the spectrum and the pure point spectrum of the percolation Schrödinger operator. The determination of the absolutely continuous and the singular continuous spectrum are left as open problems, but possibly solvable by methods indicated.

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List of Symbols

The numbers refers to the page on which the notation is explained.

Calligraphic

\mathcal{B}	The Bloch transformation, $\mathcal{B} : \mathcal{H} \rightarrow \int_B^\oplus \mathcal{H}_k d\mu(k)$.	9
\mathcal{B}_n	The Borel field from the norm topology.	25
\mathcal{B}_w	The Borel field from the weak topology.	25
$\mathcal{C}(X)$	The set of closed operators defined in X .	78
$\mathcal{C}_{sa}(\mathcal{H})$	The set of selfadjoint operators defined in \mathcal{H} .	80
\mathcal{F}	Fourier transformation, $f \mapsto \hat{f} = \mathcal{F}f$.	4
\mathcal{F}	A σ -algebra.	22
\mathcal{F}_w	The initial σ -algebra from the dual E^* .	25
\mathcal{F}_t	The σ -algebra of trivial sets.	30
\mathcal{F}_I	The σ -algebra of invariant sets.	30
\mathcal{H}	A Hilbert space.	90
\mathcal{H}_k	A Bloch Hilbert space.	8
\mathcal{H}_p	The atomic subspace in \mathcal{H} .	82
\mathcal{H}_c	The continuous subspace in \mathcal{H} .	82
\mathcal{H}_s	The singular subspace in \mathcal{H} .	82
\mathcal{H}_{ac}	The absolutely continuous subspace in \mathcal{H} .	82
\mathcal{H}_{sc}	The singular continuous subspace in \mathcal{H} .	82
\mathcal{M}	The set of complex measures on a given σ -algebra.	61
\mathcal{M}_{ac}	The set of complex absolutely continuous measures on a given σ -algebra.	62
\mathcal{M}_p	The set of complex pure point measures on a given σ -algebra.	61
\mathcal{M}_c	The set of complex continuous measures on a given σ -algebra.	62
$\mathcal{P}(X)$	The set of subsets of X .	xiii
$\mathcal{P}_F(X)$	The set of closed sets in X .	39
$\mathcal{P}_e(X)$	The set of closed essential sets in X .	44

Greek

δ_x	The Kronecker delta function.	90
$\hat{\delta}_x$	The function $k \mapsto \hat{\delta}_x(k) = e^{-ikx}$.	4
$\mu(A)$	The measure of the set A .	23
$\mu[f]$	The integral of the function f .	23
μ_a	The atomic part of the measure μ .	54
μ_c	The continuous part of the measure μ .	62
$\rho(\lambda)$	The resolvent set of the measure λ .	62
$\rho(T)$	The resolvent set of the closed operator T .	78
ρ_x	The local spectral measure at x .	120
$\hat{\rho}_x$	The local characteristic function at x .	120

$\sigma(G)$	The σ -algebra generated by G .	22
$\sigma(\lambda)$	The spectrum of the measure λ .	62
$\sigma(T)$	The spectrum of the closed operator T .	78
$\sigma_c(T)$	The continuous spectrum of the closed operator T .	81
$\sigma_d(T)$	The discrete spectrum of the closed operator T .	81
$\sigma_e(T)$	The essential spectrum of the closed operator T .	81
$\tilde{\sigma}(T)$	The core spectrum of the closed operator T .	78
$\sigma_p(\lambda)$	The point spectrum of the measure λ .	62
$\sigma_p(T)$	The point spectrum of the closed operator T .	81
$\sigma_r(T)$	The residual spectrum of the closed operator T .	81
$\sigma_{pp}(\lambda)$	The pure point spectrum of the measure λ .	62
$\sigma_{pp}(T)$	The pure point spectrum of the operator T .	82
$\sigma_{ac}(\lambda)$	The absolutely continuous spectrum of the measure λ .	62
$\sigma_{ac}(T)$	The absolutely continuous spectrum of the operator T .	82
$\sigma_{sc}(\lambda)$	The singular continuous spectrum of the measure λ .	62
$\sigma_{sc}(T)$	The singular continuous spectrum of the operator T .	82
τ	A σ -algebra homomorphism.	30
τ	A topology.	38
$\tau(G)$	The topology generated by G .	22
Γ	A subgroup of the transformation group of X .	99
$\Gamma(T)$	The regularity domain of the closed operator T .	78
Δ	The Laplace operator.	5
Δ_Λ	The Dirichlet Laplace operator.	7
Δ_V	The Dirichlet Laplace operator.	112
Λ	An index set for a net.	38
$\Pi = \Pi[V]$	A projection operator.	118
Σ	A closed random set.	44
Σ_x	The spectrum of V at x .	102
Ω	A probability space.	24
Ω_a	Atomic part of a probability space.	54
Ω_c	Continuous part of a probability space.	54

Latin

$c : x \rightarrow y$	A path from x to y .	123
$g_z = g_z[V]$	The diagonal Green's function.	118
id	A function, $id(x) = x$.	xiv
\lim	Limit. $\lim x_\lambda$ is a limit of x_λ .	38
$\overline{\lim}$	Upper pointwise limit of a net of sets.	38
$\underline{\lim}$	Lower pointwise limit of a net of sets.	38
$l\text{-}\lim$	Lower limit of a net of closed sets.	39
$l\text{-}\lim$	Lower limit of a net of closed operators.	68
$n\text{-}\lim$	Norm limit of a net of complex measures.	59
$p\text{-}\lim$	Pointwise limit of a net of complex measures.	59
$p_{\mathcal{A}}\text{-}\lim$	Pointwise convergence of measures on a generating algebra.	59
$u\text{-}\lim$	Upper limit of a net of closed sets.	41
$\tilde{u}\text{-}\lim$	Weak upper limit.	42

$v\text{-lim}$	Vague limit of a net of Borel measures.	59
$v\text{-lim}$	Vietoris limit of a net of closed sets.	43
$w\text{-lim}$	Weak limit of a net of Borel measures.	59
$L_c\text{-lim}$	The closure of the lower limit of a net of closed sets.	40
$L_i\text{-lim}$	The individual limit of a net of closed sets.	40
$L_m\text{-lim}$	The maximal lower limit of a net of closed sets.	40
$U_c\text{-lim}$	The closure of the pointwise upper limit of a net of closed sets.	41
$U_m\text{-lim}$	The modified upper limit of a net of closed sets.	41
$\tilde{v}\text{-lim}$	Weak Vietoris limit.	43
$l^2(X)$	The Hilbert sequence space.	90
$n_z(c)$	The number of times the path c visits z .	123
A_V	The set of V -admissible generalized potentials.	102
B_k	The Bloch projection, $B_k : \mathcal{H} \rightarrow \mathcal{H}_k$.	9
$D(f)$	The domain of the function f .	xv
$E[u]$	The operator from a function u and a spectral family E .	81
$E(A)$	The projection from a set A and a spectral family E .	81
$E[X]$	The expectation of the random variable X .	24
$E_X[f]$	The expectation of f from the distribution of X .	24
$E[X \mathcal{F}]$	A conditional expectation.	33
$G(f)$	The graph of the function f .	xv
$G(X)$	The set of generalized potentials on X .	91
$G_p(X)$	The set of potentials on X .	91
$G_c(X)$	The set of complex potentials on X .	91
$G_z = G_z[V]$	The Green's function.	118
$H = H[V]$	A Schrödinger operator.	112
$J = J[\Lambda]$	The nearest neighbor operator.	123
J_n	Bessel function of integer order n of the first kind.	124
$M = M[V]$	A multiplication operator.	91
N_γ	The count potential.	97
N_Λ	An approximate density of states measure.	104
N	A density of states measure.	104
$P(A)$	The probability of the event A .	24
P_X	The probability distribution of X .	24
Q_γ	Coupling constant at site γ .	96
$R(f)$	The range of the function f .	xv
$\text{supp } P$	The topological support of the measure P .	46
S_i	A unit shift operator, $S_i f(x) = f(x - e_i)$.	5
S^i	A unit shift operator, $S^i f(x) = f(x + e_i)$.	5
T_0	T_0 -separation property.	43
T_1	T_1 -separation property.	43
T_γ	The composition mapping $T_\gamma V(x) = V(\gamma(x))$.	97
U_x	The unitary shift operator, $U_y f(x) = f(x + y)$.	5,97
V	A generalized potential.	91
$V(\infty)$	The value of V at infinity, if defined.	92
X	A countable set.	90
X	A random variable.	24
$X[V]$	The proper domain of the generalized potential V .	91
\overline{X}	Time average of $t \mapsto X(t)$.	29

Miscellaneous

$ \cdot $	The Lebesgue measure $ A $ of A in \mathbb{R}^d .	xvi
$ \cdot $	The length $ c $ of the path c .	123
$:=$	$A := (\dots)$, A is defined by (\dots) .	xvi
\leq	A preorder on an index set.	38
\ll	Absolute continuity of the measure ν : $\nu \ll \mu$.	55
\sim	Equivalence of the measures ν and μ : $\nu \sim \mu$.	63
\perp	Mutual singularity of two measures: $\mu \perp \nu$.	55
\circ	Composition, $(f \circ g)(t) := f(g(t))$.	xvi
\oplus	The Lie sum $S \oplus T$ of two selfadjoint operators.	76
$\langle \cdot, \cdot \rangle$	The inner product $\langle f, g \rangle$ of f and g .	90
\mapsto	Mapsto. The function $t \mapsto f(t)$ which maps t to $f(t)$.	xvi
\rightarrow	$f : X \rightarrow Y$, f maps X into Y .	xvi
1_A	Indicator function for the set A .	22
∂_i	Derivation operator.	5
∂^i	Derivation operator.	5
\mathbb{C}	The set of complex numbers, $e.g. 1 + i$.	xvi
$\hat{\mathbb{C}}$	The extended set of complex numbers, $e.g. \infty$.	xvi
\mathbb{Z}	The set $\{\dots, -1, 0, 1, \dots\}$ of integers.	xvi
\mathbb{N}	The set $\{1, 2, 3, \dots\}$ of natural numbers.	xvi
\mathbb{Q}	The set of rational numbers, $e.g. 1/5$.	xvi
\mathbb{R}	The set of real numbers, $e.g. \pi$.	xvi

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Part I
Introductory Part

Chapter 1

Discrete Schrödinger Operators and Bloch Theory

We consider the discrete Schrödinger operator. The Hilbert space $l^2(\mathbb{Z}^d)$ is decomposed into a direct integral of Bloch Hilbert spaces. This construction is used to compute the explicit band structure for the chessboard Schrödinger operator. We present a method which determines all possible new eigenvalues in the gaps of the spectrum of an unperturbed Schrödinger operator, when the potential is perturbed on a finite number of sites.

1.1 Introduction

Partial difference operators are simpler than partial differential operators in some respects. Problems which seem untractable for the continuous Schrödinger operator have been solved for the corresponding discrete Schrödinger operator. Partial difference operators are also a subject which deserves attention without reference to partial differential operators. The purpose here is to give the necessary frame for a discussion of the discrete Schrödinger operator, and in particular the parts which are relevant for solid state physics.

The Fourier transformation gives a convenient tool for the analysis of the continuous Laplace operator. This is also so for the discrete Laplace operator. We define a convenient Fourier transformation as a Hilbert space isomorphism from $\mathcal{H} := l^2(\mathbb{Z}^d)$ to $\hat{\mathcal{H}} := L^2([-\pi, \pi]^d, dk/(2\pi)^d)$. A finite linear combination of shift operators on \mathcal{H} is transformed to a multiplication operator on $\hat{\mathcal{H}}$, and we give the explicit formula.

Discrete derivation operators from positive and negative unit shifts are defined. These derivation operators are not derivations in the algebraic sense, since they do not obey the product rule. This makes “analysis” for functions defined on \mathbb{Z}^d more complicated in some respects, compared to the continuous case. The Laplace operator is defined from the derivation operators. From the Fourier transformation we obtain the multiplication operator corresponding to the Laplace operator. We refer to this multiplication operator as the band function or dispersion relation—inspired by the language of solid state physics. Formulas for the density of states, the resolvent, and the spectral family follow readily from the band function.

The Dirichlet Laplacian for an arbitrary subset in \mathbb{Z}^d is defined. The spectrum of this

Dirichlet Laplacian is not easy to characterize for general unbounded domains. The Laplace operator may be obtained as a strong limit of Dirichlet Laplacians on finite domains. We prove that the corresponding spectra converges in the Vietoris sense towards the spectrum of the Laplace operator. Lower convergence of the spectra follows from general theory, and the upper convergence follows from bounds on the involved spectra. The density of states is obtained likewise from similar limits.

We turn next to the discrete Schrödinger operator with a periodic potential. A proper generalization of the Fourier transformation for this operator is given by the Bloch transformation. This is a Hilbert space isomorphism from \mathcal{H} to the direct integral $\hat{\mathcal{H}} = \int_B^\oplus \mathcal{H}_k d\mu(k)$ of Bloch spaces \mathcal{H}_k . The transformation of the Schrödinger operator is a direct integral $\int_B^\oplus H_k d\mu(k)$. This makes a complete analysis of the periodic Schrödinger operator possible, since each operator H_k may be represented by a finite matrix.

We apply the Bloch theory to explicit models in two dimensions. The models are given by potentials which are periodic corresponding to a 2×2 fundamental domain. One model is given by a Dirichlet condition, or more explicitly by the removal of one of the four members in each lattice translation of the fundamental domain. This is a three band model, where one band has collapsed to a constant value. The result is an eigenvalue with infinite multiplicity, with corresponding localized solutions of the Schrödinger equation. This gives in particular a counterexample to the arguments found in (most) physics textbooks, when the Bloch theory is explained. The chessboard model is defined by a potential with zero or α strength on positions in \mathbb{Z}^2 in a chessboard pattern. The four band functions are found explicitly for the chessboard model.

We consider finite rank perturbations of selfadjoint operators in \mathcal{H} . The possible eigenvalues in the gaps of the spectrum are found to be determined completely by a polynomial equation given by the resolvent of the unperturbed operator. The eigenvectors are also found.

1.2 Difference Operators and Fourier Series

Let X be the additive group \mathbb{Z}^d , with an integer d . We represent the dual group by the torus $\hat{X} = [-\pi, \pi]^d$, where opposite sides in the cube are identified. For each x in X we define the exponential function $\hat{\delta}_x$ on \hat{X} from

$$\hat{\delta}_x(k) := e^{-ikx}. \quad (1.1)$$

Equip \hat{X} with the conventional Haar measure given by Lebesgue measure divided by $(2\pi)^d$. The family $\{\hat{\delta}_x\}$ is then an orthonormal basis for $\hat{\mathcal{H}} := L^2(\hat{X})$. The family $\{\delta_x\}$ of Kronecker delta functions is an orthonormal basis for $\mathcal{H} := l^2(X)$. The Fourier transformation \mathcal{F} is the Hilbert space isomorphism determined by $\delta_x \mapsto \hat{\delta}_x$, so

$$\hat{f} := \mathcal{F}f = \mathcal{F}\left(\sum_x f(x)\delta_x\right) = \sum_x f(x)\hat{\delta}_x, \quad \hat{f}(k) = \sum_x f(x)e^{-ikx}. \quad (1.2)$$

The Parseval formula reads

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle = \frac{1}{(2\pi)^d} \int_{\hat{X}} \overline{\hat{f}(k)} \hat{g}(k) dk, \quad (1.3)$$

and the inversion formula follows

$$f(x) = \langle \delta_x, f \rangle = \langle \hat{\delta}_x, \hat{f} \rangle = \frac{1}{(2\pi)^d} \int_{\hat{X}} e^{ikx} \hat{f}(k) dk. \quad (1.4)$$

The unitary shift operator U_x is defined by

$$D(U_x) := \mathcal{H}, \quad U_x f(y) := f(y + x). \quad (1.5)$$

The corresponding operator on $\hat{\mathcal{H}}$ is easily found

$$\hat{U}_x \hat{f}(k) := (U_x f)^\wedge(k) = \hat{\delta}_{-x}(k) \hat{f}(k), \quad \hat{U}_x = M(\hat{\delta}_{-x}), \quad (1.6)$$

a multiplication operator. This generalizes directly to

$$\left(\sum_x c_x U_x \right)^\wedge = M \left(\sum_x c_x \hat{\delta}_{-x} \right) \quad (1.7)$$

for a general shift operator given by a finite linear combination of unitary shifts.

1.3 The Discrete Laplace Operator

Let e_1, \dots, e_d be the standard unit vectors in \mathbb{R}^d . To simplify notation we define unit shift operators

$$S^i := U_{e_i} \text{ and } S_i := U_{-e_i} = (S^i)^{-1} = (S^i)^*. \quad (1.8)$$

Two analogs to the usual derivation operators are given by

$$\partial^i := S^i - I \text{ and } \partial_i := I - S_i = (-\partial^i)^*. \quad (1.9)$$

We refer to these operators as discrete derivation operators. The product rules are

$$\partial^i(fg) = \partial^i f g + S^i f \partial^i g, \quad \partial_i(fg) = \partial_i f g + S_i f \partial_i g, \quad (1.10)$$

so the operators are not proper derivations. A consequence is that analysis for functions on \mathbb{Z}^d is more complicated in some respects, compared to analysis for functions on \mathbb{R}^d .

The Laplace operator is defined from

$$\Delta := \sum_i \partial^i \partial_i = \sum_i (S^i + S_i - 2I). \quad (1.11)$$

We notice the mean value property of harmonic functions

$$\Delta f = 0 \Rightarrow f(x) = \sum_{\|y-x\|_1=1} f(y) / (2d). \quad (1.12)$$

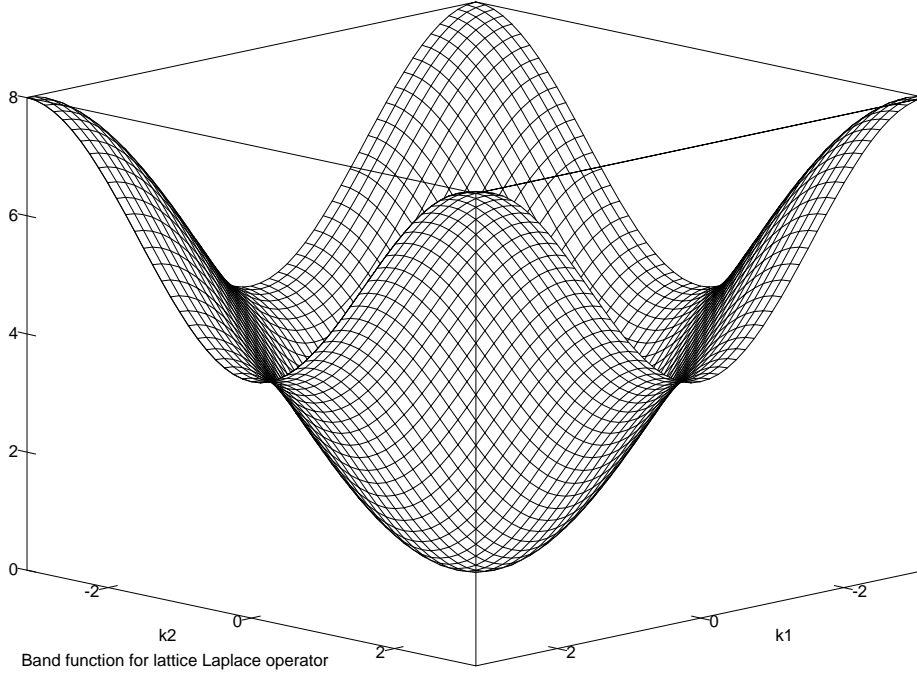


Figure 1.1: *The dispersion relation $\hat{H}_0(k)$ for the lattice Laplace operator in two dimensions. The horizontal axis give the components of the Bloch momentum k and the vertical axis gives the energy.*

The multiplication operator corresponding to the Laplace operator is given by the function

$$\hat{\Delta}(k) = \sum_i (\hat{\delta}_{e_i} + \hat{\delta}_{-e_i} - 2)(k) = 2 \sum_i (\cos(k_i) - 1). \quad (1.13)$$

The function $\hat{H}_0(k) = -\hat{\Delta}(k)$ is the dispersion relation for the free lattice electron. Figure 1.1 is a plot of the dispersion relation in the two dimensional case. Immediate consequences are

$$\sigma(-\Delta) = \sigma_{ac}(-\Delta) = [0, 4d], \quad \sigma_{pp}(-\Delta) = \sigma_{sc}(-\Delta) = \emptyset \quad \text{and} \quad \|-\Delta\| = 4d. \quad (1.14)$$

The resolvent of $-\Delta$ follows from the Parseval equation

$$G_z(x, y) := \langle \delta_x, (z + \Delta)^{-1} \delta_y \rangle = (2\pi)^{-d} \int_{\hat{X}} dk \frac{e^{ik(x-y)}}{z - 2 \sum_i (1 - \cos(k_i))}. \quad (1.15)$$

The spectral resolution likewise

$$E[f](x, y) = (2\pi)^{-d} \int_{\hat{X}} dk e^{ik(x-y)} f(-\hat{\Delta}(k)), \quad (1.16)$$

which also gives the density of states measure

$$N[f] = \lim \sum_{x \in \Lambda} E[f](x, x) / \#\Lambda = (2\pi)^{-d} \int_{\hat{X}} dk f(-\hat{\Delta}(k)). \quad (1.17)$$

1.4 The Discrete Dirichlet Laplace Operator

Let Λ be a subset of X and define the projection $\Pi_\Lambda = M(1_\Lambda)$, from multiplication with the indicator function 1_Λ . The Dirichlet Laplace operator is

$$\Delta_\Lambda := \Pi_\Lambda \Delta \Pi_\Lambda. \quad (1.18)$$

The matrix for this operator is

$$\Delta_\Lambda(x, y) := \langle \delta_x, \Delta_\Lambda \delta_y \rangle = \begin{cases} -2d & x = y \in \Lambda \\ 1 & \|x - y\|_1 = 1, x, y \in \Lambda \\ 0 & \text{otherwise} \end{cases}, \quad (1.19)$$

where $\|x\|_1 := |x_1| + \dots + |x_d|$. The space $l^2(\Lambda)$ is a reducing subspace for Δ_Λ , and Δ_Λ equals zero on $l^2(\Lambda^c)$. The Dirichlet Laplacian on $l^2(\Lambda)$ is the restriction of Δ_Λ .

Let Λ_n be such that given any x in X there is an N such that x is in $\Lambda_N, \Lambda_{N+1}, \dots$, and abuse notation with $\Delta_n := \Delta_{\Lambda_n}$. This assumption gives $\lim \Delta_n \delta_x = \Delta \delta_x$ and $\|\Delta_n\| \leq 4d$. We conclude that Δ is a strong limit (so also a strong resolvent limit) of the sequence $\{\Delta_n\}$. On the other hand $\|\Delta_n - \Delta\| \geq 2d$, so norm convergence (or norm resolvent convergence) is impossible. From this we can only conclude $\sigma(-\Delta) = l\text{-}\lim \sigma(-\Delta_n)$. We will also prove $\sigma(-\Delta) = u\text{-}\lim \sigma(-\Delta_n)$, from explicit computation for cubes $\Lambda_n = \{1, \dots, n-1\}^d$.

Consider the $(n-1)^d$ functions

$$\phi_k(x) = \begin{cases} 0 & x \notin \Lambda_n \\ c_k \prod_{i=1}^d \sin(k_i x_i) & x \in \Lambda_n, \end{cases} \quad (1.20)$$

with constants c_k such that $\{\phi_k\}$ is an orthonormal set for

$$k \in \hat{X}_n := \{1, \dots, n-1\}^d \pi/n. \quad (1.21)$$

Direct verification gives

$$-\Delta_n \phi_k = \hat{H}_0(k) \phi_k. \quad (1.22)$$

The spectrum of $-\Delta_n$ is pure point with $(n-1)^d$ eigenvalues, counting multiplicities, given by $\hat{H}_0(k) = 2 \sum_i (1 - \cos(k_i))$, $k \in \hat{X}_n$, and the eigenvalue 0 with infinite multiplicity. We have $\sigma(-\Delta_n) \subset \sigma(-\Delta)$, which in particular implies $\sigma(-\Delta) = u\text{-}\lim \sigma(-\Delta_n)$. This conclusion does also follow from $4d \geq \|\Delta_n\| = \sup\{|z| \mid z \in \sigma(\Delta_n)\}$, combined with the positivity of $-\Delta_n$. The formula for the norm holds since Δ_n is selfadjoint, and so also normal.

The distribution function for the density of states approximant is

$$N_n(E) := \frac{1}{n-1} \#\{k \in \hat{X}_n \mid \hat{H}_0(k) \leq E\}. \quad (1.23)$$

This agrees with the previously found density of states, because

$$\lim N_n(E) = N(E) = \frac{1}{(2\pi)^d} \left| \{k \in \hat{X} \mid \hat{H}_0(k) \leq E\} \right|, \quad (1.24)$$

where $|\cdot|$ denotes Lebesgue measure. Each k in \hat{X}_n corresponds to a volume $(\pi/n)^d$, and \hat{X}_n contains only positive values.

1.5 Periodic Potentials and Bloch Waves

Let $X = \mathbb{Z}^d$ and $\hat{X} = [-\pi, \pi]^d$ as before. Let L be a lattice in X , so

$$L = \left\{ \sum_{n=1}^d k_n L_n \mid k_n \in \mathbb{Z} \right\}, \quad (1.25)$$

for some set $\{L_1, \dots, L_d\}$ of independent elements in X . We will analyze the periodic Schrödinger operator H

$$H = -\Delta + V, \quad (1.26)$$

obtained from an L -periodic potential $V : X \rightarrow \mathbb{R}$ on X ,

$$V(x) = V(x + l), \quad \forall l \in L. \quad (1.27)$$

This condition forces V to be bounded, and H is then a bounded selfadjoint operator with domain $D(H) = \mathcal{H}$. The symmetry from L makes it possible to give a complete treatment of H .

First we will establish a Hilbert space isomorphism \mathcal{B} from \mathcal{H} onto a direct integral sum $\int_B^\oplus \mathcal{H}_k d\mu(k)$.

The dual lattice K is the set of $k \in \mathbb{R}^d$ such that

$$X \ni x \mapsto e^{-ikx} \in \mathbb{C}, \quad (1.28)$$

is L -periodic. Let B be a measurable subset of \mathbb{R}^d which represents \mathbb{R}^d/K , so the mapping $B \ni k \mapsto [k] = \{k + k' \mid k' \in K\} \in \mathbb{R}^d/K$ is bijective. Fix a quasi momentum $k \in B$. The Bloch space \mathcal{H}_k is the set of complex valued functions ψ on X with the property

$$\psi(x + l) = e^{ikl} \psi(x), \quad \forall l \in L. \quad (1.29)$$

Any ψ in \mathcal{H}_k has a unique representation

$$\psi(x) = e^{ikx} u(x), \quad (1.30)$$

for some L periodic function u . It is straightforward to verify that \mathcal{H}_k is a vector space. Let F be a subset of X representing X/L , so $F \ni x \mapsto \{x + l \mid l \in L\} \in X/L$ is bijective. A function in \mathcal{H}_k is determined by its values on F , and since F is finite it follows that \mathcal{H}_k is finite dimensional. Each \mathcal{H}_k has dimension N given by the number of points in F . An inner product is defined on \mathcal{H}_k by

$$\langle \psi, \phi \rangle := \sum_{x \in F} \overline{\psi(x)} \phi(x). \quad (1.31)$$

Notice that this inner product is independent of the choice of F .

Define the Bloch projection B_k for each k in B by

$$B_k f(x) := \sum_{l \in L} e^{-ikl} f(x+l), \quad (1.32)$$

so B_k maps \mathcal{H} into \mathcal{H}_k . The Bloch transformation $\mathcal{B} : \mathcal{H} \rightarrow \int_B^\oplus \mathcal{H}_k d\mu(k)$ is defined from $\mathcal{B}f(k) := B_k f$. We will verify that \mathcal{B} is a Hilbert space isomorphism—from the action of \mathcal{B} on the Kronecker delta basis $\{\delta_x\}$ in \mathcal{H} . Define

$$\tilde{\delta}_x := \mathcal{B}\delta_x, \quad (1.33)$$

and note

$$B_k \delta_y(x) = \sum_{l \in L} e^{-ikl} \delta_y(x+l) = \begin{cases} e^{ik(x-y)} & \text{if } x-y \in L \\ 0 & \text{otherwise.} \end{cases} \quad (1.34)$$

Let the measure μ be given by Lebesgue measure on \mathbb{R}^d divided by the Lebesgue measure of B , so μ is a probability measure on B . It follows that $\{\tilde{\delta}_x\}$ is an orthonormal system in $\int_B^\oplus \mathcal{H}_k d\mu(k)$. Observe that each element in $\int_B^\oplus \mathcal{H}_k d\mu(k)$ may be identified with an element in $\bigoplus_{x \in F} L^2(B, d\mu)$. This gives that $\{\tilde{\delta}_x\}$ is also complete, from the completeness of the functions

$$B \ni k \mapsto e^{ikl} \in \mathbb{C}, \quad l \in L, \quad (1.35)$$

in $L^2(B, d\mu)$. The mapping \mathcal{B} is then a Hilbert space isomorphism, since it gives a bijective correspondence between two orthonormal bases.

A consequence of this is the Parseval formula

$$\langle f, g \rangle = \langle \tilde{f}, \tilde{g} \rangle = \int_B \sum_{x \in F} \overline{B_k f(x)} B_k g(x) d\mu(k). \quad (1.36)$$

In particular we find the inversion formula

$$f(x) = \langle \delta_x, f \rangle = \langle \tilde{\delta}_x, \tilde{f} \rangle = \int_B \sum_{y \in F} \overline{B_k \tilde{\delta}_x(y)} B_k f(y) d\mu(k) = \int_B e^{ik(x-y)} B_k f(y) d\mu(k), \quad (1.37)$$

where y is the unique element in F such that $x-y \in L$.

The unitary operator U_l given by $U_l f(x) = f(x+l)$ commutes with the Schrödinger operator H for all l in L . This has the consequence

$$B_k(Hf)(x) = \sum_{l \in L} e^{-ikl} U_l Hf(x) = \sum_{l \in L} e^{-ikl} H U_l f(x) = (H_k B_k f)(x), \quad (1.38)$$

where H_k is defined by

$$H_k : \mathcal{H}_k \rightarrow \mathcal{H}_k, \quad H_k f = (-\Delta + V)f. \quad (1.39)$$

The operator H is unitarily equivalent with the operator $\tilde{H} := \int_B^\oplus H_k d\mu(k)$ on $\tilde{\mathcal{H}} = \int_B^\oplus \mathcal{H}_k d\mu(k)$.

Now observe that each H_k is given by a selfadjoint $N \times N$ matrix, since \mathcal{H}_k has dimension N . This makes a complete analysis of H possible. We could continue this general outline to obtain explicit formulas for the spectral family, the resolvent, and the density of states, but prefer to consider some explicit examples. The key to the general case is given by the formula

$$f\left(\int_B^\oplus H_k d\mu(k)\right) = \int_B^\oplus f(H_k) d\mu(k), \quad (1.40)$$

which is valid for any bounded measurable function f on the spectrum of H .

1.6 A Periodic Potential in Two Dimensions

In this section we will consider an explicit model in two dimensions, $d = 2$. The Hilbert space is $\mathcal{H} = l^2(X)$, with $X = \mathbb{Z}^2$. The sublattice is $L = (2\mathbb{Z})^2$. We choose a fundamental cell

$$F = \{F_1, \dots, F_4\}, \quad F_1 = (0, 0), F_2 = (1, 0), F_3 = (1, 0), F_4 = (1, 1). \quad (1.41)$$

The potential is L -periodic and determined by

$$V(F_i) = \alpha_i, \quad i = 1, 2, 3, 4. \quad (1.42)$$

where α_i are parameters—coupling constants.

The dual lattice is $L' = (\pi\mathbb{Z})^2$ and we represent \mathbb{R}^2/L' with

$$B = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2. \quad (1.43)$$

For each k in B we have the Hilbert space \mathcal{H}_k of functions $\psi : \mathbb{Z}^2 \rightarrow \mathbb{C}$ restricted by the boundary conditions

$$\psi(x + l) = e^{ikl} \psi(x), \quad \forall l \in L. \quad (1.44)$$

Any $\psi \in \mathcal{H}_k$ is determined by the values $z_i = \psi(F_i)$, $i = 1, \dots, 4$. In $d = 2$

$$H_k \psi(x) = (4 + V(x)) \psi(x) - \sum_{i=1}^2 (\psi(x + e_i) + \psi(x - e_i)). \quad (1.45)$$

We will first consider the special case $\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \infty$, defined by Dirichlet conditions at the places where the potential is infinite. The component $z_4 = 0$ drops out. Then

$$\begin{aligned} H_k \psi(F_1) &= 4z_1 - z_3 - z_2 - e^{-i2k_1} z_2 - e^{-i2k_2} z_3, \\ H_k \psi(F_2) &= 4z_2 - z_1 - e^{i2k_1} z_1, \quad \text{and} \\ H_k \psi(F_3) &= 4z_3 - z_1 - e^{i2k_2} z_1, \end{aligned} \quad (1.46)$$

and H_k is represented by the matrix

$$A_k = \begin{pmatrix} 4 & -1 - e^{-i2k_1} & -1 - e^{-i2k_2} \\ -1 - e^{i2k_1} & 4 & 0 \\ -1 - e^{i2k_2} & 0 & 4 \end{pmatrix}. \quad (1.47)$$

The eigenvalues are

$$E_1(k) = 4 - \sqrt{2 \sum_{i=1}^2 (1 + \cos 2k_i)}, \quad E_2(k) = 4, \quad E_3(k) = 4 + \sqrt{2 \sum_{i=1}^2 (1 + \cos 2k_i)}. \quad (1.48)$$

This gives the spectrum of the Schrödinger operator H

$$\sigma_{ac}(H) = [4 - 2\sqrt{2}, 4 + 2\sqrt{2}], \quad \sigma_{pp}(H) = \{4\}, \quad \sigma_{sc}(H) = \emptyset. \quad (1.49)$$

This case, $\alpha_4 = \infty$, is special. Let $X[V]$ be the subset of X where V is finite. The operator H is selfadjoint as an operator in $l^2(X[V])$. If we regard H as an operator in $l^2(X)$, then $\sigma(H) = \mathbb{C}$, but the core of the spectrum is $\tilde{\sigma}(H) = [4 - 2\sqrt{2}, 4 + 2\sqrt{2}]$. Regarded as a spectral operator, that is as an operator given by a spectral family, we find the above decomposition of the spectrum. This is independent of the choice of $l^2(X)$ or $l^2(X[V])$ as the Hilbert space.

The eigenvalue 4 has infinite multiplicity. A corresponding eigenvector for A_k is given by

$$\psi_k = [0, -1 - e^{-i2k_2}, 1 + e^{-i2k_1}]^T. \quad (1.50)$$

The inversion formula (1.37) gives the corresponding eigenfunction for H

$$\psi(1, 0) = -1, \quad \psi(0, 1) = 1, \quad \psi(2, 1) = 1, \quad \psi(1, 2) = -1, \quad \text{and } \psi = 0 \text{ otherwise.} \quad (1.51)$$

From figure 1.2 and equation (1.45) it is easy to verify that ψ is an eigenvector with eigenvalue 4. From the given eigenfunction ψ one may obtain new eigenfunctions $U_l \psi$ from shifts given by $l \in L$. Superposition increases the possibilities further.

Now return to the general case with four real coupling constants. The operator H_k is then represented by the matrix

$$A_k = \begin{pmatrix} 4 + \alpha_1 & -1 - e^{-i2k_1} & -1 - e^{-i2k_2} & 0 \\ -1 - e^{i2k_1} & 4 + \alpha_2 & 0 & -1 - e^{-i2k_2} \\ -1 - e^{i2k_2} & 0 & 4 + \alpha_3 & -1 - e^{-i2k_1} \\ 0 & -1 - e^{i2k_2} & -1 - e^{i2k_1} & 4 + \alpha_4 \end{pmatrix}. \quad (1.52)$$

For convenience we introduce new variables

$$\beta_i := \alpha_i + 4 \quad \text{and} \quad K_j := 2(1 + \cos(2k_j)). \quad (1.53)$$

0	0	0	0	0	0	0
0		0		0		0
0	0	0	-1	0	0	0
0		1		1		0
0	0	0	-1	0	0	0
0		0		0		0
0	0	0	0	0	0	0

Figure 1.2: The shaded squares represents positions in \mathbb{Z}^2 where a potential V is infinite. The potential is zero elsewhere. The numbers are the values of an eigenfunction $\psi \in l^2(\mathbb{Z}^2)$ corresponding to the discrete Schrödinger equation $(-\Delta + V)\psi = 4\psi$. The mean value is zero around each allowed coordinate. This verifies that ψ is an eigenfunction.

The eigenvalues $E_n(k)$ of A_k are determined by

$$0 = \det(A_k - E) = (E - \beta_1)(E - \beta_2)(E - \beta_3)(E - \beta_4) - K_1[(E - \beta_1)(E - \beta_2) + (E - \beta_3)(E - \beta_4)] - K_2[(E - \beta_1)(E - \beta_3) + (E - \beta_2)(E - \beta_4)] + (K_1 - K_2)^2. \quad (1.54)$$

We will not discuss this general case further, but specialize to the chessboard case

$$V(0,0) = V(1,1) = \alpha, \quad V(1,0) = V(0,1) = 0. \quad (1.55)$$

We refer to the corresponding Schrödinger operator as the chessboard Schrödinger operator. Equation (1.54) can then be explicitly solved. The band functions are

$$\begin{aligned} E_1(k) &= \frac{1}{2} \left\{ 8 + \alpha - \sqrt{\alpha^2 + 8 \left(\sqrt{1 + \cos(2k_1)} + \sqrt{1 + \cos(2k_2)} \right)^2} \right\}, \\ E_2(k) &= \frac{1}{2} \left\{ 8 + \alpha - \sqrt{\alpha^2 + 8 \left(\sqrt{1 + \cos(2k_1)} - \sqrt{1 + \cos(2k_2)} \right)^2} \right\}, \\ E_3(k) &= \frac{1}{2} \left\{ 8 + \alpha + \sqrt{\alpha^2 + 8 \left(\sqrt{1 + \cos(2k_1)} - \sqrt{1 + \cos(2k_2)} \right)^2} \right\}, \text{ and} \\ E_4(k) &= \frac{1}{2} \left\{ 8 + \alpha + \sqrt{\alpha^2 + 8 \left(\sqrt{1 + \cos(2k_1)} + \sqrt{1 + \cos(2k_2)} \right)^2} \right\}. \end{aligned} \quad (1.56)$$

A plot of these four dispersion relations in the case $\alpha = -4$ is given in figure 1.3.

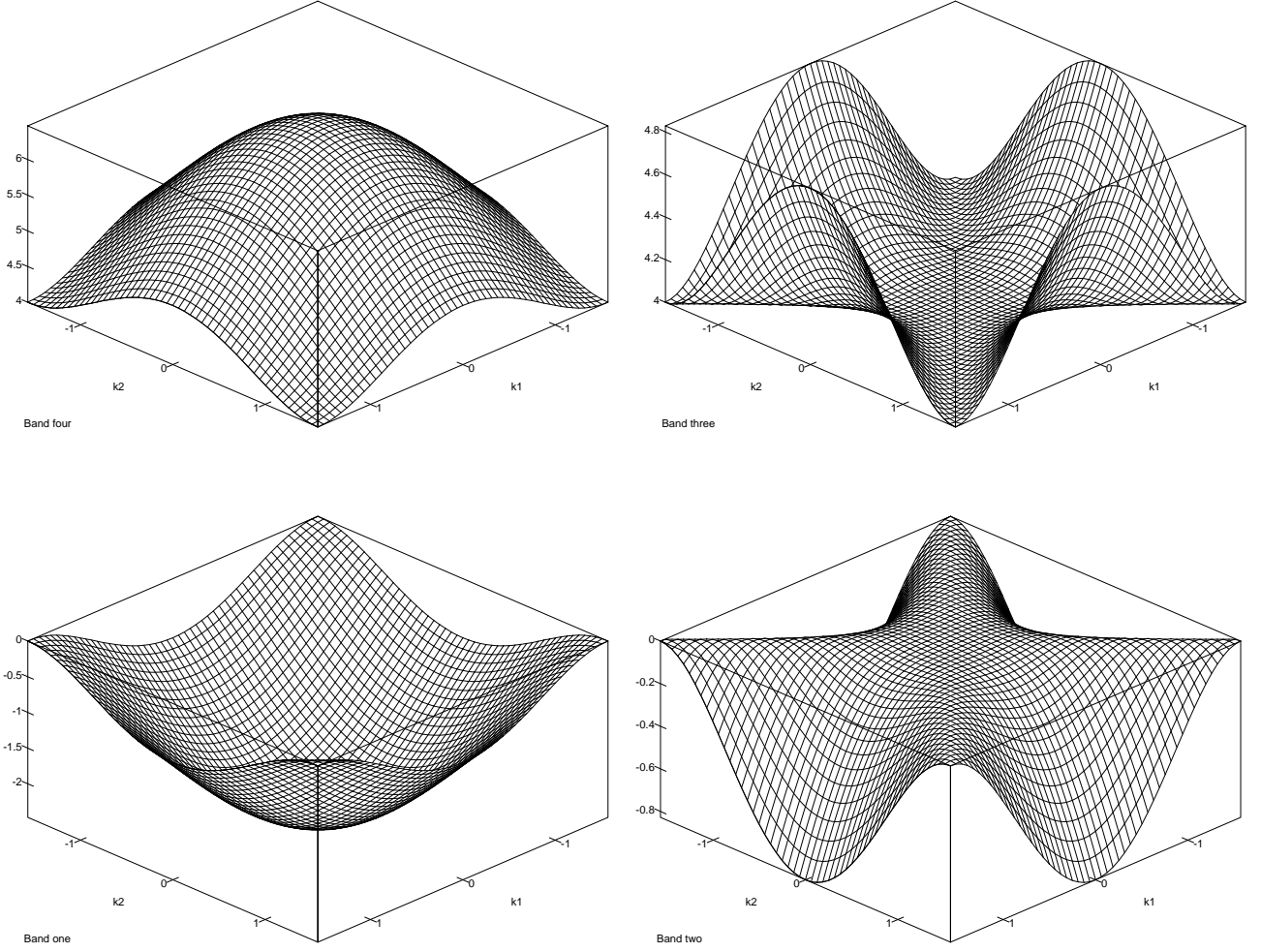


Figure 1.3: *The four dispersion relations for the chessboard Schrödinger equation with coupling constant equal to -4 . The vertical axis gives the energy, but with different scales for each band. The two horizontal axis give the two components of the Bloch momentum.*

The dispersion relations give four allowed energy bands

$$\begin{aligned}
 B_1 &= [(8 + \alpha - \sqrt{\alpha^2 + 64})/2, (8 + \alpha - \sqrt{\alpha^2})/2], \\
 B_2 &= [(8 + \alpha - \sqrt{\alpha^2 + 16})/2, (8 + \alpha - \sqrt{\alpha^2})/2], \\
 B_3 &= [(8 + \alpha + \sqrt{\alpha^2})/2, (8 + \alpha + \sqrt{\alpha^2 + 16})/2], \text{ and} \\
 B_4 &= [(8 + \alpha + \sqrt{\alpha^2})/2, (8 + \alpha + \sqrt{\alpha^2 + 64})/2],
 \end{aligned} \tag{1.57}$$

see figure 1.4.

The spectrum for the chessboard model is finally found to be

$$\begin{aligned}
 \sigma_{ac}(H) = & [(8 + \alpha - \sqrt{\alpha^2 + 64})/2, (8 + \alpha - \sqrt{\alpha^2})/2] \cup \\
 & [(8 + \alpha + \sqrt{\alpha^2})/2, (8 + \alpha + \sqrt{\alpha^2 + 64})/2],
 \end{aligned} \tag{1.58}$$

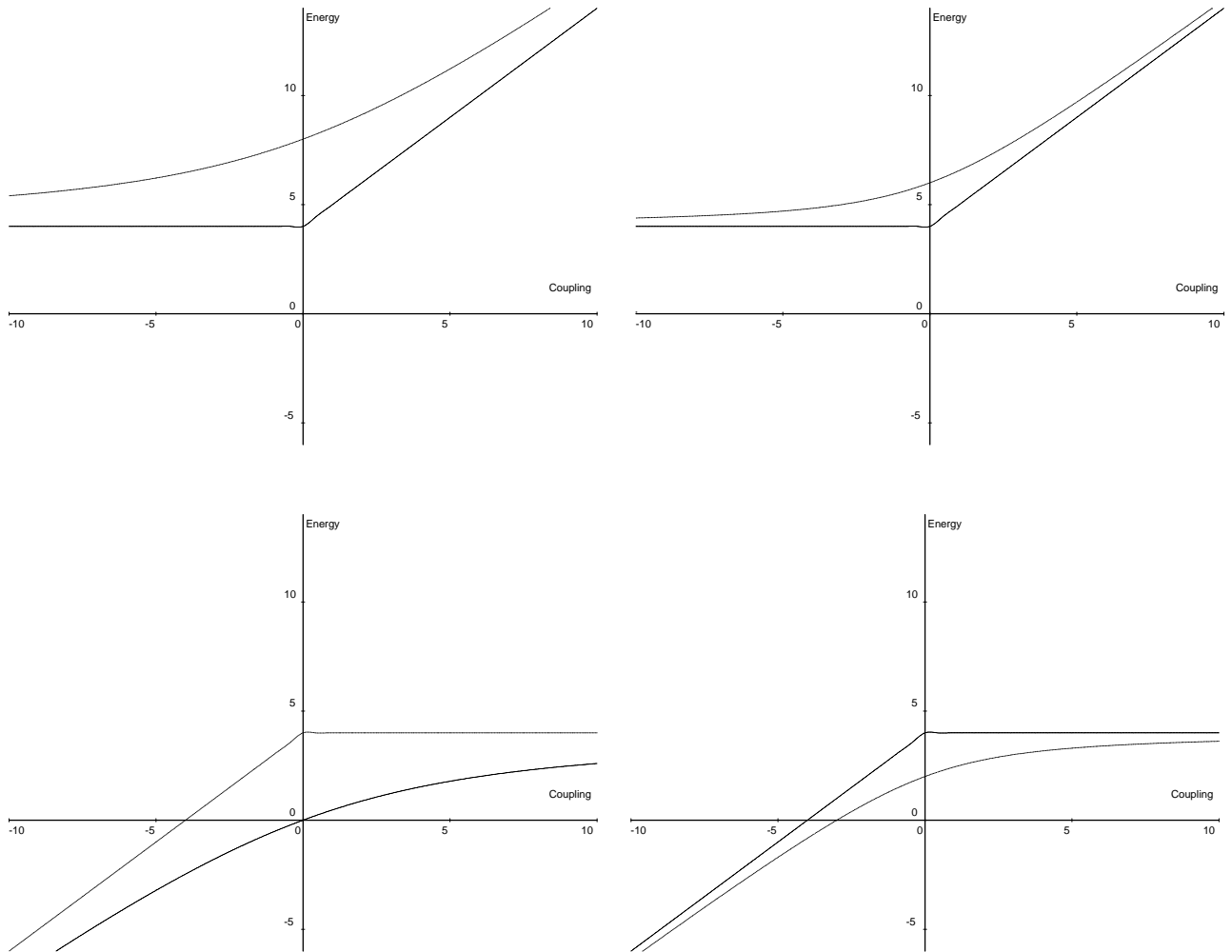


Figure 1.4: *The four allowed energy bands for the chessboard Schrödinger equation as a function of the coupling constant. The vertical axis gives the energy. The horizontal axis gives the coupling constant. The spectrum is given from the two leftmost figures, which correspond to band one and four.*

compare figure 1.5.

