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# Einstein Gravity with Non-minimally Coupled Scalar Fields

Alexander Christian Phili Flo

Physics
Submission date: March 2018
Supervisor: Jan Myrheim, IFY
Co-supervisor: Kåre Olaussen, IFY

Norwegian University of Science and Technology
Department of Physics

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# Chapter 1

# Introduction

This thesis started with the intention to investigate a possible alternative explanation as to why the observations of galaxies did not fit with the commonly accepted theory of gravity. As time went on, our focus shifted to cosmology. We decided to investigate the relations between the models of Hilbert–Einstein and Huggins [1] seeking to understand the effect the expansion of the universe during inflation would have on the energy-momentum tensor. The Huggins term was used by Callan, Coleman and Jackiw to introduce an improved energy-momentum tensor [2]. In Chapter 2 the Huggins term will be explored through adding it to the Hilbert–Einstein field theory. In chapter 3 we solve the equations numerically through a python script and in chapter 4 we investigate the results of the python script.

We begin with a brief overview of the theoretical foundation for the thesis and the work that has led to what it has become. The Einstein equation of gravity with a cosmological constant $\Lambda$ reads [3]

$$G^{\mu\nu} - g^{\mu\nu}\Lambda = \kappa T^{\mu\nu}, \tag{1.1}$$

where $g^{\mu\nu}$ is the metric tensor, $G^{\mu\nu}$ is the Einstein tensor, $T^{\mu\nu}$ is the energy-momentum tensor, $G_N$ is the gravitational constant, $c$ is the speed of light and

$$\kappa = \frac{8\pi G_N}{c^4}\,. \tag{1.2}$$

We use here the metric signature $(+,-,-,-)$. The cosmological constant $\Lambda$ may be moved to the other side of the equation and reinterpreted as a vacuum energy density.

Taking the divergence of (1.1) and using the mathematical identities ${G^{\mu\nu}}_{;\nu} = 0$ and ${g^{\mu\nu}}_{;\nu} = 0$ we get as a consistency condition for the Einstein equation that

$$\Lambda_{,\rho} = -\kappa {T^{\nu}}_{\rho;\nu}\,. \tag{1.3}$$

Thus, if $\Lambda$ is really constant, the energy-momentum conservation equation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{1.4}$$

is a consistency condition.

In standard cosmology the universe is assumed to be homogeneous and isotropic. Then the line element of the spacetime geometry may be written as

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2\right] \tag{1.5}$$

where $a(t)$ is a time dependant scale factor and $k = 0, \pm 1$ is a measure of the curvature of space. The matter content is described by the energy-momentum tensor

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\epsilon, -P, -P, -P) \tag{1.6}$$

where the energy density $\epsilon$ and the pressure $P$ depend only on time. The Einstein equation then reduces to the two Friedmann equations

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = \frac{8\pi G_N}{3c^2}\,\epsilon \tag{1.7a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}\,(\epsilon + 3P). \tag{1.7b}$$

The energy-momentum conservation law $T^{\mu\nu}{}_{;\nu} = 0$ reduces to

$$\dot{\epsilon} + 3(\epsilon + P)\,\frac{\dot{a}}{a} = 0. \tag{1.8}$$

A further relation between $\epsilon$ and $P$ is the equation of state. This is usually defined by an equilibrium condition for instantaneous states of the universe, the same as one would have in thermodynamic equilibrium in a static universe. The equation of state may change as the universe evolves but is not assumed to depend on the rate of expansion of the universe. This description can be expected to be good as long as the universe changes slowly compared to the rates of thermal equilibrium. As a first correction one may proceed to *viscous cosmology*. In recent years attempts have been made to incorporate the universe's rate of expansion by adding a term

$$P_{\mathrm{visc}} = -3\eta_v\left(\frac{\dot{a}}{a}\right) \tag{1.9}$$

to the pressure. Here $\eta_v$ is the coefficient of bulk viscosity, also known as volume viscosity, sometimes denoted by the symbol $\xi$, which again will depend on the instantaneous state of the universe.

This provides a simple way to model the backreaction of expansion on the behaviour of matter. It may be a valid approach when the rate of expansion is comparable to but still smaller than the rate of thermal equilibration. Here, we will consider a different approach to the backreaction, considering effective field theory models where the Lagrangian depends on the geometry beyond its obvious dependence on the metric.

# Chapter 2

# General relativity with a Huggins term

In this Chapter, after establishing the general equations and conventions used, we will discuss the effects of adding a Huggins term (2.13c) to the Hilbert–Einstein and matter Lagrangian.

## 2.1 Conventions and general equations

The infinitesimal line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}y^\nu \tag{2.1}$$

is used here with a metric signature of $(+,-,-,-)$. The covariant derivative is defined as

$$D_\lambda = \partial_\lambda + \Gamma_\lambda \tag{2.2}$$

where $\Gamma_\lambda$ is a matrix, this derivative should be metric compatible, i.e.

$$D_\lambda g_{\mu\nu} = 0 \tag{2.3}$$

implying that the matrix elements are

$$(\Gamma_\lambda)^\mu{}_\nu = \Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}g^{\mu\sigma}\left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\lambda\nu,\sigma}\right). \tag{2.4}$$

The Riemann tensor is defined as

$$R_{\lambda\rho} = [D_\lambda, D_\rho] = \partial_\lambda\Gamma_\rho - \partial_\rho\Gamma_\lambda + [\Gamma_\lambda, \Gamma_\rho] \tag{2.5}$$

with the property

$$(R_{\lambda\rho})^\alpha{}_\beta = R^\alpha{}_{\beta\lambda\rho}\,. \tag{2.6}$$

It follows that the Riemann tensor can be written as

$$R^{\alpha}{}_{\beta\lambda\rho} = \partial_\lambda \Gamma^{\alpha}{}_{\beta\rho} - \partial_\lambda \Gamma^{\alpha}{}_{\beta\rho} + \partial_\rho \Gamma^{\alpha}{}_{\beta\lambda} + \Gamma^{\alpha}{}_{\gamma\lambda}\Gamma^{\gamma}{}_{\beta\lambda} - \Gamma^{\alpha}{}_{\gamma\rho}\Gamma^{\gamma}{}_{\beta\rho}. \tag{2.7}$$

The Ricci tensor is defined as

$$R_{\beta\rho} = R^{\alpha}{}_{\beta\alpha\rho} \,, \tag{2.8}$$

and the Ricci scalar is

$$R = g^{\beta\rho} R_{\beta\rho} \,. \tag{2.9}$$

The Einstein tensor is

$$G_{\beta\rho} = R_{\beta\rho} - \frac{1}{2} R g_{\beta\rho} \,. \tag{2.10}$$

The functional derivative is defined as

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon \delta(x-y)] - F[f(x)]}{\epsilon} \,. \tag{2.11}$$

