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Periodic Orbits for Transcendental Hénon Maps

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To my father and mother:

Nguyen Van Hoa (Duong Duc Phuong) and Nguyen Thi Dao.

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Abstract

This master thesis deals with periodic points of transcendental Hénon maps, a subject in complex dynamics. In particular, we investigate the existence of periodic points and the discreteness of the set of k-periodic points for certain values of k. The simplest case is $k = 1$, the fixed points. We employ known results from the theory of entire functions to show that transcendental Hénon maps $(z, w) \mapsto (f(z) - \delta w, z)$, where f has finite and non-integer-valued order, admit infinitely many fixed points. We also give a complete description for the existence of fixed points in the case f is a general entire function. For values of k greater than 1, it is of interest to determine when a k-periodic point (z, w), fails to be an m-periodic point for all $m < k$. That is, when (z, w) is a *genuine* k-periodic point. We provide complete characterizations for the cases $k = 2$ and $k = 3$. A simple characterization in both cases when $\delta \neq 1$, is that such points are genuine if and only if they lie off the diagonal $\Delta = \{(z, w) \in \mathbb{C}^2 : z = w\}$.

Let F be a transcendental Hénon map, and denote by $\mathrm{Fix}(F^k)$, the set of k-periodic points of F. It follows from elementary properties of the zero set of holomorphic functions of a single variable, that $\mathrm{Fix}(F)$ and $\mathrm{Fix}(F^2)$ are discrete sets. Under the additional assumption that the order of f is strictly less than $1/2$, Leandro Arosio, Anna Miriam Benini, John Erik Fornæss, and Han Peters have further shown that $\mathrm{Fix}(F^k)$ is discrete for all $k \geq 1$ [8]. Their proof is based on a result by Wiman on the minimum modulus of entire functions with small order. This raises the question whether there are more general transcendental Hénon maps, where the order of f is greater than or equal to 1/2, for which $\mathrm{Fix}(F^k)$ is discrete for some $k \geq 3$. Using the implicit mapping theorem, we show the existence of such a map in the case $k = 3$, where $f(z) = f_\delta(z)$ is dependent on δ, and where f_δ has order equal to 1. For the case $k = 4$, using elementary properties of analytic sets in $\mathbb{C}^2$, we are also able to show that the transcendental Hénon map $(z, w) \mapsto (e^z - \delta w, z)$, has a discrete set of 4-periodic points when $\delta^2 = 1$.

We give several existence results. For instance, we prove the existence of infinitely many genuine 4-periodic points for the specific type of transcendental Hénon maps of the form $(z, w) \mapsto (e^{g(z)} + w, z)$, where g is some non-constant entire function. Our technique is an estimate method which leads to almost explicit formulae. We start with the case $g(z) = z^d$, a monomial of degree $d \geq 2$, and then generalize to the case when g is a transcendental entire function, using the Wiman-Valiron method (Theorem 1.4.4) to look for solutions near points where g looks like a polynomial of high degree. For the corresponding symplectic maps $(z, w) \mapsto (e^{g(z)} - w, z)$, using a completely different approach, we are able to show the existence of infinitely many genuine 4-periodic points under the additional assumption that g has a non-zero period p: $g(z) = g(z + p\mathbb{Z})$ for all z. We also give existence results on k-periodic points for more general values of k. We prove the following main result: let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendenal

Hénon map where f is odd and has non-zero period p. Then F_1 admits infinitely many genuine k-periodic points for all $k \geq 2$ prime. The case $k = 5$ is special and the assumption that f be p-periodic, $p \neq 0$, can be disposed of. Finally, we give two special results for k not prime: $k = 6$ and $k = 8$.

Sammendrag på norsk

Denne masteroppgaven handler om periodiske punkter til transcendente Hénonavbildninger, et tema i kompleks dynamikk. Mer spesifikt, undersøker vi eksistensen av periodiske punkter og diskretheten av mengden av k-periodiske punkter for utvalgte verdier av k. De enkleste periodiske punktene er fikspunktene. Dette er tilfellet $k = 1$. Vi benytter oss av kjente resultater fra teorien om hele funksjoner[1] til å vise at transcendente Hénonavbildninger $(z, w) \mapsto (f(z) - \delta w, z)$, der f har endelig og ikke heltallig orden, har uendelig mange fikspunkter. Vi gir også en fullstendig karakterisering for tilfellet der f er en generell hel funksjon. For verdier av k større enn 1, er det interessant å spørre når et k-periodisk punkt (z, w) ikke er et m-periodisk punkt for alle $m < k$. Det vil si når (z, w) er et *genuint* k-periodisk punkt. Vi gir fullstendige karakteriseringer i tilfellene $k = 2$ og $k = 3$. I begge tilfellene, når $\delta \neq 1$, er en enkel karakterisering at slike punkter er genuine hvis og bare hvis de ikke ligger på diagonalen $\Delta = \{(z, w) \in \mathbb{C}^2 : z = w\}$.

La F være en transcendent Hénonavbildning. Vi betegner med Fix(F^k), mengden av k-periodiske punkter til F. Det følger fra elementære egenskaper til nullmengden til en holomorf funksjon av én variabel, at Fix(F) og Fix(F^2) er diskrete mengder. Under antakelsen at f har orden ekte mindre enn $1/2$, har Leandro Arosio, Anna Miriam Benini, John Erik Fornæss og Han Peters, videre vist at Fix(F^k) er diskret for alle $k \geq 1$ [8]. Beviset deres bruker Wiman sitt resultat om minimum-modulus-funksjonen til hele funksjoner av lav orden. Dette tar opp spørsmålet om det finnes andre mer generelle transcendente Hénonavbildninger hvor ordenen til f større eller lik 1/2, der mengden Fix(F^k) er diskret for en eller annen $k \geq 3$. Ved bruk av implisitt avbildningsteoremet, viser vi eksistensen av en slik avbilding i tilfellet $k = 3$, der $f(z) = f_\delta(z)$ avhenger av δ og f_δ har orden like 1. For tilfellet $k = 4$ klarer vi også å vise, ved bruk av elementære egenskaper til analytiske mengder i $\mathbb{C}^2$, at den transcendente Hénonavbildningen gitt ved $(z, w) \mapsto (e^z - \delta w, z)$, har en diskret mengde med 4-periodiske punkter når $\delta^2 = 1$.

Vi gir flere eksistensresultater. Blant annet viser vi at de spesifikke transcendente Hénonavbildningene på formen $(z, w) \mapsto (e^{g(z)} + w, z)$, der g er en ikke-konstant hel funksjon, har uendelig mange genuine 4-periodiske punkter. Teknikken vi benytter oss av, er en estimerings metode som leder til nesten eksplitte formler. Vi starter med tilfellet $g(z) = z^d$, et monom med grad $d \geq 2$, for så å generalisere til tilfellet der g er en transcendent hel funksjon. For generaliseringen, benytter vi oss av Wiman-Valiron metoden (Teorem 1.4.4) til å lete etter løsninger i nærheten av der g ser ut som et polynom med høy grad. For de tilsvarende symplektiske avbildningene $(z, w) \mapsto (e^{g(z)} - w, z)$, beviser vi eksistensen av uendelig mange genuine 4-periodiske punkter under tilleggsantakelsen at g har en periodie

[1]På engelsk: *entire functions.*

p ulik 0: $g(z) = g(z + p\mathbb{Z})$ for alle z. Vi gir også eksistensresultater for mer generelle verdier av k. Vi beviser følgende hovedresultat: la $F_1(z, w) = (f(z) - w, z)$ være en symplektisk transcendent Hénonavbildning der f er en odde funksjon med periode $p \neq 0$. Da har F_1 uendelig mange genuine k-periodiske punker for alle primtall $k \geq 2$. Et spesielt tilfelle, er $k = 5$. Her kan en tillate perioden $p = 0$ (ingen periode) og fremdeles ha uendelig mange genuine 5-periodiske punkter. Tilslutt gir vi to spesielle resultater der k ikke er et primtall: $k = 6$ og $k = 8$.

Contents

Chapter 0

Introduction

0.1 Motivation and background

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. One writes $f^2 = f \circ f$, and inductively $f^m = f \circ f^{m-1}$ for $m \geq 3$. The map f^m is said to be the m-th iterate of f. Complex dynamics is a branch of complex analysis in which one studies the properties of iterates of holomorphic maps. Of interest, is the behaviour of the sequence of iterates $\{f^m\}_m$ as m tends to infinity. One speaks of the dynamics of the map f. Given f, one divides the space $\mathbb{C}^n$ in two disjoint sets: the Fatou set of f: $F(f)$, and Julia set of f: $J(f)$. On the Fatou set, the dynamics of f behaves nicely and is predictable, while on the Julia set, the dynamics of f is chaotic. For $\varepsilon > 0$ small and $z_0, z_1 \in F(f)$ with $0 < |z_1 - z_0| < \varepsilon$, $f^m(z_1)$ and $f^m(z_0)$ remain close for large m. Iterates of nearby points in the Fatou set behave in the long run, similarly. On the other hand, for arbitrarily small $\varepsilon > 0$, if $z_0, z_1 \in J(f)$ and $0 < |z_0 - z_1| < \varepsilon$, the iterates of z_1 and z_0 can behave drastically different. There has been and still is, extensive research on the properties of $F(f)$ and $J(f)$. Although our study will not directly be involved with the Fatou set and the Julia set, it is not completely unrelated.

In this master thesis, we investigate periodic behaviour and in particular, we aim to say something interesting regarding periodic points of the type of maps of the form $F : \mathbb{C}^2 \to \mathbb{C}^2,\ (z, w) \mapsto (f(z) - \delta w, z)$, where $\delta \in \mathbb{C} \backslash \{0\}$, and where f is a transcendental entire function. A Periodic point of any map F, is a point z such that $F^k(z) = z$ for some $k \in \mathbb{N}$. Thus, the dynamics of F for such a point, follows an orbit: $z, F(z), \ldots, F^{k-1}(z), z$ and is predictable. Maps of the form $(z, w) \mapsto (f(z) - \delta w, z)$ are called Hénon maps. When f is a transcendental entire function, these are called transcendental Hénon maps, and when f is a polynomial, these are called polynomial Hénon maps. While there has been done extensive research on polynomial Hénon maps, very little is known about transcendental Hénon maps. Most recent work has been done by Leandro Arosio, Anna Miriam Benini, John Erik Fornaess, and Han Peters [8]. Their paper [8] also discusses results aside from those related to periodic behaviour, but we will restrict attention to the parts concerned with periodic points.

The main focus, and indeed the goal, in this master thesis, has been to provide *new results* related to periodic orbits for transcendental Hénon maps. As has already been pointed out, very little is known about the dynamics of transcendental Hénon maps as of today, and this is also true for periodic behaviour of transcendental Hénon maps. What is quite intriguing, or at least so the author feels, is that although the notion of periodic orbits, is an elementary one in the field of (complex) dynamics, there is still very little that is known even here when it comes to transcendental Hénon maps. Thus, albeit being elementary, there is a lot to be discovered regarding periodic behaviour of transcendental Hénon maps. The author believes this makes research in this particular area exciting and interesting, and indeed it is this that has been the main motivation of the author for this project. There are some results related to periodic orbits of transcendental Hénon maps, given in [8]. When it comes to these results, mainly only the cases $k = 1$ and $k = 2$ are treated. As a result, of *primary interest* has therefore naturally been the investigation of cases where k takes other values than 1 and 2, values for which [8] already provides interesting results. However, of concern has also been whether a k-periodic point is genuine or not. In this regard, it is natural to consider also the case of lower values for k, such as $k = 1$ and $k = 2$, and therefore thus appropriately, these two cases have been investigated as well.

Of course, in a study of periodic orbits of maps, the natural question which surfaces, is the question concerned with the existence of periodic points. That being said, there certainly are other interesting problems as well which the author does believe deserve further investion. For instance, in [8], Leandro Arosio, Anna Miriam Benini, John Erik Fornæss, and Han Peters, give, what the author believes to be, an interesting result on the discreteness of the set of periodic points of transcendental Hénon maps $(f(z) - \delta w, z)$. However the authors of [8] restricts here to the case when the order of f is strictly less than 1/2. This naturally raises the question whether there are more general transcendental Hénon maps, where the order of f is greater than or equal to 1/2, with discrete set of k-periodic points for some $k \geq 3$. Motivated by this, in addition to focusing on the natural question of existence of periodic points, there has also been made an effort in investigating the discreteness of the set of periodic points of transcendental Hénon maps.

Finally, in addition to being interesting in its own right, the study of periodic points also relates to other topics in dynamics, such as to the the study of the Fatou and Julia set. Indeed, in [8], it is for instance, shown that a transcendental Hénon map $F(z, w) = (f(z) - \delta w, z)$ in the case $\delta \neq -1$, has infinitely many saddle points of period 1 or 2. This in turn implies the non-emptiness of the Julia set of F: $J(F) \neq \emptyset$. See Corollary 3.6 in [8].

The author hopes that what has been said, provides motivation for studying transcendental Hénon maps, complex dynamics, and what to come. Further motivation for the specific study of the dynamics of transcendental Hénon maps can be found in the introduction of [8].

0.2 Overview of content

Chapter 1 deals with the theory of entire functions and gives preliminary results referred to and used in later chapters. We introduce the concept of a transcendental entire function and discuss two interpretations of such functions. The first is to view these as holomorphic functions on all of $\mathbb{C}$ with a single essential singularity at ∞, and the second is the interpretation of transcendental entire functions as non-algebraic entire functions. We define the maximum modulus of entire functions and discuss the concept of the order of an entire function. We state and prove an elementary, but important, result that relates the maximum modulus of an arbitrary transcendental entire function to that of any polynomial, and discuss the cardinality of the solution set of the equation $f(z) = \lambda P(z)$ where $\lambda \in \mathbb{C}$, P is any non-constant polynomial, and f is an entire function with finite order. We also state a result concerned with the possibilities for entire functions g and h when it is known that the composite entire function $h \circ g$ has finite order. Finally, we state without proof, the Wiman-Valiron method (Theorem 1.4.4) for approximating any transcendental entire function near points of maximum modulus by polynomials of high degree, and Rosenbloom's theorem (Theorem 1.4.1) on the cardinality of the set of fixed points of a composition of two entire functions. All the results in chapter 1, are known results.

Chapter 2 deals with fixed points of Hénon maps. We define what is meant by Hénon maps and what is meant by a k-periodic point of a general map. Then, we employ some results on entire functions with finite order from chapter 1 to provide some special existence results on fixed points. Finally, we give a complete characterization of the existence of fixed points for Hénon maps.

Chapter 3 deals with 2-periodic points of Hénon maps. We discuss the concept of *genuine* periodic points, and give a characterization for 2-periodic points of Hénon maps. We also discuss some existence results and describe the simple dynamics of 2-periodic points of Hénon maps. In particular, we use the theorem of Rosenbloom (Theorem 1.4.1) stated in chapter 1, to show that all transcendental Hénon maps of the form $(f(z) - \delta w, z)$ with $\delta \neq -1$, have infinitely many 2-periodic points. The latter result origins from [8].

In chapter 4, we consider the discreteness of periodic points of transcendental Hénon maps. Using elementary properties of the zero set of holomorphic functions of a single variable, we determine that the set of fixed points and the set of 2-periodic points of transcendental Hénon maps are discrete sets. This is a known result already in [8]. We state and prove the result in [8] which says that for all $k \geq 1$, $\text{Fix}(F^k)$ is discrete for all transcendental Hénon maps $F(z, w) = (f(z) - \delta w, z)$ where f has order strictly less than 1/2. Our proof is completely analogous to the one given in [8]. We then proceed to show the existence of a transcendental Hénon map $H_\delta(z, w) = (h_\delta(z) - \delta w, z)$ where $\text{Fix}(H_\delta^3)$ is discrete and the order of $h_\delta(z)$ is equal to 1. Our argument uses the implicit mapping theorem. Using also elementary properties of analytic sets in $\mathbb{C}^2$, we show that the transcendental Hénon map $F(z,w) = (e^z - \delta w, z)$ has discrete set of 4-periodic points when $\delta^2 = 1$. These two results provide an affirmative answer to the open question mentioned in the previous section, at the end of paragraph 4.

Chapter 5 deals with transcendental Hénon maps with infinitely many genuine k-periodic points for certain values of k. It is the first part on the subject and is considerably longer than all the other chapters. Chapter 6 is the second part. In chapter 5, we deal with the cases $k = 3$ and $k = 4$. We consider genuine 3-periodic points and give a complete characterization for these. Then we construct examples of transcendental Hénon maps with infinitely many genuine 3-periodic points. We then move on to consider results for 4-periodic points of transcendental Hénon maps. Our main result in this chapter, is an existence result for 4-periodic points for a certain type of Hénon maps: using an estimate method and the Wiman-Valiron method (Theorem 1.4.4), we show the existence of infinitely many genuine 4-periodic points for the family of transcendental Hénon maps of the form $(z,w) \mapsto (e^{g(z)} + w, z)$ where g is a non-constant entire function. We first consider the case $g(z) = z^d$, a monomial with degree $d \geq 2$, and then generalize to the case when g is a transcendental entire function by using the Wiman-Valiron method (Theorem 1.4.4) to approximate g by a polynomial of high degree near points where $|g|$ attains the maximum modulus of g. The case when g is a general polynomial is completely analogous to the case when g is a monomial. Finally, we prove that the family of symplectic transcendental Hénon maps of the form $(z,w) \mapsto (f(z) + w, z)$ where f has a non-zero period p: $f(z) = f(z + p\mathbb{Z})$ for all z, admits infinitely many genuine 4-periodic points.

Chapter 6, the final chapter, is a continuation of chapter 5 and the second part on transcendental Hénon maps with infinitely many genuine k-periodic points. We provide a systematic way of reducing the system of equations determining k-periodic points of symplectic transcendental Hénon maps $(f(z) - w, z)$ where f is odd, by half. We then use this method to show that such maps admit infinitely many genuine 5-periodic points. We further use this method to get the main result of the chapter: let $F_1(z,w) = (f(z) - w, z)$ be any symplectic transcendental Hénon map where f is odd and p-periodic with $p \neq 0$. Then F_1 admits infinitely many genuine k-periodic points for all $k \geq 2$ prime. The proof of the main result uses a Rosenbloom-type result which we provide (Theorem 6.2.3): if $L(z)$ is any first order polynomial and $g(z)$ is a transcendental entire and periodic function, then the equation $P(z) = g(z)$, has infinitely many solutions. Finally, we give two special results in the case k is not prime for $k = 6$ and $k = 8$.

Chapter 1

Some Theory of Entire Functions

Our main objectives of study in this master thesis, are transcendental Hénon maps. These are holomorphic maps of of the form $F : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (f(z) - \delta w, z)$ where δ is a non-zero complex number and f a transcendental entire function. Thus, it is natural to expect that the study of entire functions will play an important role, and provide useful information. In this chapter, we define what is meant by *transcendental entire functions*, study the *maximum modulus* and the *order* of entire functions, and give some preliminary results from the theory of entire functions which will be used in the later chapters. References for the material we present in the first three subsections, are for instance [2] and [1]. In the fourth and last subsection, which concerns the Wiman-Valiron method (Theorem 1.4.4) and the theorem of Rosenbloom (Theorem 1.4.1) on fixed points of composites of entire functions, separate references will be given. We note here that z is called a fixed point of the map h, if $h(z) = z$. We will denote the set of fixed points of h, by $\text{Fix}(h)$.

1.1 Transcendental entire functions

The first step in a study of transcendental Hénon maps must be to understand what is meant by transcendental entire functions.

Definition 1.1.1. An **entire function** is said to be a function which is holomorphic on the whole complex plane $\mathbb{C}$.

Thus, if f is an entire function, it can be represented by its Taylor series about the origin (or any other point):

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1.1}$$

Example 1.1.2. Any polynomial $P(z) = \sum_{j=1}^{n} a_j z^j$ is entire. Also, the trigonometric functions $\sin(z)$ and $\cos(z)$ are entire functions, and so is the exponential function e^z.

A known result is Liouville's theorem:

Theorem 1.1.3. Any bounded entire function is constant.

By definition, the only possible singularity of an entire function f, must be at ∞. Liouville's theorem 1.1.3, shows that in the case that f has no singularity at ∞, or in the case that f has a removable singularity at ∞, f is constant. There are two other cases: the case that f has a pole at ∞, and the case that f has an essential singularity at ∞. In the case that f has a pole at ∞, to determine the nature of f, we can consider the pole of $f(1/z)$ at $z = 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then: $f(1/z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$, and there exists some N such that $a_n = 0$ for all $n > N$. Hence: $f(z) = \sum_{n=0}^{N} a_n z^n$, and f is a polynomial of degree at most N. In the case that ∞ is an essential singularity (not removable, nor a pole), we say by definition that f is a **transcendental entire function**.

Remark 1.1.4. The term *transcendental* also generalizes to meromorphic functions, but we will only be concerned with entire functions. For those interested in the general definition, we give it here. Recall that a meromorphic function, is one that is holomorphic everywhere except possibly at isolated singularities, all of which are poles.

Definition 1.1.5. Let f be a meromorphic function. Then, if $z = \infty$ is a regular point or a pole, f is said to be **rationally meromorphic** or a **rational meromorphic function**. If $z = \infty$ is an essential singularity, then f is said to be a **transcendental meromorphic function**.

Thus, the essential thing about transcendental functions, is that they have an essential singularity at ∞. We will soon also give an interpretation of the term *transcendental* as being *non-algebraic*. However, for our purposes, it suffices to think of transcendental entire functions as non-polynomial entire functions. For holomorphic functions with essential singularities, we recall that there is a deep result in single-variable complex analysis: the Great Picard theorem. For transcendental entire functions, we get the following version of the Great Picard theorem:

Theorem 1.1.6. Let f be a transcendental entire function. Then f assumes all complex values infinitely often with the exception of at most one exceptional value.

Example 1.1.7. A good example is the exponential function e^z, which has an essential singularity at ∞. Its exceptional value is 0. For any other complex value A, we have $e^z = A$ for any $z \in S$, where $S := \ln|A| + i\,(\text{Arg}(A) + 2\pi\mathbb{Z})$, and where $\text{Arg}(A)$ denotes the principal argument of A. We notice that S has infinite cardinality, so there are infinitely many z for which $e^z = A$.

1.2 The order of an entire function

Let f be an entire function and let $r > 0$. We will use the following notation:

$$M(f,r) := \sup_{|z|=r} |f(z)| = \max_{|z|=r} |f(z)|. \tag{1.2}$$

The function $M(f,r)$ is called the **maximum modulus** of $f(z)$ for $|z| = r$. We first consider the simple case when f is non-transcendental. That is, when f is a polynomial. We will use the following asymptotic notation:

$$f(z) \sim g(z) \text{ as } z \to z_0 \iff \lim_{z \to z_0} \frac{f(z)}{g(z)} = 1. \tag{1.3}$$

Theorem 1.2.1. Let $P(z) = \sum_{j=0}^{n} b_j z^j$ be a polynomial of degree n. That is, $b_n \neq 0$. Then:

$$|P(z)| \sim |b_n| \cdot |z|^n \text{ as } |z| \to \infty. \tag{1.4}$$

Proof. By the triangle inequality, we have:

$$|P(z)| \leq \sum_{j=0}^{n} |b_j| \cdot |z|^j = |b_n| \cdot |z|^n \left(1 + \frac{|b_{n-1}|}{|b_n| \cdot |z|} + \cdots + \frac{|b_0|}{|b_n| \cdot |z|^n}\right). \tag{1.5}$$

Let $\varepsilon > 0$. Then for $|z|$ sufficiently large, $\sum\limits_{j=0}^{n-1} \frac{|b_j|}{|b_n| \cdot |z|^{n-j}} \leq \varepsilon$. Thus, for $|z|$ sufficiently large, we have:

$$|P(z)| \leq |b_n| \cdot |z|^n (1 + \varepsilon). \tag{1.6}$$

Similarly, using the reverse-triangle inequality, we get:

$$|P(z)| \geq |b_n| \cdot |z|^n \left(1 - \frac{|b_{n-1}|}{|b_n| \cdot |z|} - \cdots - \frac{|b_0|}{|b_n| \cdot |z|^n}\right), \tag{1.7}$$

and for sufficiently large $|z|$, we then have $|P(z)| \geq |b_n| \cdot |z|^n (1 - \varepsilon)$. We conclude that for sufficiently large $|z|$, we have:

$$1 - \varepsilon \leq \frac{|P(z)|}{|b_n| \cdot |z|^n} \leq 1 + \varepsilon. \tag{1.8}$$

Because this is true for any $\varepsilon > 0$, the assertion follows. □

Corollary 1.2.2. Let $P(z) = \sum_{j=0}^{n} b_j z^j$ be a polynomial of degree n. That is, $b_n \neq 0$. Then:

$$M(P,r) \sim |b_n| r^n \text{ as } r \to \infty. \tag{1.9}$$

Proof. Let $|z| = r$ in Theorem 1.2.1. We can replace $|P(z)|$ with $M(P,r)$ in the double inequality (1.8). □

We now discuss the maximum modulus of any transcendental entire function relative that of any polynomial. We will show that the former grows considerably faster than the latter as $r \to \infty$.

We will also use the following asymptotic notation:

$$f(z) \ll g(z) \text{ as } z \to z_0 \iff \lim_{z \to z_0} \frac{f(z)}{g(z)} = 0. \tag{1.10}$$

With this, we have the following result:

Theorem 1.2.3. Let f be a transcendental entire function and let P be any polynomial. Then:

$$M(P, r) \ll M(f, r), \quad \text{as } r \to \infty. \tag{1.11}$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and let $P(z) = \sum_{j=0}^{m} b_j z^j$, with say $b_m \neq 0$. Let $\varepsilon > 0$. Then it follows from Corollary 1.2.2 that for $r > 0$ sufficiently large, we have $M(P, r) \leq |b_m| r^m (1 + \varepsilon)$. On the other hand, we know that $a_n = f^{(n)}(0)/n!$. By Cauchy's integral formula, we have:

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz. \tag{1.12}$$

Hence:

$$|a_n| \leq \frac{M(f, r)}{r^n}. \tag{1.13}$$

Thus:

$$M(f, r) \geq |a_n| r^n. \tag{1.14}$$

Or equivalently:

$$\frac{1}{M(f, r)} \leq \frac{1}{|a_n| r^n}. \tag{1.15}$$

This is true for all n. By assumption, as f is transcendental, there are infinitely many n such that $|a_n| \neq 0$. We choose one such n with $n > m$. Then we get for r sufficiently large:

$$\frac{M(P, r)}{M(f, r)} \leq \frac{|b_m| r^m (1 + \varepsilon)}{|a_n| r^n}. \tag{1.16}$$

Because $n > m$, letting $r \to \infty$, we get the desired result. □

It follows that for any m, no matter how large, $M(f, r) \gg M(z^m, r)$ as $r \to \infty$. That is, the maximum modulus of a transcendental entire function grows much faster than that of any monomial.

A transcendental entire function can also be interpreted as being a non-algebraic entire function (Proposition 1.2.6). The definition of an algebraic function is as follows:

Definition 1.2.4. A function f is said to be **algebraic** in some domain D[1], if there

[1] A **domain** is an open and connected set.

exist polynomials $P_0, \ldots, P_n$ (of arbitrary degrees) with $P_n \not\equiv 0$ and $n \geq 1$, such that:

$$P_0(z) + P_1(z)f(z) + P_2(z)[f(z)]^2 + \cdots + P_n(z)[f(z)]^n = 0 \tag{1.17}$$

for all $z \in D$.

Remark 1.2.5. The condition $n \geq 1$ is important, otherwise all functions are algebraic. Indeed, we could have chosen any f and let $P_0(z) \equiv 0$. Note also that if $f(z) = P(z)$ is a polynomial, then f is algebraic. Indeed, for $n = 1$, we can let $P_0(z) = -f(z)$ and $P_1(z) \equiv 1$.

Proposition 1.2.6. Let f be a transcendental entire function. Then f is non-algebraic.

Proof. Suppose for contradiction that f is algebraic. Then there are polynomials $P_0, \ldots, P_n$ with $P_n \not\equiv 0$ and $n \geq 1$, such that:

$$P_0(z) + P_1(z)f(z) + P_2(z)[f(z)]^2 + \cdots + P_n(z)[f(z)]^n = 0 \tag{1.18}$$

for all $z \in \mathbb{C}$. It follows by Theorem 1.2.3, that $M(f, r) \to \infty$ as $r \to \infty$. Let $z(r)$ be such that $|z| = r$ and $|f(z(r))| = M(f, r)$ with $r > 0$. Then there exists some $r_0 > 0$ such that for all $r \geq r_0$, we have $f(z(r_0)) \neq 0$. We choose $r \geq r_0$, divide (1.18) evaluated at $z = z(r)$ by $[f(z(r))]^n$, and let $r \to \infty$. By Theorem 1.2.3, we get:

$$\lim_{r\to\infty} P_n(z(r)) = 0. \tag{1.19}$$

However, this is impossible by Theorem 1.2.1. This contradiction proves the assertion. ☐

We now discuss the notion of the *order* of an entire function, an important concept in the theory of entire functions. The order of an entire function provides a way of measuring its maximum modulus relative to that of the exponential function.

Definition 1.2.7. Let f be an entire function. Then its **order**, denoted by $\rho(f)$, is said to be given by:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(f, r)}{\log(r)}. \tag{1.20}$$

To motivate this definition, we consider as a standard reference, the transcendental entire function $f(z) = e^z$. It follows that $M(f, r) = e^r$. We would like to compare it to the maximum modulus of the function $g(z) = e^{z^k}$ for some $k > 1$. It is easy to see that $M(g, r) = e^{r^k}$. To compare the growth of the maximum moduli of the two, it is natural to look at:

$$\limsup_{r\to\infty} \frac{M(g, r)}{M(f, r)} = \limsup_{r\to\infty} \frac{e^{r^k}}{e^r}. \tag{1.21}$$

However, this is equal to ∞ and therefore does not provide a very good comparison between the two maximum moduli. We would like to get a result involving k as this is really what sets one of the functions apart from the other. In hope of getting a better comparison, we instead first the take logarithm of each of the maximum moduli. Thus, we consider:

$$\limsup_{r\to\infty} \frac{\log(M(g,r))}{\log(M(f,r))} = \limsup_{r\to\infty} \frac{r^k}{r}. \tag{1.22}$$

This is still ∞. To accommodate for this, we take the logarithm once more before taking the limit superior. Thus, we are finally led to consider:

$$\limsup_{r\to\infty} \frac{\log\log(M(g,r))}{\log\log(M(f,r))} = \limsup_{r\to\infty} \frac{\log(r^k)}{\log(r)}. \tag{1.23}$$

Now, this is equal to k, and we have a better comparison of the growth of the maximum moduli of the two functions.

From our considerations, we also get the following useful interpretation of the order of of an entire function: if $\rho(f) < \infty$, the growth of the maximum modulus of f is similar to that of the function $z \mapsto e^{z^{\rho(f)}}$.

We will use the value of $\rho(\sin(z))$ later in an example dealing with fixed points of a transcendental Hénon map (see Example 2.3.4) and therefore consider the following example here:

Example 1.2.8. Let $f(z) = \sin(z)$. We want to show that $\rho(f) = 1$. Because we have:

$$f(z) = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \tag{1.24}$$

we can intuitively understand this by using the interpretation that $M(f,r)$ grows similar to the maximum modulus of $e^{z^{\rho(f)}}$. It is intuitively clear that the maximum modulus of f grows similar to that of e^{iz}. To do this rigorously, we determine $M(f,r)$. We recall the following Taylor series:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - \cdots \tag{1.25}$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \tag{1.26}$$

The triangle inequality gives, for $|z| = r$:

$$|f(z)| \leq r + \frac{r^3}{3!} + \frac{r^5}{5!} + \frac{r^7}{7!} + \cdots = \sinh(r) = \frac{e^r - e^{-r}}{2}. \tag{1.27}$$

It follows that $M(\sin(z), r) \leq \frac{e^r - e^{-r}}{2}$. However, for $z = -ir$, we have $|z| = r$ and:

$$\sin(ir) = \frac{e^r - e^{-r}}{2i}. \tag{1.28}$$

Therefore:

$$|\sin(ir)| = \frac{e^r - e^{-r}}{2} \leq M(\sin(z), r). \tag{1.29}$$

We conclude that $M(\sin(z), r) = \sinh(r) = \frac{e^r - e^{-r}}{2}$. Thus:

$$\begin{aligned}
\rho(\sin(z)) &= \limsup_{r\to\infty} \frac{\log\log\left(\frac{e^r - e^{-r}}{2}\right)}{\log(r)} = \limsup_{r\to\infty} \frac{\log\log\left(e^r\left(\frac{1-e^{-2r}}{2}\right)\right)}{\log(r)} \\
&= \limsup_{r\to\infty} \frac{\log\left(r + \log\left(\frac{1-e^{-2r}}{2}\right)\right)}{\log(r)} = \limsup_{r\to\infty} \frac{\log\left(r\left(1 + \frac{1}{r}\log\left(\frac{1-e^{-2r}}{2}\right)\right)\right)}{\log(r)} \\
&= \limsup_{r\to\infty} \frac{\log(r) + \log\left(1 + \frac{1}{r}\log\left(\frac{1-e^{-2r}}{2}\right)\right)}{\log(r)} \\
&= 1.
\end{aligned} \tag{1.30}$$

In a similar manner, we can show that $\rho(\cos(z)) = 1$.

1.3 Results on functions of finite order

Our starting point is the following two simple observations regarding the zero set of an entire function.

Proposition 1.3.1. Let f be an entire function. Then f is never-vanishing if and only if there exists some entire function g such that:

$$f(z) = e^{g(z)} \tag{1.31}$$

for all z.

Proof. Assume f is never-vanishing. Because f is never-vanishing, we can define $g(z) = \log(f(z))$ for any analytical branch of the logarithm. Conversely, if we have $f(z) = e^{g(z)}$, then clearly f is never-vanishing. □

We will use the following notation. Let f be any map and let D_f denote its domain of definition. We denote the zero set of f by $Z(f)$. That is:

$$Z(f) = \{z \in D_f : f(z) = 0\}. \tag{1.32}$$

Corollary 1.3.2. Let $f(z)$ be an entire function. Then $Z(f)$ has finite cardinality if and only if there exist some non-zero polynomial P and some entire function g, such that:

$$f(z) = P(z)e^{g(z)} \tag{1.33}$$

for all z.

Proof. Suppose that $Z(f)$ has cardinality N and let $z_1, \ldots, z_n$ be the finitely-many zeros of f with respective multiplicities $m_1, \ldots, m_n$. That is, $\sum_{i=1}^{n} m_i = N$. Then we can write:

$$f(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} h(z), \tag{1.34}$$

where h is a never-vanishing entire function. Hence, by Proposition 1.3.1, there is some entire function g such that $h(z) = e^{g(z)}$. Finally, we note that $\prod_{i=1}^{n}(z - z_i)^{m_i}$ is a polynomial. We may denote it by $P(z)$. It follows that $f(z) = P(z)e^{g(z)}$. Conversely, if $f(z) = P(z)e^{g(z)}$, then $Z(f) = Z(P)$, which has finite cardinality by the fundamental theorem of algebra. □

We note that Proposition 1.3.1 provides the case $|Z(f)| = \infty$.

Corollary 1.3.3. Let f be an entire function. Then $Z(f)$ has infinite cardinality if and only if there is no polynomial P and no entire function g such that we have $f(z) = P(z)e^{g(z)}$ for all z.

We state the following three lemmas without proof. The proofs can be found for instance in [2].

Lemma 1.3.4 ([2], Lemma 2.7.3)**.** Let f be a transcendental entire function and let P and Q be polynomials with $P \not\equiv 0$. Then:

$$\rho(f(z)P(z) + Q(z)) = \rho(f(z)). \tag{1.35}$$

This is intuitively easy to understand. We know from Theorem 1.2.3, that the maximum modulus of f has a growth rate much greater than that of both P and Q. Therefore, in the expression $f(z)P(z) + Q(z)$, it will be f that is crucial, and the growth of the maximum modulus of $f(z)P(z) + Q(z)$ is determined by that of f. Thus, $f(z)P(z) + Q(z)$ and $f(z)$ should have the same order.