1.7 Impurities in a Discrete Model

Let

$$H(\lambda) = H(0) + \lambda W, \quad D(H(\lambda)) = D(H(0)) = D \subset \mathcal{H} = \ell^2(\mathbb{Z}^d). \quad (1.59)$$

where $H(0)$ is selfadjoint, λ is a real coupling constant, and W is a potential supported on a finite set F in \mathbb{Z}^d . We will find conditions which give all possible eigenvalues of $H(\lambda)$ in the spectral gaps of $H(0)$.

The model we have in mind is where $H(0)$ is a periodic Schrödinger operator and λW represents an impurity in the lattice.

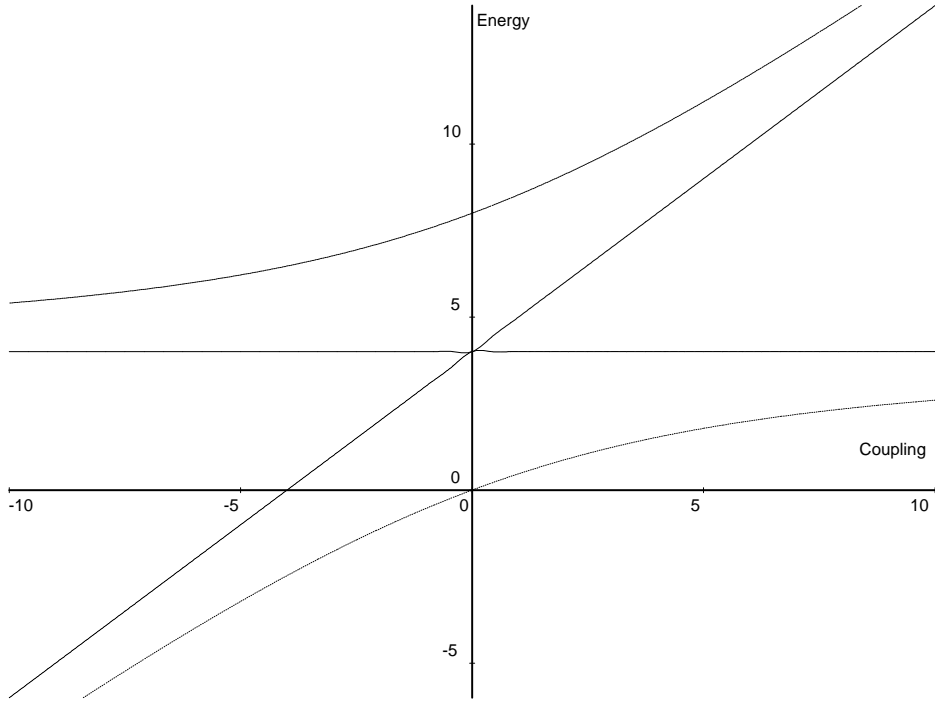


Figure 1.5: *The spectrum of the chessboard Schrödinger operator as a function of the coupling constant. The vertical axis gives the energy. The horizontal axis gives the coupling constant.*

Fix a real number E in a spectral gap of $H(0)$. The resolvent

$$G = (E - H(0))^{-1}, \quad (1.60)$$

is then a bounded selfadjoint operator with integral kernel

$$G(x, y) = \langle \delta_x, G\delta_y \rangle. \quad (1.61)$$

An element $\psi \neq 0$ in D is an eigenvector for $H(\lambda)$ with eigenvalue E if and only if

$$H(\lambda)\psi = E\psi, \text{ or equivalently iff} \quad (1.62)$$

$$(E - H(0))\psi = \lambda W\psi, \text{ or equivalently iff} \quad (1.63)$$

$$\psi(x) = \lambda \sum_{y \in F} G(x, y)W(y)\psi(y). \quad (1.64)$$

Let y_1, \dots, y_M be an enumeration of the M points in F . Define the matrix B by

$$B_{ij} = G(y_i, y_j)W(y_j), \quad (1.65)$$

which is an $M \times M$ matrix. Let

$$\lambda_1(E), \lambda_2(E), \dots, \lambda_M(E) \quad (1.66)$$

be the solutions of the M 'th order polynomial equation

$$D(\lambda) := \det(\lambda B - I) = 0. \quad (1.67)$$

Equation (1.64) gives that the possible eigenvalues of $H(\lambda)$ in spectral gaps of $H(0)$ are determined by $\lambda = \lambda_i(E)$. The corresponding eigenfunction ψ is determined from the null space of $\lambda B - I$ and equation (1.64). This method determines the spectrum of $H(\lambda)$. We note that the point spectrum of $H(\lambda)$ is not determined completely from our discussion. The possibility of embedded eigenvalues has to be considered.

The above method works equally well for complex values of the coupling constant and W may be replaced by a nondiagonal matrix. It is however possible to make some simplifications in special cases. In our diagonal case we may choose an enumeration such that $W(y_1), \dots, W(y_N)$ are negative and $W(y_{N+1}), \dots, W(y_M)$ are positive. Observe that the determinant of the diagonal matrix W is nonzero. Let U, V be matrices such that $W = UV$, and observe that both U and V are then invertible. A consequence is that we can replace the matrix B in the above discussion with the matrix $S = VGU$. If the perturbation W is positive, so $N = 0$, then we may choose $U = V = \sqrt{W}$. The result is that S is selfadjoint and the roots of $0 = \det(\lambda S - I)$ are all real, which is a simplification. If the perturbation W changes sign, so $0 < N < M$, then it is not possible to decompose W into two diagonal matrices as above. We will not discuss this general case further, but instead turn to a simple example.

A Kronig-Penney impurity is given by

$$W(x) = \delta_0(x). \quad (1.68)$$

In this case

$$D(\lambda) = \lambda G(0, 0) - 1, \quad (1.69)$$

so the possible eigenvalues are determined by $G(0, 0)$. Consider $H(0) = -\Delta$ and $E \notin [0, 4d]$. Then

$$G(0, 0) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{E - 2 \sum_{i=1}^d (1 - \cos(k_i))} dk. \quad (1.70)$$

In the one dimensional case, $d = 1$, the integral is computable and $D(\lambda) = 0$ gives

$$\lambda = \pm 2\sqrt{(E/2 - 1)^2 - 1}, \quad E = 2 \left(1 \pm \sqrt{1 + (\lambda/2)^2} \right), \quad (1.71)$$

which is the eigenvalue E as a function of the impurity strength λ .

1.8 Notes

The theory of ordinary difference equations is developed in for instance [11], but see also [131], [1]. Some elements of the theory of partial difference operators may be found in [20], but see also [73], [121], [39], and [101]. The theory is perhaps most completely treated in texts on the numerical solution of partial differential equations, but there the focus is on subjects differing from ours. Some of this theory may be found in [74, p.172-182] and references cited there.

Rudin [116] and Katznelson [76] give excellent accounts of the theory of Fourier analysis. The properties of the discrete Laplace operator are consequences of the elementary part of this theory.

The Bloch theory for the difference operator is modeled after the corresponding theory for the differential operator [124, p.279-315, p.358-360, vol.IV], [21, p.198-222]. A treatment of the physics of an electron in a periodic potential is found in [10, p.131-311], which in particular considers band structure of real world solids.

Our discussion is only preliminary and could easily be extended. The sections on analytic perturbation theory of finite matrices found in [75] are a good starting point for a more extensive treatment.

The results for the explicit models may be a novelty.

The discussion of perturbation by a finitely supported potential may be seen as a special case of parts of the material found in [5, p.360-364]. The material is strongly related to the similar continuous case as discussed in [42],[60],[61], but with simplifications. We observe that it seems possible to recover the general spectral theory associated with relatively compact perturbations [136, p.273-282] by consideration of norm limits of the case we have considered.

Part II

General Theory

Chapter 2

Randomness and Ergodicity

We consider abstract valued random variables. This abstract setting is motivated from the study of ergodic random Schrödinger operators. We prove nonrandomness of a random variable, invariant with respect to a metrically transitive system. This is a generalization from real valued variables to variables with a σ -separated range.

2.1 Introduction

The Schrödinger operator with an ergodic random potential is a formally ergodic random operator. Any quantity defined from the operator is also formally random. The purpose of this chapter is to provide some of the necessary background from probability theory, sufficient for a rigorous treatment of ergodic random operators.

The concept of abstract random variables is introduced. This definition is sufficient for any random quantity derived from a random operator. In particular, the definition is general enough to allow random numbers, random measures, random vectors, random operators, and random sets.

We next deal with some measure theoretic - fundamental - technicalities. For a separable normed vector space the norm Borel σ -algebra is equal to the initial σ -algebra from the topological dual space. This is of importance for linear random operators. Measurability of a limit of a sequence of metric space valued functions is proved. In particular this gives measurability of a strongly measurable function, which - by definition - is a pointwise limit of measurable simple functions. We do also prove sequential completeness for strongly measurable functions, defined on a probability space. A strongly measurable function has essentially separable range. Separable range turns out to be sufficient for the strong measurability of a random element in a metric space. Pettis' theorem is a corollary of the above. A function from a probability space into a normed vector space is strongly measurable iff it is weakly measurable and almost separably valued.

We give a standard construction of a sequence of i.i.d. variables. A class of random Schrödinger operators is obtained from i.i.d. coupling constants, and the given construction is sufficient to give highly non trivial random operators. We prove the strong law of large numbers, as a corollary of Birkhoff's ergodic theorem.

We give some examples of ergodic systems, and give the link to dynamical systems. This

section is more informal and contains no proofs.

The term metrical transitive is often used synonymously with the term ergodic. We will use the term ergodic system to denote a system which is stationary and metricaly transitive. We consider metrical transitivity as an isolated property. The main theorem - possibly also in this chapter - gives non randomness of a random variable, invariant with respect to a metricaly transitive system. A σ -separated range for the variable is sufficient. The theorem in this generality is possibly new. At least, the theorem seems to be unknown in the context of ergodic random operators.

Measure preserving transformations are treated next. The weak and strong ergodic theorems are beautiful consequences. We have included J. von Neumann's proof of the weak ergodic theorem.

2.2 Random Variables

The following definitions are necessary for the definition of random variables.

Definition 2.2.1 (σ -algebra) *A nonempty family \mathcal{F} of subsets from a set X is a σ -algebra in X if it is closed under complements and countable unions:*

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \text{ and } A_i \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}. \quad (2.1)$$

A set A is \mathcal{F} -measurable if $A \in \mathcal{F}$. A measurable space is a set equipped with a σ -algebra. The Borel field in a topological space is the smallest σ -algebra containing all the open sets.

Remarks. (i) A trivial consequence is that X and \emptyset are members of \mathcal{F} . Likewise the fact that \mathcal{F} is also closed under countable intersections. The case $X = \emptyset$ is not excluded in the above.

(ii) A Borel field is a σ -algebra generated by a topology. If we are given a family G of subsets then $\sigma(G)$ is the σ -algebra generated by G . It is hard to characterize $\sigma(G)$, in contrast with the situation for the topology $\tau(G)$ generated by G . Equip \mathbb{R} with the Borel field from the usual topology in \mathbb{R} . It is easy to find non open sets in \mathbb{R} , but non measurable sets are harder to point out.

Definition 2.2.2 (Measurable Functions) *Let X and Y be measurable spaces. A function $f : X \rightarrow Y$ is measurable if $f^{-1}(A)$ is measurable for every measurable set A in Y .*

Remarks. (i) Let \mathcal{G} be the σ -algebra in Y . The family $f^{-1}\mathcal{G}$ is a σ -algebra, and a sub σ -algebra of the σ -algebra in X if f is measurable.

(ii) If X and Y are topological spaces, it follows that any continuous function is Borel. A Borel function is a function which is measurable with respect to the Borel fields.

(iii) Let A be a subset of X . Define $1_A(x) = 1$ for x in A and $1_A(x) = 0$ for x in A^c . The characteristic function $1_A : X \rightarrow \mathbb{R}$ is a measurable function iff A is measurable.

Definition 2.2.3 (Measure) Let \mathcal{F} be a σ -algebra in X . A function $\mu : \mathcal{F} \rightarrow [0, \infty]$ is a measure if $\mu(\emptyset) = 0$ and if

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i) \quad (\text{countable additive}), \quad (2.2)$$

for all disjoint measurable A_1, A_2, \dots . A measure space is a set equipped with a σ -algebra and a measure. The measure μ is finite if $\mu(X) < \infty$, and σ -finite if X is a countable union of sets with finite measure.

Remarks. (i) It is convenient to refer to a measure space as a triple (X, \mathcal{F}, μ) .

(ii) A measure algebra is a Boolean σ -algebra with a countably additive, positive definite, \mathbb{R}_+ -valued function. Let $A \triangle B$ be the symmetric difference between two sets A and B . If one identifies sets A, B when $\mu(A \triangle B) = 0$, then (\mathcal{F}, μ) is a measure algebra. A measure algebra has a natural metric given by $d(A, B) = \mu(A \triangle B)$. (\mathcal{F}, d) is a complete metric space.

(iii) The μ -integral of the characteristic function 1_A is defined by $\mu[1_A] := \mu(A)$, for a measurable A . The notation $\mu[f]$ will be used for the μ integral of a function f .

We are not at all aiming at a complete presentation of general elementary measure theory. None the less we will present some typical results. We now turn to an elementary result. In the previous definition it is prudent to ask why one demands countable additivity in place of just finite additivity. One answer to this question is given by:

Theorem 2.2.4 (Continuity of Finite Measures) Let μ be a finite measure and let A_1, A_2, \dots be measurable sets. Then

$$\mu(\underline{\lim} A_i) \leq \underline{\lim} \mu(A_i) \leq \overline{\lim} \mu(A_i) \leq \mu(\overline{\lim} A_i), \quad (2.3)$$

where $\underline{\lim} A_i := \bigcup_n \bigcap_{k \geq n} A_k$ and $\overline{\lim} A_i := \bigcap_n \bigcup_{k \geq n} A_k$. The sequence $\{A_i\}$ is said to be convergent if $\underline{\lim} A_i = \overline{\lim} A_i$, and then $\underline{\lim} \mu(A_i) = \mu(\underline{\lim} A_i)$.

Proof. By considering complements, it is sufficient to prove the first inequality. We define $B_i := \bigcap_{j > i} A_j$, so that $\underline{\lim} A_i = \bigcup B_i$ and $B_i \subset B_{i+1}$. $\lim \mu(B_i) = \mu(\bigcup B_i)$ is a consequence of countable additivity if we consider the disjoint family C_i of sets given by: $C_1 = B_1$ and $C_{n+1} = B_{n+1} \setminus C_n$. This is so because $B_i = \bigcup_{j \leq i} C_j$. $B_i \subset A_i$ gives $\mu(B_i) \leq \mu(A_i)$ and finally $\mu(\underline{\lim} A_i) = \lim \mu(B_i) \leq \underline{\lim} \mu(A_i)$. \square

Remarks. (i) The first inequality is a special case of Fatou's Lemma. In fact, the theorem may be used as a starting point for a proof of Fatou's Lemma, which gives Lebesgue's monotone and dominated convergence theorems.

(ii) It should be clear that this continuity property is equivalent with the countable additivity. The continuity property is essential when one constructs a measure by extension of an additive set function defined on an algebra.

The previous theorem gives that countable additivity is equivalent with a continuity property of the measure. The continuity property is one essential reason for the usefulness of measures in analysis. This continuity gives rise to similar continuity properties of the integral. We provide a simple example to show that the integral is not a continuous mapping in general. Let (Ω, \mathcal{F}, P) be given by $\Omega := [0, 1]$ equipped with Lebesgue measure P .

Consider the functions $g_n(x) = (n+1)(n+2)x^n(1-x)$. The pointwise limit is $g = 0$, so we have $P[g] = 0$. On the other hand we have $P[g_n] = 1$, and therefore $\lim P[g_n] = 1 \neq P[g]$. This example is a reminder that interchange of limits and integration is a possible error, even in simple cases.

Definition 2.2.5 (Probability) *A probability space is a measure space (Ω, \mathcal{F}, P) , with $P(\Omega) = 1$. An event A is a measurable set, and has a probability $P(A)$. A realization is a point in Ω .*

Definition 2.2.6 (Random Variables) *Let (Ω, \mathcal{F}, P) be a probability space and let (E, \mathcal{G}) be a measurable space. An E random variable X is a measurable function $X : \Omega \rightarrow E$, and X is then a random element in E . The distribution of X is the image measure $P_X := P \circ X^{-1}$.*

Remarks. (i) A more general definition of a random variable would be as a homomorphism, compare X^{-1} , from a Boolean σ -algebra \mathcal{G} to a probability algebra (\mathcal{F}, P) .

(ii) An indicator function 1_A of a measurable set A is a simple example of a real random variable. A simple function is a function that takes a finite number of values. A fundamental result is that any Borel random variable with values in a separable metric space is a pointwise a.e. limit of measurable simple functions. It is even a uniform limit of functions which take a countable number of values.

(iii) Let P be a probability measure and X a complex random variable. The expectation of X , if it exists, is the integral $E[X] := P[X]$.

(iv) A formally random variable is a function $X : \Omega \rightarrow E$, without measurability demands.

The distribution P_X is a key point in the previous definition. We have the fundamental result

$$E_X[f] := P_X[f] = P[f \circ X] = E[f \circ X], \quad f \in L^1(P_X). \quad (2.4)$$

In applications it is usually the distribution of the random variable that is given. Equation (2.4) gives results independent of the mathematical realization of a random variable. The short notation $\{X \in A, Y \in B\} := \{\omega \mid X(\omega) \in A, Y(\omega) \in B\}$, with obvious generalizations, together with the above integral formula, makes it possible to do calculations without reference to the probability space (Ω, \mathcal{F}, P) .

2.3 Pettis' Theorem and Measurability

Measurability of an abstract valued function may often be reduced to measurability of a family of real valued functions. The following is elementary, but useful.

Theorem 2.3.1 (Measurability and Initial σ -algebras) *Equip a set E with the initial σ -algebra from a family F of functions with domain E and values in possibly distinct measurable spaces. A map $g : X \rightarrow E$ is measurable iff the function $f \circ g : X \rightarrow Y_f$ is measurable for each $f \in F$.*

Proof. Assume g to be measurable. Let A be a measurable set in Y_f . By definition $f^{-1}(A)$ is measurable and $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$ is measurable, so $f \circ g$ is measurable.

Assume $f \circ g$ to be measurable for all $f \in F$. Let \mathcal{N} be the set of $A \subset E$ such that $g^{-1}(A)$ is measurable, so \mathcal{N} is the final σ -algebra for g . We must show that $\mathcal{F} \subset \mathcal{N}$. By assumption $G = \{f^{-1}(B_f) \mid B_f \text{ measurable}, f \in F\} \subset \mathcal{N}$ and G generates \mathcal{F} . Therefore $\mathcal{F} \subset \mathcal{N}$, so g is measurable. \square

Remarks. (i) This is an exact analog of the corresponding theorem for continuous functions.

The most common application of this theorem is when all the measurable spaces Y_f are equal to the set of complex numbers. Consider the case when E is a normed vector space. A variable X is a weak E valued random variable iff $\langle y, X \rangle$ is a random variable for all y in the topological dual E^* of E . The above theorem states that this is equivalent with X being measurable with respect to the initial σ -algebra \mathcal{F}_w from E^* . There are many other possible σ -algebras on a normed space. We note in particular that any topology on E generates a corresponding Borel σ -algebra. The following theorem reduces the number of possibilities.

Theorem 2.3.2 (σ -algebras on a Separable Normed Space) *Let E be a separable, normed vector space. Let \mathcal{F}_w be the initial σ -algebra from the topological dual of E , let \mathcal{B}_w be the Borel σ -algebra from the weak topology, and let \mathcal{B}_n be the Borel σ -algebra from the norm topology. We have $\mathcal{F}_w = \mathcal{B}_w = \mathcal{B}_n$.*

Proof. $\mathcal{F}_w \subset \mathcal{B}_w \subset \mathcal{B}_n$ is clear. We prove $\mathcal{F}_w \supset \mathcal{B}_n$. Because E is separable it is sufficient to show that every closed ball B is in \mathcal{F}_w . Assume C to be norm closed and convex. Let $\{B_i\}$ be a countable family of norm open balls such that $C^c = \bigcup_i B_i$. Choose, by the Hahn-Banach separation theorem, a $\phi_i \in E^*$ and t_i such that $\text{Re } \phi_i(x) < t_i \leq \text{Re } \phi_i(y)$ for all $x \in B_i, y \in C$. $F_i = \{y \mid t_i \leq \text{Re } \phi_i(y)\}$ belongs to \mathcal{F}_w and $C \subset F_i$. We have in fact $C = \bigcap_i F_i \in \mathcal{F}_w$. It remains to prove $C \supset \bigcap_i F_i$. Take an $x \in C^c$. We have an i such that $x \in B_i$, so $x \notin F_i$. \square

Remarks. (i) A particular consequence is that X is a norm random variable iff $\langle y, X \rangle$ is a complex random variable for all y in the topological dual of E .

An additional possibility is given by the Baire σ -algebra, which is the smallest σ -algebra such that real valued continuous functions are measurable. The Baire σ -algebra is always contained in the Borel σ -algebra, but they are equal on a metric space. The proof follows from the continuity of $x \mapsto d(x, F)$ for a closed set F .

A vital property of measurable functions is that the class is closed under the operation of pointwise limits.

Theorem 2.3.3 (Sequential Completeness of Borel Measurable Functions) *A pointwise limit of a sequence of Borel functions from a measurable space into a metric space is a Borel function.*

Proof. Let $f = \lim f_n$ and let U be open. We will prove that:

$$f^{-1}(U) = \bigcup_{k \geq 1} \bigcup_{N \geq 1} \bigcap_{n \geq N} f_n^{-1}(C_k)$$

for suitably chosen measurable C_k , which proves Borel measurability.

▷: Assume that each C_k is closed and contained in U . Let ω be such that there is a k and a N such that $f_n(\omega) \in C_k$ for all $n \geq N$. We have $f(\omega) = \lim f_n(\omega) \in C_k \subset U$, because C_k is closed.

◁: Assume that every point in U has at least one C_k as a neighborhood. Let ω be such that $y = f(\omega) \in U$ and let C_k be a neighborhood to y . The convergence gives an N such that $f_n(\omega) \in C_k$ for all $n \geq N$.

The construction of C_k in a metric space is standard. Let $(C_k)^c$ be given by the union of all open balls with center in U^c and radius $1/k$. \square

Remarks. (i) We did not assume the metric space to be separable, nor complete.

(ii) One important application is the construction of random variables.

Weak measurability, combined with the Hahn-Banach theorem, may be used to define integration of Banach space valued functions. A stronger type of integral, the Bochner integral, may be defined for functions with a stronger measurability property. A function which is the pointwise a.e. limit of a sequence of measurable simple functions is said to be strongly measurable. A simple function is a function taking a finite number of values.

Theorem 2.3.4 (Sequential Completeness of Strongly Measurable Functions) *A pointwise limit of a sequence of strongly measurable functions from a probability space into a metric space is a strongly measurable function.*

Proof. Let each f_n be the pointwise a.e. limit of simple functions $g_{n,j}$. We may assume that we have pointwise convergence everywhere on Ω , obtained after subtraction of a null set. Let A_n and K_n be such that $P((A_n)^c) < 2^{-n}$ and $d(g_{n,k}(\omega), f_n(\omega)) < 2^{-n}$ when $k \geq K_n$ and $\omega \in A_n$. The existence of A_n and K_n follows from the pointwise convergence of $g_{n,k}$ towards f_n , and is one version of Egoroff's theorem. We prove that g_{n,K_n} converges pointwise a.e. towards f . The set $A = \bigcup_n \bigcap_{k \geq n} A_k$ has measure one, compare the Borel-Cantelli Lemma. Let $\omega \in A$. Choose N_1 so that $\omega \in A_n$ when $n \geq N_1$. Let $\epsilon > 0$ be arbitrary. Choose N_2 so that $d(f(\omega), f_n(\omega)) < \epsilon/2$ when $n \geq N_2$. Let N_3 be such that $2^{-n} < \epsilon/2$ when $n \geq N_3$, which implies $d(g_{n,K_n}(\omega), f_n(\omega)) < \epsilon/2$ when $n \geq N = \max(N_1, N_2, N_3)$. The triangle inequality gives $d(f(\omega), g_{n,K_n}(\omega)) < \epsilon$ for all $n \geq N$.

For completeness we prove existence of A_n, K_n as above. Let $g_n \rightarrow g$ pointwise. Fix $\mu > 0$. We prove existence of K and B , $P(B^c) < \mu$, and $d(g_n(\omega), g(\omega)) < \mu$ for all $n \geq K, \omega \in B$. Define $B_n = \bigcap_{k \geq n} \{\omega \mid d(g_k(\omega), g(\omega)) < \mu\}$ and notice $B_{n+1} \supset B_n$. The assumption gives that $\Omega = \bigcup_n B_n$ is the entire space. Because $P(\Omega)$ is finite we have a K so that $P((B_K)^c) < \mu$, and we are done with $B = B_K$. \square

The (essential) range of a strongly measurable function is separable. This follows from $f(\Omega) \subset \overline{\bigcup_n f_n(\Omega)}$, when $f_n \rightarrow f$ pointwise. This observation suggests the following.

Theorem 2.3.5 (Measurability and Strong Measurability) *Borel measurability and strong measurability are equivalent properties for a function defined on a finite measure space having separable range in a metric space.*

Proof. A strongly measurable function is a Borel measurable function, since it is a pointwise limit of Borel measurable functions.

Let X be the range of the Borel measurable function f . Pick a dense set $\{x_1, x_2, \dots\}$ in X . Let $B_n(i) = \{x \mid d(x, x_i) < 1/n\}$, $M_n(1) = B_n(1)$, and $M_n(i+1) = B_n(i+1) \setminus (\bigcup_{j=1}^i B_n(j))$.

Now we have, for each n , a disjoint cover of X of measurable sets $M_n(j)$ with $d(m, x_j) < 1/n$ for m in $M_n(j)$. Define $f_n = x_i$ on $f^{-1}(M_n(i))$. We have $\lim f_n = f$ uniformly, because $d(f_n(\omega), f(\omega)) < 1/n$. Strong measurability is preserved under pointwise limits, so we are done if f_n is strongly measurable. $g_{n,j}(\omega) = f_n(\omega)$ when $f_n(\omega) \in \{x_1, \dots, x_j\}$ and $g_{n,j}(\omega) = x_1$ otherwise proves strong measurability. \square

Remarks. (i) It is clearly sufficient in the above to assume that f is almost separably valued.

(ii) It is tempting to conclude as in the theorem without finiteness conditions on the measure space, since the approximants f_n converge uniformly. The difficulty with this is that the simple functions does not converge uniformly towards the approximants.

We have obtained conditions which give equivalence of strong measurability and Borel measurability. Equivalence of norm Borel measurability and weak measurability for a function with separable range in a normed space follows from the equality of the weak and norm σ -algebras. Pettis' theorem is the conclusion: Weak and strong measurability are equivalent for functions from a probability space into a normed space, if they have separable range.

2.4 Independence

A stochastic process or a stochastic field is by definition a family $\{X_\nu\}$ of random variables. By considering the direct product space one may also view a stochastic process as a single random variable. The interest in stochastic processes comes when there are some relations between the variables in the process. Independence, stationarity, and ergodicity are typical examples of such relations.

The idea of independent events is a source to many of the beautiful results and directions in probability theory. It makes probability theory to more than a special case of general measure theory.

Definition 2.4.1 (Independence) *A family M of random variables is a family of independent variables if $P(X_1 \in A_1, \dots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n)$ for every finite collection of variables $X_i \in M$ and measurable sets A_i . A family \mathcal{E} of events is a family of independent events if $P(A_1 \cap \dots \cap A_n) = P(A_1) \cdots P(A_n)$ for every finite collection of events A_i from \mathcal{E} .*

Remarks. (i) The variables in M may take values in distinct spaces, but they must be defined on the same probability space.

(ii) The term i.i.d. is short for independent, identically distributed. X and Y are identically distributed if $P_X = P_Y$.

We will now provide an example of how a sequence of i.i.d. variables can be constructed. Let (Ω, \mathcal{F}, P) be $[0, 1)$ equipped with Lebesgue measure P . Any $\omega \in \Omega$ has a binary representation given by a sequence B_i of random variables. To be concrete, choose $B_1 = 1_{[.5, 1)}$, $B_2 = 1_{[0.25, .5) \cup [.75, 1)}$ etc. We have

$$\omega = \sum_{i \geq 1} 2^{-i} B_i(\omega). \tag{2.5}$$

The B_i are i.i.d. random variables corresponding to the tossing of a coin. In each tossing, each i , there is equal probability ($= .5$) for head ($B_i = 1$) and tail ($B_i = 0$). Let $\{b_{i,j}\}$, $i, j \geq 1$, be a renumbering of the variables B_i . Now define the pointwise limits

$$X_i := \sum_{j \geq 1} 2^{-j} b_{i,j}. \quad (2.6)$$

The resulting X_i are i.i.d. , random variables with distributions equal to Lebesgue measure on $[0, 1)$. Starting with this family it is easy to obtain other families of i.i.d. random variables given by $Y_i = Y \circ X_i$, where Y is any random variable on $[0, 1]$. For p in $[0, 1]$ and $Y = 1_{[0,p]}$ the resulting sequence is a Bernoulli sequence. The variables Y_i are independent and $P(Y_i = 1) = p$, $P(Y_i = 0) = q = 1 - p$.

A fundamental result for i.i.d. , real, random variables is the strong law of large numbers.

Theorem 2.4.2 (The Strong Law of Large Numbers) *Let X_1, X_2, \dots be i.i.d. , real, random variables. Assume that $E[X_i] = E[X_1]$ exists. The expectation is also given by*

$$E[X_1] = \lim_{N \rightarrow \infty} \frac{X_1 + \dots + X_N}{N}. \quad (2.7)$$

The convergence on the right is pointwise convergence almost everywhere.

Proof. The process $\{X_i\}$ is stationary, since

$$\begin{aligned} P(X_{i_1} \in A_1, \dots, X_{i_n} \in A_n) &= P(X_{i_1} \in A_1) \cdots P(X_{i_n} \in A_n) = \\ P(X_{i_1+k} \in A_1) \cdots P(X_{i_n+k} \in A_n) &= P(X_{i_1+k} \in A_1, \dots, X_{i_n+k} \in A_n), \end{aligned} \quad (2.8)$$

so the limit exists a.e., according to Birkhoff's ergodic theorem. Consider an event B such that

$$B = \{X_{i_1} \in A_1, \dots, X_{i_n} \in A_n\} = T^{-1}B := \{X_{i_1+1} \in A_1, \dots, X_{i_n+1} \in A_n\}. \quad (2.9)$$

Since B is independent of $T^{-n}B$ for large n , it follows that $P(B) = P(B \cap T^{-n}B) = P(B)^2$, and $P(B)$ equals zero or one. The σ -algebra of invariant events is generated by sets like B , so any invariant set has probability zero or one, so the process $\{X_i\}$ is ergodic. The claim follows from Birkhoff's ergodic theorem. \square

Remarks. (i) If the X_i takes values in a more general space E , then the same statement is true with $f(X_i)$ replacing X_i , where $f : E \rightarrow \mathbb{R}$ is any measurable function so that $E[f(X_0)]$ exists.

2.5 Abstract Dynamical Systems and Ergodicity

We will now take a detour to explain some general ideas related to ergodicity. Assume a system to be described by a state ω in the set Ω of all possible states. The dynamics of the system is governed by a mapping T , which takes a state ω to a new state $T\omega$. $T^n\omega$

is the state at time n , given an initial state ω . If we define $U(n) = T^n$, then we see that U represents the semigroup \mathbb{N}_0 from the relation $U(n+m) = U(n)U(m)$. We may also consider a continuous time parameter, in which case we obtain a flow, if we insist on the group property. (Ω, T) is an abstract dynamical system.

Depending on the structure on Ω , it is natural to divide the subject into three major cases: (i) Differentiable dynamics, (ii) Topological dynamics, or (iii) Ergodic theory (measurable dynamics ?!). In any case it is most rewarding to consider the case when T is an automorphism ((i) diffeomorphism, (ii) homeomorphism, (iii) measure preserving transformation) on Ω . We will consider case (iii), with a given probability space (Ω, \mathcal{F}, P) as the state space.

A physical system is studied through a set of observables. A real observable X is a real valued, measurable function on Ω . In an actual measurement of the value of X it is reasonable to assume that we only can decide if the variable is in a given interval (a, b) . If our theory for the system is to predict outcomes of measurements, it should be able to assign probabilities to any event of the form $\{a < X < b\}$. This suggests a probabilistic model.

The observable changes in time according to $X_n\omega := X(T^n\omega)$. A basic problem in ergodic theory is the study of the time average

$$\bar{X} := \lim_{N \rightarrow \infty} \frac{X_1 + \dots + X_N}{N}. \quad (2.10)$$

It is not trivial to prove existence of this limit, from scratch. T is ergodic if this time average is equal to the mean $E[X]$, which in our context will be a theorem. The definition of ergodicity will be made in terms of two other concepts; metrical transitivity and stationarity. Before we proceed further with these definitions, we describe some concrete examples.

The classical example is from statistical mechanics. The space Ω is then a constant energy surface in phase space given by $H(p, q) = E$. The mapping $T(t)$ is given from Newton's laws of mechanics. Liouville's theorem states that the flow $T(t)$ is measure preserving with respect to the measure $d\mu_S := dS / \|\nabla H\|$, where dS is the element of surface area. Ergodicity is a more difficult question. Sinai proved ergodicity of a simplified version of a hard sphere gas in 1963.

The word 'ergodic' comes from the Greek for 'energy path'. Historically the idea of ergodicity came from Boltzmann's ergodic hypothesis. He assumed that each orbit filled out all of the phase space, but this is impossible on topological grounds. The assumption was soon replaced by the quasi-ergodic hypothesis; each orbit is dense in phase space. This weaker hypothesis does not imply ergodicity.

A simpler example is given by the unit interval equipped with Lebesgue measure. The transformation $Tx = x + \alpha \pmod{1}$ is then clearly measure preserving. If α is irrational, then T is ergodic. The nice conclusion of the ergodic theorem is in this case

$$\int_0^1 f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(x + n\alpha), \text{ (pointwise a.e. and in } L^1[0, 1]). \quad (2.11)$$

This example can be generalized to a compact, topological group with Haar measure.

Another example is given by the transformation $Tx = 2x \pmod{1}$, which is measure preserving, but not invertible, nor ergodic.

A real valued, stationary process gives another example. The state space is then chosen as the space of paths, and the shifts $T(x)$ defined by $T(x)X(y) := X(x + y)$, are measure preserving. We have the representation $X(x) = T(x)X(0)$. If one insists on keeping the original probability space, then the solution is given by considering mappings of measurable sets into measurable sets. A mapping is induced on the σ -algebra generated by the cylinder sets by the formula $\tau(x)\{\omega \mid X(y_1) \in A_1, \dots, X(y_n) \in A_n\} := \{\omega \mid X(y_1 + x) \in A_1, \dots, X(y_n + x) \in A_n\}$. Stationarity gives that τ is well defined and that $P(\tau A) = P(A)$. τ may also be considered as an operator acting on random variables by consideration of indicator functions. Define $\tau(x)1_A = 1_{\tau(x)A}$ and extend by linearity and continuity. We have again the representation $X(x) = \tau(x)X(0)$.

2.6 Metrical Transitivity

A strong property of an abstract dynamical system is metrical transitivity. We choose the following definition.

Definition 2.6.1 (Metrical Transitivity) *Let (X, \mathcal{F}, μ) be a measure space, and let Γ be a family of σ -algebra homomorphism $\tau : \mathcal{F} \rightarrow \mathcal{F}$. A measurable set A is Γ -invariant if $A = \tau A$ for all τ in Γ . A measurable set A is μ -trivial if $\mu(A) = 0$ or $\mu(A^c) = 0$. The quadruple $(X, \mathcal{F}, \mu, \Gamma)$ is a metrically transitive system if all Γ -invariant sets are μ -trivial.*

Remarks. (i) Let $T(\gamma) : X \rightarrow X$ be measurable for all γ in I . It is convenient to define $\{T(\gamma) \mid \gamma \in I\}$ to be a family of metrically transitive mappings if $(X, \mathcal{F}, \mu, \Gamma)$, with $\Gamma := \{T(\gamma)^{-1}\}$, is a metrically transitive system.

(ii) If $f : X \rightarrow X$ is measurable, then $f^{-1} : \mathcal{F} \rightarrow \mathcal{F}$ is a σ -algebra homomorphism; $f^{-1}A^c = (f^{-1}A)^c$ and $f^{-1} \bigcup A_i = \bigcup f^{-1}A_i$.

(iii) It is natural to define a slightly stronger concept of metrical transitivity. A metric may be defined by $d(A, B) := \mu(A \Delta B)$. One may demand every measurable set A such that $d(\tau(\gamma)A, A) = 0$ for all γ to be trivial, which explains the term ‘metrical transitivity’.

(iv) Another approach is to replace the measure space with a measure algebra. The introduction of a measure algebra is a coherent way to ignore sets of measure zero.

(v) Metrical transitivity of a single mapping $T : X \rightarrow X$ is understood in the sense given by considering the family $\{T\}$ consisting of one element. This is equivalent with metrical transitivity of the family $\{T(n) := T^n \mid n = 0, 1, \dots\}$. The set Γ is often a semigroup in applications, with a representation T .

(vi) The sets \mathcal{F}_I and \mathcal{F}_t of respectively invariant and trivial sets are σ -algebras. Metrical transitivity is the relation $\mathcal{F}_I \subset \mathcal{F}_t$.

Weak coupling of points far apart implies metrical transitivity.

Theorem 2.6.2 (Mixing and Metrical Transitivity) *Let (Ω, \mathcal{F}, P) be a probability space, let Γ be a countable infinite set, and let $\tau(\gamma) : \mathcal{F} \rightarrow \mathcal{F}$ be a σ -algebra homomorphism for all γ in Γ . The quadruple $(\Omega, \mathcal{F}, P, \tau)$ is a metrically transitive system if there exists an increasing sequence $\{\Gamma_n\}$ of finite sets, with $\bigcup \Gamma_n = \Gamma$, so that*

$$\lim_{n \rightarrow \infty} \frac{\sum_{\gamma \in \Gamma_n} P(A \cap \tau(\gamma)B)}{\#(\Gamma_n)} = P(A)P(B), \quad \forall A, B \in \mathcal{F}. \quad (2.12)$$

This holds in particular if $\lim_{n \rightarrow \infty} P(A \cap \tau(\gamma_n)B) = P(A)P(B)$ for all sequences $\{\gamma_i\}$.

Proof. Consider an invariant set $B = \tau(\gamma)B$ and set $A = B$. Then $P(B) = P(B)^2$, and B is P trivial. The last claim follows from an enumeration of the elements in Γ corresponding to the sequence Γ_n . \square

Remarks. (i) This theorem has a more interesting converse. Application of the Birkhoff ergodic theorem to the ergodic process $X_n = 1_B \circ T^n$, integration over A , and use of the dominated convergence theorem, implies (2.12) in the case $\{\tau(\gamma)\} = \{T^{-n} \mid n \in \mathbb{N}_0\}$.

(ii) Strong mixing is the property $\lim P(A \cap T^{-n}B) = P(A)P(B)$. Independence implies strong mixing implies ergodicity.

Metrical transitivity is usually hard to prove for a given system, and has correspondingly strong consequences. A main consequence of metrical transitivity is the characterization of the corresponding invariant functions as being the constants. This is not true without some restrictions on the functions under consideration, in particular on the range of the function.

Definition 2.6.3 (Invariant Measurable Functions) Let (X, \mathcal{F}) and (Y, \mathcal{G}) be measurable spaces, and let M be a family of σ -algebra homomorphisms on \mathcal{F} . A measurable function $f : X \rightarrow Y$ is said to be M -invariant if $f^{-1} = \tau \circ f^{-1}$ for all τ in M .

Remarks. (i) If $T : X \rightarrow X$ is measurable, and $f = f \circ T$, then f is invariant with respect to T^{-1} in the above sense. We will say that f is invariant with respect to T in this case.

(ii) If X is a measure space, then it is reasonable to interpret the equality as a.e. equality.

(iii) An invariant f is measurable with respect to \mathcal{F}_T .

(iv) The comment in (i) has a converse if the σ -algebra \mathcal{G} separates points. If $g^{-1}A = f^{-1}A$ for all A in \mathcal{G} , then $f = g$. This holds in particular for $g = f \circ T$.

(v) Observe the case where $f = 1_A$. It follows that f is invariant iff A is invariant.

Let $(\Omega, \mathcal{F}, P, \{T_\nu\})$ be a metrically transitive system. It is well known that a measurable function $f : \Omega \rightarrow \mathbb{R}$ is a constant iff it is invariant, that is, iff $f = f \circ T_\nu$ for all ν . For the proof one first notices that $f^{-1}(A)$ becomes an invariant set for any given $A \subset \mathbb{R}$. Next one defines the function $g(t) = P\{\omega \mid f(\omega) < t\}$. The invariance and measurability of f imply that $g(t) \in \{0, 1\}$. Because $g(t)$ clearly is monotone increasing, it must be the characteristic function of some interval (y, ∞) . The conclusion is that $f = y$ P-a.s.. We will need this theorem for more general invariant functions. The following gives a sufficient structure on the image space.

Definition 2.6.4 (Countable T_0 -separation) A set X is countably T_0 -separated iff there is a countable T_0 -separating family $\{F_n\}$ of subsets of X : Given any two different points $x, y \in X$, there is an F_n such that $x \in F_n, y \in F_n^c$ or $y \in F_n, x \in F_n^c$. A measurable space is σ -separated iff there exists a countable, T_0 -separating family of measurable sets.

We are now able to prove a generalization of the theorem concerning real valued, invariant functions.

Theorem 2.6.5 (Invariant Functions) Let $(X, \mathcal{F}, \mu, \Gamma)$ be a metrically transitive system and let Y be a σ -separated measure space. A Γ -invariant, measurable function $f : X \rightarrow Y$ is equal to a constant μ -a.e..

Proof. Define the image measure $\mu_f = \mu \circ f^{-1}$. Let $\{F_i\}$ be a countable family of T_0 -separating, measurable sets in Y . The invariance and metrical transitivity implies that the sets $f^{-1}(F_i)$ are μ -trivial. Define $Y_i = F_i$ or $Y_i = F_i^c$, such that $\mu_f(Y_i^c) = 0$ and set $Y_\infty := \bigcap_i Y_i$. Y_∞ is not empty, because $\mu_f(Y_\infty^c) = 0$. Y_∞ contains exactly one point x : Let $y \neq x \in Y_\infty$. We have two possibilities from the T_0 -separation. Assume that $x \in F_i, y \in F_i^c$ for some i . Then we have $F_i = Y_i$. This implies that $y \in Y_i^c \subset Y_\infty^c$. The other possibility is similar. The conclusion is that $Y_\infty = \{x\}$, and $f = x$ μ -a.e. \square

Remarks. (i) If Y is a second countable, T_0 -separated, topological space and f a measurable, ergodically invariant function, then the theorem applies. We choose $F = \{F_i\}$ to be a basis for Y given by second countability. This family is separating. Let $x \neq y$ and pick an open $U \ni x, y \notin U$. U is a union of sets from F and we have $x \in F_i \subset U$ for a suitable i and $y \in U^c \subset F_i^c$.

(ii) If $X = \mathbb{R} \cup \{\pm\infty\}$ we may take F to be the intervals $(-\infty, r)$ with $r \in \mathbb{Q}$ together with $\{\infty\}$.

(iii) The measure μ need not be σ -finite.

2.7 Measure Preserving Transformations and Ergodic Theorems

Now we turn to the concept of stationarity. A stochastic process is stationary if the finite dimensional distributions are independent of shifts. In statistical mechanics stationarity is Liouville's theorem. A stochastic process indexed by space variables is more properly denoted a stochastic field. The term homogeneous is then used in stead of stationary.

Definition 2.7.1 (Measure Preserving Transformations) *Let (X, \mathcal{F}, μ) be a measure space. A σ -algebra homomorphism $\tau : \mathcal{F} \rightarrow \mathcal{F}$ is measure preserving if $\mu \circ \tau = \mu$, and measure reducing if $\mu \circ \tau \leq \mu$, pointwise. A measurable mapping $T : X \rightarrow X$ is measure preserving or measure reducing if T^{-1} is measure preserving or measure reducing. An ergodic system is a metrically transitive system in which all the σ -algebra homomorphisms are measure preserving.*

Remarks. (i) We use the term "transformation" if T is bijective.

If we consider measurable functions, then we can generalize even further. Let T be a measurable mapping on X . T defines a linear transformation, also called T , on the function space by the formula $Tf(x) := f(Tx)$. Let τ be a measure reducing σ -algebra homomorphism, and set $T1_A := 1_{\tau A}$. Extension by linearity and continuity gives a linear contraction T on the Banach space $E = L^p(X)$, $p \geq 1$. A generalization of a measure reducing transformation is therefore a linear contraction on a Banach space E . In the function space case we may get back to our original concept by consideration of indicator functions, if indicator functions are mapped to indicator functions.

Any (strictly) stationary process on \mathbb{N}_0 may be represented in the form $X_n = T^n X_0$, with a measure preserving transformation T on a probability space Ω . If we only have an isometry T on $L^2(\Omega)$, then we get the concept of a weakly stationary process. This corresponds to the property $E[(X_i - E[X_0])(X_{i+j} - E[X_0])] = E[(X_0 - E[X_0])(X_j - E[X_0])]$.

The weak ergodic theorem is credited to J. von Neumann. It was published in 1932, but known prior to the Birkhoff ergodic theorem. The following is a more general theorem formulated in a Hilbert space.

Theorem 2.7.2 (The Weak Ergodic Theorem) *Let T be a linear contraction on a Hilbert space \mathcal{H} . Then*

$$M := N(I - T) = N(I - T^*) = R(I - T)^\perp, \quad (2.13)$$

and the projection P on M is given by the strong limit

$$P = s\text{-}\lim T_n, \quad T_n := \frac{1}{n} \sum_{k=1}^n T^k. \quad (2.14)$$

Proof. First we prove that $N(I - T) = N(I - T^*)$. Assume that $f \in N(I - T^*)$. $\|Tf\| \leq \|f\|$ gives

$$\|Tf - f\|^2 \leq 2\|f\|^2 - \langle Tf, f \rangle - \langle f, Tf \rangle = 0. \quad (2.15)$$

This proves $N(I - T^*) \subset N(I - T)$. The other inclusion follows from $T = T^{**}$ and the fact that T^* is also a contraction,

$$\|T^*f\|^2 = \langle TT^*f, f \rangle \leq \|T^*f\| \|f\|. \quad (2.16)$$

Now we have established $M := N(I - T) = N(I - T^*)$. $R(I - T)^\perp = N(I - T^*)$ follows from

$$\langle f, (I - T)g \rangle = \langle (I - T^*)f, g \rangle. \quad (2.17)$$

Any $x \in \mathcal{H}$ can be decomposed uniquely as $x = y + z$ with $y \in M$ and $z \in M^\perp = \overline{R(I - T)}$. We prove that $\lim T_n x = y$. Because $T_n y = y$ we are left with the proof of $\lim T_n z = 0$. We have $T_n(I - T) = (T - T^{n+1})/n$, so $\lim T_n(I - T)w = 0$. This proves the claim, since $\|T_n\| \leq 1$. \square

Remarks. (i) The theorem may be looked upon as a solution to the equation $Tf = f$. Such an equation may also be solved by the Banach fixed point theorem, with solution $f = \lim T^n g$, but there the assumption is that T is a (possibly nonlinear) strict contraction.