## 2.2 The action

The action $S$ is defined as a space-time integral

$$S = \int \mathrm{d}^4 x \, \sqrt{-g} \, \mathcal{L} \,. \tag{2.12}$$

The lagrangian $\mathcal{L}$ is a sum of three terms, the Hilbert–Einstein term with a cosmological constant, the matter term with the scalar field $\varphi$, and the Huggins term,

$$\mathcal{L}_{\mathrm{HE}} = -\frac{c^4}{16\pi G_N}(R + 2\Lambda) = -\frac{1}{2\kappa}(R + 2\Lambda) \,, \tag{2.13a}$$

$$\mathcal{L}_{\mathrm{matter}} = \frac{1}{2} \, g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi) \,, \tag{2.13b}$$

$$\mathcal{L}_{\mathrm{Huggins}} = \frac{1}{2} \xi \varphi^2 R \,. \tag{2.13c}$$

Here $V(\varphi)$ is potential of the scalar field $\varphi$. The Euler-Lagrange equations obtained by varying the metric become

$$-\frac{1}{\kappa}\left(G_{\mu\nu} - g_{\mu\nu}\Lambda\right) + T_{\mu\nu} = 0 \tag{2.14}$$

with the energy momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta(S_{\text{matter}} + S_{\text{Huggins}})}{\delta g^{\mu\nu}} \\ &= \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\,\partial^\alpha\varphi\,\partial_\alpha\varphi + g_{\mu\nu}\,V(\varphi) \\ &\quad + \xi\left[G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right]\varphi^2 . \end{aligned} \tag{2.15}$$

The Euler–Lagrange equation we get by varying $\varphi$ is

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\,\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi + V'(\varphi) - \xi R\varphi = 0 \tag{2.16}$$

For the potential $V(\varphi) = \lambda^2\varphi^2/2$ and no coupling with $\xi = 0$ this gives the Klein–Gordon equation

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\,\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi + \lambda^2\varphi = 0 \tag{2.17}$$

and the Klein–Gordon energy-momentum tensor

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2} g_{\mu\nu}\varphi^{,\alpha}\varphi_{,\alpha} + g_{\mu\nu}\frac{\lambda^2}{2}\,\varphi^2 . \tag{2.18}$$

We will call (2.16) the generalized Klein–Gordon equation.

## 2.3 Specialization to the Friedmann–Lemaître–Robertson–Walker universe

In this section we will see how the general equations reduce to the the Friedmann equations for the FLRW model.

The line element (1.5) results in the metric tensor

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\dfrac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{bmatrix} \tag{2.19}$$

and its inverse

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\dfrac{1-kr^2}{a^2} & 0 & 0 \\ 0 & 0 & -\dfrac{1}{a^2r^2} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{a^2r^2\sin^2\theta} \end{bmatrix} \tag{2.20}$$

with the square root of the metric tensor determinant being

$$\sqrt{-g} = \frac{a^3 r^2 \sin\theta}{\sqrt{1-kr^2}} \tag{2.21}$$

and the nonzero connection coefficients

$$\begin{aligned}
&\Gamma^0_{11} = \frac{1}{c}\frac{a\dot{a}}{1-kr^2}\,, \qquad \Gamma^0_{22} = -\frac{1}{c}\,a\dot{a}r^2\,, \qquad \Gamma^0_{33} = -\frac{1}{c}\,a\dot{a}r^2\sin^2\theta\,,\\
&\Gamma^1_{01} = \frac{\dot{a}}{a}\,, \qquad \Gamma^1_{11} = \frac{kr}{1-kr^2}\,, \qquad \Gamma^1_{21} = \frac{1}{r}\,, \qquad \Gamma^1_{31} = \frac{1}{r}\,,\\
&\Gamma^2_{02} = \frac{\dot{a}}{a}\,, \qquad \Gamma^2_{12} = \frac{1}{r}\,, \qquad \Gamma^3_{03} = \frac{\dot{a}}{a}\,, \qquad \Gamma^3_{13} = \frac{1}{r}\,, \qquad \Gamma^3_{23} = \cot\theta
\end{aligned} \tag{2.22}$$

The connection is symmetric, $\Gamma^\lambda_{\mu\nu} = \Gamma^\nu_{\lambda\mu}$. The Einstein tensor (2.10) is diagonal with elements

$$G^0{}_0 = \frac{3}{c^2}\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) \tag{2.23a}$$

$$G^1{}_1 = G^2{}_2 = G^3{}_3 = \frac{1}{c^2}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right). \tag{2.23b}$$

The Ricci scalar is

$$R = -G^\mu{}_\mu = -\frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right). \tag{2.24}$$

### The energy-momentum tensor

We define the four vector $W$ to be the gradient of the time dependent $\varphi^2(t)$,

$$W_\nu = \nabla_\nu\varphi^2 = \partial_\nu\varphi^2 = \left(\frac{2}{c}\,\varphi\dot{\varphi}, 0, 0, 0\right) \tag{2.25}$$

Applying another derivative to $W_\nu$ we obtain the covariant derivative

$$\nabla_\mu\nabla_\nu\varphi^2 = \nabla_\mu W_\nu = \partial_\mu W_\nu - \Gamma^\rho_{\mu\nu}W_\rho = \partial_\mu\partial_\nu\varphi^2 - \Gamma^0_{\mu\nu}\partial_0\varphi^2\,. \tag{2.26}$$

Only the diagonal elements are nonzero

$$\nabla_0\nabla_0\varphi^2 = \partial_0^2\varphi^2, \tag{2.27}$$

$$\nabla_i \nabla_i \varphi^2 = -\Gamma^0_{ii} \frac{2}{c} \varphi\dot{\varphi}. \tag{2.28}$$

The wave operator in curved space-time applied to $\varphi^2$ gives

$$\begin{aligned} \Box\varphi^2 &= \frac{1}{\sqrt{-g}}\,\partial_\mu \sqrt{-g}\, g^{\mu\nu} \partial_\nu \varphi^2 = \frac{1}{\sqrt{-g}}\,\partial_0 \sqrt{-g}\,\partial_0\,\varphi^2 \\ &= \frac{2}{c^2}\left(\dot{\varphi}^2 + \varphi\ddot{\varphi} + 3\frac{\dot{a}}{a}\varphi\dot{\varphi}\right). \end{aligned} \tag{2.29}$$

