Split-Plot Designs for Multistage Experimentation

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ABSTRACT

Most of today's complex systems and processes involve several stages through which input or the raw material has to go before the final product is obtained. Also in many cases factors at different stages interact. Therefore a holistic approach for experimentation that considers all stages at the same time will be more efficient. However there have been only a few attempts in the literature to provide an adequate and easy-to-use approach for this problem. In this paper, we present a novel methodology for constructing two-level split-plot and multistage experiments. The methodology is based on the Kronecker product representation of orthogonal designs and can be used for any number of stages, for various numbers of subplots and for different number of subplots for each stage. The procedure is demonstrated on both regular and nonregular designs and provides the maximum number of factors that be accommodated in each stage. Furthermore split-plot designs for multistage experiments with good projective properties are also provided.

KEY WORDS: Kronecker product, Mirror image pairs, Projectivity, Restrictions on randomization, Two-level designs.

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1. INTRODUCTION

Experimenters are usually recommended to execute their experiments in random order. In many experimental situations, however, complete randomization is often not feasible and sometimes not even possible due to for example the difficulty in randomly changing the levels of certain factors. In general, it is not unusual to have some factors that are harder to change than others. A sensible strategy is then to put restrictions on randomization and run several experiments in the easy to change factors for a given level (or combination of levels) of the hard to change factor(s). As in many methods in experimental design, the original suggestion of running experiments in this fashion dates back to agricultural experiments. For example Yates (1935) recommends "splitting of plots for subsidiary treatments" when applications of some combinations of treatments in small plots becomes exceedingly difficult. Such experiments are therefore called *split-plot experiments* where combinations of levels of hard to change factors form *whole plots* for which experiments for various combinations of levels of easy to change factors are run to form the *subplots*.

Various studies in industrial experimentation also show that putting restrictions on randomization is very common in industrial setting and usually provide more efficient experiments as discussed in Daniel (1976), Box and Jones (1992) and Goos and Vandebroek (2004). One of the main industrial applications of split plot designs is in robust product experimentation. While there are some early examples of this as in Michaels (1964), particularly in the 1980's the popularization of the concept of robust products was mainly due to the work of Dr. Taguchi through his inner and outer arrays (Taguchi (1986)). A great expose of the use of split-plot designs for robust product experimentation can be found in Box and Jones (1992).

As mentioned earlier, having some factors that are harder to change than others is not an exception but a norm in many experimental situations. However, for a long period of time, practitioners received only little help from the literature in properly designing these experiments except for a few sources such as the tables of split-plot designs provided in the article by Addelman (1964). Starting two decades ago, however, we have seen a flurry of activities in this area resulting in various works in the design and analysis of split-plot experiments (Letsinger et al. (1996), Bisgaard and Steinberg (1997), Bisgaard (2000), Bingham and Sitter (1999, 2001, 2003), Goos and Vandebroek (2001)). In many cases the research revolved around 2-level factorial split plot designs with various numbers of whole plot and subplot factors (Huang et al. (1998), Bingham and Sitter (1999), Bingham et al. (2004)). The number of subplots in these designs will naturally be a power of 2. For more flexibility in the number subplots required in a split plot design, Kulahci and Bisgaard (2006) and Kulahci (2007b) offer possibilities with 3, 5, 6, 10, etc. subplots for each whole plot based on Plackett and Burman designs. Similarly Kowalski (2002) proposes designs with 6 subplots for each whole plot by adding 2 follow-up experiments to the original 4 subplots.

Tyssedal et al. (2011) provide two-level split-plot designs constructed from both regular and nonregular designs where for each whole plot two subplots were run as mirror image pairs. Such designs have the appealing property that they divide the estimated effects into two orthogonal subspaces separating subplot main effects and subplot by whole plot interactions from the rest, see Tyssedal and Kulahci (2005). Tyssedal et al. (2011) further emphasize the importance of taking the projection properties of the design into account.