The probably most famous ergodic theorem is due to Birkhoff and dates back to 1931.

Theorem 2.7.3 (The Birkhoff Ergodic Theorem) *Let T be a measure preserving transformation on a probability space Ω . For any $X \in L^1(\Omega)$*

$$\bar{X}(\omega) := \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n X(T^k \omega) = E[X | \mathcal{F}_I](\omega), \quad (2.18)$$

where \mathcal{F}_I is the σ -algebra of all T invariant sets $A = T^{-1}(A)$. The limit holds both pointwise a.e. and in L^1 sense. If T is ergodic, we have $\bar{X} = E[X]$.

Proof. See the Notes section. □

Remarks. (i) The first part says that the a.e. pointwise limit that defines \bar{X} exists under the assumption of stationarity of $\{X \circ T^k\}$. The second part identifies \bar{X} with a conditional expectation.

2.8 Notes

Section 2.2 Definition 2.2.1 of a σ -algebra is equivalent with the definition in Rudin [115, p.8], and I follow his conventions whenever possible.

After my “discovery” of theorem 2.2.4, I found this theorem also in Dunford [46, vol. I, p.128-129]. Dunford is possibly my favorite reference on abstract measure theory, not because of readability, but because of completeness. Convergence of sets is also considered in Feller [57, p.105, volume II]. I have included this theorem because convergence of sets is of interest for random operators in other settings.

Lamperti [93, chap.1], Feller [57, chap.IV] [56], and Doob [43, chap.I] give excellent introductions to the theory of probability. Definitions 2.2.5 and 2.2.6 are standard. The definition of a measure algebra, in the remarks to Definition 2.2.6, is as in Petersen [112, p.15].

Section 2.3 Measurability with respect to an initial σ -algebra, Theorem 2.3.1, is standard.

The proof of Theorem 2.3.2, equality of weak and norm σ -algebras, is almost identical to the proof of the fact that every norm closed convex set is also weakly closed, compare Pedersen [110, p.66]. I acknowledge suggestions from O. Bratteli concerning the proof of this theorem. A related fact is that the Baire σ -algebra \mathcal{F}_B is equal to the Borel σ -algebra when X is a metric space, see Yamasaki [139, p.42-43].

The term “sequential completeness”, used in Theorem 2.3.3, is taken from Pedersen [110, p.228]. The first part of the theorem followed by addition of some details in the proof of a similar proposition found in [124, p.116, vol.I]. It may be possible to construct the C_k 's in a more general setting than for a metric space. A minimum demand is that the space is regular. For each $x \in U$ we must have disjoint open sets $A \ni x$, $B \supset \sim U$. It is worth noting that if a space is regular and second countable, then it is also metrizable Kelley [77, p.113, p.127]. A similar construction of closed C_k 's that approximate an open U from the inside is found in Kuratowski [92, vol. II, p.8, Corollary 4], and is relevant for set valued semicontinuous functions [92, vol. II, p.70, Theorem 1].

Yosida [140, p.131-132] has a different version of Pettis' theorem. It is weaker in the sense that the range space is assumed to be complete, but this extension is trivial. It is stronger since the measure is not assumed to be finite. Yosida [140, p.132, line 2-3] uses sequential completeness of strongly measurable functions without any assumption on the measure space, and without any proof. I needed some finiteness conditions on the measure for this, compare Theorem 2.3.4, to obtain approximate uniform convergence. Dunford [46, p.147-149, vol.I] has also finiteness conditions on the measure in order to prove an analogues theorem.

Section 2.4 An elementary, but not easy, proof of the strong law of large numbers, Theorem 2.4.2, is found in Lamperti [93, p.28-]. Lamperti also discusses many instructive variations of this theorem, in a most readable manner.

The construction of i.i.d. variables on $[0, 1)$ is well known.

Section 2.5 Lamperti [94], Feller [57], and Doob [43] consider ergodic processes. Ergodic theory from a more general point of view is treated by Petersen [112] and Sinai [37], but mostly (only?) the one parameter case. See [112, p.5-, p.49-] for more examples of ergodic systems. Our source for comments on the ergodic hypothesis is [112, p.42]. A short introduction to ergodic theory is also found in [124, vol. I: p.54-60, p.62-63].

Section 2.6 Definition 2.6.1 of metrical transitivity is more general than the one found in [109, p.3-], in [31, p.247], and in any of the other quoted sources. I choose this definition since it gives a sufficient assumption in (the possibly new) Theorem 2.6.5. I invented Theorem 2.6.5, [130], in a successful (!) attempt at a direct proof of the non-randomness of the spectrum of ergodic, linear, closed, random operators. I acknowledge useful discussions with H. Hanche-Olsen concerning the assumptions in the theorem. The theorem is obvious when all the assumptions are given. The problem was to determine suitable assumptions. The theorem is a generalization of Proposition 9.1 in Cycon [39, p.167] for the case $X = \mathbb{R}$. See also Sinai [37, p.14], Billingsley [24, p.13], Petersen [112, p.42], and Yamasaki [139, p.144] for this less general case.

Another approach to the problem is possible, with less general assumptions. One observes that an invariant function is measurable with respect to the σ -algebra of invariant sets, and invariance gives measurability with respect to the trivial σ -algebra. If f takes values in a separable, metric space, then it is the pointwise limit of simple functions. Because simple functions, measurable with respect to trivial sets, must be multiples of indicator functions, we conclude as in the theorem.

A proof of Theorem 2.6.2 (mixing and ergodicity) is found in Lamperti [94, p.95-], for the one parameter case.

Definition 2.6.4 is similar to the definition in Aversa [12, p.142]. This concept is also used to define a basis for a measure space, compare Sinai [37, p.449]. Christensen [36, p.6] defines countably separated to mean T_1 -separation, but this is equivalent in a measurable space. T_r -separation is defined as in Kelley [77, p.56, p.67, p.112].

Section 2.7 Stationarity in the sense of point and set transformations is discussed in Doob [43, p.452-464], and is recommended. Definition 2.7.1 of a measure preserving mapping is as in Lamperti [94, p.87], but he uses the term measure preserving transformation, and invertible transformation if it is invertible.

A proof of the weak ergodic theorem, Theorem 2.7.2, may be found in Petersen [112, p.24]. A Banach space version is found in Yosida [140, p.213]. It is tempting to define an operator U that satisfies the conclusion of the theorem to be a weak ergodic operator. An operator U such that $\lim U^n$ exists may likewise be called a strongly ergodic operator, see for instance reference 90e47005 in Mathematical Reviews. We will not follow these conventions. Since the theorem solves an eigenvalue problem, it has many rewarding corollaries. We mention one. Let H be a selfadjoint operator and consider the semigroup $U(s) = \exp(i(t - H)s)$. The strong limit as $T \rightarrow \infty$ of $(\int_0^T U)/T$ is the projection on $N(t - H)$. The eigenvalue problem $Hf = tf$ is solved and the theorem provides therefore a complete characterization of the point spectrum. A direct proof of this is found in Kato [75, p.517]. Wiener's theorem on time means of Fourier transforms and the RAGE theorem, Cycon [39, p.98-101], are strongly related to this. A state corresponding to the continuous spectrum of $-\Delta + V$ will, according to Cycon, "infinitely often leave" a ball with radius R .

A proof of the Birkhoff ergodic theorem is found in Petersen [112, p.30]. Many generalizations exist. Ergodic theorems for multidimensional subadditive processes are found in Akcoglu [3]. The proof is instructive, so I was tempted to include it.

Chapter 3

Topologies and σ -algebras on Hyperspaces

Some properties of topologies and σ -algebras on spaces of closed sets are studied. Upper and lower topologies are introduced and corresponding convergence concepts are discussed. We prove that the lower topology in the set of closed sets in a second countable space gives a second countable T_0 -space. Our discussion is motivated by the spectral theory of linear operators, but the topologies and σ -algebras introduced are most general.

3.1 Introduction

A formally random operator $T(\omega)$ gives the following set valued function:

$$\Omega \ni \omega \mapsto \Sigma(\omega) := \sigma(T(\omega)) \in \mathcal{P}_F(\mathbb{C}). \quad (3.1)$$

Ω is a probability space, $\sigma(T)$ is the spectrum of the operator T , and $\mathcal{P}_F(\mathbb{C})$ is the family of closed sets in the complex plane. Motivated by this we study topologies and σ -algebras on power sets in a more general situation. Topologies and σ -algebras on sets of unbounded linear operators may also be defined naturally in this more general setting. The concepts have many other applications.

We define pointwise convergence of a net of sets. This concept of convergence is derived from the corresponding pointwise convergence of indicator functions, and as such indicates several other possible definitions of convergence of sets.

We introduce the lower topology, the upper topology, and the compact upper topology. We obtain formulas for a set larger than any lower limit and for a set smaller than any upper limit. These sets are respectively also lower and upper compact limits. An explicit formula for the smallest upper limit is missing, and we do not even know if there exists a smallest upper limit. The Vietoris topologies, which are combinations of upper and lower topologies, are also defined.

Separation properties are discussed next, in particular countable T_0 -separation. We give formulas for a nonrandom random set in the case where the random set is either a lower or an upper (compact) semicontinuous function.

3.2 Pointwise Convergence

For completeness, we recall the basic definitions for point set topology and convergence.

Definition 3.2.1 (Topological Space) *A topology τ in a set X is a collection of subsets of X such that*

$$\begin{aligned} \{\emptyset, X\} &\subset \tau, \\ \{A, B\} &\subset \tau \Rightarrow A \cap B \in \tau \text{ (closed under finite intersections), and} \end{aligned} \quad (3.2)$$

$$F \subset \tau \Rightarrow \bigcup_{A \in F} A \in \tau \text{ (closed under arbitrary unions).}$$

A set A is τ -open if A is in τ . A set is τ -closed if the complement is τ open. A topological space is a set equipped with a topology.

Remarks. (i) It is sometimes convenient to refer to a topological space as an ordered pair (X, τ) .

Definition 3.2.2 (Net) *A net in a set X is a family $\{x_\lambda\}$ of elements from X with an upward-filtering preordered index set Λ*

$$\begin{aligned} u, v \in \Lambda &\Rightarrow \exists w \in \Lambda, u, v \leq w \text{ (upward-filtering),} \\ \lambda &\leq \lambda \quad \forall \lambda \in \Lambda \text{ (a reflexive relation), and} \end{aligned} \quad (3.3)$$

$$u \leq v, v \leq w \Rightarrow u \leq w \text{ (a transitive relation).}$$

Remarks. (i) It is sometimes convenient to refer to a preordered set as an ordered pair (Λ, \leq) . The preorder is an order if the relation is also antisymmetric, $u \leq v$ and $v \leq u$ imply $u = v$.

(ii) If the index set is the set of natural numbers with the usual order, then the net is a sequence. Nets are also known as generalized sequences.

Definition 3.2.3 (Convergent Net) *A net $\{x_\lambda\}$ in a topological space has a limit x if given any open set $U \ni x$, there is a λ_U such that $x_\lambda \in U$ when $\lambda \geq \lambda_U$. We write $x = \lim x_\lambda$, if x is a limit of the net $\{x_\lambda\}$. A net is convergent if it has a limit.*

Remarks. (i) We will use a prefix, as for instance in $x = \tau - \lim x_\lambda$, if we want to emphasize the topology.

(ii) A net may have many limits.

A set may be considered as a point in the space of all subsets of a given larger set. A possible convergence concept for a net of sets is given by pointwise convergence.

Definition 3.2.4 (Pointwise Convergence of Sets) *Let $\{A_\lambda\}$ be a net of sets. The pointwise upper and lower limit of this net is then*

$$\overline{\lim} A_\lambda := \bigcap_N \bigcup_{\lambda \geq N} A_\lambda, \text{ and } \underline{\lim} A_\lambda := \bigcup_N \bigcap_{\lambda \geq N} A_\lambda. \quad (3.4)$$

If the upper and lower limits are equal, then the net of sets is pointwise convergent. Any subset of the pointwise lower limit is by definition again a pointwise lower limit. Any superset of the pointwise upper limit is likewise a pointwise upper limit.

Remarks. (i) Pointwise convergence of a net of sets is equivalent with the pointwise convergence of the corresponding indicator functions. This proves that pointwise convergence corresponds to a topology.

(ii) The definition does not depend on any underlying structure on the set where the subsets are taken from. In particular the convergence definition does not depend on any topological structure.

(iii) Note the identity $\overline{\lim} A_\lambda = (\underline{\lim} (A_\lambda)^c)^c$.

(iv) If pointwise lower convergence comes from a topology, then the limits are non unique, and every net is pointwise lower convergent.

Any pointwise lower limit is contained in any pointwise upper limit of a given net of sets—a support of the naming conventions.

Theorem 3.2.5 (Upper and Lower Pointwise Limits) *If $\{A_\lambda\}$ is a net of sets, then*

$$\underline{\lim} A_\lambda \subset \overline{\lim} A_\lambda. \quad (3.5)$$

Proof. Let $x \in \underline{\lim} A_\lambda$. Then we have a λ_0 such that $x \in A_\lambda$ for all $\lambda \geq \lambda_0$. Chose an arbitrary λ_1 . Because the index set filtrates upwards, we can pick a λ_2 which dominates both λ_0 and λ_1 . In particular we have $x \in A_{\lambda_2}$ and so also $x \in \overline{\lim} A_\lambda$. \square

Other concepts of convergence of sets may be defined in a natural way from convergence of the indicator functions. One example of this is to go from pointwise convergence to pointwise convergence almost everywhere, with respect to some given measure. The set definition should then correspond to convergence if the symmetric difference of the upper and lower limits has measure zero. In this case we are dealing with convergence in a space of equivalence classes of measurable sets. Two sets are equivalent if their symmetric difference has measure zero.

3.3 Convergence of Closed Sets

Definition 3.3.1 (The Lower Topology) *Let (X, τ) be a topological space, and let Y be the closed sets in X . The lower topology τ_l in Y is the coarsest topology such that*

$$U_l := \{A \in Y \mid A \cap U \neq \emptyset\} \quad (3.6)$$

is open when U is open. We write $A = l\text{-}\lim A_\lambda$, if the net $\{A_\lambda\}$ converges towards A in the lower topology.

Remarks. (i) The lower topology is defined likewise in any set Y of subsets of a topological space. We will consider mostly $Y := \mathcal{P}_F(X)$.

Theorem 3.3.2 (Closure of a Union of Lower Limits) *The closure of an arbitrary union of lower limits of a given net of closed sets, is itself a lower limit for the net.*

Proof. The net $\{F_\lambda\}$ of closed sets converges towards F iff we can find a λ_U such that F_λ intersects U when $\lambda \geq \lambda_U$, for any open set U which intersects the closed set F . This is sufficient because any finite subset of the index set has a majorant, and a preorder is transitive. Let F be the closure of a union of limits of the net $\{F_\lambda\}$. If the closure of a set intersects an open set U , then the set itself intersects U , since the closure is the smallest closed set containing the set. We conclude that U also intersects the union of lower limits, so in particular it intersects one of the lower limits, and this gives the claim. \square

Remarks. (i) The first part of the proof is true for convergence in any given topology. It is sufficient to consider sets from a generator of the topology, to decide if a point is a limit of a net.

(ii) The closure of a set equals the intersection of the complements of open sets, disjoint from the set, and this gives the second part in the proof.

Definition 3.3.3 (Three Lower Limits) *Let $\{F_\lambda\}$ be a net of closed sets. We define*

$$L_c\text{-}\lim F_\lambda := \overline{\underline{\lim} F_\lambda} \text{ (the closure of the lower limit),} \quad (3.7)$$

$$L_i\text{-}\lim F_\lambda := \overline{\{x \mid \forall \lambda, \exists x_\lambda \in F_\lambda, x = \lim x_\lambda\}} \text{ (the individual limit) and,} \quad (3.8)$$

$$L_m\text{-}\lim F_\lambda := \bigcup_{F \in M} F \text{ (the maximal lower limit),} \quad (3.9)$$

where M is the set of F such that $F = l\text{-}\lim F_\lambda$.

Theorem 3.3.4 (Convergence in the Lower Topology) *A net of closed sets converges towards a closed set in the lower topology iff the closed set is a subset of the maximal lower limit. The empty set is a lower limit of any net of closed sets. The closure of the lower limit is contained in the individual lower limit, and*

$$L_c\text{-}\lim F_\lambda \subset L_i\text{-}\lim F_\lambda \subset L_m\text{-}\lim F_\lambda. \quad (3.10)$$

The closure of the lower limit may be a strict subset of the individual lower limit. The individual lower limit is the maximal lower limit for a sequence of closed sets in a first countable space, and the closure in the definition of the individual lower limit is superficial.

Proof. The maximal lower limit is a lower limit, since it is the closure of the union of all lower limits, and it is clearly maximal. The empty set is a closed subset of the maximal lower limit, so it is a lower limit. Let $x \in \underline{\lim} F_\lambda$, so there exist a λ_0 such that $x \in F_\lambda$ for all $\lambda \geq \lambda_0$. Define $x_\lambda = x$ if $\lambda \geq \lambda_0$ and pick $x_\lambda \in F_\lambda$ otherwise. This gives $x = \lim x_\lambda$ and $L_c\text{-}\lim F_\lambda \subset L_i\text{-}\lim F_\lambda$ - after closure. Now we prove that $F_L = L_i\text{-}\lim F_\lambda$ is a limit. Let U be open and intersecting F_L . Pick $x \in U$, with $x_\lambda \in F_\lambda$, and $x = \lim x_\lambda$. This implies existence of a λ_U such that $x_\lambda \in U$, and so also $F_\lambda \cap U \neq \emptyset$, for all $\lambda \geq \lambda_U$, and the individual lower limit is a lower limit. Let $x \in F$ and let U_i be a countable basis of decreasing open sets around x . We prove that $x = \lim x_n$, for some $x_n \in F_n$, when $F = l\text{-}\lim F_n$. Because F is a lower limit we have an increasing infinite sequence of integers N_i , such that F_n intersects U_i when $n \geq N_i$. Select $x_n \in F_n \cap U_i$ when $N_i \leq n <$

N_{i+1} , and $x_n \in F_n$ when $n < N_1$. This give $x_n \in U_i$ for all $n \geq N_i$, so $x_n \rightarrow x$, and $x_n \in F_n$. This proves that $L_m\text{-lim } F_n = L_i\text{-lim } F_n = \{x \mid \forall n, \exists x_n \in F_n, x = \lim x_n\}$, in particular also that the last set is closed, when the space is first countable. Let $x_i = 1/i$, so x_i converges towards 0 in the usual topology on the real line. Define $F_i = \{x_i\}$, so $L_c\text{-lim } F_i = \emptyset$, and $L_m\text{-lim } F_i = \{0\}$. \square

Remarks. (i) In view of the proof of the possibility of $L_i\text{-lim } F_n \neq L_c\text{-lim } F_n$, it is tempting to consider only closed sets without isolated points.

Definition 3.3.5 (The Upper Topology) Let (X, τ) be a topological space, and let Y be the closed sets in X . The upper topology τ_u in Y is the coarsest topology such that

$$U_u := \{A \in Y \mid A \subset U\} \quad (3.11)$$

is open when U is open. We write $A = u\text{-lim } A_\lambda$, if the net $\{A_\lambda\}$ converges towards A in the upper topology.

Theorem 3.3.6 (Intersection of Upper Limits) A finite intersection of upper limits of a net of closed sets in a normal topological space is an upper limit.

Proof. We prove that the intersection $F \cap G$ is an upper limit if $F = u\text{-lim } F_\lambda$ and $G = u\text{-lim } F_\lambda$. Let W be an open set containing $F \cap G$. The normality gives two disjoint open sets U and V , which separates the two disjoint closed sets $F \cap W^c (\supset U)$ and $G \cap W^c (\supset V)$. This gives $F \subset U \cup W$, $G \subset V \cup W$, and $W = (U \cup W) \cap (V \cup W)$. The convergence gives a λ_U such that $F_\lambda \subset U \cup W$ when $\lambda \geq \lambda_U$, a λ_V such that $F_\lambda \subset V \cup W$ when $\lambda \geq \lambda_V$, and finally a $\lambda_W (\geq \lambda_U, \lambda_V)$ such that $F_\lambda \subset W$ when $\lambda \geq \lambda_W$. \square

Remarks. (i) A metric space is normal.

Definition 3.3.7 (Upper Limits?) Let $\{F_\lambda\}$ be a net of closed sets. We define

$$U_c\text{-lim } F_\lambda := \overline{\lim F_\lambda} \text{ (the closure of the pointwise upper limit), and} \quad (3.12)$$

$$U_m\text{-lim } F_\lambda := \bigcap_{\lambda} \overline{\bigcup_{\mu \geq \lambda} F_\mu} \text{ (the modified upper limit).} \quad (3.13)$$

Theorem 3.3.8 (Upper and Lower Limits) If $\{F_\lambda\}$ is a net of closed sets, then

$$L_c\text{-lim } F_\lambda \subset L_i\text{-lim } F_\lambda \subset L_m\text{-lim } F_\lambda \subset U_c\text{-lim } F_\lambda \subset U_m\text{-lim } F_\lambda. \quad (3.14)$$

Proof. The first two inclusions are proved in Theorem 3.3.4. The last inclusion follows from $\bigcap_{\mu} \bigcup_{\lambda \geq \mu} F_\lambda \subset \bigcap_{\mu} \overline{\bigcup_{\lambda \geq \mu} F_\lambda}$, and closure. We assume $F = l\text{-lim } F_\lambda$ and prove $F \subset U_c\text{-lim } F_\lambda$. Let U be an open set which intersects F . It is sufficient to prove that U intersects $\overline{\lim F_\lambda}$. Let μ_1 be arbitrary, and let λ_U be such that F_λ intersects U when $\lambda \geq \lambda_U$. Let μ be a majorant for $\{\mu_1, \lambda_U\}$, so F_μ intersects U . Then there is an $x \in U \cap F_\mu$, and this x is also in $\overline{\lim F_\lambda}$. \square

Remarks. (i) The closure of the upper limit contains any lower limit.
(ii) The set $\overline{\bigcup_{\lambda \geq \mu} F_\lambda}$ contains any lower limit and is also an upper limit.

Theorem 3.3.9 (Convergence in the Upper Topology) *Assume that $F = u\text{-}\lim F_\lambda$ for a net $\{F_\lambda\}$ of closed sets in a topological space X . Any closed set containing F is also an upper limit, and X is in particular an upper limit. If F is given by the intersection of the closures of open sets containing F , then $U_m\text{-}\lim F_\lambda \subset F$. The modified upper limit is contained in any limit in the upper topology, if the space X is a metric space.*

Proof. The only open set containing X is X , so $X = u\text{-}\lim F_\lambda$. Any open set containing a superset of the limit contains the limit, and this gives the first claim. Let U be an open set containing F . Then there exists a λ_U such that $F_\lambda \subset U$ for all $\lambda \geq \lambda_U$. So we have $\bigcup_{\lambda \geq \lambda_U} F_\lambda \subset U$. The second assumption implies $U_m\text{-}\lim F_\lambda \subset F$, by taking intersections. Let \bar{F} be a closed set in a metric space. For any $x \in F^c$ chose an open ball U_x containing x and so that \bar{U}_x does not intersect F . Then $F \subset V_x := (\bar{U}_x)^c$, V_x is open, and F equals $\bigcap_{x \in F^c} \bar{V}_x = \bigcap_{x \in F^c} (U_x)^c$. \square

Remarks. (i) We do not claim that the modified upper limit is a limit in the upper topology.

(ii) It is not clear from the above if there exists a unique minimal upper limit.

(iii) If $F = u\text{-}\lim F_\lambda$ is a closed set in a metric space, then $F \supset \bigcap_\lambda F_\lambda$.

Definition 3.3.10 (The Upper Compact Topology) *Let (X, τ) be a topological space, and let Y be the closed sets in X . The upper compact topology τ_{uc} in Y is the coarsest topology such that*

$$U_u := \{A \in Y \mid A \subset U\} \tag{3.15}$$

is open when U is open and U^c is compact. We write $A = \tilde{u}\text{-}\lim A_\lambda$, if the net $\{A_\lambda\}$ converges towards A in the upper compact topology.

Remarks. (i) If X is a Hausdorff space, then U is open if U^c is compact.

Theorem 3.3.11 (Convergence in the Upper Compact Topology) *Let $\{F_\lambda\}$ be a net of closed sets in a topological space X . If F is an upper limit of $\{F_\lambda\}$, then F is also an upper compact limit, in particular $X = \tilde{u}\text{-}\lim F_\lambda$. Any closed superset of a limit in the upper compact topology is itself an upper compact limit. The modified upper limit is an upper compact limit. Assume $F = \tilde{u}\text{-}\lim F_\lambda$ to be the intersection of the closure of open sets containing F , with the restriction that each of these open sets have compact complement. Then $U_m\text{-}\lim F_\lambda \subset F$. The modified upper limit is the minimal upper compact limit, if the space X is a locally compact metric space.*

Proof. The proof is almost identical with the proof of Theorem 3.3.9. We prove that $F := U_m\text{-}\lim F_\lambda$ is an upper compact limit. Set $G(\lambda) := \overline{\bigcup_{\mu \geq \lambda} F_\mu}$. Let U^c be compact and so that $U \supset F = \bigcap_\lambda G(\lambda)$, and so also $U^c \subset \bigcup_\lambda G(\lambda)^c$. The compactness gives $\lambda_1, \dots, \lambda_n$ such that $\bigcap_{i=1}^n G(\lambda_i) \subset U$. Chose a $\lambda_U \geq \lambda_1, \dots, \lambda_n$, so $G(\lambda_U) \subset U$. Finally we conclude that $F_\lambda \subset U$ for all $\lambda \geq \lambda(U)$. \square

Remarks. (i) This gives an essential difference between the upper and the upper compact topology. The upper compact topology is weaker than the upper topology, and this gives that the modified upper limit is an upper compact limit, but possibly not an upper limit.

(ii) A closed set is an upper compact limit iff it contains the modified upper limit, when X is a locally compact metric space.

(iii) If $F = \tilde{u}\text{-}\lim F_\lambda$ is a closed set in a locally compact metric space, then $F \supset \bigcap_\lambda F_\lambda$.

Definition 3.3.12 (The Vietoris Topology) Let Y be the set of closed sets in a topological space. The Vietoris topology τ_v in Y is the coarsest topology containing the lower topology and the upper topology. The weak Vietoris topology $\tau_{\tilde{v}}$ is the coarsest topology containing the lower topology and the upper compact topology. Corresponding limits will be denoted by $v\text{-}\lim F_\lambda$ and $\tilde{v}\text{-}\lim F_\lambda$.

Theorem 3.3.13 (The Vietoris Topology is Hausdorff) Let Y be the set of closed sets in a regular topological space X . The Vietoris topology in Y is a Hausdorff topology. If X is also locally compact, then the weak Vietoris topology in Y is a Hausdorff topology.

Proof. Let F, G be two closed sets in X , with a point $x \in F \cap G^c$. Since X is regular, there exists an open set U containing x , and with $\bar{U} \subset G^c$. If X is locally compact, we may ensure that \bar{U} is compact. The two open sets $U_l, ((\bar{U})^c)_u$ are disjoint and separates F, G . \square

Remarks. (i) A metric space is regular.

3.4 Countable T_0 -Separation

Theorem 3.4.1 (T_0 -separation) Let Y be the set of closed sets in a topological space X . The lower topology in Y is a T_0 -topology. If X is a T_1 -space, then the upper topology and the upper compact topology are T_0 -topologies. The lower, the upper compact, and the upper topologies are not T_1 -topologies, if X contains a nonempty closed proper subset.

Proof. Let F, G be two closed sets in the topological space, with an $x \in F \cap G^c$. The set $(G^c)_l$ is open in the lower topology, contains F but not G . This proves that the lower topology is a T_0 -topology. Assume now that the space X is a T_1 space, so that the set $\{x\}$ is a closed set. The set $(\{x\}^c)_u$ is open both in the upper compact topology and the upper topology, contains G , but not F , so we have T_0 -separation. The closed set X is an element in any nonempty lower open set. A closed nonempty proper subset of X can then not be separated from X with a set in the lower topology. The set of closed sets is the only upper open set which contains X , so X can not be separated from any other closed set in the upper topology. \square

Remarks. (i) The empty set is an element in any nonempty upper open set.

(ii) The set of closed sets is the only lower open set which has the empty set as an element.

The result for the lower topology motivates a look on a trivial example. Let $X = \{0, 1\}, \tau = \{X, \emptyset\}$, so X is not a T_0 -space. We get $Y = \mathcal{P}_F = \tau$ and $\tau_l = \{Y, \emptyset, \{X\}\}$. In essence the procedure suggests the topology $\tau_0 = \{\{0, 1\}, \emptyset, \{0\}\}$ for X , which is a T_0 -topology, but not a T_1 -topology.

Theorem 3.4.2 (The Lower Topology is Second Countable) The set of closed sets in a second countable space is second countable in the lower topology.

Proof. Let B be a basis for the topology in X . Let \mathcal{P}_F be the set of closed sets in X . Let B_l be the set of lower open sets in \mathcal{P}_F given by

$$[U_1, \dots, U_n] = \{F \in \mathcal{P}_F \mid \forall i F \cap U_i \neq \emptyset\}, \quad (3.16)$$

for finite subsets $\{U_1, \dots, U_n\} \subset B$. The set B_i is a countable basis for the lower topology, since $U = \bigcup_i U_i$ implies

$$\{F \mid F \cap U \neq \emptyset\} = \bigcup_i \{F \mid F \cap U_i \neq \emptyset\}. \quad (3.17)$$

□

Theorem 3.4.3 (Separating Lower Open Sets) *If Y is the set of closed sets in a second countable topological space, then Y is a second countable T_0 -separated space. The Borel field in Y gives Y as a σ -separated measurable space.*

Proof. Second countability is inherited and the lower topology is always T_0 -separating. The Borel field of a second countable T_0 -space gives a σ -separated measurable space. □

Theorem 3.4.4 (Separating Upper Compact Open Sets) *If Y is the set of closed sets in a σ -compact, separable, metric space, then Y is T_0 -separated by a countable family of open sets from the upper compact topology in Y .*

Proof. Let K_1, K_2, \dots be compact increasing sets, whose union is the metric space. Let D_1, D_2, \dots be the sets given by the union of one K_j^c and a finite union of open balls with rational radius and center in a countable dense set. If $U_i := \{F \in Y \mid F \subset D_i\}$, then $\{U_i\}$ is a countable T_0 -separating family in Y . Countability of $\{D_i\}$ follows from countability of $\{J \subset \mathbb{N} \mid J \text{ is finite}\}$. Let $F \neq G$ be closed sets with an $x \in F \cap G^c$. Set $\epsilon := d(x, G)$. Chose a positive rational $r < \epsilon$, and a K_i containing $B := \{y \mid d(x, y) < \epsilon\}$. Let D be the countable dense set. Cover $\hat{K} := K_i \setminus B$ with open balls with center in $D \cap \hat{K}$ and radius r . Compactness gives a finite subcover. Let D_j be the union of this finite subcover together with K_i^c . We have $G \subset D_j$ and $F \not\subset D_j$, because $x \notin D_j$. We conclude that $G \in U_j$ and $F \notin U_j$, and T_0 -separation is demonstrated. □

The last theorem can be simplified in a more general setting if we consider the space $Y = \mathcal{P}_e(X)$, consisting of closed sets equal to the closure of their interior. The subscript e indicates the word essential. Assume that the space X is a T_1 space with a countable dense set D . $\{(\{x\}^c)_u\}$, with $x \in D$, is a countable family of sets from τ_{uc} that T_0 separates Y .

3.5 Random and Nonrandom Random Sets

Definition 3.5.1 (Random Set) *A closed, formally random set is a mapping from a probability space into the set of closed sets in a topological space. A closed, formally random set Σ is random if $\{\omega \mid \Sigma(\omega) \cap U \neq \emptyset\}$ is measurable for any open set U .*

Remarks. (i) $\{\Sigma \cap U \neq \emptyset\}$ is an event if Σ is a closed random set.

(ii) A closed random set is a Borel measurable mapping from a probability space into the set of closed sets, equipped with the lower topology.

Theorem 3.5.2 (Events from Closed Random Sets) *If Σ is a compact, random set in a metric space, then $\{\Sigma \subset U\}$ is an event when U is open. If Σ is a closed, random set in a metric space, then $\{\Sigma \subset U\}$ is an event when U^c is compact. If the metric space is σ -compact, then $\{\Sigma \subset U\}$ is an event when U is open.*

Proof. Let U be open and set $U_i = \{x \mid d(x, U^c) > 1/i\}$, so $\overline{U_i} \subset U_{i+1}$ and $U = \bigcup_i U_i$. If Y is a set of closed sets, then

$$\begin{aligned} U_u & := \{F \in Y \mid F \subset U\} = \{F \in Y \mid F \subset \bigcup_i \overline{U_i}\} \\ & \supset \bigcup_i \{F \in Y \mid F \subset \overline{U_i}\} = \bigcup_i \{F \in Y \mid F \cap (\overline{U_i})^c \neq \emptyset\}^c. \end{aligned} \quad (3.18)$$

If we consider only compact sets, then the first claim follows because the \supset can be reversed. This claim comes from $F \subset \bigcup_{i=1}^{N_F} U_i \subset \overline{U_{N_F}}$, which follows from the compactness of F . The second claim follows likewise because $d(F, U^c) > 0$, when U^c is compact. If the space X is σ -compact, then the upper open sets are σ -algebra generated by the lower open sets, compare:

$$\begin{aligned} \{F \in Y \mid F \subset U\} & = \{F \in Y \mid F \cap U^c \neq \emptyset\} = \{F \in Y \mid F \cap (\bigcup_i F_i) \neq \emptyset\} \\ & = \bigcup_i \{F \in Y \mid F \cap F_i \neq \emptyset\} = \bigcup_i \{F \in Y \mid F \subset (F_i)^c\}^c \\ & = \bigcup_i \bigcap_j \{F \in Y \mid F \cap (\overline{U_{ij}})^c \neq \emptyset\}. \end{aligned} \quad (3.19)$$

We used that any closed set U^c can be written as a union of compact sets F_i , and the previous construction. \square

- Remarks.* (i) The generators of the upper compact topology are F_σ sets, relative to the lower topology in the set of closed sets in a metric space.
(ii) The generators of the upper topology are $G_{\sigma\delta}$ sets, relative to the lower topology in the set of closed sets in a σ -compact, metric space.
(iii) Possibility of the representation $U = \bigcup_i U_i = \bigcup_i \overline{U_i}$ can possibly give generalizations beyond metric spaces.
(iv) It is easy to construct two closed (noncompact !) sets in the plane which are disjoint, but not separated by a finite distance.

Theorem 3.5.3 (Upper Measurability Implies Lower Measurability) *Let Y be the set of closed sets in a topological space, where every open set is a countable union of closed sets. Let Σ be a function from a probability space into Y . If $\{\Sigma \subset U\}$ is an event when U is open, then Σ is a closed, random set. If the topological space is σ -compact and $\{\Sigma \subset U\}$ is an event when U^c is compact, then Σ is a closed, random set.*

Proof. With the representation $U = \bigcup_i F_i$ we find:

$$U_l := \{F \mid F \cap (\bigcup_i F_i) \neq \emptyset\} = \bigcup_i \{F \mid F \cap F_i \neq \emptyset\} = \bigcup_i \{F \mid F \subset F_i^c\}^c \quad (3.20)$$

The second part in the claim holds because we can chose compact F_i 's. \square

Remarks. (i) The generators of the lower topology are F_σ sets relative to the upper topology, if every open set is a F_σ set.

Definition 3.5.4 (Continuous Set Valued Mappings) *A mapping from a topological space into the set of closed sets in a topological space is said to be lower semicontinuous, upper semicontinuous, upper compact semicontinuous, (Vietoris) compact continuous, or (Vietoris) continuous if it is continuous with respect to respectively the lower topology τ_l , the upper topology τ_u , the upper compact topology τ_{uc} , the modified Vietoris topology τ_{vc} , or the Vietoris topology τ_v .*

The following result are motivated by the study of ergodic Schrödinger operators. One of the main features of these operators is that the spectrum and some of its components are nonrandom, despite the randomness of the operator. We are addressing the question that naturally comes up: Can we determine the nonrandom set? The following theorem gives explicit formulas.

Definition 3.5.5 (Nonrandom Set) *A closed random set Σ is nonrandom if there exists a fixed closed set Σ_0 such that $P(\Sigma = \Sigma_0) = 1$.*

Theorem 3.5.6 (Locating a Nonrandom Random Set) *Let Σ be a function from a Borel probability space into the set of closed sets in a topological space. Assume that the probability measure has a topological support $\text{supp } P$ with $P(\text{supp } P) > 0$. Let D be a subset of $\text{supp } P$ such that $P(\overline{D}) > 0$. Assume that Σ is nonrandom. If $\omega \mapsto \Sigma(\omega)$ is lower semicontinuous, then*

$$\Sigma = \bigcup_{\omega \in \text{supp } P} \Sigma(\omega) = \overline{\bigcup_{\omega \in D} \Sigma(\omega)}, \quad (3.21)$$

with probability one. If $\omega \mapsto \Sigma(\omega)$ is upper semicontinuous into the closed sets in a metric space, then

$$\Sigma = \bigcap_{\omega \in \text{supp } P} \Sigma(\omega) = \bigcap_{\omega \in D} \Sigma(\omega), \quad (3.22)$$

with probability one.

Proof. Set $\Omega_0 = \{\omega \mid \Sigma(\omega) = \Sigma_0\}$ for the closed set Σ_0 which gives $P(\Omega_0) = 1$.

Assume lower semicontinuity. Let $\omega_0 \in \text{supp } P$. Chose $\omega_U \in U \cap \Omega_0$ for all open $U \ni \omega_0$, so $\omega_0 = \lim \omega_U$. Lower semicontinuity gives $\Sigma(\omega_0) = l\text{-}\lim \Sigma(\omega_U)$ and $\Sigma_0 = \overline{\bigcup_U \Sigma(\omega_U)} \supset \Sigma(\omega_0)$, so $\bigcup_{\omega \in \text{supp } P} \Sigma(\omega) \subset \Sigma_0$. The reverse inclusion follows from $\text{supp } P \cap \Omega_0 \neq \emptyset$.

Upper semicontinuity gives $\Sigma(\omega_0) = u\text{-}\lim \Sigma(\omega_U)$, so $\Sigma(\omega_0) \supset \bigcap_U \Sigma(\omega_U) = \Sigma_0$, and $\bigcap_{\omega \in \text{supp } P} \Sigma(\omega) \supset \Sigma_0$. The reverse inclusion follows from $\text{supp } P \cap \Omega_0 \neq \emptyset$.

Let ω be in \overline{D} . Since D is dense in \overline{D} there is ω_λ in D with $\omega = \lim \omega_\lambda$. Lower semicontinuity gives $\Sigma(\omega) \subset \overline{\bigcup_\lambda \Sigma(\omega_\lambda)}$ and upper compact semicontinuity gives $\Sigma(\omega) \supset \bigcap_\lambda \Sigma(\omega_\lambda)$. Both cases imply the claims. \square

Remarks. (i) The proof did only rely on the relations $l\text{-}\lim F_\lambda \subset \overline{\bigcup_\lambda F_\sigma}$ and $\tilde{u}\text{-}\lim F_\lambda \supset \bigcup_\lambda F_\sigma$, so the continuity assumptions in the theorem are too strong. The last formula holds for closed sets in a σ -compact, metric space, if Σ is upper compact semicontinuous.

(ii) The set $\text{supp } P$ may be complicated, and a strengthening of this theorem is often needed in concrete applications.

(iii) The set D may be nonmeasurable or it may have zero measure.

(iv) The mapping Σ may be the identity mapping.

In the above theorem we have the assumption $P(\text{supp } P) > 0$, which may seem strange. $\text{supp } P$ is a closed set: Let $x_\lambda \rightarrow x$, with $x_\lambda \in \text{supp } P$. Let U be open and containing x . The convergence gives a $x_{\lambda_U} \in U$, so $P(U) > 0$, so $x \in \text{supp } P$, so $\text{supp } P$ is closed. $\text{supp } P$ is contained in the intersection of all closed sets F with $P(F) = 1$: F^c is open and contains

any point from F^c . Since $P(F^c) = 0$, we have $F^c \subset (\text{supp } P)^c$ and $\text{supp } P \subset F$. So far we may be in a situation where $\text{supp } P = \emptyset$. To prove that $P(\text{supp } P) = 1$, we seem to need some additional requirement. Assume that each union of open sets has a countable refinement. Then we have $P(\text{supp } P) = 1$: Around each point x in $(\text{supp } P)^c$ we have an open set U_x , contained in $(\text{supp } P)^c$ and with $P(U_x) = 0$. Since we have a countable refinement we have $(\text{supp } P)^c = \bigcup_i U_i$ with $P(U_i) = 0$, so we have $P(\text{supp } P) = 1$. The above assumption is satisfied in a second countable topological space and therefore also in a separable metric space. First countability and separability do not seem to be sufficient.

A natural question is the characterization of $\text{supp } P$. If we let $(X_i), i \in \mathbb{N}$, be a real valued stochastic process, then we get a natural measure P on the direct product $\Omega = \mathbb{R}^{\mathbb{N}}$. If the variables are i.i.d., then we get $\text{supp } P = (\text{supp } P_0)^{\mathbb{N}}$. P_0 is the common distribution of the variables X_i . Can we get a similar formula in a more general situation, say with the i.i.d. assumption replaced by ergodicity?

3.6 Concluding Remarks

We have discussed some properties of topologies and σ -algebras on power sets. Our discussion is motivated by corresponding natural questions in the spectral theory for random operators. There are many other natural questions that comes up from a more general point of view. We have not attempted to give a general systematic introduction to the subject. It is nonetheless our hope that the reader can recognize that the subject is of some interest in its own right.

The separate study of the upper compact and lower topologies is motivated by continuity properties of the spectrum of an operator. Let $T_\lambda \rightarrow T$. Let $F_\lambda := \sigma(T_\lambda)$ and $F := \sigma(T)$. If the convergence is in the gap topology, then our results give $F \supset U_m\text{-}\lim F_\lambda$, and $F = U_m\text{-}\lim F_\lambda$ is possible. If we restrict the gap topology to bounded operators we obtain the more common norm topology. If the convergence is for a sequence of selfadjoint operators in the strong resolvent topology, then we have obtained $F \subset L_i\text{-}\lim F_n \subset U_m\text{-}\lim F_n$.

The results for the topologies and the separation properties resolve similar problems for the spectrum of random operators. Of particular interest are the formulas for the nonrandom sets. The case of lower semicontinuity has been applied for the determination of the spectrum of an ergodic selfadjoint operator. The case of upper semicontinuity has not yet found an application, but it suggests a way to attack the corresponding problem for parts of the spectrum that are believed to shrink when $\text{supp } P$ increases. An example is the absolutely continuous spectrum.

A first course in point set topology includes generic methods of constructing new topological spaces from old ones, the prototype being the initial or final topology of a given set of functions. The construction of topologies on power sets provides another generic method.

3.7 Notes

The concept of topologies for spaces of sets, also called power sets or hyperspaces, is certainly not a new one. The first steps in this direction was perhaps taken by F. Hausdorff [67, §28] and L. Vietoris [135], which give rise to corresponding names on these topologies.

More recently the concepts have found applications in connection with image processing [15].

A standard reference for topologies on power sets is [103], which is hereby recommended for further reading. Books on general topology has little about the subject, but there are two notable exceptions: Kuratowski [92] and Bourbaki [29]. In Bourbaki the subject is covered in the following exercises: I:2.7 3.10 3.16 5.7 8.12 8.29 9.13 9.14 10.20 11.18 11.24; II: 1.5 2.6 3.7 4.11 4.15; IX: 2.6 2.21; X: 1.7. In the above I:2.7 means exercise 7 in § 2 and chapter I and so on.

In [32] the Hausdorff topology, which is Hausdorff, is treated, starting from the concept of uniform or metric topologies.

The treatment of the upper and lower topologies separately does not seem to be a standard approach, maybe because they so explicitly give non-Hausdorff spaces. The topologies we have described are related to the concept of order topologies [77, p.57, ex.I], [62]. Upper and lower limits of sequences of sets are also mentioned briefly in [71, p.100-103], in connection with connected sets.

Set valued functions are also called multifunctions by some authors. Semicontinuity and measurability of multifunctions are the subject in [98]. Measurability properties of spectra of bounded operators are discussed in [95]. [12] treats separation properties for hyperspaces. [36] contains material on topologies and σ -algebras on power sets.

The general notions of topological spaces and nets are used (almost) as defined in Pedersen [110].

Any topological space may be defined entirely from the concept of converging nets in the space. The idea is that a closure operation is defined from the limits of nets. There are restrictions on what sets of converging nets are to satisfy. A constant net must for instance always converge to the constant. The other three sufficient and necessary conditions are found in Willard [138, p.77, ex. 11D].

In Definition 3.2.4, pointwise convergence of sets, it seems reasonable that the lower and the upper pointwise convergence concepts come from two topologies. I have just not found time to check this out.

The naming conventions are my own, by necessity, since I have not found a separate treatment of the lower topology elsewhere. I do not know if the individual lower limit always is the maximal lower limit, or if the closure is needed in the definition of the individual lower limit.

Convergence of closed sets, in the sense given by the individual lower limit, is used in Weidmann [136, p.272] to describe a perturbation result for the spectrum, and the essential spectrum, of selfadjoint operators. Related problems caused my interest in the consideration of convergence of closed sets.

Theorem 3.3.6 gives that the intersection of two upper limits is an upper limit, if the space is normal. More on normal spaces is found in Kelley [77, p.112, 120, 127,132-134], in particular the claim on normality of a pseudo-metric space.

The upper compact topology, with the corresponding modified Vietoris topology, is perhaps introduced here for the first time. Kato [75, p.209, line 3-4] refers to the corresponding continuity of set valued functions by "... $\Sigma(T)$ is upper semicontinuous in a slightly weaker sense." The continuity of the mapping which maps a closed operator to its spectrum

motivated a closer look at the upper compact topology.

[75, p.208, Theorem 3.1] can be consulted to verify the claims about the spectrum of closed operators equipped with the gap topology. The spectrum is an upper compact mapping, and this is why we considered the upper compact topology in the first place. It turned out that this topology has nice properties.

The proof for the nonrandom set, in the case of lower semicontinuity, is abstracted from [79, p.305].

Chapter 4

Lebesgue-Hammer Decomposition of Random Measures

We consider random measures, motivated by the spectral theory of random Schrödinger operators. The randomness of the Lebesgue-Hammer parts of a random measure is proven. Several topologies in a set of measures are considered, including in particular the pointwise topology and the weak topology. Finally we consider how decompositions behave under perturbations.

4.1 Introduction

Let H be a random selfadjoint operator. Any vector f in the Hilbert space gives a random measure ρ_f ,

$$\rho_f(A) := \langle f, 1_A(H)f \rangle. \quad (4.1)$$

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis in the Hilbert space. A spectral measure for H is then given by

$$\rho(A) = \sum_{n=1}^{\infty} \rho_{e_n}(A)/2^n, \quad (4.2)$$

again a random measure. We mention also the density of states measure, also a random measure given from a random Schrödinger operator. There has been considerable interest in the spectral decomposition of random Schrödinger operators H , given by the Lebesgue-Hammer decomposition of the spectral measure. Motivated by this we consider Lebesgue-Hammer decomposition of random measures.

We define the concept of random measures, and give a slightly simpler condition for the randomness of a finite measure on a countably generated σ -algebra.