Lemma 1.3.5 ([2], Lemma 2.7.4)**.** Let P, g, and Q be polynomials with $P \not\equiv 0$. Then:

$$\rho(P(z)e^{g(z)} + Q(z)) = \deg(g(z)). \tag{1.36}$$

This result is also intuitively easy to understand. We know that we have $\rho(P(z)e^{g(z)} + Q(z)) = \rho(e^{g(z)})$. This is Lemma 1.3.4. Now, because g is a polynomial, it has a leading term, say $b_n z^n$. Then we know from Theorem 1.2.1, that $|g(z)| \sim |b_n| \cdot |z|^n$ as $|z| \to \infty$, so $e^{g(z)}$ can be expected to have a maximum modulus which grows like that of e^{z^n}. The order of the latter is $n = \deg(g)$.

Lemma 1.3.6 ([2], Lemma 2.7.5)**.** Let $f(z) = e^{g(z)}$ where g is an entire function. If the order of f is finite: $\rho(f(z)) < \infty$, then g is a polynomial. Consequently, $\rho(f) \in \mathbb{N}$.

It is insightful to look at what happens when g is not a polynomial. For instance, if $g(z) = e^z$. Then $f(z) = e^{e^z}$ and it is easy to verify that $\rho(f) = \infty$. Indeed, it is easy to see that $M(f, r) = e^{e^r}$. Thus, $\log\log(M(f, r)) = \log(e^r) = r$, and we get:

$$\limsup_{r\to\infty} \frac{\log\log(M(f, r))}{\log(r)} = \limsup_{r\to\infty} \frac{r}{\log(r)} = \infty. \tag{1.37}$$

We now come to the two main results of this subsection which we will use later in chapter 2.

Theorem 1.3.7. Let f be a transcendental entire function and suppose that f has finite non-integer-valued order. Let P be any non-zero polynomial, and let λ be any complex number. Then the equation:

$$f(z) = \lambda P(z), \tag{1.38}$$

admits infinitely many solutions.

Proof. Suppose for contradiction that the assertion is false. Then the equation $f(z) - \lambda P(z) = 0$ has only finitely many solutions. By Corollary 1.3.2, there exist a non-zero polynomial Q and an entire function g, such that

$$f(z) - \lambda P(z) = Q(z)e^{g(z)}. \tag{1.39}$$

That is:

$$f(z) = \lambda P(z) + Q(z)e^{g(z)}. \tag{1.40}$$

But then by Lemma 1.3.4, we have $\rho(f) = \rho(e^{g(z)})$. Because $\rho(f) < \infty$ by assumption, it follows that $\rho(e^g) < \infty$. By Lemma 1.3.6, we must have then $\rho(f) \in \mathbb{N}$. But this contradicts that f has non-integer-valued order. This completes the proof. □

The next result deals with the case that f has finite and *integer*-valued order. The conclusion is the same as that of Theorem 1.3.7 with at most one exceptional value for λ.

Theorem 1.3.8. Let f be a transcendental entire function with finite integer-valued order and let P be any non-zero polynomial. Then, for at most one exceptional value of λ, the equation:

$$f(z) = \lambda P(z), \tag{1.41}$$

admits infinitely many solutions.

Proof. Suppose for contradiction that this is false. Then there are at least two exceptional values $\lambda = a$ and $\lambda = b$ with $a \neq b$ for which the equation given by:

$f(z) - \lambda P(z) = 0$, has only finitely many solutions. Thus there exist non-zero polynomials Q_1, Q_2 and entire functions g_1, g_2 such that:

$$f(z) - aP(z) = Q_1(z)e^{g_1(z)} \tag{1.42}$$

$$f(z) - bP(z) = Q_2(z)e^{g_2(z)}. \tag{1.43}$$

As in the proof of Theorem 1.3.7, we conclude that $\rho(e^{g_i}) \in \mathbb{N}$ for $i = 1, 2$. So g_i is a polynomial of finite degree $\rho(f)$, for $i = 1, 2$. We subtract equation (1.43) from equation (1.42) and get:

$$Q_1(z)e^{g_1(z)} - Q_2(z)e^{g_2(z)} = P(z)(b-a) := R(z), \tag{1.44}$$

where $R(z)$ is non-zero because: P is non-zero and $a \neq b$. We differentiate equation (1.44) and get:

$$(Q_1'(z) + Q_1(z)g_1'(z))e^{g_1(z)} - e^{g_2(z)}(Q_2'(z) + Q_2(z)g_2'(z)) = R'(z). \tag{1.45}$$

The two equations (1.44) and (1.45) can be combined into the single equation:

$$\begin{bmatrix} Q_1(z) & -Q_2(z) \\ (Q_1'(z) + Q_1(z)g_1'(z)) & -(Q_2'(z) + Q_2(z)g_2'(z)) \end{bmatrix} \begin{bmatrix} e^{g_1(z)} \\ e^{g_2(z)} \end{bmatrix} = \begin{bmatrix} R(z) \\ R'(z) \end{bmatrix}. \tag{1.46}$$

Let us define:

$$A = A(z) = \begin{bmatrix} c(z) & d(z) \\ e(z) & f(z) \end{bmatrix} := \begin{bmatrix} Q_1(z) & -Q_2(z) \\ (Q_1'(z) + Q_1(z)g_1'(z)) & -(Q_2'(z) + Q_2(z)g_2'(z)) \end{bmatrix} \tag{1.47}$$

$$\mathbf{x} = x(z) = \begin{bmatrix} x(z) \\ y(z) \end{bmatrix} := \begin{bmatrix} e^{g_1(z)} \\ e^{g_2(z)} \end{bmatrix} \tag{1.48}$$

$$\mathbf{b} = b(z) = \begin{bmatrix} b_1(z) \\ b_2(z) \end{bmatrix} := \begin{bmatrix} R(z) \\ R'(z) \end{bmatrix}. \tag{1.49}$$

Then equation (1.46) becomes:

$$A\mathbf{x} = \mathbf{b}. \tag{1.50}$$

We want to prove that $\det(A) \not\equiv 0$. Then, when $\det(A) \neq 0$, we can solve for $\mathbf{x}$. By continuity, $\det(A) \neq 0$ holds on some open subset. The uniqueness principle then provides entire solutions. When $\det(A) \neq 0$, we get:

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{1.51}$$

We have:

$$\begin{aligned} \det(A) &= c(z)f(z) - d(z)e(z) \\ &= Q_2(z)(Q_1'(z) + Q_1(z)g_1'(z)) - Q_1(z)(Q_2'(z) + Q_2(z)g_2'(z)) \\ &= Q_2(z)Q_1'(z) - Q_1(z)Q_2'(z) \\ &\quad + Q_1(z)Q_2(z)g_1'(z) - Q_1(z)Q_2(z)g_2'(z) \\ &= Q_1(z)Q_2(z)(g_1'(z) - g_2'(z)) + Q_2(z)Q_1'(z) - Q_1(z)Q_2'(z). \end{aligned} \tag{1.52}$$

Suppose for contradiction that det$(A) \equiv 0$. We divide equation (1.52) by $Q_1(z)Q_2(z)$ and then get for z such that $Q_1(z)Q_2(z) \neq 0$:

$$g_1'(z) - g_2'(z) + \frac{Q_1'(z)}{Q_1(z)} - \frac{Q_2'(z)}{Q_2(z)} = 0. \tag{1.53}$$

Integrating equation (1.53), gives:

$$g_1(z) - g_2(z) = \ln \frac{Q_2(z)}{Q_1(z)} + C, \tag{1.54}$$

where C is some constant. Taking the exponential of both sides in equation (1.54), yields:

$$\frac{e^{g_1(z)}}{e^{g_2(z)}} = \frac{Q_2(z)}{Q_1(z)} e^C, \tag{1.55}$$

which can be rearranged to give:

$$\frac{Q_1(z)}{Q_2(z)} e^{g_1(z) - g_2(z)} = e^C. \tag{1.56}$$

However, dividing equation (1.42) by (1.43), we find when $Q_1(z)Q_2(z) \neq 0$:

$$\frac{f(z) - aP(z)}{f(z) - bP(z)} = \frac{Q_1(z)}{Q_2(z)} e^{g_1(z) - g_2(z)}. \tag{1.57}$$

Thus, we have:

$$\frac{f(z) - aP(z)}{f(z) - bP(z)} = e^C. \tag{1.58}$$

That is, after rearranging:

$$f(z)\left(1 - e^C\right) = P(z)\left[a - be^C\right], \tag{1.59}$$

for all z such that $Q_1(z)Q_2(z) \neq 0$. We have $a \neq b$ and therefore by equation (1.58), that $e^C \neq 1$. But then from equation (1.59), we get:

$$f(z) = P(z)\frac{a - be^C}{1 - e^C}. \tag{1.60}$$

This now holds everywhere by the uniqueness principle. But this contradicts that f is transcendental. Hence it follows that det$(A) \not\equiv 0$, and we can solve for $\mathbf{x}$. We get:

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} f(z) & -d(z) \\ -e(z) & c(z) \end{bmatrix}. \tag{1.61}$$

Substituting from equation (1.47) - (1.49), yields:

$$\begin{aligned}\begin{bmatrix} e^{g_1(z)} \\ e^{g_2(z)} \end{bmatrix} &= \frac{1}{\det(A)} \begin{bmatrix} -(Q_2'(z) + Q_2(z)g_2'(z)) & Q_2(z) \\ -(Q_1'(z) + Q_1(z)g_1'(z)) & Q_1(z) \end{bmatrix} \begin{bmatrix} R(z) \\ R'(z) \end{bmatrix} \\ &= \frac{1}{\det(A)} \begin{bmatrix} -R(z)(Q_2'(z) + Q_2(z)g_2'(z)) + R'(z)Q_2(z) \\ -R(z)(Q_1'(z) + Q_1(z)g_1'(z)) + R'(z)Q_1(z) \end{bmatrix}. \end{aligned} \tag{1.62}$$

But equation (1.62) is impossible because the right-hand side is a vector whose components are rational functions, while the left-hand side is a vector whose components are transcendental functions. This proves the assertion. □

Let $h(z) = g(z) = e^z$. Then $\rho(h \circ g) = \infty$. This suggests that in the case $\rho(h \circ g) < \infty$, there are restrictions on what kind of entire functions h and g can be. In fact, we know from Lemma 1.3.6 that if $\rho(e^{g(z)}) < \infty$, then g is a polynomial. Lemma 1.3.6 is a special case of the following more general result:

Theorem 1.3.9 ([2], Theorem 4.14.3)**.** Let h and g be entire functions and suppose that $\rho(h \circ g) < \infty$. Then:

(i) either g is a polynomial and $\rho(h) < \infty$, or

(ii) g is not a polynomial, but $\rho(g) < \infty$, in which case $\rho(h) = 0$.

Example 1.3.10. Let g and h be entire functions and suppose that $\rho(h) > 0$ and that g is transcendental. Then $\rho(h \circ g) = \infty$: the possible cases for when we have $\rho(h \circ g) < \infty$, are given by Theorem 1.3.9. Because g is transcendental, only (ii) can occur, but in this case, $\rho(h) = 0$ which contradicts the assumption that $\rho(h) > 0$. In particular, if $\rho(f) > 0$ where f is transcendental, we have $\rho(f^k) = \infty$ for all $k \geq 2$.

1.4 The Wiman-Valiron method and the theorem of Rosenbloom

The following results from the theory of entire functions, will play a major role in the later chapters. We state them here without proof, but provide references. The following is Rosenbloom's theorem on the fixed points of the composite of two entire functions:

Theorem 1.4.1. Let f and g be entire functions, set $h(z) = f(g(z))$, and suppose that Fix(h) has finite cardinality. That is, suppose that h only has a finite number of fixed points. Then:

(i) either f is a polynomial, or

(ii) g is constant or the identity function $z \mapsto z$.

The proof of Rosenbloom uses techniques from the Nevanlinna theory. The original reference for Theorem 1.4.1 is [11], but the statement can also be found in for instance, the introduction of [12].

We will be interested in transcendental entire functions. An immediate consequence of Theorem 1.4.1 is then the following corollary:

Corollary 1.4.2. Let f and g be two transcendental entire functions, and set $h(z) = f(g(z))$. Then $\mathrm{Fix}(h)$ has infinite cardinality.

Example 1.4.3. Let f be a transcendental entire function. Then the equation

$$z = f(f(z) + z), \tag{1.63}$$

has infinite many solutions. Indeed, let $g(z) = f(z) + z$. Then z is a solution of equation (1.63) if and only if $z \in \mathrm{Fix}(f \circ g)$. Because f is a transcendental entire function, so is g, and the assertion follows from Corollary 1.4.2. As we shall see in chapter 6, this example actually shows that symplectic transcendental Hénon maps $F_1(z, w) = (f(z) - w, z)$ where f is odd, has infinitely many genuine 5-periodic points.

Finally, we consider the Wiman-Valiron theorem or method. Let g be a a transcendental entire function. The Wiman-Valiron theorem or method, lets us approximate g and the derivatives of g, near points where $|g|$ assumes the value of the maximum modulus of g, by a polynomial of generally high degree. Let us write for g:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1.64}$$

Because g is entire, the sequence $|a_n|r^n$ tends to 0 as $n \to \infty$ for all $r > 0$. Denote the maximum term of this sequence by $\mu(g, r)$, and let $N = N(r)$ be the largest index such that $|a_N|r^N = \mu(g, r)$. The function N increases with r, and we call it the **central index for** g. With this terminology, the following version of the Wiman-Valiron method can be found in [5]:

Theorem 1.4.4. Let $g(z)$ be a transcendental entire function and let $r > 0$. Let $N = N(r)$ denote the central index for g and let $M = M(g, r)$ denote the maximum modulus of $g(z)$ for $|z| = r$. Let $\alpha > 1/2$. Let ζ be such that $|\zeta| = r$ and $|g(\zeta)| = M$. Then for all z such that $|z - \zeta| < \frac{r}{N^\alpha}$, the functions $g(z)$ and $g'(z)$ can be written in the forms:

$$g(z) = \left(\frac{z}{\zeta}\right)^N g(\zeta)(1 + \delta_0) \tag{1.65}$$

$$g'(z) = N\frac{z^{N-1}}{\zeta} g(\zeta)(1 + \delta_1), \tag{1.66}$$

where δ_0 and δ_1 tend to 0 uniformly with respect to z as $r \to \infty$ for values of r chosen outside some exceptional set E with finite logarithmic measure:

$$\mathrm{lm}(E) = \int_E \frac{1}{t}\, dt < \infty. \tag{1.67}$$

We will use Theorem 1.4.4 when we prove the existence of infinitely many genuine 4-periodic points for transcendental Hénon maps of the form given by $F(z, w) = (e^{g(z)} + w, z)$ where g is a transcendental entire function, in chapter 5. References for Theorem 1.4.4 are for instance [5] and [6]. The latter: [6], is considerably more detailed.

Chapter 2

Existence of Fixed Points

2.1 Definition of periodic points

Let F be a map. We will use the following notation for the iterates of F:

$$F^1 \equiv F,\ F^2 := F \circ F, \quad \text{and} \quad F^m := F \circ F^{m-1} \text{ for } m \geq 3. \tag{2.1}$$

The map F^m is called the m**-th iterate of** F.

Definition 2.1.1. Let F be a map. A point z_0 is said to be a k**-periodic point for** or **of** F, if $F^k(z_0) = z_0$. A 1-periodic point of F, is called a **fixed point of** F.

We will denote the set of k-periodic points of F by $\text{Fix}(F^k)$. When $k = 1$, we simply write $\text{Fix}(F)$.

Definition 2.1.2. Let F be a map and let $k > 1$. Let $z_0 \in \text{Fix}(F^k)$. Then z_0 is said to be a **genuine** k**-periodic point for** or **of** F, if there are no $m < k$ for which $F^m(z_0) = z_0$. That is, if $z_0 \notin \text{Fix}(F^m)$ for all $m < k$.

We are interested in studying periodic points of transcendental Hénon maps. Hénon maps are holomorphic maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ of the form:

$$F(z, w) = (f(z) - \delta w, z), \tag{2.2}$$

where δ is a non-zero complex number, and f an entire function. Extensive research has been done on polynomial Hénon maps, the case when f is a polynomial. On the other hand, very little is known about transcendental Hénon maps, the case when f is a transcendental entire function. To our understanding, the only reference seems to be [8]. It is not hard to show that a Hénon map $F(z, w) = (f(z) - \delta w, z)$ is an automorphism with $\det(F') = \delta$, where F' denotes the complex Jacobian of F. An explicit expression for the inverse of F is given by:

$$F^{-1}(\zeta, \eta) = \left(\eta, \frac{f(\eta) - \zeta}{\delta}\right), \tag{2.3}$$

as can easily be verified by the reader. We will soon see that fixed points of Hénon maps lie on the diagonal in $\mathbb{C}^2$. We will reserve the symbol Δ to denote the diagonal in $\mathbb{C}^2$. That is:

$$\Delta := \left\{ (z, w) \in \mathbb{C}^2 : z = w \right\}. \tag{2.4}$$

2.2 Elementary observations for fixed points

We give some results on $\text{Fix}(F)$, where F is any Hénon map, not necessarily transcendental. Our first observation for fixed points of F, is that they lie on the diagonal.

Proposition 2.2.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then:

$$\text{Fix}(F) \subseteq \Delta. \tag{2.5}$$

Proof. Let $(z, w) \in \text{Fix}(F)$. Then $F(z, w) = (z, w)$. That is:

$$(f(z) - \delta w, z) = (z, w). \tag{2.6}$$

The second component of equation (2.6) gives $z = w$. Hence $(z, w) \in \Delta$. □

Proposition 2.2.1 gives a necessary condition for fixed points of F. We now give a sufficient and necessary condition. Recall that we denote by $Z(f)$, the **zero set of** f. That is:

$$Z(f) = \{z \in D_f : f(z) = 0\}, \tag{1.32}$$

where D_f denotes the domain of definition of f.

Proposition 2.2.2. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map and define $g_\delta(z) = f(z) - (1 + \delta)z$. Then:

$$\text{Fix}(F) = (Z(g_\delta) \times Z(g_\delta)) \cap \Delta. \tag{2.7}$$

Proof. We have $(z, w) \in \text{Fix}(F)$ if and only if $F(z, w) = (f(z) - \delta w, z) = (z, w)$. Because $\text{Fix}(F) \subseteq \Delta$, we can set $w = z$. Then the equation $F(z, w) = (z, w)$, becomes: $(f(z) - \delta z) = (z, z)$. That is: $f(z) = (1+\delta)z$. That is: $f(z) - (1+\delta)z = 0$. That is: $z \in Z(g_\delta)$. This completes the proof. □

We will feel free to use expressions such as "Hénon maps with $\delta = -1$, or more generally, "Hénon maps with $\delta = \tilde{\delta}_0$", for any fixed $\tilde{\delta}_0 \neq 0$. Then it shall always be understood, unless otherwise specified, that the δ we refer to, is the one in given in equation (2.2). For Hénon maps with $\delta = -1$, $Z(g_\delta)$ immediately reduces to $Z(f)$:

Corollary 2.2.3. Let $F(z, w) = (f(z) + w, z)$ be a Hénon map with $\delta = -1$. Then:

$$\text{Fix}(F) = (Z(f) \times Z(f)) \cap \Delta. \tag{2.8}$$

Proof. Follows from Proposition 2.2.2 and $g_{-1} = f$. □

Corollary 2.2.3 makes it easy to construct examples of Hénon maps with non-empty set of fixed points. In fact, it is not hard to even provide examples where $\text{Fix}(F)$ has infinite cardinality.

Example 2.2.4. Let $F_{-1}(z, w) = (\sin(z) + w, z)$. It follows from Corollary 2.2.3 that:

$$\text{Fix}(F_{-1}) = \left\{(z, z) \in \mathbb{C}^2 : \sin(z) = 0\right\}. \tag{2.9}$$

Because $\sin(z) = 0$ has infinitely many solutions: $z = \pi\mathbb{Z}$, it follows that $\text{Fix}(F)$ has infinite cardinality. We note that F is a transcendental Hénon map, the kind of Hénon map we are interested in studying.

We can also provide examples where $\text{Fix}(F) = \emptyset$.

Example 2.2.5. Let $F(z, w) = (e^{g(z)} + w, z)$ where g is any entire function. Then $Z(e^{g(z)}) = \emptyset$ and thus $\text{Fix}(F) = \emptyset$.

We know by now that when determining whether $(z, z) \in \text{Fix}(F)$, the decisive equation is $f(z) = (1 + \delta)z$. From this, it is easy to see that $\text{Fix}(F) \neq \emptyset$ whenever $f(0) = 0$. Thus, there are infinitely many Hénon maps F for which the set of fixed points of F is non-empty.

Proposition 2.2.6. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then there exists a Hénon map $G(z, w) = (g(z) - \delta w, z)$ such that $g(z)$ and $f(z)$ differ by a constant and $\text{Fix}(G) \neq \emptyset$.

Proof. If $\text{Fix}(F) \neq \emptyset$, we can take $g(z) = f(z)$ and so the constant they differ by is 0. Otherwise, we can take $g(z) = f(z) - f(0)$. Then $g(0) = 0$ and it follows that $(0, 0) \in \text{Fix}(G)$. □

Corollary 2.2.7. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map where f is odd. Then $\text{Fix}(F) \neq \emptyset$.

Proof. Because f is odd, we have by definition that $f(-z) = -f(z)$ for all z and hence $-f(0) = f(0)$. Thus $f(0) = 0$. □

The class of transcendental Hénon maps $F_1(z, w) = (f(z) - w, z)$ where f is odd, will be important when we later investigate the existence of k-periodic points of so-called *symplectic* transcendental Hénon maps (see the third paragraph in the introduction in chapter 5).

2.3 Fixed points when f has finite order

We can use Theorem 1.3.7 and Theorem 1.3.8 to provide interesting results for fixed points of transcendental Hénon maps $F(z,w) = (f(z) - \delta w, z)$ when $\rho(f) < \infty$. We recall that $\rho(f)$ denotes the order of f.

Theorem 2.3.1. Let $F(z,w) = (f(z) - \delta w, z)$ be a transcendental Hénon map, and suppose that $\infty > \rho(f) \notin \mathbb{N}$. That is, suppose that f has finite non-integer-valued order. Then Fix(F) has infinite cardinality.

Proof. We have $(z,z) \in$ Fix(F) if and only if $f(z) - (1+\delta)z = 0$. That is, if and only if $f(z) = (1+\delta)z$. Under the given assumptions, by Theorem 1.3.7, this equation has infinitely many solutions. Indeed, using the notation in Theorem 1.3.7, we can let $P(z) = z$ and $\lambda = \lambda_\delta = 1 + \delta$. □

This deals with the case when $\rho(f)$ is a finite non-integer number. The common transcendental entire functions like e^z, $\sin(z)$, and $\cos(z)$ however, have finite integer-valued order. Naturally, we would like a result for these as well. We can use Theorem 1.3.8 for this:

Theorem 2.3.2. Let $\{F_\delta\}_{\delta\in\mathbb{C}\setminus\{0\}}$ be a family of transcendental Hénon maps where a general member is given by $F_\delta(z,w) = (f(z) - \delta w, z)$, and suppose that $\rho(f) \in \mathbb{N}$. That is, suppose that f has finite and integer-valued order. Then all members of this family, with the exception of at most one, has the property that their set of fixed points has infinite cardinality.

Proof. The proof is completely analogous to the proof of Theorem 2.3.1, except that we use Theorem 1.3.8 instead of Theorem 1.3.7. With the notation used there, we let $\lambda = \lambda_\delta = 1 + \delta$ and $P(z) = z$. □

Example 2.3.3. Let $F_\delta(z,w) = (e^z - \delta w, z)$. Then we know that Fix$(F_{-1}) = \emptyset$. Because $\rho(e^z) = 1$, it follows then from Theorem 2.3.2, that $\delta = -1$ is the only exceptional value of δ for which Fix(F_δ) has finite cardinality. Thus we conclude that for all $\delta \neq -1$, we have that Fix(F_δ) has infinite cardinality.

The next example shows how Theorem 2.3.1 may be useful also in cases where f has finite and integer-valued order.

Example 2.3.4. Let $F_\delta(z,w) = (\sin(z) - \delta w, z)$. Because $\rho(\sin(z)) = 1$, Theorem 2.3.2 tells us that there is at most one exceptional value of δ for which Fix(F_δ) is finite. We show how Theorem 2.3.1 can be used here to show that in fact there are no exceptional values of δ. That is, Fix(F_δ) has infinite cardinality for all possible values of δ. We want to consider the cardinality of the solution set of the equation:

$$\sin(z) = (1+\delta)z. \tag{2.10}$$

Using the Taylor series for $\sin(z)$, this is the equation:

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = (1+\delta)z. \tag{2.11}$$

If we ignore the trivial solution $z = 0$, we can divide by $z \neq 0$ and get:

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots = (1+\delta). \tag{2.12}$$

This is then the equation $\frac{\sin(z)}{z} = (1+\delta)$, and we notice that $\frac{\sin(z)}{z}$ is an entire function. We now introduce the variable ζ by $z = \sqrt{\zeta}$. This gives the equation $\frac{\sin(\sqrt{\zeta})}{\sqrt{\zeta}} = (1+\delta)$, and equation (2.11) becomes:

$$1 - \frac{\zeta}{3!} + \frac{\zeta^2}{5!} - \frac{\zeta^3}{7!} + \cdots = (1+\delta). \tag{2.13}$$

The left-hand side of equation (2.13) shows that $\frac{\sin(\sqrt{\zeta})}{\sqrt{\zeta}}$ is an entire function in the variable ζ. Now, $\rho\left(\zeta \mapsto \frac{\sin(\sqrt{\zeta})}{\sqrt{\zeta}}\right) = \frac{1}{2}$ and so Theorem 2.3.1 applies to show that there are infinitely many solutions of ζ to the equation $\frac{\sin(\sqrt{\zeta})}{\sqrt{\zeta}} = (1+\delta)$ with no exceptional value of δ. Because $z = \sqrt{\zeta}$, this provides infinitely many solutions of $\frac{\sin(z)}{z} = (1+\delta)$ with no exceptional value for δ. Hence it follows that $\mathrm{Fix}(F_\delta)$ has infinite cardinality with no exceptional value for δ.

2.4 A complete characterization

We now give a complete characterization for the existence of fixed points of arbitrary Hénon maps. The results we give, provide a way of constructing a Hénon map whose fixed points are exactly any predetermined finite set of points on the diagonal Δ in $\mathbb{C}^2$.

We first recall that for any entire function f, $Z(f) = \emptyset$ if and only if there exists some entire function $g(z)$ such that $f(z) = e^{g(z)}$. This gives:

Theorem 2.4.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then $\mathrm{Fix}(F) = \emptyset$ if and only if there exists some entire function $g(z)$ such that:

$$f(z) = (1+\delta)z + e^{g(z)}. \tag{2.14}$$

Proof. The decisive equation for determining whether $(z, w) \in \mathrm{Fix}(F)$, is given by: $f(z) - (1+\delta)z = 0$. Suppose first that $\mathrm{Fix}(F) = \emptyset$. Then $f(z) - (1+\delta)z$ is never-vanishing. Thus there exists some entire function $g(z)$ such that we have $f(z) - (1+\delta) = e^{g(z)}$. That is: $f(z) = (1+\delta)z + e^{g(z)}$.

Conversely, if we have that $f(z) = (1+\delta)z + e^{g(z)}$ for some entire function $g(z)$, then $f(z) - (1+\delta)z = e^{g(z)}$ whose zero set is empty. This proves the assertion. ☐

More generally, we have that $Z(f)$ is finite if and only if $f(z) = P(z)e^{g(z)}$ for some non-zero polynomial $P(z)$ and entire function $g(z)$.

Theorem 2.4.2. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then $\text{Fix}(F)$ has finite cardinality if and only if there exist some non-zero polynomial $P(z)$ and some entire function $g(z)$ such that $f(z) = (1 + \delta)z + P(z)e^{g(z)}$.

Proof. Suppose first that $f(z) = (1 + \delta)z + P(z)e^{g(z)}$. Then:

$$Z(f(z) - (1 + \delta)z) = Z(P(z)e^{g(z)}) = Z(P(z)). \tag{2.15}$$

By the fundamental theorem of algebra $|Z(P(z))| \leq \deg(P(z))$ which is finite. Because $\text{Fix}(F) = \{(z, z) \in \mathbb{C}^2 : z \in Z(f(z) - (1 + \delta)z)\}$, it follows that $\text{Fix}(F)$ has finite cardinality.

Conversely, suppose that $\text{Fix}(F)$ has finite cardinality. Then $Z(f(z)-(1+\delta)z)$ has finite cardinality. Thus, there exist a polynomial $P(z)$ and an entire function $g(z)$ such that $f(z) - (1 + \delta)z = P(z)e^{g(z)}$. That is, after rearranging, such that $f(z) = (1 + \delta)z + P(z)e^{g(z)}$. This completes the proof. □

The proof of Theorem 2.4.2 makes it very easy to construct Hénon maps whose set of fixed points is exactly any predetermined finite set of points on Δ. Indeed, we notice simply that:

$$\text{Fix}(F) = (Z(P(z)) \times Z(P(z))) \cap \Delta. \tag{2.16}$$

Theorem 2.4.3. Let $S = \{(z_1, z_1), (z_2, z_2), \ldots, (z_m, z_m)\} \subseteq \mathbb{C}^2$ be any arbitrary finite set of points on Δ. Then there exist infinitely many Hénon maps F such that $\text{Fix}(F)$ is precisely S. In fact, one such is given by $F(z, w) = (f(z) - \delta w, z)$, where:

$$f(z) = (1 + \delta)z + e^{g(z)} \prod_{j=1}^{m} \lambda_j (z - z_j)^{m_j}, \tag{2.17}$$

where $\lambda_j \in \mathbb{C}\backslash\{0\}, m_j \in \mathbb{N}$, and where $g(z)$ is any arbitrary entire function.

Proof. Let $P_m(z) = \prod_{j=1}^{m} \lambda_j (z-z_j)^{m_j}$. Then $Z(P_m) = S$ and the assertion follows immediately from equation (2.16). □

Notice that the given results provide a complete description of the existence of infinitely many fixed points for Hénon maps. Indeed, let $F(z, w) = (f(z)-\delta w, z)$ be a Hénon map. Then it follows from what we have shown, that $\text{Fix}(F)$ has infinite cardinality if and only if there exist no non-zero polynomial $P(z)$ and no entire function $g(z)$ such that $f(z) = (1 + \delta)z + P(z)e^{g(z)}$. In particular, F must be a transcendental Hénon map or a linear monomial, in which case its set of fixed points is all of $\mathbb{C}$:

Proposition 2.4.4. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map and suppose that Fix(F) has infinite cardinality. Then F is a transcendental Hénon map or f is a linear polynomial in which case Fix$(F) = \mathbb{C}$.

Proof. Fix(F) has infinite cardinality if and only if there are no non-zero polynomial P and entire function g such that $f(z) = (1 + \delta)z + P(z)e^{g(z)}$. If F is not transcendental, f is a polynomial. Say, Q. Then f is of this form with $g(z) \equiv 0$ and $P(z) = Q(z) - (1 + \delta)z$ in the case $P(z) \not\equiv 0$. Hence, in the case $P(z) \not\equiv 0$, Fix(F) cannot have infinite cardinality. It remains the case that $P(z) \equiv 0$ in which case $Q(z) \equiv (1 + \delta)z$. Then $f(z) = Q(z) = (1 + \delta)z$ is a linear polynomial and Fix$(F) = Z(P(z)) = Z(0) = \mathbb{C}$ as required. □

The non-existence of P and g however, is difficult to use in practice. Therefore we now provide, using the theorem of Rosenbloom (Theorem 1.4.1), a class of transcendental Hénon maps whose set of fixed points is infinite.

Theorem 2.4.5. There exist infinitely many transcendental Hénon maps which admit infinitely many fixed points. In fact, let $g(z)$ and $h(z)$ be any two transcendental entire functions. Then, if we let:

$$F_\delta(z, w) = (f(z) - \delta w, z), \tag{2.18}$$

where $f(z) = h(g(z))$ and $\delta \neq -1$, we have that Fix(F_δ) has infinite cardinality.

Proof. The decisive equation for determining whether $(z, z) \in$ Fix(F), is given by: $f(z) = (1 + \delta)z$. Because $\delta \neq -1$, we can divide by $(1 + \delta)$ and get: $\frac{f(z)}{1+\delta} = z$. As $f(z) = h(g(z))$, this gives:

$$\frac{h(g(z))}{1 + \delta} = z. \tag{2.19}$$

Let $k(z) = \frac{h(z)}{1+\delta}$. Then k is a transcendental entire function and $(z, z) \in$ Fix(F) if and only if:

$$z \in \text{Fix}(k(g(z))). \tag{2.20}$$

By Rosenbloom's theorem 1.4.1, Fix$(k(g(z)))$ has infinite cardinality. This completes the proof. □

Finally, we have the following result which says that Fix(F) cannot be bounded for most Hénon maps:

Proposition 2.4.6. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map where $f(z)$ is not equal to the monomial $(1 + \delta)z$, and suppose that Fix(F) has infinite cardinality. Then there must exist a sequence $\{(z_n, z_n)\}_n \subseteq$ Fix(F) such that $|z_n| \to \infty$ as $n \to \infty$.

Proof. Suppose not. Then every sequence of fixed points $\{(z_n, z_n)\}_n \subseteq \text{Fix}(F)$ is such that $|z_n|$ are bounded for all n. Hence there exists some radius R such that all z_n are contained in the open disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$. Now, let us define $g(z) = f(z) - (1 + \delta)z$. Then g is not identically equal to 0 by the assumption on the form of f. Now, each z_n satisfies $g(z_n) = 0$. Hence, it follows that g has infinitely many zeros inside a finite disc. This however, is impossible for then g would have a zero which is a point of accumulation, contradicting that g has isolated zeros. This contradiction proves the assertion. □

Chapter 3

2-Periodic Points

3.1 Genuine 2-periodic points

Let F be a map and let $(z, w) \in \mathrm{Fix}(F^k)$. The first value of k for which it becomes interesting to ask whether (z, w) is a genuine k-periodic point, is $k = 2$. It is also one of the simplest case. Indeed, we note that if $(z, w) \in \mathrm{Fix}(F^2)$ fails to be a genuine, then the only possibility is that $(z, w) \in \mathrm{Fix}(F)$. This simple observation combined with our study of fixed points in the previous chapter, provides us with the following result for Hénon maps:

Proposition 3.1.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then any 2-periodic point of F off the diagonal Δ, is genuine.

Proof. This follows because all fixed points of F lie on the diagonal Δ by Proposition 2.2.1. □

Actually more is true. A priori it is possible that there exist genuine 2-periodic points which do lie on Δ. We show now that this is in fact impossible. All genuine 2-periodic points of Hénon maps must lie off the diagonal.

Proposition 3.1.2. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map and suppose that $(z, w) \in \mathrm{Fix}(F^2)$. Then if $(z, w) \in \Delta$ as well, we have that $(z, w) \in \mathrm{Fix}(F)$. That is:

$$\mathrm{Fix}(F^2) \cap \Delta = \mathrm{Fix}(F). \tag{3.1}$$

Proof. Because $(z, w) \in \mathrm{Fix}(F^2)$, we have that $F^2(z, w) = (z, w)$. Let us define $(z_k, w_k) = F^k(z, w)$. Thus, the condition that $(z, w) \in \mathrm{Fix}(F^2)$ is given by $(z_2, w_2) = (z, w)$. We find:

$$(z_1, w_1) = (f(z) - \delta w, z) \tag{3.2}$$

$$\begin{aligned}(z_2, w_2) &= (f(z_1) - \delta w_1, z_1) \\ &= (f[f(z) - \delta w] - \delta z, f(z) - \delta w).\end{aligned} \tag{3.3}$$

For $(z, w) \in \Delta$, we have $z = w$ and equation (3.3) together with the condition $(z_2, w_2) = (z, w)$, becomes:

$$(z, z) = (f[f(z) - \delta z] - \delta z, f(z) - \delta z). \tag{3.4}$$

The second component of equation (3.4) gives:

$$z = f(z) - \delta z. \tag{3.5}$$

That is:

$$f(z) = (1+\delta)z. \tag{3.6}$$

Thus by Proposition 2.2.2, $(z, z) \in \text{Fix}(F)$. Notice that the first component of equation (3.4) follows from $f(z) = (1+\delta)z$. This proves the assertion. □

Up till now, we have assumed from the outset that $(z, w) \in \text{Fix}(F^2)$. However, it is easy to give a necessary and sufficient condition for when any $(z, w) \in \mathbb{C}^2$ actually is a 2-periodic point for F:

Theorem 3.1.3. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then we have that $(z, w) \in \text{Fix}(F^2)$ if and only if:

$$f(z) = (1+\delta)w \tag{3.7}$$
$$f(w) = (1+\delta)z. \tag{3.8}$$

Proof. We look at the proof of Proposition 3.1.2. Then equation (3.3) and the fact that $(z, w) \in \text{Fix}(F^2)$ if and only if $(z, w) = (z_2, w_2)$, give:

$$(z, w) \in \text{Fix}(F^2) \iff (z, w) = (f[f(z) - \delta w] - \delta z, f(z) - \delta w). \tag{3.9}$$

The first and second component of equation (3.9) gives respectively the equations (3.10) and (3.11):

$$f[f(z) - \delta w] - \delta z = z \tag{3.10}$$
$$f(z) - \delta w = w, \tag{3.11}$$

which can be rewritten as the desired set of equations, noticing that $f(z) - \delta w$ in equation (3.10), can be replaced by w due to equation (3.11). This completes the proof. □

In the case $\delta = -1$, the equations (3.7) and (3.8) become particularly simple.