Split-plot designs have a natural extension to processes with more than two stages, see Tobias et al. (2013). For three stages such designs are called split-split-plot designs. An example of a four stage split-split-split-plot design for identifying factors causing rancidity of stored meat loaf is given in Baardseth et al. (2005). Tyssedal and Kulahci (2014) provide designs for multistage processes, hereafter called multistage experiments, which are direct generalizations of the designs introduced by Tyssedal et al. (2011) and where only two experiments are run as mirror image pairs at each stage for each experiment from the previous stage. D-optimal designs of split-split-plot experiments are considered in Jones and Goos (2009).

While the literature on how to obtain two-level split-plot designs is rich, it seems to be fairly limited for their generalizations to more than two stages and we are not aware of any established strategy for the generation of such designs. However, most modern manufacturing nowadays involve processes where the raw material goes through several stages before the final product is obtained. This is valid for both parts manufacturing and continuous processes alike. Moreover the factors at an early stage may (and often do) interact with another factor in a later stage and have subsequently an impact on the final product. Therefore it is no longer viable to consider each stage of the production on its own and perform experiments that seem to be important particularly for that stage. A more holistic approach where the interdependencies among the stages are taken into account, is a more effective approach. The purpose of this paper is to present a methodology for the construction and the understanding of design properties of two-level designs used for both split-plot and multi-stage experiments. The methodology is based on the Kronecker product representation of orthogonal designs and allows for the number of subplots to vary from stage to stage. The designs obtained are saturated in the sense that they provide the maximum number of factors that can be allocated at each stage. In the literature, if the number of factors in one or several stages is less than the maximum number of factors allowed, the allocation of these factors to the appropriate columns of the proposed design seems often to be based on the minimum aberration criterion (Fries and Hunter (1980), see e.g. Huang et al. (1998), Bingham and Sitter (1999)). We will not follow that track. Also Bisgaard (2000) and Kulahci et al. (2006), point out that other design criteria can yield more desirable split-plot designs for a given situation, see also Tyssedal et al. (2011). Instead as in Tyssedal and Kulahci (2014), we will focus on how we can obtain designs where runs at different stages can be constructed as mirror image pairs and how to provide designs with good projection properties. This is accomplished by restricting the number of factors that can be allocated at each stage. Such design properties are very appealing in screening situations. It should be noted that Tyssedal and Kulahci (2014) focus on multistage experiments where the number of experiments (“subplots”) at each stage is always two for each experiment (“whole plot”) from the previous stage. In that regard, in this paper we propose a more general methodology that will allow for any number of experiments as a multiple of 2 at any given stage.

This paper is organized as follows. In Section 2 we introduce the Kronecker product representation used in the construction of general split-plot designs from regular two-level designs. Section 3 is devoted to the discussion of some important designs and their projection properties. The construction of two-level split-plot designs with various designs and projection properties is dealt with in Section 4 and generalized to design for multistage processes in Section 5. In Section 6 we show some examples of how the

Kronecker product framework can also be used to construct split-plot and multistage experiments from nonregular designs. Concluding remarks are given in section 7.

2. CONSTRUCTION OF TWO LEVEL SPLIT-PLOT EXPERIMENTS FROM REGULAR DESIGNS USING KRONECKER PRODUCT REPRESENTATION

A general matrix representation of a 2^k factorial design using the Kronecker product representation is provided in the appendix. This follows from the proof given in Dey and Mukerjee (1999) where it is shown that the Kronecker product of two Hadamard matrices of orders N_1 and N_2 respectively is also a Hadamard matrix of order N_1N_2. Similar arguments and a detailed discussion on the orthogonal arrays can also be found in chapter 11 of Hedayat, Sloane and Stufken (1999). For further information on the Kronecker product operation see also Rao (1973).