The Hammer and the Lebesgue decompositions of a measure are defined. The Hammer decomposition represents a σ -finite measure as an orthogonal sum of an atomic measure and a continuous measure. We prove that the atomic and the continuous part of a finite random measure on a countably generated σ -algebra are random measures. Let m be a

reference measure. Any measure μ is then the unique sum of an m -continuous part μ_{ac} and an m -singular part μ_s . This is the Lebesgue m -decomposition of μ . We prove that λ_{ac} and λ_s are random measures, if λ is a finite measure on a countably generated σ -algebra. We prove that an atom may be represented by a single point in a σ -separated space. The section closes with an example of a family of singular continuous measure with topological support equal to $[0, 1]$.

Several topologies for sets of measures are defined. We state and prove the Vitali-Hahn-Saks theorem for the pointwise convergence of a sequence of complex measures. Weak and vague convergence are compared. We recall some necessary and sufficient conditions for weak convergence of a sequence of finite Radon measures on a metric space.

The behavior of the spectral decomposition of a measure under perturbation is best described as “wild”. First we prove denseness of the pure point measures \mathcal{M}_p in the product topology. This means that a pure point measure may easily appear if any measure is perturbed slightly. On the other hand \mathcal{M}_p is closed in the norm topology and also sequentially closed in the product topology. Given a continuous reference measure we prove that the set of absolutely continuous measures is a closed set with empty interior in the product topology. Then the spectrum of a measure is defined to be the topological support of the measure. We prove that the Lebesgue-Hammer parts of the spectrum only depend on the measure class of the measure. Finally we give conditions which ensure the randomness of the Lebesgue-Hammer parts of the spectrum of a random measure.

4.2 Random Measures

Definition 4.2.1 (Random Measures) *A random measure μ is a mapping from a probability space into the set of measures on a given σ -algebra, so that $\mu(A)$ is a random number in $[0, \infty]$ for all measurable sets A .*

Remarks. (i) Complex random measures, finite random measures, and random probability measures are defined likewise.

Let E denote a suitable set of abstract measures, e.g. the space of complex measures on a given σ -algebra \mathcal{F} . A σ -algebra \mathcal{G} may be defined on E as the initial σ -algebra of the family of mappings given by $E \ni \mu \mapsto \mu(A)$, $A \in \mathcal{F}$. The E is then a measurable space of measures. The definition of a random measure coincide with the definition of E valued random variables, as given by \mathcal{G} . We note that the image measure of a random measure realizes the space of measures as a probability space.

It is sufficient to consider measurability on a generating algebra.

Theorem 4.2.2 (Finite Random Measures) *A finite formally random measure μ is a random measure iff $\mu(A)$ is a random number for all A in an algebra of sets which generate the σ -algebra.*

Proof. Let \mathcal{F} be the σ -algebra generated by the algebra \mathcal{A} . Define \mathcal{G} to be the set of measurable A , such that $\mu(A)$ is a Borel random variable. The assumption is that $\mathcal{A} \subset \mathcal{G}$. Let A_n be a monotonely decreasing sequence from \mathcal{G} . Because μ is finite, we get $\mu(\bigcap_n A_n) = \lim_n \mu(A_n)$, which is a Borel random variable, so $\bigcap_n A_n \in \mathcal{G}$. Let A_n be a

monotonely increasing sequence from \mathcal{G} . We get $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$, which is a Borel random variable, so $\bigcup_n A_n \in \mathcal{G}$. The conclusion is that \mathcal{G} is a monotone class and $\mathcal{G} \supset \mathcal{F}$, which gives the claim. \square

4.3 Decomposition of Measures

In this section we will study decomposition of measures associated with the names Hammer and Lebesgue. We will in particular give explicit formulas which determine the decompositions.

Definition 4.3.1 (Atom) *A measurable set is an atom if it is not empty and the empty set is the only measurable subset. Let μ be a measure. A measurable set A is a μ -atom if $\mu(A) \neq 0$ and if $F \subset A$ implies $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure μ is continuous, or equivalently it is nonatomic, if there are no μ -atoms. A measure μ is atomic if any measurable set F with $\mu(F) > 0$ contains a μ -atom.*

The Dirac measure δ_x is defined by $\delta_x(A) = 1_A(x)$. If $\{x\}$ is measurable, then it is a δ_x -atom and also an atom. The usual Lebesgue measure on $[0, 1]$ is continuous.

Assume that $\mu(F) < \infty$ and that F does not contain any μ -atom. Then, given any $\epsilon > 0$, there is a set $A \subset F$ such that $0 < \mu(A) < \epsilon$. Furthermore, given any $0 \leq c \leq \mu(F)$, there is a set $A \subset F$ with $\mu(A) = c$, which follows from an application of Zorn's lemma. These properties justify the definition of a continuous measure as being a nonatomic measure.

If \mathcal{F} is the discrete σ -algebra of an uncountable set and we consider the measure given by $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ otherwise, then we have an uncountable number of μ -atoms with measure ∞ . This is in some sense the worst possible case, as indicated by the following theorem.

Theorem 4.3.2 (Number of Atoms) *Let μ be a σ -finite measure. Any family of disjoint sets with positive μ -mass is countable. Any family of disjoint μ -atoms is countable. A μ -atom has finite mass.*

Proof. Let E be a set with finite measure and let \mathcal{A} be a family of disjoint subsets of E with positive measure. We prove that \mathcal{A} is countable. The set $\{A \in \mathcal{A} \mid \mu(A) > \epsilon\}$ is finite for any given $\epsilon > 0$, since otherwise we would get $\mu(E) = \infty$. This implies that $\mathcal{A} = \bigcup_n \{A \in \mathcal{A} \mid \mu(A) > 1/n\}$ is countable.

Now let \mathcal{A} be a family of disjoint sets with positive measure. We will prove that \mathcal{A} is countable. Since μ is σ -finite, the space is a countable union of disjoint sets K_i with finite measure. The previous result gives that $\mathcal{A}_i = \{A \in \mathcal{A} \mid \mu(A \cap K_i) > 0\}$ is countable. The relation $A = \bigcup_i (A \cap K_i)$ and the countable additivity proves $\mathcal{A} = \bigcup_i \mathcal{A}_i$, which is countable. In particular we also have that any family of distinct μ -atoms is countable, since we may assume that the atoms are disjoint, after a modification with zero sets.

We finally prove that a μ -atom has finite mass. Assume that A is a set with infinite mass. The σ -finiteness implies that we can find disjoint sets K_1, K_2, \dots with finite mass such that $A = \bigcup_j K_j$. The countable additivity gives a j such that $0 < \mu(K_j) < \infty$, which implies that A is not a μ -atom. \square

We will abuse the language slightly and speak of “the” family of atoms. It is only unique when we consider the family of equivalence classes $[A]$ of atoms, two atoms A and B being equivalent when $\mu(A \triangle B) = 0$. If A and B are μ -atoms, then we have either $\mu(A \triangle B) = 0$ or $\mu(A \cap B) = 0$. In the last case $[A]$ and $[B]$ may be represented by $\tilde{A} = A$ and $\tilde{B} = B \setminus A$, two disjoint μ -atoms.

Definition 4.3.3 (Hammer Decomposition) *Let $(\Omega, \mathcal{F}, \mu)$ be a σ -finite measure space with μ -atoms A_1, A_2, \dots . The set $\Omega_a = \bigcup_i A_i$ is the μ -atomic part of Ω and the set $\Omega_c = \Omega_a^c$ is the μ -continuous part of Ω . We define the continuous part μ_c and the atomic part μ_a of μ by $\mu_c(E) = \mu(E \cap \Omega_c)$ and $\mu_a(E) = \mu(E \cap \Omega_a)$.*

Remarks. (i) Clearly $\mu = \mu_a + \mu_c$, and the decomposition is unique.

(ii) Ω_a and Ω_c are not unique.

(iii) Two projections π_a and π_c are defined on the set of σ -finite measures on a given σ -algebra, from $\pi_a(\mu) := \mu_a$ and $\pi_c(\mu) := \mu_c$.

(iv) Note $\pi_a(c\mu) = c\pi_a(\mu)$ and $\pi_a(\mu + \lambda) = \pi_a(\mu) + \pi_a(\lambda)$, and similar for π_c .

The following characterization is more constructive.

Theorem 4.3.4 (Continuous Part of a Finite Measure) *Let μ be a finite measure. The continuous part of μ is given by*

$$\mu_c(A) = \lim |\mu|_n(A), \quad |\mu|_n(A) := \sup_{\{A_i\}} \sum_i \mu(A_i), \quad (4.3)$$

where the supremum is taken over all finite families of disjoint measurable subsets A_i of A , each subset satisfying $\mu(A_i) < 1/n$.

Proof. Set $f_n(A) := |\mu|_n(A)$. We have $\mu(A) \geq f_1(A)$ and $f_n(A) \geq f_{n+1}(A) \geq 0$, so the limit $f(A) = \lim_n f_n(A)$ exists. This means that f is a well defined function from the σ -algebra into $[0, \infty)$. Each f_n is finitely additive, since μ is finitely additive, so f is also finitely additive. We also get $f(\emptyset) = 0$ and f is continuous at \emptyset , so f is a finite measure. Each $|\mu|_n$ is also a finite measure.

Let A be a subset of the continuous part Ω_c of the measure space, so A contains no μ -atoms. We may partition A into N sets, each with measure $\mu(A)/N$, with an arbitrary N . This follows, because we may select one subset of A with measure $\mu(A)/N$, and continue this process on the relative complement. The conclusion is that $\mu(A) = f_n(A) = f(A)$.

Let A be a subset of the atomic part Ω_a of the measure space. If $\mu(A) > 0$, A is a countable union of μ -atoms A_i and we get $f_n(A) = \sum_{i, \mu(A_i) < 1/n} \mu(A_i)$. In any case we get $f(A) = \lim f_n(A) = 0$, since the tail in a converging series converges to zero.

Finally we get $f(A) = f(A \cap \Omega_c) + f(A \cap \Omega_a) = \mu(A \cap \Omega_c)$, so $f = \mu_c$. \square

This theorem does also give a formula for the atomic part of the measure, from $\mu_a = \mu - \mu_c$. The formula is not valid for σ -finite measures, as can be seen from the measure on the integers given by $\mu(\{n\}) = 1/n$. In the above notation we get $\lim f_n(\{k\}) = 0$ for all integers k , but $f_n(\mathbb{N}) = \infty$, so the limit of f_n can not define a measure.

Theorem 4.3.5 (Atomic and Continuous Part of a Finite Random Measure) *The atomic and the continuous part of a finite random measure on a countably generated σ -algebra, are random measures.*

Proof. This follows from Theorem 4.3.4, when we observe that the supremum can be taken over a countable set, given by the countable algebra which generates the σ -algebra. \square

Definition 4.3.6 (Singular Measures) *A set A is μ -thick if all disjoint measurable sets have zero μ -mass. A measure is concentrated on a set if this set is measurable and thick. Two measures μ and ν are mutually singular, written $\mu \perp \nu$, if they are concentrated on two disjoint sets. A measure ν is μ -singular if ν is concentrated on a set A , with $\mu(A) = 0$.*

If A is μ -thick, then A can be realized as a measure space by $\mu(A \cap B) := \mu(B)$. We note that the continuous and atomic parts of a σ -finite measure are mutually singular. The Hahn decomposition of a measure space into two measurable sets states that the positive and negative variations of a real measure are mutually singular. We have that ν is μ -singular iff $\mu \perp \nu$. If given any $\epsilon > 0$, we can find a set F with $\mu(F) < \epsilon$ and $\nu(F^c) < \epsilon$, then we have $\mu \perp \nu$. This result makes sense to the term “uniform singularity” of a family of measures with respect to a given measure.

Definition 4.3.7 (Absolute Continuity) *Let μ and ν be measures. The measure ν is absolutely continuous with respect to μ , written $\nu \ll \mu$, if $\mu(F) = 0$ implies $\nu(F) = 0$ for all measurable sets F . A measure ν is μ -continuous if for any given $\epsilon > 0$ there exists a $\delta > 0$ such that $\nu(A) < \epsilon$ for all measurable A such that $\mu(A) < \delta$.*

If ν is μ -continuous, then $\nu \ll \mu$, because $\nu(A)$ is bounded by all positive numbers when $\mu(A) = 0$. The reverse implication holds if ν is a complex measure. Let μ be a measure. The set of μ -singular complex measures and the set of μ -continuous complex measures are vector spaces. The Lebesgue decomposition of a finite measure λ gives the measure as a unique sum of a μ -continuous part λ_{ac} and a μ -singular part λ_s . If f is a μ -integrable function, we denote the integral by $\mu[f]$. This notation emphasizes that a measure μ gives a linear functional on the vector space of μ -integrable functions. In this notation we have in particular $\mu(A) = \mu[1_A]$. If μ is σ -finite, then any μ -continuous complex measure λ is given by $\lambda(A) = \mu[f1_A]$, which is the Radon-Nikodym theorem. The function f is integrable and uniquely (μ -a.e.) given by λ . The Lebesgue decomposition is also given by the following formula.

Theorem 4.3.8 (Singular Part of a Finite Measure) *Let μ be a measure and let λ be a finite measure. The μ -singular part of λ is given by*

$$\lambda_s(A) = \lim f_n(A), \quad f_n(A) := \sup\{\lambda(A \cap B) \mid \mu(B) < 1/n\}. \quad (4.4)$$

Proof. We have $\lambda(A) \geq f_1(A)$ and $f_n(A) \geq f_{n+1}(A) \geq 0$, so the limit $f(A) = \lim_n f_n(A)$ exists, so f is a well defined function from the σ -algebra into $[0, \infty)$. The f_n 's and f are monotone in the sense that $F \subset G$ implies $f(F) \leq f(G)$. We prove $f = \lambda_s$.

Let $\lambda = \lambda_s + \lambda_{ac}$ be the Lebesgue μ -decomposition of λ . This gives the inequality $\lambda(A \cap B) \leq \lambda_s(A) + \lambda_{ac}(B)$. Let $\epsilon > 0$ and choose n so that $\lambda_{ac}(B) < \epsilon$ for all B such that $\mu(B) < 1/n$. This gives $f(A) \leq f_n(A) \leq \lambda_s(A) + \epsilon$, which implies $f \leq \lambda_s$.

Let F be such that $\mu(F) = 0$, which implies $f(F) = f_n(F) = \lambda(F) = \lambda_s(F)$. Choose S such that $\mu(S) = 0$ and $\lambda_s(S^c) = 0$. We have $\lambda_s(A) = \lambda_s(A \cap S) = f(A \cap S) \leq f(A)$, so $\lambda_s \leq f$. The conclusion is $f = \lambda_s$. \square

Remarks. (i) Fix a reference measure on a σ -algebra \mathcal{F} and consider the family F of finite measures on \mathcal{F} . Two projections π_s and π_{ac} are defined on F by $\pi_s(\lambda) = \lambda_s$ and $\pi_{ac} = id - \pi_s$.

(ii) Note $\pi_s(c\lambda) = c\pi_s(\lambda)$, $\pi_s(\lambda + \mu) = \pi_s(\lambda) + \pi_s(\mu)$, and likewise for π_{ac} .

Theorem 4.3.9 (Random Singular and Absolutely Continuous Part) *Fix a reference measure μ on a countably generated σ -algebra. The μ -singular and the μ -continuous part of a finite random measure are random measures.*

Proof. This follows from Theorem 4.3.4, when we observe that the supremum can be taken over a countable set, given by the countable algebra which generates the σ -algebra. \square

Remarks. (i) The above holds also with a random reference measure.

Let μ be a measure and λ a finite measure. If we combine the Hammer decomposition and the Lebesgue decomposition we have a decomposition of λ into four parts, concentrated on four disjoint sets. Atoms may be complex sets. To arrive at a simpler description of atoms we need less general σ -algebras.

Theorem 4.3.10 (Atoms are Points) *Let Ω be a σ -separated measurable space. If $x \in \Omega$, then $\{x\}$ is a measurable set. If $A \subset \Omega$ is a μ -atom, then there is a unique point $a \in A$ such that $\{a\}$ is a μ -atom.*

Proof. Let F_i be separating measurable sets. Define $X_i = F_i$ or $X_i = F_i^c$ so that $x \in X_i$. We have $\{x\} = \bigcap_i X_i$, which is measurable.

We may assume that $A = \Omega$, by consideration of the restriction to A . Define $A_i = F_i$ or $A_i = F_i^c$ so that $\mu(A_i) = \mu(A)$. There is then at most one element a in $\bigcap_i A_i$, and there is one element because $\mu((\bigcap_i A_i)^c) = 0$. \square

Let μ be a continuous measure on a σ -separated measurable space. We note that a μ -continuous measure is continuous. Any finite measure λ may be uniquely decomposed in a μ -continuous part λ_{ac} , a μ -singular and continuous part λ_{sc} , and a μ -singular and atomic part λ_p . In light of the previous formulas for λ_c and λ_s we note that $\lambda_p = \lambda_a = \lambda - \lambda_c$, $\lambda_{ac} = \lambda - \lambda_s$, and $\lambda_{sc} = \lambda_s - \lambda_a = \lambda_s + \lambda_c - \lambda = (\lambda_s)_c$. If a measure is equal to one of its components, then we say that it is a pure absolutely continuous measure, a pure point measure or a pure singular continuous measure, depending on the component in question.

Pure point measures are trivial to construct. The Radon-Nikodym theorem gives pure absolutely continuous measures. Pure singular continuous measures are more exotic and we will therefore give two examples. We consider the interval $[0, 1]$ with Lebesgue measure m on the Borel sets. Let F be a right continuous nondecreasing real valued function on $[0, 1]$. This F is the distribution function of a unique Borel measure λ on $[0, 1]$, $\lambda([0, x]) = F(x)$. A point a is a λ -atom iff $F(a) - F(a-) \neq 0$, and λ is continuous iff F is continuous. The λ is relatively singular to Lebesgue measure iff $F' = 0$ Lebesgue almost everywhere, and then F is also said to be singular.

Let F be the Cantor function which is continuous, is constant on a set of Lebesgue measure 0, and increases from 0 to 1 on the Cantor middle third set E . The corresponding measure is singular continuous and concentrated on E . We note that E is closed, and in particular it is not dense in $[0, 1]$.

Let E be the set of numbers y in $[0, 1]$ in which the frequency of the number 1 in the dyadic expansion $y = \sum_i x_i/2^i$ tends to $p \in (0, 1) \setminus \{1/2\}$. The set E has Lebesgue measure 0,

is uncountable, and is dense in $[0, 1]$. We will construct a singular continuous measure λ concentrated on E from Bernoulli trials. Let X_1, X_2, \dots , be i.i.d. random variables, which take the value 1 with probability p and the value 0 with probability $q = 1 - p$. Define λ to be the distribution of $Y = \sum_i X_i/2^i$. The claim follows from the law of large numbers, where we note that we have excluded the case $p = 1/2$, because this gives the Lebesgue measure. We will also describe the distribution function of λ . Define $f_0(x) = x$. Let f_n be piecewise linear between points $\alpha = k2^{-n}$ and $\beta = (k+1)2^{-n}$. Let $f_{n+1}(x) = f_n(x)$ at these points, whereas at the midpoint of these segments, that is, at the new points of division, let $f_{n+1}((\alpha+\beta)/2) = pf_n(\alpha) + qf_n(\beta)$. Define F_p as the pointwise limit of f_n . This F_p is continuous, strictly increasing, and F_p is the distribution function of λ . We finally note that any $h \in L^1([0, 1], \lambda)$ gives a singular continuous measure from $A \mapsto \lambda[1_A h]$. The conclusion is that singular continuous measures are quite typical. A plot of some of the singular continuous Bernoulli distributions F_p is given in Figure 4.1. This example does also give λ , a singular continuous measure, as a weak limit of absolutely continuous measures λ_n , corresponding to f_n .

4.4 Convergence of Measures

Let \mathcal{M} be the set of complex measures on a given σ -algebra \mathcal{F} . We observe that \mathcal{M} is a subset of the direct product $\mathbb{C}^{\mathcal{F}}$. The product topology on \mathcal{M} is defined by demanding continuity of the mapping $\mu \mapsto \mu(A)$ for any $A \in \mathcal{F}$. Convergence of measures in the product topology is pointwise convergence.

Theorem 4.4.1 (Vitali-Hahn-Saks) *Let ν_n be a pointwise convergent sequence of complex measures on a σ -algebra \mathcal{F} . The limit $\lim_n \nu_n(A)$ defines a complex measure. Let μ be a measure on \mathcal{F} . If all the ν_n are μ -continuous, then they are uniformly μ -continuous and the limit is μ -continuous.*

Proof. Assume first μ -continuity of all ν_n . The measure μ defines a complete metric space $X(\mu)$ consisting of equivalence classes of measurable sets. The distance is given by the measure of the symmetric difference, and two sets are equivalent if this distance is zero. A measure is μ -continuous if and only if it gives a well defined continuous function on $X(\mu)$. The convergence and continuity give X as a countable increasing union of closed sets

$$X = \bigcup_n \bigcap_{i,j \geq n} \{A \in X \mid |(\nu_i - \nu_j)(A)| \leq \epsilon\}, \quad (4.5)$$

for any given $\epsilon > 0$. The Baire category theorem gives that one of these closed sets contains an open ball. Therefore there exist an $r > 0$, a measurable set A , and an n so that $\mu(A \Delta C) < r$ implies that $|(\nu_i - \nu_j)(C)| \leq \epsilon$ for all $i, j \geq n$. The identity $B = [A \cup B] \setminus [A \setminus B]$ gives the identity

$$\begin{aligned} \nu_i(B) &= \nu_n(B) + [\nu_i(B) - \nu_n(B)] \\ &= \nu_n(B) + (\nu_i - \nu_n)(A \cup B) - (\nu_i - \nu_n)(A \setminus B). \end{aligned} \quad (4.6)$$

Choose $\delta < r$ such that $\nu_i(B) < \epsilon$ for $i = 1, \dots, n$ for all B fulfilling $\mu(B) < \delta$. The identity implies that we have $\nu_i(B) < 3\epsilon$ for all i and such B , so the ν_i are uniformly μ -continuous.

Bernoulli Distributions

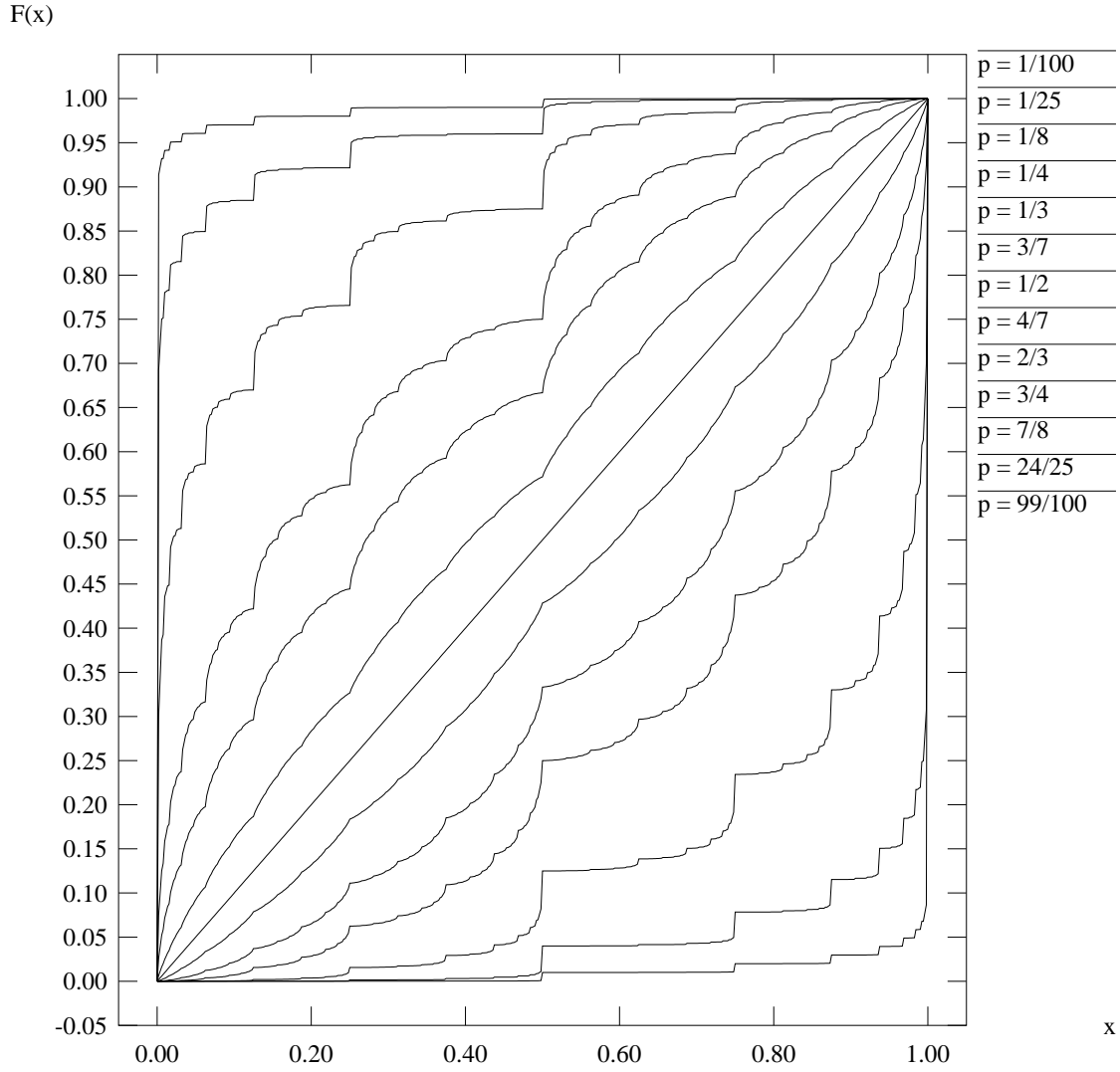


Figure 4.1: Bernoulli distributions are mutually singular, continuous, and strictly increasing. The graph located in the upper left corner corresponds to $p = 1/100$ and the graph located in the lower right corner corresponds to $p = 99/100$.

The first statement in the theorem follows from the trick $\mu(A) := \sum_i 2^{-i} |\nu_i|(A) / |\nu_i|(\Omega)$. This defines a finite measure μ and all ν_n are μ -continuous. Let ν denote the complex valued additive set function defined by $\nu(A) = \lim \nu_i(A)$. The claimed σ -additivity follows from continuity at \emptyset . Let A_n converge monotonically towards \emptyset . We will prove that $\lim \nu(A_n) = 0$. Let $\epsilon > 0$ be given. Because μ is finite we have $\lim \mu(A_i) = 0$. This and the uniform μ -continuity give an n such that $\nu_i(A_j) < \epsilon$ for all $i, j \geq n$. The convergence implies $\nu(A_j) \leq \epsilon$ for all $j \geq n$, so ν is a measure.

The μ -continuity of ν is equivalent with absolute continuity of ν with respect to μ , because ν is finite. That $\nu \ll \mu$ follows from the implications: $\mu(A) = 0$ implies $\nu_n(A) = 0$ for all n which implies $\nu(A) = \lim \nu_n(A) = 0$. \square

The Vitali-Hahn-Saks theorem is powerful, but it does not imply that \mathcal{M} is closed in $\mathbb{C}^{\mathcal{F}}$, since it only involves sequences. The product topology on $\mathbb{C}^{\mathcal{F}}$ contains many open sets, because \mathcal{F} typically is uncountable, and then it is not first countable. In particular

this implies that sequences is insufficient for the description of convergence. The product topology on \mathcal{M} is also typically not first countable. This is not a self evident fact, but follows from facts below.

Measures on topological spaces are important and deserve special names.

Definition 4.4.2 (Baire, Borel, and Radon Measures) *A Borel σ -algebra is the smallest σ -algebra containing a specified topology. A Baire σ -algebra is the smallest σ -algebra on a topological space, such that the real valued continuous functions are measurable. A Borel space is a topological space equipped with the Borel σ -algebra. A Baire space is a topological space equipped with the Baire σ -algebra. A Borel measure μ is a measure on a Borel σ -algebra, and is inner regular iff $\mu(A) = \sup\{\mu(K) \mid K \text{ compact subset of } A\}$. A finite Radon measure is a finite, inner regular, Borel measure.*

Remarks. (i) The Baire σ -algebra is contained in the Borel σ -algebra, but in a metric space there is no difference.

Convergence of measures is a fundamental concept. We single out the following possibilities for convergence of measures.

Definition 4.4.3 (Convergence of Measures)

- (a) Norm Convergence. *Let λ be a complex measure and define $\|\lambda\| := \sup |\lambda(A)|$, with supremum taken over all measurable sets A . We write $\lambda = n\text{-}\lim \lambda_\nu$, if $\lim \|\lambda_\nu - \lambda\| = 0$.*
- (b) Pointwise Convergence. *Consider measures or complex measures. We write $\lambda = p\text{-}\lim \lambda_\nu$, if $\lim \lambda_\nu(A) = \lambda(A)$ for all measurable A .*
- (c) Pointwise Convergence on a Generating Algebra. *Consider measures or complex measures on a σ -algebra generated by the algebra \mathcal{A} . We write $\lambda = p_{\mathcal{A}}\text{-}\lim \lambda_\nu$, if $\lim \lambda_\nu(A) = \lambda(A)$ for all $A \in \mathcal{A}$.*
- (d) Weak Convergence. *Consider complex measures on a Baire or a Borel σ -algebra. We write $\lambda = w\text{-}\lim \lambda_\nu$, if $\lim \lambda_\nu[f] = \lambda[f]$ for all bounded continuous functions f .*
- (e) Vague Convergence. *Consider measures or complex measures on a Baire or a Borel σ -algebra. We write $\lambda = v\text{-}\lim \lambda_\nu$, if $\lim \lambda_\nu[f] = \lambda[f]$ for all continuous functions f with compact support.*

Weak convergence is sometimes referred to as weak* convergence, which is natural when the complex measures are viewed as linear continuous functionals on C_b equipped with supremum norm. The same view also leads to the norm topology, from the norm on the functionals.

Let \mathcal{M} be the set of complex measures on a given σ -algebra. The above $\|\cdot\|$ makes \mathcal{M} into a Banach space. Let Ω be the measurable space. The finiteness of $|\mu|(\Omega)$ proves that the

norm is well defined, and the total variation may in fact be used to define an equivalent norm on \mathcal{M} . That $\|\cdot\|$ actually is a norm follows from the injection of \mathcal{M} into $l^\infty(\mathcal{F})$. The completeness may be proven directly, but follows easily from the Vitali-Hahn-Saks theorem, since norm convergent or norm Cauchy sequences have corresponding pointwise properties.

Any norm convergent net of complex measures is also pointwise convergent, so the norm topology is stronger than the pointwise topology. Evidently pointwise convergence implies pointwise convergence on any generating family. Pointwise convergence on a generating family \mathcal{A} does not imply pointwise convergence on the algebra generated by \mathcal{A} . This is seen from the example given by the sequence of Dirac measures δ_n , which converges pointwise to zero on $\mathcal{A} = \{\{n\} \mid n \in \mathbb{N}\}$, but $\delta_n(\{m\}^c) \rightarrow 1$ for all $m \in \mathbb{N}$. Clearly weak convergence implies vague convergence. This implication can be reversed with additional assumption.

Theorem 4.4.4 (Weak and Vague Convergence) *Let μ, μ_1, μ_2, \dots be finite Radon measures on a locally compact Hausdorff space. If $\mu = v\text{-}\lim \mu_n$ and $\mu[1] = \lim \mu_n[1]$, then $\mu = w\text{-}\lim \mu_n$.*

Proof. Let $f \in C_b$ and $\epsilon > 0$. Chose $u \in C_c$, $0 \leq u \leq 1$, such that $\mu[1 - u] < \epsilon$. We get

$$\begin{aligned} |\mu_n[f] - \mu[f]| &= |\mu_n[fu] - \mu[fu] + \mu_n[(1-u)f] - \mu[(1-u)f]| \\ &\leq |\mu_n[fu] - \mu[fu]| + 2\epsilon \|f\|_\infty, \quad n \geq N_\epsilon. \end{aligned} \tag{4.7}$$

This proves the claim, because $fu \in C_c$. We used $|\mu[f(1-u)]| \leq \|f\|_\infty \mu[1-u]$, and similarly for μ_n . Existence of N_ϵ follows from $\mu_n[1-u] = \mu_n[1] - \mu_n[u] \rightarrow \mu[1-u]$ and the assumption on u . Existence of a proper u follows from inner regularity and Urysohn's Lemma. Inner regularity gives a compact K such that $\mu[1 - 1_K] < \epsilon$ and Urysohn's Lemma gives a $u \in C_c$ with $u = 1$ on K and $0 \leq u \leq 1$. \square

In a metric space we can give an explicit formula for a suitable u as required above. Define g by $g(x) = 1$ for $x = 0$, $g(x) = 0$ for $x \geq 1$, and connect linearly between $(0, 1)$ and $(1, 0)$. Let d be the metric and set $u(x) := g(d(x, K)/t)$, with any positive number t . This u is supported in any given neighborhood of K if we select a small t .

Regularity of a finite Borel measure on a locally compact Hausdorff space follows from the Riesz representation theorem, if we assume that every open set is σ -compact. Any finite Borel measure on a separable, complete metric space is a Radon measure.

Let the real sequence x_n converge towards x . The sequence δ_{x_n} of Dirac measures converges weakly towards δ_x , but δ_{x_n} does not in general converge pointwise. That pointwise convergence implies weak convergence is a consequence of the following.

Theorem 4.4.5 (Weak Convergence) *Let μ, μ_1, μ_2, \dots be finite Radon measures on a metric space. Let \mathcal{A} be a family of measurable sets which is closed under finite intersections and is such that any open set is a countable union of sets from \mathcal{A} . If $\lim \mu_n[1] = \mu[1]$ and $\lim \mu_n(A) = \mu(A)$ for all A in \mathcal{A} , then $w\text{-}\lim \mu_n = \mu$. Convergence $\mu = w\text{-}\lim \mu_n$ is equivalent with any of the following:*

- (i) $\lim \mu_n[1] = \mu[1]$ and $\overline{\lim} \mu_n(F) \leq \mu(F)$ for all closed sets F .
- (ii) $\lim \mu_n[1] = \mu[1]$ and $\underline{\lim} \mu_n(U) \geq \mu(U)$ for all open sets U .

(iii) $\lim \mu_n(A) = \mu(A)$ for all measurable A with $\mu(\partial A) = 0$.

If given $\epsilon > 0$ there exists a compact K such that $\sup_n \mu_n(K^c) < \epsilon$ and $\sup_n \mu_n[1] < \infty$, then there exists a weakly convergent subsequence.

Proof. See the references in the notes. The last part is known as Prokhorov's theorem. \square

Remarks. (i) The first part in the theorem describes a convergence determining class of sets.

(ii) The sequence in Prokhorov's theorem is said to be uniformly Radon, or tight. The converse does also hold: Any weakly convergent series of finite Radon measures is uniformly Radon.

Consider now the set \mathcal{M} of complex measures on a σ -algebra generated by a countable algebra \mathcal{A} . A complex formally random measure μ is a random measure iff $\mu(A)$ is a Borel random variable for all $A \in \mathcal{A}$. We equip \mathcal{M} with a topology given by the metric

$$d(\mu, \nu) = \sum_n 2^{-n} \min(1, |(\mu - \nu)(A_n)|), \quad \mathcal{A} = \{A_1, A_2, \dots\}. \quad (4.8)$$

That $d(\mu, \nu) = 0$ implies $\mu = \nu$ follows because a complex measure is determined from its values on a generating algebra. Convergence $p_{\mathcal{A}}\text{-}\lim \mu_n = \mu$ is equivalent with convergence in the above metric. The metric makes \mathcal{M} into a separable metric vector space, and the corresponding Borel σ -algebra defines the set of random measures. If μ_1, μ_2, \dots are complex random measures and $\mu = p_{\mathcal{A}}\text{-}\lim \mu_n$ for almost all realizations, then μ is a random measure.

4.5 Decompositions and Perturbations

Theorem 4.5.1 (Denseness of Pure Point Measures) *Let \mathcal{M} be the set of complex measures on a σ -separated space. In the product topology, the set \mathcal{M}_p of complex pure point measures is dense in \mathcal{M} , but sequentially closed. \mathcal{M}_p is closed in \mathcal{M} in the norm topology. \mathcal{M} is separable in the product topology, and therefore also in any weaker topology.*

Proof. Let A be a finite family of measurable sets. Then there exists a finite family B of disjoint measurable sets, such that any member in A is a union of some members in B . The neighborhood of a complex measure μ is completely described by open sets of the form $U_A = \{\nu \mid |\nu(E) - \mu(E)| < \epsilon, \forall E \in A\}$, where ϵ is a positive number. This follows directly from the definition of the product topology. Choose B as above and define $\nu := \sum_{E \in B} \mu(E) \delta_e$, where different e 's in different E 's are picked for each nonempty E in B . Given E in A , we have E_1, \dots, E_n from B , with $E = \bigcup_i E_i$, so $\mu(E) = \sum_i \mu(E_i) = \sum_i \nu(E_i) = \nu(E)$, and ν is in U_A . This proves that the set of pure point measures is dense. Since we only needed a finite number of atoms, it follows also that the set of complex measures is separable in the product topology.

Let μ_1, μ_2, \dots be complex pure point measures, concentrated respectively on countable sets M_1, M_2, \dots . Let $M = \bigcup_i M_i$, which is countable, and observe that $\lim \mu_i(M^c) = 0$, so any possible pointwise limit is concentrated on a countable set, and is therefore a pure point measure. Since norm convergence implies pointwise convergence, this also implies that the set of complex pure point measures is closed in the norm topology. \square

- Remarks.* (i) If there exists a complex measure on the given σ -algebra, that is not pure point, then it follows that the product topology on the corresponding set of complex measures cannot be first countable.
- (ii) If the family A above has n members, then B may be chosen with $2^n - 1$ members, some of which may be the empty set.
- (iii) The approximating pure point measures above have a finite number of atoms, and may be chosen with total mass equal to the mass of the measure being approximated, which in particular is relevant for probability measures.
- (iv) Any finite measure may be perturbed into a pure point measure, arbitrary close in the product topology.

Theorem 4.5.2 (The Space of Absolutely Continuous Measures) *Let \mathcal{M} be the set of complex measures on a σ -separated measurable space. Fix a reference measure m . In the product topology, the set \mathcal{M}_{ac} of absolutely continuous complex measures and the set \mathcal{M}_c of continuous complex measures are closed vector spaces in \mathcal{M} . If m is continuous, then $\mathcal{M}_{ac} \subset \mathcal{M}_c$, and the interiors of both are empty.*

Proof. Let m be the measure with respect to which absolute continuity is defined. Let $\{\mu_\iota\}$ be a net of complex measures, which converges towards μ . Let $m(E) = 0$, and let A be any measurable subset of E . Then $\mu(A) = \lim \mu_\iota(A) = 0$, if $\mu_\iota \ll m$, so $|\mu|(E) = 0$, and $\mu \ll m$. This proves closedness of \mathcal{M}_{ac} in the product topology. Now assume $\mu_\iota(\{a\}) = 0$ for all ι and a given point a . Then $\mu(\{a\}) = \lim \mu_\iota(\{a\}) = 0$, so \mathcal{M}_c is closed, where we have used that the underlying measurable space is σ -separated. The vector space claim is clear.

When m is continuous and $\mu \ll m$, then $m(\{a\}) = 0 = \mu(\{a\})$, so μ is continuous, and $\mathcal{M}_{ac} \subset \mathcal{M}_c$. Let U be an open subset of the set of continuous measures. Since the pure point measures are dense and not absolutely continuous, U is empty. \square

Remarks. (i) If we restrict the product topology to \mathcal{M}_c , then \mathcal{M}_{ac} will still be a closed subspace. It is not clear from the above whether the interior of \mathcal{M}_{ac} in \mathcal{M}_c is empty or not.

Definition 4.5.3 (Spectrum of a Measure) *Let λ be a finite measure on a σ -separated Borel space X . The spectrum $\sigma(\lambda)$ and the resolvent set $\rho(\lambda)$ of λ is defined by*

$$\sigma(\lambda) := \text{supp}(\lambda) := \{x \in X \mid \lambda(U) > 0, \text{ for any open } U \ni x\} \quad \text{and} \quad (4.9)$$

$$\rho(\lambda) := \{x \in X \mid \text{there exists an open } U \ni x, \lambda(U) = 0\}. \quad (4.10)$$

Fix a continuous reference measure. The point spectrum, the pure point spectrum, the absolutely continuous spectrum, and the singular continuous spectrum are defined respectively by

$$\sigma_p(\lambda) := \{x \in X \mid \lambda(\{x\}) \neq 0\}, \quad (4.11)$$

$$\sigma_{pp}(\lambda) := \sigma(\lambda_{pp}), \quad (4.12)$$

$$\sigma_{ac}(\lambda) := \sigma(\lambda_{ac}), \text{ and} \quad (4.13)$$

$$\sigma_{sc}(\lambda) := \sigma(\lambda_{sc}). \quad (4.14)$$

- Remarks.* (i) All the spectra above are closed sets, except for the set $\sigma_p(\lambda)$ of eigenvalues.
(ii) Note $\sigma_{pp}(\lambda) = \overline{\sigma_p(\lambda)}$.
(iii) If the topological space is second countable, it follows that $\lambda((\sigma(\lambda))^c) = 0$.
(iv) Equality $\sigma_{ac}(\lambda) = \sigma_{sc}(\lambda) = \sigma_{pp}(\lambda)$ is possible.

Definition 4.5.4 (Measure Class of a Measure) *Let F be the set of finite measures on a given σ -algebra. The finite measures λ and ν are equivalent, written $\lambda \sim \nu$, iff $\lambda \ll \nu$ and $\nu \ll \lambda$. The measure class $[\lambda]$ of $\lambda \in F$ is defined by*

$$[\lambda] := \{\nu \in F \mid \nu \sim \lambda\}. \quad (4.15)$$

The set $[F]$ of measure classes is defined by

$$[F] := \{[\lambda] \mid \lambda \in F\}. \quad (4.16)$$

- Remarks.* (i) The relation \sim is an equivalence relation and $[F] = F / \sim$.
(ii) The $[\lambda]$ obeys $\mu, \nu \in [\lambda] \Rightarrow a\mu + b\nu \in [\lambda]$ for positive scalars a, b , so $[\lambda]$ is a positive cone.

Theorem 4.5.5 (Spectrum of a Measure Class) *Let F be the set of finite measures on a σ -separated Borel space. The spectra $\sigma, \sigma_p, \sigma_{pp}, \sigma_{ac}, \sigma_{sc}$ give well defined functions on $[F]$, the set of measure classes in F .*

Proof. We prove that $\sigma([\lambda]) := \sigma(\lambda)$ is well defined. Let $\lambda \ll \mu$. If $z \in \rho(\mu)$, then there is an open $U \ni z$ such that $\mu(U) = 0$, so $z \in \rho(\lambda)$, since $\lambda(U) = 0$, so $\sigma(\lambda) \subset \sigma(\mu)$. Now $\sigma([\lambda])$ is well defined, since $\lambda \sim \mu \Rightarrow \sigma(\lambda) = \sigma(\mu)$.

Let $\lambda \ll \mu$. If $\lambda(\{x\}) > 0$, then $\mu(\{x\}) > 0$, so $\sigma_p(\lambda) \subset \sigma_p(\mu)$, and $\sigma_{pp}(\lambda) \subset \sigma_{pp}(\mu)$. We conclude also that $\lambda \sim \mu$ implies $\lambda_p \sim \mu_p$.

Let m be the measure with respect to which singularity and absolute continuity are defined. We can find a measurable A such that $m(A^c) = 0$, $\lambda_{ac}(U) = \lambda(U \cap A)$, and $\mu_{ac}(U) = \mu(U \cap A)$. This gives $\lambda \ll \mu \Rightarrow \sigma_{ac}(\lambda) \subset \sigma_{ac}(\mu)$, and $\sigma_{ac}([\mu]) := \sigma_{ac}(\mu)$ is well defined.

Let $\lambda \sim \mu$. Define $C := \sigma_p(\mu)^c$, and $S = S_\lambda \cup S_\mu$, where λ_s is concentrated on S_λ , μ_s is concentrated on S_μ , and $m(S) = 0$. The formulas $\lambda_{sc}(U) = \lambda(U \cap S \cap C)$ and $\mu_{sc}(U) = \mu(U \cap S \cap C)$ give the claim. \square

Theorem 4.5.6 (Random Spectra) *Fix a continuous reference measure μ on a σ -separated Borel σ -algebra \mathcal{B} on a second countable space. Let λ be a finite random measure on \mathcal{B} . The spectra $\sigma(\lambda)$, $\sigma_{pp}(\lambda)$, $\sigma_{ac}(\lambda)$, and $\sigma_{sc}(\lambda)$ are closed random sets.*

Proof. We need only prove that $E = \{U \cap \sigma(\lambda) \neq \emptyset\}$ is an event for any open U , since the components are random measures. The measurability of E follows from

$$U \cap \sigma(\lambda) \neq \emptyset \iff \lambda(U) > 0. \quad (4.17)$$

The only if part follows from the definition. The if part follows because $\lambda((\sigma(\lambda))^c) = 0$ in a second countable space. \square

A strengthening of this theorem is given by the lower semicontinuity of the spectrum.

Theorem 4.5.7 (Lower Semicontinuity of the Spectrum) *Let R be the set of finite Radon measures on a separable metric space. Equip R with the weak topology. The mapping $R \ni \lambda \rightarrow \sigma(\lambda)$ is lower semicontinuous.*

Proof. Assume $\lambda = w\text{-}\lim \lambda_n$. We prove $l\text{-}\lim \sigma(\lambda_n) = \sigma(\lambda)$, which is sufficient since R can be realized as a metric space. Let $U \cap \sigma(\lambda) \neq \emptyset$ for an open set U . Then $\underline{\lim} \lambda_n(U) \geq \lambda(U) > 0$, and $\lambda_n(U) > 0$ for all n larger than some N_U . This implies $\sigma(\lambda_n) \cap U \neq \emptyset$ for $n > N_U$, which proves the claim. \square

Remarks. (i) The mapping $\lambda \mapsto \sigma(\lambda)$ is then lower semicontinuous in any stronger topology in R .

Let m be Lebesgue measure on $[0, 1]$. Set $\lambda_n(A) := n \cdot m([0, 1/n] \cap A)$. Then $w\text{-}\lim \lambda_n = \lambda = \delta_0$, so $\sigma_{pp}(\lambda) \subset \overline{\bigcup_n \sigma_{pp}(\lambda_n)}$ is false. This means in particular that $F \ni \lambda \mapsto \sigma_{pp}(\lambda)$ can not be lower semicontinuous when F is equipped with the weak topology. If F is equipped with the pointwise topology, then σ_{pp} is lower semicontinuous.

Theorem 4.5.8 (Lower Semicontinuity of the Point Spectrum) *If λ is the pointwise limit a net $\{\lambda_\iota\}$ of measures, then $\sigma_{pp}(\lambda) = l\text{-}\lim \sigma_{pp}(\lambda_\iota)$.*

Proof. Let U be open with $p \in U$ and $\lambda(\{p\}) > 0$. Since $\lim \lambda_\iota(\{p\}) = \lambda(\{p\}) > 0$, it follows that $\lambda_\iota(\{p\}) > 0$ for all ι greater than some N_U , which proves the claim. \square

4.6 Notes

A general comment on the references is that they have unequivalent definitions of common concepts, so one must compare with care.

I have defined random measures as by Kirsch in [79, p.284] and by Carmona in [31, p.245, p.262-264]. I have not found any literature on Lebesgue-Hammer decomposition of random measures. In my mind, the associated problems are fundamental for an understanding of Lebesgue-Hammer decomposition of selfadjoint, random operators.