Corollary 3.1.4. Let $F(z, w) = (f(z) + w, z)$ be a Hénon map with $\delta = -1$. Then:

$$\text{Fix}(F^2) = Z(f) \times Z(f). \tag{3.12}$$

In particular, if $|Z(f)| > 1$, then F admits genuine 2-periodic points.

Proof. The equations (3.7) and (3.8) reduce to:

$$f(z) = f(w) = 0, \tag{3.13}$$

from which the assertion follows. □

We can also use Theorem 2.4.2 to relate the cardinality of Fix(F^2) to the form of f in the case that $\delta = -1$:

Corollary 3.1.5. Let $F(z,w) = (f(z)+w,z)$ be a Hénon map with $\delta = -1$. Then Fix(F^2) has finite cardinality if and only if there exist some non-zero polynomial $P(z)$ and some entire function $g(z)$ such that:

$$f(z) = P(z)e^{g(z)} \tag{3.14}$$

for all z. Furthermore:

(i) Fix$(F^2) = \emptyset$ if and only if $P(z)$ is a constant.

(ii) Fix(F^2) contains genuine 2-periodic points if and only if $|Z(P)| > 1$.

Proof. Follows immediately from Theorem 2.4.2 and Corollary 3.1.4. □

Example 3.1.6. Let $F_{-1}(z,w) = (\sin(z) + w, z)$. Because $|Z(\sin(z))| = \infty$, it follows from Corollary 3.1.4 that Fix(F^2) contains infinitely many genuine 2-periodic points. In fact, combining Corollary 3.1.4 and Proposition 3.1.2, we see that for all Hénon maps $H(z,w) = (h(z) + w, z)$ with $\delta = -1$, we have that the set of genuine 2-periodic points for H is precisely the set $(Z(h) \times Z(h))\backslash\Delta$, and thus all the genuine 2-periodic points of F_{-1} are given by $\pi(n,m)$, where $(n,m) \in \mathbb{Z} \times \mathbb{Z}$ and $n \neq m$.

3.2 Characterizing dynamics of 2-periodic points

We start with a simple example. We have previously seen that for $\delta = -1$, the equations (3.7) and (3.8) become particularly simple.

Example 3.2.1. Consider any Hénon map $F(z,w) = (f(z) + w, z)$ with $\delta = -1$. Let $(z,w) \in$ Fix(F^2). Then we know that $(z,w) \in Z(f) \times Z(f)$. We can consider iterates of (z,w) under F. We set $(z_k, w_k) = F^k(z,w)$. As $f(z) = f(w) = 0$, we get:

$$(z_1, w_1) = F(z,w) = (f(z) + w, z) = (w, z) \tag{3.15}$$

$$(z_2, w_2) = F(w,z) = (f(w) + z, w) = (z, w), \tag{3.16}$$

and because (z,w) is 2-periodic, this pattern repeats. Hence, the dynamics of any 2-periodic point of F is given simply by swapping the first and second component of the point per iteration, and the dynamics takes the form:

$$(z,w) \mapsto (w,z) \mapsto (z,w) \mapsto (w,z) \mapsto (z,w) \mapsto \cdots \tag{3.17}$$

This simple behaviour for the dynamics of 2-periodic points for F, is easy to imagine when $\delta = -1$ because all terms $f(z), f(w)$ vanish. Motivated by this seemingly simple behaviour in the case $\delta = -1$, it is natural to ask when such behaviour may occur for general values of δ. As it turns out, the behaviour is completely independent of values of δ and only depends on whether we consider 2-periodic points. In fact, as we will show, this behaviour completely characterizes any 2-periodic point for F:

Theorem 3.2.2. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map. Then we have that $(z, w) \in \text{Fix}(F^2)$ if and only if the dynamics of (z, w) under F is given by:

$$(z, w) \mapsto (w, z) \mapsto (z, w) \mapsto (z, w) \mapsto \cdots \tag{3.18}$$

That is, if and only if the dynamics of (z, w) under iterates of F, is given by swapping the first and second components of (z, w) per iteration.

Proof. Suppose first that we have $(z, w) \in \text{Fix}(F^2)$. It then suffices to show that $F(z, w) = (w, z)$. That is, that the first iterate swaps the components. The rest of the required behaviour of the dynamics follows then from the periodicity of (z, w). It follows from Theorem 3.1.3 that $f(z) = (1 + \delta)w$ and $f(w) = (1 + \delta)z$. That is, it follows that $f(z) - \delta w = w$ and $f(w) - \delta z = z$. Thus:

$$F(z, w) = (f(z) - \delta w, z) = (w, z), \tag{3.19}$$

as required.

Conversely, suppose that $F(z, w) = (w, z)$ and $F(w, z) = (z, w)$. We show that then $(z, w) \in \text{Fix}(F^2)$. By assumption:

$$(w, z) = (f(z) - \delta w, z). \tag{3.20}$$

Only the first component of equation (3.20) matters. This gives: $w = f(z) - \delta w$, or equivalently: $f(z) = (1 + \delta)w$. This is equation (3.7). Similarly, from the assumption that $F(w, z) = (z, w)$, we get: $f(w) = (1 + \delta)z$, which is equation (3.8). Hence $(z, w) \in \text{Fix}(F^2)$ by Theorem 3.1.2. Alternatively, we may simply note that $F(z, w) = (w, z)$ and $F(w, z) = (z, w)$, imply $F^2(z, w) = (z, w)$ from which it immediately follows that $(z, w) \in \text{Fix}(F^2)$. □

3.3 Existence results

We now turn to the question of existence of 2-periodic points. Because any fixed point is trivially a 2-periodic point, and because we already have dealt with the existence question in great detail for fixed points, we better restrict to genuine 2-periodic points. Combining Proposition 3.1.2 and Theorem 3.1.3, we see that the existence of genuine 2-periodic points reduces to solutions of equation (3.7) and (3.8) with $(z, w) \notin \Delta$. In order to simplify the equations, we try and look for solutions where w is some function of z. A simple example would be $w = z$, but then $(z, w) \in \Delta$ which we wanted to avoid. Not very far from this approach, is to look for solutions of the form $w = -z$. Then (3.7) and (3.8) become:

$$f(z) = -(1 + \delta)z \tag{3.21}$$
$$f(-z) = (1 + \delta)z. \tag{3.22}$$

If we add the two equations (3.21) and (3.22), we get the necessary condition:

$$f(z) + f(-z) = 0, \tag{3.23}$$

or: $f(z) = -f(-z)$. A certain class of functions for which this is true for all z comes to mind. Because we want genuine 2-periodic points, we need only make sure that not all solutions of the equations (3.21) and (3.22), satisfy $z = -z$. That is: $z = 0$.

Proposition 3.3.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map, and suppose that f is odd. Let $g(z) = f(z) + (1 + \delta)z$, and suppose that $|Z(g)| > 1$. Then F admits genuine 2-periodic points. In fact:

$$Z(g) \times (-Z(g)) \subseteq \text{Fix}(F^2). \tag{3.24}$$

Proof. From the discussion above, we have that $(z, -z) \in \text{Fix}(F^2)$ if and only if z is a solution to both the equations (3.21) and (3.22). Because f is odd, the two equations are the same and we really only have one equation:

$$f(z) = -(1 + \delta)z, \tag{3.25}$$

or equivalently: $f(z) + (1 + \delta)z = g(z) = 0$. This proves (3.24). Now, because $|Z(g)| > 1$ by assumption, it follows that there exists some $z \neq 0$ such that $g(z) = 0$. Then $z \neq -z$ and therefore $(z, -z) \notin \Delta$. Thus, $(z, -z)$ is a genuine 2-periodic point for F. This completes the proof.

□

Example 3.3.2. Let $P(z)$ be an odd polynomial and let $g(z)$ be a non-constant even entire function. Fix $\delta \neq 0$ and set $f(z) = P(z)e^{g(z)} - (1 + \delta)z$. Then f is an odd transcendental entire function. Because P is odd, $P(0) = 0$. From Proposition 3.3.1, it follows that the transcendental Hénon map $F(z, w) = (f(z) - \delta w, z)$, admits at least $|Z(P)| - 1$ genuine 2-periodic points.

We can do even better. Because of the simple form of equations (3.7) and (3.8), we can, in the case $\delta \neq -1$, solve for w in equation (3.7), and substitute in equation (3.8) to get an equation for z. We can then use the theorem of Rosenbloom (Theorem 1.4.1) and Theorem 2.4.2, to construct examples of transcendental Hénon maps with infinitely many genuine 2-periodic points. This approach frees us from the condition that f be odd. If we allow for f to be odd, we can apply the theorem of Rosenbloom (Theorem 1.4.1) directly combined with Proposition 3.3.1, to immediately provide a class of transcendental Hénon maps with infinitely many genuine 2-periodic points:

Proposition 3.3.3. Let g and h be any two odd transcendental entire functions and let $0 \neq \delta \neq -1$. Let $f(z) = h(g(z))$ and let $F(z, w) = (f(z) - \delta w, z)$. Then F is a transcendental Hénon map with infinitely many genuine 2-periodic points.

Proof. Because g and h are transcendental entire functions, so is $f = h \circ g$. Therefore F is a transcendental Hénon map. Now, as g and h are odd, we find that:

$$f(-z) = h(g(-z)) = h(-g(z)) = -h(g(z)). \tag{3.26}$$

So f is an odd transcendental entire function, and Proposition 3.3.1 applies. That is, if z is any solution to:

$$f(z) = -(1+\delta)z, \tag{3.25}$$

then $(z, -z) \in \text{Fix}(F^2)$. Now, let $\tilde{f}(z) = -\frac{h(z)}{1+\delta}$. Then $\tilde{f}$ is a transcendental entire function. Equation (3.25) becomes:

$$\tilde{f}(g(z)) = z. \tag{3.27}$$

Then z is a solution to equation (3.27) if and only if $z \in \text{Fix}(\tilde{f}(g(z))$, which has infinite cardinality by the theorem of Rosenbloom (Theorem 1.4.1). In particular, equation (3.25) has infinitely many solutions $z \neq 0$. Then all of these are genuine 2-periodic points for F. This completes the proof. □

We now proceed to solve for w in equation (3.7) and substitute this in (3.8). Using the theorem of Rosenbloom (Theorem 1.4.1), we can then easily show that $\text{Fix}(F^2)$ has infinite cardinality for any transcendental Hénon map $F(z,w) = (f(z) - \delta w, z)$ with $\delta \neq -1$. The following result is given in [8]:

Theorem 3.3.4 (Proposition 3.3, [8])**.** Let $F(z,w) = (f(z) - \delta w, z)$ be a transcendental Hénon map where $\delta \neq -1$. Then $\text{Fix}(F^2)$ has infinite cardinality.

Proof. As $\delta \neq -1$, from equation (3.7), we get:

$$w = \frac{f(z)}{1+\delta}. \tag{3.28}$$

Substituted into equation (3.8), yields:

$$\frac{f\left(\frac{f(z)}{1+\delta}\right)}{1+\delta} = z. \tag{3.29}$$

Let us define $g(z) = \frac{f(z)}{1+\delta}$. Then equation (3.29) becomes:

$$g^2(z) = z. \tag{3.30}$$

Thus $z \in \text{Fix}(g^2(z))$. Because g is a transcendental entire function, it follows from the theorem of Rosenbloom (Theorem 1.4.1), that there are infinitely many solutions for z of equation (3.29). Then equation (3.28) provides w given z. It follows from Theorem 3.1.3 that $\text{Fix}(F^2)$ has infinite cardinality. □

Although Theorem 3.3.4 gives that $\text{Fix}(F^2)$ has infinite cardinality for F a transcendental Hénon map with $\delta = -1$, it does not answer the interesting question whether infinitely many of these are genuine 2-periodic points. This subject is not directly treated in [8] either. Below, we provide a result. A simple, but efficient way of dealing with the problem, is to make sure that $\text{Fix}(F)$ only has finite cardinality. Then it follows from Theorem 3.3.4 that F has infinitely many genuine 2-periodic points. Theorem 2.4.2 is applicable.

Corollary 3.3.5. Let $F(z, w) = (f(z) - \delta w, z)$ be a transcendental Hénon map with $\delta \neq -1$. Let $P(z)$ be any non-zero polynomial and let $g(z)$ be any non-constant entire function. Then, if $f(z) = (1 + \delta)z + P(z)e^{g(z)}$, we have that $\text{Fix}(F^2)$ consists of infinitely many genuine 2-periodic points.

Proof. Under the assumption on f, by Theorem 2.4.2, we have $|\text{Fix}(F)| < \infty$. Thus there are at most finitely many non-genuine 2-periodic points. But by Theorem 3.3.4, it follows that F admits infinitely many 2-periodic points. It follows that infinitely many of these must be genuine. This completes the proof. □

Corollary 3.3.5 provides us with a class of infinitely many transcendental Hénon maps with infinitely many genuine 2-periodic points.

Chapter 4

Some Maps with Discrete set of k-Periodic Points, $k \leq 4$

We investigate the discreteness of the set of k-periodic points for $k = 1, 2, 3, 4$, of transcendental Hénon maps. The cases $k = 1$ and $k = 2$ for general transcendental Hénon maps are both immediate consequences of the discreteness of the zero set $Z(g)$ for non-constant holomorphic functions g of a single variable, and already given in [8]. In [8], the authors also show that for transcendental Hénon maps $F(z, w) = (f(z) - \delta w, z)$ where $\rho(f) < 1/2$, the set Fix(F^k) is discrete for all $k \geq 1$. This raises the question about the discreteness of the set of k-periodic points for more general transcendental Hénon maps $H(z, w) = (h(z) - \delta w, z)$ with $\rho(h) \geq 1/2$ and $k \geq 3$. In particular, it is an open question whether the set of k-periodic points of the transcendental Hénon map $H(z, w) = (e^z - \delta w, z)$, is discrete for any $k \geq 3$. We note that $\rho(e^z) = 1 > 1/2$, so the result in [8] does not apply to this map. Below, we use the implicit mapping theorem to show the following result: for any $\delta \in \mathbb{C}\backslash\{0\}$, there exists a transcendental Hénon map $H_\delta(z, w) = (h_\delta(z) - \delta w, z)$ with the property that Fix(H_δ^3) is discrete and $\rho(h_\delta) = 1$. We also use elementary properties of analytic sets in $\mathbb{C}^2$, to show that Fix(G^4) is discrete where $G(z, w) = (e^z - \delta w, z)$ and $\delta^2 = 1$.

4.1 The cases $k = 1$ and $k = 2$

For completeness, we first show the discreteness of the set of fixed points and the set of 2-periodic points for transcendental Hénon maps. Both of the results we give below, Theorem 4.1.1 and Theorem 4.1.3, origin from [8].

Theorem 4.1.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a transcendental Hénon map. Then Fix(F) is discrete.

Proof. We know that $(z, w) \in$ Fix(F) if and only if $z = w$ and:

$$f(z) = (1 + \delta)z. \tag{4.1}$$

Thus it suffices to show that $Z(z \mapsto f(z) - (1 + \delta)z)$ is discrete. But this follows because f is transcendental. □

Remark 4.1.2. Note that the proof of Theorem 4.1.1, shows that more general Hénon maps have discrete set of fixed points. Indeed, the given proof works for all Hénon maps $F(z,w) = (f(z) - \delta w, z)$ where f is *not* equal to the monomial $z \mapsto (1+\delta)z$.

Analogously, we can show that Fix(F^2) is discrete:

Theorem 4.1.3. Let $F(z,w) = (f(z) - \delta w, z)$ be a transcendental Hénon map. Then Fix(F^2) is discrete.

Proof. We know that $(z,w) \in$ Fix(F^2) if and only if:

$$f(z) = (1+\delta)w \tag{3.7}$$
$$f(w) = (1+\delta)z. \tag{3.8}$$

Suppose first that $\delta \neq -1$. Then, we can solve for w in equation (3.7). We get: $w = \frac{f(z)}{1+\delta}$. Substituted into equation (3.8), gives then:

$$f\left(\frac{f(z)}{1+\delta}\right) - (1+\delta)z = 0. \tag{4.2}$$

Let $g(z) = f\left(\frac{f(z)}{1+\delta}\right) - (1+\delta)z$. Then z solves equation (4.2) if and only if $z \in Z(g)$. Because f is transcendental, $Z(g)$ is discrete. Thus, there can only be discrete z. But then from $w = \frac{f(z)}{1+\delta}$, there can only be discrete w as well. We conclude that Fix(F^2) is discrete in the case $\delta \neq -1$. In the case $\delta = -1$, we have that $(z,w) \in$ Fix(F^2) if and only if $(z,w) \in Z(f) \times Z(f)$, and the discreteness follows immediately. This completes the proof. □

Remark 4.1.4. As with the set of fixed points, the discreteness in Theorem 4.1.3 is true also for more general Hénon maps. Indeed, the given proof works for all Hénon maps $F(z,w) = (f(z) - \delta w, z)$ for which $f\left(\frac{f(z)}{1+\delta}\right) - (1+\delta)z \not\equiv 0$ when $\delta \neq -1$, and for all $f \not\equiv 0$ in the case $\delta = -1$.

4.2 Hénon maps $(f(z) - \delta w, z)$ with $\rho(f) < 1/2$

The following discreteness result is given in [8]:

Theorem 4.2.1 ([8], Proposition 3.2)**.** Let $F(z,w) = (f(z) - \delta w, z)$ be a Hénon map with $\rho(f) < 1/2$. Then for all $k \geq 1$, Fix(F^k) is discrete.

Before stating and proving Theorem 4.2.1, the authors of [8], comment: "Without making further assumptions it is not clear to the authors that Fix(F^k) is discrete when $k \geq 3$. However, we can show discreteness when we assume that the function f has small order of growth." This comment became the underlying reason for the investigation of the case $k = 3$.

We will use the following preliminary result due to Wiman for the proof of Theorem 4.2.1. We follow the same argument given in [8]. Given an entire function f and $r > 0$, we denote by $m(f, r)$, the **minimum modulus of** f for $|z| = r$, given by:

$$m(f, r) = \inf_{|z|=r} |f(z)|. \tag{4.3}$$

Theorem 4.2.2 ([2], Theorem 7.9.1)**.** Suppose f is an entire function with order $\rho(f) < 1/2$. Then there exists a sequence of values of r tending to infinity through which $m(r) \to \infty$.

For the proof, see for instance [2], page 134. Before we prove Theorem 4.2.1, it is convenient to understand the general structure of the set of equations determining whether a point (z, w) is an element of the set Fix(F^k) for general values of k. We define the following iterative points: $(z_k, w_k) := F^k(z, w)$. For instance, for $k = 1$, we get:

$$(z_1, w_1) = (f(z) - \delta w, z) = (z, w), \tag{4.4}$$

and for $k = 2$, we get:

$$\begin{align} (z_1, w_1) &= (f(z) - \delta w, z) \tag{4.5} \\ (z_2, w_2) &= (f(z_1) - \delta w_1, z_1) = (z, w). \tag{4.6} \end{align}$$

For general k, we get:

$$\begin{align} (z_1, w_1) &= (f(z) - \delta w, z) \tag{4.7} \\ (z_2, w_2) &= (f(z_1) - \delta w_1, z_1) \tag{4.8} \\ (z_3, w_3) &= (f(z_2) - \delta w_2, z_2) \tag{4.9} \\ &\vdots \nonumber \\ (z_{k-1}, w_{k-1}) &= (f(z_{k-2}) - \delta w_{k-2}, z_{k-2}) \tag{4.10} \\ (z_k, w_k) &= (f(z_{k-1}) - \delta w_{k-1}, z_{k-1}) = (z, w). \tag{4.11} \end{align}$$

It is convenient to think of (z, w) as (z_0, w_0) and further w as z_{-1}. Then, looking at the second component of the equations (4.7) - (4.11), we see that $w_m = z_{m-1}$ for general $0 \leq m \leq k$. Thus, we can replace all occurrences of w_m, excluding $w_0 = w$, with z_{m-1}. If we do this, we need only consider the first components of the equations, which then become the following; note that $w_k = z_{k-1} = w$:

$$z_1 = f(z) - \delta w \tag{4.12}$$
$$z_2 = f(z_1) - \delta z \tag{4.13}$$
$$z_3 = f(z_2) - \delta z_1 \tag{4.14}$$
$$\vdots$$
$$z_{k-2} = f(z_{k-3}) - \delta z_{k-4} \tag{4.15}$$
$$w = f(z_{k-2}) - \delta z_{k-3} \tag{4.16}$$
$$z = f(w) - \delta z_{k-2}. \tag{4.17}$$

We will refer to the set /system of equations (4.12) - (4.17) as **the set/system of determining equations for k-periodic points** or **the set/system of equations determining k-periodic points for** or **of** F. Let us refer to the line which takes the form $z_1 = f(z) - \delta w$ as the first line. By examination, we then see that the m-th line generally takes the form:

$$z_m = f(z_{m-1}) - \delta z_{m-2}. \tag{4.18}$$

This, together with $z_k = z_0 = z$ and $z_{k-1} = w_0 = w = z_{-1}$, let us immediately write the system of determining equations for k-periodic points of F.

Example 4.2.3. We can immediately write down the equations determining 3-periodic points of F. We start with the first line, which is: $z_1 = f(z) - \delta w$, and have further that $z_2 = w$, and $z_3 = z$. Thus we get the equations:

$$z_1 = f(z) - \delta w \tag{4.19}$$
$$w = f(z_1) - \delta z \tag{4.20}$$
$$z = f(w) - \delta z_1 \tag{4.21}$$

We now come to the proof of Theorem 4.2.1:

Proof of Theorem 4.2.1. Assume for contradiction that the theorem assertion is false. Let $(z, w) \in \text{Fix}(F^k)$. Then we know that the system of determining equations for k-periodic points of F, is given by:

$$z_1 = f(z) - \delta w \tag{4.12}$$
$$z_2 = f(z_1) - \delta z \tag{4.13}$$
$$z_3 = f(z_2) - \delta z_1 \tag{4.14}$$
$$\vdots$$
$$w = f(z_{k-2}) - \delta z_{k-3} \tag{4.16}$$
$$z = f(w) - \delta z_{k-2}. \tag{4.17}$$

The equations (4.12) - (4.17) can be viewed as a system of k equations with k unknowns: $z, w, z_1, \ldots, z_{k-2}$. Let $G : \mathbb{C}^k \to \mathbb{C}^k$ be defined by:

$$G(z, w, z_1, \ldots, z_{k-2}) = \begin{bmatrix} z_1 + \delta w - f(z) \\ z_2 + \delta z - f(z_1) \\ z_3 + \delta z_1 - f(z_2) \\ \vdots \\ w + \delta z_{k-3} - f(z_{k-2}) \\ z + \delta z_{k-2} - f(w) \end{bmatrix}. \tag{4.22}$$

Then $(z, w, z_1, \ldots, z_{k-2})$ solves the system of equations (4.12) - (4.17) if and only if it lies in $Z(G)$. This is a closed analytic set. By assumption, this analytic set is non-discrete. Because there are no compact such sets (e.g. Proposition 6.1 in [7]), there must then exist an unbounded connected component V of the solution set $\{(z, w, z_1, \ldots, z_{k-2}) : G(z, w, \ldots, z_{k-2}) = 0\}$. Thus, for all sufficiently large $r > 0$, V intersects the boundary of the ball centred at the origin in $\mathbb{C}^k$ with radius r. Say in some point $(z, w, z_1, \ldots, z_{k-2})$. Then there is some index $j \in \{1, 2, \ldots, k-1, k\}$ depending on r and V, where $z_{k-1} = w$, and $z_k = z$, for which we have that $r \geq |z_j| > r_j$ with r_j large. In particular, given any $r_0 > 0$, by choosing $r > 0$ sufficiently large, we can find an index j such that $r_j > r_0$. Now, let $g(z) = \frac{f(z)-f(0)}{z}$. Then g is entire with $\rho(g) < 1/2$, and we can apply Theorem 4.2.2 to g. We can choose r sufficiently large and part of the sequence in Theorem 4.2.2 such that by Theorem 4.2.2, $|f(z_j)| > |z_j| \cdot |(1+\delta)|$. Indeed, we have $\frac{f(z)}{z} = g(z) + \frac{f(0)}{z}$, and so choosing r sufficiently large such that we have $|g(z_j)| \geq m(g, r) > |1+\delta|$, we see that $\frac{|f(z_j)|}{|z_j|} \geq |g(z_j)| - \frac{|f(0)|}{|z_j|} \geq |g(z_j)| > |1+\delta|$. This almost gives the contradiction we need. To complete the proof, we simply note that we may choose j such that z_j has the greatest modulus among all z_m's. Then we get:

$$|z_j + \delta z_{j-2}| \leq |z_j| \cdot |(1+\delta)| \tag{4.23}$$

Now, looking at the jth component of the equation $G(z, w, \ldots, z_{k-2}) = 0$, we have:

$$z_j + \delta z_{j-2} = f(z_j). \tag{4.24}$$

Comparing (4.23) and (4.24), we then get:

$$|f(z_j)| \leq |z_j| \cdot |1+\delta|. \tag{4.25}$$

But this is impossible for $|f(z_j)| > |z_j| \cdot |(1+\delta)|$. This contradiction proves the assertion.

□

Theorem 4.2.2 provides us with a way of obtaining a large lower bound of the growth of an entire function f, when f has low order. In general, we do not have this when f has order greater than or equal to 1/2. For instance, if we consider the transcendental Hénon map $F(z, w) = (f(z) - \delta w, z)$ where $f(z) = e^z$, then we have $\rho(f) = 1$, but $m(f, r) = e^{-r}$. Thus, $m(f, r)$ becomes arbitrarily small for large r. This example raises the question whether $\text{Fix}(F^k)$ is discrete for $k \geq 3$. Below, we show that we can prove discreteness of the set of 3-periodic points if we instead replace $f(z) = e^z$ with $f_\delta(z) = e^z - \delta^2 z$.

4.3 3-periodic points

Our starting point is the following result in the case that the set of 3-periodic points of a transcendental Hénon map is non-discrete:

Theorem 4.3.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a transcendental Hénon map and denote for points in $\text{Fix}(F^3)$, (z, w). Then, if $\text{Fix}(F^3)$ is not discrete, $w = w(z)$ is some holomorphic function of z on some open set D. That is, $\text{Fix}(F^3)$ contains the graph of a holomorphic function $w = w(z)$ over some open set D:

$$\Gamma_w^D := \left\{(z, w(z)) \in \mathbb{C}^2 : z \in D\right\} \subseteq \text{Fix}(F^3). \tag{4.26}$$

Our proof uses the implicit mapping theorem.

Proof. The set of determining equations for 3-periodic points of F, is given by:

$$z_1 + \delta w = f(z) \tag{4.19}$$
$$w + \delta z = f(z_1) \tag{4.20}$$
$$z + \delta z_1 = f(w). \tag{4.21}$$

We can regard this as a system of 3 equations with 3 unknowns: z, w and z_1. Suppose one of the unknowns are allowed to be discrete in the solution set of this system. By symmetry, we may assume z_1 to be discrete. By eliminating w, using equations (4.19) and (4.20), we get:

$$f(z) - z_1 = \delta f(z_1) - \delta^2 z. \tag{4.27}$$

That is, rearranged:

$$f(z) + \delta^2 z = \delta f(z_1) + z_1. \tag{4.28}$$

Because the set of possible z_1 is discrete, equation (4.28) and the fact that f is transcendental, show that the set of possible z must be discrete as well: let Z_1 denote the set of possible values of z_1 and consider the following family of maps: $\{g_{z_1}\}_{z_1 \in Z_1}$ where each $g_{z_1}(z) = f(z) - \delta^2 z - \delta f(z_1) - z_1$. Let $\tilde{Z}$ denote the set of possible values of z. Then: $\tilde{Z} = \bigcup_{z_1 \in Z_1} Z(g_{z_1})$. Because f is transcendental, for each possible z_1, the set $Z(g_{z_1})$ is discrete. Then, because Z_1 is discrete, it follows that $\tilde{Z}$ is discrete. By a similar argument, the set of possible w is discrete as well. Thus, if the set of possible values of at least one of the unknowns: z, w, or z_1, is allowed to be discrete, it follows that all must be discrete and consequently, Fix(F^3) must be discrete. Thus we may assume none of these sets to be discrete.

Let us define the holomorphic map $G : \mathbb{C}^3 \to \mathbb{C}^2$ by:

$$G(z, w, z_1) = \begin{bmatrix} z_1 + \delta w - f(z) \\ w + \delta z - f(z_1) \end{bmatrix}. \tag{4.29}$$

We differentiate G with respect to the last two variables. Denote the result by $G'_{(w,z_1)}(z, w, z_1)$. We get:

$$G'_{(w,z_1)}(z, w, z_1) = \begin{bmatrix} \delta & 1 \\ 1 & -f'(z_1) \end{bmatrix}. \tag{4.30}$$

Calculating the determinant of $G'_{(w,z_1)}(z, w, z_1)$, gives:

$$\det(G'_{(w,z_1)}(z, w, z_1)) = -\delta f'(z_1) - 1. \tag{4.31}$$

Suppose that $\det(G'_{(w,z_1)}(z, w, z_1))$ is zero. This happens if and only if we have $z_1 \in Z(\delta f' + 1)$. Because f is transcendental, the function $\delta f' + 1$ is not identically 0. Hence $Z(\delta f' + 1)$ is discrete and it follows that the set Z_1 must be discrete. By our previous discussion, we conclude that Fix(F^3) is discrete. It follows that we may assume from the outset that $\det(G'_{(w,z_1)}(z, w, z_1)) \neq 0$. Thus, by the implicit mapping theorem, there exists some holomorphic function from $\mathbb{C}$ to $\mathbb{C}^2$, say g, defined near z, say on some open set $D \ni z$, such that for all $z \in D$:

$$G(z, w, z_1) = 0 \iff (w, z_1) = g(z). \tag{4.32}$$

If we look at the definition of G, we see that the two equations (4.19) and (4.20) are equivalent to $(z, w, z_1) \in Z(G)$. We have that $(z, w) \in$ Fix(F^3) implies $(z, w, z_1) \in Z(G)$ which then happens if and only if $(z, w) = g(z)$ for all $z \in D$. In particular, it follows that $w = w(z)$ is some holomorphic function of z defined on D, and therefore $\Gamma_w^D \subseteq$ Fix(F^3) as required. This completes the proof. □

Remark 4.3.2. We note that the proof of Theorem 4.3.1 actually works for more general Hénon maps $F(z, w) = (f(z) - \delta w, z)$ where f is *not* a linear polynomial. Indeed, the only places the specific form of f was used in the proof, is when we determined that Z_1 being discrete, implies that $\tilde{Z}$ is discrete, and when we reasoned that $\delta f + 1$ cannot be identically zero. If f is *not* a linear polynomial, the latter follows. The first follows from $f(z) - \delta^2 z - \delta f(z_1) - z_1$ not being identically zero for fixed z_1. This is true if f is not a linear polynomial.

Let $H_\delta(z, w) := (h_\delta(z) - \delta w, z)$, where $h_\delta(z) = e^z - \delta^2 z$. We want to show that Fix(H_δ^3) is discrete. We first consider the case $\delta^3 = 1$, which is significantly less tedious than the general case.

Lemma 4.3.3. For $\delta^3 = 1$, Fix(H_δ^3) is discrete.

Proof. The system of determining equations for 3-periodic points for H, is given by:

$$z_1 + \delta w = h_\delta(z) \tag{4.33}$$
$$w + \delta z = h_\delta(z_1) \tag{4.34}$$
$$z + \delta z_1 = h_\delta(w). \tag{4.35}$$

Suppose for contradiction that the lemma assertion is false. Then, by Theorem 4.3.1, we may assume z_1 and w to be functions of z on some open set D. We write $w = w(z)$ and $z_1 = z_1(z)$. We can eliminate $z_1(z)$ by using equations (4.33) and (4.35). This gives:

$$h_\delta(w(z)) - z = \delta h_\delta(z) - \delta^2 w(z). \tag{4.36}$$

That is:

$$h_\delta(w(z)) + \delta^2 w(z) = \delta h_\delta(z) + z. \tag{4.37}$$

Recall that $h_\delta = \exp -\delta^2 \cdot \mathbf{1}$ where $\mathbf{1}$ denotes the identity function on $\mathbb{C}$. Hence we get:

$$e^{w(z)} = \delta h_\delta(z) + z = \delta e^z + z(1 - \delta^3). \tag{4.38}$$

Because $\delta^3 = 1$ by assumption, we get:

$$e^{w(z)} = \delta e^z. \tag{4.39}$$

Consequently:

$$w'(z) = \frac{\delta e^z}{\delta e^z} = 1, \tag{4.40}$$

and we conclude that $w(z) = z + A$ for some constant A. Similarly, we can use equations (4.34) and (4.35) to eliminate z in terms of $z_1(z)$ and $w(z)$. This gives:

$$h_\delta(z_1(z)) - w(z) = \delta h_\delta(w(z)) - \delta^2 z_1(z). \tag{4.41}$$

That is, rearranged:

$$h_\delta(z_1(z)) + \delta^2 z_1(z) = e^{z_1(z)} = \delta h_\delta(w(z)) + w(z) = \delta e^{w(z)} + w(z)(1 - \delta^3). \tag{4.42}$$

Again, because $\delta^3 = 1$, the last term on the right-hand side vanishes, and we are left with:

$$e^{z_1(z)} = \delta e^{w(z)}. \tag{4.43}$$

Consequently:

$$z_1'(z) = \frac{\delta e^{w(z)}}{\delta e^{w(z)}} \cdot w'(z) = 1. \tag{4.44}$$

Thus we have $z_1(z) = z + B$ for some constant B. Substituting $w(z) = z + A$ and $z_1(z) = z + B$ in equation (4.33), yields:

$$z(1+\delta) + B + \delta A = h_\delta(z) = e^z - \delta^2 z. \tag{4.45}$$

This can be rearranged to give:

$$z(1+\delta+\delta^2) + B + \delta A = e^z. \tag{4.46}$$

This now holds everywhere because the left-hand side function is entire, and the right-hand side function is entire. However, this is impossible because the left-hand side function is a polynomial, while the right-hand side function is transcendental. This contradiction completes the proof. □

Lemma 4.3.3 takes care of the discreteness of Fix(H_δ^3) in the case $\delta^3 = 1$. Next, we would like to consider the discreteness of Fix(H_δ^3) for $\delta^3 \neq 1$. The proof for this case involves much more tedious computations, but the idea is analogous to that in the proof of Lemma 4.3.3: we use the special form of $h_\delta(z)$ to obtain an explicit expression for $w'(z)$ or $z_1'(z)$, and then use the determining equations to get a contradiction. Differentiation is available, courtesy of Theorem 4.3.1.

Theorem 4.3.4. Let $F_\delta(z,w) = (e^z - \delta^2 z - \delta w, z)$. Then Fix$(F_\delta^3)$ is discrete.