Using the Kronecker product representation, the design matrix for a 2^k design fully expanded with interaction columns and a first column of only 1's can be written as

$$\begin{aligned}\mathbf{2}^{\mathrm{k}} &= \underbrace{\mathbf{2}^1 \otimes \mathbf{2}^1 \otimes \ldots \otimes \mathbf{2}^1}_{\mathrm{k}} \\ &= \mathbf{2}^{\mathrm{j}} \otimes \mathbf{2}^{\mathrm{k-j}} \qquad 1 \le \mathrm{j} \le \mathrm{k}-1\end{aligned} \tag{1}$$

where $\otimes$ stands for the Kronecker product operation and $\mathbf{2}^1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ Note that the design matrix in (1) is a 2^k by 2^k array. It can be shown that this notation can be used in blocking factorial experiments in 2, 4, etc. blocks (Kulahci (2007a)).

In the following we show how split-plot experiments can be constructed using the Kronecker product representation for which the number of subplots for each whole plot is a power of 2. For illustration purposes, we consider a split-plot design with 8 runs in total and present two cases for which the number of subplots for each whole plot is 2 and 4.

2.1 Split-Plot Experiments with 2 Subplots per Whole Plot

Since the total number of runs is 8, we consider a 2^3 design as our base design and write the Kronecker product representation of its design matrix as:

$$\mathbf{2}^3 = \mathbf{2}^1 \otimes \mathbf{2}^2 = \begin{bmatrix} \mathbf{2}^2 & -\mathbf{2}^2 \\ \mathbf{2}^2 & \mathbf{2}^2 \end{bmatrix}. \tag{2}$$

In this notation, the first column block of $\begin{bmatrix} \mathbf{2}^2 \\ \mathbf{2}^2 \end{bmatrix}$ is the candidate for the allocation of whole plot factors as the $\mathbf{2}^2$ matrix remains the same meaning that each run in the 2^2 factorial design appears twice in the overall design constituting a whole plot for which 2 subplots will be run. Similarly, the second column block $\begin{bmatrix} \mathbf{-2}^2 \\ \mathbf{2}^2 \end{bmatrix}$ is the candidate for the allocation of subplot factors.

In the column block used for the subplot factors, the $\mathbf{2}^2$ matrix appears twice with opposite signs. Hence for each whole plot the subplots will be in the form of mirror image pairs. Thus this design fits well into the $\begin{bmatrix} \mathbf{W} & \mathbf{S} \\ \mathbf{W} & \mathbf{-S} \end{bmatrix}$ representation of split-plot mirror image pairs (SPMIP) designs introduced in Tyssedal and Kulahci (2005). To see this more clearly, consider the entire design in (2) expanded as

$$\begin{matrix} \mathrm{I} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}$$

$$\mathbf{2}^3 = \left[\begin{array}{c} \begin{bmatrix} +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 \end{bmatrix} \begin{bmatrix} -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \\ \hline \begin{bmatrix} +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 \end{bmatrix} \begin{bmatrix} +1 & -1 & -1 & +1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & +1 & +1 \end{bmatrix} \end{array} \right]. \tag{3}$$

In this form, the first and the fifth runs correspond to the same whole plot for which the subplots are the mirror images. From the labels of the columns in (3), we can also see that this design will accommodate up to 3 whole plot factors that can be allocated in columns 1 to 3 and 4 subplot factors that can be allocated in columns 4 to 7.

2.2 Split-Plot Experiments with 4 Subplots per Whole Plot

In this case, since the total number of runs is 8, we have 2 whole plots with 4 subplots for each. We then modify the Kronecker product representation of the 2^3 design matrix as:

$$\mathbf{2}^3 = \mathbf{2}^2 \otimes \mathbf{2}^1 = \begin{bmatrix} \mathbf{2}^1 & -\mathbf{2}^1 & -\mathbf{2}^1 & \mathbf{2}^1 \\ \mathbf{2}^1 & \mathbf{2}^1 & -\mathbf{2}^1 & -\mathbf{2}^1 \\ \mathbf{2}^1 & -\mathbf{2}^1 & \mathbf{2}^1 & -\mathbf{2}^1 \\ \mathbf{2}^1 & \mathbf{2}^1 & \mathbf{2}^1 & \mathbf{2}^1 \end{bmatrix} \tag{4}$$