Atomic sets and measures are defined as in Rao [23, p.141-], but compare also Dunford [46, part I, p.308] and Halmos [65, p.168,p.169 ex.10]. The Hammer decomposition is defined as in Rao [23, p.146,p.149]. Feller [57, p.138,142] refers to the Hammer decomposition as the Jordan decomposition, but I denote the decomposition of a real measure into a positive and a negative part as the Jordan decomposition, compare Rudin [115, p.119], Pedersen [110, p.241], or Reed [124, vol.I,p.22].

The formula for the continuous part of a finite measure is a generalization of a similar formula in Carmona [31, p.246]. A related formula for the atomic part is found in Pastur [107, p.44].

Concentration on a set and singularity of measures are defined as in Rudin [115, p.120], but see also Reed [124, p.24], Rao [23, p.164], Halmos [65, p.126], or Dunford [46, p.131]. The definition of a thick set is taken from Yamasaki [139, p.18].

For the Lebesgue decomposition of a measure I refer to [65, p.97,p.124], [23, p.159], [46, part I,p.131], [115, p.120]. These references have unequivalent definitions of absolute continuity of a measure. My choice is as in Rudin [115] and Dunford [46].

A unique Lebesgue decomposition of any finite measure with respect to a measure μ follows from Zorn's Lemma, Dunford [46, p.132,vol.I], and μ need not be σ -finite.

The formula for the singular part of a finite measure is abstracted from Cycon [39, p.170]. The presented proof seems to be a simplification. The formula is also found in Pastur [107, p.44] and Carmona [31, p.245], but without a proof.

A probabilistic interpretation of the Cantor measure on $[0, 1]$ is found in Feller [57, p.35-36]. I recommend a reading of [57, p.141-142], which clarifies the nature of singular continuous measures, and their unavoidability. They arise naturally from consideration of Bernoulli trials, as indicated in the text. A nonprobabilistic view of the same distribution is found in Nagy [114, p.48-49]. I was initially interested in singular continuous measures because they arise in the spectral theory of Sturm-Liouville operators [34],[17],[39, p.199-]. Castillo [34, p.3] uses the distribution function $F_{1/3}$ to construct a spectral function with pure singular continuous spectrum in $[0, 1]$.

The Vitali-Hahn-Saks theorem, including its proof, is in my opinion, one of the most beautiful results in measure theory. The proof of the Vitali-Hahn-Saks theorem is similar to the ones found in Dunford [46, part I,p.158], and Halmos [65, p.169, ex. 12-14]. The first reference assumes μ to be finite and the second reference assumes μ to be σ -finite. I do not see the need for these assumptions in the presented proof, which is a rearrangement of the proof in [46, part I, p.158]. The Vitali-Hahn-Sax theorem is also found in the context of finitely additive set functions (charges) and weak convergence in Rao [23, p.204]. The product topology is not treated any further in any of the given sources.

The norm topology and corresponding completeness is found in Dunford [46, part I,p.160]. The space of regular complex measures on a locally compact abelian group is a commutative Banach algebra, from the convolution product, see Rudin [116, p.14].

I have defined vague convergence as in Bauer [16, p.226]. The literature on convergence of measures is large, as it should be, see Billingsley [25], but also [26, p.335-], [96], [22], [93, p.50-], [57, p.247-], [43, p.8-9, p.128-], [106, p.178-189]. The literature on weak convergence dominates. Weak convergence of a different kind is defined in Rao [23, p.218-] and they refer to weak* convergence to cover our choice.

A probability measure is tight iff measurable sets can be approximated in measure by compact subsets, Billingsley [25, p.9].

The quoted regularity of a finite measure as a consequence of σ -compactness of open sets in a locally compact Hausdorff space is a consequence of [115, p.48, 2.18 Theorem]. The characterization of finite Borel measures on a separable, complete metric space as being the Radon measures is found in Linde [96, p.12]. The characterizations of weak convergence and Prokhorov's theorem is abstracted from Linde [96, p.14-15], and Billingsley [25, p.11-14].

The quotation: "Thus we learn from the Krein-Milman theorem that every probability measure on X can be approximated pointwise on $C(X)$ by measures with finite support on X .", taken from Pedersen [110, p.73, 2.5.8. Remark], is relevant for the denseness of pure point measures. The presented proof is elementary, and gives a more general result, since the product topology is stronger than the weak topology. For the denseness of pure point measures, see also Reed [124, vol.I, p.114, p.118-119, p.123 ex.41].

Equivalent measures are defined as in Halmos [65, p.126-127]. The measure class, or the spectral measure, of a selfadjoint operator was my motivation for the study of this concept,

see Simon [122, p.501-], but also [113, p.400], [137, p.126-], [31, p.80].

The lower semicontinuity of the spectrum of a measure given in Theorem 4.5.7 is perhaps a new observation.

Chapter 5

Unbounded Random Operators

The lower operator topology is introduced. It gives a proper generalization of strong resolvent convergence of selfadjoint operators. We prove that the set of finite matrix operators is dense. The lower topology is applied for the definition of unbounded, closed random operators. We prove nonrandomness of the spectrum of a transitive random operator, given that the spectrum is random in the sense defined by the lower topology. We prove lower semicontinuity of the spectrum of normal operators. A general formula for the spectrum of a normal, transitive random operator is given. We give conditions which imply presence of continuous spectrum or pure point spectrum.

5.1 Introduction

Classical mechanics gives a description of a one particle system by means of a Hamilton function as a function of the position and the velocity of the particle. If the environment of the particle is complex or not exactly known, then it may be reasonable to model this by a random Hamilton function. The result is a real valued stochastic process H indexed by position and velocity variables. The general theory of stochastic processes is then available. In quantum mechanics one may use similar arguments to arrive at a random Hamilton operator, but one meets a fundamental obstacle. It is not at all clear how one should define a random operator, in particular since the involved operators are unbounded. The purpose here is to discuss the notion of unbounded random operators in a general context.

For this purpose we introduce the lower operator topology in a general set of closed operators. The definition avoids domain problems completely by consideration of the graphs of the involved operators. This is not a new idea and has been used previously to define the gap topology, which gives a proper extension of the operator norm topology. We consider closed operators defined in a separable normed space, and observe that the set of finite matrix operators is dense. More generally we prove that the lower operator topology is second countable and T_0 -separating, if the underlying topological vector space is second countable. This follows without any separation demands on the underlying vector space. It is shown that convergence on a core implies convergence in the lower topology. This means in particular that the strong topology for bounded operators is stronger than the lower topology for bounded operators. If we restrict to norm bounded sets, then the two topologies coincide. We prove equality of the strong resolvent topology and the lower op-

erator topology in the set of selfadjoint operators, so we have given a proper extension of the topology from strong resolvent convergence.

We give a brief discussion of the concept of Skorohod random operators, and conclude that this is an essentially different concept compared to our concept. A general definition of the concept of closed, random operators is given. The definition is in terms of the Borel field from the lower operator topology. We compare our definition with the concepts of Pastur random operators, bounded random operators, and selfadjoint random operators, and conclude that our definition is a suitable generalization in all these cases. The sum of unbounded operators is difficult to define in a good sense, because of domain problems. One solution is given by consideration of the limit of the sum of two approximating bounded operators. The Lie sum is another, but related possibility. One advantage with a definition of the sum by means of a limit is that the sum of two random operators is again a random operator.

Spectral theory is considered next. We prove that the spectrum of a transitive random operator is nonrandom, if the spectrum is random in the sense given by the lower topology. Upper semicontinuity of the spectrum, considered as a set valued mapping, is proven when closed operators are equipped with the gap topology. The main theorem is that the spectrum of normal operators is a lower semicontinuous mapping, when the normal operators are equipped with the lower topology. We do also observe that the core spectrum seems to be more interesting than the usual spectrum, when we consider general closed operators, without conditions on dense domains.

An elementary proof of the randomness of the Lebesgue-Hammer components of the spectrum of a selfadjoint, random operator is given. We give a general formula for the spectrum of a normal transitive random operator. Finally we give theorems which imply existence of absolute continuous spectrum or pure point spectrum.

5.2 Perturbation of Unbounded Operators

Let $G(T)$ denote the graph of a linear operator $T : D(T) \rightarrow Y$, with domain $D(T)$ in X . The operator T is closed if and only if $G(T)$ is closed in $X \times Y$. A graph G is a subspace in $X \times Y$, with the additional property that $(0, y) \in G \Rightarrow y = 0$. In the following we will only consider the case $X = Y$.

Definition 5.2.1 (The Lower Operator Topology) *Let \mathcal{C} be a set of closed, linear operators $T : D(T) \rightarrow X$ defined in a topological vector space X . The lower operator topology τ_l in \mathcal{C} is the weakest topology where*

$$[U, V] := \{T \in \mathcal{C} \mid \exists f \in D(T) \bigcap U \text{ such that } Tf \in V\} \quad (5.1)$$

is open when U and V are open in X . If the net $\{T_\lambda\}$ in \mathcal{C} converges towards T in the lower operator topology, we write $T = l\text{-}\lim T_\lambda$.

Remarks. (i) The lower operator topology is the relative topology in the set of operator graphs from the lower topology in the set of closed sets in $X \times X$.

(ii) The operators in \mathcal{C} are not required to have equal domain, nor dense domain. The

domain of a linear operator is a linear subspace.

(iii) The lower operator topology is typically not a Hausdorff topology, but this depends on \mathcal{C} .

(iv) The maximal mappings $D_z \ni T \mapsto z - T$ and $D_I \ni T \mapsto T^{-1}$ are continuous.

The lower operator topology is defined from the lower topology for the set of graphs, so $T = l\text{-}\lim T_\lambda$ is equivalent with $G(T) = l\text{-}\lim G(T_\lambda)$. If $\{E_\lambda\}$ is a net of closed subspaces, then the closure of the lower limit and the individual lower limit are closed subspaces. The closure of the lower limit of a net $\{G_\lambda\}$ of closed graphs is a graph, but typically equal to the trivial graph $\{(0, 0)\}$, since the intersection of two graphs typically is the trivial graph. The individual lower limit $L_i\text{-}\lim G_\lambda$ may not be a graph. It is not clear from these comments whether there exists a unique maximal limit graph.

The gap topology for closed operators is given by the Hausdorff distance between the unit spheres of the operator graphs. This topology is equal to the norm topology if it is restricted to bounded operators. The lower topology is weaker, and this is reflected in the following theorems.

Theorem 5.2.2 (The Lower Operator Topology is Separable) *If \mathcal{C} is the set of closed operators defined in a separable normed vector space, then \mathcal{C} is separable in the lower operator topology.*

Proof. Let X be an infinite dimensional normed space, with a dense countable subset D . Let $U_1, V_1, \dots, U_n, V_n$ be nonempty open balls in X . Let M_D be the set of matrix operators M , each M defined by a choice of a finite independent set $\{x_1, \dots, x_k\}$ and a finite set $\{y_1, \dots, y_k\}$ from D , by setting $M(\sum_i c_i x_i) = \sum_i c_i y_i$. The set M_D is countable. We will construct a closed operator M in $\bigcap_i [U_i, V_i]$ which also lies in M_D . Select x_1 from $D \cap U_1$ and y_1 from $D \cap V_1$. Assume that $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ is given. Let L_k be the linear span of $\{x_1, \dots, x_k\}$, which is a closed set. The set $\tilde{U}_{k+1} = U_{k+1} \setminus L_k$ is open and nonempty, since X is an infinite dimensional normed space. Select x_{k+1} from $D \cap \tilde{U}_{k+1}$ and y_{k+1} from $D \cap V_{k+1}$. This procedure gives the required M as above. \square

Remarks. (i) This theorem will also follow more generally from second countability.

Theorem 5.2.3 (The Lower Operator Topology is Second Countable) *If \mathcal{C} is a set of closed operators defined in a second countable topological vector space, then \mathcal{C} is a second countable T_0 -space in the lower operator topology.*

Proof. The relative topology in any subset of a second countable topological space is second countable, and the lower topology in the set of closed sets in a second countable space is second countable. The separation property is also inherited generally. \square

Remarks. (i) If B is a subbasis for the topology in the vector space, then a subbasis for the lower operator topology is given by sets $[U, V]$ for U, V in B .

The infinitesimal generator of a strongly continuous semigroup is defined from a convergence concept which is related to the concept of strong convergence of bounded operators. The connection with lower convergence of closed operators is given in the following theorems.

Theorem 5.2.4 (Convergence on a Core) *Let D be an operator core for the closed linear operator T . If $Tf = \lim T_\lambda f$ for all f in $D \subset \underline{\lim} D(T_\lambda)$, then the net $\{T_\lambda\}$ of closed linear operators converges towards T in the lower operator topology.*

Proof. Let U and V be open with $f \in U \cap D(T)$ and $Tf \in V$. Since D is an operator core, we may assume $f \in D$, because the restricted graph $G(T_D)$ is dense in the graph $G(T)$. The assumption $D \subset \varinjlim D(T_\lambda)$ gives an N_1 such that $f \in D(T_\lambda)$ for all $\lambda \geq N_1$. The assumed convergence gives an $N \geq N_1$ such that $T_\lambda f \in V$ for all $\lambda \geq N$, which proves the claim. \square

Remarks. (i) A set D is a core if $T = \overline{T_D}$.

(ii) A special case is given by letting T be the trivial operator with graph $\{(0, 0)\}$. This T is a lower limit of any net of closed linear operators, but it may not belong to the set of closed operators under consideration.

Theorem 5.2.5 (The Lower Operator Topology for Bounded Operators) *Let B be the set of bounded, closed, linear operators defined on a normed space. The lower operator topology in B is weaker than the strong operator topology. The restriction of the two topologies to a norm bounded set in B are identical.*

Proof. The lower operator topology is weaker than the strong operator topology, from Theorem 5.2.4. Assume $T = l\text{-}\lim T_\lambda$ in B with $\sup \|T_\lambda\| < K$. For any x and x_λ we have

$$\|T_\lambda x - Tx\| \leq \|T_\lambda x_\lambda - Tx\| + K \|x - x_\lambda\|. \quad (5.2)$$

Let $\epsilon > 0$. Set $U := \{y \mid \|y - x\| < \epsilon/(2K)\}$ and $V := \{y \mid \|y - Tx\| < \epsilon/2\}$. The convergence gives a λ_ϵ such that $T_\lambda \in [U, V]$ for all $\lambda \geq \lambda_\epsilon$, which assures existence of x_λ 's such that the above estimate proves the claim. \square

Remarks. (i) It is not clear from the above whether the lower topology in B is Hausdorff.

The concept of strong resolvent convergence gives a well defined topology for selfadjoint operators. The lower operator topology is a proper generalization of this convergence concept, as stated in the following theorem.

Theorem 5.2.6 (The Lower Operator Topology for Selfadjoint Operators) *The strong resolvent topology for selfadjoint operators in a Hilbert space is identical with the lower operator topology.*

Proof. Let $\{T_\lambda\}$ be a net of selfadjoint operators. Assume $(i - T)^{-1} = s\text{-}\lim (i - T_\lambda)^{-1}$ for some selfadjoint T . Then $(i - T)^{-1} = l\text{-}\lim (i - T_\lambda)^{-1}$, so $G((i - T)^{-1}) = l\text{-}\lim G((i - T_\lambda)^{-1})$, so $G(T) = l\text{-}\lim G(T_\lambda)$, and $T = l\text{-}\lim T_\lambda$. Assume $T = l\text{-}\lim T_\lambda$ for some selfadjoint T . Let z be any complex number with nonzero imaginary part $\text{Im}(z)$. Theorem 5.2.5 gives $(z - T)^{-1} = s\text{-}\lim (z - T_\lambda)^{-1}$, since $\|(z - S)^{-1}\| \leq 1/\text{Im}(z)$ for any selfadjoint S . \square

Remarks. (i) The lower topology for selfadjoint operators is Hausdorff.

(ii) If the Hilbert space is separable, then there is a metric which gives the strong resolvent topology.

(iii) The formula $\|(z - S)^{-1}\| = 1/d(z, \sigma(S))$ holds for any normal operator S with spectrum $\sigma(S)$.

One might suspect equivalence between strong convergence and lower convergence for bounded selfadjoint operators, but this is wrong as the following example demonstrates. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis in a Hilbert space. Define the closed operator $T_i f = \lambda_i e_i \langle e_i, f \rangle$, which is selfadjoint if λ_i is real. Let D be the set of finite linear combinations from $\{e_i\}$. This D is an operator core for the bounded, selfadjoint operator $T = 0$. We conclude $T = l\text{-}\lim T_n$, since $Tf = \lim T_n f$ for all f in D . The sequence $\{T_n\}$ is strongly convergent if and only if $\sup_i |\lambda_i| < \infty$.

The following is relevant for spectral theory.

Theorem 5.2.7 (Bounded Invertibility of Lower Limits) *Assume that $T = l\text{-}\lim T_\lambda$ for closed operators T, T_λ . If $\|T_\lambda^{-1}\| \leq K$, then T is invertible and $\|T^{-1}\| \leq K$. Furthermore $T^{-1}f = \lim T_\lambda^{-1}f$ for any $f \in D(T^{-1}) \cap \underline{\lim} D(T_\lambda^{-1})$.*

Proof. Assume $Tx = 0$ and let $\epsilon > 0$ be arbitrary. The convergence gives an x_λ such that $\|x_\lambda - x\| < \epsilon$ and $\|T_\lambda x_\lambda\| < \epsilon$. Then $\|x_\lambda\| \leq \|T_\lambda^{-1}\| \|T_\lambda x_\lambda\|$ and $\|x\| \leq \|x_\lambda\| + \|x_\lambda - x\| \leq (1 + K)\epsilon$. Since ϵ was arbitrary we conclude that T^{-1} exists. Let $y = Tx$ and let $\epsilon > 0$ be arbitrary. The convergence gives an x_λ such that $\|x_\lambda - x\| < \epsilon$ and $\|T_\lambda x_\lambda - y\| < \epsilon$. From $\|x_\lambda\| \leq K \|T_\lambda x_\lambda\| \leq K(\epsilon + \|y\|)$ we get $\|T^{-1}y\| = \|x\| \leq \epsilon + K(\epsilon + \|y\|)$, and $\|T^{-1}y\| \leq K \|y\|$. The last claim follows from the estimate $\|T_\lambda^{-1}f - T^{-1}f\| \leq K \|f - f_\lambda\| + \|T_\lambda^{-1}f_\lambda - T^{-1}f\|$. \square

Remarks. (i) None of the above inverse operators need be defined on the whole space.

Theorem 5.2.8 (Continuity of The Sum) *If $T = l\text{-}\lim T_\lambda$ for closed operators T, T_λ defined in a normed space X and $B = l\text{-}\lim B_\lambda$ for closed, uniformly bounded operators B, B_λ defined on X , then $T + B = l\text{-}\lim (T_\lambda + B_\lambda)$.*

Proof. Assume $\|B_\lambda\| \leq K < \infty$. Let $\epsilon > 0$ and $h \in D(T)$ be given. The lower convergence gives $f_\lambda \in D(T_\lambda)$, $g_\lambda \in X = D(B_\lambda)$, and λ_ϵ such that $\|f_\lambda - h\| < \epsilon$, $\|g_\lambda - h\| < \epsilon$, $\|T_\lambda f_\lambda - Th\| < \epsilon$, and $\|B_\lambda g_\lambda - Bh\| < \epsilon$ for all $\lambda \geq \lambda_\epsilon$. The estimate

$$\begin{aligned} \|(T_\lambda + B_\lambda)f_\lambda - (T + B)h\| &\leq \|T_\lambda f_\lambda - Th\| + \|B_\lambda g_\lambda - Bh\| + \|B_\lambda\| \|f_\lambda - g_\lambda\| \\ &\leq 2(1 + K)\epsilon, \end{aligned} \tag{5.3}$$

proves the claim. \square

Remarks. (i) Observe the case where the bounded operators are given by complex numbers.

(ii) The mapping $(t, T) \mapsto (t + i\epsilon - T)^{-1}$ is jointly continuous for real t and selfadjoint T .

5.3 Unbounded Random Operators

We begin with a digression on Skorohod random operators. The following is abstracted from Skorohod's book "Random Linear Operators", including the notation. We will not use any of this notation, nor the concepts, in the remainder of this text. We include this material to make clear that there is an essential distinction between the two concepts.

Let $f \in L^2[0, 1] = \mathcal{H}$, let B_t be Brownian motion, and define a mapping A on the Hilbert space \mathcal{H} by the Ito integral

$$Af(t) = \int_0^t f(s) dB_s. \quad (5.4)$$

The function Af is continuous with probability 1. The mapping A depends on $B_t(\omega)$ and is formally a random variable $A(\omega)$. The properties

$$(S1) \quad P(A(\alpha f + \beta g) = \alpha Af + \beta Ag) = 1 \text{ and}$$

$$(S2) \quad \lim_{f_n \rightarrow f} P(\|Af - Af_n\| > \epsilon) = 0,$$

may be verified. The above should hold for all $f, g \in \mathcal{H}$, all complex α, β , and all real $\epsilon > 0$. Let $\mathcal{H}(\Omega)$ denote the set of \mathcal{H} valued random variables based on the probability space (Ω, P) . A mapping $f \mapsto A(\omega)f$ of \mathcal{H} into $\mathcal{H}(\Omega)$ satisfying (S1) and (S2) defines a strong random operator in the sense of Skorohod. The set of strong random operators is denoted by $L_s(\Omega, \mathcal{H})$. (S1) says that A is linear with probability one, in the given sense. This does not imply that $A(\omega)$ is a linear operator on \mathcal{H} for all ω in a set of measure one. $A(\omega)f$ is defined for almost all ω for any fixed f , but not the other way around with a fixed ω . (S2) says that $A(\omega)x$ is continuous in x in probability. Let $L(\Omega, \mathcal{H})$ denote the set of mappings $\omega \mapsto A(\omega)$ from Ω into the set of bounded linear operators on \mathcal{H} , such that (f, Ag) is a random variable. The set $L_w(\Omega, \mathcal{H})$ of weak random operators is given by mappings $(f, g) \mapsto (f, A(\omega)g)$ of $\mathcal{H} \times \mathcal{H}$ into $\mathbb{C}(\Omega)$ satisfying

$$(W1) \quad P((\alpha_1 f_1 + \alpha_2 f_2, A(\beta_1 g_1 + \beta_2 g_2)) = \sum_{i,j} \overline{\alpha_i} \beta_j (f_i, Ag_j)) = 1 \text{ and}$$

$$(W2) \quad \lim_{(f_n, g_n) \rightarrow (f, g)} P(|(f_n, Ag_n) - (f, Ag)| > \epsilon) = 0.$$

The above should hold for all $f_i, g_i, f, g \in \mathcal{H}$, all complex α_i, β_i , and all real $\epsilon > 0$. One may verify that $L(\Omega, \mathcal{H}) \subset L_s(\Omega, \mathcal{H}) \subset L_w(\Omega, \mathcal{H})$, in a natural sense.

We will not use the concept of Skorohod strong or weak random operators in the following. Our definition will ensure that $A(\omega)$ is a linear, possibly unbounded, operator with probability one. In short one may say that Skorohod random operators are well suited for the analysis of random differential equations, typically equations involving white noise. Our concept, which keeps the old definition of a linear operator, is well suited for the familiar concepts from operator theory. It is *e g* not clear how to define the spectrum of a Skorohod random operator.

Let $\mathcal{C}(X)$ be the set of closed, linear operators defined in a Banach space X . The bounded operators in $\mathcal{C}(X)$ are characterized by the property of having a closed domain. Let $\omega \mapsto A(\omega)$ be a mapping from a probability space into $\mathcal{C}(X)$. The mapping A is a Pastur random operator if $D = \bigcap_{\omega} D(A(\omega))$ is a core for all $A(\omega)$ and if (f, Ag) is a random variable when $f \in X^*$ and $g \in D$. We recall that D is a core for A if the closure of A restricted to D is A . The set D must be dense in the domain, but this is not sufficient. If A^{-1} is bounded, then D is a core iff $A(D)$ is dense in the range $A(D(A))$.

The Pastur definition is not completely satisfactory, because of the artificial assumption on existence of a common core. We will define random operators with the aid of a suitably chosen σ -algebra of operators, namely the Borel field from the lower operator topology.

Definition 5.3.1 (Closed Random Operators) *A closed, linear, random operator is a Borel measurable mapping from a probability space into a set of closed, linear operators equipped with the lower operator topology.*

Remarks. (i) If T is a random operator, then the image measure P_T realizes a set of closed operators as a probability space. On the other hand, given a probability space (M, \mathcal{B}_l, P) of closed operators, we get a random operator from the identity mapping.

(ii) The domain $D_\omega = D(T(\omega))$ is in general a formally random subspace. D_ω is not closed when $T(\omega)$ is unbounded.

(iii) A closed, linear, formally random operator T in a separable Banach space X is a random operator iff $\{\omega \in \Omega \mid \exists f \in D(T(\omega)) \cap U, T(\omega)f \in V\}$ is a measurable set for any selection of open sets U and V from a family of open sets which generates the Banach space topology in X . This follows because the lower operator topology is second countable, when the underlying vector space is second countable.

Theorem 5.3.2 (Pastur Random Operators) *Let T be a mapping from a probability space into the set of closed operators defined in a separable normed space X . If there exists a countable set D such that T is the closure of its restriction to D and (f, Tg) is a random number for all f in the dual space X^* and all $g \in D$, then T is a random operator.*

Proof. The Hahn-Banach separation theorem gives that Tg is a norm random variable for all $g \in D$. Let M be the set of realizations of T . Equip M with the lower operator topology. The mapping $\omega \rightarrow T(\omega) \ni M$ is a Borel mapping since

$$\{\omega \mid \exists f \in D(T(\omega)) \cap U, T(\omega)f \in V\} = \bigcup_{f \in D \cap U} \{\omega \mid T(\omega)f \in V\} \quad (5.5)$$

is measurable for any open U and V and the lower operator topology is second countable. \square

Remarks. (i) It is necessary that the set D is dense in $\bigcap_\omega D(T(\omega))$, but this is not sufficient.

(ii) Let D be a core for a closed operator T . Let D_1 be a dense subset of D . We can not conclude that T is the closure of its restriction to D_1 , since T may be unbounded.

(iii) Let D be a core for a closed operator T in a separable normed space. Then there exists a countable subset D_1 in D such that T is the closure of its restriction to D_1 . This follows from second countability.

(iv) Assume C to be a common core for almost all $T(\omega)$. If there exists an ω_M such that almost all restrictions $T_C(\omega)$ are $T_C(\omega_M)$ bounded, then there exists a countable set D as required in the theorem. The assumption is $\|T(\omega)f\| \leq a(\omega) \|f\| + b(\omega) \|T(\omega_M)f\|$ for all f in C .

Theorem 5.3.3 (Random Matrix Operators) *Let $\{e_1, e_2, \dots\}$ and $\{f_1, f_2, \dots\}$ be random orthonormal systems in a separable Hilbert space. Let $\{a_{ij}\}$ be random numbers such*

that $\sum_i |a_{ij}|^2$ and $\sum_j |a_{ij}|^2$ are finite with probability one. Let A be the closed formally random operator determined by $Ae_i = \sum_j a_{ij} f_j$. Let Q be a countable set of finite sequences of numbers and let $D = \{\sum_i q_i e_i \mid (q_i) \in Q\}$ be the corresponding formally random set. If A is equal to the closure of A restricted to D , then A is a random operator.

Proof. The assumption gives

$$\begin{aligned} & \{\omega \mid \exists f \in D(A(\omega)) \cap U, A(\omega)f \in V\} \\ &= \bigcup_{q \in Q} \{\omega \mid \sum_n q(n)e_n(\omega) \in U, \sum_{n,k} q(n)a_{nk}(\omega)f_k(\omega) \in V\}, \end{aligned} \tag{5.6}$$

which is measurable for any open U, V . This is sufficient, since the Hilbert space is separable. \square

Remarks. (i) Any unbounded closed operator in a Hilbert space may be represented by a matrix operator.

(ii) The set Q may for instance be the set of finite sequences with rational components. The assumption $A = \overline{A_D}$ is nontrivial.

(iii) This is a generalization of the notion of Pastur random operators, since we allow random orthonormal systems. If the systems are nonrandom and complete, then we are essentially back to Pastur random operators.

(iv) This theorem has a converse for selfadjoint random operators.

Theorem 5.3.4 (Bounded Random Operators) *Let $T : \Omega \rightarrow B(\mathcal{H})$ be a mapping from a probability space into the set of closed, bounded operators defined on a Hilbert space with an orthonormal basis $\{e_1, e_2, \dots\}$. T is a random operator if any of the following equivalent conditions are satisfied*

- (i) (e_i, Te_j) is a random number for all i, j .
- (ii) T is a random variable with respect to the weak operator topology.
- (iii) Te_i is a random variable for all i with respect to the norm topology.
- (iv) T is a random variable with respect to the strong operator topology.

If $\sup_\omega \|T(\omega)\|$ is finite, then the above conditions are also necessary. If the range $T(\Omega)$ is separable in the uniform topology, then the above are equivalent with

- (v) T is a random variable with respect to the uniform operator topology.

Proof. Assume first T to be a closed random operator with uniformly bounded norm. Since the lower operator topology is identical with the strong operator topology, we conclude that T is also a random variable with respect to the strong operator topology. This proves the last claim in the theorem.

Note that e_i and e_j in (i) and (iii) can be replaced by general $f, g \in \mathcal{H}$ from a simple limiting argument. Set $B_n := \{T \mid \|T\| \leq n\}$ and $A_n := A \cap B_n$, so $A = \bigcup_n A_n$ and $A_n \subset B_n$. Note that B_n is weakly compact, in particular weakly closed. Let $\{f_k\}$ be a countable dense set in $\{g \mid \|g\| \leq n\}$.

(i) \Rightarrow (iv): Assume A to be open in the strong operator topology. A_n is then a countable union of open sets like $\{T \mid \sum_k 2^{-k} \|(T - S)f_k\| < \epsilon\}$. Therefore it is sufficient to prove measurability of $\omega \mapsto \|T(\omega)g - h\|$. The assumption in (i) gives that Tg is a random variable with respect to the σ -algebra \mathcal{F}_w generated by the mappings $h \mapsto (f, h)$. The Borel σ -algebra \mathcal{B}_n from the inner product topology is equal to \mathcal{F}_w , because the Hilbert space is separable. The conclusion is that $\{\omega \mid T(\omega) \in A\}$ is measurable.

(i) \Rightarrow (ii): Assume A to be open in the weak operator topology. A_n is then a countable union of open sets like $\{T \mid \sum_k 2^{-k} |(f_k, (T - S)f_k)| < \epsilon\}$. Therefore it is sufficient to prove measurability of $\omega \mapsto (f_k, T(\omega)f_k)$, but this follows from (i).

(iv) \Rightarrow (v): Let A be open in the uniform operator topology. $A \cap T(\Omega)$ is a countable union of sets like $\{S \mid \|S - S_i\| < \epsilon\}$, since $T(\Omega)$ is separable. Measurability of $\omega \mapsto \|T(\omega) - S_i\| = \sup_{\|f_k\| < 1} \|(T(\omega) - S_i)f_k\|$ follows from the assumption.

In the above we used the fact that both the strong and the weak operator topologies restricted to B_n comes from separable metric spaces when \mathcal{H} is separable. The remaining part does not depend on the separability of the Hilbert space.

(v) \Rightarrow (iv) \Rightarrow (ii): The uniform operator topology is stronger than the strong operator topology which is stronger than the weak operator topology.

(iv) \Rightarrow (iii): The mapping $S \mapsto Sg$ is continuous from the definition of the strong operator topology and the mapping $\omega \mapsto T(\omega)g$ is measurable as the composition of two measurable mappings.

(iii) \Rightarrow (i): The mapping $h \mapsto (f, h)$ is continuous.

(ii) \Rightarrow (i): The mapping $S \mapsto (f, Sg)$ is continuous. □

Remarks. (i) Consider the identity mapping on $B(\mathcal{H})$. It follows that the weak operator topology and the strong operator topology have equal Borel field, when \mathcal{H} is separable. This holds also if the Hilbert space is replaced by a separable normed space.

(ii) Let T be a closed random operator and assume $f \in \bigcap_{\omega} D(T(\omega))$. The vector Tf is a random vector if $\sup_{\omega} \|T(\omega)f\|$ is finite. It would be nice to remove this last condition.

A selfadjoint random operator is a measurable mapping $\omega \mapsto T(\omega)$ from a probability space into the set $\mathcal{C}_{sa}(\mathcal{H})$ of selfadjoint operators defined in a Hilbert space. The operator $T(\omega)$ has a domain $D(\omega)$, which is a linear dense subspace in the Hilbert space. It seems reasonable that the mapping $\omega \mapsto D(\omega)$ should be measurable if the operator is measurable. This is in fact true with our definitions, since $\{\omega \mid D(T(\omega)) \cap U \neq \emptyset\} = \{\omega \mid \exists g \in D(T(\omega)) \cap U, T(\omega)g \in \mathcal{H}\}$ is measurable when U is open. The mapping $\omega \mapsto (f, T(\omega)g(\omega))$ should also be measurable for any $f \in \mathcal{H}$ and any measurable $\omega \mapsto g(\omega)$, with $g(\omega) \in D(\omega)$. This would give a definition of random operators which extends Pastur's definition. Our approach has been to realize $\mathcal{C}_{sa}(\mathcal{H})$ as a measurable space with the introduction of a σ -algebra. The choice is in accordance with the general notion of a random variable. Selfadjoint random operators may be characterized in many other equivalent ways.

Theorem 5.3.5 (Selfadjoint Random Operators) *A mapping T from a probability space into the set of selfadjoint operators defined in a Hilbert space \mathcal{H} is a random operator if and only if one of the following conditions is satisfied*

(i) $(z_0 - T)^{-1}$ is a random operator for one z_0 , $Im(z_0) \neq 0$.

(ii) $(z - T)^{-1}$ is a random operator for all complex z , $Im(z) \neq 0$.

Proof. This is obvious, since the randomness is defined in terms of the randomness of the graph. □

Remarks. (i) If the Hilbert space has an orthonormal basis $\{e_1, e_2, \dots\}$, then measurability of $\omega \mapsto \langle e_i, (i - T(\omega))^{-1}e_j \rangle$ is sufficient for the randomness of T .
(ii) The inequality $\|(i - T(\omega))^{-1}\| \leq 1$ gives the connection to the strong operator topology.

Let $g \in \mathcal{H}$ and set $f(\omega) = (i - T(\omega))^{-1}g$. This f is a random vector and $f(\omega) \in D(T(\omega))$. Any $f(\omega) \in D(T(\omega))$ may be obtained like this. We have $T(\omega)f(\omega) = (i - (i - T(\omega)))f(\omega) = if(\omega) - g$, which is a random vector. Let $\{g_j\}$ be an orthonormal basis in \mathcal{H} and set $f_j = (i - T)^{-1}g_j$. The family $\{f_j\}$ is linearly independent, is total in \mathcal{H} , and consists of random vectors in $D(T)$. We apply the Gram-Schmidt orthogonalization process on $\{f_j\}$ and obtain a random orthogonal system $\{e_j\}$: $v_1 = f_1, e_1 = v_1 / \|v_1\|$ and $v_n = f_n - \sum_{k=1}^{n-1} e_k \langle e_k, f_n \rangle$, $e_n = v_n / \|v_n\|$. Each e_j is a random vector in $D(T)$ since it is a random linear combination of vectors in $\{f_j\}$ and Te_j is therefore also a random vector. If we define $a_{ij} = \langle e_i, Te_j \rangle$ we obtain the representation $Tf = \sum_i e_i (\sum_j a_{ij} \langle e_j, f \rangle)$ with $f \in D(T)$ if the sums involved converge. The matrix operator will in general be an extension of the given closed operator. We note that a closed symmetric extension of a selfadjoint operator is nothing but the operator itself. The conclusion is that a random selfadjoint operator may be represented by a random orthonormal system and a random matrix. It seems impossible to revert this implication, so it seems to give a more general concept of a selfadjoint random operator.

The Trotter-Kato theorem has an immediate generalization for strong resolvent convergence of selfadjoint random operators.

Theorem 5.3.6 (Limits of Selfadjoint Random Operators) *Let T_n be a sequence of selfadjoint random operators. Let Ω_c be a fixed set with measure one. Assume that $(i + T_n(\omega))^{-1}$ and $(-i + T_n(\omega))^{-1}$ converges strongly for all $\omega \in \Omega_c$ and that one of the two limiting operators has a dense range. Then there exists a selfadjoint random operator T such that $T_i(\omega)$ converges towards $T(\omega)$ in the strong resolvent sense for all $\omega \in \Omega_c$.*

Proof. The Trotter-Kato theorem gives the claim for each $\omega \in \Omega_c$. The measurability of $T(\omega)$ follows from the measurability of $(i - T(\omega))^{-1}$, which is the strong limit of $(i - T_n(\omega))^{-1}$. \square

Remarks. (i) We repeat that existence of T is a part of the conclusion in the theorem, so it gives a way to define random operators.

Consider the formula

$$\exp(it(S + T)) = s - \lim (\exp(itS/n) \exp(itT/n))^n = U(t). \quad (5.7)$$

If S and T are finite dimensional matrices this formula is known as the Lie product formula. If S and T are selfadjoint operators and $S + T$ is essentially selfadjoint on $D(S) \cap D(T)$, then the formula is known as the Trotter product formula. If the limit $U(t)$ exists, we define the Lie sum $S \oplus T$ as the infinitesimal generator of $U(t)$. The right hand side is a unitary operator since it is the strong limit of unitary operators. The group property follows from the functional equation $U(nt) = U(t)^n$, valid for integer n . Strong continuity in t does also follow, but this is harder to prove, see the references at the end. Strong continuity of $t \mapsto U(t)$ follows from weak measurability, if the underlying Hilbert space is separable. If $S \oplus T$ exists, then it is a selfadjoint extension of $S + T$.

Theorem 5.3.7 (The Lie Sum of Two Selfadjoint Random Operators) *The Lie sum of two selfadjoint random operators, if it exists, is a selfadjoint random operator.*

Proof. The measurability follows because the Lie sum is defined from a strong limit. \square

Remarks. (i) The Feynman-Kac path integral formula gives a useful specialization for random Schrödinger operators.

5.4 Spectral Theory for Random Operators

Definition 5.4.1 (Transitive Random Operator) *A closed random operator T in a Hilbert space is a transitive random operator if*

$$T(\Theta_\nu\omega) = U_\nu T(\omega) U_\nu^{-1}, \quad (5.8)$$

for a metrically transitive system $\{\Theta_\nu\}_{\nu \in I}$ of transformations of the probability space and a corresponding family $\{U_\nu\}_{\nu \in I}$ of unitary operators. T is an ergodic random operator if the system $\{\Theta_\nu\}_{\nu \in I}$ is an ergodic system.

Remarks. (i) The equation above means in particular that $D(T(\Theta_\nu\omega)) = U_\nu D(T(\omega))$.

(ii) The main point in this definition is that functions of operators that are invariant under unitary equivalence results in corresponding transitively invariant functions. In particular, unitary equivalence preserves norm, selfadjointness, normality, unitarity and spectrum. The corresponding measurability, needed to conclude that the invariant function is a constant, may however be nontrivial.

(iii) The mapping $T \mapsto U_\nu T U_\nu^{-1}$ corresponds to a dynamical system where the “phase-space” consists of operators. One might also consider more general flows on operator spaces.

Theorem 5.4.2 (Nonrandom Spectrum) *Let T be a transitive random operator. If the spectrum $\sigma(T)$ is a random set, then there exists a closed set Σ such that $\sigma(T) = \Sigma$ with probability one.*

Proof. The function $\omega \mapsto \sigma(T(\omega))$ is invariant with respect to a metrically transitive system of transformations of the probability space, and it is measurable by assumption. Since this function takes values in a σ -separated space, the conclusion follows. \square

Remarks. (i) The spectrum of a random operator should be a random set, but this seems difficult to prove in this generality.

The previous theorem requires randomness of the spectrum $\sigma(T)$ of a random operator T . This means measurability of the set $\{\omega \mid \sigma(T(\omega)) \cap U \neq \emptyset\}$ for any given open set U . This is equivalent with measurability of $\{\omega \mid \sigma(T(\omega)) \cap K = \emptyset\}$ for any compact set K . Continuity of $T \mapsto \sigma(T)$ is sufficient.

Theorem 5.4.3 (Upper Semicontinuity of Spectrum I) *Let \mathcal{A} be a complex unital Banach algebra. The spectral mapping $\mathcal{A} \ni a \mapsto \sigma(a)$ is upper semicontinuous with respect to the norm topology in \mathcal{A} .*

Proof. Assume $\sigma(a) \subset U$ for an open set U and an element $a \in \mathcal{A}$. We will find an open set containing a such that any element in this open set has spectrum contained in U . Since

the spectrum is a compact set we may assume U to be contained in a compact set.

Choose an open $B \ni a$ and a compact $K \supset U$ such that $b \in B \Rightarrow \sigma(b) \subset K$. This is possible because of the bound $r(b) \leq \|b\|$ for the spectral radius $r(b)$.

Let \mathcal{B} be the Banach algebra $C(K \cap U^c, \mathcal{A})$ with $\|f\|_\infty = \sup_x \|f(x)\|$. $\lambda \mapsto \lambda - a$ and $\lambda \mapsto (\lambda - a)^{-1}$ are members of \mathcal{B} , since $K \cap U^c$ is a compact subset of a 's resolvent set.

Choose an open $D \ni a$ such that $d \in D \Rightarrow \lambda - d$ is invertible in \mathcal{B} . This is possible because the group $GL(\mathcal{B})$ of invertible elements is open in \mathcal{B} and $\|(\lambda - d) - (\lambda - a)\|_\infty = \|d - a\|$.

For $d \in D$ we have $K \cap U^c \subset \rho(d)$, so $\sigma(d) \subset K^c \cup U$.

Finally we have $\sigma(c) \subset (K^c \cup U) \cap K \subset U$, when $c \in B \cap D$, which concludes the proof. \square

Remarks. (i) The Banach algebra \mathcal{B} is introduced to get an open set D which does the purpose for all λ in $K \cap U^c$.

(ii) The set B forces the spectrum to be away from the exterior of K and the set D forces the spectrum to be away from the annulus $K \cap U^c$, so the spectrum is forced to be in U .

Definition 5.4.4 (The Spectrum of a Closed Operator) *Let T be a closed operator in a Banach space X . The resolvent set $\rho(T)$ is*

$$\rho(T) := \{z \in \mathbb{C} \mid N(z - T) = \{0\}, (z - T)^{-1} \in B(X)\}. \quad (5.9)$$

The spectrum $\sigma(T)$ is

$$\sigma(T) := \mathbb{C} \setminus \rho(T). \quad (5.10)$$

Remarks. (i) The resolvent set is open and $\rho(T) \ni z \mapsto (z - T)^{-1} \in B(X)$ is analytic.

Theorem 5.4.5 (Upper Semicontinuity of Spectrum II) *Let $\mathcal{C}(X)$ be the closed operators in the Banach space X equipped with the gap topology. The spectral mapping $\mathcal{C}(X) \ni T \mapsto \sigma(T)$ is upper compact semicontinuous.*

Proof. Let $\sigma(T) \cap K = \emptyset$ for a compact K and $T_\lambda \rightarrow T$.

Let $z \in K$, so $(z - T)^{-1} \in B(X)$. There exist a λ_z and an open $U_z \ni z$ such that $(w - T_\lambda)^{-1} \in B(X)$ when $w \in U_z$ and $\lambda \geq \lambda_z$, which we will prove shortly. The compactness gives $K \subset \cup_{i=1}^k U_{z_i}$ and a $\lambda_M \geq \lambda_{z_i}$ solves our problem. We have $\sigma(T_\lambda) \cap K = \emptyset$ for $\lambda \geq \lambda_M$.

The general case $z \in K$ is reduced to the case $z = 0$, by the introduction of shifted operators. Assume $T^{-1} \in B(X)$. The convergence gives a λ_0 and an M such that $T_\lambda^{-1} \in B(X)$ and $\|T_\lambda^{-1}\| \leq M$ for all $\lambda \geq \lambda_0$. The relations $\sigma(T_\lambda) = \{1/z \mid z \in \sigma(1/T_\lambda)\}$ and $z \in \sigma(1/T_\lambda) \Rightarrow |z| \leq M$ give $\sigma(T_\lambda) \cap \{z \mid |z| < 1/M\} = \emptyset$ for all $\lambda \geq \lambda_0$. \square

Remarks. (i) The gap convergence gives $(z - T)^{-1}$ equal to the norm limit of $(z - T_\lambda)^{-1}$ for all z in the resolvent of T . The proof above depends only on a uniform bound on $\|(z - T_\lambda)^{-1}\|$ for large λ .

(ii) The proof of Theorem 5.4.3 can be simplified with the use of this theorem.

The previous results are not sufficient for the proof of randomness of the spectrum of a random operator, because the gap topology is too strong. The core spectrum seems to be more interesting than the spectrum, when we consider general closed operators.

Definition 5.4.6 (The Core Spectrum) Let T be a closed operator defined in a Banach space X . The regularity domain is

$$\Gamma(T) := \{z \in \mathbb{C} \mid (z - T)^{-1} \text{ is a bounded operator}\}, \quad (5.11)$$

and the core spectrum is $\tilde{\sigma}(T) := \mathbb{C} \setminus \Gamma(T)$.

Remarks. (i) The core spectrum is a closed set.

(ii) The core spectrum consists of the point spectrum, the continuous spectrum, and parts of the residual spectrum.

(iii) A normal operator has empty residual spectrum, so the core spectrum is equal to the spectrum in this case.

Theorem 5.4.7 (The Core Spectrum and Approximate Eigenvectors) A $\lambda \in \mathbb{C}$ is in the core spectrum of the closed linear operator T if and only if there exist $f_n \in D(T)$, $\|f_n\| = 1$, such that $\lim(\lambda - T)f_n = 0$.

Proof. Let S be the core spectrum and let S_0 be the set of λ such that there exist approximate eigenvectors as in the theorem. We prove $S \subset S_0$ by proving $S_0^c \subset S^c$. Let $z \notin S_0$. It is sufficient to consider $z = 0$. This assumption gives a $K > 0$ such that $\|Tf\| \geq K \|f\|$ for all $f \in D(T)$. The first conclusion is then that T^{-1} exists. The second conclusion is that $\|T^{-1}g\| = \|f\| \leq K^{-1} \|g\|$, so 0 is in the regularity domain. Assume next $0 \in S_0$. If we assume $0 \in S^c$, we get the contradiction $\|T^{-1}(Tf_n)/\|Tf_n\|\| = 1/\|Tf_n\| \rightarrow \infty$, so $0 \in S$. \square

Remarks. (i) The essential spectrum of a selfadjoint operator is given by the set of λ such that there exists an orthonormal system of approximate eigenvectors as in the theorem.

Theorem 5.4.8 (Monotonicity of The Core Spectrum) If S is a closed operator contained in the closed operator T , $S \subset T$, then the core spectrum of S is a subset of the core spectrum of T , $\tilde{\sigma}(S) \subset \tilde{\sigma}(T)$. Similar results for the point spectrum and the residual spectrum are that $S \subset T$ implies $\sigma_p(S) \subset \sigma_p(T)$ and $\sigma_r(S) \supset \sigma_r(T)$.