Proof. Let us define $f_\delta(z) = e^z - \delta^2 z$, so that $F_\delta(z,w) = (f_\delta(z) - \delta w, z)$. By Lemma 4.3.3, we may assume $\delta^3 \neq 1$. If the theorem statement is false, we may, by Theorem 4.3.1, assume $z_1 = z_1(z)$ and $w = w(z)$ to be holomorphic functions of z on some open set, where z_1 and w are given by the set of determining equations for 3-periodic points of F_δ:

$$z_1 + \delta w = f_\delta(z) \tag{4.47}$$
$$w + \delta z = f_\delta(z_1) \tag{4.48}$$
$$z + \delta z_1 = f_\delta(w). \tag{4.49}$$

As in the proof of Lemma 4.3.3, we can eliminate $z_1 = z_1(z)$ and get:

$$f_\delta(w(z)) + \delta^2 w(z) = e^{w(z)} = \delta f_\delta(z) + z = \delta e^z + z(1-\delta^3). \tag{4.50}$$

Consequently:

$$w'(z) = \frac{\delta e^z + 1 - \delta^3}{\delta e^z + z(1-\delta^3)}. \tag{4.51}$$

Using equation (4.47), we can solve for $z_1(z)$ and get:

$$z_1(z) = f_\delta(z) - \delta w(z). \tag{4.52}$$

We differentiate equation (4.48) and get:

$$\begin{aligned} w'(z) + \delta &= f_\delta'(z_1(z)) \cdot z_1'(z) \\ &= \frac{d}{dz_1}\left\{e^{z_1(z)} - \delta^2 z_1(z)\right\} \cdot z_1'(z) \\ &= (e^{z_1(z)} - \delta^2) z_1'(z). \end{aligned} \tag{4.53}$$

We have $e^{z_1(z)} = f_\delta(z_1(z)) + \delta^2 z_1(z)$. With equation (4.48), we thus have that $e^{z_1(z)} = w(z) + \delta z + \delta^2 z_1(z)$. Combining this with equation (4.52), yields then:

$$\begin{aligned} e^{z_1(z)} &= w(z) + \delta z + \delta^2 (f_\delta(z) - \delta w(z)) \\ &= w(z)(1 - \delta^3) + \delta z + \delta^2 f_\delta(z). \end{aligned} \tag{4.54}$$

Substituting this and $z_1'(z) = f_\delta'(z) - \delta w'(z)$, which follows from equation (4.52), into equation (4.53), gives:

$$w'(z) + \delta = \left(w(z)(1 - \delta^3) + \delta z + \delta^2 f_\delta(z) - \delta^2\right)\left(f_\delta'(z) - \delta w'(z)\right). \tag{4.55}$$

We have that $f_\delta'(z) = e^z - \delta^2 \not\equiv \delta w'(z)$, where $w'(z)$ is given by equation (4.51), so we can divide by it in equation (4.55) to get:

$$\frac{w'(z) + \delta}{f_\delta'(z) - \delta w'(z)} = w(z)(1 - \delta^3) + \delta z + \delta^2 f_\delta(z) - \delta^2. \tag{4.56}$$

Thus:

$$w(z) = \frac{1}{1 - \delta^3}\left(\frac{w'(z) + \delta}{f_\delta'(z) - \delta w'(z)} - \delta z - \delta^2 f_\delta(z) + \delta^2\right). \tag{4.57}$$

Differentiating equation (4.57) and working tediously to simplify the result, gives Expression 4.1a. We note that the denominator in Expression 4.1a, is zero when

$$\Big(-(\delta-1)^2(\delta^2+\delta+1)^2\big((-3z^2-2z)\delta^6+(-6z-2)\delta^5-3\delta^4+2\delta^3z^2+2\delta^2z - \delta z^2 - z - 1\big)e^z - 3\delta(\delta-1)\Big(\Big(\Big(z^2+2z+\frac{1}{3}\Big)\delta^6+(2z+2)\delta^5+\delta^4+\Big(-\frac{4}{3}z^2 - \frac{4}{3}z\Big)\delta^3+\Big(-\frac{4z}{3}-\frac{2}{3}\Big)\delta^2+\frac{2z\delta}{3}+\frac{z^2}{3}+1\Big)(\delta^2+\delta+1)e^{2z}+\delta^2\big((z^2 + 6z+3)\delta^4+(-z^2-4z+3)\delta^3+(-2z-5)\delta^2+(-z^2-2z)\delta+(z+1)^2\big)(\delta^2 + \delta+1)e^{3z}-2\delta^3\Big(\Big(z+\frac{3}{2}\Big)\delta^3+\delta^2-z-\frac{1}{2}\Big)e^{4z}+e^{5z}\delta^4\Big) - (\delta-1)^3(\delta^2 + \delta+1)^3\delta^3(z\delta+1)^2\Big) \Big/ \Big((\delta-1)\big(e^{2z}\delta+\big((-z-1)\delta^3-\delta^2+z\big)e^z+\delta^5 z + \delta^4-\delta^2 z-\delta\big)^2(\delta^2+\delta+1)\Big) \tag{1}$$

Expression 4.1a : $w'(z)$ obtained by differentiating equation (4.57).

$\delta = 1$ or $\delta^2 + \delta + 1 = 0$. However, by assumption $\delta^3 \neq 1$, so this never happens.

Subtracting from $w'(z)$ in Expression 4.1a, $w'(z)$ in equation (4.51), should give identically zero. Otherwise, we have a contradiction. The result of the subtraction is given in Expression 4.1b. As can be verified by examination, this is not identically zero. In fact, looking at the denominator in Expression 4.1b, it is clear that this difference is defined for $z = r$ for all sufficiently large $r > 0$. Suppose for contradiction that Expression 4.1b is identically zero where defined. Then, the

$$
\begin{aligned}
&\Big(-(\delta-1)^3\,\big((-3z^3-3z^2)\,\delta^6+(-9z^2-6z)\,\delta^5+(-9z-3)\,\delta^4+(2z^3-3)\,\delta^3 \qquad \textbf{(1)}\\
&\quad+4z^2\delta^2+(-z^3+2z)\,\delta-z^2-z\big)\,(\delta^2+\delta+1)^3\,e^z-3\,(\delta-1)^2\,\Big((z^3+3z^2\\
&\quad+z)\,\delta^7+(3z^2+6z+1)\,\delta^6+(3z+3)\,\delta^5+\Big(-\frac{4}{3}z^3-2z^2+1\Big)\,\delta^4+\Big(-\frac{8}{3}z^2\\
&\quad-\frac{8}{3}z\Big)\,\delta^3+\Big(z^2-\frac{4}{3}z-\frac{2}{3}\Big)\,\delta^2+\Big(\frac{1}{3}z^3+\frac{4}{3}z+\frac{1}{3}\Big)\,\delta+\frac{z^2}{3}\Big)\,(\delta^2+\delta\\
&\quad+1)^2\,e^{2z}+\delta\,(\delta-1)\,\big((z^3+9z^2+9z+1)\,\delta^5+(-z^3-6z^2+9z+8)\,\delta^4+(\\
&\quad-3z^2-15z)\,\delta^3+(-z^3-3z^2-7)\,\delta^2+(z^3+2z^2-z+3)\,\delta+z^2+2z\big)\,(\delta^2\\
&\quad+\delta+1)^2\,e^{3z}-3\,\delta^2\,\Big((z^2+3z+1)\,\delta^5+(-z^2-z+2)\,\delta^4+(-2z-2)\,\delta^3+\Big(-z^2\\
&\quad-z-\frac{2}{3}\Big)\,\delta^2+\Big(z^2+\frac{1}{3}z\Big)\,\delta+\frac{2z}{3}+\frac{1}{3}\Big)\,(\delta^2+\delta+1)\,e^{4z}+\big((3z+3)\,\delta^7+3\,\delta^6\\
&\quad+(-3z-1)\,\delta^4-\delta^3\big)\,e^{5z}-\boldsymbol{e^{6z}\,\delta^5}-(\delta-1)^4\,(\delta^2+\delta+1)^4\,(z\,\delta+1)^3\,\delta^2\Big)\Big/\Big((\delta\\
&\quad-1)\,(\delta^3z-e^z\,\delta-z)\,\big(e^{2z}\,\delta+\big((-z-1)\,\delta^3-\delta^2+z\big)\,e^z+\delta^5z+\delta^4-\delta^2z\\
&\quad-\delta\big)^2\,(\delta^2+\delta+1)\Big)
\end{aligned}
$$

Expression 4.1b : the difference between $w'(z)$ in Expression 4.1a and $w'(z)$ in equation (4.51). In particular, this difference is not identically 0.

numerator must vanish. By examination, we find that the highest power of the exponential function, is 6. Thus the most dominating term for $z = r > 0$ large, is e^{6z}. Its coefficient is $-\delta^5$. We divide the whole numerator by e^{6z}, evaluate at $z = r$ and let $r \to \infty$. It follows that we then get $-\delta^5 = 0$, which is impossible because $\delta \neq 0$ for Hénon maps. This contradiction proves the theorem assertion. □

4.4 4-periodic points

For 3-periodic points we were able to use the implicit mapping theorem to prove discreteness for the Hénon maps $(e^z - \delta^2 z - \delta w, z)$. For 4-periodic points, we will use an elementary property of analytic sets in $\mathbb{C}^2$. The implicit mapping theorem and the study of analytic sets, are of course not unrelated. For instance, the Weierstrass's preparation theorem, a central result in the study of analytic sets, can be seen as a generalization of the implicit mapping theorem.

We will use the following: let A be an analytic set in $\mathbb{C}^2$. Then A is a countable union of curves which are locally graphs of holomorphic functions in $\mathbb{C}$ and single points which are either isolated or singular points to these curves. In particular, A does not contain a sequence of points with an accumulation point which does mnot belong to some such curve. The curves and points of which A is a union of, are called the **irreducible components** of A. If A has a single irreducible component, it is called **irreducible**. Otherwise, A is called **reducible**. In the proof of Theorem 4.2.1, we used that any irreducible compact analytic set reduces to a single point. For these properties of analytic sets, see any introductory book on several complex variables which include the topic of analytic sets in the contents, for instance [7] chapter 3, or [4] chapter 2, section 8.

First, we show that Fix(F^4) cannot contain a non-discrete set of points (z, w) where z is constant and F is a transcendental Hénon map. In particular, Fix(F^4) cannot contain a vertical line in $\mathbb{C}^2$.

Lemma 4.4.1. Let $F(z, w) = (f(z) - \delta w, z)$ be any transcendental Hénon map. Then Fix(F^4) cannot contain any non-discrete set of points (z, w) where z is constant.

Proof. The set of determining equations for 4-periodic points of F, is given by:

$$z_1 + \delta w = f(z) \tag{4.58}$$
$$z_2 + \delta z = f(z_1) \tag{4.59}$$
$$w + \delta z_1 = f(z_1) \tag{4.60}$$
$$z + \delta z_2 = f(w). \tag{4.61}$$

Suppose for contradiction that the lemma assertion is false. Because z is constant, equation (4.58), shows that $z_1 = z_1(w) = A - \delta w$, where $A = f(z)$ is constant. Substituted into equation (4.60), it follows that:

$$w + \delta(A - \delta w) = f(A - \delta w) \tag{4.62}$$

That is:

$$w(1 - \delta^2) + \delta A = f(A - \delta w). \tag{4.63}$$

We define a new variable $\eta := A - \delta w$. Note that by the uniqueness principle and the assumption of non-discreteness, equation (4.63) holds for all w in the plane. We have $w = \frac{A-\eta}{\delta}$. Substituted into equation (4.63), we then get:

$$\frac{A - \eta}{\delta}(1 - \delta^2) + \delta A = f(\eta). \tag{4.64}$$

This now holds for all η in the plane. In particular, this means that $f(\eta)$ a linear polynomial in η. But this contradicts that f is transcendental and completes the proof. □

Thus, it follows from what we said in the introduction of this section, that in the case F is a transcendental Hénon map and Fix(F^4) fails to be discrete, the set Fix(F^4) must contain the graph of some holomorphic function $w = w(z)$. This provides the analogue of Theorem 4.3.1 for 4-periodic points:

Theorem 4.4.2. Let $F(z,w) = (f(z) - \delta w, z)$ be a transcendental Hénon map and denote by points in Fix(F^4), (z,w). Then, if Fix(F^4) is not discrete, $w = w(z)$ is some holomorphic function of z on some open set D_w. Thus, Fix(F^4) contains the graph of $w = w(z)$ over D_w:

$$\Gamma_w^{D_w} = \{(z, w(z)) \in \mathbb{C}^2 : z \in D_w\} \subseteq \text{Fix}(F^4). \tag{4.65}$$

We use Theorem 4.4.2 to show that Fix(F^4) is discrete, where F is the transcendental Hénon map given by $F(z,w) = (e^z - \delta w, z)$ with $\delta^2 = 1$:

Theorem 4.4.3. Let $F(z,w) = (e^z - \delta w, z)$ with $\delta^2 = 1$. Then Fix(F^4) is discrete.

Proof. Suppose for contradiction that the theorem assertion is false. By Theorem 4.4.2, we may then assume $w = w(z)$, $z_1 = z_1(z)$, and $z_2 = z_2(z)$ to be holomorphic functions of z on some open set, and where w, z_1, and z_2 are given from the set of determining equations for 4-periodic points of F:

$$z_1 + \delta w = f(z) \tag{4.58}$$
$$z_2 + \delta z = f(z_1) \tag{4.59}$$
$$w + \delta z_1 = f(z_2) \tag{4.60}$$
$$z + \delta z_2 = f(w). \tag{4.61}$$

We eliminate z_2 using equations (4.59) and (4.61), and get:

$$f(w(z)) - z = \delta f(z_1(z)) - \delta^2 z. \tag{4.66}$$

That is:

$$e^{w(z)} = \delta e^{z_1(z)} + z(1 - \delta^2). \tag{4.67}$$

Because $\delta^2 = 1$, we get:

$$e^{w(z)} = \delta e^{z_1(z)}. \tag{4.68}$$

Consequently:

$$w'(z) = z_1'(z). \tag{4.69}$$

We differentiate equation (4.58). This gives, substituting $z_1'(z) = w'(z)$:

$$w'(z)(1 + \delta) = f'(z) = f(z). \tag{4.70}$$

If $\delta = -1$, this gives $f(z) = 0$ which is false. This contradiction proves the assertion in the case $\delta = -1$. Thus, we may assume $\delta = 1$. Then:

$$w'(z) = \frac{f'(z)}{2}. \tag{4.71}$$

That is:

$$w(z) = \frac{f(z)}{2} + A, \tag{4.72}$$

where A is some constant. Then, solving for z_2 in equation (4.61), we get:

$$z_2(z) = f(w(z)) - z = f\left(\frac{f(z)}{2} + A\right) - z. \tag{4.73}$$

We recall that $z_1'(z) = w'(z)$. Therefore, $z_1(z) = w(z) + B$ for some constant B. We substitute this and the expression for $z_2(z)$ in equation (4.73), into equation (4.60). This gives:

$$2w(z) + B = f\left(f\left(\frac{f(z)}{2} + A\right) - z\right). \tag{4.74}$$

Because $w(z) = \frac{f(z)}{2} + A$, we get:

$$f(z) + 2A + B = f\left(f\left(\frac{f(z)}{2} + A\right) - z\right), \tag{4.75}$$

and because $f(z) = e^z$, we finally have:

$$e^z + 2A + B = e^{e^{\frac{e^z}{2} + A} - z}. \tag{4.76}$$

This now holds for all z in the plane. But this is impossible because the left-hand side is an entire function with order 1, while the right-hand side is an entire function with infinite order. This contradiction proves the assertion in the case $\delta = 1$. Together with the case $\delta = -1$ from before, this completes the proof. □

Chapter 5

Classes of Maps with Infinitely Many Periodic Points, Part 1

This chapter is the first part of an investigation on transcendental Hénon maps with infinitely many k-periodic points for certain values of k.

The cases $k = 1$ and $k = 2$ have already been dealt with in great detail in previous chapters, and we therefore start off with the case $k = 3$. We discuss the topic of genuine 3-periodic points and give a complete characterization. We also construct examples of transcendental Hénon maps with infinitely many 3-periodic points. In particular, we show that any transcendental Hénon map $F_1(z, w) = (f(z) - w, z)$ with $\delta = 1$, admits infinitely many 3-periodic points. Finally, we impose certain conditions to get classes of transcendental Hénon maps with infinitely many *genuine* 3-periodic points.

We proceed to consider the case $k = 4$. We give two main results. The first, is the existence of infinitely many genuine 4-periodic points of the class of Hénon maps $F_{-1}(z, w) = (e^{g(z)} + w, z)$ with $\delta = -1$, where g is a non-constant entire function. For this, we use an estimate method which leads to almost explicit formulae. We start with the case where g is a monomial of degree at least 2, and then generalize to the case where g is a transcendental entire function by use of the Wiman-Valiron method (Theorem 1.4.4). The second main result, is the existence of infinitely many genuine 4-periodic points for the class of transcendental Hénon maps $F_1(z, w) = (f(z) - w, z)$ with $\delta = 1$, and f having a non-zero period p: $f(z + \mathbb{Z}p) = f(z)$ for all z. For this, we use that the equation $f(z) = L(z)$, where $L(z)$ is a linear polynomial, admits infinitely many solutions.

We will use the following terminology. Let F be a holomorphic map. We will say that F is **symplectic**, if $\det(F') = 1$. We know that for a general Hénon map $F(z, w) = (f(z) - \delta w, z)$, we have $\det(F') = \delta$, and therefore symplectic Hénon maps take the form:

$$F : (z, w) \mapsto (f(z) - w, z). \tag{5.1}$$

With the notation above, we thus have that F_1 is a symplectic transcendental Hénon map.

The symplectic maps are volume-preserving and form an important class of maps. Historically, symplectic maps date back to Poincaré and the study of periodic orbits in *Celestial Mechanics*, a branch of physics dealing with the motions of celestial objects. A central result here is the *Poincaré-Birkhoff theorem* concerned with the existence of fixed points of so-called area-preserving twist homeomorphism on an annulus. See for instance [3] and [10]. Other applications in physics where symplectic maps arise naturally, include for instance, *accelerator, chemical, condensed-matter*, and *fluid physics*. The symplectic maps also play an important part in *symplectic topology* and *Kolmogorov-Arnold-Moser (KAM) theory*. See for instance [9].

As the reader will come to realize, the set of determining equations for periodic points of symplectic Hénon maps, possess a certain symmetry which makes them easier to work with than general Hénon maps. Many of the results we give, are concerned with namely symplectic Hénon maps.

5.1 3-periodic points

We wish to investigate the existence of 3-periodic points of Hénon maps, and we are especially interested in genuine 3-periodic points. It follows that any non-genuine 3-periodic point must lie on the diagonal Δ. Indeed, because all fixed points of Hénon maps lie on the diagonal Δ, and because any non-genuine prime-periodic point must be a fixed point, this is true also for any k-periodic point where k is prime.

Let $F(z,w) = (f(z) - \delta w, z)$ be a Hénon map. We recall that the set of determining equations for 3-periodic points of F, is given by:

$$z_1 + \delta w = f(z) \tag{4.19}$$
$$w + \delta z = f(z_1) \tag{4.20}$$
$$z + \delta z_1 = f(w). \tag{4.21}$$

From equation (4.19), we get: $z_1 = f(z) - \delta w$. Substituted into the two remaining equations, equation (4.20) and equation (4.21), gives then:

$$w + \delta z = f(f(z) - \delta w) \tag{5.2}$$
$$z + \delta f(z) - \delta^2 w = f(w). \tag{5.3}$$

To get further, a possibility at this point, is to look for solutions where w is some function of z. It would then be natural to look for solutions where w is a not too complicated function of z, for example $w = z$ or $w = -z$. Perhaps a more interesting choice for a function for w, would be something involving the function f. After some considerations, we see that if we choose $f(z) - \delta w = w$, or equivalently in the case $\delta \neq -1$: $w = \frac{f(z)}{1+\delta}$, the right-hand sides in equations (5.2) and (5.3), coincide. If the right-hand sides coincide, so must the left-hand sides. Consequently, we end up with the necessary condition:

$$w + \delta z = z + \delta f(z) - \delta^2 w. \tag{5.4}$$

With our choice for w, we have $f(z) = (1+\delta)w$. Substituting this in the condition (5.4), gives:

$$w + \delta z = z + \delta(1+\delta)w - \delta^2 w \tag{5.5}$$

That is, after simplifying:

$$z(\delta - 1) = (\delta - 1)w. \tag{5.6}$$

If $\delta \neq -1$, then we can substitute $\frac{f(z)}{1+\delta}$ for w. This gives:

$$z(\delta - 1) = \frac{\delta - 1}{1+\delta} f(z). \tag{5.7}$$

If additionally $\delta \neq 1$, that is, if $\delta^2 \neq 1$, then recalling also that $f(z) = (1+\delta)w$, this gives:

$$z(1+\delta) = f(z) = (1+\delta)w. \tag{5.8}$$

Consequently, $(z, w) \in \Delta$ and $f(z) = (1+\delta)z$. That is, $(z, w) \in \text{Fix}(F)$ by Proposition 2.2.2.

By considering the case $\delta = -1$ separately, we get the following proposition:

Proposition 5.1.1. Let $F(z, w) = (f(z) - \delta w, z)$ be a Hénon map with $\delta \neq 1$. Then, the point (z, w) with $w(1+\delta) = f(z)$, is a 3-periodic point for F if and only if it is a fixed point for F.

Proof. The assertion follows from our discussion in the case $\delta^2 \neq 1$. Thus it remains to consider the single case $\delta = -1$. Then, equation (5.7) is no longer valid. However, equation (5.6) is. Because $\delta = -1$, we get $-2z = -2w$. That is: $z = w$. Hence $(z, w) \in \Delta$. Either of the two equations (5.2) and (5.3), now gives then that $(z, w) \in \text{Fix}(F^3)$ if and only if $f(z) = 0$. That is, if and only if $z \in Z(f)$. Because $w = z$, we then get that $(z, w) \in \text{Fix}(F^3)$ if and only if $(z, w) \in (Z(f) \times Z(f)) \cap \Delta$. From Corollary 2.2.3, it follows that $\text{Fix}(F) = (Z(f) \times Z(f)) \cap \Delta$. This completes the proof. ☐

Because we are interested in genuine 3-periodic points of Hénon maps, Proposition 5.1.1 tells us that, in looking for 3-periodic points of F of the form (z, w) with $f(z) = (1+\delta)w$, we should focus on the case $\delta = 1$. That is, the case when F is symplectic. By using Rosenbloom's theorem (Theorem 1.4.1), we can show the existence of infinitely many 3-periodic points in the case F is also transcendental:

Theorem 5.1.2. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map. Then $\text{Fix}(F_1^3)$ contains infinitely many points of the form $\left(z, \frac{f(z)}{2}\right)$. In particular, $\text{Fix}(F_1^3)$ has infinite cardinality.

Proof. Because $\delta = 1$, equation (5.6) is automatically satisfied. Therefore, whether $\left(z, \frac{f(z)}{2}\right) \in \text{Fix}(F_1^3)$, is determined by either of the two equations (5.2) and (5.3), which then gives:

$$\frac{f(z)}{2} + z = f\left(\frac{f(z)}{2}\right). \tag{5.9}$$

That is, by:

$$z = f\left(\frac{f(z)}{2}\right) - \frac{f(z)}{2}. \tag{5.10}$$

Let us define $g(z) = \frac{f(z)}{2}$ and $h(z) = f(z) - z$. Then both g and h are transcendental entire functions, and we have that z solves equation (5.10) if and only if $z \in Z(h \circ g)$. By Rosenbloom's theorem (Theorem 1.4.1), $Z(h \circ g)$ has infinite cardinality. This proves the assertion. $\square$

Although Theorem 5.1.2 guarantees that F_1 admits infinitely many 3-periodic points, it does not say anything whether $\text{Fix}(F_1^3)$ contains, if any at all, *genuine* 3-periodic points. However, we know that the only non-genuine 3-periodic points, must be the fixed points. Thus, we at least have the following result:

Corollary 5.1.3. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map and suppose that $|\text{Fix}(F_1)| < \infty$. Then F_1 admits infinitely many genuine 3-periodic points. Furthermore, infinitely many of these can be chosen of the form (z, w) where $z = f\left(\frac{f(z)}{2}\right) - \frac{f(z)}{2}$ and $w = \frac{f(z)}{2}$.

The results we have given up to this point, are based on the choice of w such that $f(z) = (1 + \delta)w$. We mentioned that two other natural choices are: $w = z$ and $w = -z$. Because we are interested in genuine 3-periodic points, and because all non-genuine such points lie on the diagonal Δ, it is natural to proceed with the choice $w = -z$. We will later see that this choice provides interesting results also for other periodic points than 3-periodic points. These, as well as the next result we give, are based on the following simple observation:

Proposition 5.1.4. Let F be a Hénon map and let $k > 1$ be prime. Suppose that $(z, -z) \in \text{Fix}(F^k)$. Then either this is a genuine k-periodic point of F, or it is the origin.

Proof. Because $k > 1$ is prime, $(z, -z) \in \text{Fix}(F^k)$ is a non-genuine k-periodic point for F if and only if it is a fixed point. Then, because F is a Hénon map, $(z, -z)$ must lie on the diagonal. Thus, $z = -z = 0$. □

Consequently, if we are able to show that a Hénon map F, has infinitely many k-periodic points of the form $(z, -z)$, where $k > 1$ is prime, then it has infinitely many genuine ones. We can make this happen by imposing appropriate conditions on F.

Lemma 5.1.5. Let $F(z, w) = (f(z) - w, z)$ be a symplectic Hénon map where f is odd. Then $(z, -z) \in \text{Fix}(F^3)$ if and only if $f(z) = -z$.

Proof. For $\delta = 1$ and $w = -z$, the set of determining equations for 3-periodic points of F, (4.19) - (4.21), becomes:

$$z_1 - z = f(z) \tag{5.11}$$

$$-z + z = f(z_1) \tag{5.12}$$

$$z + z_1 = f(-z). \tag{5.13}$$

Adding equations (5.11) and (5.13), and using that $f(-z) = -f(z)$ for all z because f is odd, we get:

$$2z_1 = f(z) + f(-z) = f(z) - f(z) = 0. \tag{5.14}$$

That is: $z_1 = 0$. Because f is odd, we always have $f(0) = 0$, and so equation (5.12) becomes the trivial equation $0 = 0$. Furthermore, substituting $z_1 = 0$ and using that f is odd, we see that equation (5.11) is the same as equation (5.13). Thus, we are only left with a single equation determining whether $(z, -z) \in \text{Fix}(F^3)$. Namely, either one of (5.11) and (5.13), which then gives:

$$f(z) = -z. \tag{5.15}$$

This completes the proof. □

We can now use Rosenbloom's theorem (Theorem 1.4.1) together with Lemma 5.1.5, to give infinitely many examples of symplectic transcendental Hénon maps with infinitely many genuine 3-periodic points:

Corollary 5.1.6. Let g and h be two transcendental entire and odd functions. Let $f = h \circ g$. Then $F_1(z, w) = (f(z) - w, z)$ is a symplectic transcendental Hénon map which admits infinitely many genuine 3-periodic points of the form $(z, -z)$.

Proof. Because g and h are transcendental entire functions, so is f. Therefore F_1 is a symplectic transcendental Hénon map. Furthermore, because g and h are odd, so is f:

$$f(-z) = h(g(-z)) = h(-g(z)) = -h(g(z)) = -f(z). \tag{5.16}$$

Thus, by Lemma 5.1.5, $(z, -z) \in \text{Fix}(F_1^3)$ if and only if $f(z) = -z$. That is, if and only if $z \in Z(-h \circ g)$. By Rosenbloom's theorem (Theorem 1.4.1), the latter has infinite cardinality. Hence, by Proposition 5.1.4, F_1^3 admits infinitely many *genuine* 3-periodic points of the form $(z, -z)$. □

Example 5.1.7. Let $g(z) = \sin(z)$, $h(z) = ze^{z^2}$, and let $f(z) = h(g(z))$. Then $F_1(z,w) = (f(z) - w, z)$ is a symplectic transcendental Hénon map which admits infinitely many genuine 3-periodic points of the form $(z, -z)$.

Example 5.1.8. Let $f(z) = \sin(z)$, and consider the symplectic transcendental Hénon map $F_1(z,w) = (f(z) - w, z)$. Then, Corollary 5.1.6 does not apply. However, Lemma 5.1.5 does. From Example 2.3.4, we know that the equation $\sin(z) = \lambda P(z)$ for all non-zero polynomial P, admits infinitely many solutions for all complex values λ without exception. If we take $\lambda = -1$ and $P(z) = z$, it therefore follows that Fix(F_1) contains infinitely many genuine 3-periodic points of the form $(z, -z)$.

We will later see that the result in Example 5.1.8, can also be obtained by utilizing the periodicity of $\sin(z)$ and the fact that it is a transcendental entire function. Using this approach, the fact that $\sin(z) = -z$ admits infinitely many solutions, follows then as a particular case of a much more general result which we give in chapter 6. See Theorem 6.2.3.

Finally, we consider the question of genuine 3-periodic points of Hénon maps on the diagonal, corresponding to the choice $w = z$ mentioned earlier. A priori, it is possible that there exist genuine 3-periodic points of Hénon maps on the diagonal. We now show that this cannot be the case unless $\delta = 1$. That is, unless the Hénon map is symplectic:

Proposition 5.1.9. Let $F(z,w) = (f(z) - \delta w, z)$ be a Hénon map. Then if F is not symplectic, that is, if $\delta \neq 1$, all 3-periodic points on the diagonal Δ, are in fact fixed points. That is:

$$\text{for all } \delta \neq 1 : \text{Fix}(F^3) \cap \Delta = \text{Fix}(F). \tag{5.17}$$

Proof. Suppose $\delta \neq 1$. It suffices to show that Fix$(F^3) \cap \Delta \subseteq$ Fix(F), because we always have Fix$(F) \subseteq$ Fix$(F^k) \cap \Delta$ for all $k \geq 1$. The set of determining equations for 3-periodic points of F, when $w = z$, is given by:

$$z_1 + \delta z = f(z) \tag{5.18}$$
$$z + \delta z = f(z_1) \tag{5.19}$$
$$z + \delta z_1 = f(z). \tag{5.20}$$

Comparing equation (5.18) and equation (5.20), we get from $f(z) = f(z)$:

$$z_1(1 - \delta) = z(1 - \delta). \tag{5.21}$$

Because $\delta \neq 1$, this means $z = z_1$. Hence all three equations (5.18) - (5.20) collapse to the single equation given by: $f(z) = (1+\delta)z$, which is the determining equation for fixed points of F by Proposition 2.2.2. This completes the proof. □

It follows that if we want to look for genuine 3-periodic points of Hénon maps on the diagonal Δ, we must restrict to the symplectic Hénon maps. We start with a lemma:

Lemma 5.1.10. Let $F(z,w) = (f(z) - w, z)$ be a symplectic Hénon map. Then:

$$\text{Fix}(F^3) \cap \Delta = \left\{(z,z) \in \mathbb{C}^2 : 2z = f(f(z) - z)\right\}. \tag{5.22}$$

Proof. For $\delta = 1$, it is easily seen that equation (5.18) and equation (5.20), coincide. Solving for z_1 in either of them, and substituting in equation (5.19), then yields: $2z = f(f(z) - z)$. This proves the assertion. □

Using Rosenbloom's theorem (Theorem 1.4.1), we then get the following result for symplectic transcendental Hénon maps:

Corollary 5.1.11. Let $F_1(z,w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map. Then F_1 admits infinitely many 3-periodic points on the diagonal. In particular, if $|\text{Fix}(F_1)| < \infty$, infinitely many of these are genuine. More generally, $(z,z) \in \text{Fix}(F_1^3) \cap \Delta$ is a genuine 3-periodic point of F_1 if and only if:

$$2z = f(f(z) - z) \text{ and } f(z) \neq 2z. \tag{5.23}$$

Proof. That $(z,z) \in \text{Fix}(F_1^3) \cap \Delta$ if and only if $2z = f(f(z) - z)$, follows from Lemma 5.1.10. Let $g(z) = f(z) - z$ and let $h(z) = \frac{f(z)}{2}$. Then both g and h are transcendental entire functions because f is a transcendental entire function, and z solves the equation $2z = f(f(z) - z)$, if and only if $z \in Z(h \circ g)$. The latter has infinite cardinality by Rosenbloom's theorem (Theorem 1.4.1). We conclude that $\text{Fix}(F_1^3) \cap \Delta$ has infinite cardinality. Then $(z,z) \in \text{Fix}(F_1^3) \cap \Delta$ is a genuine 3-periodic point for F_1 if and only if it is not a fixed point. This happens if and only if $f(z) = 2z$. □

5.2 4-periodic points

We investigate the existence of infinitely many 4-periodic points for certain classes of transcendental Hénon maps. The first section treats transcendental Hénon maps of the form $F_{-1}(z,w) = (e^{g(z)} + w, z)$ where g is some non-constant entire function. Let $f(z) = e^{g(z)}$. Because F_{-1} is a transcendental Hénon map with $\delta = -1$, we know from Corollary 2.2.3 and Corollary 3.1.4 respectively, that the set of fixed points and 2-periodic points for F_{-1}, is given by respectively $Z(f) \times Z(f) \cap \Delta$ and $Z(f) \times Z(f)$. We have $Z(f) = \emptyset$, and so F has no fixed points nor 2-periodic points. We show that $\text{Fix}(F_{-1}^4)$ however, has infinite cardinality. Because 4 is not prime, the problem whether a 4-periodic point of a general Hénon map, is a genuine 4-periodic point, becomes more complicated than our previous two cases of 2-periodic and 3-periodic points. Indeed, now non-genuine 4-periodic points may also be genuine 2-periodic points in addition to being fixed points. However, in our case, because $\text{Fix}(F_{-1}) = \text{Fix}(F_{-1}^2) = \emptyset$, we can immediately say that *all our 4-periodic points are genuine.*

In the second section, we give a result on the existence of infinitely many genuine 4-periodic points for symplectic transcendental Hénon maps of the form $F_1(z,w) = (f(z) - w, z)$ where f has a non-zero period p: $f(z + \mathbb{Z}p) = f(z)$ for all z. In particular, our result works for $f(z) = e^{g(z)}$ where g is periodic, or where $g(z) = z$, as e^z is periodic.

5.2.1 The class of maps of the form $F_{-1}(z,w) = (e^{g(z)} + w, z)$

An approximate equation on the diagonal

Let $F_{-1}(z,w) = (e^{g(z)} + w, z)$ where g is some non-constant entire function. We consider the system of determining equations for 4-periodic points for F_{-1}. We get:

$$z_1 - w = e^{g(z)} \tag{5.24}$$
$$z_2 - z = e^{g(z_1)} \tag{5.25}$$
$$w - z_1 = e^{g(z_2)} \tag{5.26}$$
$$z - z_2 = e^{g(w)}. \tag{5.27}$$

We see that equation (5.24) gives z_1 in terms of z and w. Indeed, by solving for z_1, we find that: $z_1 = e^{g(z)} + w$. Similarly, equation (5.27) gives z_2 expressed in terms of z and w: $z_2 = z - e^{g(w)}$. We can now substitute both these expressions for z_1 and z_2 in the two remaining equations (5.25) and (5.26). This yields:

$$-e^{g(w)} = e^{g\left(e^{g(z)}+w\right)} \tag{5.28}$$
$$-e^{g(z)} = e^{g\left(z-e^{g(w)}\right)}. \tag{5.29}$$

These can be rewritten as:

$$e^{g\left(e^{g(z)}+w\right)-g(w)} = -1 \tag{5.30}$$
$$e^{g\left(z-e^{g(w)}\right)-g(z)} = -1. \tag{5.31}$$

It is then easily seen that if (z,w) solves the two equations (5.32) and (5.33) below, then it solves the two equations (5.30) and (5.31):

$$g\left(e^{g(z)} + w\right) - g(w) = \pi i \tag{5.32}$$
$$g\left(z - e^{g(w)}\right) - g(z) = -\pi i. \tag{5.33}$$

The reason that we chose $-\pi i$ for the right-hand side of equation (5.33), instead of πi, will be apparent when we soon use the Taylor expansion to approximate each of the equations. The idea is that the choice $-\pi i$, will allow us to use a single approximative equation in g' for *both* equations (5.32) and (5.33). Restricting to solutions on the diagonal Δ is a natural simplification, especially now that we know that we can have no fixed points for F_{-1}.