In (4), the first block column $\begin{bmatrix} \mathbf{2}^1 \\ \mathbf{2}^1 \\ \mathbf{2}^1 \\ \mathbf{2}^1 \end{bmatrix}$ is the candidate for the allocation of the whole plot factor since it is the only block column that has a repeated entry; i.e. each row of the $\mathbf{2}^1$ matrix is repeated exactly 4 times corresponding to each subplot. The rest of the block columns can be used to allocate the subplot factors. Hence this design will allow for only 1 whole plot factor and 6 subplot factors.

The generalization of the proposed approach can be done by answering the following two questions:

1. What is the total number of runs?
2. What is the number of subplots (or similarly the number of whole plots)?

The answer to the first question, *N*, determines the base design $N = 2^k$ whereas the answer to the second question, e.g. the number of subplots, $n = 2^s$ determines the Kronecker product representation:

$$2^k = 2^s \otimes 2^{k-s} = \left[\begin{array}{c|cccc} \overbrace{2^{k-s}}^{\text{WP}} & \partial_{12}\, 2^{k-s} & \cdots & \overbrace{}^{\text{SP}} & \partial_{1N}\, 2^{k-s} \\ 2^{k-s} & \partial_{22}\, 2^{k-s} & \cdots & & \partial_{2N}\, 2^{k-s} \\ \vdots & \vdots & \vdots & & \vdots \\ 2^{k-s} & \partial_{N2}\, 2^{k-s} & \cdots & & \partial_{NN}\, 2^{k-s} \end{array}\right] \tag{5}$$

where ∂_{ij} is the sign in the *i*-th row and *j*-th column in the design matrix for a 2^s design in standard form fully expanded with interaction columns with the first column consisting of only 1's. From (5), we can see that in this design it is possible to allocate up to $2^{k-s}-1=(N/n)-1$ whole plot factors and $2^k-2^{k-s}=$ N- N/n subplot factors. Note that designs in which more than one factor is allocated to the same column are not considered.

3. SOME IMPORTANT DESIGN CONSIDERATIONS

In the previous section, we showed that the Kronecker product representation can be used to obtain the maximum number of whole plot and subplot factors that can be tested in a split-plot design for a given number of experiments and number of subplots (or whole plots). This of course results in saturated split-plot designs. However designs with several desirable properties can be obtained if they are not saturated, i.e. the numbers of whole plot and subplot factors are less than the maximum allowable numbers given in the previous section. This is achieved by the proper allocation of the factors to the available columns for whole plot and subplot factors.

As a motivational example, we consider two designs, D1 and D2, both with 8 experimental runs and both with four subplots for each whole plot. Furthermore we assume that D1 has 6 subplot factors labeled 2 through 7 whereas D2 has 4 subplot factors, labeled 4 through 7 as shown in Figure 1. We will now investigate the aliasing between two-factor interactions and main effects for these designs. Let w and s denote whole plot main effects and subplot main effects and let ww, ws, ss denote whole plot by whole plot, whole plot by subplot and subplot by subplot interactions respectively. Also let $\{\text{w,ww,ss}\}$ for instance represent the subgroup consisting of w,ww and ss effects. It can be shown that for D1, the ws interactions are fully aliased with the $\{\text{s,ss}\}$ effects, while ss interactions are fully aliased with the $\{\text{w,s,ss}\}$ effects as shown in the alias matrix in Table 1. For D2, the ws interactions are fully aliased with s effects only while ss interactions are fully aliased with the $\{\text{w,ss}\}$ effects as shown in the alias matrix in Table 2. Hence a much simpler alias structure is obtained by only allowing four factors at the subplot level. It can further be shown that for D2, the space defined by the columns for the s effects and the columns for the ws interaction effects is orthogonal to the space defined by the columns for w, ww and ss effects and therefore as pointed out in Tyssedal and Kulahci (2005), a variable search for active factors can be performed in two independent steps. This is a direct result of the fact that we at the subplot level of D2 have two mirror image pairs. That is, for each whole plot in D2, the first two and the last two subplots are pairs of mirror image runs. Note that this is not the case for D1.