Proof. Assume that 0 is in the regularity domain of T , so T^{-1} is defined and bounded. The relation $S \subset T$ gives $N(S) \subset N(T) = \{0\}$, so S^{-1} is defined. For $g \in D(S^{-1})$ we have $\|S^{-1}g\| = \|T^{-1}g\| \leq \|T^{-1}\| \|g\|$, so $\|S^{-1}\| \leq \|T^{-1}\|$. This verifies the claim for the core spectrum. If $g \in D(S)$ and $Sg = \lambda g$, then $S \subset T$ gives $Tg = Sg = \lambda g$, so $\sigma_p(S) \subset \sigma_p(T)$. Assume that T^{-1} exists, but is not densely defined, so 0 is in the residual spectrum of T . Then $S \subset T$ gives that S^{-1} exists, but is not densely defined. This proves the reverse monotonicity for the residual spectrum. \square

Remarks. (i) The spectrum itself is not monotone in the sense given by this theorem.

(ii) A lower semicontinuous set valued mapping $T \mapsto \Sigma(T)$ has to be monotone.

For selfadjoint or more generally for normal random operators the situation is satisfactory, because the spectrum is lower semicontinuous with respect to the lower operator topology. The following well known formula is the key point.

Theorem 5.4.9 (Norm of Resolvent and Spectrum) *Let T be a normal operator in a Hilbert space. When $z \in \rho(T)$ we have*

$$\|(z - T)^{-1}\| = 1/d(z, \sigma(T)). \quad (5.12)$$

Proof. If $R = (z - T)^{-1}$, then

$$\begin{aligned} \|R\| &\stackrel{(1)}{=} r(R) = \sup\{|x| \mid x \in \sigma(R)\} \stackrel{(2)}{=} \sup\{|x| \mid x = \frac{1}{z-t}, t \in \sigma(T)\} \\ &= 1/\inf\{|z - t| \mid t \in \sigma(T)\} = 1/d(z, \sigma(T)) \end{aligned} \quad (5.13)$$

(1) follows because R is normal. (2) is true because T is a closed operator. It is a special case of the spectral mapping theorem, $\sigma(f(T)) = f(\sigma(T))$. \square

Remarks. (i) (5.12) follows as an \leq -inequality if we only know that the operator is closed. It would be nice to have an inequality the other way in more general cases.

Lower semicontinuity for the selfadjoint case follows from the above.

Theorem 5.4.10 (Lower Semicontinuity of the Spectrum I) *Let $\mathcal{C}_{sa}(\mathcal{H})$ be the self-adjoint operators in the Hilbert space \mathcal{H} equipped with the lower operator topology. The spectral mapping $\mathcal{C}_{sa}(\mathcal{H}) \ni T \mapsto \sigma(T)$ is lower semicontinuous.*

Proof. The key point is the formula

$$\|R(z, T)\| = 1/d(z, \sigma(T)), \quad (5.14)$$

which is valid when T is normal. Let $x \in \sigma(T)$ and $T_\lambda \rightarrow T$. Let $\epsilon > 0$ and choose an $f \in \mathcal{H}$, $\|f\| = 1$, such that $\|R(x + i\epsilon, T)f\| > 1/(2\epsilon)$. Then there exists a λ_ϵ such that $\|R(x + i\epsilon, T_\lambda)f\| > 1/(3\epsilon)$ for $\lambda \geq \lambda_\epsilon$. For $\lambda \geq \lambda_\epsilon$ we have $1/(3\epsilon) < \|R(x + i\epsilon, T_\lambda)\| \leq 1/d(x + i\epsilon, \sigma(T_\lambda))$, so $d(x + i\epsilon, \sigma(T_\lambda)) < 3\epsilon$, so $\sigma(T_\lambda) \cap \{z \mid |z - x| < 4\epsilon\} \neq \emptyset$. This proves the claim, since 4ϵ can be chosen arbitrary small. \square

Remarks. (i) The gap topology equals the norm resolvent topology in $\mathcal{C}_{sa}(\mathcal{H})$. Theorem 5.4.5 implies that $T \mapsto \sigma(T)$ is weak Vietoris continuous, when we use the norm resolvent topology for selfadjoint operators.

(ii) The above proof depends on the fact that all selfadjoint operators have real spectrum.

(iii) The above proof depends on $\lim \|R(z, T_\lambda)f\| = \|R(z, T)f\|$. This is weaker than the assumption $\lim R(z, T_\lambda)f = R(z, T)f$.

The following observation makes it possible to extend the previous result from selfadjoint operators to include all normal operators.

Theorem 5.4.11 (Spectrum and Norm) *If T is a normal operator, then*

$$d(z, \sigma(T)) \leq \|(z - T)f\|, \quad \forall z \in \mathbb{C}, \quad \forall f \in D(T) \quad \|f\| = 1. \quad (5.15)$$

Proof. It is sufficient to consider the case $z = 0$. In the case $0 \in \sigma(T)$ there is nothing to prove so we assume $0 \in \rho(T)$. Now $\|f\| / \|Tf\| = \|T^{-1}Tf\| / \|Tf\| \leq \|T^{-1}\|$, so $\|T^{-1}\|^{-1} \leq \|Tf\| / \|f\|$. The normality gives $d(0, \sigma(T)) \leq \|T^{-1}\|^{-1} \leq \|Tf\| / \|f\|$ which proves the claim. \square

Theorem 5.4.12 (Lower Semicontinuity of the Spectrum II) *If the closed operator T is a lower limit of the net $\{T_\lambda\}$ of normal operators in a Hilbert space, then $\tilde{\sigma}(T) = l\text{-}\lim \sigma(T_\lambda)$.*

Proof. Let $q \in \sigma(T)$, let $r > 0$, and set $U := \{z \in \mathbb{C} \mid |z - q| < r\}$. To simplify notation we assume $q = 0$. Let $0 < \epsilon < 1/2$. Since 0 is in the core spectrum of T , there exists an $f \in D(T)$ such that $\|f\| = 1$ and $\|Tf\| < \epsilon$. The operator convergence gives a λ_ϵ such that there exists a $f_\lambda \in D(T_\lambda)$ with $\|f_\lambda - f\| < \epsilon$ and $\|T_\lambda f_\lambda - Tf\| < \epsilon$ for all $\lambda \geq \lambda_\epsilon$. The normality gives $d(0, \sigma(T_\lambda)) \leq \|T_\lambda f_\lambda\| / \|f_\lambda\| \leq 2\epsilon / (1 - \epsilon) < r$ for all $\lambda \geq \lambda_\epsilon$, for a proper choice of ϵ . We conclude that $U \cap \sigma(T_\lambda) \neq \emptyset$ for all $\lambda \geq \lambda_\epsilon$. \square

Remarks. (i) If T is a normal operator, then $\tilde{\sigma}(T) = \sigma(T)$.

5.5 Decompositions of the Spectrum

The spectrum of a general closed operator is naturally divided into parts depending on the properties of the resolvent operator.

Definition 5.5.1 (Point-, Continuous-, and Residual Spectrum) *Let T be a closed operator in a Banach space X . The point spectrum $\sigma_p(T)$, the continuous spectrum $\sigma_c(T)$, and the residual spectrum $\sigma_r(T)$ are*

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid N(z - T) \neq \{0\}\}, \quad (5.16)$$

$$\sigma_c(T) := \{\lambda \in \mathbb{C} \mid (z - T)^{-1} \text{ is densely defined and unbounded}\}, \text{ and} \quad (5.17)$$

$$\sigma_r(T) := \{\lambda \in \mathbb{C} \mid (z - T)^{-1} \text{ exists, but is not densely defined}\}. \quad (5.18)$$

The numbers in $\sigma_p(T)$ are the eigenvalues of T . A vector f in $D(T)$ is an eigenvector of T iff $f \neq 0$ and there exists a number λ such that $Tf = \lambda f$.

Remarks. (i) The spectrum is the disjoint union of the point spectrum, the continuous spectrum, and the residual spectrum.

Definition 5.5.2 (Discrete- and Essential Spectrum) *Let T be a closed operator in a Banach space X . The discrete spectrum $\sigma_d(T)$ is the set of isolated eigenvalues with finite multiplicity. The essential spectrum is $\sigma_e(T) := \sigma(T) \setminus \sigma_d(T)$.*

It seems difficult to say anything in general about the randomness of the previous parts of the spectrum, given a random operator. Because of this we restrict attention to the case where the operator is given by a spectral family.

Definition 5.5.3 (Spectral Family) *Let a Hilbert space \mathcal{H} and a σ -algebra \mathcal{F} be given. A spectral family E is a projection valued function E such that $\mathcal{F} \ni A \mapsto \langle f, E(A)g \rangle$ is a complex measure for all f, g in \mathcal{H} .*

Remarks. (i) This definition is more general than usual, since we do not require $E(\emptyset^c) = I$.

Definition 5.5.4 (Spectral Measure) *Let E be a spectral resolution of a projection in a Hilbert space and let the operator T be given by*

$$T = E[id] = \int zdE(z). \quad (5.19)$$

The measure $A \mapsto \langle f, E(A)f \rangle$ is the spectral measure for the vector f . A measure ρ is a spectral measure for T if it is equivalent with E ; $\rho \ll E$ and $E \ll \rho$.

Remarks. (i) The integral $E[u] = u(T)$ is defined from the spectral measures of the vectors in the Hilbert space.

(ii) If $\{e_n\}$ is an orthonormal basis, then a spectral measure is given by

$$\rho(A) = \sum_{n=1}^{\infty} \langle e_n, E(A)e_n \rangle / 2^n.$$

(iii) $\rho \ll E$ means that $E(A) = 0 \Rightarrow \rho(A) = 0$ for all measurable sets A .

(iv) If T is a selfadjoint operator, then the characteristic function $F(t) := \langle f, e^{itT}f \rangle$ determines the spectral measure of the vector f and so also the spectral family of T .

(v) The spectral measure ρ of f is determined by

$$-\pi\rho((-\infty, t]) = \lim_{\delta \rightarrow 0^+} \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{t+\delta} \text{Im} \langle f, (s + i\epsilon - T)^{-1}f \rangle ds.$$

Definition 5.5.5 (Lebesgue Decomposition of The Hilbert Space) *If T is a self-adjoint operator in a Hilbert space \mathcal{H} , and ρ_f denotes the spectral measure of an vector $f \in \mathcal{H}$, then $\mathcal{H}_{ac} = \{f \mid \rho_f \text{ is absolutely continuous}\}$, $\mathcal{H}_{sc} = \{f \mid \rho_f \text{ is singular continuous}\}$, $\mathcal{H}_p = \{f \mid \rho_f \text{ is atomic}\}$, $\mathcal{H}_c = \{f \mid \rho_f \text{ is continuous}\}$, and $\mathcal{H}_s = \{f \mid \rho_f \text{ is singular}\}$, where absolute continuity and singularity is defined relative to Lebesgue measure.*

Remarks. (i) All of the above subspaces are reducing subspaces for T .

Definition 5.5.6 (The Lebesgue-Hammer Decomposition of the Spectrum) *Let ρ be a Borel spectral measure for T . The pure point spectrum and the Hammer continuous spectrum of T are given by $\sigma_{pp}(T) := \sigma(\rho_a)$ and $\sigma_{hc}(T) := \sigma(\rho_c)$, where $\rho = \rho_a + \rho_c$ is the Hammer decomposition. Let ρ_{ac} , ρ_s , and ρ_{sc} be the absolutely continuous component, the singular component, and the singular continuous component of ρ relatively to a continuous reference measure. This defines the absolutely continuous spectrum $\sigma_{ac}(T) := \sigma(\rho_{ac})$, the singular spectrum $\sigma_s(T) := \sigma(\rho_s)$, and the singular continuous spectrum $\sigma_{sc}(T) := \sigma(\rho_{sc})$ of T .*

Remarks. (i) The notation σ_{hc} is not standard, but used here to avoid confusion with the continuous spectrum σ_c of a closed operator.

(ii) It is possible to have $\sigma_{pp}(T) = \sigma_{ac}(T) = \sigma_{sc}(T)$.

(iii) The equalities $\sigma_{pp}(T) = \overline{\sigma_p(T)}$, $\sigma_{hc}(T) = \sigma_{sc}(T) \cup \sigma_{ac}(T)$, and $\tilde{\sigma}(T) = \sigma_{pp} \cup \sigma_{ac} \cup \sigma_{sc}(T)$ hold, but none of the unions need be disjoint unions.

(iii) The definition gives in particular the above decompositions for a selfadjoint operator, in which case the decomposition is with respect to Lebesgue measure on the real line.

(iv) The decomposition of the spectrum is independent of the chosen spectral measure.

(v) $\sigma(\mu) := \text{supp } \mu := \{x \mid \mu(U) > 0 \text{ for all open } U \ni x\}$.

Theorem 5.5.7 (Spectral Family) *Let T be a selfadjoint random operator in a Hilbert space. The measure given by $A \mapsto \langle f, 1_A(T)g \rangle$ is a random Borel measure on \mathbb{R} for any vectors f, g .*

Proof. Fix $f, g \in \mathcal{H}$. Set $G(z) := \langle f, (z - T)^{-1}g \rangle$. Now

$$\langle f, 1_{(a,b)}(T)g \rangle = \lim_{\delta \rightarrow 0^+} \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [G(x - i\epsilon) - G(x + i\epsilon)] dx, \quad (5.20)$$

for real numbers $a < b$. Observe that $(z, T) \mapsto G(z)$ is continuous and that $|G(z)| \leq \|f\| \|g\| |\operatorname{Im} z|^{-1}$. Observe furthermore that the distribution P_T of T and Lebesgue measure on \mathbb{R} both are σ -finite measures. Fubini's theorem may then be applied to conclude that the above integral gives a random variable, which implies the claimed weak measurability of the spectral family. \square

Theorem 5.5.8 (Random Spectrum) *The spectrum of a normal, random operator is a random set. The Lebesgue-Hammer components of the spectrum of a selfadjoint, random operator in a separable Hilbert space are random sets.*

Proof. The first statement follows from the lower semicontinuity of the spectrum. Let E be the spectral family of the selfadjoint random operator. The spectral measure given by

$$\rho(A) = \sum_{n=1}^{\infty} \langle e_n, E(A)e_n \rangle / 2^n, \quad (5.21)$$

for an orthonormal basis $\{e_n\}$, is a random measure, from the randomness of E . The Lebesgue-Hammer components of a random measure are random measures. \square

For selfadjoint, transitive random operators it is possible to give a direct proof of the nonrandomness of the spectrum.

Theorem 5.5.9 (Pastur) *If T is a selfadjoint, transitive random operator in a separable Hilbert space, then there is a closed set $\Sigma \subset \mathbb{R}$ such that $\sigma(T) = \Sigma$ with probability one.*

Proof. Define the function $f(r, \omega) = d(i + r, \sigma(T(\omega)))$. The value of this function for rational r determines the spectrum of $T(\omega)$. This is so because $z \mapsto d(z, F)$ is continuous when F is closed, and because $x \in \sigma(T) \Leftrightarrow d(i + x, \sigma(T)) = 1$ for a selfadjoint T and a real x . We will prove existence of an A such that $P(A) = 1$ and $f(r, \omega) = g(r)$ for $\omega \in A$. The formula $\|R(z, T)\| = d(z, \sigma(T))^{-1}$ implies that $f(r, \omega)$ is a transitively invariant measurable function for each $r \in \mathbb{Q}$. This gives an A_r and a $g(r)$ such that $f(r, \omega) = g(r)$ for $\omega \in A_r$, $P(A_r) = 1$. $A = \bigcap_{r \in \mathbb{Q}} A_r$ gives the required A .

The invariance follows from the unitary equivalence of the resolvents $R(\omega)$ and $R(\Theta_\nu \omega)$, which follows from the unitary equivalence of $T(\omega)$ and $T(\Theta_\nu \omega)$.

The measurability of $\omega \mapsto \|R_\omega\|$ follows directly from

$$\|R_\omega\| = \sup\{\|R_\omega f\| \mid f \text{ is in a countable dense subset of the unit ball}\}, \text{ and}$$

$$\|R_\omega f\|^2 = \sum_{i=1}^{\infty} \langle R_\omega f, e_i \rangle \langle e_i, R_\omega f \rangle.$$

Here we used the separability of the Hilbert space and preservation of measurability in limits. \square

A more general proof is also possible.

Theorem 5.5.10 (Nonrandom Spectrum) *If T is a normal, transitive random operator in a separable Hilbert space, then*

$$\sigma(T) = \overline{\bigcup_{S \in D} \sigma(S)} \text{ with probability one,} \quad (5.22)$$

for any set $D \subset \text{supp } P_T$ with $P(T \in \overline{D}) > 0$. The Lebesgue-Hammer components of the spectrum of T , are given by nonrandom sets, if T in addition is selfadjoint.

Proof. The spectrum and its Lebesgue-Hammer components are given by nonrandom sets, since they all are transitively invariant measurable functions into a σ -separated measurable space. The lower operator topology is second countable, so the topological support of T 's distribution has measure one, so there exists a set $D \in \text{supp } P_T$ with $P(T \in \overline{D}) > 0$. This and the lower semicontinuity of the mapping $T \mapsto \sigma(T)$ give the explicit formula for the spectrum. \square

Remarks. (i) The spectrum of a normal, transitive random operator is nonrandom, even if the Hilbert space is nonseparable.

(ii) The nonrandom spectrum increase when the topological support of T 's distribution increase.

The localization of the Lebesgue-Hammer components of the spectrum is a delicate matter. Existence or absence of Lebesgue-Hammer components in the spectrum may be determined by the following theorems.

Theorem 5.5.11 (The Continuous Subspace) *If T is a selfadjoint operator, then f is in the continuous subspace of T if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N |\langle f, e^{-itT} f \rangle|^2 dt = 0. \quad (5.23)$$

Proof. Let μ be the Borel measure determined by $\mu(A) = \langle f, 1_A(T) f \rangle$. The vector f is in the continuous subspace of T iff μ is a continuous measure. Observe

$$F(t) := \langle f, e^{-itT} f \rangle = \int e^{-itx} d\mu(x), \quad (5.24)$$

which is the characteristic function of μ . A theorem of Wiener gives

$$\sum_{x \in \mathbb{R}} |\mu(\{x\})|^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N |F(t)|^2 dt, \quad (5.25)$$

which implies the theorem. \square

Remarks. (i) If $\lim_{t \rightarrow \infty} \langle f, e^{-itT} f \rangle = 0$, then f is in the continuous subspace of T .

(ii) The characteristic function of a finite measure determines the measure completely.

Theorem 5.5.12 (The Atomic Subspace) *If T is a selfadjoint operator, and f is in the atomic subspace of T , then*

$$t \mapsto F(t) := \langle f, e^{-itT} f \rangle \quad (5.26)$$

is almost periodic.

Proof. Let μ be the Borel measure determined by $\mu(A) = \langle f, 1_A(T)f \rangle$. The vector f is in the atomic subspace of T iff μ is an atomic measure. Now

$$F(t) = \int e^{-itx} d\mu(x), \quad (5.27)$$

which is the Fourier transform of μ . If μ is an atomic measure with weights μ_1, μ_2, \dots , then $F(t) = \sum_j \mu_j e^{it\lambda_j}$, which is an almost periodic function. \square

Remarks. (i) A more direct proof follows from an expansion of f in the eigenvectors of T .

(ii) If $F(t) = \sum_j \mu_j e^{it\lambda_j}$, then f is in the atomic subspace of T . The frequencies λ_j of F are eigenvalues of T .

Theorem 5.5.13 (The Absolutely Continuous Subspace) *If T is a selfadjoint operator, and if*

$$t \mapsto F(t) := \langle f, e^{itT} f \rangle \quad (5.28)$$

is integrable, then the spectral measure of f is absolutely continuous with a bounded continuous density given by

$$t \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-its} ds. \quad (5.29)$$

If $F(t) \geq 0$, then F is integrable if and only if the spectral measure of f has a bounded density.

Proof. This is a corollary of theorems concerning the characteristic function of a finite measure. \square

5.6 Notes

Topologies for closed unbounded operators are introduced in [104], [75, p.197-], [124, vol.I,p.283-]. The strong resolvent topology in the set of selfadjoint operators is not actually defined in any of these sources, but rather the notion of strong resolvent convergence of sequences. It is necessary to generalize from sequences to nets to get a topology. Strong convergence of $(z - T_\lambda)^{-1}$ from strong convergence of $(z_0 - T_\lambda)^{-1}$ for a sequence of selfadjoint operators is proven in [136, p.283, Theorem 9.15], but the same proof does also work for nets. We have not found any references on σ -algebras in sets of (selfadjoint) closed operators, and only a few on topologies. The lower operator topology is perhaps introduced here for the first time, and it seems to give some simplifications.

We have used the fact that a bounded set in the space of bounded operators on a separable Hilbert space is a separable metric space in both the weak and the strong operator topology. Proof of this may be found in [110, p.172].

The definitions of random operators in the sense of Skorohod are adapted from [125, p.1-3]. See also [49].

Pastur random operators are defined almost as in [107, p.13],[109, p.9]. See [107, Remark on p.35] for a short discussion on the choice of definition. Pastur does not assume existence of a common countable core in his definition, but this is an additional assumption in important theorems [107, p.37-, (2.4) Theorem]. A weak Skorohod random operator is also a Pastur random operator, if it obeys an additional requirement [125, p.33].

The representation of closed operators by matrix operators is found in [136, p.149-]. Closed random operators from random matrices and random orthonormal systems, are not considered in any of the given sources. It gives a proper view of the difference between Pastur random operators and the more general definition we have introduced. A class of random matrix operators which are essentially selfadjoint on the set of finite sequences is given in [107, p.67-71].

Measurability of selfadjoint operators is also relevant for direct integral decompositions of operators, [124, p.283, vol.IV]. A random selfadjoint operator may be used to define a new operator, the direct integral product of the realizations.

Randomness of $(z - T)^{-1}$ from randomness of $(z_0 - T)^{-1}$ is claimed in [79, p.286], but they refer to [80] for the proof. We can not find this claim in [80, p.146]. Another reference for this claim is [5, p.334], but the proof is missing.

Limits of selfadjoint random operators are treated in [31, p.244], [80, p.146]. The result follows directly from the Trotter-Kato theorem [124, vol.I,p.288]. The sum of two selfadjoint random operators is also considered in [80, p.146]. The Trotter product formula is proved in [124, vol.I,p.295-,p.308], [30, p.192-,p.304]. The idea of using the Trotter product formula for the definition of the sum of unbounded operators is not new [35], and the same idea applies also to the Schrödinger operator and the Feynmann-Kac formula [102].

The core spectrum is defined in [27, p.83], but compare also [136, p.229-230]. Continuity properties of the spectrum as a set valued mapping is considered in [105], [75, p.208, p.431], [136, p.217-218,p.272-273,p.286-287]. The result for normal operators in Theorem 5.4.12 seems to be new. The lower semicontinuity of the spectrum for selfadjoint operators and the upper compact semicontinuity for closed operators is well known, compare [105], [75, p.208, p.431]. The presented proof of upper compact semicontinuity seems to be a

simplification.

The definition of a transitive random operator is similar to the one found in [79, p.287], [39, p.167], [31, p.252] in the same context. The definition goes back to L.A. Pastur. See [109, p.9] [107] and the literature cited there. The term metrically transitive operator (*mto*) is used in [109] for closed operators in a Hilbert space. Our term ergodic operator corresponds to Pastur's definition of a *mto*.

Halmos [66, p.58] uses the term spectral measure to denote what we have defined to be a spectral family. We follow closely the conventions in [136, p.180-228].

Theorem 5.5.9 is essentially due to Pastur [108]. Measurability of the components of the spectrum is treated nicely in [31, p.245]. Alternative approaches are found in [80, p.148-], [107, p.44],[39, p.170].

The idea of the proof of Theorem 5.5.10 is as in [79, p.305, Theorem 2], but our result is more general. Less general results are found in [107, p.85-86] and [31, p.264-267].

The stated Wiener theorem used in the proof of Theorem 5.5.11 is found in [39, p.98], and is related to the RAGE theorem [124, p.340-344, vol.III] known in scattering theory. H. Martens suggested to use the theory of almost periodic functions, when I asked him a question strongly related to Theorem 5.5.12. I acknowledge his helpful hint. Theorem 5.5.13 follows from the results in [57, p.509-510].

Part III

Models

Chapter 6

Discrete Random Multiplication Operators

A random sequence is interpreted as a random multiplication operator. We investigate the properties of this random operator. The motivation for the study is the electron theory of disordered solids, and more generally the theory of linear random operators. We gain insight into general concepts from this elementary model.

6.1 Introduction

The discrete Schrödinger operator

$$H = -\Delta + V, \tag{6.1}$$

is central in the electron theory for solids. When the potential V is neither short range, nor periodic, the analysis is difficult. The seemingly best approach, both from a practical and a fundamental point of view, is given by a statistical description of V .

The discrete Laplacian Δ is a bounded operator, and some of the properties of H are given directly from properties of the multiplication operator. On the other hand the random Schrödinger operator is so complicated that it is an invitation to a study of the general concept of a linear random operator. The general theory is to a large extent still missing, and one approach is to learn from the simplest possible examples. Most of the questions posed for the random Schrödinger operator are completely solvable for the random multiplication operator.

Convergence of unbounded operators is more delicate than convergence of bounded operators. This is because unbounded closed operators can never act on the whole space, which in turn makes it impossible in general to compare two operators by means of their difference. The intersection of the operator domains may easily be $\{0\}$. This domain problem is even worse for random operators. A solution to these problems is given by the introduction of a proper topology for unbounded closed operators. The gap topology, which is a proper generalization of the norm topology, is studied in some detail in the literature, but it is not well suited for the definition of linear random operators. A weaker topology is needed.

The lower topology seems to be a good choice, but a thorough study of this topology is missing. For the multiplication operator we prove continuity of the mapping which maps the potential to the multiplication operator, continuity in the sense given by the pointwise convergence topology and the lower topology. We do also prove continuity of the mapping of the multiplication operator to its spectrum. A corollary of these two results is that the spectrum of the random multiplication operator is a random set. We identify operator topologies with more elementary topologies for the potentials. An example is given by the strong resolvent topology which is shown to be equivalent with the pointwise convergence topology, when we consider real valued potentials.

Our discussion includes generalized random potentials, which simply means that the potential may take the value ∞ . The multiplication operator is defined by a Dirichlet boundary condition at the points where the potential equals ∞ . The corresponding multiplication operators can still be compared, since we have avoided the domain problem by the introduction of a proper topology. We consider random potentials given by random coupling constants. In particular we demonstrate that a singular continuous distribution for $V(x)$ may result from reasonable distributions for the coupling constants. The possibility of a singular continuous single site distribution seems to have attracted no attention in the literature yet. It is in fact usual to assume the single site distribution to be absolutely continuous.

The density of states is important in solid state physics. For the ergodic multiplication operator it is shown to be equal to the single site distribution, so it may in particular be singular continuous. The spectrum of the ergodic multiplication operator is equal to the union of the supports of the single site distributions.

6.2 The Hilbert Space

To avoid possible confusion we start with an elementary definition.

Definition 6.2.1 (Countable Sets) *A set is countable iff it is finite or is in one-one correspondence with the set of natural numbers.*

Let X be a countable set. The Hilbert sequence space $\mathcal{H} = l^2(X)$ will be used in the following. The elements in \mathcal{H} are complex valued functions f with domain X and finite norm, $\|f\| = \sqrt{\langle f, f \rangle} < \infty$. The inner product is given by

$$\langle f, g \rangle = \sum_{x \in X} \bar{f}(x)g(x). \quad (6.2)$$

Any sum over a countable set is to be interpreted as given by the counting measure $\#$, if not explicitly stated otherwise. In particular this implies that the series is assumed to be absolutely convergent.

Let the Kronecker delta function be defined by $\delta_x = 1_{\{x\}}$. A natural orthonormal basis for \mathcal{H} is given by $\{\delta_x\}, x \in X$. The Kronecker delta vector does also give a reproducing kernel for \mathcal{H} , as expressed by

$$\langle \delta_x, f \rangle = f(x). \quad (6.3)$$

We define the subspace $\mathcal{H}_0 = \{f \mid \#(f \neq 0) < \infty\}$. The subspace \mathcal{H}_0 of finitely supported functions is dense in \mathcal{H} , and the Kronecker delta functions give an algebraic basis for \mathcal{H}_0 . For any subset Y of X , we will identify $l^2(Y)$ with $l^2(Y) \oplus \{f \in l^2(Y^c) \mid f(x) = 0\}$, a subspace of $l^2(X)$.

6.3 Perturbation of Discrete Multiplication Operators

When considering convergence of unbounded potentials, it is natural to extend the class of potentials, and allow the potential to take the value ∞ .

Definition 6.3.1 (Generalized Potentials) *Let X be a countable set. A potential on X is a real valued function defined on X . A complex potential on X is a complex valued function defined on X . A generalized potential V on X is a function, $V : X \rightarrow \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. The set of potentials on X , the set of complex potentials on X , and the set of generalized potentials on X are denoted by respectively $G_p(X)$, $G_c(X)$, and $G(X)$. The proper domain of V in $G(X)$ is the set $X(V) := \{x \in X \mid V(x) \neq \infty\}$.*

Remarks. (i) The sets $G_p(X)$ and $G_c(X)$ are vector spaces in a natural way, but $G(X)$ is not a vector space.

(ii) The inclusions $G_p(X) \subset G_c(X) \subset G(X)$ hold, with obvious (necessary) identifications.

(iii) The natural topology in $\hat{\mathbb{C}}$ is given from the natural topology in \mathbb{C} with one addition: The complements of compact sets in \mathbb{C} give the neighborhood basis for the point ∞ .

Definition 6.3.2 (Discrete Multiplication Operators) *Let V be a generalized potential on a countable set X . The discrete multiplication operator $M = M(V)$ is defined by*

$$D(M) := \{f \in l^2(X) \mid f(x) = 0, \text{ when } x \notin X(V), \sum_{x \in X(V)} |f(x)V(x)|^2 < \infty\}$$

$$Mf(x) = \begin{cases} V(x)f(x) & x \in X(V) \\ 0 & \text{otherwise.} \end{cases} \quad (6.4)$$

Remarks. (i) The multiplication operator is closed, but not densely defined when $X(V) \neq X$. The subspace $\mathcal{H}_0(V) := \{f \in \mathcal{H}_0 \mid \text{supp } f \subset X(V)\}$ is an operator core.

(ii) The space $\mathcal{H}(V) := l^2(X(V))$, viewed as a subspace of $l^2(X)$, satisfies $M(\mathcal{H}(V) \cap D(M)) \subset \mathcal{H}(V)$ and $M(\mathcal{H}(V)^\perp \cap D(M)) \subset \mathcal{H}(V)^\perp$, so $\mathcal{H}(V)$ is a reducing subspace for M . The M_n , defined by restricting M to $l^2(X(V))$, is a normal operator, and $M = M_n \oplus 0_t$, where $0_t = 0_t(V)$ is the trivial operator given by $D(0_t) := \{f \in l^2(X(V)^c) \mid f = 0\}$, $0_t 0 = 0$.

The mapping $V \mapsto M(V)$ gives a one-one correspondence between generalized potentials V and multiplication operators M . The set of multiplication operators inherits operator topologies, as a subset of the linear operators on the Hilbert space \mathcal{H} . Likewise, the set of potentials, has many natural topologies, as a set of functions.

Definition 6.3.3 (Bounded and Decaying Potentials on a Countable Set) *A complex potential V on a countable set is bounded iff $\|V\|_\infty := \sup_x |V(x)| < \infty$. The complex potential equals z at infinity, written $V(\infty) = z$, iff $\{x \mid |V(x) - z| > \epsilon\}$ is finite for any given positive ϵ . The complex potential is decaying iff $V(\infty) = 0$.*

Remarks. (i) The notation $V(\infty) = z$ may be misleading, since ∞ is not a point in the domain of V .

(ii) If $\{x \mid |V(x)| \leq K\}$ is finite for any given positive K , we write $V(\infty) = \infty$.

Theorem 6.3.4 (Multiplication Operators from Bounded Potentials) *The l^∞ norm of a complex potential on a countable set is equal to the operator norm of the corresponding multiplication operator. A complex potential is decaying iff the multiplication operator is compact. The B^p norm of $M(V)$, from the trace, is equal to the l^p norm of V .*

Proof. The operator M is bounded iff V is bounded, because

$$\|Mf\|^2 = \sum_x |V(x)f(x)|^2 \leq (\|V\|_\infty \|f\|)^2, \text{ so } \|M\| \leq \|V\|_\infty. \quad (6.5)$$

Equality of the norms follows from $\|M\delta_x\| = |V(x)|$.

The operator M is the strong limit of finite range operators $M(V1_A)$, and it is the norm limit of such finite range operators if the potential is decaying. In the last case M is compact. If we assume that M is compact, then it follows that the eigenvalues can only accumulate at 0, which means that the potential is decaying. The B^p norm is given by

$$(\|M\|_p)^p = \text{tr}([MM^*]^{p/2}) = \sum_x \langle \delta_x, |V|^p \delta_x \rangle = \sum_x |V(x)|^p = (\|V\|_p)^p, \quad (6.6)$$

so we have one-one correspondence between the l^p spaces and the set of multiplication operators in B^p . \square

Remarks. (i) The space $B^1(\mathcal{H})$ is the set of trace class operators, the space $B^2(\mathcal{H})$ is the set of Hilbert-Schmidt operators, and the space $B^\infty(\mathcal{H})$ is the set of compact operators. The inclusions $B^1 \subset B^p \subset B^\infty \subset B$, the triangle (or Minkowski) inequality $\|S + T\|_p \leq \|S\|_p + \|T\|_p$, and the Hölder (Schwarz, when $p = 2$) inequality $\|ST\|_1 \leq \|S\|_p \|T\|_q$ hold for $p \geq 1 = 1/p + 1/q$.

Now we turn to a weaker convergence concept, valid for all generalized potentials. The notation $T = l\text{-}\lim T_\lambda$ means that the net $\{T_\lambda\}$ converges towards the closed operator T , in the lower topology for closed operators. This topology is the restriction of the lower topology for closed sets to the set of graphs of closed operators in a given Banach space. This topology gets an easier description when it is restricted to the set of multiplication operators.

Theorem 6.3.5 (Lower Graph Convergence of Multiplication Operators) *Equip the set of generalized potentials on a countable set with the direct product topology and equip the corresponding set of multiplication operators with the lower topology for closed operators. The mapping $V \mapsto M = M(V)$, which maps a generalized potential V to the multiplication operator M , is a continuous mapping. Conversely, if $M(V) = l\text{-}\lim M(V_\lambda)$, then $V(x) = \lim V_\lambda(x)$ for all x in the proper domain of V .*

Proof. Let $\{V_\lambda\}$ be a convergent net of generalized potentials, with limit V in the product topology, $V = p\text{-}\lim V_\lambda$. The claim is that $M := M(V) = l\text{-}\lim M_\lambda$, with $M_\lambda := M(V_\lambda)$. Let $\epsilon > 0$ and $f \in D(M)$ be given. Choose a finite $K \subset X(V)$ such that $\|f1_{K^c}\| < \epsilon$ and $\|Vf1_{K^c}\| < \epsilon/2$. Choose N such that $\|(V_\lambda - V)f1_K\| < \epsilon/2$ for all $\lambda \geq N$. The conclusions are that $g = f1_K \in D(M_\lambda)$, $\|g - f\| < \epsilon$, and $\|V_\lambda g - Vf\| < \epsilon$, for all $\lambda \geq N$. This proves that for a given open U which intersects the graph $G(M)$, there exists an N_U such that U intersects the graph $G(M_\lambda)$ when $\lambda \geq N_U$, so $G(M) = l\text{-}\lim G(M_\lambda)$, which proves the first claim.

Fix an x such that $V(x) \neq \infty$ and let δ be an arbitrary positive number. The pair $(\delta_x, \delta_x V(x))$ belongs to the graph of M . The function $\phi(a, b) := a/b$ is continuous at $(V(x), 1)$. Choose $\epsilon > 0$ such that $|\phi(a, b) - \phi(V(x), 1)| < \delta$ when $|a - V(x)|, |b - 1| < \epsilon$. The set U of pairs (f, h) with $f, h \in l^2(X)$, $|f(x) - 1|, |h(x) - V(x)| < \epsilon$ intersects the graph of M . This U is open, because $g \mapsto g(x)$ is continuous. The assumption $M = l\text{-}\lim M_\lambda$ gives a λ_δ such that the graph G_λ intersects U when $\lambda \geq \lambda_\delta$. Let $\lambda \geq \lambda_\delta$. Then we have an f_λ in $D(M_\lambda)$ such that both $|f_\lambda(x) - 1|$ and $|V_\lambda(x)f_\lambda(x) - V(x)|$ are less than ϵ . Now $|V_\lambda(x) - V(x)|$ equals $|\phi(V_\lambda(x)f_\lambda(x), f_\lambda(x)) - \phi(V(x), 1)|$, which is less than δ . This proves the claim $V(x) = \lim V_\lambda(x)$. \square

Remarks. (i) In the last part we assumed convergence $G(M) = l\text{-}\lim G(M_\lambda)$ as given by the norm topology on $[l^2(X)]^2$. The proof shows that the conclusion also holds if we define graph convergence from the restriction of the product topology to $l^2(X)$.

(ii) The lower topology is not a Hausdorff topology on the set of closed operators, so the statement $M = l\text{-}\lim M_\lambda$ may be true for many multiplication operators. Note that the trivial multiplication operator $M(\infty)$, with graph $G(M(\infty)) = \{(0, 0)\}$, is a limit of any net of multiplication operators. The operator $M(p\text{-}\lim V_\lambda)$ is a unique operator, because the set of generalized potentials may be identified with a compact, complete metric space.

(iii) The restriction of the lower topology to the set of selfadjoint operators, is equal to the strong resolvent topology, and is in particular a Hausdorff topology. The set of potentials with the product topology is therefore homeomorphic with the corresponding set of selfadjoint multiplication operators equipped with the strong resolvent topology.

The lower operator topology restricted to $B(\mathcal{H})$, the bounded operators on \mathcal{H} , is slightly weaker than the strong operator topology on $B(\mathcal{H})$. Now we turn to the gap topology for closed multiplication operators, which is equal to the norm topology, when restricted to bounded operators. First we consider only complex potentials.

Definition 6.3.6 (The V -bound of a Complex Potential) *Let V and W be complex potentials on a countable set X . The potential W is V -bounded if there exist numbers $a, b \geq 0$ such that*

$$|W(x)| \leq a + b|V(x)|, \quad \forall x \in X. \quad (6.7)$$

The infimum of all such b 's is the V -bound of W .

Remarks. (i) One may consider also generalized potentials, but then it seems natural only to consider the case $X(V) = X(W)$, since this concept is tailored for consideration of $V + W$, regarding W as a perturbation of V .

Theorem 6.3.7 (Relatively Bounded Multiplication Operators) *Let V and W be complex potentials on a countable set. The multiplication operator $M(W)$ has $M(V)$ -bound α iff W has V -bound α .*

Proof. Assume first that $M(W)$ has $M(V)$ -bound α , so

$$\|M(W)f\| \leq a\|f\| + b\|M(V)f\|, \quad \forall f \in D(M(V)) \subset D(M(W)), \quad (6.8)$$

for an arbitrary $b > \alpha$ and a suitable a , depending on b . Let $x \in X$ and consider $f = \delta_x$. It follows that

$$|W(x)| \leq a + b|V(x)|, \quad \forall x \in X, \quad (6.9)$$

so W has V -bound α .

Now assume that W has V -bound α and let a, b be as in the previous equation. We prove

$$\|M(W)f\| \leq \sqrt{a^2 + ab/t}\|f\| + \sqrt{b^2 + abt}\|M(V)f\|, \quad t > 0, \forall f \in D(M(V)), \quad (6.10)$$

which proves the claim, in particular also $D(M(V)) \subset D(M(W))$, since t is arbitrary. The above estimate follows from

$$\|Wf\| \leq A + B \text{ if } \|Wf\|^2 \leq A^2 + B^2, \quad A, B \geq 0 \quad (6.11)$$

$$\|Wf\|^2 \leq a^2\|f\|^2 + b^2\|Vf\|^2 + R \quad (6.12)$$

$$R = 2ab \sum_x |f(x)|(|V(x)f(x)|) \leq ab(\|f\|^2/t + \|Vf\|^2 t), \text{ from} \quad (6.13)$$

$$qr \leq (q^2t + r^2/t)/2, \quad q, r \geq 0, t > 0. \quad (6.14)$$

□

The convergence concept corresponding to relative boundedness is given by the following.

Definition 6.3.8 (V -convergence) *The sequence V_1, V_2, \dots of complex potentials on a countable set V -converges towards the complex potential V if*

$$|V_i(x) - V(x)| \leq a_i + b_i|V(x)|, \quad \forall x \in X \quad (6.15)$$

for positive numbers a_i, b_i , converging towards 0.

Remarks. (i) If we consider a bounded potential V , then V -convergence is equivalent with l^∞ convergence. If $V_i - V$ is bounded, then uniform convergence can be defined. The V -convergence can take place even when $V_i - V$ is unbounded.

(ii) Gap convergence seems to be the natural extension of norm convergence, when dealing with generalized potentials. Gap convergence is defined from the Hausdorff distance between the unit spheres of the multiplication operator graphs. Norm resolvent convergence is defined in the case where $\sigma(M(V)) \neq \mathbb{C}$, and is then equivalent with gap convergence. Norm resolvent convergence or V -convergence implies gap convergence.

Gap convergence does not imply V -convergence, as can be seen from the following example. Let $X = \mathbb{N}$, $V(x) = x$ and $V_i(x) = x + \alpha_i x^2$, with positive numbers α_i converging towards 0. We find

$$\|M(V_i)^{-1} - M(V)^{-1}\| = \alpha_i / (1 + \alpha_i) \rightarrow 0, \quad (6.16)$$

so $M(V_i)$ converges towards $M(V)$ in norm resolvent sense and therefore also in the gap topology. On the other hand we have $|V_i(x) - V(x)| = \alpha_i x^2$, so $|V_i(x) - V(x)| \leq a_i + b_i |V(x)|$ is impossible to satisfy. If we consider the involved spectra we conclude that upper semicontinuity fails, but upper compact semicontinuity holds.

6.4 Discrete Ergodic Multiplication Operators

Armed with the previous topological notions we are ready to deal with random multiplication operators, given by generalized random potentials.

Definition 6.4.1 (Generalized Random Potentials) *Let X be a countable set, and equip the sets $G(X)$, $G_c(X)$, and $G_p(X)$ with the Borel σ -algebra from the direct product topologies. A generalized random potential on X , a complex random potential on X , or a random potential on X is a measurable mapping from a probability space into respectively $G(X)$, $G_c(X)$, or $G_p(X)$.*

Remarks. (i) A random potential on X is, from the above, nothing but a real valued stochastic process, indexed by X .

The prototype for a linear random operator is given by a random potential, which is the content of the following.

Theorem 6.4.2 (Random Multiplication Operators) *Let V be a generalized random potential on a countable set. The multiplication operator $M(V)$ is then a linear random operator, selfadjoint iff V is a random potential.*

Proof. Let U be a lower open set of closed operators defined on $l^2(X)$. The set $\{V \mid M(V) \in U\}$ is open in the product topology, since $V \mapsto M(V)$ is continuous. The assumed measurability of $\omega \mapsto V_\omega$ implies then that $\{\omega \mid M(V_\omega) \in U\}$ is measurable, so M is a random operator. The operator $M(V)$ is selfadjoint iff V is real valued, so $M(V)$ is selfadjoint iff V is a potential. \square

The applications in solid state physics give an effective potential as a result of an infinity of individual atomic potentials. Randomness in the effective potential is obtained, in the simplest cases, by introducing random coupling constants. From a mathematical point of view the procedure gives a construction of a more interesting process from an elementary one. The following theorem is a starting point.

Theorem 6.4.3 (Potentials from Random Coupling Constants) *Let X be a countable set and Γ a family of mappings $\gamma : X \rightarrow X$ such that $\Gamma \ni \gamma \mapsto \gamma(x) \in X$ is injective for each $x \in X$. Let Q_γ , for γ in Γ , be a real valued stochastic variable, such that $\sup_\gamma E[|Q_\gamma|^r] < \infty$ for some $r \geq 1$. Let a potential $v \in l^1(X)$ be given. The pointwise limit*

$$V(x) = \sum_{\gamma \in \Gamma} Q_\gamma v(\gamma(x)) \quad (6.17)$$

exists with probability one, and defines a selfadjoint, random multiplication operator $M(V)$.

Proof. For $r \geq 1$

$$E[|Q_\gamma|] = E[(1_{|Q_\gamma| \leq 1} + 1_{|Q_\gamma| > 1})|Q_\gamma|] \leq 1 + E[|Q_\gamma|^r], \quad (6.18)$$

so $\sup_\gamma E[|Q_\gamma|] < \infty$.

Fix x . The injectivity of $\gamma \mapsto \gamma(x)$ implies that Γ is countable, in fact in one-one correspondence with a subset of X . Let $\gamma_1, \gamma_2, \dots$ be an enumeration of the elements in Γ and set $Q_i := Q_{\gamma_i}$, $v_i := v(\gamma_i(x))$. We prove existence of

$$\sum_{i=1}^{\infty} |Q_i v_i| \quad (6.19)$$

with probability one, so that the sum in equation (6.17) is absolutely convergent for all x with probability one. It follows then that V is a random potential, since it is a pointwise limit of a sequence of random potentials and $M(V)$ is then a selfadjoint random operator. The monotone convergence theorem gives

$$E\left[\sum_{i=1}^{\infty} |Q_i v_i|\right] = \lim_N \sum_{i=1}^N E[|Q_i| |v_i|] \leq \sup_\gamma E[|Q_\gamma|] \|v\|_\infty < \infty. \quad (6.20)$$

This proves the claim, since otherwise we get a contradiction. \square

Remarks. (i) The Q_γ represents, in applications, a coupling constant and v an effective atomic potential. The potential V is then the resulting potential from charges distributed according to a lattice Γ .

(ii) The nonrandom case is included in the above and the assumption becomes $\sup_\gamma |Q_\gamma| < \infty$, so the coupling constants must be uniformly bounded. The random case gives a generalization which includes possibly unbounded coupling constants. All of the Q_γ 's may be Gaussian variables with a common finite variance securing the assumption in the theorem.

(iii) If the coupling constants decay at infinity, then v need not decay so fast as when in l^1 . Assume that $(E[|Q_i|^r])^{1/r}$ is in l^q and v is in l^p for some $r, p \geq 1$, $1/q + 1/p = 1$. The Minkowski inequality may then be applied to prove $L^r(P)$ convergence of the stochastic variables, which is sufficient for the definition of the random potential. The above theorem corresponds to $q = \infty$ and $p = 1$, so we may add $L^r(P)$ convergence in the conclusion of the theorem.

Consider the example given by $X = \mathbb{N}_0$, $\Gamma = \mathbb{N}$, $\gamma(x) = \gamma + x$,

$$v(x) = 2^{-|x|}, \text{ and} \quad (6.21)$$

$$V(x) = \sum_{\gamma \geq 1} Q_\gamma v(x + \gamma), \quad (6.22)$$

with Q_1, Q_2, \dots i.i.d., $P(Q_1 = 1) = p \in (0, 1)$, $P(Q_1 = 0) = q$, and $p + q = 1$, so $\{Q_\gamma\}$ is a Bernoulli sequence. The resulting potential is

$$V(x) = V(0)v(x), \quad (6.23)$$

and the single random parameter $V(0)$ has a distribution function $F(x) = P(V(0) \leq x)$. A nice application of the ergodic theorem gives that the distribution of $V(0)$ is a continuous measure with support $\text{supp } dF = [0, 1]$, and singular with respect to Lebesgue measure iff $p \neq 1/2$. The references contains a proof of this statement.