Lemma 5.2.1. Suppose that z satisfies the equation:

$$g'(z)e^{g(z)} = \pi i, \tag{5.34}$$

with $|e^{g(z)}| < 1$ small. Then the error $E_g(z)$, in using the point $(z, z) \in \Delta$ as a solution for equations (5.32) and (5.33), instead of the correct solution, which generically may lie off the diagonal Δ, satisfies the inequality:

$$E_g(z) \leq \sum_{m=2}^{\infty} \frac{\pi^m}{m!} \left| \frac{g^{(m)}(z)}{(g'(z))^m} \right|. \tag{5.35}$$

Proof. Suppose $|e^{g(\zeta)}|$ is small. Then, using the Taylor expansion for g near the point z, for equations (5.32) and (5.33), yields respectively:

$$g(z + e^{g(\zeta)}) - g(z) = g'(z)e^{g(\zeta)} + \sum_{m=2}^{\infty} \frac{g^{(m)}(z)}{m!} e^{mg(\zeta)} \tag{5.36}$$

$$g\left(z - e^{g(\zeta)}\right) - g(z) = -g'(z)e^{g(\zeta)} + \sum_{m=2}^{\infty} \frac{g^{(m)}(z)(-1)^m}{m!} e^{mg(\zeta)}. \tag{5.37}$$

Now, by assumption, $g'(z)e^{g(z)} = \pi i$, and so setting $\zeta = z$, and noting that $e^{mg(z)} = \frac{(\pi i)^m}{(g'(z))^m}$, we get respectively:

$$g(z + e^{g(z)}) - g(z) = \pi i + \sum_{m=2}^{\infty} \frac{\pi^m i^m}{m!} \frac{g^{(m)}(z)}{(g'(z))^m} \tag{5.38}$$

$$g(z - e^{g(z)}) - g(z) = -\pi i + \sum_{m=2}^{\infty} (-1)^m \frac{\pi^m i^m}{m!} \frac{g^{(m)}(z)}{(g'(z))^m}. \tag{5.39}$$

Comparing with equations (5.32) and (5.33), we see that the error in using the approximate solution (z, z) in each of the two equations (5.32) and (5.33), is given by:

$$E_g^{\pm}(z) = |g(z \pm e^{g(z)}) - g(z) \mp \pi i| = \left| \sum_{m=2}^{\infty} (\pm 1)^m \frac{\pi^m i^m}{m!} \frac{g^{(m)}(z)}{(g'(z))^m} \right|. \tag{5.40}$$

Finally, using the triangle inequality, we get:

$$E_g^{\pm}(z) \leq \sum_{m=2}^{\infty} \frac{\pi^m}{m!} \left| \frac{g^{(m)}(z)}{(g'(z))^m} \right|. \tag{5.41}$$

We may now replace $E_g^{\pm}(z)$ with $E_g(z)$ because both $E_g^{+}(z)$ and $E_g^{-}(z)$ satisfy the same inequality (5.41). This completes the proof. $\square$

The case $g(z)$ is a linear polynomial

The case where $g(z)$ is a linear polynomial is particularly simple for we can easily solve the equations (5.32) and (5.33) directly. Indeed, if we write $g(z) = az + b$ with $a \neq 0$, we get:

$$ae^{az+b} = \pi i \tag{5.42}$$
$$-ae^{aw+b} = -\pi i. \tag{5.43}$$

That is:

$$e^{az} = \frac{\pi i}{ae^b} = e^{aw}. \tag{5.44}$$

Let $\mathrm{Log}(x)$ denote the principal logarithm of x. Then the solutions of (5.44), are given by:

$$(z, w) = \frac{1}{a}\left(\mathrm{Log}\left(\frac{\pi i}{ae^b}\right)(1,1) + 2\pi i(m,n)\right), \tag{5.45}$$

where $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. In particular, it follows that we have infinitely many solutions. Because we know that F_1 cannot have fixed points, nor 2-periodic points, it follows that all these, even those that lie on the diagonal Δ, are genuine 4-periodic points of F_1. We have shown:

Theorem 5.2.2. Let $F_1(z,w) = (e^{az+b} + w, z)$ be a transcendental Hénon map where $a \neq 0$. Then, F_1 admits infinitely many genuine 4-periodic points of the form given by equation (5.45).

The case $g(z) = z^d, d \geq 2$

We consider now the case $g(z) = z^d, d \geq 2$. That is, where g is a monomial of degree $d \geq 2$. This case will be illustrative for the generalization we do later when considering more general entire functions $g(z)$. The idea is to look for solutions of equation (5.34) in a sector-like region given by:

$$D := \left\{z \in \mathbb{C} : r < |z| < r+1, |\arg(z)| < \frac{\pi}{d}\right\}, \tag{5.46}$$

where $r > 0$ is large. We then use the form of g to directly give estimates for $|z|$ and $|\arg(z)|$. Finally, we show how these estimates give almost explicit formulae.

First, let us show that the error in using the approximate equation (5.34) in place of the equations (5.32) and (5.33), is small. We use Lemma 5.2.1. We will use the following asymptotic notation:

Let f and g be two functions. We will write:

$$f(z) = \mathcal{O}(g(z)) \text{ as } z \to z_0, \tag{5.47}$$

if and only if there exist some constant M and some $\delta > 0$ such that for all z with $|z - z_0| < \delta$, we have $|f(z)| \leq M|g(z)|$. If F and G are vector-valued maps, then by $F(z) = \mathcal{O}(G(z))$ as $z \to z_0$, we will mean that $f_j(z) = \mathcal{O}(g_j(z))$ as $z \to z_0$, where f_j is the jth component of F, and g_j is the jth component of G.

In particular, if $f(x)$ is some smooth function and $|\Delta x| < 1$ is small, the Taylor expansion of f near some point x_0, can be written to second order, using this asymptotic notation, as:

$$f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \mathcal{O}(f''(x_0)\Delta x^2) \tag{5.48}$$

as $\Delta x \to 0$.

Corollary 5.2.3. Let $g(z) = z^d$ with $d \geq 2$, and suppose that z solves the approximate equation (5.34) with $|z| = r$ and $r > 0$ large. Then the error in using the approximate solution of (5.34), is $\mathcal{O}\left(\frac{1}{r^d}\right)$ as $r \to \infty$. In particular, the error tends to 0 as $r \to \infty$.

Proof. By assumption we have: $e^{g(z)} = \frac{\pi i}{g'(z)}$. Because $g(z) = z^d$, it follows that: $g'(z) = dz^{d-1}$. Thus: $e^{mg(z)} = \frac{\pi^m i^m}{(g'(z))^m} = \frac{\pi^m i^m}{d^m z^{m(d-1)}}$. By further differentiation, we also have that: $g^{(m)}(z) = d(d-1)\cdots(d-m+1)z^{d-m}$. Of course, we here assume $m \leq d$. By Lemma 5.2.1, the error then satisfies:

$$\begin{aligned} E_g(z) &\leq \sum_{m=2}^{d} \frac{\pi^m}{m!} \left| \frac{d(d-1)\cdots(d-m+1)z^{d-m}}{d^m z^{m(d-1)}} \right| \\ &= \sum_{m=2}^{d} \frac{\pi^m}{m!} \frac{(d-1)\cdots(d-m+1)}{d^{m-1}} r^{d(1-m)}. \end{aligned} \tag{5.49}$$

Because $m \geq 2$, it follows that $1 - m \leq -1$, and therefore that:

$$E_g(z) = \mathcal{O}\left(r^{d(1-2)}\right) = \mathcal{O}\left(\frac{1}{r^d}\right) \tag{5.50}$$

as $r \to \infty$. This completes the proof. □

Lemma 5.2.4. Let $g(z) = z^d$ with $d \geq 2$ and consider the approximate equation:

$$g'(z)e^{g(z)} = \pi i. \tag{5.37}$$

Then there exists $r_0 > 0$ large such that for all $r \geq r_0$, there exists $k \in \mathbb{N}$ such that:

$$\frac{r^d}{2\pi} - \frac{1}{4d} + 1 < k < \frac{(r+1)^d}{2\pi} - \frac{1}{4d} - 1, \tag{5.51}$$

and with such a k chosen and with $R := R(k) = 2\pi k + \frac{\pi}{2d}$, there exists some small $\varepsilon > 0$ such that equation (5.37) has solutions z of the form:

$$|z|^d = \frac{R + \varepsilon(1/d - 1)}{\cos(\varepsilon)} \tag{5.52}$$

$$d \arg(z) = \frac{\pi}{2} + \varepsilon. \tag{5.53}$$

In fact, $\varepsilon = \mathcal{O}\left(\frac{\log(R)}{R}\right)$ as $R \to \infty$.

Proof. The idea is to look for solutions $z \in D$, where D is given in equation (5.46). Because $g(z) = z^d$, the equation we want to solve, (5.37), becomes:

$$dz^{d-1}e^{z^d} = \pi i. \tag{5.54}$$

That is:

$$e^{z^d} = \frac{\pi i}{dz^{d-1}}. \tag{5.55}$$

Let $g(z) = u(z) + iv(z)$, so that $u = \Re(g)$ and $v = \Im(v)$. We take the modulus of equation (5.55) and get:

$$e^u = \frac{\pi}{d|z|^{d-1}}. \tag{5.56}$$

We note that equation (5.55) can be written:

$$e^{u(z)+iv(z)} = \frac{\pi}{d|z|^{d-1}} e^{i \arg\left(\frac{\pi i}{dz^{d-1}}\right)}. \tag{5.57}$$

Using equation (5.56), this gives:

$$v(z) = \arg\left(\frac{\pi i}{dz^{d-1}}\right) = \arg\left(\frac{i}{z^{d-1}}\right). \tag{5.58}$$

The idea is to find estimates for equation (5.56) and equation (5.58). Let us start with equation (5.56). From the definition of D, we have an estimate on $|z|$, namely: $r < |z| < r + 1$. This therefore gives an estimate for $d|z|^{d-1}$. Namely:

$$dr^{d-1} < d|z|^{d-1} < d(r+1)^{d-1}. \tag{5.59}$$

Hence:

$$\frac{\pi}{d(r+1)^{d-1}} < \frac{\pi}{d|z|^{d-1}} < \frac{\pi}{dr^{d-1}}. \tag{5.60}$$

As we are considering equation (5.56), this provides an estimate for e^u. Namely:

$$\frac{\pi}{d(r+1)^{d-1}} < e^u < \frac{\pi}{dr^{d-1}}. \tag{5.61}$$

Taking logarithms we get:

$$\log\frac{\pi}{d} - (d-1)\log(r+1) < u < \log\frac{\pi}{d} - (d-1)\log r. \tag{5.62}$$

Now, recall that $u(z) = \Re(g(z)) = \Re(z^d)$. Let us write $z = |z|e^{i\arg(z)}$. Then we get: $g(z) = z^d = |z|^d e^{id\arg(z)}$. Thus, $u(z) = |z|^d\cos(d\arg(z))$. We substitute this into equation (5.62) and find:

$$\log\frac{\pi}{d} - (d-1)\log(r+1) < |z|^d\cos(d\arg(z)) < \log\frac{\pi}{d} - (d-1)\log r. \tag{5.63}$$

We have $|z|^d \in (r^d, (r+1)^d)$. Therefore, for sufficiently large r, we have that $|z|^d$ is significantly greater than $\log r$. It follows that in order for the estimate in equation (5.63) to hold, we must have $\cos(d\arg(z))$ very close to 0, but with negative sign. The negative sign comes from $-(d-1)\log r < 0$, which again follows from $(d-1)\log r > 0$. Hence $d\arg(z)$ should be very close to $\frac{\pi}{2}$, but just slightly above. Say, $d\arg(z) = \frac{\pi}{2} + \varepsilon$ where $\varepsilon > 0$ is very small and to be specified later. With $d\arg(z) = \frac{\pi}{2} + \varepsilon$, we find then:

$$v(z) = |z|^d\sin(d\arg(z)) = |z|^d\sin\left(\frac{\pi}{2} + \varepsilon\right) = |z|^d\cos(\varepsilon). \tag{5.64}$$

Substituting this into equation (5.58) and solving for $|z|^d$, yields:

$$|z|^d = \frac{\arg\left(\frac{i}{z^{d-1}}\right)}{\cos(\varepsilon)} = \frac{\arg(i) - (d-1)\arg(z)}{\cos(\varepsilon)}. \tag{5.65}$$

We have that $(d-1)\arg(z) = \frac{d-1}{d}d\arg(z) = \frac{d-1}{d}\left(\frac{\pi}{2} + \varepsilon\right)$, and we may set $\arg(i) = \frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$. Substituting these expressions in equation (5.65), gives:

$$\begin{aligned} |z|^d &= \frac{\frac{\pi}{2} + 2\pi k - \frac{d-1}{d}\left(\frac{\pi}{2} + \varepsilon\right)}{\cos(\varepsilon)} = \frac{\frac{\pi}{2} + 2\pi k + \frac{1-d}{d}\left(\frac{\pi}{2} + \varepsilon\right)}{\cos(\varepsilon)} \\ &= \frac{\frac{\pi}{2}\left(1 + \frac{1}{d} - 1\right) + 2\pi k + \frac{1-d}{d}\varepsilon}{\cos(\varepsilon)} \\ &= \frac{\frac{\pi}{2d} + 2\pi k + \varepsilon\left(\frac{1}{d} - 1\right)}{\cos(\varepsilon)}. \end{aligned} \tag{5.66}$$

This makes sense as long as k is chosen sufficiently large and positive. In fact, we may even provide an estimate for k. Recall that $|z|^d \in (r^d, (r+1)^d)$. Using equation (5.66), we then find the following estimate for $2\pi k$:

$$r^d\cos(\varepsilon) - \frac{\pi}{2d} + \varepsilon\left(1 - \frac{1}{d}\right) < 2\pi k < (r+1)^d\cos(\varepsilon) - \frac{\pi}{2d} + \varepsilon\left(1 - \frac{1}{d}\right). \tag{5.67}$$

That is, the following estimate for k:

$$\frac{r^d}{2\pi}\cos(\varepsilon) - \frac{1}{4d} + \frac{\varepsilon}{2\pi}\left(1 - \frac{1}{d}\right) < k < \frac{(r+1)^d}{2\pi}\cos(\varepsilon) - \frac{1}{4d} + \frac{\varepsilon}{2\pi}\left(1 - \frac{1}{d}\right). \tag{5.68}$$

It is clear that for sufficiently large r, k becomes large as well. Indeed, the length of the interval in which k lies according to (5.68), is given by:

$$\frac{(r+1)^d - r^d}{2\pi}\cos(\varepsilon) \geq \frac{dr^{d-1}}{2\pi}\cos(\varepsilon), \tag{5.69}$$

which is large for large r. We would like to find an estimate for k which is independent of ε.

For sufficiently small ε, $\cos(\varepsilon)$ is very close to 1, and so it is clear that for sufficiently small $\varepsilon > 0$, we have:

$$\frac{r^d}{2\pi}\cos(\varepsilon) + \frac{\varepsilon}{2\pi}\left(1 - \frac{1}{d}\right) < \frac{r^d}{2\pi} + 1 \tag{5.70}$$

$$\frac{(r+1)^d}{2\pi}\cos(\varepsilon) + \frac{\varepsilon}{2\pi}\left(1 - \frac{1}{d}\right) > \frac{(r+1)^d}{2\pi} - 1. \tag{5.71}$$

Using (5.70) and (5.71), we propose the following estimate for k:

$$\frac{r^d}{2\pi} - \frac{1}{4d} + 1 < k < \frac{(r+1)^d}{2\pi} - \frac{1}{4d} - 1. \tag{5.72}$$

We need to check that k stays inside the interval determined by (5.68). That is, that adding the 1 on the left-hand side of (5.72), does not make k jump past the right-hand side in (5.68), and similarly that subtracting 1 on the right-hand side of (5.72), does not make k fall below the left-hand side of (5.68). It suffices to show that the length of interval determined by (5.72), is greater than 2, say 4 for good measure. The length of the interval determined by (5.72), is given by:

$$\frac{(r+1)^d - r^d}{2\pi} - 2 > \frac{dr^{d-1}}{2\pi} - 2. \tag{5.73}$$

This is clearly larger than 4 for sufficiently large r. We have found an estimate for k coinciding with (5.51) in the statement of the lemma. We also note that our choice for $|z|^d$ and $d\arg(z)$ up to now, coincide with (5.52) and (5.53) in the lemma statement.

We need to to prove that our z solves $g'(z)e^{g(z)} = \pi i$ for some small ε with $\varepsilon = \mathcal{O}(\log(R)/R)$ as $R \to \infty$, and where R is as defined in the lemma statement.

The equation we want to solve, is equation (5.57). By construction of $|z|^d$ and $d\arg(z)$, the part $e^{iv(z)} = e^{i\arg\left(\frac{\pi i}{dz^{d-1}}\right)}$, is fulfilled. Hence we are left with:

$$e^u = \frac{\pi}{d|z|^{d-1}}. \tag{5.56}$$

Taking logarithms, and invoking $u = u(z) = z^d\cos(d\arg(z))$, equation (5.66), $d\arg(z) = \frac{\pi}{2} + \varepsilon$, and $R(k) = 2\pi k + \frac{\pi}{2d}$, we find:

$$\begin{aligned} u &= |z|^d\cos(d\arg(z)) \\ &= \frac{R(k) - \varepsilon(1-1/d)}{\cos(\varepsilon)}\cos\left(\frac{\pi}{2} + \varepsilon\right) \\ &= \frac{R(k) - \varepsilon\,(1-1/d)}{\cos(\varepsilon)}(-1)\sin(\varepsilon) \\ &= \log\frac{\pi}{d} - (d-1)\log|z| \\ &= \log\frac{\pi}{d} - \frac{d-1}{d}\log|z|^d \\ &= \log\frac{\pi}{d} - \frac{d-1}{d}\log\left(\frac{R(k) - \varepsilon(1-1/d)}{\cos(\varepsilon)}\right). \end{aligned} \tag{5.74}$$

We note that $-\frac{\sin(\varepsilon)}{\cos(\varepsilon)}$ can be changed to $-\tan(\varepsilon)$. We get:

$$-(R(k) - \varepsilon(1-1/d))\tan(\varepsilon) = \log\frac{\pi}{d} - \frac{d-1}{d}\log\left(\frac{R(k) - \varepsilon(1-1/d)}{\cos(\varepsilon)}\right) \tag{5.75}$$

We assume r very large. Then k is also very large and hence so is R. We want to show that there exists some $\varepsilon > 0$ which solves equation (5.75). By construction, we know that ε should be small. Let us define the two functions:

$$f(x) := -(R(k) + x(1-1/d))\tan(x) \tag{5.76}$$

$$h(x) := \log\frac{\pi}{d} - \frac{d-1}{d}\log\left(\frac{R(k) + x(1-1/d)}{\cos(x)}\right). \tag{5.77}$$

We want to consider roots of $f(\varepsilon) - h(\varepsilon)$ for $\varepsilon \geq 0$. It is easy to see that f and h are well-defined for $\varepsilon = 0$. Also, $f(0) = 0$ and $-h(0) > 0$. On the other hand, as $R(k)$ is much greater than $\log(R(k))$ for sufficiently large R, and as $\tan(\varepsilon) \approx \varepsilon$ for small ε, it follows that for not too small $0 < \varepsilon < \pi/2$, we have that $f(\varepsilon) - h(\varepsilon) < 0$. Clearly, f and h are continuous functions with respect to ε on $\varepsilon \in [0, \pi/2)$. Hence, by the intermediate-value theorem, there is some $\varepsilon \in (0, \pi/2)$ small, such that $f(\varepsilon) - h(\varepsilon) = 0$. That is, $f(\varepsilon) = h(\varepsilon)$. But this is precisely equation (5.75). This proves the existence of a solution.

It remains to show $\varepsilon = \mathcal{O}(\log(R)/R)$ as $R \to \infty$. We may assume, by choosing r, and hence R, large, that ε is so small that $\tan(\varepsilon)$ can be replaced by ε, $\cos(\varepsilon)$ by 1, and ε^2 by 0. Then (5.75) becomes:

$$-\varepsilon R = \log \frac{\pi}{d} - \frac{d-1}{d} \log(R). \tag{5.78}$$

For large R, we can neglect the term $\log \frac{\pi}{d}$. This gives $\varepsilon = \mathcal{O}\left(\frac{\log(R)}{R}\right)$ as $R \to \infty$, as required. This finally proves the lemma. □

As a corollary to Lemma 5.2.4, we have that the approximate equation (5.34), has infinitely many solutions in the case that $g(z) = z^d$ with $d \geq 2$. The idea is that we can choose solutions with arbitrarily large modulus:

Corollary 5.2.5. Let $g(z) = z^d, d \geq 2$. Then the approximate equation (5.34), has infinitely many solutions with arbitrarily large modulus.

Proof. Let $r > 0$. For r sufficiently large, we know by Lemma 5.2.4 that there exists a solution z with $r < |z| < r + 1$. This is true for also larger r. Hence, we can choose a sequence $r_n \uparrow \infty$, say with $n \geq 1$ and $r_1 \geq r$. For each r_n, Lemma 5.2.4 provides a solution z_n. Thus we get an infinite sequence $\{z_n\}_{n \geq 1}$ of solutions. This proves the assertion. □

However, the approximate equation $g'(z)e^{g(z)} = \pi i$, is *not* the solution we really want to solve. What we really want to solve, is the system of equations given by the two equations (5.32) and (5.33). Furthermore, our solutions up to now all lie on the diagonal Δ. We must expect that the general solutions of the equations (5.32) and (5.33), may lie off the diagonal Δ. Let us define:

$$G(z, w) := \begin{bmatrix} G_1(z, w) \\ G_2(z, w) \end{bmatrix} = \begin{bmatrix} g\left(w + e^{g(z)}\right) - g(w) \\ g\left(z - e^{g(w)}\right) - g(z) \end{bmatrix}. \tag{5.79}$$

Then we want to search for solutions (z, w) not necessarily on Δ, such that:

$$G(z, w) = (\pi i, -\pi i)^T. \tag{5.80}$$

Here, x^T denotes the *transpose* of x. The idea is the following:

let z_0 be such that $g'(z_0)e^{g(z_0)} = \pi i$. Then, although $G(z_0, z_0)$ is not equal to $(\pi i, -\pi i)^T$, it comes quite close. Indeed, by Corollary 5.2.3, we have:

$$|G(z_0, z_0) - (\pi i, -\pi i)^T| = \mathcal{O}\left(\frac{1}{|z_0|^d}\right) \tag{5.81}$$

as $|z_0| \to \infty$. The expression $\frac{1}{|z_0|^d}$ is small for $|z_0| := r > 0$ large. Let $\delta > 0$ be small and let $(z, w) \in B_\delta(z_0, z_0)$, where $B_\rho(x)$ denotes the open ball centred at x

with radius ρ. For small δ, (z, w) is close to (z_0, z_0). If now the image under G of the small ball $B_\delta(z_0, z_0)$, is not too small, say, so large that $G(B_\delta(z_0, z_0))$ contains a ball centred at $G(z_0, z_0)$ with radius R greater than $\mathcal{O}\left(\frac{1}{r^d}\right)$ as $r \to \infty$, then the image $G(B_\delta(z_0, z_0))$ "swallows" the point $(\pi i, -\pi i)^T$. Thus there must exist some point $(z, w) \in B_\delta(z_0, z_0)$ such that $G(z, w) = (\pi i, -\pi i)^T$. This provides the existence of a solution to equation (5.80). In fact, infinitely many with courtesy of Corollary 5.2.5.

Instead of the image of $B_\delta(z_0, z_0)$ under G, we consider the image of $B_\delta(z_0, z_0)$ under the *linear part* of G at (z_0, z_0), which we denote by $G_L^{(z_0,z_0)}$, and which is defined by:

$$G_L^{(z_0,z_0)}(z, w) = G(z_0, z_0) + G'(z_0, z_0) \begin{bmatrix} z - z_0 \\ w - z_0 \end{bmatrix}. \tag{5.82}$$

We check that the images $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$ and $G(B_\delta(z_0, z_0))$, are not far from each other when δ is small. To do this, we use a Taylor expansion for each of the components of G about (z_0, z_0). Let $j \in \{1, 2\}$. Then we get:

$$\begin{aligned} G_j(z, w) = G_j(z_0, z_0) &+ \frac{\partial G_j(z_0, z_0)}{\partial z}(z - z_0) + \frac{\partial G_j(z_0, z_0)}{\partial w}(w - z_0) \\ &+ \frac{1}{2}\mathcal{O}\left(\frac{\partial^2 G_j(z_0, z_0)}{\partial z^2}(z - z_0)^2 + \frac{\partial^2 G_j(z_0, z_0)}{\partial w^2}(w - z_0)^2\right) \\ &+ \mathcal{O}\left(\frac{\partial^2 G_j(z_0, z_0)}{\partial z \partial w}(z - z_0)(w - z_0)\right) \end{aligned} \tag{5.83}$$

as $\delta \to 0$. Thus:

$$
\begin{aligned}
G(z,w) &= G(z_0,z_0) + \begin{bmatrix} \frac{\partial G_1(z_0,z_0)}{\partial z}(z-z_0) + \frac{\partial G_1(z_0,z_0)}{\partial w}(w-z_0) \\ \frac{\partial G_2(z_0,z_0)}{\partial z}(z-z_0) + \frac{\partial G_2(z_0,z_0)}{\partial w}(w-z_0) \end{bmatrix} \\
&\quad + \frac{1}{2}\mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_1(z_0,z_0)}{\partial w^2}(w-z_0)^2 \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_2(z_0,z_0)}{\partial w^2}(w-z_0)^2 \end{bmatrix} \right) \\
&\quad + \mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \end{bmatrix} \right), \qquad \text{as } \delta \to 0 \\
&= G(z_0,z_0) + G'(z_0,z_0) \begin{bmatrix} z - z_0 \\ w - z_0 \end{bmatrix} \\
&\quad + \frac{1}{2}\mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_1(z_0,z_0)}{\partial w^2}(w-z_0)^2 \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_2(z_0,z_0)}{\partial w^2}(w-z_0)^2 \end{bmatrix} \right) \\
&\quad + \mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \end{bmatrix} \right), \qquad \text{as } \delta \to 0 \\
&= G_L^{(z_0,z_0)}(z,w) + \frac{1}{2}\mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_1(z_0,z_0)}{\partial w^2}(w-z_0)^2 \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z^2}(z-z_0)^2 + \frac{\partial^2 G_2(z_0,z_0)}{\partial w^2}(w-z_0)^2 \end{bmatrix} \right) \\
&\quad + \mathcal{O}\left(\begin{bmatrix} \frac{\partial^2 G_1(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \\ \frac{\partial^2 G_2(z_0,z_0)}{\partial z \partial w}(z-z_0)(w-z_0) \end{bmatrix} \right), \qquad \text{as } \delta \to 0. \qquad (5.84)
\end{aligned}
$$

Because $\overline{B_\delta(z_0,z_0)}$ is compact, by continuity, the derivatives $\frac{\partial G_j}{\partial z}$, $\frac{\partial G_j}{\partial w}$, $\frac{\partial^2 G_j}{\partial z^2}$, $\frac{\partial^2 G_j}{\partial w^2}$, and $\frac{\partial^2 G_j}{\partial z \partial w}$, $j = 1, 2$, are all bounded at (z_0, z_0). Hence we get:

$$
|G(z,w) - G_L^{(z_0,z_0)}(z,w)| = \mathcal{O}\left(\max\left\{ \left\| \begin{bmatrix} (z-z_0)^2 \\ (z-z_0)^2 \end{bmatrix} \right\|, \left\| \begin{bmatrix} (w-w_0)^2 \\ (w-w_0)^2 \end{bmatrix} \right\|, \left\| \begin{bmatrix} (z-z_0)(w-z_0) \\ (z-z_0)(w-z_0) \end{bmatrix} \right\| \right\} \right) \qquad (5.85)
$$

as $\delta \to 0$. Now, we have:

$$
\left\| \begin{bmatrix} (z-z_0)^2 \\ (z-z_0)^2 \end{bmatrix} \right\| = \sqrt{2|z-z_0|^4} = \sqrt{2}|z-z_0|^2 \qquad (5.86)
$$

$$
\left\| \begin{bmatrix} (w-z_0)^2 \\ (w-z_0)^2 \end{bmatrix} \right\| = \sqrt{2|w-z_0|^2} = \sqrt{2}|w-z_0|^2 \qquad (5.87)
$$

$$
\left\| \begin{bmatrix} (z-z_0)(w-z_0) \\ (z-z_0)(w-z_0) \end{bmatrix} \right\| = \sqrt{2|z-z_0|^2 \cdot |w-z_0|^2} = \sqrt{2}|z-z_0| \cdot |w-z_0|. \qquad (5.88)
$$

Because $(z, w) \in B_\delta(z_0, z_0)$, we find:

$$
\sqrt{|z-z_0|^2 + |w-z_0|^2} < \delta. \qquad (5.89)
$$

That is, after squaring:

$$|z - z_0|^2 + |w - z_0|^2 < \delta^2. \tag{5.90}$$

In particular, we have $|z - z_0|^2 < \delta^2$ and $|w - z_0|^2 < \delta^2$. Consequently, all the right-hand sides in the equations (5.86) - (5.88), are less than $\sqrt{2}\delta^2$. Then equation (5.85) gives:

$$|G(z, w) - G_L^{(z_0,z_0)}(z, w)| = \mathcal{O}\left(\sqrt{2}\delta^2\right) \tag{5.91}$$

as $\delta \to 0$. This shows that the distance between points with common pre-images, in the images $G(B_\delta(z_0, z_0))$ and $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$, are $\mathcal{O}(\sqrt{2}\delta^2)$ as $\delta \to 0$. In other words, we can consider the image $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$ instead of the image $G(B_\delta(z_0, z_0))$, and the error in doing so, is $\mathcal{O}(\sqrt{2}\delta^2)$ as $\delta \to 0$. As the distance from $G(z_0, z_0)$ to the point $(\pi i, -\pi i)^T$, is $\mathcal{O}(1/r^d)$ as $r \to \infty$, it suffices to show that $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$ contains a ball centred at $G(z_0, z_0)$ with radius greater than $\mathcal{O}\left(\frac{1}{r^d} + \sqrt{2}\delta^2\right)$ as $(r, \delta) \to (\infty, 0)$. Now, we can choose δ so small that:

$$\sqrt{2}\delta^2 < \frac{1}{r^d}. \tag{5.92}$$

That is:

$$\delta < \sqrt{\frac{1}{\sqrt{2}r^d}}. \tag{5.93}$$

Then it suffices to show that $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$ contains a ball centred at $G(z_0, z_0)$ with radius greater than $\mathcal{O}\left(\frac{2}{r^d}\right)$ as $r \to \infty$. We calculate:

$$\begin{aligned} |G_L^{(z_0,z_0)}(z, w) - G(z_0, z_0)| &= \left| G'(z_0, z_0) \begin{bmatrix} z - z_0 \\ w - z_0 \end{bmatrix} \right| \\ &= \left| \begin{bmatrix} \frac{\partial G_1(z_0,z_0)}{\partial z}(z - z_0) + \frac{\partial G_1(z_0,z_0)}{\partial w}(w - z_0) \\ \frac{\partial G_2(z_0,z_0)}{\partial z}(z - z_0) + \frac{\partial G_2(z_0,z_0)}{\partial w}(w - z_0) \end{bmatrix} \right|. \end{aligned} \tag{5.94}$$

Let us write $G_{L,j}^{(z_0,z_0)}$ for the jth component of $G_L^{(z_0,z_0)}$, and define:

$$W_j := G_{L,j}^{(z_0,z_0)} - G_j(z_0, z_0). \tag{5.95}$$

We also define:

$$a := \frac{\partial G_1(z_0, z_0)}{\partial z} \tag{5.96}$$
$$b := \frac{\partial G_1(z_0, z_0)}{\partial w} \tag{5.97}$$
$$c := \frac{\partial G_2(z_0, z_0)}{\partial z} \tag{5.98}$$
$$d := \frac{\partial G_2(z_0, z_0)}{\partial w}. \tag{5.99}$$

Then equation (5.94) gives:

$$|W_1(z,w)| = |a(z-z_0) + b(w-z_0)| \tag{5.100}$$
$$|W_2(z,w)| = |c(z-z_0) + d(w-z_0)|. \tag{5.101}$$

We use the reverse triangle inequality on the two equations (5.100) and (5.101). This yields:

$$\begin{aligned} |W_1(z,w)| &\geq ||a|\cdot|z-z_0| - |b|\cdot|w-z_0|| \\ &= \max\{|a|\cdot|z-z_0| - |b|\cdot|w-z_0|, |b|\cdot|w-z_0| - |a|\cdot|z-z_0|\} \end{aligned} \tag{5.102}$$
$$\begin{aligned} |W_2(z,w)| &\geq ||c|\cdot|z-z_0| - |d|\cdot|w-z_0|| \\ &= \max\{|c|\cdot|z-z_0| - |d|\cdot|w-z_0|, |d|\cdot|w-z_0| - |c|\cdot|z-z_0|\}. \end{aligned} \tag{5.103}$$

Let $(z,w) \in \partial B_\delta(z_0,z_0)$, and write $|z-z_0| = k\delta$ and $|w-z_0| = l\delta$. Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Then $(k,l) \in \partial\mathbb{D}$. Here we of course, think of $\mathbb{D}$ as a subset in the plane $\mathbb{R}^2$: $\mathbb{R}^2 \supseteq \partial\mathbb{D} \ni (k,l)$. We also note that $(k,l) \in \mathbb{R}_+^2$ where $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. That is, $k,l \geq 0$. Then the two inequalities (5.102) and (5.103), become:

$$|W_1(z,w)| \geq ||a|k - |b|l|\,\delta \tag{5.104}$$
$$|W_2(z,w)| \geq ||c|k - |d|l|\,\delta. \tag{5.105}$$

Suppose now that one of the members of $\{|a|,|b|\}$ and one of the members of $\{|c|,|d|\}$, is so large that we are able to show: $|W_1(z,w)|^2 + |W_2(z,w)|^2 > \mathcal{O}\left(\frac{2}{r^d}\right)^2$ as $r \to \infty$, where we recall that $(z,w) \in \partial B_\delta(z_0,z_0)$. Then, because the images of compact sets under continuous functions, are compact sets, and because holomorphic maps are open maps, it follows that:

$$G_L^{(z_0,z_0)}(\partial B_\delta(z_0,z_0)) = \partial G_L^{(z_0,z_0)}(B_\delta(z_0,z_0)). \tag{5.106}$$

Thus, the boundary of the image set $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$, lies at a distance greater than $\mathcal{O}\left(\frac{2}{r^d}\right)$ as $r\to\infty$, from the point $G(z_0,z_0)$. We must make sure that the *whole* image set $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$, contains a ball centred at $G(z_0,z_0)$ of a sufficiently large radius greater than $\mathcal{O}\left(\frac{2}{r^d}\right)$ as $r\to\infty$. A priori this may not be the case. There are two things which may go wrong. The first potential problem is that the image set $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$, is an annulus-like region centred at $G(z_0,z_0)$. See Figure 5.1.

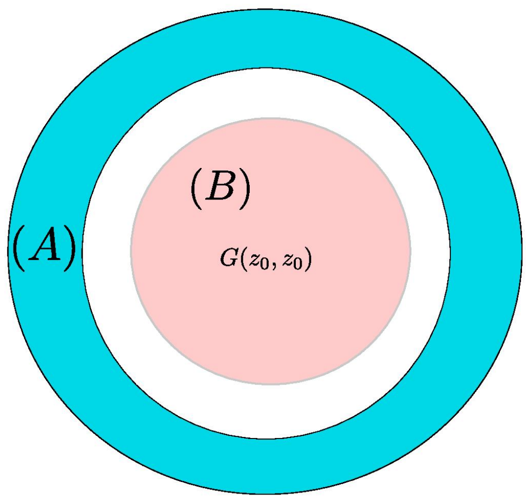

Figure 5.1: The figure illustrates the case when the image set $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$, is an annulus-like region. The annulus-like region is given by (A) in blue. The (B) in light red depicts a ball of some sufficiently large radius greater than $\mathcal{O}(2/r^d)$ as $r\to\infty$ centred at the center point $G(z_0,z_0)$. We see that (A) does not contain (B), but the distance from the center point $G(z_0,z_0)$ to the boundary of (A) is greater than the radius if (B).