[Insert Fig 1 here]

[Insert Table 1 here]

[Insert Table 2 here]

Another important design consideration particularly for screening experiments is the projective properties. Box and Tyssedal (1996) defined projectivity for a two-level design as follows: *A* $N \times k$ *design with* N *runs and* k *factors each at two levels is said to be of projectivity* P *and is called a* (N, k, P) *screen if every subset of* P *factors out of possible k contains a complete* 2^P *factorial design, possibly with some runs replicated.*

Projectivity P implies that all main effects and all interactions of any P factors are estimable with no bias if the other factors are inert, and it is empirically well documented that they are well-suited for identifying the active factors if no more than P factors are active. This is in particular true for nonregular designs where effects normally are not fully aliased. It is also well known that for $P \geq 3$ designs it is possible to de-alias main effects from two-factor interactions. For $P \geq 3$ regular designs, there is even no aliasing between these two types of effects. In nonregular designs it is normally also possible to de-alias two-factor interactions from each other while regular designs need to be of $P \geq 4$ in order to do so.

For a completely randomized design, four factors assigned to the columns labeled 1, 2, 4 and 7 in (3) give a projectivity $P = 3$ design. The same columns can be used to construct projectivity $P = 3$ split-plot designs with 2 or 4 subplots for each whole plot. For the former, it follows from (3) that we can have at most two subplot factors assigned to the columns 4 and 7. The subplots will then be mirror image pairs. For the latter, it follows from (4) that one whole plot factor can be assigned to column 1 and three sub-plot factors can be assigned to the columns 2, 4 and 7.

4. SPLIT-PLOT DESIGNS WITH MIRROR IMAGE PAIRS AS SUBPLOTS

In the previous section, we discussed the advantages of using mirror image pairs as subplots. If regular two-level designs are used as base designs in generating split-plot designs using the Kronecker product representation, the number of subplots for each whole plot will be a power of 2. For 2 subplots, the suggested mirror image pairs will simply be two subplots for each whole plot where the levels of the subplot main effects are reversed. For 4 or more subplots for each whole plot, we consider pairs of subplot runs each of which is a mirror image pair as in design D2 in Figure 1. The following result can be used to determine the maximum number of factors allowed in a split-plot design where the subplots are run as mirror image pairs.

Proposition 1. *The maximum allowable number of subplot factors in a split-plot design constructed from a regular design such that the subplots are run as mirror image pairs is* $2^{k-1} = N/2$ *where* N *is the total number of runs.*

Proof. For any number of subplots $n = 2^s, s < k$ we can write

$$\mathbf{2}^s \otimes \mathbf{2}^{k-s} = \mathbf{2}^k = \mathbf{2}^1 \otimes \mathbf{2}^{k-1} = \begin{bmatrix} \mathbf{2}^{k-1} & -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} & \mathbf{2}^{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} & -\mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} \\ \mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} & \mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} \end{bmatrix}.$$

Clearly it is possible to allocate up to 2^{k-1} subplot factors to the columns in $\begin{bmatrix} -\mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} \\ \mathbf{2}^{s-1} \otimes \mathbf{2}^{k\text{-}s} \end{bmatrix} = \begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ such that these runs are mirror image pairs. If more subplot factors are to be used, they have to be allocated to the columns in the first block $\begin{bmatrix} \mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{2}^{s-1} \otimes \mathbf{2}^{k-s} \\ \mathbf{2}^{s-1} \otimes \mathbf{2}^{k-s} \end{bmatrix}$. But this obviously violates the mirror image pair requirement.

As for the projectivity, the following result provides the maximum allowable number of subplot factors with corresponding experiments run as mirror image pairs in order to have a projectivity $P = 3$ split-plot design.