Consider next the potential

$$W(x) = \sum_{n \in \mathbb{Z}} Q_n v(x + n), \quad (6.24)$$

with a doubly infinite Bernoulli sequence $\{Q_n\}$. The potential W is ergodic, in particular stationary, but more exotic than the Bernoulli sequence. The single site distribution is, with notation as in the previous example,

$$\begin{aligned} P(W(0) \leq t) &= qG(t) + pG(t - 1) \\ G(t) &= \int_{x=0}^t F(t - x) dF(x). \end{aligned} \quad (6.25)$$

The distribution of $W(0)$ is concentrated at $[0, 2]$, is continuous, and singular with respect to Lebesgue measure iff $p \neq 1/2$, which follows from the properties of F .

A sketch of the proof of equation (6.25) is given by

$$\begin{aligned} W(0) &= Q_0 + \sum_{n \geq 1} Q_n / 2^n + \sum_{n \geq 1} Q_{-n} / 2^n \\ P(W(0) \leq t) &= \sum_{k=0}^1 P(Q_0 = k) P(Y_1 + Y_2 \leq t - k) \\ P(Y_1 + Y_2 \leq t) &= \int F(t - x) dF(x) \\ F(x) &= P(Y_i \leq x), \end{aligned} \quad (6.26)$$

where independence is the main point.

Definition 6.4.4 (Induced Transformations) *Let G be the set of generalized potentials on a countable set X . A mapping $\gamma : X \rightarrow X$ defines the composition T_γ , the composition operator U_γ , and the count N_γ :*

$$\begin{aligned} T_\gamma : G &\rightarrow G, \quad T_\gamma V(x) := V(\gamma(x)) \\ U_\gamma : D(U_\gamma) &\rightarrow l^2(X), \quad D(U_\gamma) := \{f \in l^2(X) \mid f \circ \gamma \in l^2(X)\}, \quad U_\gamma f := f \circ \gamma \\ N_\gamma : X &\rightarrow \hat{\mathbb{C}}, \quad N_\gamma(y) := \#\{x \in X \mid \gamma(x) = y\}. \end{aligned} \quad (6.27)$$

A transformation on X is an invertible mapping from X onto X .

Remarks. (i) It is sometimes convenient to use the same symbol, say γ , to denote all of the three different mappings U_γ , T_γ , and γ . Then $\gamma V = T_\gamma V$, $\gamma f = U_\gamma f$, and $\gamma(x) = \gamma(x)$. (ii) The count N_γ is a generalized potential on X .

The following definition is used in the next theorem, and also when we consider functions of multiplication operators.

Definition 6.4.5 (Functions of a Generalized Potential.) *Let V be a generalized potential on a countable set X . Let f be a complex valued function with a domain containing $V(X(V))$. The generalized potential $f(V)$ is defined by*

$$[f(V)](x) := \begin{cases} \infty & V(x) = \infty \\ f(V(x)) & V(x) \neq \infty \end{cases} . \quad (6.28)$$

Remarks. (i) If V is a generalized random potential, and $f : \mathbb{C} \rightarrow \mathbb{C}$ is measurable, then $f(V)$ is a generalized random potential.

The unitarity of U_γ for a transformation γ is elementary, and is all we will need in the following. It is however possible to treat the general case also, without too much effort. The following theorem does this, with the aid of the generalized potential given by the count N_γ .

Theorem 6.4.6 (The Composition Operator) *Let $\gamma : X \rightarrow X$, where X is a countable set. The composition operator U_γ is a linear, closed operator with domain $D(U_\gamma) = D(M(\sqrt{N_\gamma}))$. The range $R(U_\gamma)$ equals $\{f \in l^2(X) \mid \gamma(x) = \gamma(y) \Rightarrow f(x) = f(y)\}$. The formula $\|U_\gamma f\| = \|M(\sqrt{N_\gamma})f\|$ holds for all f in l^2 , where the value ∞ is allowed for the l^2 norm. The U_γ is bounded iff $M(N_\gamma)$ is bounded, and then $\|U_\gamma\| = \sqrt{\|M(N_\gamma)\|}$. The inverse U_γ^{-1} exists iff $\gamma(X) = X$, or equivalently, iff $N_\gamma(x) \neq 0$ for all x in X . If U_γ^{-1} is densely defined, then γ is a transformation of X . The U_γ is unitary iff γ is a transformation of X , and then $U_\gamma^* = U_\gamma^{-1} = U_{\gamma^{-1}}$.*

Proof. The linearity of U_γ follows from $(\alpha f) \circ \gamma = \alpha(f \circ \gamma) \in l^2$ if $f \circ \gamma \in l^2$ and $(f + g) \circ \gamma = f \circ \gamma + g \circ \gamma \in l^2$ if $f \circ \gamma, g \circ \gamma \in l^2$.

The norm equalities follow from $A_y := \{x \mid a(x) = y\}$ and $\|U_\gamma f\|^2 = \sum_x |f(\gamma(x))|^2 = \sum_y \sum_{x \in A_y} |f(y)|^2 = \sum_y N(y) |f(y)|^2$. This also gives the domain and boundedness claim.

If $\gamma(x) = \gamma(y)$, then $f(x) = g(\gamma(x)) = f(y)$. Given f we can define g on $\gamma(X)$ so $f(x) = g(\gamma(x))$, and set $g = 0$ elsewhere. This gives the range claim.

Let $0 = \lim f_i$, with $f_i \in D(U_\gamma)$ and so that $\lim U_\gamma f_i$ exists. Since $M(\sqrt{N_\gamma})$ is closed it follows that $\lim \|U_\gamma f_i\| = \lim \|M(\sqrt{N_\gamma})f_i\| = 0$, so U_γ is closed.

Assume that $\gamma(X) = X$. If $f(\gamma(x)) = 0$ for all x , then clearly $f = 0$, so U_γ^{-1} exists.

If y is in $X \setminus \gamma(X)$, then $U_\gamma \delta_y = 0$, so U_γ^{-1} does not exist.

If U_γ^{-1} is densely defined, then $R(U_\gamma) = l^2(X)$, from the formula for $R(U_\gamma)$, so γ is injective. Since U_γ^{-1} exists, γ is also surjective.

Let γ be a transformation. The formula $U_\gamma^{-1} = U_{\gamma^{-1}}$ follows from $f = f \circ \gamma \circ \gamma^{-1}$. Unitarity of U_γ follows from rearrangement of an absolutely convergent sum.

Set $e_x = U_\gamma \delta_x$. Unitarity of U_γ gives that $\{e_x\}$ is an orthonormal basis. Then $1 = \|e_x\| = \#\{y \mid \gamma(y) = x\}$, so γ is injective and surjective, which proves that γ is a transformation. \square

Given a measure preserving transformation T on a probability space, a stationary process is given by $n \mapsto Y(T^n \omega)$ for any stochastic variable Y . If T is also metrically transitive,

the process is an ergodic process. In the applications in solid state physics, the notion needs to be generalized in two respects. Firstly we need more than one generator T to deal with more than one direction in space. Secondly we need to consider measure preserving transformations given by subgroups of the index group. In the above example the index group is \mathbb{Z} and the subgroup is also \mathbb{Z} , but an application may give measure preserving transformations corresponding to the proper subgroup $2\mathbb{Z}$.

Definition 6.4.7 (Ergodic Generalized Potentials) *Let Γ be a subgroup of the transformation group of a countable set X , such that $\Gamma \ni \gamma \mapsto \gamma(x) \in X$ is injective for each $x \in X$. A generalized potential V on X is Γ -invariant if $V = T_\gamma V$ for all γ in Γ . A generalized random potential V on X is Γ -stationary if the distribution P_V equals $P_{T_\gamma V}$ for all γ in Γ . A generalized random potential on X is metrically transitive, with respect to Γ , if $T_\gamma A = A$ for all γ in Γ , implies $P_V(A) \in \{0, 1\}$. A V is Γ -transitive if it is metrically transitive with respect to Γ . A generalized random potential on X is Γ -ergodic if it is both Γ -stationary and Γ -transitive.*

Remarks. (i) If X is a group we may consider the transformation group Γ given by the left action of a subgroup of X . A model is given by $X = \mathbb{Z}^d$ and a lattice Γ in X .

(ii) A measurable set is trivial if it or the complement has measure 0.

(iii) All the statements involving “for all γ in Γ ”, may be replaced by “for all γ in a generator for Γ ”, without a change in the definition. This is why Γ is assumed to be a group.

(iv) The random potential $T_\gamma V = V \circ \gamma$ is given by the random coordinates $V(\text{gamma}(x))$ for x in X .

The abstract example of a Γ -invariant complex potential is given by the pointwise limit

$$V = \sum_{\gamma \in \Gamma} T_\gamma v, \quad (6.29)$$

where v is in $l^1(X)$. This V may also be considered to be ergodic. Any generalized potential may be identified with a metrically transitive potential, from the corresponding Dirac measure, so metrical transitivity and stationarity are independent properties. The following observation is the motivation for the extension of the ergodicity notions from stochastic processes to linear random operators.

Theorem 6.4.8 (Ergodic Multiplication Operators) *Let V be a generalized potential on a countable set X , and let γ be a transformation of X . Then*

$$M(T_\gamma V) = U_\gamma M(V) U_\gamma^{-1}. \quad (6.30)$$

If a generalized random potential is stationary, metrically transitive, or ergodic, then the multiplication operator has the same property.

Proof. We need only prove $M(T_\gamma V) U_\gamma = U_\gamma M(V)$, since the rest is obvious from the definitions. The domain of $U_\gamma M(V)$ equals the domain of $M(V)$, because U_γ is unitary. The proper domain $X(T_\gamma V)$ is given by $\gamma^{-1} X(V)$, and since f in l^2 is equivalent with $f \circ \gamma$ in l^2 , the domain of $M(T_\gamma V) U_\gamma$ equals the domain of $M(V)$. Let f in $D(M(V))$. Then $(U_\gamma M(V) f)(x) = V(\gamma(x)) f(\gamma(x))$, and $(M(T_\gamma V) U_\gamma f)(x) = V(\gamma(x)) f(\gamma(x))$, which proves the claim. \square

6.5 The Spectrum

The spectrum of a linear operator T is given by a characterization of the invertibility of the operator $z - T$ for complex z . The inverse of a multiplication operator is easily described.

Theorem 6.5.1 (The Inverse of a Multiplication Operator) *Let $M(V)$ be a discrete multiplication operator. The inverse M^{-1} exists iff $V(x) \neq 0$ for all x , in which case $M^{-1} = M(1/V)$.*

Proof. The inverse exists iff $Mf = 0$ implies $f = 0$ for all f in $D(M)$. This reduces to $V(x)f(x) = 0$ for all x in $X(V)$, so the claim follows. \square

Remarks. (i) The inverse M^{-1} , if it exists, is unbounded if 0 is an accumulation point for $V(X)$.

Since $z - M(V)$ is a multiplication operator, the spectrum of $M(V)$ may be completely characterized. The following theorem gives the spectrum, the point spectrum, the continuous spectrum, the residual spectrum, the discrete spectrum, the core of the essential spectrum, the core of the spectrum, and the pure point spectrum.

Theorem 6.5.2 (The Spectrum of a Multiplication Operator) *Let $M = M(V)$ be a discrete multiplication operator. The spectrum is characterized by*

$$\sigma(M) = \begin{cases} \overline{V(X)} & X(V) = X \\ \mathbb{C} & X(V) \neq X \end{cases}, \quad (6.31)$$

$$\sigma_p(M) = V(X), \quad (6.32)$$

$$\sigma_c(M) = \begin{cases} \overline{V(X)} \setminus V(X) & X(V) = X \\ \emptyset & X(V) \neq X \end{cases}, \quad (6.33)$$

$$\sigma_r(M) = \begin{cases} \emptyset & X(V) = X \\ V(X)^c & X(V) \neq X \end{cases}, \quad (6.34)$$

$$\sigma_d(M) = \{\lambda \in \mathbb{C} \mid \exists x, \lambda = V(x), \exists \text{ open } U \ni \lambda, \#(U \cap V(X)) < \infty\}, \quad (6.35)$$

$$\tilde{\sigma}_e(M) = \{\lambda \in \mathbb{C} \mid \forall \text{ open } U \ni \lambda, \#(U \cap V(X)) = \infty\}, \text{ and} \quad (6.36)$$

$$\tilde{\sigma}(M) = \sigma_{pp}(M) = \overline{V(X)}. \quad (6.37)$$

Proof. Let z be a complex number. The range $R(z - M)$ is a subspace of $l^2(X(V))$, the later regarded as a subspace of $l^2(X)$. We conclude that the resolvent set $\rho(M)$ is empty if $X(V) \neq X$, since the inverse never is defined for all f in $l^2(X)$. Assume $X(V) = X$. In this case the range $R(z - M)$ is a dense subset of $l^2(\{x \mid z \neq V(x)\})$. The operator $z - M$ has an inverse iff z is not in $V(X)$. This inverse is bounded iff z is not in $\overline{V(X)}$, and so the first equation is established.

The z is an eigenvalue of M iff there exist an x with $z = V(x)$, and then δ_x is a corresponding

eigenvector. This gives the point spectrum.

Then consider the residual spectrum. The range $R(z - M)$ is dense in $l^2(X)$ if $X = X(V)$, in which case the residual spectrum is empty. Consider $X(V) \neq X$. The range $R(z - M)$ is then not dense in $l^2(X)$ for any z , and $z - M$ is injective for any z not in $V(X)$, so the residual spectrum is given by $V(X)^c$.

The number z is in the continuous spectrum iff $z - M$ is injective, with a densely defined, unbounded inverse. The range $R(z - T)$ is not dense if $X(V) \neq X$, so in this case the continuous spectrum is empty. The continuous spectrum in the case $X(V) = X$ follows by elimination.

The operator $z - M$ has a bounded inverse iff z is not in $\overline{V(X)}$, and this is then the core of the spectrum. The pure point spectrum is the closure of the point spectrum.

The rest of the proof is also elementary verification of the definitions. \square

Remarks. (i) The multiplicity of a given eigenvalue z is the cardinality of $\{x \mid z = V(x)\}$. This cardinality is either the cardinality of the integers or represented by an integer.

(ii) For any given subset of the complex plane there are many multiplication operators with dense point spectrum in this set. This follows because the set of complex numbers has a dense, countable subset D and it is trivial to construct V such that $V(X) = D$.

The multiplication operator is a spectral operator, in the sense that it is given uniquely by a spectral resolution of a projection. The spectral family, a spectral measure, and the Lebesgue-Hammer decomposition of the spectrum are found in the following.

Theorem 6.5.3 (The Spectral Family) *Let $M(V)$ be a discrete multiplication operator. The operator $E(A) := M(1_A(V))$ defines a spectral family E . The multiplication operator is given by $M = E[id]$. More generally $M(u(V)) = E[u]$ holds for any Borel measurable function $u : D(u) \rightarrow \mathbb{C}$, $D(u) \supset \overline{V(X(V))}$. A spectral measure for M is given by*

$$\rho(A) := \sum_i 2^{-i} 1_A(V(x_i)), \quad (6.38)$$

where $X = \{x_1, x_2, \dots\}$ is the countable set. The Lebesgue-Hammer decomposition of the spectrum is given by $\sigma_{pp}(M) = \overline{V(X)}$ and $\sigma_{ac}(M) = \sigma_{sc}(M) = \emptyset$.

Proof. Direct verification gives that $E(A)$ is the orthogonal projection on $l^2(\{x \mid V(x) \in A\})$. Furthermore

$$\rho_f(A) := \langle f, E(A)f \rangle = \sum_{z \in \mathbb{C}} \delta_z(A) \left(\sum_{x \in X} |f(x)|^2 1_{\{z\}}(V(x)) \right), \quad (6.39)$$

so E is an atomic projection valued measure. A z is an atom iff $\#(\{x \mid z = V(x)\}) > 0$. Next we verify that $E[u]$ is the multiplication operator $M(u(V))$. A vector f is in $D(E[u])$ iff f is in $R(E(\mathbb{C}))$ and u is in $L^2(\rho_f)$ that is, iff

$$\infty > \sum_z |u(z)|^2 \left(\sum_x |f(x)|^2 \delta_z(V(x)) \right) = \sum_{x \in X(u(V))} |u(V(x))f(x)|^2, \quad (6.40)$$

so $D(E[u]) = D(M(u(V)))$. Equality $E[f] = M(u(V))$ follows from a similar verification. The given spectral measure is obtained from

$$\rho(A) = \sum_n 2^{-n} \langle \delta_{x_n}, E(A) \delta_{x_n} \rangle, \quad (6.41)$$

and this does also give the characterization of the spectrum. \square

The starting point for perturbation theory for linear operators is to analyze properties of the operator $T + \delta T$ from knowledge about the operator T , when δT is “small”. In applications the “small” term may very well be an unbounded operator, and even the definition of the sum may give fundamental problems. This motivates the introduction of topologies for unbounded operators. A typical problem is to determine if the spectra of two closed operators are close if the two operators are close. This question motivates the introduction of topologies for sets of closed sets, since the spectrum is a closed set. A study of the multiplication operator indicates the kind of results one may obtain.

Theorem 6.5.4 (Perturbation of The Spectrum) *Equip the set of discrete multiplication operators with the lower topology for closed operators. The core of the spectrum, $M \mapsto \tilde{\sigma}(M)$, is lower semicontinuous, but not upper compact semicontinuous. Equip the set of generalized potentials with the direct product topology. The mapping $V \mapsto \tilde{\sigma}(M(V))$ is lower semicontinuous, but not upper compact semicontinuous.*

Proof. Let $M = l\text{-}\lim M_\lambda$, $\Sigma := \tilde{\sigma}(M) = \overline{V(X(V))}$, and $\Sigma_\lambda := \tilde{\sigma}(M_\lambda)$. Let U be an open set which intersects Σ . Then there is an x with $V(x) \in U$. Fix this x . The last part of Theorem 6.3.5 gives $\lim V_\lambda(x) = V(x)$, so there is a λ_U such that $V_\lambda(x)$ is in U when $\lambda \geq \lambda_U$. The conclusion is that Σ_λ intersects U for $\lambda \geq \lambda_U$.

Upper compact semicontinuity is impossible from the last part of the theorem.

The lower semicontinuity of $V \mapsto \tilde{\sigma}(M(V))$ follows from the first part of Theorem 6.3.5 and the claim just proven.

Let $V_n(m) := 1_{\{k|k>n\}}(m)$ for natural numbers n, m , so $0 = p\text{-}\lim V_n = V$. The compact set $\{1\}$ is disjoint from $\tilde{\sigma}(M(V)) = \{0\}$, but $1 \in \tilde{\sigma}(M(V_n))$ for all n , so $V \mapsto \tilde{\sigma}(M(V))$ is not upper compact semicontinuous. \square

Remarks. (i) The first claim may be written $\tilde{\sigma}(l\text{-}\lim M_\lambda) = l\text{-}\lim \tilde{\sigma}(M_\lambda)$, but read with care since the involved topologies are not Hausdorff topologies.

(ii) Lower semicontinuity of $V \mapsto \tilde{\sigma}(M(V))$ may also be verified directly.

The crudest measure of the degree of disorder in a random potential V is given by the set $R(V) = \{v \mid \exists \omega \in \Omega, V_\omega = v\}$ of possible realizations. If $R(V) \subset R(W)$, then it is reasonable to consider W to be more disordered than V . The set of realizations has little importance, mainly because the distribution of the potential is not properly reflected. A better choice is to measure the degree of disorder by the topological support $\text{supp } P_V$, so V is less disordered than W iff $\text{supp } P_V \subset \text{supp } P_W$. This measure of disorder is perhaps the crudest that has any significance, but the definition has a serious drawback. It may be practically impossible to verify, for a nontrivial distribution, if a given potential v is in $\text{supp } P_V$. This motivates the definition of the set of admissible potentials.

Definition 6.5.5 (Admissible Generalized Potentials) Let V be a generalized random potential on a countable set X . The set of V -admissible generalized potentials is

$$A_V := \prod_{x \in X} \text{supp } P_x, \quad (6.42)$$

where P_x is the distribution of $V(x)$. The spectrum of V at x is

$$\Sigma_x := \{z \in \mathbb{C} \mid P(|V(x) - z| < \epsilon) > 0, \forall \epsilon > 0\}. \quad (6.43)$$

Remarks. (i) The difference between Σ_x and $\text{supp } P_x$ is that ∞ may be in $\text{supp } P_x$.
(ii) The spectrum Σ_x at x is a closed set.
(iii) The set A_V of V -admissible potentials is closed and contains $\text{supp } P_V$.

Nonrandom spectrum is a basic property of ergodic Schrödinger operators. In good cases the spectrum is even computable. The multiplication operator represents a good case, and allows for an explicit formula.

Theorem 6.5.6 (The Random Spectrum) Let V be a generalized random potential on a countable set, with distribution P_V . The core $\Sigma := \tilde{\sigma}(M(V))$ of the spectrum is a random set. If the generalized random potential has a metrically transitive distribution, then

$$\Sigma = \bigcup_{v \in \text{supp } P_V} \tilde{\sigma}(M(v)) = \bigcup_{v \in A_V} \tilde{\sigma}(M(v)) = \overline{\bigcup_x \Sigma_x}, \quad (6.44)$$

with probability one. If V is Γ -ergodic, with an infinite Γ , then

$$\Sigma = \tilde{\sigma}_e(M(V)) = \overline{\bigcup_{[x] \in X/\Gamma} \Sigma_x}, \quad (6.45)$$

with probability one.

Proof. Let (Ω, \mathcal{F}, P) be the probability space. By assumption $\Omega \ni \omega \mapsto V_\omega$ is measurable. The measurability of $\omega \mapsto \Sigma = \tilde{\sigma}(M(V_\omega))$ follows from the continuity of $V \mapsto M(V)$ and the continuity of $M \mapsto \tilde{\sigma}(M)$. This proves that Σ is a random set.

Let G be the set of generalized potentials equipped with the product topology. The distribution $P_V(A) := P(V \in A)$ is a Borel measure on G . The sets $\Sigma_x \subset \mathbb{C}$ and $\text{supp } P_x \subset \hat{\mathbb{C}}$ are supports of Borel measures and are therefore closed. The measures P_V and P_x are Borel probability measures, and second countability gives that the supports have measure one. From $A_V = \bigcap_x \{v \mid v(x) \in \text{supp } P_x\}$ it follows that A_V is closed and has measure one. The inclusion $\text{supp } P_V \subset A_V$ holds because $\text{supp } P_V$ is contained in any closed set of full measure.

The single point set $\{\Sigma_0\}$ is Borel measurable in the set of closed sets of complex numbers, since $\mathcal{P}_{\mathbb{F}}(\mathbb{C})$ is σ -separated in the natural σ -algebra. The set $\{v \mid \tilde{\sigma}(v) = \Sigma_0\}$ is measurable from continuity of $v \mapsto \tilde{\sigma}(v)$, and $\Omega_0 := \{v \mid \tilde{\sigma}(v) = \Sigma_0, v \in \text{supp } P_V\}$ is measurable since $\text{supp } P_V$ is measurable. Set

$$\Sigma_0 := \bigcup_{v \in \text{supp } P_V} \tilde{\sigma}(v). \quad (6.46)$$

The metrical transitivity of V , the second countability of G , and the lower semicontinuity of $v \mapsto \tilde{\sigma}(v)$ implies that the set Ω_0 has measure one.

Let $z = v(x) \in \mathbb{C}$ for an x and a $v \in \text{supp } P_V$. Let U in \mathbb{C} be open, with $z \in U$. From $v \in \text{supp } P_V$ it follows that $P(|V(x) - z| < \epsilon) > 0$, so $z \in \Sigma_x$, and $\Sigma_0 \subset \overline{\bigcup_x \Sigma_x}$. Now assume $z \in \Sigma_x$ and set $C_n := \{v \mid |v(x) - z| < 1/n\}$, so $P(V \in C_n) > 0$. Choose $v_n \in \Omega_0 \cap C_n$ for $n = 1, 2, \dots$, so $\lim v_n(x) = z \in \overline{\bigcup_n v_n(X)}$, and $\overline{\bigcup_x \Sigma_x} \subset \Sigma_0$.

Finally we prove $\overline{\bigcup_x \Sigma_x} = \overline{\bigcup_{v \in A_V} \tilde{\sigma}(M(v))}$. From the above $\text{supp } P_V \subset A_V$, so \subset is clear. The opposite inclusion \supset follows because $z = v(x) \in \mathbb{C}$ for an V -admissible v gives $z \in \Sigma_x$, by definition.

Fix x and let U be open with $U \cap \Sigma_x \neq \emptyset$, so $p := P(V(x) \in U) > 0$. The function $v \mapsto t(v) = \sum_x 1_U(v(x))$ is Γ invariant, $t(v \circ b) = t(v)$ for all b in Γ . From this, with probability one, $\#\{x \mid V(x) \in U\}$ equals $E[t(V)] \geq \sum_{a \in \Gamma} p = \infty$, since $\Gamma \ni b \mapsto b(x) \in X$ is injective and Γ is infinite, so the spectrum can not be discrete. \square

Remarks. (i) It may be hard to verify if a given potential is in the support of a random potential.

(ii) That $\overline{V(X(V))}$ equals $\overline{\bigcup_x \Sigma_x}$ with probability one, may be proven directly. The set $\overline{v(X(v))}$ is clearly a subset of $\overline{\bigcup_x \Sigma_x}$ for any admissible potential v . Let z be in Σ_x , so $P(|V(x) - z| < \epsilon) > 0$ for any positive ϵ . The set $A_n = \{v \mid \exists b \in \Gamma, |v(b(x)) - z| < 1/n\}$ is Γ invariant, so the metrical transitivity gives $P_V(\bigcap_n A_n) = 1$. Let v in $\bigcap_n A_n$. For each n there is a b with $|v(b(x)) - z| < 1/n$, and we set $z_n := v(b(x))$. Now z_n is in $\overline{v(X(v))}$, and $z = \lim z_n$ gives $z \in \overline{v(X(v))}$.

The previous theorem gives in particular two properties of generic nature. Firstly the spectrum increases with increasing disorder, and secondly the discrete spectrum is empty, if the potential is stationary.

6.6 The Density of States

Let $T = E[id]$, where E is a resolution of the identity. A spectral measure for T is given by

$$\rho(A) = \sum_{n=1}^{\infty} 2^{-n} \langle e_n, E(A)e_n \rangle, \quad (6.47)$$

where $\{e_n\}$ is an arbitrary orthonormal basis. The density of states measure may be considered as an analogue to spectral measures, but each dimension is to get equal weight. Let P be a finite dimensional orthogonal projection. A probability measure is then defined by

$$N_P(A) = \text{Tr}(PE(A)P) / \text{Tr}(P). \quad (6.48)$$

Finite dimensional projections have a natural order, so $\{N_P\}$ is a net of probability measures. A weak limit $N = w\text{-}\lim N_P$ is the density of states measure, but the limit may not exist. A related approach is to look for a normalized trace τ with domain so that $N(A) := \tau(E(A))$ is possible. The absolute continuity $N \ll E$ would be a consequence.

It seems that none of the above approaches is sufficient for the applications in solid state physics. We will define a density of states which will be sufficient for discrete multiplication

operators and discrete Schrödinger operators. In particular we will find examples where the density of states measure is not absolutely continuous with respect to the resolution of the identity.

Definition 6.6.1 (The Discrete Density of States Measure) *Let E be a spectral resolution of a projection on $l^2(X)$, where X is an infinite countable set. An approximate density of states measure N_Λ is defined by*

$$N_\Lambda(A) := \sum_{x \in \Lambda} \langle \delta_x, E(A)\delta_x \rangle / \#\Lambda, \quad (6.49)$$

for any finite subset Λ of X . Let x_1, x_2, \dots be an enumeration of the elements in X . The weak limit

$$N = w\text{-}\lim N_{\{x_1, \dots, x_{n_k}\}}, \quad (6.50)$$

if it exists for a subsequence $\{n_k\} \subset \mathbb{N}$, is a density of states measure for the operator $E[id]$.

Remarks. (i) The existence of N and N itself depends on the chosen sequence in X , so N is not uniquely given from E .

(ii) A density of states measure exists iff the countable set $\{N_\Lambda\}$ of approximate density of states measures is relatively compact. There seems to be no gain by consideration of convergent nets or subnets, because the weak topology for the measures is given by a metric. Sequential compactness is equivalent with compactness for a metric space.

The density of states measure is connected to the spectrum. In applications the spectrum is given by the support of the density of states measure. This is not true in general, not even for the multiplication operator.

Theorem 6.6.2 (The Density of States Measure) *The density of states approximants for the discrete multiplication operator $M(V)$ are given by*

$$N_\Lambda[f] = \sum_{x \in \Lambda} f(V(x)) / \#\Lambda, \quad (6.51)$$

where $f(V(x)) := 0$ for $V(x) = \infty$. The approximant N_Λ is a pure point measure with $N_\Lambda(\mathbb{C}) \leq 1$. The inclusion

$$\text{supp } N \subset \tilde{\sigma}_e(M(V)) \quad (6.52)$$

holds if N exists.

Proof. The formula for N_Λ follows from the spectral family, $E[f] = M(f(V))$.

Abuse notation by $N_n = N_{\{x_1, \dots, x_{k_n}\}}$. The assumption $N = w\text{-}\lim N_n$ implies $\underline{\lim} N_n(U) \geq N(U)$ for any open U . Let U be an open neighborhood of $z \in \text{supp } N$, so $N(U) > 0$, and

$$0 < \underline{\lim} \frac{1}{k_n} \sum_{i=1}^{k_n} 1_U(V(x_i)). \quad (6.53)$$

This implies $\#\{x \mid V(x) \in U\} = \infty$, so z is in $\tilde{\sigma}_\epsilon(M(V))$. \square

The density of states is a nontrivial quantity, and even so for the existence. The following proof depends on Prokhorov's tightness theorem for compactness of a set of finite Radon measures. It is also noteworthy to single out potentials where the density of states measure is approximated in a stronger topology.

Theorem 6.6.3 (Existence of The Density of States) *A discrete multiplication operator $M(V)$ has a density of states measure iff given any $\epsilon > 0$ there is a compact set K with*

$$\sup\left\{\sum_{x \in Y} 1_{K^c}(V(x)) / \#Y \mid \#Y < \infty\right\} < \epsilon. \quad (6.54)$$

If V equals z at infinity, then $N = \delta_z$ is a unique density of states. If V takes only the values z_1, \dots, z_m, ∞ and

$$w_i = \lim \sum_{j=1}^n \delta_{z_i}(V(x_j)) / n \quad (6.55)$$

exists, then

$$N = p\text{-}\lim N_k = \sum_i w_i \delta_{z_i} \quad (6.56)$$

is a density of states measure. A density of states exists if $V(X) \cap \mathbb{C}$ is bounded.

Proof. The first part follows from Prokhorov's theorem.

We prove $\lim N_n[f] = f(z)$ for all functions f , continuous at z , given $V(\infty) = z$. Let $\epsilon > 0$ and choose $\delta > 0$ so $|f(w) - f(z)| < \epsilon$ when $|w - z| < \delta$. Pick M so $|V(x_i) - z| < \delta$ when $i \geq M$ and $V(x_i) \neq \infty$. This gives

$$\lim N_n(f) = \lim \frac{1}{n} \left[\sum_{i=1}^{M-1} + \sum_{i \geq M} \right] f(V(x_i)) = 0 + f(z) + R, \quad |R| \leq \epsilon, \quad (6.57)$$

which proves the second claim.

We have $N_n(\{z_k\}) = \sum_{i=1}^n 1_{\{z_k\}}(V(x_i)) / n$, so $\lim N_n(\{z_k\}) = w_k$. This gives the third claim, $\lim N_n(A) = N(A)$, from a finite decomposition of a general measurable set A .

If $V(X) \cap \mathbb{C}$ is bounded, chose a compact $K \supset V(X) \cap \mathbb{C}$, and the first part of the theorem is fulfilled. \square

Remarks. (i) The weights w_i , and their existence, depend on the enumeration.

(ii) The case $V(\infty) = z$ shows that the product topology for measures is insufficient, since $N_n(\{z\}) = 0$ is a possible case. This also shows that $N \ll E$ is wrong in general, even when N is a unique pure point measure.

(iii) In the case where V takes a finite number of values the product topology was sufficient, and in particular $N \ll E$ is true. It may seem that it is sufficient to assume that $V(X)$ is without complex limit points, but the tempting interchange of limit and summation is not

correct.

(iv) Consider the example $V(x) = x$ on the integers. There is no density of states measure, because $\sup_Y N_Y(K^c) = 1$ for any compact K .

The final theorem depends on the Birkhoff ergodic theorem, and is perhaps a proper place to end the list of explicitly computable quantities for the ergodic multiplication operator.

Theorem 6.6.4 (Nonrandom Density of States) *Let d be a natural number. Let V be a generalized L ergodic potential on \mathbb{Z}^d for a d -dimensional lattice L in \mathbb{Z}^d , with a fundamental cell of points x_1, \dots, x_m . Let P_i be the distribution of $V(x_i)$ restricted to \mathbb{C} . A density of states measure is given by*

$$N = \sum_{i=1}^m P_i/m. \quad (6.58)$$

Proof. Consider first $L = \mathbb{Z}^d$, so $m = 1$. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be bounded and measurable, so in particular $f(V(x_1)) \in L^1(P)$. For boxes Λ the Birkhoff ergodic theorem gives

$$P[f(V(x_1))] = E(f(V(x_1))) = \lim \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(V(x)) = N[f], \quad (6.59)$$

with probability one. Consideration of $f = 1_A$, for all dyadic boxes A , gives the claim, since the family of dyadic boxes is countable and convergence determining. In the general case X is the disjoint union of the sets $X_i = \{x_i + l \mid l \in L\}$. Equation (6.58) follows from

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(V(x)) = \sum_i \frac{|\Lambda_i|}{|\Lambda|} \frac{1}{|\Lambda_i|} \sum_{x \in \Lambda_i} f(V(x)), \quad (6.60)$$

where $\Lambda_i = \Lambda \cap X_i$. □

Remarks. (i) The expression for the approximate density of states measures is a good reason by itself for consideration of ergodic potentials.

(ii) The measures P_i may not be probability measures, but they are defective probability measures in the sense used by Feller.

(iii) The convergence in equation (6.59) is both pointwise and in $L^p(P)$ sense, for $f(V(x_1))$ in $L^p(P)$.

6.7 Notes

There is apparently no literature in which the random multiplication operator is discussed, possibly because it is truly elementary. The aim in the previous discussion is not to get an understanding of the multiplication operator, but rather to use the simplicity to get an understanding of the (missing!) general theory. It is surprisingly many of the results for this model which can be generalized. On the other hand, many “reasonable” assumptions regarding the general theory find counterexamples.

Countability is defined in Definition 6.2.1 in accordance with Kreyszig [88, p.612]. Ebbinghaus [47, p.11] excludes the finite case in the definition of countable, and uses the term “at most countable” to include also the finite case.

Consideration of generalized potentials, see Definitions 6.3.1, is maybe a new twist. It allows for instance random Dirichlet boundary conditions.

The classification of the compact multiplication operators in Theorem 6.3.4 may be seen as a special case of more general theorems on compact operators, see [88, p.405-458], [136, p.129-141, p.166-179], [110, p.105-125].

I introduced the lower topology for closed operators in [129], and it seems to be a natural generalization of the notion of generalized strong convergence found in Kato’s book [75, p.426-462], in the context of asymptotic perturbation theory for closed operators. It does also generalize the concept of strong graph convergence, see [124, vol.I p.293-294, vol.II p.268-274], [30, p.186-188, p.281]. The example given by convergence of unbounded multiplication operators in Theorem 6.3.5 does also indicate that the lower topology for closed operators deserves further study.

The V -bound of a complex potential in Definition 6.3.6 is perhaps not a standard notion, and Theorem 6.3.7 may be a new observation. Definition 6.3.8 of V -convergence is motivated by for instance [75, p.206], [136, p.272]. Gap convergence is defined in the context of analytic perturbation theory in [75, p.197-208, p.364-426].

The random multiplication operator in Theorem 6.4.2 is more general than the usual random multiplication operators, since we allow the value ∞ for the potential.

Theorem 6.4.3, which gives a random potential from random coupling constants, should be compared with the corresponding continuous case [79, p.302]. I constructed the potential W in the second example after Theorem 6.4.3, to demonstrate that singular continuous single site distributions may easily occur. Just how typical singular distributions are, in the context of applications in solid state physics, seems to be an interesting question. The singular continuous distribution function F in the first example is found in [57, p.35-36], [114, p.48-49], [128], where the last reference also gives a plot of F .

The composition operator U_γ in Definition 6.4.4, or at least special cases of it, is used in a variety of problems in mathematical physics. The count N_γ is maybe not so commonly used.

Definition 6.4.5 of functions of generalized potentials seems to be a natural choice, and is what is needed in following theorems.

Theorem 6.4.6 for the composition operator U_γ may be “new”, and the proof gives a nice application of the concept of generalized potentials with the use of the count N_γ .

Definition 6.4.7 of ergodic generalized potentials is modeled after similar definitions found in [94, p.87, p.95]. Sometimes the term metrically transitive is used synonymously with the term ergodicity [37, p.14], but I see no reason for this.

Theorem 6.4.8 is elementary, but fundamental. It leads to the complete analysis of the Schrödinger operator with a periodic potential, and it is the motivation for the generalization of ergodicity concepts to include linear random operators.

Theorem 6.5.2 gives in particular examples of operators with nonempty residual spectrum, bounded operators in finite dimensional spaces with spectrum equal to \mathbb{C} , and it indicates that the core of the spectrum is of special interest for the spectral theory of general closed

operators. The definitions used for the components of the spectrum are found in [136, p.99, p.202-203] and [110, p.134, p.207]. There is a conflict in the definition of the continuous spectrum. One definition holds for general closed operators, which is my choice also, and another is based on the continuous part of a spectral operator. Sinai [37, p.453] defines the discrete part of the spectrum to be the closure of the set of eigenvalues, which is far from the definition I have chosen.

It is perhaps nonstandard to consider spectral families for operators which are not densely defined, as in Theorem 6.5.3.

I do not know about any references where lower topologies for both operators and spectra are used, so Theorem 6.5.4 may be new.

The set A_V of admissible potentials in Definition 6.5.5 is used also in [5, p.343], [81, p.334], but in the context of continuous Schrödinger operators as functions of random coupling constants. In the i.i.d. case, as in [81, p.330], there is no difference between A_V and the support Ω_V of the potential.

Theorem 6.5.6 determines the spectrum of an ergodic operator. For this special case it may be “new”, in particular if the set of admissible potentials is strictly larger than the support of the random potential. Empty discrete spectrum is the generic case, as shown in [79, p.295].

The density of states measure is of vital importance in solid state physics, by which for instance the difference between metals and insulators can be understood [10, p.562, Figure 28.1]. Kirsch [79, p.307-334, p.338, p.343-344, p.363] considers several equivalent definitions for the density of states for the continuous, random Schrödinger operator. The original definition of the density of states uses finite cubes and approximate density of states measures for these cubes. Definition 6.6.1 is more general than usual, and allows for subsequences of increasing cubes.

Theorem 6.6.2 gives the approximate density of states measure directly as a space average, and may be a new observation. It suggests strongly to consider ergodic potentials.

Prokhorov’s theorem, as used in the proof of Theorem 6.6.3, is found in [96, p.15].

Theorem 6.6.4 gives the density of states explicitly, and it may be possible to use this as a starting point for the Schrödinger operator case.

The problem of existence of the density of states is similar to the existence of the thermodynamic limit in statistical mechanics [117, p.13-28].

Chapter 7

Random Schrödinger Operators on Random Graphs

The lower topology is applied to define random Schrödinger operators on a random graph, defined by generalized potentials. We prove that the spectrum is a lower semicontinuous function of the generalized real potential. This gives an explicit formula for the spectrum, when the potential is a metrically transitive, random process. The significance of the Lebesgue decompositions of a selfadjoint Hamiltonian is discussed in terms of the time dependent wave function. Local spectral measures are defined and applied. Firstly for the proof of randomness of the Lebesgue decomposition of the spectrum, and secondly in an explicit formula for the integrated density of states measure. We determine the spectrum and the pure point spectrum of the percolation Schrödinger operator. The determination of the absolutely continuous and the singular continuous spectrum are left as open problems, but possibly solvable by methods indicated.

7.1 Introduction

The general theory of closed random operators would be of little interest if there were no concrete models available. The first aim here is to present and extend the model which has been most extensively studied. The model is known as the tight binding model or the Anderson model. We will call it the discrete Schrödinger operator model.

The discrete Schrödinger operator is defined from a finite difference Laplace operator and a potential. We allow the potential to take the value ∞ , and use the term generalized potential when we allow this possibility. This gives a natural way to model a random graph by random removal of points. By means of the lower operator topology we prove that the Schrödinger operator from a generalized random potential is a random operator in a proper sense. We prove also that the mapping of a generalized real potential to the core spectrum of the Schrödinger operator is lower semicontinuous. The general theory for metrically transitive closed random operators may then be applied. The result is an explicit formula for the nonrandom spectrum of the Schrödinger operator from a metrically transitive generalized real random potential.

In the Anderson model it is known that large disorder implies absence of diffusion. We

discuss how the time behavior of the wave function is related to the Lebesgue decomposition of the Hilbert space into an absolutely continuous part, a singular continuous part, and a pure point part. This gives a most beautiful and rigorous interpretation of some deep mathematical results. The point is that any quantum state gives an associated measure, and the Fourier transformation of this measure is directly related to the time development of the state given by Schrödinger's time dependent equation.

The Green's function has been the most important tool in the existing proofs of pure point spectrum. We prove that the Green's function is a jointly continuous function of the generalized real potential and the resolvent set parameter. A consequence is that the spectral family of the Schrödinger operator is random in a proper sense. General theory does then imply nonrandomness of the Lebesgue decomposition of the spectrum when the generalized real random potential is metrically transitive. We give an explicit formula for the integrated density of states measure in terms of mean values of local spectral measures, which is valid for a general ergodic potential. We need the stationarity assumption on the potential to ensure existence of the integrated density of states measure.

We determine the spectrum and the pure point spectrum of the percolation Schrödinger operator. The existence of continuous spectrum remains as an open challenging problem, but is determined completely by a corresponding knowledge about the nearest neighbor operator on an infinite connected domain. We find explicit formulas for operators defined from functions of the nearest neighbor operator.

7.2 The Lower Operator Topology

The discrete Schrödinger operator is possibly the most fundamental second order partial difference operator. We will need a slightly more general definition than the usual one. We include the possibility of a complex valued potential, which is quite natural. The most important extension is that we allow the potential to take the value ∞ . This is not so interesting when we only consider one fixed potential. The main point is that it will allow random operators which is defined on a random graph given by random removal of points.

Definition 7.2.1 (The Discrete Schrödinger Operator) *Let V be a generalized potential on $X := \mathbb{Z}^d$, with a proper domain $X[V] := \{x \in X \mid V(x) \neq \infty\}$. We define the Laplace operator Δ_V in $l^2(X)$ by*

$$\Delta_V : l^2(X[V]) \rightarrow l^2(X[V]), \quad \Delta_V f(x) := \sum_{y \in n(x)} (f(y) - f(x)), \quad (7.1)$$

where $n(x) := \{y \in X[V] \mid \|y - x\|_1 = 1\}$. The discrete Schrödinger operator $H = H[V]$ in $l^2(X)$ is defined by

$$\begin{aligned} D(H) &:= \{f \in l^2(X[V]) \mid Vf \in l^2(X[V])\}, \quad H : D(H) \rightarrow l^2(X[V]), \\ Hf(x) &:= -\Delta_V f(x) + V(x)f(x). \end{aligned} \quad (7.2)$$

Remarks. (i) We identify $\mathcal{H}[V] := l^2(X[V])$ with a subspace in $\mathcal{H} := l^2(X)$.

(ii) The restriction H_n of $H[V]$ to $l^2(X[V])$ is a normal operator, if the imaginary part of V is zero. This means in particular that H is given by a spectral family, so $H = E[id]$.

It is not clear a priori how to compare two operators which may be defined in orthogonal subspaces. The lower operator topology gives one possibility. The following theorem gives partly the correspondence between pointwise convergence of potentials and lower convergence of Schrödinger operators.

Theorem 7.2.2 (Continuity of the Generalized Schrödinger Operator) *The mapping $V \mapsto H[V]$ of a generalized potential V to the Schrödinger operator $H[V]$ is continuous, when the set of generalized potentials is equipped with the product topology and the set of generalized Schrödinger operators is equipped with the lower operator topology.*

Proof. We prove that $H[V] = l\text{-}\lim H[V_\lambda]$, when $V = p\text{-}\lim V_\lambda$. Set $H = H[V]$ and $H_\lambda = H[V_\lambda]$. Let $f \in D(H)$ and $\epsilon > 0$ be arbitrary. Let $X = \mathbb{Z}^d$ and $X[V]$ be the subset in X where V is finite. From the definition of $D(H)$ we can find a finite $K_0 \subset X[V]$ such that

$$\|1_{K_0^c} \cdot f\| < \epsilon, \quad \text{and} \quad \|1_{K_0^c} \cdot Hf\| < \epsilon. \quad (7.3)$$

Set $K := \overline{K_0} \cap X[V]$, where $\overline{K_0} := \{x \in X \mid \exists y \in K_0, \|x - y\|_1 \leq 1\}$. The convergence $V_\lambda \rightarrow V$ gives a λ_1 such that $K \subset X[V_\lambda]$ for $\lambda \geq \lambda_1$, a $\lambda_2 \geq \lambda_1$ such that

$$\|1_K \cdot (V_\lambda - V)f\| < \epsilon, \quad \text{when } \lambda \geq \lambda_2, \quad (7.4)$$

and a $\lambda_3 \geq \lambda_2$ such that

$$|2d + V_\lambda(x)| > \|f\|/\epsilon, \quad \text{when } \lambda \geq \lambda_3, \quad x \in \overline{K} \cap X[V]^c. \quad (7.5)$$

Consider a $\lambda \geq \lambda_3$. Set $Z := \overline{K} \cap X[V_\lambda] \cap X[V]^c$ and $Y_x := \{y \in K \mid \|y - x\|_1 = 1\}$. Define $f_\lambda \in D(H_\lambda)$ by

$$f_\lambda(x) := \begin{cases} 0 & x \in (K \cup Z)^c \\ f(x) & x \in K \\ \sum_{y \in Y_x} f(y)/(2d + V_\lambda(x)) & x \in Z \end{cases}. \quad (7.6)$$

Note that each point in K has at most $2d$ neighbors which belong to Z . This and (7.5) give the estimate

$$\|1_{K^c} \cdot f_\lambda\| < \epsilon\sqrt{2d}, \quad (7.7)$$

which implies

$$\|f_\lambda - f\| < \epsilon(1 + \sqrt{2d}). \quad (7.8)$$

Next consider

$$\|H_\lambda f_\lambda - Hf\| \leq \|1_K \cdot (H_\lambda f_\lambda - Hf)\| + \|1_{K^c} \cdot (H_\lambda f_\lambda - Hf)\|. \quad (7.9)$$

The inequality (7.4), equality of f and f_λ on K , and the estimate (7.8) give

$$\begin{aligned} \|1_K \cdot (H_\lambda f_\lambda - Hf)\| &\leq \|1_K \cdot (V_\lambda - V)f\| + \|1_K \cdot (\Delta_{V_\lambda} f_\lambda - \Delta_V f)\| \\ &< \epsilon(1 + \sqrt{2d}(1 + \sqrt{2d})). \end{aligned} \quad (7.10)$$

Split K^c into a disjoint union of sets by

$$K^c = \overline{K}^c \cup (\overline{K} \setminus K). \quad (7.11)$$

Observe furthermore that

$$X[V_\lambda] \cap (\overline{K} \setminus K) = Z \cup [X[V] \cap X[V_\lambda] \cap K^c \cap \overline{K}], \quad (7.12)$$

again a disjoint union, which follows from the splitting

$$K^c = X[V]^c \cup (X[V] \cap K^c). \quad (7.13)$$

Inequality (7.3), the previous splitting of $X[V_\lambda] \cap K^c$, and the above estimates give

$$\begin{aligned} \|1_{K^c} \cdot (H_\lambda f_\lambda - Hf)\| &\leq \|1_{K^c} \cdot Hf\| + \|1_{K^c} \cdot H_\lambda f_\lambda\| \\ &< \epsilon(1 + \sqrt{2d} + 2d + \sqrt{2d(2d+1)}). \end{aligned} \quad (7.14)$$

The estimates (7.10) and (7.14) give finally

$$\|H_\lambda f_\lambda - Hf\| < \epsilon(2 + 4d + 2\sqrt{2d} + \sqrt{2d(2d+1)}). \quad (7.15)$$

Let G and G_λ be the graphs of H and H_λ . Let U be open in $l^2(X) \times l^2(X)$ and such that $U \cap G \neq \emptyset$. The above estimates give a λ_U such that $U \cap G_\lambda \neq \emptyset$ for $\lambda \geq \lambda_U$, which proves the theorem. \square

Remarks. (i) The set of generalized Schrödinger operators is then compact, as the image of a compact set.