Then the boundary of the image set $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$ can still be sufficiently far away from $G(z_0,z_0)$, while the image set itself fails to contain a whole ball centred at $G(z_0,z_0)$ of radius greater than $\mathcal{O}\left(\frac{2}{r^d}\right)$ as $r\to\infty$. However, this is impossible, for $G_L^{(z_0,z_0)}$ is continuous and $B_\delta(z_0,z_0)$ is connected, and therefore so must $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$ be. An annulus-like region would disconnect the center point $G(z_0,z_0)$ from the rest of the image set. Hence, at worst, $G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$ is almost an annulus-like region centred at $G(z_0,z_0)$, with a small sector-like region which "swoops" in to include the center point $G(z_0,z_0)$. See Figure 5.2.

This is the second potential problem. But also this is impossible, for the boundary of such a sector-like region would belong to the boundary $\partial G_L^{(z_0,z_0)}(B_\delta(z_0,z_0))$,

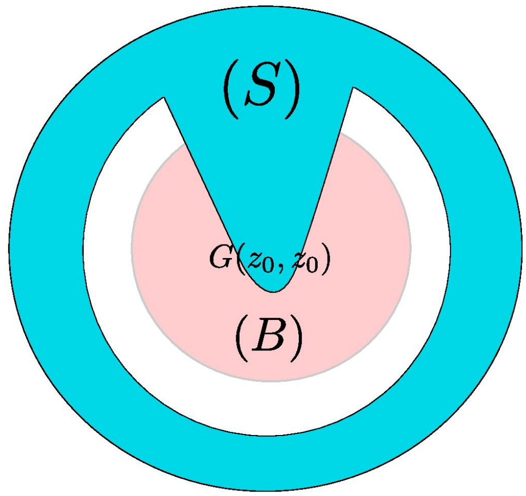

Figure 5.2: The figure illustrates the case that the image set $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$, is an annulus-like region with a small sector that swoops in to include the center point $G(z_0, z_0)$. This region is depicted as (S) in blue. The (B) in light red is as in Figure 5.1, and depicts a ball centred at $G(z_0, z_0)$. We see that either the small sector fails to envelope (B), in which case parts of its boundary will be at a distance closer to the center points $G(z_0, z_0)$ that the radius of (B), or it follows that (S) does contain (B).

which by the assumption that:

$$|W_1(z, w)|^2 + |W_2(z, w)|^2 > \mathcal{O}\left(\frac{2}{r^d}\right)^2 \tag{5.107}$$

as $r \to \infty$ and $(z, w) \in \partial B_\varepsilon(z_0, z_0)$, cannot come too close to the center point $G(z_0, z_0)$. We conclude that the image set $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$, must be, *close to the center point* $G(z_0, z_0)$, a ball-like region centred at $G(z_0, z_0)$. Consequently, if the inequalities (5.104) and (5.105) hold on the boundary $\partial B_\delta(z_0, z_0)$ in a such a way that also (5.107) holds, then the whole image set $G_L^{(z_0,z_0)}(B_\delta(z_0, z_0))$, ***does*** contain a ball centred at $G(z_0, z_0)$ with sufficient radius larger than $\mathcal{O}\left(\frac{2}{r^d}\right)$ as $r \to \infty$. Thus it remains to verify the inequality (5.107). We will need estimates on $|a|, |b|, |c|, |d|$:

Lemma 5.2.6. Let a, b, c, d be defined according to equations (5.96) - (5.99), where G is defined in equation (5.79). Then:

$$|a| = \mathcal{O}\left(|z_0|^{d-1}\right) = |d|, \qquad \text{as } |z_0| \to \infty \tag{5.108}$$

$$|b| = \mathcal{O}\left(\frac{1}{|z_0|}\right) = |c|, \qquad \text{as } |z_0| \to \infty. \tag{5.109}$$

Proof. We simply compute $|a|, |b|, |c|$, and $|d|$ using the definition for a, b, c, d, and the fact that $g'(z_0)e^{g(z_0)} = \pi i$, where $g(z) = z^d$. That is:

$$dz_0^{d-1}e^{z_0^d} = \pi i. \tag{5.110}$$

Recall the definition for G_j given by (5.79), $j = 1, 2$. We get:

$$\begin{aligned}
a &= \frac{\partial G_1(z_0, z_0)}{\partial z} = \frac{\partial}{\partial z}\Big|_{(z_0,z_0)} \left\{g\left(w + e^{g(z)}\right) - g(w)\right\} \\
&= \left[g'(w + e^{g(z)})e^{g(z)}g'(z)\right]\Big|_{(z_0,z_0)} \\
&= g'\left(z_0 + e^{g(z_0)}\right) e^{g(z_0)} g'(z_0) && (5.111) \\
b &= \frac{\partial G_1(z_0, z_0)}{\partial w} = \frac{\partial}{\partial w}\Big|_{(z_0,z_0)} \left\{g\left(w + e^{g(z)}\right) - g(w)\right\} \\
&= \left[g'(w + e^{g(z)}) - g'(w)\right]\Big|_{(z_0,z_0)} \\
&= g'\left(z_0 + e^{g(z_0)}\right) - g'(z_0) && (5.112) \\
c &= \frac{\partial G_2(z_0, z_0)}{\partial z} = \frac{\partial}{\partial z}\Big|_{(z_0,z_0)} \left\{g\left(z - e^{g(w)}\right) - g(z)\right\} \\
&= \left[g'\left(z - e^{g(w)}\right) - g'(z)\right]\Big|_{(z_0,z_0)} \\
&= g'\left(z_0 - e^{g(z_0)}\right) - g'(z_0) && (5.113) \\
d &= \frac{\partial G_2(z_0, z_0)}{\partial w} = \frac{\partial}{\partial w}\Big|_{(z_0,z_0)} \left\{g\left(z - e^{g(w)}\right) - g(z)\right\} \\
&= \left[g'\left(z - e^{g(w)}\right)(-e^{g(w)})g'(w)\right]\Big|_{(z_0,z_0)} \\
&= -g'\left(z_0 - e^{g(z_0)}\right) e^{g(z_0)} g'(z_0). && (5.114)
\end{aligned}$$

Because we have $g(z) = z^d$, $g'(z) = dz^{d-1}$, and $e^{g(z_0)} = \frac{\pi i}{g'(z_0)}$, we find:

$$\begin{aligned}
a &= d\left(z_0 + \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} \cdot \frac{\pi i}{dz_0^{d-1}} \cdot dz_0^{d-1} \\
&= \pi i d\left(z_0 + \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} && (5.115) \\
b &= d\left(z_0 + \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} - dz_0^{d-1} && (5.116) \\
c &= d\left(z_0 - \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} - dz_0^{d-1} && (5.117) \\
d &= d\left(z_0 - \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} \cdot (-1) \cdot \frac{\pi i}{dz_0^{d-1}} \cdot dz_0^{d-1} \\
&= -\pi i d\left(z_0 - \frac{\pi i}{dz_0^{d-1}}\right)^{d-1} && (5.118)
\end{aligned}$$

Recall that $|z_0| = r > 0$ is large. Hence from equations (5.115) - (5.118), we get:

$$|a| = \mathcal{O}\left(|z_0|^{d-1}\right) = |d|, \qquad \text{as } |z_0| \to \infty, \tag{5.108}$$

$$|b| = \mathcal{O}\left(\left|\frac{z_0^{d-2}}{z_0^{d-1}}\right|\right) = \mathcal{O}\left(\frac{1}{|z_0|}\right) = |c|, \qquad \text{as } |z_0| \to \infty. \tag{5.119}$$

These are the required estimates. Alternatively, we could have used Taylor expansion. We illustrate this for b:

$$b = g'(z_0 + e^{g(z_0)}) - g'(z_0) = g''(z_0)e^{g(z_0)} + \mathcal{O}\left(g'''(z_0)e^{2g(z_0)}\right), \quad \text{as } |z_0| \to \infty. \tag{5.120}$$

We have $e^{g(z_0)} = \frac{\pi i}{g'(z_0)}$, so equation (5.120) becomes:

$$\begin{aligned} b &= g''(z_0)\frac{\pi i}{g'(z_0)} + \mathcal{O}\left(g'''(z_0)\frac{(-1)\pi^2}{[g'(z_0)]^2}\right), \quad \text{as } |z_0| \to \infty \\ &= \frac{\pi i d(d-1)z_0^{d-2}}{dz_0^{d-1}} + \mathcal{O}\left(-\frac{\pi^2 d(d-1)(d-2)z_0^{d-3}}{d^2 z_0^{2d-2}}\right), \qquad \text{as } |z_0| \to \infty \\ &= \frac{\pi i(d-1)}{z_0} + \mathcal{O}\left(-\frac{\pi^2(d-1)(d-2)}{dz_0^{d+1}}\right), \quad \text{as } |z_0| \to \infty, \end{aligned} \tag{5.121}$$

from which it is easy to see that $|b| = \mathcal{O}\left(\frac{1}{|z_0|}\right)$ as $|z_0| \to \infty$. In any event, the equations (5.108) and (5.119) already proves the assertion and we are done. □

Thus, for $|z_0|$ large, $|a|$ and $|d|$ are large, while $|b|$ and $|c|$ are small. Looking at (5.104) and (5.105), we therefore get:

$$|W_1(z, w)| \geq (|a|k - |b|l)\delta \tag{5.122}$$

$$|W_2(z, w)| \geq (|d|l - |c|k)\delta. \tag{5.123}$$

Because $|z_0| = r$ can be chosen as large as we want, we expect that we can make it so that $|W_1(z, w)|^2 + |W_2(z, w)|^2 > \mathcal{O}\left(\frac{2}{r^d}\right)^2$ as $r \to \infty$. We finally have our existence result:

Theorem 5.2.7. Let $g(z) = z^d, d \geq 2$, fix $\lambda \in (0, 1)$, and let z_0 be such that $g'(z_0)e^{g(z_0)} = \pi i$ and $|z_0| = r$, with $r > 0$ sufficiently so that we have $r^{3d-2} > \frac{4\sqrt{2}}{\lambda}$. Then equation (5.80): $G(z, w) = (\pi i, \pi i)^T$, where G is defined by equation (5.79), has a solution in $B_\delta(z_0, z_0)$ with $\sqrt{2}\delta^2 = \frac{\lambda}{r^d}$.

Proof. Using Lemma 5.2.6, we calculate:

$$
\begin{aligned}
|W_1(z,w)|^2 + |W_2(z,w)|^2 \geq & (|a|k - |b|l)^2\delta^2 + (|d|l - |c|k)^2\delta^2 \\
& = \left(|a|^2k^2 - 2|a|\cdot|b|kl + |b|^2l^2\right)\delta^2 \\
& \quad + \left(|c|^2k^2 - 2|c|\cdot|d|kl + |d|^2l^2\right)\delta^2 \\
& = \mathcal{O}\left(\left(|z_0|^{2d-2}k^2 - 2|z_0|^{d-2}kl + \frac{1}{|z_0|^2}l^2\right)\delta^2\right) \\
& \quad + \mathcal{O}\left(\left(|z_0|^{2d-2}l^2 - 2|z_0|^{d-2}kl + \frac{1}{|z_0|^2}k^2\right)\delta^2\right)
\end{aligned}
\tag{5.124}
$$

as $|z_0| \to \infty$. Because $2d-2 = 2(d-1) > d-1 > d-2$ for $d \geq 2$, and because $(k,l) \in \partial\mathbb{D}$, it follows that we have:

$$
|W_1(z,w)|^2 + |W_2(z,w)|^2 \geq \mathcal{O}\left(|z_0|^{2d-2}\right)(k^2+l^2)\delta^2 = \mathcal{O}(|z_0|^{2d-2})\delta^2 \tag{5.125}
$$

as $|z_0| \to \infty$. Thus, it is clear that we can choose $|z_0| = r$ sufficiently large such that:

$$
|W_1(z,w)|^2 + |W_2(z,w)|^2 > \mathcal{O}\left(\frac{2}{r^d}\right)^2 \tag{5.126}
$$

as $r \to \infty$. Indeed, we have $\delta^2 < \frac{1}{\sqrt{2}r^d}$ by assumption. Let us write $\delta^2 = \frac{\lambda}{\sqrt{2}r^d}$ where $\lambda \in (0,1)$ is fixed. Then, using equation (5.125), it suffices to choose r such that:

$$
r^{2d-2} \cdot \frac{\lambda}{\sqrt{2}r^d} > \frac{4}{r^{2d}}. \tag{5.127}
$$

That is:

$$
r^{2d-2-d+2d} = r^{3d-2} > \frac{4\sqrt{2}}{\lambda}. \tag{5.128}
$$

This is precisely the assumption in the theorem statement, and the proof is done. □

Corollary 5.2.8. Let $g(z) = z^d, g \geq 2$, and let $F_{-1}(z,w) = (e^{g(z)} + w, z)$. Then, F_1 has infinitely many genuine 4-periodic points. Moreover, infinitely many of these can be chosen close to the points of the form $(z_0, z_0) \in \Delta$, where z_0 is the solution given by Lemma 5.2.4.

Proof. Let $r > 0$ be large. For sufficiently large r, Lemma 5.2.4 provides a z_0 such that $g'(z_0)e^{g(z_0)} = \pi i$. If necessary, we can choose to make r even larger such that Theorem 5.2.7 provides a solution $(z, w) = (z_r, w_r) \in B_\delta(z_0, z_0)$, where $\sqrt{2}\delta \in (0, 1/r^d)$, and where $|z_0| = r$ with z_0 provided by Lemma 5.2.4 for this larger r. We assume we have chosen any fixed $\lambda \in (0, 1)$ as given in Theorem 5.2.7. But then we can repeat this argument with any larger r. Thus, let $\{r_n\}_{n\geq 1} \uparrow \infty$ be an infinite sequence with $r_1 > r$. Then for each r_n, Lemma 5.2.4 and Theorem 5.2.7 provides a solution $(z_n, w_n) \in \text{Fix}(F_{-1}^4)$. Thus we get an infinite sequence of 4-periodic points for F_{-1}, $\{(z_n, w_n)\}_{n\geq 1}$. To complete the proof, we notice that all these must be genuine 4-periodic points because $\text{Fix}(F_{-1}) = \text{Fix}(F_{-1}^2) = \emptyset$, and the only way any point $(z, w) \in \text{Fix}(F_{-1}^4)$ fails to be a genuine 4-periodic point, is that $(z, w) \in \text{Fix}(F_{-1}) \cup \text{Fix}(F_{-1}^2)$. □

The case $g(z)$ is a general non-constant entire function

We just showed that for $g(z) = z^d$ with $d \geq 2$, the transcendental Hénon map $F_{-1}(z, w) = (e^{g(z)} + w, z)$, admits infinitely many genuine 4-periodic points. We now use a similar approach and similar estimate methods used in proving this, to show that the same is true when g is allowed to be any general non-constant entire function. The polynomial case where g has degree $d \geq 2$, is completely analogous to the case where g is a monomial with degree d. In fact, we need only recall from chapter 1, Theorem 1.2.1, that if $g(z) = \sum_{j=0}^{d} b_j z^j$, with $b_d \neq 0$, then:

$$g(z) \sim b_d z^d, \qquad \text{as } |z| \to \infty. \tag{5.129}$$

In chapter 1, the result was stated as: $|g(z)| \sim |b_d| \cdot |z|^d$ as $|z| \to \infty$, but it is easy to see that the same is true without moduli. In other words, if we let $|z| \in (r, r+1)$, then by choosing r sufficiently large, we can make $g(z)$ as close as we want to $b_d z^d$. Finally, defining then a new variable ζ by $\zeta = \frac{z}{\lambda}$, where $\lambda^d = b_d$, we can almost reduce to the case monomial case $g(\zeta) = \zeta^d$. We say almost because $g(\zeta)$ is not quite equal to ζ^d. Instead, $g(\zeta) = \zeta^d(1+\delta)$, where $\delta \to 0$ uniformly with respect to ζ as $\tilde{r} \to \infty$, and where $|\zeta| \in (\tilde{r}, \tilde{r}+1)$. The previous estimate methods, however, carry through in a completely analogous manner as with the monomial case. Additionally, most of the proof for when g is transcendental (Lemma 5.2.9 below), can also be used in the case g is polynomial. See Remark 5.2.10.

We therefore focus on the case where g is a transcendental entire function. The idea is to look for solutions of the approximate equation (5.34) on the diagonal where g looks like a polynomial of high degree. We use the Wiman-Valiron method (Theorem 1.4.4) for this.

Thus, let $r > 0$ be large and let $N = N(r)$ denote the central index for g with respect to r. Let $M = M(r) = M(g, r) = \sup_{|z|=r} |g(z)|$ and let ζ with $|\zeta| = r$ be such that $|g(\zeta)| = M$. Let $\alpha > 1/2$. Then we know that for all z such that:

$$|z - \zeta| < \frac{r}{N^\alpha}, \tag{5.130}$$

we can write:

$$g(z) = \left(\frac{z}{\zeta}\right)^N g(\zeta)(1+\delta_0), \tag{5.131}$$

where $\delta_0 \to 0$ with respect to z as $r \to \infty$. We choose r outside the exceptional set in Wiman-Valiron's theorem (Theorem 1.4.4), with finite logarithmic measure. We also have:

$$g'(z) = N\frac{z^{N-1}}{\zeta^N} g(\zeta)(1+\delta_1), \tag{5.132}$$

where again $\delta_1 \to 0$ uniformly with respect to z as $r \to \infty$. The idea is to look for solutions for z of equation (5.34) where (5.130) holds.

Lemma 5.2.9. Let $g(z)$ be a transcendental entire function and let $\alpha \in (1, 2/1]$. Then, for a sufficiently large $r > 0$ outside some exceptional set with finite logarithmic measure, there exists a solution z to the equation $g'(z)e^{g(z)} = \pi i$, with $|z|$ close to r. In fact, z can be chosen of the form:

$$|z|^N \sim r^N \frac{2\pi k + \frac{\pi}{2N} + \left(\frac{1}{N} - 1\right)(\varepsilon - \arg(g(\zeta))) + \arg(\zeta) - \arg(g(\zeta))}{M\cos(\varepsilon)} \tag{5.133}$$

$$\arg(z) \sim \frac{\frac{\pi}{2} + \varepsilon - \arg(g(\zeta))}{N} + \arg(\zeta), \tag{5.134}$$

both as $r \to \infty$, where $N = N(r)$ is the central index for g, $M = M(g,r)$, ζ is such that $|g(\zeta)| = M$, $k \in \mathbb{N}$ is chosen from the interval $(\tilde{a}, \tilde{b})$, where the following asymptotic relations are satisfied:

$$\tilde{a} \sim \frac{M}{2\pi}\left(1 - \frac{1}{N^\alpha}\right)^N - \frac{1}{4N} + \frac{\left(\frac{1}{N} - 1\right)\arg(g(\zeta))}{2\pi} + \frac{\arg(g(\zeta)) - \arg(\zeta)}{2\pi} + 1 \tag{5.135}$$

$$\tilde{b} \sim \frac{M}{2\pi}\left(1 - \frac{1}{N^\alpha}\right)^N - \frac{1}{4N} + \frac{\left(\frac{1}{N} - 1\right)\arg(g(\zeta))}{2\pi} + \frac{\arg(g(\zeta)) - \arg(\zeta)}{2\pi} - 1 \tag{5.136}$$

as $r \to \infty$, and finally where we have $\varepsilon \sim \mathcal{O}\left(\frac{\log(R)}{R}\right)$ as $R \to \infty$, with R satisfying $R = R(k) \sim 2\pi k + \frac{\pi}{2N} - \left(\frac{1}{N} - 1\right)\arg(g(\zeta)) + \arg(\zeta) - \arg(g(\zeta))$ as $r \to \infty$.

Proof. Let $|\zeta| = r$ and let $|z| = \rho$. We look for z in a region determined by (5.130). This provides an estimate on $\rho = |z|$. The triangle inequality gives:

$$\rho = |z| = |(z-\zeta) + \zeta| \le |z - \zeta| + |\zeta| < \frac{r}{N^\alpha} + r = r\left(1 + \frac{1}{N^\alpha}\right), \tag{5.137}$$

and the reverse triangle inequality gives:

$$\rho = |z - \zeta + \zeta| \geq ||z - \zeta| - |\zeta|| = |\zeta| - |z - \zeta| > r - \frac{r}{N^\alpha} = r\left(1 - \frac{1}{N^\alpha}\right). \tag{5.138}$$

Combining (5.137) and (5.138), we get the following estimate for ρ:

$$r\left(1 - \frac{1}{N^\alpha}\right) < \rho < r\left(1 + \frac{1}{N^\alpha}\right). \tag{5.139}$$

We recall that N increases as r increases. Thus, for sufficiently large r, ρ becomes very close to r. Let $\theta = \arg(z)$ and let $\phi = \arg(\zeta)$. Then, if θ is sufficiently close to ϕ, we may assume (5.130) holds. Then equations (5.131) and (5.132) are available.

The equation we want to solve, is: $g'(z)e^{g(z)} = \pi i$. We follow the same approach as for the case $g(z) = z^d, d \geq 2$, and write $g(z) = u(z) + iv(z)$. Then this single equation splits into two equations:

$$e^{u(z)} = e^{|g(z)|\cos(\arg(g(z))} = \frac{\pi}{|g'(z)|} \tag{5.140}$$

$$v(z) = |g(z)|\sin(\arg(g(z)) = \arg\left(\frac{i}{g'(z)}\right). \tag{5.141}$$

As in the case $g(z) = z^d, d \geq 2$, we consider first equation (5.140). Using equations (5.131) and (5.132), we get:

$$|g(z)| = \left(\frac{\rho}{r}\right)^N M|1 + \delta_0| \tag{5.142}$$

$$|g'(z)| = N\frac{\rho^{N-1}}{r^N}M|1 + \delta_1|. \tag{5.143}$$

As $\delta_0, \delta_1 \to 0$ uniformly with respect to z as $r \to \infty$, by choosing r sufficiently large, we may assume δ_0 and δ_1 to be constants with very small magnitude. Thus, we regard $|1 + \delta_0|$ and $|1 + \delta_1|$ as positive constants which are very close to 1. We now let $\sigma_0 := \arg(g(\zeta)) + \arg(1 + \delta_0)$, and $\sigma_1 := \arg(g(\zeta)) + \arg(1 + \delta_1)$. Then, using equation (5.131) and (5.132), we get:

$$\arg(g(z)) = N(\theta - \phi) + \sigma_0 \tag{5.144}$$

$$\arg(g'(z)) = N(\theta - \phi) - \theta + \sigma_1. \tag{5.145}$$

Dividing by r in the estimate (5.139), and raising everything to the power $N - 1$, we get an estimate on $\left(\frac{\rho}{r}\right)^{N-1}$, given by:

$$\left(1 - \frac{1}{N^\alpha}\right)^{N-1} < \left(\frac{\rho}{r}\right)^{N-1} < \left(1 + \frac{1}{N^\alpha}\right)^{N-1}. \tag{5.146}$$

Thus:

$$\frac{MN|1+\delta_1|}{r}\left(1-\frac{1}{N^\alpha}\right)^{N-1} < \frac{MN|1+\delta_1|\rho^{N-1}}{r^N} < \frac{MN|1+\delta_1|}{r}\left(1+\frac{1}{N^\alpha}\right)^{N-1}. \tag{5.147}$$

Thus, using equation (5.143):

$$\frac{\pi r}{MN|1+\delta_1|\left(1+\frac{1}{N^\alpha}\right)^{N-1}} < \frac{\pi}{|g'(z)|} < \frac{r}{MN|1+\delta_1|\left(1-\frac{1}{N^\alpha}\right)^{N-1}}. \tag{5.148}$$

Using equation (5.140), equation (5.148) then provides us with an estimate on $e^{u(z)} = e^{|g(z)|\cos(\arg(g(z))}$:

$$\frac{\pi r}{MN|1+\delta_1|\left(1+\frac{1}{N^\alpha}\right)^{N-1}} < e^{u(z)} < \frac{\pi r}{MN|1+\delta_1|\left(1-\frac{1}{N^\alpha}\right)^{N-1}}. \tag{5.149}$$

We take logarithms, and invoke $u(z) = |g(z)|\cos(\arg(g(z)))$, where $|g(z)|$ is given by equation (5.142). This gives:

$$\begin{aligned}
&\log \pi r - \log(M) - \log(N) - \log|1+\delta_1| - (N-1)\log\left(1+\frac{1}{N^\alpha}\right) \\
&< M|1+\delta_0|\left(\frac{\rho}{r}\right)^N \cos(\arg(g(z))) \\
&< \log \pi r - \log(M) - \log(N) - \log|1+\delta_1| - (N-1)\log\left(1-\frac{1}{N^\alpha}\right).
\end{aligned} \tag{5.150}$$

From the theory of entire functions (see Theorem 1.2.3), we now that $M \gg r^k$ as $r \to \infty$, for any k. Hence, in the expression $\frac{M}{r^N}$, M dominates, so said expression is $\mathcal{O}(M)$ as $r \to \infty$. The upper-hand side of (5.150) is for the same reason, dominated by $-\log(M) - \log(N)$. The same is true for the lower-hand side. We also note that $M \gg \log(M)$ as $r \to \infty$. In particular, it follows that $\cos(\arg(g(z))$ must be of negative sign and very small in magnitude. This is similar to the case when $g(z) = z^d, d \geq 2$. We therefore conclude that $\arg(g(z))$ must be very close to $\frac{\pi}{2}$, but slightly above. Thus, we set:

$$\arg(g(z)) = N(\theta - \phi) + \sigma_0 = \frac{\pi}{2} + \varepsilon, \tag{5.151}$$

for some very small $\varepsilon > 0$ to be specified. Recall that $\theta = \arg(z)$ and $\phi = \arg(\zeta)$. Equation (5.151) can be rearranged to give:

$$\arg(z) = \frac{\frac{\pi}{2} + \varepsilon - \sigma_0}{N} + \arg(\zeta). \tag{5.152}$$

From this we see that we are justified in using the Wiman-Valiron method (Theorem 1.4.4). Indeed, because N increases with r, by choosing a sufficiently large r, we can make $\arg(z)$ sufficiently close to $\arg(\zeta)$, where $\arg(z)$ is given by equation (5.152), such that when also (5.139) holds, (5.130) holds. We proceed to consider equation (5.141), which with equation (5.145), becomes:

$$M|1+\delta_0|\left(\frac{\rho}{r}\right)^N \cos(\varepsilon) = \frac{\pi}{2} + 2\pi k + \theta - N(\theta - \phi) - \sigma_1. \tag{5.153}$$

Here $k \in \mathbb{Z}$, and as before, $\theta = \arg(z)$ is given by equation (5.152). Solving for ρ^N in equation (5.153), yields then:

$$\begin{aligned} \rho^N &= \frac{\frac{\pi}{2} + 2\pi k + \frac{\frac{\pi}{2}+\varepsilon-\sigma_0}{N} + \phi - \left(\frac{\pi}{2} + \varepsilon - \sigma_0\right) - \sigma_1}{M|1+\delta_0|\cos(\varepsilon)} r^N \\ &= \frac{2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right)}{M|1+\delta_0|\cos(\varepsilon)} r^N. \end{aligned} \tag{5.154}$$

We need ρ^N close to r^N as given by (5.146). Notice that M is very large. Thus by choosing k sufficiently large, we can make this happen. In fact, we can give an estimate on k. We use (5.146) or equivalently, (5.139), for this. We then find:

$$\left(1 - \frac{1}{N^\alpha}\right)^N < \frac{2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right)}{M|1+\delta_0|\cos(\varepsilon)} < \left(1 + \frac{1}{N^\alpha}\right)^N. \tag{5.155}$$

Let us define a and b by:

$$2\pi a = \left(1 - \frac{1}{N^\alpha}\right)^N M|1+\delta_0|\cos(\varepsilon) - \frac{\pi}{2N} + \sigma_1 - \phi - (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right) \tag{5.156}$$

$$2\pi b = \left(1 + \frac{1}{N^\alpha}\right)^N M|1+\delta_0|\cos(\varepsilon) - \frac{\pi}{2N} + \sigma_1 - \phi - (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right). \tag{5.157}$$

Then (5.155) can be written:

$$a < k < b. \tag{5.158}$$

The estimate (5.158) is dependent on ε. We would like to find an estimate for k independent of ε, similar to what we did in the case $g(z) = z^d, d \geq 2$. We may assume N very large. Then $\frac{1}{N} - 1$ is very close to -1 and so $-\varepsilon\left(\frac{1}{N} - 1\right)$ is very close to ε. We may assume ε so small that:

$$\frac{M|1+\delta_0|}{2\pi}\cos(\varepsilon) - \frac{\varepsilon}{2\pi}\left(\frac{1}{N} - 1\right) < \frac{M|1+\delta_0|}{2\pi} + 1 \tag{5.159}$$

$$\frac{M|1+\delta_0|}{2\pi}\cos(\varepsilon) - \frac{\varepsilon}{2\pi}\left(\frac{1}{N} - 1\right) > \frac{M|1+\delta_0|}{2\pi} - 1. \tag{5.160}$$

Thus we propose the following ε-independent estimate for k:

$$\tilde{a} < k < \tilde{b}, \tag{5.161}$$

where:

$$\tilde{a} = \left(1 - \frac{1}{N^\alpha}\right)^N \frac{M|1+\delta_0|}{2\pi} + 1 - \frac{1}{4N} + \frac{\sigma_1 - \phi}{2\pi} + \frac{\sigma_0}{2\pi}\left(\frac{1}{N} - 1\right) \tag{5.162}$$

$$\tilde{b} = \left(1 + \frac{1}{N^\alpha}\right)^N \frac{M|1+\delta_0|}{2\pi} - 1 - \frac{1}{4N} + \frac{\sigma_1 - \phi}{2\pi} + \frac{\sigma_0}{2\pi}\left(\frac{1}{N} - 1\right). \tag{5.163}$$

It suffices to show that $[\tilde{a}, \tilde{b}]$ has length greater than 2, say 4 for good measure. Denote its length by $l([\tilde{a}, \tilde{b}])$. We find:

$$l([\tilde{a}, \tilde{b}]) = \frac{M|1+\delta_0|}{2\pi}\left[\left(1 + \frac{1}{N^\alpha}\right)^N - \left(1 - \frac{1}{N^\alpha}\right)^N\right] - 2. \tag{5.164}$$

Recall that $\lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N = e^x$. Thus, in the case $\alpha = 1$, we get:

$$\lim_{N\to\infty}\left(1 \pm \frac{1}{N^\alpha}\right)^N = e^{\pm 1}. \tag{5.165}$$

Hence, the expression:

$$\left(1 + \frac{1}{N^\alpha}\right)^N - \left(1 - \frac{1}{N^\alpha}\right)^N, \tag{5.166}$$

is close to $e - \frac{1}{e} > 2$ for large N. In the case $\alpha \in (1/2, 1)$, because $1 + \frac{1}{N^\alpha}$ becomes larger and $1 - \frac{1}{N^\alpha}$ becomes smaller, compared to the case $\alpha = 1$, it follows that the expression in (5.166), becomes even larger. Indeed, it is easy to check that for α close to 1/2, the expression becomes very large for large N. For $\alpha > 1$, the expression may in general, be very small for large N. However, because we assume $\alpha \in (1/2, 1]$, we may conclude that the expression in (5.166), is at least larger than 2 for large N. Then, because M is very large, we conclude that we can make it so that $l([\tilde{a}, \tilde{b}]) > 4$. In particular, by choosing r, and hence M, sufficiently large, we can make it so that the expression in (5.166), is much larger than $\frac{20\pi}{M|1+\delta_0|}$. Then we find from equation (5.164):

$$l([\tilde{a}, \tilde{b}]) > \frac{M|1+\delta_0|}{2\pi} \cdot \frac{20\pi}{M|1+\delta_0|} - 2 = 8 > 4. \tag{5.167}$$

Thus, k can be chosen independently of ε. With k chosen independently of ε such that $k \in (\tilde{a}, \tilde{b})$, let ρ be given according to equation (5.154), where we may now view ρ as a function of ε to be specified. We will show that there exists a sufficiently small $\varepsilon > 0$ such that with ρ chosen as described, equation (5.140) holds. Equation (5.140) becomes, after taking logarithms on both sides:

$$
\begin{aligned}
&-\left[2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right)\right]\tan(\varepsilon) \\
&= \log \frac{\pi}{N\frac{\rho^{N-1}}{r^N}M|1+\delta_1|} = \log \frac{\pi r^N}{MN|1+\delta_1|} - \frac{(N-1)}{N}\log(\rho^N) \\
&= \log \pi + N\log r - \log(M) - \log(N) - \log|1+\delta_1| \\
&\quad - \frac{N-1}{N}\log\left(\frac{2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right)}{M|1+\delta_0|\cos(\varepsilon)} r^N\right) \\
&= \log \pi + \log r - \frac{1}{N}\log(M) - \log(N) - \log|1+\delta_1| \\
&\quad + \frac{N-1}{N}\log|1+\delta_0| + \frac{N-1}{N}\log(\cos(\varepsilon)) \\
&\quad - \frac{N-1}{N}\log\left(2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right)\right).
\end{aligned} \tag{5.168}
$$

Now, let us define the functions:

$$
f(x) := -\left[2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (x - \sigma_0)\left(\frac{1}{N} - 1\right)\right]\tan(x) \tag{5.169}
$$

$$
\begin{aligned}
h(x) &:= \log \pi + \log r - \frac{1}{N}\log(M) - \log(N) - \log|1+\delta_1| \\
&\quad + \frac{N-1}{N}\log|1+\delta_0| + \frac{N-1}{N}\log(\cos(x)) \\
&\quad - \frac{N-1}{N}\log\left(2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 + (x - \sigma_0)\left(\frac{1}{N} - 1\right)\right).
\end{aligned} \tag{5.170}
$$

Notice that $f(0) = 0$. Comparing with the proof for the case when g is a monomial $g(z) = z^d, d \geq 2$, we see that we would like to have $h(\varepsilon)$ negative for all $\varepsilon \geq 0$. We need to analyse a bit. Notice that if N grows faster than r, then $\log N$ dominates $\log r$ and so $\log r - \log N$ is negative. However, if N grows slower, then this is positive as $\log r$ will dominate. Similarly, it is not at all obvious what to say about $\log r$ and $\frac{1}{N}\log M$. Even though we know that M is much greater than r, because this also involves N, whose relationship with r as suggested, is unclear, it is not apparent which of the two terms $\log(r)$ and $\frac{1}{N}\log(M)$, that will dominate. Fortunately, there is something we can do to help the situation. Namely, we recall how k was chosen. Looking at (5.161) and the two equations (5.162) and (5.163), we see that $2\pi k \sim cM$ as $r \to \infty$, for some constant $c > 0$. Hence we have that:

$$
2\pi k + \frac{\pi}{N} + \phi - \sigma_1 + (\varepsilon - \sigma_0)\left(\frac{1}{N} - 1\right) \sim cM \tag{5.171}
$$

as $r \to \infty$. So the last logarithm term in the expression for $h(\varepsilon)$, really involves M:

$$\frac{N-1}{N}\log(2\pi k + \cdots) \sim \left(1 - \frac{1}{N}\right)\log(cM). \tag{5.172}$$

as $r \to \infty$. Thus, in the expression for $h(\varepsilon)$, we have:

$$\begin{aligned} &\log(r) - \frac{1}{N}\log(M) - \log(N) - \frac{N-1}{N}\log(2\pi k \cdots) \\ &\sim \log(r) - \log(N) - \frac{1}{N}\log(M) - \left(1 - \frac{1}{N}\right)\log(cM) \end{aligned} \tag{5.173}$$

as $r \to \infty$, and this is certainly negative. So $h(\varepsilon) < 0$ for all $\varepsilon \geq 0$. Thus $f(0) - h(0) > 0$.