The energy density is

$$\epsilon = T^0{}_0 = \frac{\dot{\varphi}^2}{2c^2} + V(\varphi) + \xi\left(G^0{}_0\varphi^2 + \frac{6}{c^2}\,\frac{\dot{a}}{a}\,\varphi\dot{\varphi}\right) \tag{2.30}$$

and the pressure is

$$P = -T^1{}_1 = -T^2{}_2 = -T^3{}_3 \tag{2.31}$$

$$= \frac{\dot{\varphi}^2}{2c^2} - V(\varphi) - \xi\left(G^1{}_1\varphi^2 + \frac{2}{c^2}\left[\dot{\varphi}^2 + \varphi\ddot{\varphi} + 2\,\frac{\dot{a}}{a}\varphi\dot{\varphi}\right]\right). \tag{2.32}$$

### The scalar field equation

The Euler–Lagrange equation we get by varying $\varphi$ is

$$\frac{1}{c^2}\,\ddot{\varphi} + \frac{3}{c^2}\,\frac{\dot{a}}{a}\,\dot{\varphi} + V'(\varphi) - \xi R\varphi = 0 \tag{2.33}$$

where $R$ is given by the equation (2.24). For simplicity we will call this the Klein–Gordon equation.

## 2.4 Summary of the equations

Altogether we have the gravitational equations (1.1) with a cosmological constant term and the generalized Klein–Gordon equation (2.16). We use two of the gravitational equations,

$$G^0{}_0 = \kappa\epsilon + \Lambda\,, \qquad G^1{}_1 = -\kappa P + \Lambda\,. \tag{2.34}$$

By using (2.23a) and (2.30) the first equation takes the form

$$3\left(1 - \xi\kappa\varphi^2\right)\left(\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) = \kappa\left(\frac{1}{2}\dot{\varphi}^2 + c^2 V(\varphi) + 6\xi\,\frac{\dot{a}}{a}\,\varphi\dot{\varphi}\right) + \Lambda c^2\,. \tag{2.35}$$

To eliminate $\Lambda$ and $V(\varphi)$ we take the difference of the two equations,

$$G^0{}_0 - G^1{}_1 = \kappa(\epsilon + P)\,. \tag{2.36}$$

By using (2.23a), (2.23b), (2.30), and (2.39) this equation takes the form

$$2(1-\xi\kappa\varphi^2)\left(-\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{kc^2}{a^2}\right) = \kappa\left(\dot{\varphi}^2+2\xi\left[-\dot{\varphi}^2-\varphi\ddot{\varphi}+\frac{\dot{a}}{a}\varphi\dot{\varphi}\right]\right). \tag{2.37}$$

The generalized Klein–Gordon equation is

$$\ddot{\varphi}+3\,\frac{\dot{a}}{a}\,\dot{\varphi}+V'(\varphi) = -6\xi\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{kc^2}{a^2}\right)\varphi\,. \tag{2.38}$$

Thus we have three equations for the two unknowns $a(t)$ and $\varphi(t)$. This is consistent because the energy conservation equation (1.8) is satisfied.

### Analysis of the equations

The first equation, (2.35), contains no higher than first derivatives of the variables $a$ and $\varphi$. There is one coupling term in the equation that prevents separation of the variables. This equation is a constraint equation that has to be satisfied at all times. In particular, we have to take it into account when choosing initial values for $a, \dot{a}, \varphi, \dot{\varphi}$, and $\Lambda$ in our numerical integrations.

When we integrate the other equations numerically, we use equation (2.35) to compute $\Lambda$ as a function of $a, \dot{a}, \varphi, \dot{\varphi}$. It is an important test of the precision of our numerical integration that $\Lambda$ computed in this way should be constant in time.

The second and third equations, (2.37) and (2.38), are two linear equations for the second derivates of $a$ and $\varphi$. They do not contain $\Lambda$. It is these two equations that we integrate numerically. They have the form

$$\begin{aligned} c_{11}\ddot{a}+c_{12}\ddot{\varphi} &= A(a,\dot{a},\varphi,\dot{\varphi})\,,\\ c_{21}\ddot{a}+c_{22}\ddot{\varphi} &= B(a,\dot{a},\varphi,\dot{\varphi})\,. \end{aligned} \tag{2.39}$$

The determinant of the equation system is

$$c_{11}c_{22}-c_{12}c_{21} = -\frac{2}{a}\,(1+\xi(6\xi-1)\kappa\varphi^2)\,. \tag{2.40}$$

The determinant can vanish completely for certain values of $\varphi^2$ if $0 < \xi < 1/6$. In the two cases $\xi = 0$ and $\xi = 1/6$ the determinant is no longer dependant on $\varphi$ and is determined simply by $-2/a$.

# Chapter 3

# Solving the equations

## 3.1 Units

It is convenient to choose units such that

$$\begin{aligned} c &= 1 = 299\,792\,458 \text{ m/s}\,, \\ G_N &= 1 = 6.6738 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}\,, \\ \kappa &= 8\pi\,. \end{aligned} \tag{3.1}$$

We are still free to choose a time unit. A convenient time unit in cosmology may be

$$t_0 = 10^9 \text{ year} = 3.1558 \times 10^{16} \text{ s}\,. \tag{3.2}$$

This fixes the length unit as

$$L_0 = 10^9 \text{ light year} = 9.4607 \times 10^{24} \text{ m}\,. \tag{3.3}$$

Then we get

$$\begin{aligned} G_N = 1 &= 6.6738 \times 10^{-11}\, \frac{L_0^3}{(9.4607 \times 10^{24})^3} \text{ kg}^{-1}\, \frac{(3.1558 \times 10^{16})^2}{t_0^2} \\ &= 7.9244 \times 10^{-53}\, L_0^3 \text{ kg}^{-1}\, t_0^{-2} = L_0^3\, M_0^{-1}\, t_0^{-2}\,, \end{aligned} \tag{3.4}$$

when we introduce the mass unit

$$M_0 = 1.2619 \times 10^{52} \text{ kg}\,. \tag{3.5}$$

This is about the total mass of the Universe [4].

In these units, the value of the cosmological constant, from the observations with the Planck satellite, is

$$\Lambda = 1.11 \times 10^{-52} \text{ m}^{-2} = 0.0099\, L_0^{-2}\,. \tag{3.6}$$

The reduced Planck's constant is

$$\hbar = 1.0546 \times 10^{-34}\ \mathrm{kg\,m^2\,s^{-1}} = 2.9466 \times 10^{-120}\ M_0\ L_0^2\ t_0^{-1}\,. \tag{3.7}$$

The unit of the parameter $\lambda$ of the scalar field is inverse length. Quantization of the scalar field with the potential $V(\varphi) = \lambda^2\varphi^2/2$ gives field quanta of mass

$$m = \frac{\hbar\lambda}{c} = 2.9466 \times 10^{-120}\ M_0\ L_0\ \lambda\,. \tag{3.8}$$

Thus, if we take $\lambda = 1/L_0$ we get

$$m = \frac{\hbar\lambda}{c} = 2.9466 \times 10^{-120}\ M_0 = 3.7183 \times 10^{-68}\ \mathrm{kg}\,. \tag{3.9}$$

This is a very small mass, about $10^{-38}$ times the mass of the electron, or $10^{-33}$ times the experimental upper limit on the mass of the electron neutrino.