(ii) The corresponding theorem for the multiplication operator $M[V]$ does not depend on that $V_\lambda(x)$ goes to ∞ for x in $X[V]^c$.

(iii) It is tempting to consider the possibility of the reverse conclusion: What does $H[V] = l\text{-}\lim H[V_\lambda]$ imply about V, V_λ ?

A random generalized potential gives a random multiplication operator. It is more interesting that it also gives a random Schrödinger operator.

Theorem 7.2.3 (The Random Schrödinger Operator) *If V is a generalized random potential on \mathbb{Z}^d , then $H = H[V]$ is a closed random operator.*

Proof. Randomness of H follows from continuity of $V \mapsto H[V]$. \square

Remarks. (i) Observe the case where the potential takes only the values 0 and ∞ . The result is a Dirichlet Laplace operator defined on a random graph.

Now we have established a nontrivial example of a closed random operator. If we restrict attention to real valued potentials, then we would have the results from the general theory available. The first task now is to study the spectrum of our random operator. A fundamental question is: Is the spectrum a random set? This, and more, follows from the following theorem.

Theorem 7.2.4 (Continuity of $H \mapsto \tilde{\sigma}(H)$) *If $\{V_\lambda\}$ is a net of generalized real potentials on \mathbb{Z}^d , then*

$$\tilde{\sigma}(l\text{-}\lim H[V_\lambda]) = l\text{-}\lim \tilde{\sigma}(H[V_\lambda]) \quad (7.16)$$

holds for the lower convergence of the core spectra of the Schrödinger operators.

Proof. Let $H = l\text{-}\lim H_\lambda$, $q \in \tilde{\sigma}(H)$, let $r > 0$, and set $U := \{z \in \mathbb{C} \mid |z - q| < r\}$. To simplify notation we assume $q = 0$. Let $0 < \epsilon < 1/2$. Since 0 is in the core spectrum of H , there exists an $f \in D(H)$ such that $\|f\| = 1$ and $\|Hf\| < \epsilon$. The operator convergence gives a λ_ϵ such that there exists a $f_\lambda \in D(H_\lambda)$ with $\|f_\lambda - f\| < \epsilon$ and $\|H_\lambda f_\lambda - Hf\| < \epsilon$ for all $\lambda \geq \lambda_\epsilon$. Consider an H_λ such that 0 is in its regularity domain. Then $\|f_\lambda\| / \|H_\lambda f_\lambda\| = \|H_\lambda^{-1} H_\lambda f_\lambda / \|H_\lambda f_\lambda\|\| \leq \|H_\lambda^{-1}\|$, so $\|H_\lambda^{-1}\|^{-1} \leq \epsilon / (1 - \epsilon)$. The normality of the natural restriction of H_λ gives $d(0, \tilde{\sigma}(H_\lambda)) \leq \epsilon / (1 - \epsilon) < r$ for all $\lambda \geq \lambda_\epsilon$, for a proper choice of ϵ . We conclude that $U \cap \tilde{\sigma}(H_\lambda) \neq \emptyset$ for all $\lambda \geq \lambda_\epsilon$. \square

Remarks. (i) The trivial operator $H(\infty)$ with graph $\{(0, 0)\}$ has empty core spectrum and is a limit of any net of discrete, generalized Schrödinger operators .

(ii) The proof is valid for general normal operators.

(iii) Let Λ_n be finite boxes increasing towards X and let H_n be the Dirichlet restrictions of $H = H[V]$ to Λ_n . The result here gives $\tilde{\sigma}(H) = l\text{-}\lim \tilde{\sigma}(H_n)$, or equivalently $\tilde{\sigma}(H) \subset \{\lambda \mid \exists \lambda_n \in \tilde{\sigma}(H_n), \lambda = \lim \lambda_n\}$. Note that the H_n 's are represented by finite selfadjoint matrices.

Theorem 7.2.5 (Random Spectrum) *If V is a generalized real random potential on \mathbb{Z}^d , then the core spectrum $\Sigma := \tilde{\sigma}(H(V))$ is a closed random set.*

Proof. Randomness of Σ follows from continuity of $V \mapsto H[V]$ and $H \mapsto \tilde{\sigma}(H)$. \square

The mathematical theory associated with Schrödinger operators defined from periodic or almost periodic potentials is very rich. This theory is also of fundamental importance in solid state physics. The following observation is the starting point for the deep knowledge obtained in this theory.

Theorem 7.2.6 (Shift Transformations) *Let $\gamma \in \mathbb{Z}^d$ and let U_γ be the unitary operator on $l^2(\mathbb{Z}^d)$ determined by $U_\gamma f(x) := f(x + \gamma)$. If T_γ is the shift transformation defined by $T_\gamma V(x) := V(x + \gamma)$ and $T_\gamma H(x, y) = H(x + \gamma, y + \gamma)$, then*

$$H[T_\gamma V] = U_\gamma H[V] U_\gamma^{-1}, \text{ and} \quad (7.17)$$

$$H[T_\gamma V] = T_\gamma H[V].$$

Proof. It is straightforward to verify $U_\gamma(-\Delta_V + M(V)) = (-\Delta_{T_\gamma V} + M(T_\gamma V))U_\gamma$, which proves the first claim. If B and A are two matrix operators, such that $B = U_\gamma A U_\gamma^{-1}$, then

$$B(x, y) = \langle \delta_x, U_\gamma A U_\gamma^{-1} \delta_y \rangle = \langle \delta_{x+\gamma}, A \delta_{y+\gamma} \rangle = (T_\gamma A)(x, y), \quad (7.18)$$

which implies the second claim. \square

Remarks. (i) An essential part of this theorem is the implicit relation between the domains of the involved operators.

(ii) Similar properties follows for the Green's function G .

(iii) Observe that $x \mapsto G(x, x)$ is periodic if V is periodic. Likewise; metrical transitivity, stationarity, or ergodicity of V gives similar properties for derived operators.

(iv) The matrix $H(x, y) = \langle \delta_x, H \delta_y \rangle$ is defined only for y in the proper domain of V .

A generalization of the case given by a periodic potential is given by a random potential which is ergodic. Nonrandomness of the spectrum follows even when we remove the stationarity assumption. The following explicit formula for the spectrum is possibly the highlight of the general theory.

Theorem 7.2.7 (Metrically Transitive Random Potentials) *Let V be a generalized real random potential, which is metrically transitive with respect to a sublattice in \mathbb{Z}^d . If D is a subset of the topological support of the probability distribution of V such that $P(V \in \overline{D}) > 0$, then*

$$\Sigma := \tilde{\sigma}(H[V]) = \overline{\bigcup_{v \in D} \tilde{\sigma}(H(v))}, \quad (7.19)$$

with probability one.

Proof. The metrical transitivity gives that the event $\{\Sigma \cap U \neq \emptyset\}$ has probability zero or one for any given open set U , so Σ is given by a nonrandom set. The lower semicontinuity of $V \mapsto \Sigma$ gives the explicit formula for Σ . \square

Remarks. (i) The theorem does not require the potential to be stationary, so it is impossible to prove absence of discrete spectrum.

(ii) It would be nice to have a strengthening of this theorem, given by a replacement of $\text{supp } P_V$ by the set of admissible potentials.

(iii) A consequence is that $\tilde{\sigma}(H(v)) \subset \Sigma$ with probability one for any $v \in \text{supp } P_V$, from the case $D = \text{supp } P_V$.

The second task in a study of spectral properties of our random Schrödinger operator is to consider the Lebesgue decompositions of the spectrum. These questions are not at all only of mathematical interest. In the following section we will consider the relevance in physics of the Lebesgue decompositions.

7.3 Large Time Behavior of Wave Functions

In this section we will give a description of some of the physical effects associated with the Lebesgue decomposition of a selfadjoint operator.

The state of a quantum electron at a given time, say $t = 0$, is given by a vector ψ_0 in the Hilbert space \mathcal{H} , with $\|\psi_0\| = 1$. The vector ψ_0 is also referred to as the wave function of the electron. Let H be the selfadjoint Hamilton operator which is given from the energy of the electron. The dynamics of the electron is then given by a time dependent state

$$\psi_t = e^{-itH/\hbar}\psi_0. \quad (7.20)$$

This is equivalent with the time dependent Schrödinger equation, and $\hbar = 1.054589 \cdot 10^{-34} Js$ is Planck's constant.

Consider the function defined by

$$t \mapsto F(t) := \langle \psi_0, \psi_t \rangle, \quad (7.21)$$

for positive times t . This function F is continuous and takes complex values in $\{z \in \mathbb{C} \mid |z| \leq 1\}$. It is continuous, because $t \mapsto \psi_t$ is continuous. The selfadjointness of H gives that $\|\psi_t\| = 1$, and the Schwarz inequality gives the claim on the range of F . Observe that if $F(t)$ is close to one, then ψ_t is close to ψ_0 . More precisely is

$$\|\psi_t - \psi_0\| = \sqrt{2 - 2 \operatorname{Re} F(t)} \leq \sqrt{2|F(t) - 1|}. \quad (7.22)$$

Furthermore $F(t)$ is close to zero if and only if ψ_t is close to the orthogonal complement space of ψ_0 , since $F(t)$ equals the ψ_0 component of ψ_t .

The Hamilton operator H has a set of eigenvectors, and the closure of the linear span of these vectors defines the atomic subspace \mathcal{H}_p . Assume now that ψ_0 is in the atomic subspace of H . It follows that

$$F(t) = \sum_{n \geq 1} e^{-itE_n/\hbar} |\langle \psi_0, \phi_n \rangle|^2, \quad (7.23)$$

where $\{\phi_n\}$ is a chosen orthonormal set of eigenvectors with associated eigenvalues $\{E_n\}$. For simplicity we have assumed that the Hilbert space is separable. The function F in equation (7.23) is a uniform limit of trigonometric polynomials and is then an almost periodic function. A consequence is that ψ_t returns arbitrary close to ψ_0 for arbitrary large times. The proof of this is far from obvious, if the theory of almost periodic functions is unknown.

The continuous subspace \mathcal{H}_c is the orthogonal complement of the atomic subspace of H . Now we want to know what happens if ψ_0 is in the continuous subspace. The vector ψ_0 is associated with a measure ρ determined by its Fourier transformation

$$F(t) = \langle \psi_0, e^{-itH/\hbar}\psi_0 \rangle = \int e^{it\omega} d\rho(\omega). \quad (7.24)$$

It turns out that ψ_0 is in the continuous subspace if and only if the measure ρ is continuous. Another nontrivial fact is that

$$L := \lim_{M \rightarrow \infty} \frac{1}{M} \int_0^M |F(t)|^2 dt = \sum_{\omega \in \mathbb{R}} |\rho(\{\omega\})|^2, \quad (7.25)$$

where existence of the limit is part of the claim. We conclude that $L = 0$ if and only if ψ_0 is in the continuous subspace of H . In particular it follows that if the ψ_0 component of ψ_t goes to zero as time increases, then ψ_0 is in the continuous subspace of H .

Any nonzero vector in the Hilbert space gives an associated measure as in equation (7.24). A vector is in the absolutely continuous subspace \mathcal{H}_{ac} of H if and only if the associated measure is absolutely continuous with respect to Lebesgue measure. It turns out that the absolutely continuous subspace is a closed subspace of the continuous subspace. Now we observe that if F is integrable, then the vector ψ_0 is in the absolutely continuous subspace. In particular it follows that if the ψ_0 component of ψ_t goes to zero fast enough as time increases, then ψ_0 is in the absolutely continuous subspace of H .

The singular continuous subspace \mathcal{H}_{sc} of H is defined as the orthogonal complement of the absolutely continuous subspace relatively to the continuous subspace. The vector ψ_0 is in the singular continuous subspace if and only if the associated measure ρ is singular continuous with respect to Lebesgue measure. This case is implicitly described in the above by elimination. The time mean L must be zero, but the ψ_0 component of ψ_t can not go too fast to zero, if at all. It seems possible that ψ_t can return closely to ψ_0 for arbitrary large times, but not too frequently. It seems that this case is the most interesting one.

7.4 Lebesgue Decomposition of the Random Schrödinger Operator

The first aim in this section is to prove randomness of the spectral family of the Schrödinger operator defined from a generalized real, random potential. The spectral family is given explicitly from Cauchy's integral formula and the Green's function. We will therefor first consider properties of the Green's function.

Definition 7.4.1 (The Green's Function) *Let V be a generalized potential on $X := \mathbb{Z}^d$ and set $\mathcal{H} := l^2(X)$. The projection operator $\Pi = \Pi[V]$ is defined by*

$$\Pi : \mathcal{H} \rightarrow \mathcal{H}, \quad \Pi f(x) := 1_{X[V]}(x)f(x), \quad X[V] := \{x \in X \mid V(x) \neq \infty\}. \quad (7.26)$$

Let z be in the regularity domain of the Schrödinger operator $H = H[V]$. The Green's function $G_z = G_z[V]$ is defined by

$$X \times X \ni (x, y) \mapsto G_z(x, y) := \langle \delta_x, (z - H)^{-1} \Pi \delta_y \rangle, \quad (7.27)$$

and the diagonal Green's function $g_z = g_z[V]$ is defined by

$$X \ni x \mapsto g_z(x) := G_z(x, x). \quad (7.28)$$

Remarks. (i) We will sometimes write G instead of G_z , when the value of z is irrelevant, and likewise for g .

(ii) Let A be any matrix operator defined in $l^2(X[V])$. We extend $A(x, y)$ to be defined for all x, y in X by setting it equal to zero where it is initially undefined. The operator defined from a proper extension like this will get zero as an eigenvalue.

Theorem 7.4.2 (Continuity of the Green's Function) *Let $z = \lim z_\nu$ in $\mathbb{C} \setminus \mathbb{R}$, and let $V = p\text{-}\lim V_\lambda$ for generalized real potentials on \mathbb{Z}^d . Let $G_z[V]$ be the Green's function, and let $g_z[V]$ be the diagonal Green's function. Consider nets given by the product order, so $(\nu_1, \lambda_1) \leq (\nu_2, \lambda_2)$ iff $\nu_1 \leq \lambda_1$ and $\nu_2 \leq \lambda_2$. If $x \in X[V]$, then*

$$\|G_{z_\nu}[V_\lambda](\cdot, x) - G_z[V](\cdot, x)\|_2 \rightarrow 0, \quad \|G_{z_\nu}[V_\lambda](x, \cdot) - G_z[V](x, \cdot)\|_2 \rightarrow 0, \text{ and} \quad (7.29)$$

$$G_z[V] = p\text{-}\lim G_{z_\nu}[V_\lambda], \quad g_z[V] = p\text{-}\lim g_{z_\nu}[V_\lambda].$$

Proof. Fix x in X and set $K := |\text{Im}(z)|$. Let ν_1 be such that $|\text{Im}(z_\nu)| \geq K/2$ when $\nu \geq \nu_1$. Note that $\|(z_\nu - H[V_\lambda])^{-1}\| \leq 2/K$ for $\nu \geq \nu_1$. Consider first $V(x) \neq \infty$. It follows that $\delta_x \in D((z - H[V])^{-1}) \cap \underline{\lim} D((z_\nu - H[V_\lambda])^{-1})$. The continuity of $(z, V) \mapsto z - H(V)$ and the bound on the resolvents give convergence of $G_{z_\nu}[V_\lambda](\cdot, x)$ in l^2 towards $G_z[V](\cdot, x)$. The identity $\langle \delta_x, (z - H(V))^{-1} \Pi(V) \delta_y \rangle = \langle (z^* - H(V))^{-1} \Pi(V) \delta_x, \delta_y \rangle$ gives similarly the second claim on l^2 convergence. Next consider $V(x) = \infty$, so $g(x) = 0 = G(x, y_1) = G(y_2, x)$. Set $h_{(\nu, \lambda)} := G_{z_\nu}[V_\lambda]$, so

$$\Pi(V_\lambda) \delta_y(x) = (z_\nu - 2d - V_\lambda(x)) h_{(\nu, \lambda)}(x, y) + \sum_{\|a\|_1=1} h_{(\nu, \lambda)}(x + a, y), \quad (7.30)$$

so $V_\lambda(x) \rightarrow \infty$ gives $h_{(\nu, \lambda)}(x, y) \rightarrow 0$, since $|h_{(\nu, \lambda)}(x, y)| \leq 2/K$ when $\mu \geq \mu_1$. This, together with the previous l^2 convergence, proves the required pointwise convergence. \square

Remarks. (i) It is actually sufficient to consider just one index set.

The joint continuity of the Green's function makes it possible to establish the randomness of the spectral family.

Theorem 7.4.3 (The Spectral Family) *If V is a generalized real, random potential, then*

$$E(a, b) = 1_{(a, b)}(H[V]) \Pi[V] \quad (7.31)$$

is a random orthogonal projection for any real numbers a, b with $a < b$.

Proof. Since $\langle \delta_y, 1_{(a, b)}(H) \Pi \delta_x \rangle$ equals

$$\lim_{\delta \rightarrow 0^+} \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (G_{t-i\epsilon}(y, x) - G_{t+i\epsilon}(y, x)) dt, \quad (7.32)$$

the randomness follows from the joint continuity of $(t, V) \mapsto G_{t \pm i\epsilon}(y, x)$, the estimate $|G_{t \pm i\epsilon}(y, x)| \leq 1/\epsilon$, and Fubini's theorem. \square

Remarks. (i) Any operator $f(H) = E[f]$ from the spectral family E of H is by definition only defined in $l^2(X[V]) = E(\mathbb{R})l^2(X)$.

It is noteworthy that the Schrödinger operator gives a natural assignment of a spectral measure at each lattice point.

Definition 7.4.4 (The Local Spectral Measure) Let V be a generalized real potential on \mathbb{Z}^d . The local spectral measure ρ_x at x is defined by

$$\rho_x(A) := E(A)(x, x) = \langle \delta_x, 1_A(H[V])\Pi[V]\delta_x \rangle. \quad (7.33)$$

Remarks. (i) The definition is valid for any selfadjoint, or normal, operator defined in $l^2(\mathbb{Z}^d)$, and more generally in any Hilbert space after a choice of basis.

The Fourier transformation of a probability measure gives a one-one correspondence between finite measures and characteristic functions. This means that we can associate a characteristic function to each lattice site.

Theorem 7.4.5 (The Local Characteristic Function) The characteristic function of the spectral measure at x is given by

$$\hat{\rho}_x(t) = e^{itH}(x, x). \quad (7.34)$$

Proof. From $\hat{\rho}_x(t) := \int e^{it\omega} d\rho_x(\omega) = e^{itH}(x, x)$. \square

Remarks. (i) The large time behavior of the characteristic function at x for each x determines the Lebesgue decomposition of the spectrum completely.

The previous section gives an interpretation of the characteristic function $\hat{\rho}_x$ at a given site x . We have

$$\hat{\rho}_x(t) = \psi_t(x), \quad \psi_0 = \delta_0, \quad (7.35)$$

where ψ_t is the time dependent wave function of an electron which initially where sitting in position x .

Theorem 7.4.6 (Nonrandom Lebesgue Components) Let V be a generalized real random potential on \mathbb{Z}^d . The Schrödinger operator $H = H[V]$ has a random spectral measure. The pure point spectrum, the absolutely continuous spectrum, and the singular continuous spectrum of H are closed random sets in \mathbb{R} , and they are nonrandom if the potential is metrically transitive with respect to a sublattice in \mathbb{Z}^d .

Proof. Let x_1, x_2, \dots be some enumeration of the points in \mathbb{Z}^d . A spectral measure for H is given by

$$\rho(A) := \sum_n \frac{1}{2^n} \rho_{x_n}(A). \quad (7.36)$$

The measure ρ is a random measure if V is a generalized real random potential, from the randomness of ρ_x . Since the Lebesgue-Hammer components of a random measure are random measures, it follows that the pure point spectrum, the absolutely continuous spectrum, and the singular continuous spectrum are closed random sets. Metrical transitivity gives that the event $\{\Sigma_{pp} \cap U \neq \emptyset\}$ has probability zero or one for any given open set U , so Σ_{pp} is then given by a nonrandom set. The same applies to the absolutely continuous spectrum and the singular continuous spectrum. \square

Remarks. (i) Observe that the measure class of the given spectral measure is transitively invariant, but it is not given by a nonrandom measure, since this would imply a nonrandom point spectrum. The set of measure classes is not σ -separated in a proper sense.
(ii) The spectral measure above is independent of the enumeration, since the series is absolutely convergent.

The periodic table of the elements can be explained to a surprisingly large extent from consideration of the energy levels given from the Hydrogen atom model. The given number of available electrons is filled into a corresponding number of levels, starting from the lowest energy level. This procedure is also adapted to solids, but there is a complication due to a continuum of available energy states. The integrated density of states is introduced to resolve this counting problem. We have defined a slightly more general concept of a density of states measure than usual, but the following consequence remains valid.

Theorem 7.4.7 (The Essential Spectrum) *Let V be a generalized potential on \mathbb{Z}^d . If N is a density of states measure for the Schrödinger operator $H = H[V]$, then $\text{supp } N \subset \tilde{\sigma}_e(H)$.*

Proof. Let $z \in \text{supp } N$ and let U be an open set in \mathbb{C} which contains z . The assumption gives an enumeration x_1, x_2, \dots of the elements in $X = \mathbb{Z}^d$ and a subsequence $\{n_k\} \subset \mathbb{N}$ such that

$$0 < N(U) \leq \lim_k \frac{1}{n_k} \sum_{i=1}^{n_k} \langle \delta_{x_i}, 1_U(H[V]) \Pi[V] \delta_{x_i} \rangle, \quad (7.37)$$

where the last inequality follows from the assumed weak convergence. In particular we conclude that $1_U(H[V]) \Pi[V]$ has infinite trace, so it is a projection onto an infinite dimensional subspace, which proves that z is in the essential spectrum. \square

The spectral measure class of an ergodic Schrödinger operator is not nonrandom. The integrated density of states measure is fortunately nonrandom.

Theorem 7.4.8 (A Nonrandom Density of States Measure) *Let V be a generalized real L -ergodic potential on \mathbb{Z}^d for a d -dimensional lattice L in \mathbb{Z}^d , with a fundamental cell of points x_1, \dots, x_m . Let P_i be the Borel measure on \mathbb{R} defined by the expectation*

$$P_i(A) := E[\rho_{x_i}(A)] \quad (7.38)$$

of the local spectral measure. A density of states measure is given by

$$N := \frac{1}{m} \sum_{i=1}^m P_i. \quad (7.39)$$

Furthermore

$$\tilde{\sigma}(H[V]) = \tilde{\sigma}_e(H[V]) = \text{supp } N, \quad (7.40)$$

with probability one.

Proof. Let Λ_n be an increasing sequence of cubes in $X = \mathbb{Z}^d$ such that $X = \bigcup_n \Lambda_n$. Let $A = (k2^{-M}, l2^{-M})$ for integers $k < l$ and a natural number M . Observe that the set of such dyadic intervals is countable and furthermore determines weak convergence of measures. Consider density of states approximants given by

$$N_n(A) := \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} h(x), \quad h(x) := \rho_x(A). \quad (7.41)$$

Observe that $0 \leq h(x) \leq 1$ and that h is an L -ergodic process. Set $X_i := \{x_i + l \mid l \in L\}$ and $\Lambda_{n,i} := \Lambda_n \cap X_i$. The Birkhoff ergodic theorem gives

$$N(A) = \lim N_n(A) = \lim \sum_{i=1}^m \frac{|\Lambda_{n,i}|}{|\Lambda_n|} \frac{1}{|\Lambda_{n,i}|} \sum_{x \in \Lambda_{n,i}} h(x) = \sum_{i=1}^m \frac{1}{m} E[h(x_i)] \quad (7.42)$$

with probability one, which proves the formula for the density of states measure. The relations $\text{supp } N \subset \tilde{\sigma}_e(H) \subset \tilde{\sigma}(H)$ hold in general, so we only need to prove $\tilde{\sigma}(H) \subset \text{supp } N$. Let $z \notin \text{supp } N$, so there exists an open set $U \ni z$ with $N(U) = 0$. The formula for N , and the stationarity, imply $\langle \delta_x, 1_U(H) \Pi \delta_x \rangle = 0$ for all x with probability one, so $1_U(H) = 0$, and $z \notin \tilde{\sigma}(H)$. \square

Remarks. (i) If the above density of states measure has a nonzero singular component, then the Schrödinger operator has a nonzero singular component with probability one. (ii) If the Schrödinger operator has purely absolutely continuous spectrum with probability one, then the density of states measure is purely absolutely continuous.

A particular consequence of the previous theorem is noteworthy. Assume that the potential is periodic and let ρ_1, \dots, ρ_m be the local spectral measures associated to each point in a fundamental cell for the potential. The density of states measure is then given by

$$N = \frac{1}{m} \sum_{j=1}^m \rho_m, \quad (7.43)$$

and this is also a spectral measure for the periodic Schrödinger operator.

7.5 The Percolation Model

One good reason for the consideration of generalized random potentials is given by the possibility of the following model.

Definition 7.5.1 (The Percolation Model) *Let $\{V(x)\}$, $x \in \mathbb{Z}^d$, be an i.i.d. stochastic process defined by $P(V(0) = 0) = p \in (0, 1)$ and $P(V(0) = \infty) = q = 1 - p$. The percolation model is given by the generalized random Schrödinger operator $H = H[V]$ defined in $l^2(X)$.*

Remarks. (i) This is possibly the simplest nontrivial model representing an ergodic closed operator.

The pure point spectrum and the spectrum of the percolation model are easily found.

Theorem 7.5.2 (Pure Point Spectrum) *The pure point spectrum and the spectrum of the d -dimensional percolation model are*

$$\Sigma_{pp} = \Sigma = [0, 4d], \quad (7.44)$$

with probability one.

Proof. The event $A = \{V(x_i) = v_i, i = 1, \dots, N\}$ has positive probability for any given finite sequence $\{v_i\} \subset \{0, \infty\}$ and $\{x_i\} \subset \mathbb{Z}^d$. The event $B = \bigcup T_x A$ is then $\{T_x\}$ -invariant, so $P(B) = 1$. Now the pure point spectrum follows from the spectra of the Dirichlet Laplace operator for arbitrary large cubes. Because the norm of the percolation Schrödinger operator is bounded by $4d$, it follows that $\Sigma \subset [0, 4d]$ for all realizations, and the opposite inclusion follows from the point spectrum. \square

Remarks. (i) The absolutely continuous spectrum Σ_{ac} and the singular continuous spectrum Σ_{sc} are nonrandom and contained in $[0, 4d]$, but this is essentially all we know.

(ii) If p is so small that there exists no infinite connected graph, then $\Sigma_{ac} = \Sigma_{sc} = \emptyset$. This is always the case in the one dimensional case.

(iii) We conjecture that the absolutely continuous spectrum is empty.

(iv) We conjecture that the continuous spectrum increases with increasing p .

The nontrivial part of the spectral theory of the percolation model is given by the possibility of an infinite connected graph. It seems reasonable that the Dirichlet Laplace operator for an infinite connected graph must contain some continuous spectrum, but a proof of this is missing. In the following we present some material which might be useful for a study of this problem.

The essential part of the Dirichlet operator is given by the nearest neighbor operator.

Definition 7.5.3 (The nearest neighbor Operator) *Fix a subset Λ in \mathbb{Z}^d . The neighbors of $x \in \Lambda$ is the points in $n(x) := \{y \in \Lambda \mid \|y - x\|_1 = 1\}$. The nearest neighbor operator $J = J[\Lambda]$ is defined by*

$$J : l^2(\Lambda) \rightarrow l^2(\Lambda), \quad Jf(x) := \sum_{y \in n(x)} f(y). \quad (7.45)$$

Definition 7.5.4 (Paths) *Fix a subset Λ in \mathbb{Z}^d . Let $x, y \in \Lambda$. A path c from x to y , written $c : x \rightarrow y$, is a function $c : \{0, 1, \dots, N\} \rightarrow \Lambda$ such that $c(0) = x$, $c(N) = y$, and $c(k+1) \in n(c(k))$ for all $k \in \{0, \dots, N-1\}$. A discontinuous path from x to y is a function c as above without the nearest neighbor restriction. The number N is the length $|c|$ of c . The number $n_z(c)$ is the number of times the path c visits the point z , $n_z(c) := |\{k \in D(c) \mid c(k) = z\}|$.*

Theorem 7.5.5 (Powers of The nearest neighbor Operator) *If J is the nearest neighbor operator for a set $\Lambda \subset \mathbb{Z}^d$, then*

$$J^N(x, y) = \sum_{\substack{c: x \rightarrow y \\ |c|=N}} 1, \quad N \in \{0, 1, \dots\}. \quad (7.46)$$

Proof. This is clear for $N = 0$, in which case $J^N = I$. Observe $J^{k+1}(x, y) = \sum_z J^k(x, z)J(z, y)$, which proves the claim by induction. The number of paths from x to y of length $k + 1$ equals the number of paths from x to the possible neighbors z of y of length k . \square

Remarks. (i) The matrix element $J^N(x, y)$ equals the number of paths of length N from x to y .

A consequence of the previous theorem is that the characteristic function of the spectral measure at x for the nearest neighbor operator is given by

$$\hat{\rho}_x(t) = \sum_{n \geq 0} (-1)^n \frac{J_{2n}}{(2n)!} t^{2n}, \quad (7.47)$$

where J_m equals the number of paths $c : x \mapsto x$ of length m . We repeat that the large time behavior of the characteristic function determines existence of continuous spectrum.

The characteristic function at x is given by the (x, x) element of the matrix of the unitary semigroup. In the free case this semigroup can be explicitly computed from Fourier theory.

Theorem 7.5.6 (The Unitary Free Semigroup) *If J is the nearest neighbor operator for \mathbb{Z}^d , then*

$$e^{itJ}(x, y) = \sum_{n \geq 0} \frac{(it)^n}{n!} J^n(x, y) = \prod_{j=1}^d (i)^{x_j - y_j} J_{x_j - y_j}(2t), \quad t \in \mathbb{C}, \quad (7.48)$$

where $z \mapsto J_n(z)$ is the Bessel function of integer order n of the first kind.

Proof. The first part is obvious since J is bounded. Fourier transformation and an integral representation of the Bessel function give

$$\begin{aligned} e^{itJ}(x, y) &= \left\langle \hat{\delta}_x, \hat{J} \hat{\delta}_y \right\rangle = \frac{1}{(2\pi)^d} \int dk \left[e^{ik(x-y) + 2it \sum_j \cos k_j} \right] \\ &= \prod_{j=1}^d \frac{1}{\pi} \int_0^\pi e^{2it \cos \theta} \cos(\theta(x_j - y_j)) d\theta = \prod_{j=1}^d (i)^{x_j - y_j} J_{x_j - y_j}(2t). \end{aligned} \quad (7.49)$$

\square

Expansions by means of the J operator is also valuable when we consider the full Schrödinger operator. As an example we consider the Green's function.

Theorem 7.5.7 (Path Expansion for The Green's Function) *Let V be a generalized complex potential on \mathbb{Z}^d . The Green's function is determined by*

$$G_z[V](x, y) = - \sum_{c: x \rightarrow y} \prod_{t \in X[V]} D(t)^{n_t(c)}, \quad \text{where} \quad (7.50)$$

$$D(t) := (V(t) + 2d - z)^{-1}, \quad (7.51)$$

$c : x \rightarrow y$ is a path in the set $X[V]$ where V is finite, and $n_t(c)$ equals the number of times c visits t . The convergence is absolute and valid for z such that

$$\|J\| < \inf_x |V(x) + 2d - z|, \quad (7.52)$$

where J is the nearest neighbor operator corresponding to the set $X[V]$. If the potential V is constant on $X[V]$, then

$$G(x, y) = - \sum_{k=0}^{\infty} J^k(x, y) D^{k+1}. \quad (7.53)$$

Proof. Fix the Hilbert space $\mathcal{H} = l^2(X[V])$. Let D denote the multiplication operator defined from the function also denoted by D above. The assumption in inequality (7.52) ensures that D is a bounded operator on \mathcal{H} , with a possibly unbounded inverse D^{-1} . Observe the identity $H[V] - z = D^{-1} - J = (1 - JD)D^{-1}$, which holds for the involved unbounded operators. Inequality (7.52) gives $\|JD\| \leq \|J\| \|D\| < 1$, so $(z - H[V])^{-1} = -D(1 - JD)^{-1} = -\sum_n D(JD)^n$, with absolute convergence in norm. Now

$$\begin{aligned} D(JD)^n(x, y) &= \sum_{y_1, \dots, y_{n-1}} D(x)J(x, y_1)D(y_1) \cdots J(y_{n-1}, y)D(y) \\ &= \sum_{\substack{c: x \rightarrow y \\ |c|=n}} \prod_{t \in X[V]} D(t)^{n_t(c)}, \end{aligned} \quad (7.54)$$

and $|D(JD)^n(x, y)| \leq \|D(JD)^n\|$ proves the absolute convergence and the first formula. When the potential is constant, the second formula for G follows from the first formula, $\sum_t n_t(c) = |c| + 1$, and the fact that $J^k(x, y)$ equals the number of paths from x to y of length k . \square

Remarks. (i) Recall that $\|J\| \leq 2d$.

(ii) The formula is correct for all $x, y \in \mathbb{Z}^d$, but trivial when $V(x) = \infty$ or $V(y) = \infty$.

(iii) Formula (7.53) may also be proven directly without application of (7.50) and the known property of J^k .

7.6 Notes

The term ‘‘Anderson model’’ is related to the famous publication [7] by P.W.Anderson in 1958. We quote the abstract of this paper

This paper presents a simple model for such processes as spin diffusion or conduction in the ‘‘impurity band’’. These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

The term “tight binding model” comes from the tight binding approximation in solid state physics. This approximation explains the band structure of the energy spectrum of the electron by consideration of the atomic energy states [10, p.175-190].

The theory of almost periodic functions was founded by Harald Bohr [28]. We note that he is the brother of Niels Bohr, the founder of quantum mechanics. The discussion of the significance of the Lebesgue decomposition of the Schrödinger operator does not seem to be contained in [31] or [107]. It gives a physical interpretation to some rather deep mathematical results, compare the theory of almost periodic functions. I seems to me that the significance of singular continuous spectrum is very important. It is tempting to claim that presence of singular continuous spectrum can give models with rigorously finite conductivity, but this is so far nothing but a qualified guess.

It makes sense to speak about local spectral measures also for continuous Schrödinger operators. A local spectral density is defined in [122, p.117-119] and it is used to define the integrated density of states measure in [122, p.120-127].

The conjecture about absence of absolutely continuous spectrum may possibly be attacked by application of the formulas found in [31, p.38-39,p.424].

The formula for the free semigroup of the nearest neighbor operator is adapted from [90, p.294].

Part IV

Appendix

Appendix A

What Next?

The following is a list of questions and suggestions which came to mind as a result of a quick reading of the previous chapters in reverse order. Some of the suggestions are very general. As a consequence there is no hope for a “solution” to these suggestions. On the other hand some of the questions are quite specific, and a definite answer seems possible. The answer may also already exist somewhere in the huge amount of literature available on related subjects.

1. Investigate the nature of the spectrum of the nearest neighbor operator.
2. Study the connection between the characteristic function and the Lebesgue decomposition of a measure. Similar for other transformations of the measure, and in particular the Stieltjes transformation.
3. Does an increase of the graph increase the continuous subspace of the nearest neighbor operator? Does the continuous spectrum increase?
4. Does the absolutely continuous spectrum decrease with increasing disorder?
5. Is singular continuous spectrum typical in some sense?
6. Is the spectrum of the discrete Schrödinger operator a lower semicontinuous function of the generalized complex potential?
7. Consider the relevance of the methods in [2] for the nearest neighbor operator.
8. Is it possible to use acceptor/donor levels of an impure Hamiltonian to prove existence of pure point spectrum of corresponding random Hamilton operators? Absence of absolutely continuous spectrum?
9. To what extent can results for the discrete multiplication operator be used to obtain results for the corresponding Schrödinger operator?
10. Investigate further into the properties of the lower operator topology.
11. Is the lower operator topology Hausdorff when restricted to the set of bounded operators defined on a Banach space?
12. Is the essential spectrum of a closed operator a lower semicontinuous function? A measurable function? What about the various other decompositions of the spectrum?

13. The case of random matrix operators from a random orthonormal system should be investigated further.
14. The possibility of defining the sum of two unbounded operators from approximating bounded operators should be investigated further.
15. What is the class of obtainable continuous Schrödinger operators from limits of discrete Schrödinger operators? What kind of convergence is possible?
16. Is “the” lower limit of closed random operators again a closed random operator?
17. Consider further the Lebesgue decomposition of random measures.
18. Is the weak limit of random measures again a random measure?
19. Find suitable convergence concepts for random measures. Convergence in probability?
20. Does the absolutely continuous spectrum of an ergodic measure valued process decrease with increasing disorder? What does disorder mean in this setting?
21. Investigate further into the properties of lower and upper topologies in hyperspaces.
22. Existence of a minimal upper (compact) limit of a net of closed sets?
23. Study further the concept of random sets.
24. Is the pointwise limit of a sequence of Borel random variables with values in a second countable T_0 -space again a random variable?
25. Other convergence concepts for abstract valued random variables?
26. Are there interesting examples of metrically transitive systems that are not stationary?
27. What are the relevance of other concepts from ergodic theory for ergodic random operators? What about entropy?
28. Study partial difference operators in general. Analogues and extensions of the results from the continuous case?
29. Further results for Bloch theory in the discrete case? Impurities?

Appendix B

Notes on Related Literature

The list of references below are divided into different sections depending on the category. This division is sometimes quite artificial and should not be taken too literally.

B.1 Physics

[7] In this paper the author observes that absence of diffusion is a possible in certain random lattices. The Anderson model was introduced here.

[40] Strong and weak localization is discussed briefly from a physicists point of view. An experiment with microwave localization caused by two-dimensional random scattering is described.

[97, 111] The possibility of Anderson localization of light is considered.

[64] A review of quantum chaos is given.

[48] The authors consider the effective-mass Hamiltonian for abrupt heterostructures.

[119] Unitary operator bases are considered for quantum systems. Weyl pairs are central.

[118] Operators with continuous spectra are realized as limits of operators with a finite number of states. The approximants are given by Weyl pairs at each stage.

[120] A brief review of some features of quasicrystals is given.

B.2 Mathematics From Other Fields

[3] The authors prove a maximal ergodic lemma for superadditive processes. The proof of the multidimensional ergodic theorem is a corollary of the results obtained. The proof is most instructive.

[12] The author considers separation properties in hyperspaces.

[35] Addition of unbounded operators is defined from consideration of the Lie product formula.

[63] The author gives a variant of the Feynman-Kac formula. The main result is that approximation by integral operators are replaced by approximation directly by finite difference operators. This is deduced as a variant of the Chernoff product formula.

- [72] The author gives a version of the Trotter product formula where the perturbation is given merely as a sesquilinear form by means of a regularization procedure.
- [53] The author considers the Lie product formula for semigroups defined by Friedrichs extensions.
- [85] A Trotter product formula for the form sum of selfadjoint operators that may be unbounded from below is proven.
- [102] A Schrödinger operator is defined by means of the Feynman-Kac formula.
- [126] Nonstandard analysis is used to define nonlinear functions of distributions. A generalized Trotter product formula is obtained as an application.
- [132] Trotter proves the Trotter product formula.
- [141] It is shown that the Trotter-Lie product formula holds in the Hilbert-Schmidt norm for certain Gibbs semigroups by means of Grümme's convergence theorem.
- [95] Borel measurability of the spectrum in topological algebras is studied. In particular it is proven that the mapping of a Banach algebra element to its spectrum is continuous on a dense G_δ .
- [105] Continuity properties of the spectra of operators are investigated.
- [104] A topology for closed operators is defined.
- [98] Measurability of multifunctions is considered.
- [103] This is a standard reference for topologies on spaces of subsets.
- [113] The spectrum and its relation to generalized eigenfunctions is studied. The results are applied to Schrödinger operators.

B.3 C^* -algebras

The following references do all rely on results from the theory related to C^* -algebras.

- [9, 8] The author considers the possibility of computing the spectrum of a bounded selfadjoint operator from the spectra of approximating finite matrices.
- [19] Continuity properties of the electronic spectrum of 1D quasicrystals are investigated.
- [17] K-theory is applied to prove gap labeling theorems for the spectrum of some 1D tight binding Hamiltonians.
- [18] The trace map for the transfer matrix is used to investigate the spectrum of a 1D Hamiltonian with period doubling potential.
- [50, 51] The author uses C^* -algebra technics to investigate the spectrum and the density of states for an almost periodic 1D Schrödinger operator. The almost Mathieu operator is in particular considered.

B.4 Mathematical Physics

- [2] The authors present a new and simplified proof of localization in the Anderson model for large disorder or extreme energies. The multiscale analysis in previous known proofs is

- avoided. The results are general enough to handle some models with off-diagonal disorder.
- [54] The main result is that at sufficiently high frequency the waves in a discrete random medium with off-diagonal coupling are localized.
- [44, 45] The probabilistic part of the multiscale proof of localization in the Anderson model is simplified. The methods are general enough to handle some models with correlated potentials. Nelson's best possible hypercontractive estimate is applied.
- [58] This paper contains the possibly first rigorous proof of localization in the multidimensional Anderson model.
- [59] This paper contains the possibly first rigorous proof of absence of diffusion in the multidimensional Anderson model. The main result is a multiscale proof of exponential decay of the Green's function.
- [42, 4] The authors study the eigenvalue branches of a perturbed Schrödinger operator in the spectral gaps of the unperturbed operator. This corresponds to a study of donor or acceptor levels resulting from impurities in a crystal. They do in particular address the following gap completeness problem: Is every energy value in a spectral gap a possible eigenvalue for a suitable coupling constant?
- [61] An alternative solution to the gap completeness problem is given.
- [68] The author applies the Birman-Schwinger principle for the consideration of eigenvalues in gaps of an unperturbed Schrödinger operator as a result of a local perturbation.
- [55] The author proves existence of eigenvalues in gaps of the Kronig-Penney Hamiltonian as a result of a definite sign perturbation. The Gelfand expansion is used to get a convenient expression for the Birman-Schwinger kernel.
- [60] The authors study the eigenvalue branches of a perturbed Schrödinger operator in the spectral gaps of the unperturbed operator in the limit of an infinite coupling constant. A cascade phenomenon for the eigenvalues is observed. This is related to the Zel'dovich phenomenon known in the physics literature.
- [69] The relation between the distance between an eigenvalue and the essential spectrum and the decay of the corresponding eigenfunction is explored.
- [70] The authors give explicit formulas for the spectrum of certain Schrödinger operators obtainable by consideration of generalized eigenvectors. Quite general conditions which ensure essential self-adjointness are given.
- [6] The authors prove almost periodicity of random Jacobi matrices which have purely absolutely continuous spectrum and a finite number of bands.
- [14] The authors consider the analytic properties of band functions. It is in particular proven that the lowest band of the periodic attractive screened Coulomb potential has a nondegenerate lowest band.
- [13] The authors consider the Lyapunov index and the integrated density of states measure for almost periodic Schrödinger operators. Examples of operators with purely singular continuous spectrum are given.
- [38] The trace formula for Schrödinger operators on the line is used to discuss inverse spectral problems. The results are generalizations of similar results for the Hill's equation.
- [41] This is a review of some of the relations between the KdV equation and the one dimensional Schrödinger equation. The discrete analogue is also discussed.

- [52] The nonrandomness of the spectrum of ergodic Schrödinger operators is proven by means of the notion of infinite representability of one selfadjoint operator in another. The authors do also comment on the Saxon-Hütner conjecture.
- [80] It is proven that the Lebesgue components of the spectrum of an ergodic random selfadjoint operator are nonrandom sets. This general result is then applied to a class of Schrödinger operators defined in $L^2(\mathbb{R}^d)$.
- [81] The nonrandom spectrum of certain ergodic selfadjoint operators is studied. A formula for the nonrandom spectrum is given by the union of the spectra of deterministic admissible operators. Conditions which imply essential selfadjointness with probability one are also given.
- [78] The asymptotic behavior of the density of states near band edges is investigated for a class of ergodic Schrödinger operators.
- [84] The authors prove absolute continuity of the spectrum of a large class of almost periodic 1D Schrödinger operators defined in $L^2(\mathbb{R})$ given by certain random potentials.
- [82] It is shown that the Ljapunov indices determine the absolutely continuous spectrum of a stationary random 1D Schrödinger operator.
- [83] It is shown that the topological support of the potential for an ergodic Schrödinger operator in $L^2(\mathbb{R}^d)$ determines the nonrandom spectrum. In one dimension the absolutely continuous spectrum is also determined by the topological support of the potential.
- [86] The asymptotics of the density of states for a general elliptic self-adjoint operator in $L^2(\mathbb{R}^d)$ with ergodic coefficients is studied. The main tool is given by a formula for the density of states given as an appropriate trace of the spectral family.
- [87] Nonlinear random equations with operators of monotone type is studied. A main tool is given by the existence theorem of measurable selections for multivalued measurable mappings.
- [89] A general scheme for construction of periodic and almost periodic solutions of certain nonlinear partial differential equations is presented. Methods from algebraic geometry is central. The methods are related to Lax's method of determining periodic solutions of the KdV equations.
- [91] It is shown that a class of Schrödinger operators in $l^2(\mathbb{Z}^d)$ with $d \geq 4$ have some absolutely continuous spectrum. The class is given by certain sparsely supported potentials.
- [90] It is shown that certain Anderson models with decaying potential has absolutely continuous spectrum.
- [100] The authors prove absence of absolutely continuous spectrum given that the Green's function decay exponentially.
- [99] Absence of diffusion for certain Schrödinger operators in $L^2(\mathbb{R}^d)$ is proven.
- [108] Spectral properties of ergodic Schrödinger operators are considered. Nonrandomness of the spectrum is proven. Absence of absolutely continuous spectrum in certain 1D models is proven. The conductivity is also considered.
- [33] The behavior of embedded eigenvalues under local perturbations of the Sturm Liouville operator is studied. It is shown that a local perturbation can add infinitely many eigenvalues.
- [34] The author considers stability of singular continuous spectrum under local perturba-

tions of the Sturm Liouville operator.

[123] The author reviews regularity properties of the density of states for stochastic Jacobi matrices. In particular it is proven that the density of states measure is continuous. It is noted that this is not yet proven for the continuous ergodic Schrödinger operator.

[127] The author reviews some results on localization for random and almost periodic Schrödinger operators.

[133] A solution of the inverse problem for the Hill's equation is given.

[134] The author reviews ideas behind quantization and deformation. The role of unitary representations in quantum mechanics is pointed out.

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