Proposition 2. *For* $N \geq 8$, *the maximum allowable number of subplot factors in a* $P = 3$ *split-plot design constructed from a regular design such that the subplots are run as mirror image pairs is* $2^{k-2} = N/4$ *where* N *is the total number of runs.*

Proof. A two-level fractional factorial design of projectivity 3 with a total number of N runs can be obtained by allocating the factors to the main effects columns and odd-factor interaction columns of the $2^k(=N)$ full factorial design. Since there are only $N/2$ such columns, the maximum number of factors that can be used in a fractional factorial design of projectivity 3 is $N/2$. Let $\boldsymbol{i} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{2^{k-1}\times 1}$. The columns in $\begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ are just the entry-wise product between the columns in $\begin{bmatrix} \mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ and the column $\begin{bmatrix} \boldsymbol{-i} \\ \boldsymbol{i} \end{bmatrix}$ which itself is a column in $\begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$. In order to have a $P=3$ design, interaction columns between $\begin{bmatrix} \boldsymbol{-i} \\ \boldsymbol{i} \end{bmatrix}$ and main effects and odd factor interactions in $\begin{bmatrix} \mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ need to be avoided. Thereby we are left with exactly $N/4$ columns in $\begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$. This number cannot be augmented since the rows in $\begin{bmatrix} \mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ corresponding to the mirror image pairs in $\begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$ are identical.

We note that the above results do not depend on s which determines the number of subplot factors. Also the simplification of the alias pattern when subplots are run as mirror image pairs will hold independently of s. To show that, consider the general representation of a saturated regular design with 2^k-1 factors as $\begin{bmatrix} \mathbf{2}^{k-1} & -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} & \mathbf{2}^{k-1} \end{bmatrix}$. With the subplot factors allocated to $\begin{bmatrix} -\mathbf{2}^{k-1} \\ \mathbf{2}^{k-1} \end{bmatrix}$, it can be easily shown that ws interactions can only be aliased with s effects while ss and ww interactions can only be aliased w effects. The possibility of splitting the main effects and the two-factor interaction columns into two orthogonal subspaces where searches for active factors can be done separately, has the benefit that more active factors can be identified than one would expect from the projective properties of the design as pointed out in Tyssedal and Kulahci (2005).

For split-plot designs with N runs, f_1 whole plot factors and f_2 subplot factors we will use the abbreviations $\mathrm{SP}(N, f_1, f_2)(n)$ and $\mathrm{SPMIP}(N, f_1, f_2)(n)$ to denote split-plot designs with $n = 2^s$ subplots and split-plot designs with $n = 2^s$ subplots run as mirror image pairs respectively. We will further denote split plot designs of projectivity P as $\mathrm{SP}(N, f_1, f_2)(n)_P$ and $\mathrm{SPMIP}(N, f_1, f_2)(n)_P$. If otherwise obvious the number of runs and factors are left out. The following rules apply on the maximum allowable number of whole plot and subplot factors for the various types of designs constructed from regular designs.

I. In $\mathrm{SP}(N, f_1, f_2)(n)$ and $\mathrm{SPMIP}(N, f_1, f_2)(n)$ designs, the maximum allowable number of whole plot factors is $(N/n)-1$

II. In $\mathrm{SP}(N \geq 8, f_1, f_2)(n)_3$ and $\mathrm{SPMIP}(N \geq 8, f_1, f_2)(n)_3$ designs, the maximum allowable number of whole plot factors is $2^{k\text{-}s-1} = N/2n$.

III. The maximum allowable number of subplot factors in $\mathrm{SP}(N, f_1, f_2)(n)$, $\mathrm{SPMIP}(N, f_1, f_2)(n)$, $\mathrm{SP}(N \geq 8, f_1, f_2)(n)_3$ and $\mathrm{SPMIP}(N \geq 8, f_1, f_2)(n)_3$ designs is $N\text{-}N/n$, $N/2$, $N/2-N/2n$ and $N/4$ respectively.