## 3.2 Choosing the initial conditions

For the sake of simplicity we set $k = 0$, $c = 1$ and $V(\varphi) = \lambda^2\varphi^2/2$ thus (2.35) takes the form

$$3\left(1 - \xi\kappa\varphi^2\right)\frac{\dot{a}^2}{a^2} = \kappa\left(\frac{\dot{\varphi}^2}{2} + \frac{\lambda^2\varphi^2}{2} + 6\,\xi\frac{\dot{a}}{a}\varphi\dot{\varphi}\right) + \Lambda. \tag{3.10}$$

and equation (2.38) takes the form

$$\ddot{\varphi} + 3\,\frac{\dot{a}}{a} + \lambda^2\varphi = -6\xi\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\varphi\,. \tag{3.11}$$

Equation (2.37) takes the form

$$2(1 - \xi\kappa\varphi^2)\left(-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \kappa\left(\dot{\varphi}^2 + 2\xi\left[-\dot{\varphi}^2 - \varphi\ddot{\varphi} + \frac{\dot{a}}{a}\varphi\dot{\varphi}\right]\right). \tag{3.12}$$

Now we can choose the initial values of $a$, $\varphi$, $\dot{a}$, and $\dot{\varphi}$, in order to simplify as much as possible. We choose $\varphi(0) = 0$ and $a(0) = 1$ and reduce the initial value condition (3.10) to the simple relation between $\dot{a}(0)$ and $\dot{\varphi}(0)$

$$(\dot{a}(0))^2 = \kappa\,\frac{(\dot{\varphi}(0))^2}{6} + \frac{\Lambda}{3}\,. \tag{3.13}$$

We observe from equation (3.10) that the equation for $\dot{a}(0)$ becomes a more general second order equation when $\varphi(0) \neq 0$ and $\xi \neq 0$.

We choose to investigate the two values $\xi = 0$ and $\xi = 1/6$ where the determinant (2.40) is guaranteed to be non-zero.

We take the constraint equation, (2.35), as a definition of $\Lambda(t)$. It is a test of the accuracy of the calculations that $\Lambda(t)$ should be constant. We therefore plot $\Lambda(t) - \Lambda(0)$. This difference is seen to fluctuate below $10^{-7}$ which is a reasonable numerical accuracy. It is also a non-trivial verification that the analytical formulas are correct.

# Chapter 4

# Numerical examples

The calculations done in the Python script use the units from section 3.1 where $c = 1$ and $G_N = 1$. The time unit is $t_0 = 10^9$ years. We choose to set $a(0) = 1$, $\varphi(0) = 0$, and $\dot{\varphi}(0) = 0.1$ for all plots and only vary the constants $\xi$, $\Lambda$, and $\lambda$.

We present here plots of the scale parameter $a(t)$, the scalar field $\varphi(t)$, the Hubble parameter $\dot{a}(t)/a(t)$, and the numerical error parameter $\Lambda(t) - \Lambda(0)$ for 12 different cases. Note that $\Lambda(t) - \Lambda(0)$ is always multiplied by $10^9$.

Note also that when we take $\lambda = 1$, which means $\lambda = 1/L_0$, more realistic values in terms of known particles would be $10^{30}/L_0$ or $10^{40}/L_0$, which would give extremely rapid oscillations of $\varphi$ on cosmological time scales.