For simplicity, let us assume $c = 1$; what we now discuss is not affected by the exact value of c, only by its sign. Then, for not too small $\varepsilon \in (0, \pi/2)$, we see that $f(\varepsilon) \sim -M\tan(\varepsilon)$ as $r \to \infty$, while $h(\varepsilon) \sim \log(\pi) + \log(r) - \log(M) - \log(N)$ as $r \to \infty$. *Now*, we will need to discuss the relation between M and N, for we want to consider $f(\varepsilon) - h(\varepsilon)$. In order to do so, we will exploit the relation between $M(P, r)$ and M for P any polynomial. From the theory of entire functions (see Theorem 1.2.3), we know that $\lim_{r\to\infty} \frac{M(P,r)}{M} = 0$. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$. We choose $P(z) = a_N z^N$. Hence $M(P, r) = |a_N| r^N$ is equal to the maximal modulus term of the series of g. But then it follows that $r^N \ll M$ as $r \to \infty$. In particular, it follows that $N \ll \frac{\log(M)}{\log(r)} \ll M$ as $r \to \infty$. So we can certainly assert that $\log(M)$ dominates $\log(N)$ for large r. We remark that figuring this out, was not completely trivial to the author. Hence $|f(\varepsilon)|$ dominates $|h(\varepsilon)|$. But then, it follows that for not too small $\varepsilon \in (0, \pi/2)$, we have $f(\varepsilon) - h(\varepsilon) < 0$, for we have that $f(\varepsilon) < 0$ for $\varepsilon \in (0, \pi/2)$. As with the case $g(z) = z^d, d \geq 2$, continuity and the intermediate-value theorem, certifies the existence of some small $\varepsilon > 0$ such that $f(\varepsilon) - h(\varepsilon) = 0$. That is, such that equation (5.140) holds as required.

We are almost done, it remains to give the estimate for ε. This is similar to as before: the case when $g(z) = z^d,\ d \geq 2$. Let us define:

$$R(k) = 2\pi k + \frac{\pi}{2N} + \phi - \sigma_1 - \sigma_0\left(\frac{1}{N} - 1\right). \tag{5.174}$$

Then for small $\varepsilon > 0$, we replace ε^2 by 0, $\tan(\varepsilon)$ by ε, and finally $\cos(\varepsilon)$ by 1. Thus, we get $f(\varepsilon) = -\mathcal{O}(R\varepsilon)$ as $r \to \infty$, while $h(\varepsilon) = \mathcal{O}(-\log(R))$ as $r \to \infty$. Because we have $f(\varepsilon) = h(\varepsilon)$ and because $R \to \infty$ as $r \to \infty$, this gives $\varepsilon = \mathcal{O}\left(\frac{\log(R)}{R}\right)$ as $R \to \infty$, as required. This finally completes the proof of the lemma. □

Remark 5.2.10. In the case that g is polynomial, most of the proof of Lemma 5.2.9 can be used. The exceptions lie in interval for $|z|$; this now instead takes the form $(r, r+1)$ as with the case when g is a monomial, and in the growth of M

relative N and r. If g is a polynomial, it is no longer true that $M(g, r)$ grows faster than any power of r. A special case is when g is a first order polynomial, which we already have dealt with. If g is any other non-constant polynomial of degree $\deg(g) \geq 2$, then $M(g, r)$ grows faster than r, although not faster than any power of r. Moreover, $N = \deg(g)$, so M grows faster than N as well, but presumably a lot slower than what it did when g was transcendental. In any event, we can use the same reasoning as was done when g is transcendental and the proof goes through without significant change.

As in the case $g(z) = z^d, d \geq 2$, Lemma 5.2.9 actually provides us with infinitely many solutions of the approximate equation $g'(z)e^{g(z)} = \pi i$:

Corollary 5.2.11. Let $g(z)$ be a non-constant entire function. Then the equation $g'(z)e^{g(z)} = \pi i$ admits infinitely many solutions with increasing modulus.

Proof. We assume g is transcendental, and note that the case g is a polynomial, is analogous. The proof is now completely analogous to the proof of Corollary 5.2.5, only that we use Lemma 5.2.9 instead of Lemma 5.2.4. Also, we make sure to form a sequence with members outside the exceptional set with finite logarithmic measure given in the Wiman-Valiron theorem (Theorem 1.4.4). □

This takes care of the approximate equation $g'(z)e^{g(z)} = \pi i$. *However*, this is *not* the original system of equations we really want to solve, which is the one given by equations (5.32) and (5.33). The existence of solutions of this system is proven in a completely analogous manner as we did for the case when $g(z) = z^d, d \geq 2$. The only difference, is that we allow g to be any transcendental entire function. As before, it suffices to say that the polynomial case is completely analogous.

Let G be defined as in equation (5.79). We want to solve equation (5.80). As before, we will consider the image of the linear part of G, denoted by $G_L^{(z_0,z_0)}$, and defined by equation (5.82), where z_0 solves the approximate equation $g'(z)e^{g(z)} = \pi i$. We let W_j be defined by equation (5.95), $j = 1, 2$, and a, b, c, d defined by equations (5.96) - (5.99). Our first step is to determine the error in using the solution $(z_0, z_0) \in \Delta$, which solves the approximate equation $g'(z)e^{g(z)} = \pi i$ instead of the original equation (5.80). This will determine the distance between $G(z_0, z_0)$ and $(\pi i, -\pi i)^T$.

The following lemma is the transcendental analogue to Corollary 5.2.3:

Lemma 5.2.12. Let g be a transcendental entire function and let z_0 be the solution of $g'(z)e^{g(z)} = \pi i$ given by Lemma 5.2.9. Let G be defined by g according to equation (5.79). Then:

$$|G(z_0, z_0) - (\pi i, -\pi i)^T| = \mathcal{O}\left(\frac{1}{M}\right) \tag{5.175}$$

as $|z_0| \to \infty$, where M is the same as in Lemma 5.2.9.

Proof. If we use $(z_0, z_0) \in \Delta$ where z_0 is the solution obtained from Lemma 5.2.9, then for z close to z_0, the following expressions for $g'(z)$ and $g''(z)$ are valid due to the Wiman-Valiron method (Theorem 1.4.4):

$$g'(z) = N\frac{z^{N-1}}{\zeta^N}g(\zeta)(1+\delta_1) \tag{5.133}$$

$$g''(z) = N(N-1)\frac{z^{N-2}}{\zeta^N}g(\zeta)(1+\delta_2) \tag{5.134}$$

where $\delta_i \to 0$ uniformly with respect to z as $|z_0| \to \infty$, for both $j = 1, 2$, where $|\zeta| = r$, where $|g(\zeta)| = M = \max_{|z|=r}|g(z)|$, and where $N = N(r)$ is the central index for g. We use equation (5.40) and get:

$$\left|g\left(z_0 \pm e^{g(z_0)}\right) - g(z_0) \mp \pi i\right| = \mathcal{O}\left(\frac{|g''(z_0)e^{2g(z_0)}|}{2}\right), \quad \text{as } |z_0| \to \infty. \tag{5.176}$$

Because $g'(z_0)e^{g(z_0)} = \pi i$ by assumption, we find: $e^{2g(z_0)} = \frac{-\pi^2}{[g'(z_0)]^2}$. Using equation (5.133), we calculate:

$$[g'(z_0)]^2 = \frac{N^2 z_0^{2N-2}}{\zeta^{2N}}[g(\zeta)]^2(1+\delta_1)^2. \tag{5.177}$$

Using also (5.134), we then get:

$$\begin{aligned} g''(z_0)e^{2g(z_0)} &= -\frac{\pi^2 N(N-1)\frac{z_0^{N-2}}{\zeta^N}g(\zeta)(1+\delta_2)}{\frac{N^2 z_0^{2N-2}}{\zeta^{2N}}[g(\zeta)]^2(1+\delta_1)^2} \\ &= -\frac{\pi^2(N-1)[1+\delta_2]}{N(g(\zeta))[1+\delta_1]^2}\left(\frac{\zeta}{z_0}\right)^N, \quad \text{as } |z_0| \to \infty. \end{aligned} \tag{5.178}$$

We now take the modulus and get:

$$|g''(z_0)e^{2g(z_0)}| = \frac{\pi^2(N-1)|1+\delta_2|}{MN|1+\delta_1|^2} \cdot \left|\frac{\zeta}{z_0}\right|^N = \mathcal{O}\left(\frac{1}{M}\right). \tag{5.179}$$

We have used that M is the largest factor in the denominator for $|z_0|$ large, and that $\left|\frac{\zeta}{z_0}\right|$ is close to 1. Hence we get:

$$|G(z_0, z_0) - (\pi i, -\pi i)^T| = \mathcal{O}\left(\left|\left[\begin{matrix}\frac{1}{M} \\ \frac{1}{M}\end{matrix}\right]\right|\right) = \sqrt{2}\mathcal{O}\left(\frac{1}{M}\right) = \mathcal{O}\left(\frac{1}{M}\right) \tag{5.180}$$

as $|z_0| \to \infty$. This completes the proof. □

Remark 5.2.13. Notice that if $g(z) = z^d, d \geq 2$. Then $M = M(g, r) = r^d$, and so Lemma 5.2.12 gives the same order relation for the error as Corollary 5.2.3.

It follows that equation (5.91) is still valid. Indeed, in deriving this, we did not use any specific form for g. We use the same notation as before: $B_\delta(z_0, z_0)$ for the open ball centred at (z_0, z_0) with radius δ. In equation (5.91), we assume that we have $(z, w) \in B_\delta(z_0, z_0)$. Instead of equation (5.92), we propose to make δ so small that:

$$\sqrt{2}\delta^2 < \frac{1}{M}. \tag{5.181}$$

That is:

$$\delta < \sqrt{\frac{1}{\sqrt{2}M}}. \tag{5.182}$$

This is the general analogue to equation (5.93). Thus, we want to show that:

$$|W_1(z,w)|^2 + |W_2(z,w)|^2 > \mathcal{O}\left(\frac{2}{M}\right)^2, \quad \text{as } |z_0| \to \infty, \tag{5.183}$$

and with $(z, w) \in \partial B_\delta(z_0, z_0)$. This is the general analogue of (5.107). We need a result analogous to Lemma 5.2.6:

Lemma 5.2.14. Let z_0 be the solution of $g'(z)e^{g(z)} = \pi i$ given by Lemma 5.2.9 where g is a non-constant entire function, and let a, b, c, d be defined as in equations (5.96) - (5.99). Also, let M and N be as in Lemma 5.2.9. Then we have:

$$|a| = \mathcal{O}(MN|z_0|^{-1}) = |d|, \quad \text{as } |z_0| \to \infty \tag{5.184}$$

$$|b| = \mathcal{O}\left(\frac{1}{|z_0|}\right) = |c|, \quad \text{as } |z_0| \to \infty. \tag{5.185}$$

Proof. We calculate a, b, c, d using their definitions in equations (5.96) - (5.99), and where G_j is defined by equation (5.79). Doing so, gives equations (5.111) - (5.114):

$$a = g'(z_0 + e^{g(z_0)})e^{g(z_0)}g'(z_0) \tag{5.186}$$

$$b = g'(z_0 + e^{g(z_0)}) - g'(z_0) \tag{5.187}$$

$$c = g'(z_0 - e^{g(z_0)}) - g'(z_0) \tag{5.188}$$

$$d = -g'(z_0 - e^{g(z_0)})e^{g(z_0)}g'(z_0). \tag{5.189}$$

Of course, because g no longer necessarily is $z^d, d \geq 2$, the equations (5.115) - (5.118) are not necessarily available. Instead we use Wiman-Valiron's theorem (Theorem 1.4.4) and Taylor expansion. We let N and ζ mean the same as before. Then, for z sufficiently close to z_0, it follows from Wiman-Valiron's theorem (Theorem 1.4.4), that we can write:

$$g^{(m)} = \frac{N(N-1)\cdots(N-m+1)}{\zeta^N} z^{N-m} g(\zeta)(1+\delta_m), \tag{5.190}$$

where $\delta_m \to 0$ uniformly with respect to z as $|z_0| \to \infty$. We prove the assertion for $|a|$ and $|b|$, and remark that the respective cases $|d|$ and $|c|$, are completely analogous to the respective cases $|a|$ and $|b|$. This is also easy to see directly from the equations (5.186) - (5.189).

By assumption, we have $g'(z_0)e^{g(z_0)} = \pi i$. That is, using equation (5.190) with $m = 1$: $e^{g(z_0)} = \frac{\pi i}{\frac{N z_0^{N-1}}{\zeta^N} g(\zeta)(1+\delta_1)}$. Because $|g(\zeta)| = M$, we have $|e^{g(z_0)}| = \mathcal{O}\left(\frac{1}{M|z_0|}\right)$ as $|z_0| \to \infty$. In particular, $|e^{g(z_0)}|$ is small for large $|z_0|$. Thus we can use a Taylor expansion for $g'(z_0 + e^{g(z_0)})$. We get:

$$g'(z_0 + e^{g(z_0)}) = g'(z_0) + \sum_{m=2}^{\infty} \frac{g^{(m)}(z_0)}{m!} e^{(m-1)g(z_0)}. \tag{5.191}$$

It is convenient to take a closer look at the sum in equation (5.191). Using equation (5.190) and the expression for $e^{g(z_0)}$ from earlier, we find:

$$\begin{aligned}
\sum_{m=2}^{\infty} \frac{g^{(m)}(z_0)}{m!} e^{(m-1)g(z_0)} &= \sum_{m=2}^{\infty} \Bigg\{ \frac{\pi^{m-1} i^{m-1}}{m!} \frac{N(N-1)\cdots(N-m+1)}{\zeta^N} z_0^{N-m} \\
&\qquad \cdot g(\zeta)(1+\delta_m) \cdot \left[\frac{1}{\frac{N z_0^{N-1}}{\zeta^N} g(\zeta)(1+\delta_1)} \right]^{m-1} \Bigg\} \\
&= \sum_{m=2}^{\infty} \Bigg\{ \frac{\pi^{m-1} i^{m-1}}{m!} \frac{N(N-1)\cdots(N-m+1)}{\zeta^N} z_0^{N-m} \\
&\qquad \cdot g(\zeta)(1+\delta_m) \\
&\qquad \cdot \left(\frac{\zeta^{N(m-1)}}{N^{m-1} z_0^{(N-1)(m-1)} [g(\zeta)]^{m-1} (1+\delta_1)^{m-1}} \right) \Bigg\}. \\
&= \sum_{m=2}^{\infty} \Bigg\{ \frac{\pi^{m-1} i^{m-1}}{m!} \frac{(N-1)\cdots(N-m+1)}{(1+\delta_1)^{m-1} N^{m-2}} \frac{1+\delta_m}{[g(\zeta)]^{m-2}} \\
&\qquad \cdot z_0^{N-m-(N-1)(m-1)} \zeta^{N(m-1)-N} \Bigg\} \\
&= \sum_{m=2}^{\infty} \frac{(N-1)\cdots(N-m+1)(1+\delta_m)}{m! \left(\frac{(1+\delta_1)}{\pi i}\right)^{m-1} (N g(\zeta))^{m-2} z_0} \left(\frac{\zeta}{z_0}\right)^{N(m-2)}.
\end{aligned} \tag{5.192}$$

Then, because $|a| = \pi i g'(z_0 + e^{g(z_0)}) = \pi i \left(g'(z_0) + \sum_{m=2}^{\infty} \frac{g^{(m)}(z_0) e^{(m-1)g(z_0)}}{m!} \right)$, and because $g'(z_0) = \frac{N z_0^{N-1}}{\zeta} g(\zeta)(1+\delta_1)$, with $M = |g(\zeta)|$ and $|z_0/\zeta|$ close to 1, it follows that $|a| = \mathcal{O}(MN|z_0|^{-1})$ as $|z_0| \to \infty$. This takes care of the order relation for $|a|$.

We proceed to the order relation for $|b|$. Using equation (5.187) and equation (5.190), we see that $|b| = \sum_{m=2}^{\infty} \frac{g^{(m)}(z_0)}{m!} e^{(m-1)g(z_0)}$. Looking at equation (5.192), we see that the dominating term is given by $m = 2$. That is, the first term in the large sum in (5.192). Taking the modulus of this term and using again that $|z_0/\zeta|$ is close to 1, we see that $|b| = \mathcal{O}\left(\frac{1}{|z_0|}\right)$ as $|z_0| \to \infty$. This concludes the proof.

□

Note that due to the estimates in Lemma 5.2.14, the two inequalities (5.122) and (5.123) *hold*; recall also that $(k, l) \in \partial\mathbb{D}$. We finally have the following general analogue of Theorem 5.2.7:

Theorem 5.2.15. Let g be a non-constant entire function and let G be defined by equation (5.79). Then the equation $G(z, w) = (\pi, -\pi i)^T$, admits some solution (z, w). Furthermore, if $r > 0$ sufficiently large and z_0 are given by Lemma 5.2.9, then for fixed $\lambda \in (0, 1)$, with $\sqrt{2}\delta^2 = \frac{\lambda}{M} \in \left(0, \frac{1}{M}\right)$, where $M = \max_{|z|=r} |g(z)|$ as given in Lemma 5.2.9, (z, w) can be chosen close to $(z_0, z_0) \in \Delta$ in the sense that $(z, w) \in B_\delta(z_0, z_0)$ provided r is also chosen so large that $\frac{M^3 N^2}{|z_0|^2} > \frac{4\sqrt{2}}{\lambda}$, where $N = N(r)$ is the central index for g as given in Lemma 5.2.9.

Proof. We need only show that (5.183) holds. We know that the inequalities (5.122) and (5.123) hold. We use the estimates for a, b, c, d in Lemma 5.2.14. Thus we get:

$$
\begin{aligned}
|W_1(z,w)|^2 + |W_2(z,w)|^2 &\geq (|a|k - |b|l)^2\delta^2 + (|d|l - |c|k)^2\delta^2 \\
&= \left(|a|^2k^2 - 2|a|\cdot|b|kl + |b|^2l^2\right)\delta^2 \\
&\quad + \left(|d|^2l^2 - 2|d|\cdot|c|kl + |c|^2k^2\right)\delta^2 \\
&= \mathcal{O}\left(\frac{M^2N^2}{|z_0|^2}k^2 - \frac{2MN}{|z_0|^2}kl + \frac{1}{|z_0|^2}l^2\right)\delta^2 \\
&\quad + \mathcal{O}\left(\frac{M^2N^2}{|z_0|^2}l^2 - \frac{2MN}{|z_0|^2}kl + \frac{1}{|z_0|^2}k^2\right)\delta^2, \\
&\qquad\qquad \text{as } |z_0| \to \infty \\
&= \mathcal{O}\left(\frac{M^2N^2}{|z_0|^2} - 4\frac{MN}{|z_0|^2}kl + \frac{1}{|z_0|^2}\right)\delta^2, \text{ as } |z_0| \to \infty \\
&= \mathcal{O}\left(\frac{M^2N^2}{|z_0|^2}\right)\delta^2, \qquad \text{as } |z_0| \to \infty.
\end{aligned}
\tag{5.193}
$$

Now, $M^2N^2 \gg r^2$ as $r \to \infty$. Also, we have that r^2 is close to $|z_0|^2$. Thus, $\frac{M^2N^2}{|z_0|^2}$ is very large. We know that $\delta^2 < \frac{1}{\sqrt{2}M}$, so let us write $\delta^2 = \frac{\lambda}{\sqrt{2}M}$ for some $\lambda \in (0, 1)$. We can imagine fixing λ first which then provides δ. Then, in order for (5.183) to hold, it suffices to require that:

$$
\frac{MN^2\lambda}{\sqrt{2}|z_0|^2} > \frac{4}{M^2}. \tag{5.194}
$$

That is:

$$\frac{M^3N^2}{|z_0|^2} > \frac{4\sqrt{2}}{\lambda}. \tag{5.195}$$

This is precisely the assumption in the theorem statement, so we are done. This proves the assertion. □

Remark 5.2.16. Notice that in the case $g(z) = z^d, d \geq 2$, we have $N = d$ and $M = r^d$. Hence (5.195) becomes:

$$d^2 r^{3d-2} > \frac{4\sqrt{2}}{\lambda}, \tag{5.196}$$

which agrees with Theorem 5.2.7 except for the constant factor d^2 on the left-hand side of (5.196). Note that although there is an extra factor d^2, (5.126) is still satisfied because the right-hand side of (5.126) is independent of such constants by definition of the asymptotic notation in (5.47). Indeed, the constant d^2 becomes insignificant for sufficiently large r.

Finally, we have the general analogue to Corollary 5.2.8. This provides us with infinitely many genuine 4-periodic points for transcendental Hénon maps of the form $F(z, w) = (e^{g(z)} + w, z)$, where g is a non-constant entire function:

Corollary 5.2.17. Let g be a non-constant entire function and define the transcendental Hénon map F by $F(z, w) = (e^{g(z)} + w, z)$. Then F admits infinitely many genuine 4-periodic points. Furthermore, infinitely many of these can be chosen close to points of the form $(z_0, z_0) \in \Delta$ where $|z_0|$ has large modulus and solves the equation $g'(z)e^{g(z)} = \pi i$.

Proof. The proof is identical to the proof of Corollary 5.2.8, except that we use Theorem 5.2.15 instead of Theorem 5.2.7, and except that we have $\sqrt{2}\delta \in \left(0, \frac{1}{M}\right)$ instead of $\sqrt{2}\delta \in \left(0, \frac{1}{r^d}\right)$. □

5.2.2 Maps $F(z, w) = (f(z) - w, z)$ where f is periodic

This section will be considerably shorter than the previous one. The result we give here can be applied to maps which in their form, look like those in the previous section, under the additional assumption that g now be periodic. More precisely, the result we give here can, in particular, be applied to symplectic transcendental Hénon maps of the form $F(z, w) = (e^{g(z)} - w, z)$, where g is periodic. More generally, our result will work for symplectic transcendental Hénon maps $F_1(z, w) = (f(z) - w, z)$ where f is periodic.

Theorem 5.2.18. Let $F(z, w) = (f(z) - w, z)$ be a symplectic Hénon map where f has non-zero period p. That is: $f(z) = f(z + pk)$ for all z and for all $k \in \mathbb{Z}$. Then:

$$\{(z, z + pk) : f(z) = 2z, k \in \mathbb{Z}\} \subseteq \text{Fix}(F^4). \tag{5.197}$$

Proof. The system of determining equations for 4-periodic points for F, is given by:

$$z_1 + w = f(z) \tag{5.198}$$
$$z_2 + z = f(z_1) \tag{5.199}$$
$$w + z_1 = f(z_2) \tag{5.200}$$
$$z + z_2 = f(w). \tag{5.201}$$

From equation (5.198), we get $z_1 = f(z) - w$, and from equation (5.201), we get $z_2 = f(w) - z$. Substituting these in the remaining two equations (5.199) and (5.200), we are left with the following two equations:

$$f(w) = f(f(z) - w) \tag{5.202}$$
$$f(z) = f(f(w) - z). \tag{5.203}$$

Now, we propose to look for solutions of the form $w = z + pk$. The reason being that then $f(z) = f(w)$ for all k. Indeed, substituting this for w, and using the periodicity of f, we get:

$$f(z) = f(f(z) - z) \tag{5.204}$$

for both equations. So we only have one equation which needs to be solved for z. Namely, equation (5.204). It is now easy to see that the set $\{z \in \mathbb{C} : f(z) = 2z\}$ is a subset of the solution set of (5.204). The assertion follows. □

Remark 5.2.19. Notice that the solutions $(z, z + pk)$ in Theorem 5.2.18 does not lie on the diagonal Δ for $k \neq 0$. Thus, these solutions are not fixed points of F.

An immediate consequence of Theorem 5.2.18 which is sufficient for Fix(F^4) to have infinite cardinality, is the following result.

Corollary 5.2.20. Let f be a periodic entire function. Say, with period $p \neq 0$. That is: $f(z) = f(z + p\mathbb{Z})$ for all z. Let $g(z) = f(z) - 2z$ and suppose that $Z(g) \neq \emptyset$. Then, Fix(F^4) has infinite cardinality and contains infinitely many non-fix points for F, where F is the symplectic Hénon map given by $F(z, w) = (f(z) - w, z)$.

Proof. Because $Z(g) \neq \emptyset$, there exists some $z \in Z(g)$. That is, some z such that $f(z) = 2z$. Then, by Theorem 5.2.18 we have:

$$\{(z, z + p\mathbb{Z})\} \subseteq \text{Fix}(F^4). \tag{5.205}$$

Because the cardinality of $\mathbb{Z}$ is infinite, so is the cardinality of the set $\{(z, z+p\mathbb{Z})\}$. Therefore, so is the cardinality of Fix(F^4). Finally, the assertion regarding fixed points follows from the previous remark. □

Thus, we can assure that the set Fix(F^4), has infinite cardinality if we can find *a single* point z such that $f(z) = 2z$. Now, we have a good grasp on when this does not happen. Indeed, the only times we fail to find a single such solution, is when f is of the form:

$$f(z) = 2z + e^{g(z)}, \tag{5.206}$$

where g is some entire function.

Corollary 5.2.21. Let g be any periodic entire function, and let F be the symplectic Hénon map given by $F(z, w) = (e^{g(z)} - w, z)$. Then Fix$(F^4)$ has infinite cardinality.

Proof. Let $f(z) = e^{g(z)}$. Then f is not of the form in equation (5.206). The assertion follows. □

Example 5.2.22. Let g be a transcendental entire function with the period p and let h be any transcendental entire function. Let $f(z) = h(g(z))$. Then Fix(F_1^4) has infinite cardinality, where F_1 is the symplectic transcendental Hénon map given by $F_1(z, w) = (f(z) - w, z)$. Indeed, f has period p:

$$f(z + p\mathbb{Z}) = h(g(z + p\mathbb{Z})) = h(g(z)), \tag{5.207}$$

and the equation $f(z) = 2z$ means $z \in$ Fix $\left(\frac{f}{2}\right)$. This has infinite cardinality by Rosenbloom's theorem (Theorem 1.4.1).

Example 5.2.23. Consider the symplectic transcendental Hénon map given by $F_1(z, w) = (e^z - w, z)$. Then Fix(F_1^4) has infinite cardinality. Indeed, we know that the only exceptional value for λ to the equation $e^z = \lambda P(z)$, where $P(z)$ is any non-zero polynomial, is $\lambda = 0$. So the equation $e^z = 2z$ has infinitely many solutions.

Example 5.2.24. Suppose that $f(0) = 0$ and let f be a periodic transcendental entire function. Then, the equation $f(z) = 2z$ has the trivial solution $z = 0$. Thus Fix(F^4) has infinite cardinality, where F is the symplectic Hénon map given by $F(z, w) = (f(z) - w, z)$. Concrete examples are any odd and periodic f. For instance $f(z) = \sin(z)$. Of course, for the particular case $f(z) = \sin(z)$, we already know from Example 2.3.4, that the equation $\sin(z) = P(z)$, has in fact, infinitely many solutions for any odd polynomial $P(z)$.

Example 5.2.25. Let g be any periodic entire function, say with period p. Let $f(z) = g(z) - g(0) + pl$ where $l \in \mathbb{Z}$. Of course, l must be fixed. Then f has the same period p as g and moreover, $f(0) = pl$. That is, by periodicity:

$$f(p\mathbb{Z}) = pl. \tag{5.208}$$

We consider the equation $f(z) = 2z$. We propose a solution $z = pk$ where $k \in \mathbb{Z}$. Using equation (5.208), we get $f(z) = f(pk) = f(0) = pl$. On the other hand, $2z = 2pk$. Hence in order for $z = pk$ to be a solution of $f(z) = 2z$, we get:

$$pl = 2pk. \tag{5.209}$$

That is: $l = 2k$. Thus, fix any $k \in \mathbb{Z}$. This then provides $l = 2k$. So we define $f(z) = g(z) - g(0) + 2pk$. Then the equation $f(z) = 2z$ has the solution $z = pk$. So Fix(F^4) has infinite cardinality where F is the symplectic Hénon map given by $F(z, w) = (f(z) - w, z)$.

Actually, in the next chapter, we will prove that the equation:

$$L(z) = f(z), \tag{5.210}$$

where f is a periodic *transcendental* entire function, admits infinitely many solutions for any first order polynomial L. With $L(z) = 2z$, this means that the equation $f(z) = 2z$ admits infinitely many solutions. As a consequence, we then actually get the following result:

Theorem 5.2.26. Let f be a periodic transcendental entire function. Then the symplectic transcendental Hénon map $F_1(z, w) = (f(z) - w, z)$, has infinitely many 4-periodic points.

Finally, we have already remarked that infinitely many of the 4-periodic points given in Theorem 5.2.18 are not fixed points. Thus, the only way these may fail to be genuine 4-periodic points, is if they are 2-periodic points. Recall that $(z, w) \in$ Fix(F^2), where $F(z, w) = (f(z) - w, z)$is a Hénon map, if and only if:

$$f(z) = 2w \tag{5.211}$$

$$f(w) = 2z. \tag{5.212}$$

Our 4-periodic points take the form $(z, z + p\mathbb{Z})$ where z solves $f(z) = 2z$. By periodicity of f, (5.212) is satisfied for such points. However, (5.211) becomes:

$$f(z) = 2(z + p\mathbb{Z}) = 2z + 2p\mathbb{Z} \tag{5.213}$$

But $f(z) = 2z$ and so this means $2p\mathbb{Z} = 0$. So $p = 0$ and $w = z$. By assumption $p \neq 0$. We conclude that all solutions of the form $(z, z + pk)$ where $k \neq 0$ and $f(z) = 2z$, are *not* 2-periodic points for F. Hence, in all the above, we can actually replace "infinitely many 4-periodic points" by "infinitely many *genuine* 4-periodic points." Our most general result of this section thus becomes:

Theorem 5.2.27. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map where f has non-zero period p. Then, F_1 admits infinitely many genuine 4-periodic points of the form $(z, z + pk)$, where $k \in \mathbb{Z}$ and $f(z) = 2z$.

Chapter 6

Classes of Maps with Infinitely Many Periodic Points, Part 2

In this chapter we continue the investigation on transcendental Hénon maps with infinitely many k-periodic points for certain values of k. We will restrict our considerations to transcendental Hénon maps $F_\delta = (f(z) - \delta w, z)$ where $\delta^2 = 1$. Then, by imposing appropriate conditions on f, we give a systematic way of reducing the system of determining equations by half. Our technique is based on exploiting the symmetry of the system of equations which appear for $\delta^2 = 1$, to show that many of the equations are the same. We then use this to get existence results. For general values of δ, exploiting the symmetry is more challenging, and the author has yet to find any existence results for k-periodic points with $k \geq 3$ for general values of δ. To easily deal with the problem whether a k-periodic of F_δ is genuine, we restrict attention to the cases where k is prime. Then it follows that non-genuine k-periodic points, must be fixed points.

The last chapter dealt with 3-periodic and 4-periodic points, so the natural next step is 5-periodic points. Suppose that f is odd when $\delta = 1$, and that f is even and $f(0) = 0$, when $\delta = -1$. Then we prove that Fix(F_δ^5) admits infinitely many genuine 5-periodic points. For $k > 5$ we show the following main result: let f be periodic. That is, there exists some $p \neq 0$ such that $f(z + \mathbb{Z}p) = f(z)$ for all z. In the case that $\delta = 1$, let f be odd. In the case that $\delta = -1$, let f be even with $f(0) = 0$. Then, Fix(F_δ^k) contains infinitely many genuine k-periodic points for all prime k.

Finally, we provide two special results in the case that k is not prime: namely the cases $k = 6$ and $k = 8$. We give infinitely many examples of transcendental Hénon maps with infinitely many genuine 6-periodic points, and infinitely many examples of transcendental Hénon maps with infinitely many genuine 8-periodic points.

6.1 Infinitely many genuine 5-periodic points

As suggested, the natural next step going by the previous chapter, is to consider 5-periodic points. Our considerations here also suggests the more general method we propose later for more general values of k. By experience, symplectic transcendental Hénon maps give a system of determining equations which comes with a symmetry that has proven to simplify things considerably compared to the general case. Thus, we first consider a symplectic transcendental Hénon map $F_1 = (f(z) - w, z)$. The system of determining equations for 5-periodic points of F_1, becomes:

$$z_1 + w = f(z) \tag{6.1}$$
$$z_2 + z = f(z_1) \tag{6.2}$$
$$z_3 + z_1 = f(z_2) \tag{6.3}$$
$$w + z_2 = f(z_3) \tag{6.4}$$
$$z + z_3 = f(w). \tag{6.5}$$

The most straight-forward approach is to eliminate z_1, z_2, and z_3 and end up with a system of equations involving only the variables z and w. To eliminate z_1, we use equation (6.1), and to eliminate z_3, we use equation (6.5). This gives respectively $z_1 = f(z) - w$ and $z_3 = f(w) - z$. Substituting these expressions in the remaining equations (6.2) - (6.4), yields:

$$z_2 + z = f(f(z) - w) \tag{6.6}$$
$$f(w) - z + f(z) - w = f(z_2) \tag{6.7}$$
$$w + z_2 = f(f(w) - z). \tag{6.8}$$

We now come to our first obstacle: a compatibility condition. This is what in general happens when we take the straight-forward approach as above. Looking at equations (6.6) and (6.8), we see that each of both the equations can be used to solve for z_2 in terms of z and w. Because $z_2 = z_2$, we thus end up with the compatibility condition which needs to be satisfied by both equations, given by:

$$f(f(z) - w) - z = f(f(w) - z) - w. \tag{6.9}$$

In addition, equation (6.7) must also be satisfied. Under the assumption that the compatibility condition holds, we can choose either of equations (6.6) and (6.8) to get an expression for z_2. Say we choose the first and get $z_2 = f(f(z) - w) - z$. Then, with this substituted into equation (6.7), we get:

$$f(w) + f(z) - z - w = f\left[f(f(z) - w) - z\right]. \tag{6.10}$$

In conclusion: we need to solve both equations (6.9) and (6.10). An immediate way of satisfying the compatibility condition (6.9), is to propose that we look for solutions of the form $z = w$. That is, to look for solutions on the diagonal Δ. The danger of being on the diagonal Δ, is that it might turn out that (z, w) is a fixed point for F_1. We are of course, mainly interested in *genuine* 5-periodic points for F_1. However, we have good control over when a point $(z, w) \in \Delta$ is a fixed point for F_1. Indeed, $(z, z) \in \text{Fix}(F_1)$ if and only if $f(z) = 2z$. A priori, we might end up with some z which does not satisfy this, in which case we have found a genuine 5-periodic point. Substituting $w = z$ in (6.10), yields:

$$2f(z) - 2z = f\left[f(f(z) - z) - z\right]. \tag{6.11}$$

The author finds it difficult to see any solution to equation (6.11) aside from the one which gives fixed points: $f(z) = 2z$. In fact, we could even try to do the following which fails. Suppose f is periodic, say with non-zero period p. We can easily arrange for $f(0) = A$ for any constant A we choose by replacing f with $f - f(0) + A$. Let $A = pl$, where we fix $l \in \mathbb{Z}$. Because f is periodic, we have $f(p\mathbb{Z}) = f(0) = pl$. We now propose a solution $z = pk$ where $k \in \mathbb{Z}$. Substituting this into equation (6.11), we get:

$$2f(0) - 2pk = f(0) \tag{6.12}$$

which implies

$$2pk = f(0) = pl. \tag{6.13}$$

That is: $2k = l$. Thus, fix any $k \in \mathbb{Z}$. We choose $l = 2k$. Our proposed solution is $z = pk$, which indeed solves equation (6.11). However, we find then that $f(z) = f(pk) = f(0) = pl = 2pk = 2z$ and so this actually gives a fixed point for F_1.

Thus, the danger of ending up with fixed points, is very much there. What seems to be an effective way of dealing with this problem, is to propose looking for solutions of the form $w = -z$. In this case, the only possible fixed point solution, is the origin. Pursuing this, we have the following result which gives infinitely many genuine 5-periodic points. The proof is worth looking at carefully as it suggests the method we will use for the more general results later.