The first rule follows from the general Kronecker representation given in (5). Running the subplots as mirror image pairs affects only the number of subplot factors. The second rule also follows from (5) since the maximum allowable number of factors in order to have a projectivity $P=3$ design is half the number of runs, i.e. $2^{k\text{-}s-1}$ for the whole plot factors. The maximum allowable number of subplot factors for $\mathrm{SP}(n)_3$ designs given in rule III is due to the fact that the total number of factors in a projectivity $P=3$ design cannot exceed $N/2$. A list of possible number of whole plot and subplot factors for $N \leq 64$ are shown in Table 3.

[Insert Table 3 here]

In Table 4, we provide various projectivity 3 split-plot designs with proper allocation of whole plot and subplot factors. Note that the labeling of the columns follows the effect columns of the corresponding base design. For example for a 16 run SPMIP design, the main effects columns of the 2^4 base design are labeled as 1, 2, 3 and 4.

According to Table 4, a 16 run $\text{SPMIP}(2)_3$ can then be generated by allocating 4 whole plot factor to the 1, 2, 3 and 123 (the interaction of 1,2 and 3) columns and 4 subplot factors to the 4, 124, 134 and 234 columns of the 2^4 design respectively. Note that this complies with the maximum allowable number of whole plot and subplot factors for $\text{SPMIP}(2)_3$ design given in Table 3 as (4,4). It is also possible to construct designs with projectivity $P > 3$. A fairly extensive list of such designs with two subplots for each whole plot can be found in Tyssedal et al. (2011). We will therefore only consider designs with 4 and 8 subplots for each whole plot. For the total number of runs equal to 16, 32 and 64, the maximum allowable number of factors is 5, 6 and 8 respectively. A list of possible designs is given in Table 5. Once again the column labels are given as the columns of the corresponding base design.

[Insert Table 4 here]

[Insert Table 5 here]

5. MULTISTAGE EXPERIMENTS

In many industrial settings, processes consist of several stages before the final product is obtained. This is particularly true for the chemical and the process industries. At each stage there may potentially be several factors affecting the quality characteristic(s) of the final product. Moreover factors from different stages may interact and affect the output accordingly.

The Kronecker product representation can be used to generate designs for an *l*-stage process by rewriting the base design as

$$\mathbf{2}^k = \mathbf{2}^{s_l} \otimes \mathbf{2}^{s_{l-1}} \otimes \cdots \otimes \mathbf{2}^{s_1} \qquad (6)$$

where $\sum_{i=1}^{l} s_i = k$ and the total number of runs is $N = 2^k$.The number of subplots for stage i, $i > 1$, is then 2^{s_i}. The properties of such designs can be determined in the same way as for a two-stage design starting from the last stage and joining the $l-1$ first stages into one stage, and using the rules I-III in section 4. To illustrate this with an example, consider a three-stage process with $N = 16$. The possible designs are given as

1. $\mathbf{2}^4 = \mathbf{2}^2 \otimes \mathbf{2}^1 \otimes \mathbf{2}^1$

2. $\mathbf{2}^4 = \mathbf{2}^1 \otimes \mathbf{2}^2 \otimes \mathbf{2}^1$

3. $\mathbf{2}^4 = \mathbf{2}^1 \otimes \mathbf{2}^1 \otimes \mathbf{2}^2$

The first case corresponds to a design with 2 runs in stage 1, 2 runs in stage 2 for each run of stage 1 and 4 runs in stage 3 for each run of stage 2. Similarly the second case corresponds to a design with 2 runs in stage 1, 4 runs in stage 2 for each run of stage 1 and 2 runs in stage 3 for each run of stage 2. Finally the third case corresponds to 4 runs in stage 1, 2 runs in stage 2 for each run of stage 1 and 2 runs in stage 3 for each run of stage 2. Consequently the maximum number of factors that are allowed in each stage varies in these three cases. To investigate this we first consider each design as a two-stage design where the first and second stages are joined into one stage. We get for the three respective cases as