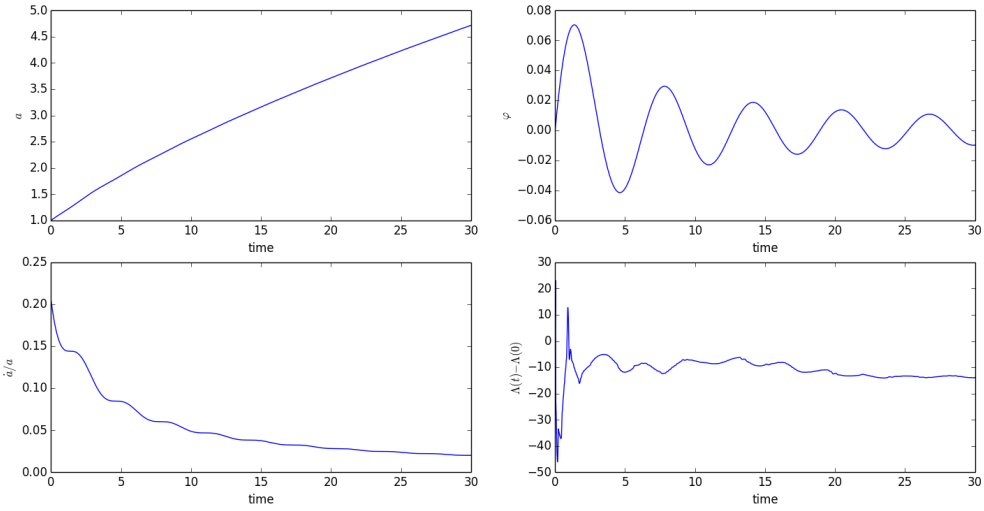


Figure 4.1: $\xi = 0$, $\Lambda = 0$, $\lambda = 1$

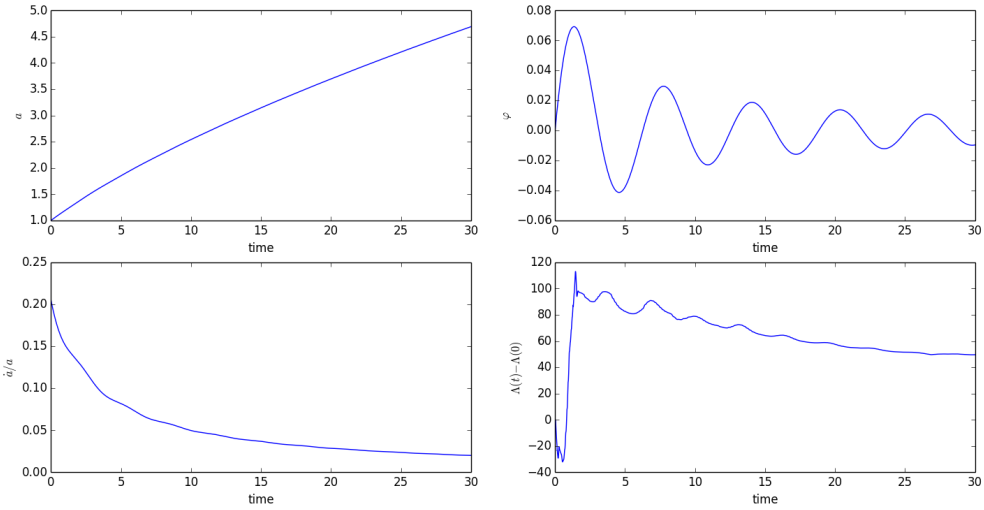


Figure 4.2: $\xi = 1/6$, $\Lambda = 0$, $\lambda = 1$

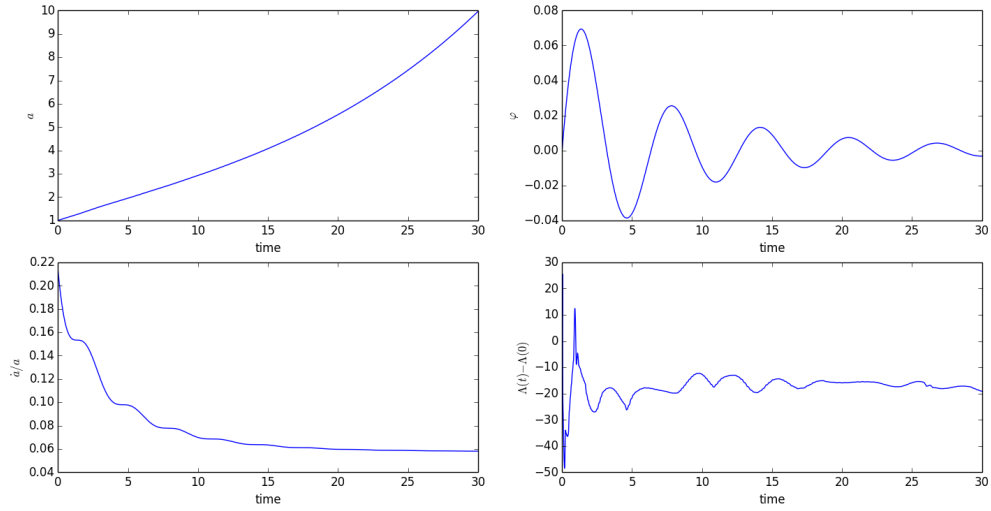


Figure 4.3: $\xi = 0$, $\Lambda = 0.01$, $\lambda = 1$

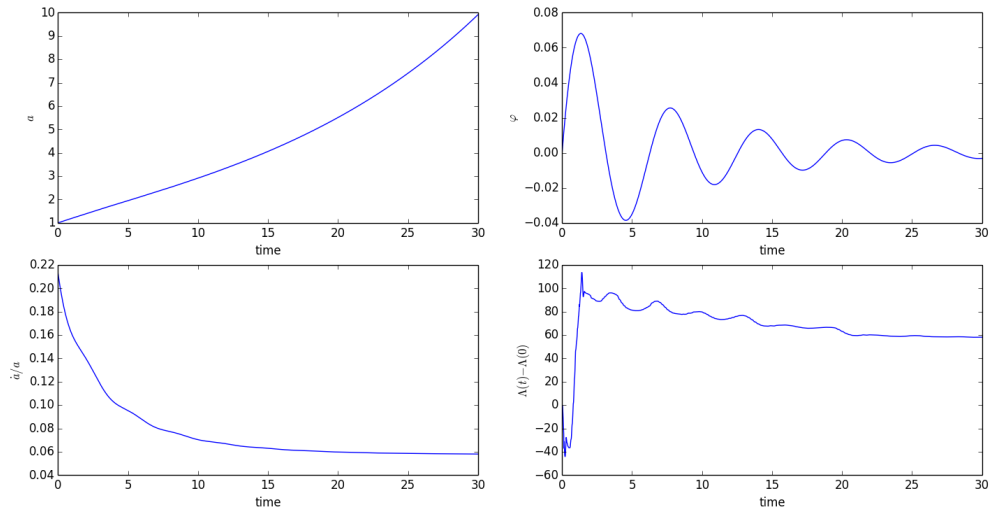


Figure 4.4: $\xi = 1/6$, $\Lambda = 0.01$, $\lambda = 1$

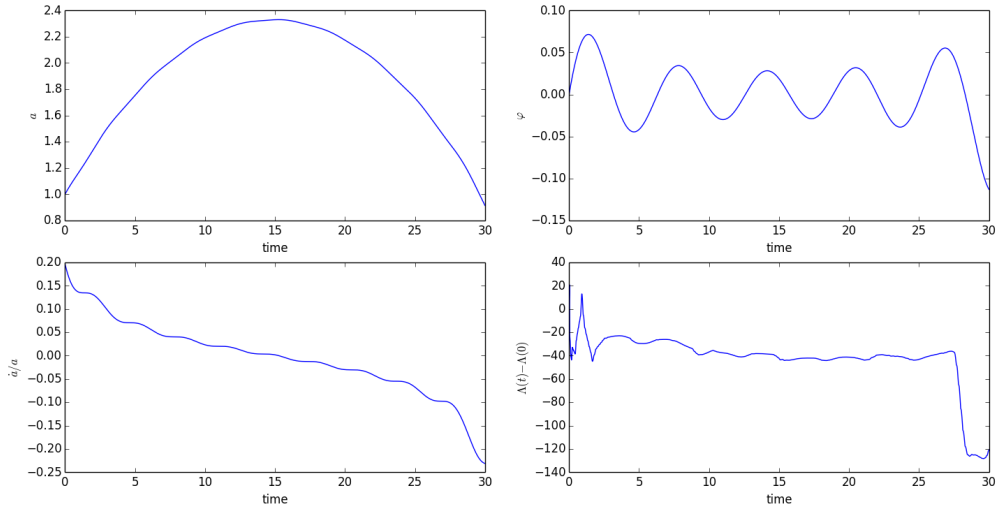


Figure 4.5: $\xi = 0$, $\Lambda = -0.01$, $\lambda = 1$

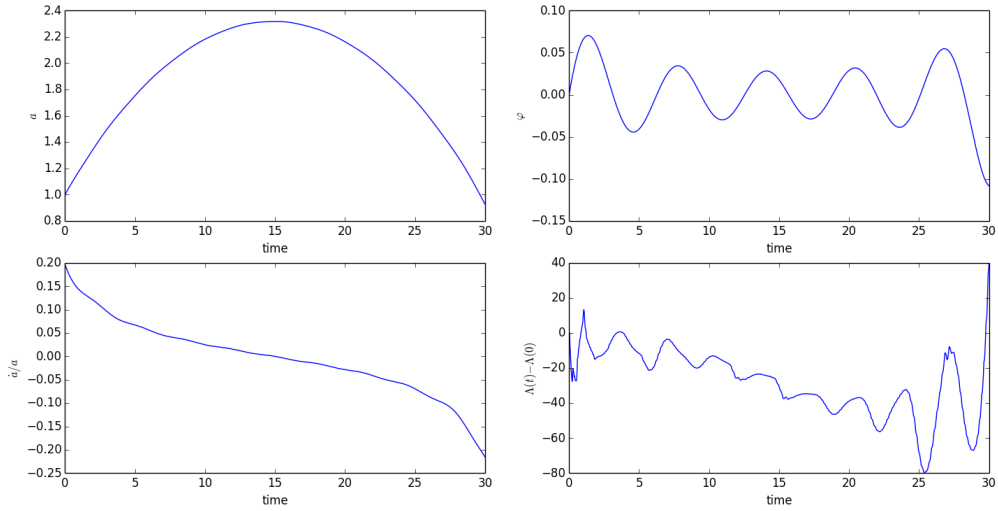


Figure 4.6: $\xi = 1/6$, $\Lambda = -0.01$, $\lambda = 1$

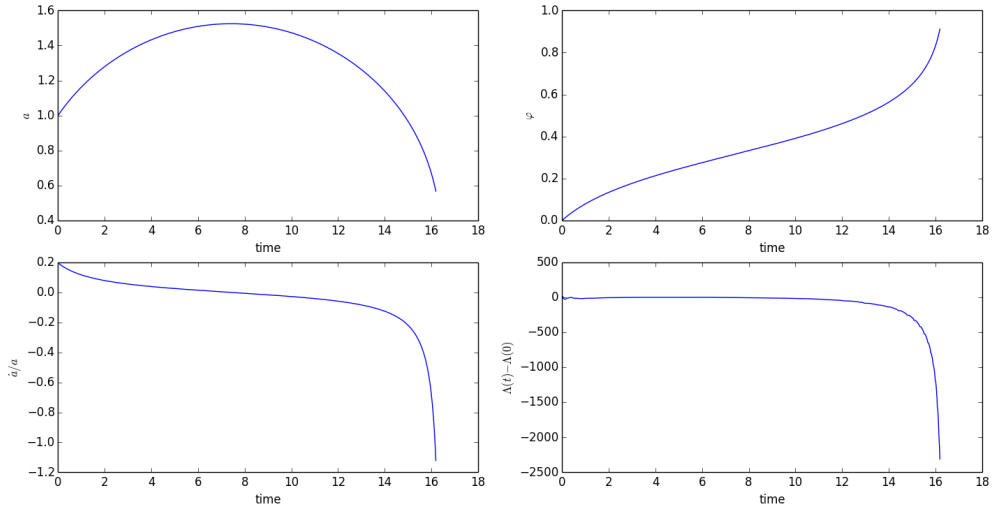


Figure 4.7: $\xi = 0$, $\Lambda = 0$, $\lambda = 0$

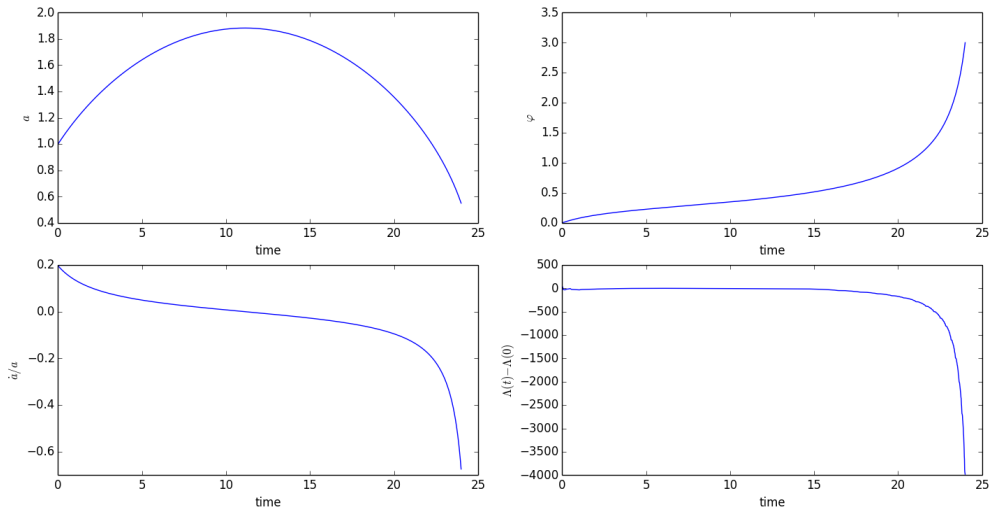