Theorem 6.1.1. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map where f is odd. Then F_1 admits infinitely many genuine 5-periodic points. Furthermore, infinitely many of these can be chosen of the form $(z, -z)$ where:

$$z = f(f(z) + z). \tag{1.63}$$

Proof. We pursue 5-periodic points for F_1 of the form $(z, w) = (z, -z)$. Substituting $w = -z$ in the equations (6.1) - (6.5), we get the following determining equations for 5-periodic points of F_1:

$$z_1 - z = f(z) \tag{6.14}$$
$$z_2 + z = f(z_1) \tag{6.15}$$
$$z_3 + z_1 = f(z_2) \tag{6.16}$$
$$-z + z_2 = f(z_3) \tag{6.17}$$
$$z + z_3 = f(-z). \tag{6.18}$$

Now, because f is odd, we have $f(-z) = -f(z)$ for all z, and equation (6.18) becomes: $z + z_3 = -f(z)$. That is: $z_3 = -f(z) - z = -[f(z) + z]$. Comparing with equation (6.14) which gives $z_1 = f(z)+z$, we see that $z_1 = -z_3$. Substituting for z_3, $-z_1$ and using that f is odd, in the equations (6.15) - (6.18), we get:

$$z_1 = f(z) + z \tag{6.19}$$
$$z_2 = f(z_1) - z \tag{6.20}$$
$$0 = f(z_2) \tag{6.21}$$
$$z_2 = -f(z_1) + z \tag{6.22}$$
$$-z_1 = -f(z) - z \tag{6.23}$$

Comparing, we see that equation (6.23) and (6.19) coincides, so we keep only one of them. Say the latter: (6.19). Then comparing equation (6.20) and (6.22), we find that $z_2 = -z_2$. Hence $z_2 = 0$. Because f is odd, $f(0) = 0$ and so the equation (6.21) becomes trivially true. Notice also that the equations (6.20) and (6.22) are the same, so we also keep only one of these, say the first: (6.20). It follows that we are left with the following two equations:

$$z_1 = f(z) + z \tag{6.24}$$
$$0 = f(z_1) - z. \tag{6.25}$$

Because z_1 is directly given as expressed in terms of z by equation (6.24), we can substitute this into equation (6.25), and end up with the single equation:

$$z = f(f(z) + z), \tag{1.63}$$

for z. Now, let $g(z) = f(z) + z$. Then g is transcendental and entire. Equation (1.63) means $z \in \text{Fix}(f \circ g)$, which by Rosenbloom's theorem (Theorem 1.4.1) has infinite cardinality. It follows that there are infinitely many solutions z of equation (1.63). Hence, there are infinitely many 5-periodic points $(z, -z)$ of the form given in the statement. The only possible non-genuine 5-periodic points of all these, are the fixed points. But the only possible fixed point of the form $(z, -z)$, is the origin. It follows that $\text{Fix}(F_1^5)$ contains infinitely many genuine 5-periodic points. This completes the proof.

□

Remark 6.1.2. Looking at the proof of Theorem 6.1.1, we see that we end up with a single equation in z, and that no compatibility conditions appear along the process. We also note that 4 of the equations, reduce to 2 equations because they come in pair as coinciding equations. That is, equation (6.19) coincides with (6.23) and so one of these can be removed. The same happens with equation (6.20) and (6.22). Finally, we note that the middle equation (6.21) becomes trivially true because $f(0) = 0$ for odd functions f. We will later show that what we have observed here will generally happen when we pursue solutions of the form $(z, -z) \in \text{Fix}(F_1^k)$ for k prime and f odd. Thus this provides us with a systematic way of reducing the system of determining equations for k-periodic points by about half, while also avoiding compatibility conditions. In fact, this way of reducing the system works for any value of k, however, for k not prime, it is in general considerably more difficult to determine whether $(z, -z)$ is a genuine k-periodic point. As a concrete example, let $k = 16$. Then non-genuine 16-periodic points could be fixed points, genuine 2-periodic points, genuine 4-periodic points, or genuine 8-periodic points.

Finally, we note that we can get a similar result when $\delta = -1$, that is for F_{-1}, by instead conditioning that f be even and that it satisfies $f(0) = 0$:

Theorem 6.1.3. Let $F_{-1}(z, w) = (f(z) + w, z)$ be a transcendental Hénon map with $\delta = -1$, where f is even and $f(0) = 0$. Then $\text{Fix}(F_{-1}^5)$ contains infinitely many genuine 5-periodic points.

Proof. The proof is almost identical to the proof for the case of symplectic transcendental Hénon maps with f odd. The system of equations determining 5-periodic points, now becomes:

$$z_1 - w = f(z) \tag{6.26}$$
$$z_2 - z = f(z_1) \tag{6.27}$$
$$z_3 - z_1 = f(z_2) \tag{6.28}$$
$$w - z_2 = f(z_3) \tag{6.29}$$
$$z - z_3 = f(w). \tag{6.30}$$

We look for solutions where $w = -z$. Substituting this and using that f is even, we find $z_1 = -z_3$, $z_2 = -z_2 = 0$, and that equations (6.30) and (6.29) are the same as equations (6.26) and (6.27) respectively. So we keep only the two latter. Also, equation (6.28) becomes the trivial equation $f(0) = 0$ which holds by assumption. Substituting $z_1 = f(z) - z$ from (6.26) into (6.27) and invoking $z_2 = 0$, we get the single equation:

$$z = -f(f(z) - z), \tag{6.31}$$

which has infinitely many solutions by Rosenbloom's theorem (Theorem 1.4.1). This proves the assertion. ☐

6.2 Infinitely many genuine prime periodic points

In this section, we give the main result of this chapter: given any symplectic transcendental Hénon map $F_1(z, w) = (f(z) + w, z)$ where f is odd and periodic, Fix(F_1^k) contains infinitely many genuine k-periodic points for all $k > 1$ prime.

The first step is the reduction of the system of determining equations for k-periodic points for F_1, to about half its original size, and is a direct generalization of what was done in the proof of Theorem 6.1.1 in the previous section.

Thus, fix $k > 2$ (we have already dealt with the case $k = 2$ in great detail) to be any prime number. We look for k-periodic points (z, w) of the form $w = -z$. Recall that because f is odd, $f(-z) = -f(z)$ for all z, and in particular, $f(0) = 0$. The system of determining equations for k-periodic points of F_1, becomes:

$$\begin{aligned}
z_1 - z &= f(z) && (1)\\
z_2 + z &= f(z_1) && (2)\\
z_3 + z_1 &= f(z_2) && (3)\\
&\vdots\\
z_{k-3} + z_{k-5} &= f(z_{k-4}) && (k-3)\\
z_{k-2} + z_{k-4} &= f(z_{k-3}) && (k-2)\\
-z + z_{k-3} &= f(z_{k-2}) && (k-1)\\
z + z_{k-2} &= f(-z) = -f(z). && (k)
\end{aligned}$$

Comparing equation (1) and (k), we see that $z_1 = -z_{k-2}$. Thus, equation $(k-1)$ is given by $-z + z_{k-3} = -f(z_1)$. That is: $z_{k-3} = -f(z_1) + z$. Comparing this with equation (2), we see that $z_2 = -z_{k-3}$. Substituting this into equation $(k-2)$ and comparing with equation (3), we get that $z_3 = -z_{k-4}$ and so on. Inductively, we see that $z_j = -z_{k-(j+1)}$. We also note that this relationship tells us that equation number j and equation number $k - j + 1$, coincides. Indeed, for instance we see that equation (1) and equation (k) coincides, equation (2) and equation $(k-1)$ coincides, and that equation (3) and equation $(k-2)$ coincides. Now, because $k > 2$ and is prime, k is odd. Thus, we can write $k = 2m + 1$ where $m \in \mathbb{N}$. We consider the $(m-1)$-th equation, the m-th equation, and the $(m+1)$-th equation, which then respectively take the forms:

$$\begin{aligned}
z_{m-1} + z_{m-3} &= f(z_{m-2}) && (m-1)\\
z_m + z_{m-2} &= f(z_{m-1}) && (m)\\
z_{m+1} + z_{m-1} &= f(z_m). && (m+1)
\end{aligned}$$

Now, we have $z_{m-1} = -z_{k-(m-1)-1} = -z_{2m+1-m} = -z_{m+1}$. So z_{m+1} can be replaced with $-z_{m-1}$. Finally, we have $z_m = -z_{k-m-1} = -z_{2m+1-m-1} = -z_m$.

That is: $z_m = 0$. Substituting into the equations ($(m-1)$) - ($m+1$), we therefore find:

$$z_{m-1} + z_{m-3} = f(z_{m-2}) \qquad ((m-1))$$
$$z_{m-2} = f(z_{m-1}) \qquad ((m))$$
$$0 = f(0) = 0. \qquad ((m+1))$$

Thus the last non-repeated, non-trivial equation we have, is the m-th equation. Indeed, the next equation after the $(m+1)$-th equation, is the $(m+2)$-th equation, and this is the same as the m-th equation. Indeed, recall that equation j coincides with equation $k-j+1$. Thus, we find that equation m coincides with equation $k-m+1 = 2m+1-m+1 = m+2$. Of course, the following equation, the $(m+3)$-th equation then coincides with the $(m-1)$-th equation and so on. It follows that we have reduced the system of $k = 2m+1$ equations, to $\frac{k-1}{2} = m$ equations. That is, about half. Now, consider this. The first equation (1), gives z_1 as a function of z. Then the next equation, (2) gives z_2 as a function of z and z_1. Hence, as a function of z. The next equation after that again, equation (3), then gives z_3 as a function of z_2 and z_1. Hence as a function of z. Inductively, we see that the jth equation gives z_j as a function of z_{j-2} and z_{j-1}, hence as a function of z, as z_{j-1} and z_{j-2} already have been given as functions of z in the two preceding equations. In particular, for $j = m-1$, we see that z_{m-1} is given as a function of z. Thus the last equation, the m-th equation, being given by $z_{m-2} = f(z_{m-1})$, is an equation which only involves z. It follows that this is the equation which determines z, and that this is really the only equation we have. All the preceding equations give z_j as functions of z for $1 \leq j \leq m-1$, and leads to the concluding equation $z_{m-2} = f(z_{m-1})$. We will refer to this equation, the m-th equation, as the **decisive equation** of the system.

Example 6.2.1. We want to determine the decisive equation for $k = 5$. Writing $k = 2m+1$, this gives $m = 2$. The decisive equation is given by $z_{m-2} = f(z_{m-1})$. That is, $z_0 = f(z_1)$. Of course, here $z_0 = z = z_k$. Thus, the decisive equation in the case $k = 5$, is given by

$$z = f(z_1). \qquad (6.32)$$

Because $z_1 = f(z) + z$, this can be written in complete form as:

$$z = f(f(z) + z). \qquad (1.63)$$

This is exactly the equation given in Theorem 6.1.1, equation (1.63). Thus, the cardinality of Fix(F_1^5) is determined by the number of solutions of equation (1.63). By Rosenbloom's theorem (Theorem 1.4.1), there are infinitely many solutions.

Finally, one might wonder what happened to z_2 and z_3. Using $z_j = -z_{k-j-1}$, we find that $z_1 = -z_{5-1-1} = -z_3$, and $z_2 = -z_{5-2-1} = -z_2$. The first means that z_3 is already given by z_1, and the latter means of course that $z_2 = 0$. Notice how much faster we can get the complete system of determining equations, using $z_j = -z_{k-(j+1)}$, instead of writing up everything from start to end, and then go in and eliminate by solving. That is, what we did in the proof of Theorem 6.1.1. Indeed, we know that we only have $m = \frac{k-1}{2} = \frac{5-1}{2} = 2$ equations, so we immediately get the full system given by:

$$z_1 = f(z) + z \tag{6.33}$$

$$z = f(z_1). \tag{6.34}$$

Example 6.2.2. In the case $k = 5$, we could use Rosenbloom's theorem (Theorem 1.4.1) directly to conclude that F_1 has infinitely many genuine 5-periodic points. In particular, we did not need the additional assumption that f be periodic. Unfortunately, in general, things are a bit more subtle and Rosenbloom's theorem (Theorem 1.4.1) cannot be applied. At least, the author has not been able to see how. Let us consider for instance, $k = 7$. This is a natural place to start after dealing with the case $k = 5$. Then using $z_j = -z_{k-j-1}$, we find $z_1 = -z_{7-2} = -z_5, z_2 = -z_4$, and $z_3 = -z_3 = 0$. Writing $k = 2m + 1$ as before, for $k = 7$, we get $m = 3$. Thus we only have 3 equations for the full system of determining equations:

$$z_1 = f(z) + z \tag{6.35}$$

$$z_2 + z = f(z_1) \tag{6.36}$$

$$z_1 = f(z_2). \tag{6.37}$$

The decisive equation is of course the last of these, namely (6.37). Solving from top to bottom, we find $z_1 = f(z) + z$ and $z_2 = f(z_1) - z = f[f(z) + z] - z$. Hence the decisive equation becomes:

$$f(z) + z = f\left(f[f(z) + z] - z\right). \tag{6.38}$$

$$\tag{6.39}$$

That is:

$$z = f\left(f[f(z) + z] - z\right) - f(z). \tag{6.40}$$

The author fails to see how Rosenbloom's theorem can be applied here to give infinitely many solutions for z of equation (6.40). However, if we assume also that f is periodic, say with non-zero period p, so that $f(z + p\mathbb{Z}) = f(z)$ for all z, then we *can* show that there are infinitely many solutions:

fix any $l \in \mathbb{Z}$. We let $z = u + pl$ where u is some new variable we introduce. Substituting this in equation (6.40), and using the periodicity of f, we get:

$$u + pl = f\left(f[f(u) + u] - u\right) - f(u). \tag{6.41}$$

That is:

$$pl = \left(f[f(u) + u] - u\right) - f(u) - u. \tag{6.42}$$

Now, the right-hand side function, is a transcendental entire function in u, while the left-hand side is some constant complex number. Let $g(z)$ be a transcendental entire function and let $\lambda \in \mathbb{C}$. The Great Picard theorem for transcendental entire functions, Theorem 1.1.6, tells us that the equation:

$$\lambda = g(z), \tag{6.43}$$

has infinitely many solutions for all values of λ with the exception of at most one. Thus, either equation (6.42) admits infinitely many solutions for all $l \in \mathbb{Z}$, or there exists some exceptional $l_0 \in \mathbb{Z}$ such that the equation (6.42) fails to have infinitely many solutions for $l = l_0$. If there are no exceptional l_0, then we are done: equation (6.42) has infinitely many solutions for u. Otherwise, if there is such an exceptional l_0, then we simply repeat what we did prior to getting equation (6.41), but this time we fix any $l \in \mathbb{Z}$ such that $l \neq l_0$. It follows that we get infinitely many u which solves equation (6.42). Because $z = u + pl$, this means that we get infinitely many solutions for z solving the decisive equation (6.40). But then it follows that Fix(F_1^7) contains infinitely many genuine 7-periodic points.

In the previous example, Example 6.2.2, in the course of our discussion, we proved the following result, which can be viewed as a type of Rosenbloom result for periodic transcendental functions f:

Theorem 6.2.3. Let g be any transcendental entire and periodic function, say with period p, and let $L(z)$ be any first order polynomial. Then the equation:

$$L(z) = g(z), \tag{6.44}$$

has infinitely many solutions.

Proof. Because $L(z)$ is a first order polynomial, we can write for it: $L(z) = az + b$. Let us introduce a new variable u given by $z = u + pl$ where $l \in \mathbb{Z}$. Then equation (6.44) becomes:

$$a(u + pl) + b = g(u + pl). \tag{6.45}$$

Because g is periodic, we have $g(u + p\mathbb{Z}) = g(u)$ for all u. In particular, we get $g(u + pl) = g(u)$. Rearranging, we then get:

$$apl = g(u) - au. \tag{6.46}$$

By the Great Picard theorem for transcendental entire functions, Theorem 1.1.6, equation (6.46) has infinitely many solutions for all values of $l \in \mathbb{Z}$ with the exception of at most one exceptional value $l_0 \in \mathbb{Z}$. We can always choose $l \neq l_0$ in the outset. Then for a fixed $l \in \mathbb{Z} \backslash \{l_0\}$, the Great Picard theorem for transcendental entire functions, Theorem 1.1.6, tells us that equation (6.46) admits infinitely many solutions for u. Finally, because $z = u + pl$, this gives infinitely many solutions for z of equation (6.44). This proves the assertion. □

Remark 6.2.4. Let us see what may go wrong if $L(z)$ is replaced by a more general polynomial, say $P(z) = z^2$. Then the equation we want to solve is:

$$z^2 = g(z). \tag{6.47}$$

We do as earlier, and define $z = u + pl$. This gives then:

$$(u + pl)^2 = u^2 + 2plu + p^2l^2 = g(u). \tag{6.48}$$

This can be rearranged to give:

$$p^2l^2 = g(u) - u^2 - 2plu. \tag{6.49}$$

We would now like to apply the Great Picard theorem for transcendental entire functions, Theorem 1.1.6, to conclude that there are infinitely many solutions for u. The problem, is that the transcendental entire function on the right-hand side now depends on l. That is, different values for l give different transcendental entire functions. Thus, the same arguments as before cannot be used here. However, we remark that we *can* replace $L(z)$ by other functions and still make the previous arguments work. We see from our considerations just now, that the problem we need to avoid, is to end up with a right-hand-side transcendental entire function which depends on l. Suppose for instance $p \notin 2\pi i\mathbb{Z}$, and let $L(z) = e^z$. We consider the equation:

$$e^z = g(z). \tag{6.50}$$

We do as before and introduce u by $z = u+pl$ with $l \in \mathbb{Z}$. Equation (6.50) becomes:

$$e^u e^{pl} = g(u). \tag{6.51}$$

As $Z(e^u) = \emptyset$, we can divide by e^u and end up with a transcendental entire function on the right-hand side:

$$e^{pl} = \frac{g(u)}{e^u}. \tag{6.52}$$

The right-hand side function now does *not* involve l and so we can apply the Great Picard theorem for transcendental entire functions, Theorem 1.1.6, as before and conclude that there are infinitely many solutions z to the equation $e^z = g(z)$.

Remark 6.2.5. From the theory of entire functions (see Theorems 1.3.7 and 1.3.8), we know that the equation $\lambda P(z) = h(z)$, admits infinitely many solutions with at most one exceptional value for λ, where $P(z)$ is any non-zero polynomial and where $\rho(h(z)) < \infty$. In the case $\rho(h(z)) \in \mathbb{N}$ as well, we furthermore know that there are no exceptional values for λ. This result would not work if for example $f(z) = e^{e^z}$, for the order of the latter is not finite. However, in the case $P(z)$ is a first order polynomial, Theorem 6.2.3 *can* be applied and we *may* conclude that the equation $\lambda P(z) = e^{e^z}$ has infinitely solutions for all values of $\lambda \neq 0$. Indeed, e^{e^z} is periodic with period $2\pi i$. In fact, we can replace e^{e^z} with $h(g(z))$ where g

is any periodic non-constant entire function and h is any transcendental entire function. In particular, if g is transcendental and periodic, then for any $k \in \mathbb{N}$, the equation:

$$z = g^k(z), \tag{6.53}$$

has infinitely many solutions. Thus Fix(g^k) has infinite cardinality. This is almost Rosenbloom's theorem (Theorem 1.4.1); or more precisely: the corollary that follows from Rosenbloom's theorem (Corollary 1.3.8), for transcendental entire functions, except that we need the additional assumption that g be periodic. Another difference is that Rosenbloom's theorem allows for at most one exceptional value of k for which Fix(g^k) fails to have infinite cardinality. In our result, there are no exceptional values for k.

As a consequence, we get the following two existence results for 2-periodic and 4-periodic points. The result for 2-periodic points, is not restricted to only symplectic transcendental Hénon maps:

Corollary 6.2.6. Let $F(z, w) = (f(z) - \delta w, z)$ be a transcendental Hénon map, where f is odd and periodic. Then F has infinitely many genuine 2-periodic points.

Proof. Let $g(z) = f(z) - (1+\delta)z$. We know that $Z(g) \times (-Z(g)) \subseteq$ Fix(F^2). Thus, it suffices to show that $Z(g)$ has infinite cardinality. But this follows immediately from Theorem 6.2.3. □

The result for 4-periodic points, stated at the end of the previous chapter, does not impose the condition that f be odd, only that f be periodic.

Corollary 6.2.7. Let $F(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map, where f is periodic. Then F has infinitely many genuine 4-periodic points.

Proof. From the previous chapter, Theorem 5.2.18, we know that the following is true: $\{(z, z + p\mathbb{Z}) : f(z) = 2z\} \subseteq$ Fix(F^4). Thus, it suffices to show that there are infinitely many solutions to the equation $f(z) = 2z$. But this follows from Theorem 6.2.3 and the assumption on f. The assertion follows. □

We now arrive at our main result for this chapter. Most of the work has already been done in the previous example for $k = 7$, Example 6.2.2.

Theorem 6.2.8. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map where f is periodic and odd. Let $k > 2$ be prime. Then F_1 admits infinitely many genuine k-periodic points.

Proof. Let $k = 2m + 1$. It suffices to show that the decisive equation given by $z_{m-2}(z) = f(z_{m-1}(z))$, has infinitely many solutions in z. Let $1 \leq j \leq m - 1$ be any index. Then the equation determining z_j, the j-th equation, is given by

$$z_j = f(z_{j-1}) - z_{j-2}. \tag{6.54}$$

Let us start with $j = 1$. This then gives $z_1 = f(z) + z$. Then for $j = 2$, we get $z_2 = f(z_1) - z$. For $z = 3$, we get $z_3 = f(z_2) - z_1 = f(z_2) - f(z) - z$. For z_4, we get $z_4 = f(z_3) - z_2 = f(z_3) - f(z_1) + z$, and so on. In general, we see that $z_j = g(z) \pm z$ where g is some transcendental entire and periodic function with the same period as f. Indeed, let us prove this by strong induction. For $j = 1$, we are done. For then $z_1 = f(z) + z$, so $g(z)$ can be taken as $f(z)$. Thus, suppose this is true for all j such that $1 \leq j \leq n$ for some $1 < n \leq m - 1$. We show that this is then true for all $1 \leq j \leq n+1$. Because by assumption, this is true for all $j \leq n$, we need only show that this is true for z_{n+1}. That is, that $z_{n+1} = h(z) \pm z$ for h some periodic function with the same period as f. Now, we have $z_{n+1} = f(z_n) - z_{n-1}$. By assumption, there are periodic functions g_1 and g_2 with the same period as f such that: $z_n = g_2(z) \pm z$ and $z_{n-1} = g_1(z) \pm z$. Hence we get:

$$z_{n+1} = f(z_n) - z_{n-1} = f(g_2(z) \pm z) - g_1(z) \mp z. \tag{6.55}$$

Let us define $h(z) = f(g_2(z) \pm z) - g_1(z)$. We need only show that h is periodic with the same period as f. Suppose f has period p. Then, we need only show that $h(z + p\mathbb{Z}) = h(z)$ for all z. By assumption $g_i(z + p\mathbb{Z}) = g_i(z)$ for all z, $i = 1, 2$, and so we find, using also the periodicity for f:

$$\begin{aligned} h(z + p\mathbb{Z}) &= f(g_2(z \pm p\mathbb{Z}) \pm (z + p\mathbb{Z})) - g_1(z + p\mathbb{Z}) \\ &= f(g_2(z)) \pm z) - g_1(z) \\ &= h(z), \end{aligned} \tag{6.56}$$

as required. Thus h is periodic with the same period as f. This completes the proof by strong induction that all z_j is of the form $g(z) \pm z$ for some periodic g with same period as f for $1 \leq j \leq m - 1$. The decisive equation is given by $z_{m-2} = f(z_{m-1})$. By what we just showed, we can write $z_{m-2} = g_2(z) \pm z$ for some periodic function g_2 with the same period as f. Hence the decisive equation can be written:

$$g_2(z) \pm z = f(z_{m-1}). \tag{6.57}$$

But we also showed that $f(z_{m-1})$ is of the form $g_1(z) \pm z$ where g_1 is periodic with the same period as f as well. So equation (6.57) can be written as

$$g_2(z) \pm z = f(g_1(z) \pm z). \tag{6.58}$$

That is: after rearranging:

$$\pm z = f(g_1(z) \pm z) - g_2(z). \tag{6.59}$$

The right-hand side of equation (6.59) is a transcendental entire and periodic function. Thus by Theorem 6.2.3, equation (6.59) has infinitely many solutions. But equation (6.59) is the decisive equation, so we are done. This completes the proof. □

Example 6.2.9. Let $k = 13$. We write $k = 2m + 1$ and see that $m = 6$. Thus we have 6 equations, with the last and decisive being given by $z_4 = f(z_5)$:

$$z_1 = f(z) + z \tag{6.60}$$
$$z_2 + z = f(z_1) \tag{6.61}$$
$$z_3 + z_1 = f(z_2) \tag{6.62}$$
$$z_4 + z_2 = f(z_3) \tag{6.63}$$
$$z_5 + z_3 = f(z_4) \tag{6.64}$$
$$z_4 = f(z_5) \tag{6.65}$$

Solving from top to bottom, we find:

$$z_1 = f(z) + z \tag{6.66}$$
$$z_2 = f(z_1) - z = f[f(z) + z] - z \tag{6.67}$$
$$z_3 = f(z_2) - z_1 = f\{f[f(z) + z] - z\} - f(z) - z \tag{6.68}$$
$$z_4 = f(z_3) - z_2 = f(f\{f[f(z) + z] - z\} - f(z) - z) - f[f(z) + z] + z \tag{6.69}$$
$$\begin{aligned} z_5 = f(z_4) - z_3 = f\,[&f(f\{f[f(z) + z] - z\} - f(z) - z) - f[f(z) + z] + z] \\ &- f\{f[f(z) + z] - z\} + f(z) + z. \end{aligned} \tag{6.70}$$

Thus, the decisive equation becomes:

$$z = f(z_5) + f[f(z) + z] - f(f\{f[f(z) + z] - z\} - f(z) - z), \tag{6.71}$$

which is of the form $z = g(z)$ for g a transcendental entire and periodic function with the same period as f.

Remark 6.2.10. Finally, we remark that we can get a corresponding result to Theorem 6.2.8 for $\delta = -1$, by instead conditioning that f is even and fixes the origin. That is: if $F_{-1}(z, w) = (f(z) + w, z)$ is a transcendental Hénon map with $\delta = -1$, where f is even, fixes the origin: $f(0) = 0$, and has non-zero period p: $f(z + \mathbb{Z}p) = f(z)$ for all z, then Fix(F_{-1}^k) contains infinitely many genuine k-periodic points of the form $(z, -z)$ for all $k > 1$ prime. The proof is almost identical to that of Theorem 6.2.8, except that we use $f(-z) = f(z)$ for all z as f is even, instead for $f(-z) = -f(z)$. The decisive equation we end up with is also slightly different from the case where f is odd and $\delta = 1$, but the proof for the the existence of infinitely many solutions of the resulting equation, remains the same.

6.3 Two special results

6.3.1 6-periodic points

Our previous results do not apply to 6-periodic points as 6 is not prime. However, we can use many of the idea previously discussed to show the following:

Theorem 6.3.1. Let $F_1(z,w) = (f(z) - w, z)$ be a symplectic transcendental Hénon where f is odd and where $-\frac{f}{2}$ has a finite number of fixed points. Then F_1 admits infinitely many genuine 6-periodic points of the form $(z, -z)$.

Proof. The idea is to look for 6-periodic points of the form $(z, -z)$. Because f is odd, we can use the the method in the beginning of section 6.2, to reduce the system of equations determining 6-periodic points for F_1. The relation $z_j = -z_{k-(j+1)}$ still holds, but now k is even, and so we have $k = 2m$ for some $m \in \mathbb{N}$, instead of $k = 2m + 1$. The decisive equation also ends up different from $z_{m-2} = f(z_{m-1})$. It is not hard to determine it. We still get that the decisive equation, is the m-th equation. This has always the general form $z_m + z_{m-2} = f(z_{m-1})$. Now, $z_m = -z_{k-m-1} = z_{2m-m-1} = z_{m-1}$; recall that when k is odd, we instead get $z_m = -z_m = 0$, so the m-th equation is: $z_{m-2} = f(z_{m-1}) + z_{m-1}$. In our case, $k = 6$, so we have $m = 3$. We thus get $z_1 = -z_4$ and $z_2 = -z_3$. Therefore we have the equations:

$$z_1 - z = f(z) \tag{6.72}$$

$$z_2 + z = f(z_1) \tag{6.73}$$

$$-z_2 + z_1 = f(z_2). \tag{6.74}$$

The first, equation (6.72), gives $z_1 = f(z)+z$. Substituted into the second, equation (6.73), gives $z_2 = f(f(z) + z) - z$. Substituted into the final equation, equation (6.74), gives the the decisive equation:

$$z - f(f(z) + z) + f(z) + z = f(f[f(z) + z] - z). \tag{6.75}$$

That is:

$$2z = f(f(z) + z) - f(z) + f(f[f(z) + z] - z). \tag{6.76}$$

After staring at this for a while, we see that if $f(f(z)+z) = 2z$, then the right-hand side becomes $2z$ which is the left-hand side. Thus, if z is such that $f(f(z)+z) = 2z$, then z is a solution to equation (6.76), and therefore a 6-periodic point. By Rosenbloom's theorem 1.4.1), there are infinitely many such z.

It remains to verify that these give infinitely many *genuine* 6-periodic points. It follows that a non-genuine 6-periodic point is either a fixed point, a genuine 2-periodic point, or a genuine 3-periodic point. The only fixed point of the form $(z, -z)$ is the origin, so we can consider 2-periodic points and 3-periodic points.

We recall that $(z, -z)$ is a 2-periodic point for F_1 if and only if $f(z) = -2z$. By assumption, there can only be finitely many z for which this is true. Thus infinitely many of the 6-periodic points of F_1 of the form $(z, -z)$, are not 2-periodic points.

Finally, we consider 3-periodic points. We recall that $(z, -z)$ is a 3-periodic point for F_1, if and only if $f(z) = -z$. But then $f(f(z) + z) = f(0) = 0$. Thus from $f(f(z) + z) = 2z$, we conclude that $z = 0$. The assertion follows. □

Note that there are infinitely many symplectic transcendental Hénon maps which satisfy the conditions of the previous theorem, Theorem 6.3.1. Indeed, we can let $f(z) = -2z + P(z)e^{g(z)}$ for any odd polynomial P and any even non-constant entire function g. A concrete example is for instance:

$$f(z) = -2z + (z^3 + 6z + z^9)e^{\cos(z)}. \tag{6.77}$$

6.3.2 8-periodic points

As with 6-periodic points, our previous results do not apply to 8-periodic points because 8 is not prime. Nevertheless, we can give an interesting result:

Theorem 6.3.2. Let $F_1(z, w) = (f(z) - w, z)$ be a symplectic transcendental Hénon map where f is odd and $|Z(f)| < \infty$. Then F_1 admits infinitely many genuine 8-periodic points of the form (z, w) with $w = \frac{f(z)}{2}$.

Proof. The idea is to look for 8-periodic points of the form in the statement. Then the set of determining equations for 8-periodic points of F_1, becomes:

$$z_1 + \frac{f(z)}{2} = f(z) \tag{6.78}$$
$$z_2 + z = f(z_1) \tag{6.79}$$
$$z_3 + z_1 = f(z_2) \tag{6.80}$$
$$z_4 + z_2 = f(z_3) \tag{6.81}$$
$$z_5 + z_3 = f(z_4) \tag{6.82}$$
$$z_6 + z_4 = f(z_5) \tag{6.83}$$
$$\frac{f(z)}{2} + z_5 = f(z_6) \tag{6.84}$$
$$z + z_6 = f\left(\frac{f(z)}{2}\right). \tag{6.85}$$

The first equation, equation (6.78), gives $w = z_1 = \frac{f(z)}{2}$. Substituting this into the remaining equations, therefore gives:

$$z_2 + z = f(z_1) \tag{6.79}$$
$$z_3 + z_1 = f(z_2) \tag{6.80}$$
$$z_4 + z_2 = f(z_3) \tag{6.81}$$
$$z_5 + z_3 = f(z_4) \tag{6.82}$$
$$z_6 + z_4 = f(z_5) \tag{6.83}$$
$$z_1 + z_5 = f(z_6) \tag{6.86}$$
$$z + z_6 = f(z_1). \tag{6.87}$$

The first and last equation, respectively equation (6.79) and equation (6.87), are the same and $z_2 = z_6$. Substituting this into the remaining equations, then give that the first and last equation of the remaining ones, respectively equation (6.80) and equation (6.86), are the same, with $z_3 = z_5$, and so on. In general, we see that $z_2 = z_6, z_3 = z_5, z_4 = z_4$, and that we are left with only the equations:

$$z_2 + z = f(z_1) \tag{6.79}$$
$$z_3 + z_1 = f(z_2) \tag{6.80}$$
$$z_4 + z_2 = f(z_3) \tag{6.81}$$
$$2z_3 = f(z_4). \tag{6.88}$$

Solving downwards, we get:

$$z_2 = f(z_1) - z \tag{6.89}$$
$$z_3 = f(z_2) - z_1 = f[f(z_1) - z] - z_1 \tag{6.90}$$
$$z_4 = f(z_3) - z_2 = f\left(f[f(z_1) - z] - z_1\right) - f(z_1) + z. \tag{6.91}$$

The decisive equation is $2z_3 = f(z_4)$, which becomes then, with $z_1 = w = \frac{f(z)}{2}$:

$$\begin{aligned} &2f\left(f\left(\frac{f(z)}{2}\right) - z\right) - f(z) \\ &= f\left(f\left(f\left(f\left(\frac{f(z)}{2}\right) - z\right) - \frac{f(z)}{2}\right) - f\left(\frac{f(z)}{2}\right) + z\right) \end{aligned} \tag{6.92}$$

We glare at this for a long time. Then, finally, we try to see what happens if $f\left(\frac{f(z)}{2}\right) - z = 0$. The idea is that if we have this, then because f is odd and thus, $f(0) = 0$, many of the quantities in equation (6.92), vanish. Indeed, the left-hand side is simplified to:

$$\text{LHS} = -f(z), \tag{6.93}$$

while the right-hand side becomes:

$$\text{RHS} = f\left(f\left(-\frac{f(z)}{2}\right) - 0\right) = -f\left(f\left(\frac{f(z)}{2}\right)\right) = -f\left(z\right). \tag{6.94}$$

Thus if $z = f\left(\frac{f(z)}{2}\right)$, then z solves the decisive equation, equation (6.92), and it follows that (z, w) is a 8-periodic point for F_1. By Rosenbloom's theorem (Theorem 1.4.1), there are infinitely many such z.

It remains to verify that there are infinitely many *genuine* 8-periodic points of this form. If an 8-periodic point fails to be genuine, it must be a fixed point, a genuine 2-periodic point, or a genuine 4-periodic point.

(z, w) is a fixed point of F_1 if and only if $f(z) = 2z$ and $w = z$. We have $w = \frac{f(z)}{2}$ and $z = f(w)$. Hence we find from $z = w$, that $z = \frac{f(z)}{2} = f(z)$. That is

$\frac{f(z)}{2} = 0$ or $f(z) = 0$. There are by assumption, only finitely many such z. It follows that infinitely many of the 8-periodic points of the form (z, w) with $w = \frac{f(z)}{2}$ and $z = f(w)$, are not fixed points.

We proceed to consider 2-periodic points of F_1. We recall that $(z, w) \in \text{Fix}(F_1^2)$ if and only if $f(w) = 2z$ and $f(z) = 2w$. With $w = \frac{f(z)}{2}$, the last of these are automatically fulfilled. Thus, we consider the first: $f(w) = 2z$. By assumption, we have $z = f(w)$. Therefore we get $2z = z$ or $z = 0$. Thus the only 2-periodic point of F_1 of the form we are considering, is the origin.

Finally, we consider 4-periodic points for F_1. The system of determining equations for 4-periodic points for F_1, is given by, recall that $z_1 = w$:

$$2z_1 = f(z) \tag{6.95}$$

$$z_2 + z = f(z_1) \tag{6.96}$$

$$2z_1 = f(z_2) \tag{6.97}$$

$$z + z_2 = f(z_1). \tag{6.98}$$

The last and second, respectively equation (6.98) and equation (6.96), coincides. Thus we only have the three first equations, equations (6.95) - (6.97). We recall that: $z_1 = w = \frac{f(z)}{2}$. Thus: $z_2 = f\left(\frac{f(z)}{2}\right) - z$, and the decisive equation, equation (6.97), becomes:

$$f(z) = f\left(f\left(\frac{f(z)}{2}\right) - z\right). \tag{6.99}$$

Under our assumption, $z = f(w) = f\left(\frac{f(z)}{2}\right)$, and so this gives $f(z) = 0$. Again, by assumption, there are only finitely many such z. The assertion follows. □

Example 6.3.3. Let $P(z)$ be any odd polynomial and let $g(z)$ be any non-constant entire and even function. Then, it follows that the symplectic transcendental Hénon map $F_1(z, w) = (P(z)e^{g(z)} - w, z)$, admits infinitely many genuine 8-periodic points.

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