Figure 4.8: $\xi = 1/6$, $\Lambda = 0$, $\lambda = 0$

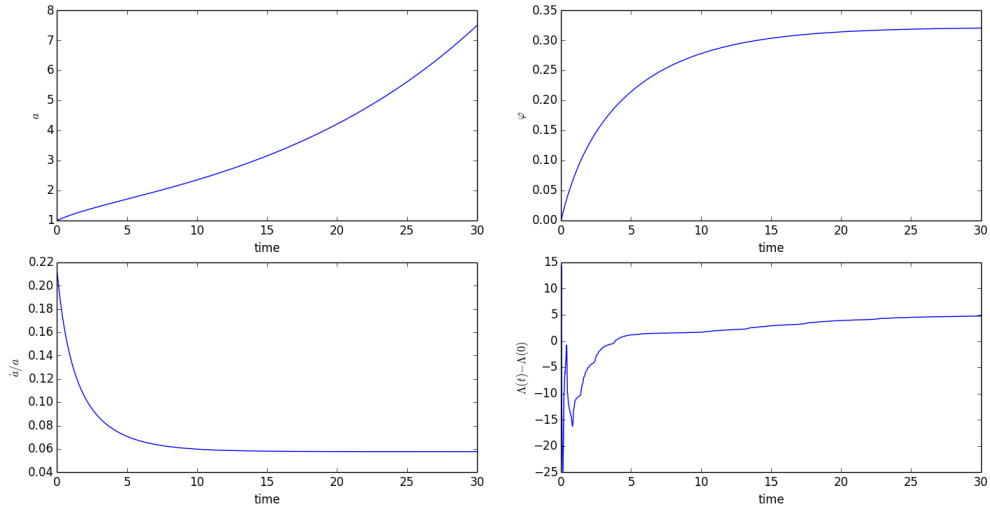


Figure 4.9: $\xi = 0$, $\Lambda = 0.01$, $\lambda = 0$

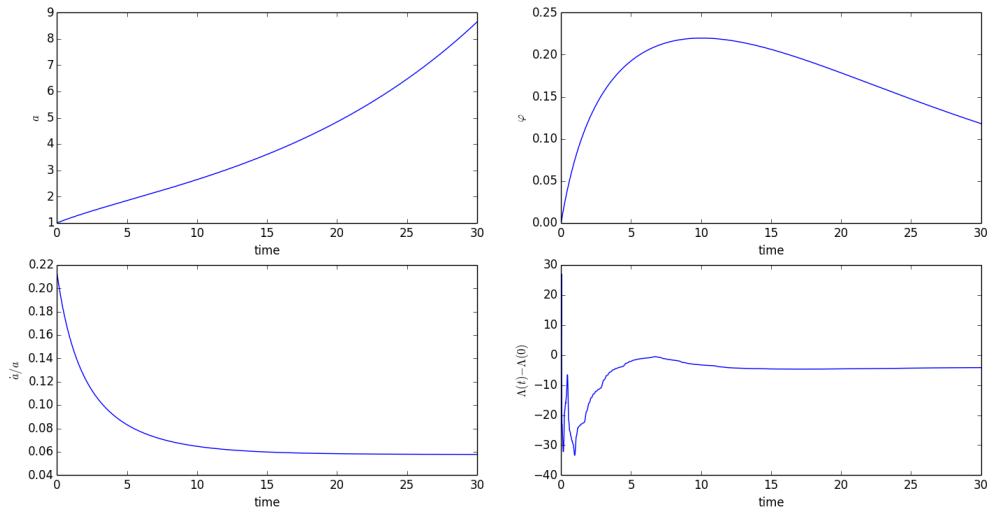


Figure 4.10: $\xi = 1/6$, $\Lambda = 0.01$, $\lambda = 0$

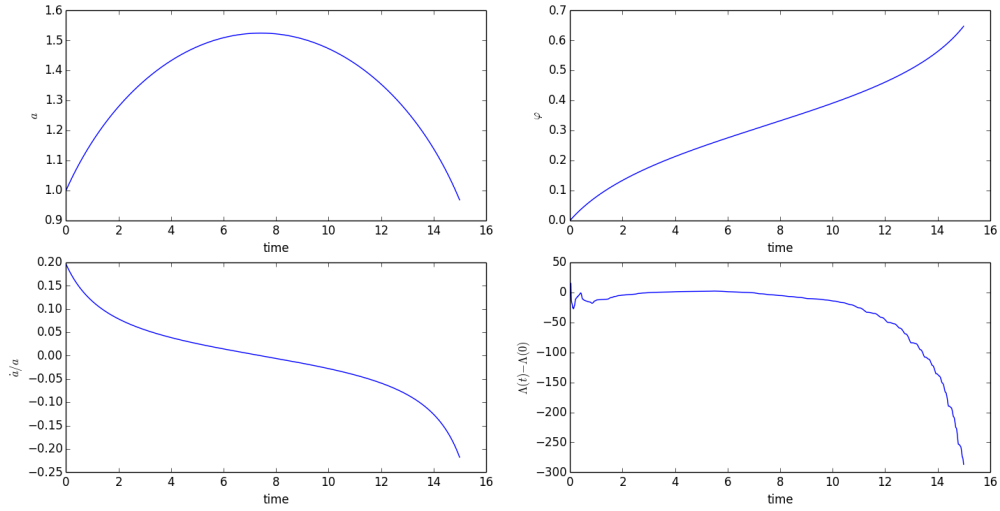


Figure 4.11: $\xi = 0$, $\Lambda = -0.01$, $\lambda = 0$

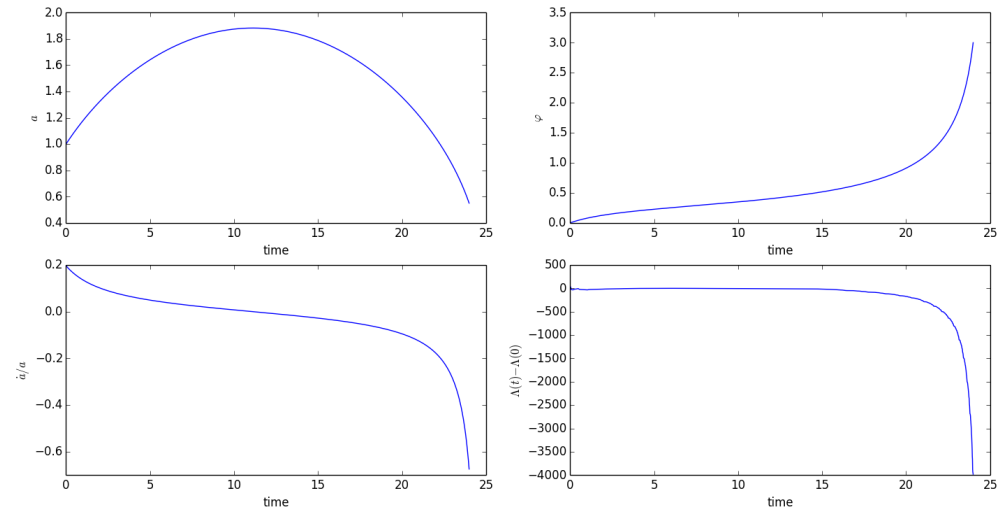


Figure 4.12: $\xi = 1/6$, $\Lambda = -0.01$, $\lambda = 0$

## 4.1 Conclusions and outlook

We see that, in the cases we have plotted, a positive value of the cosmological constant $\Lambda$ makes the universe expand exponentially for large times, whereas a negative $\Lambda$ makes the Universe collapse in a finite time. We see little difference between the cases $\xi = 0$ and $\xi = 1/6$, except that the oscillations around a smooth curve of the Hubble parameter $\dot{a}/a$, due to the oscillations of the scalar field $\varphi$, are smoothed out when $\xi = 1/6$.

For the Figures 1 to 6 we have set $\lambda = 1$, which makes $\varphi$ oscillate and go to zero as the universe expands. This allows the expansion to continue indefinitely when $\Lambda = 0$.

For the Figures 7 through 12 we have set $\lambda = 0$, which removes the oscillation of $\varphi$ and allows it to grow. In the case $\Lambda = 0$ this makes the universe collapse in a finite time. In the case $\Lambda > 0$, on the other hand, the exponential growth of the scale factor $a$ stops the growth of $\varphi$, and makes it go to zero when $\xi = 1/6$.

We have investigated here only two different values of the coupling constant $\xi$ of the Huggins term. Other cases may be more interesting, in particular $0 < \xi < 1/6$ where the determinant (2.40) becomes zero for certain values of $\varphi$, changing the second order nature of the differential equations.

In general, a nonzero value of $\xi$ may have effects simulating a variable gravitational constant. We conclude that there may be room for further investigations of the present model.

# Appendix A

# Python code

```
from __future__ import division
import sys
import time
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt
import scipy as sp

from scipy.integrate import odeint

xi=1.0/6.0
kcc=0.0
#kappa=1
kappa=8*np.pi
lmbda=0
c=1
a0=1
varphi0=0.0
dot_varphi0=0.1
#Lambdacc0=0.01
Lambdacc0=0.01
#dot_a0=np.sqrt(kappa/6)*np.abs(dot_varphi0)
dot_a0=np.sqrt((a0**2 /3)*(kappa*dot_varphi0**2/2+Lambdacc0))
print(dot_a0)
#dot_a0=0.2
#quadsolve_a=1
#quadsolve_b=(-2*xi*kappa*(a0)*(c**2)*varphi0*dot_varphi0) /
```

```
#                       (1-xi*kappa*varphi0**2)
#quadsolve_c=(-kappa*(a0**2)*(c**2))*(dot_varphi0**2 +
#                       (lmbda**2)*(varphi0**2)/2)/
#                       (3*(1-xi*kappa*(varphi0**2)))

quadsolve_aL=(1-xi*kappa*varphi0**2)*3/a0**2
quadsolve_bL=-6*(kappa*xi*varphi0*dot_varphi0/a0)
quadsolve_cL=-(kappa*(dot_varphi0**2/2 + lmbda**2*varphi0**2/2)
        +Lambdacc0)
dot_a0=(-quadsolve_bL+np.sqrt(quadsolve_bL**2-4*quadsolve_aL*
                        quadsolve_cL))/(2*quadsolve_aL)
print(dot_a0)
#dot_a0=(-quadsolve_b+np.sqrt(quadsolve_b**2-4*quadsolve_a*
#         quadsolve_c))/(2*quadsolve_a)

#dot_a0=(-quadsolve_b-np.sqrt(quadsolve_b**2-4*quadsolve_a*
#         quadsolve_c))/(2*quadsolve_a)
y0=[a0,varphi0,dot_a0,dot_varphi0]
t_stop=2000
t_inc=0.01
t=np.arange(0.,t_stop, t_inc)


# Den kosmologiske konstanten som funksjon av de andre variablene
def big_lambdacc(y):
        a = y[0]
        varphi = y[1]
        dot_a = y[2]
        dot_varphi = y[3]
        lhs=(1-xi*kappa*varphi**2)*3*(dot_a**2+kcc)/a**2
        rhs=kappa*(0.5*dot_varphi**2 +0.5*c**2*lmbda**2*varphi**2 +
                        xi*6*(dot_a/a)*varphi*dot_varphi)
        #eq 2.37
        return lhs-rhs


#def derivert(a,varphi,dot_a,dot_varphi):
def derivert(y,t):
        #dobbelt_derivert=np.empty(2)*0 #array for
        a = y[0]
```

```
        varphi = y[1]
        dot_a = y[2]
        dot_varphi = y[3]
        coeff11=-2*(1-xi*kappa*varphi**2)/a #ddot_a coeff
        coeff12=2*xi*kappa*varphi #ddot_varphi coeff
        A=kappa*(dot_varphi**2+2*xi*(-dot_varphi**2+(dot_a/a)*
                varphi*dot_varphi))-2*(1-xi*kappa*varphi**2)*
                (dot_a**2+kcc)/a**2
        coeff21=6*xi*varphi/a
        coeff22=1
        B=(-6*xi*(dot_a**2+kcc)/a**2 - c**2*lmbda**2)*varphi -
                3*(dot_a/a)*dot_varphi
        det=coeff11*coeff22-coeff12*coeff21
        ddot_a=(coeff22*A-coeff12*B)/det          #ddot_a
        ddot_varphi=(coeff11*B-coeff21*A)/det     #ddot_varphi
        return [dot_a, dot_varphi, ddot_a, ddot_varphi]


odesol=odeint(derivert, y0, t)

#fig = plt.figure(0, figsize=(8,8))
#plt.plot(t,odesol[:,0])
fig = plt.figure(1, figsize=(8,8))

#fig = plt.figure(1, figsize=(8,8))
#plt.plot(t,odesol[:,1])

#fig = plt.figure(2, figsize=(8,8))
#plt.plot(t,odesol[:,2])

#fig = plt.figure(3, figsize=(8,8))
#plt.plot(t,odesol[:,3])
#fig = plt.figure(1, figsize=(8,8))

#fig = plt.figure(0, figsize=(8,8))
#plt.plot(t,odesol[:,0])
ax1 = fig.add_subplot(221)
ax1.plot(t, odesol[:,0])
ax1.set_xlabel('time')
ax1.set_ylabel(r'$a$')
```

```
#fig = plt.figure(1, figsize=(8,8))
#plt.plot(t,odesol[:,1])

#fig = plt.figure(2, figsize=(8,8))
#plt.plot(t,odesol[:,2])
ax1 = fig.add_subplot(222)
ax1.plot(t, odesol[:,1])
ax1.set_xlabel('time')
ax1.set_ylabel(r'$\varphi_$')

#fig = plt.figure(3, figsize=(8,8))
#plt.plot(t,odesol[:,3])
ax1 = fig.add_subplot(223)
ax1.plot(t, (odesol[:,2]/odesol[:,0]))
ax1.set_xlabel('time')
ax1.set_ylabel(r'$\dot{a}/a$')
##plt.ylabel('some numbers')


test = np.matrix.copy(t)

#Lambdacc = big_lambdacc(y0)
Lambdacc=Lambdacc0
print(Lambdacc0)
print(big_lambdacc(y0))

ii=0
for tt in t:
    test[ii] = big_lambdacc(odesol[ii,:])-Lambdacc0
    ii+=1

#fig = plt.figure(5, figsize=(8,8))
#plt.plot(t,1e9*test)
#fig = plt.figure(5, figsize=(8,8))
#plt.plot(t,1e9*test)
ax1 = fig.add_subplot(224)
ax1.plot(t, 1e9*test)
ax1.set_xlabel('time')
ax1.set_ylabel(r'$\Lambda_(t)-\Lambda(0)$')
```

```
plt.show()
```

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