1. $$\mathbf{2}^4 = \mathbf{2}^2 \otimes \left(\mathbf{2}^1 \otimes \mathbf{2}^1\right) = \mathbf{2}^2 \otimes \mathbf{2}^2 = \begin{bmatrix} \mathbf{2}^2 & -\mathbf{2}^2 & -\mathbf{2}^2 & \mathbf{2}^2 \\ \mathbf{2}^2 & \mathbf{2}^2 & -\mathbf{2}^2 & -\mathbf{2}^2 \\ \mathbf{2}^2 & -\mathbf{2}^2 & \mathbf{2}^2 & -\mathbf{2}^2 \\ \mathbf{2}^2 & \mathbf{2}^2 & \mathbf{2}^2 & \mathbf{2}^2 \end{bmatrix}$$

2. $$\mathbf{2}^4 = \mathbf{2}^1 \otimes \left(\mathbf{2}^2 \otimes \mathbf{2}^1\right) = \mathbf{2}^1 \otimes \mathbf{2}^3 = \begin{bmatrix} \mathbf{2}^3 & -\mathbf{2}^3 \\ \mathbf{2}^3 & \mathbf{2}^3 \end{bmatrix}$$

3. $$\mathbf{2}^4 = \mathbf{2}^1 \otimes \left(\mathbf{2}^1 \otimes \mathbf{2}^2\right) = \mathbf{2}^1 \otimes \mathbf{2}^3 = \begin{bmatrix} \mathbf{2}^3 & -\mathbf{2}^3 \\ \mathbf{2}^3 & \mathbf{2}^3 \end{bmatrix}$$

The first block column where the signs do not change represents the joined stages and the others represent the last stage. Hence the maximum number of factors that can be allocated to the joined stages one and two is 3, 7 and 7 in the three cases. Thereby 12, 8 and 8 factors can be allocated in stage 3 for the respective cases.

To investigate stage one and two separately we simply decouple the two stages again.

1. $$\mathbf{2}^2 = \mathbf{2}^1 \otimes \mathbf{2}^1 = \begin{bmatrix} \mathbf{2}^1 & -\mathbf{2}^1 \\ \mathbf{2}^1 & \mathbf{2}^1 \end{bmatrix}$$

2. $$\mathbf{2}^3 = \mathbf{2}^2 \otimes \mathbf{2}^1 = \begin{bmatrix} \mathbf{2}^1 & -\mathbf{2}^1 & -\mathbf{2}^1 & \mathbf{2}^1 \\ \mathbf{2}^1 & \mathbf{2}^1 & -\mathbf{2}^1 & -\mathbf{2}^1 \\ \mathbf{2}^1 & -\mathbf{2}^1 & \mathbf{2}^1 & -\mathbf{2}^1 \\ \mathbf{2}^1 & \mathbf{2}^1 & \mathbf{2}^1 & \mathbf{2}^1 \end{bmatrix} \qquad (7)$$

3. $\mathbf{2}^3 = \mathbf{2}^1 \otimes \mathbf{2}^2 = \begin{bmatrix} \mathbf{2}^2 & -\mathbf{2}^2 \\ \mathbf{2}^2 & \mathbf{2}^2 \end{bmatrix}$

Hence the maximum number of factors that can be allocated to stage one is 1, 1 and 3 and to stage two it is 2, 6 and 4 for the respective cases.

Now suppose we want to run the subplots as mirror image pair. From rule III the maximum number of factors that can be allocated to stage 3 is 8 in each of the cases. For stage 2 it follows from (7) and rule II that the maximum allowable number of factors for the three cases is 2, 4 and 4 respectively.

In order to have $P = 3$ designs we can at most have a total of eight factors. From rule III the maximum number of factors in stage 3 is 4 in each of the cases for the subplots to be run as mirror image pairs. Otherwise case 1 can accommodate 6 factors. From (7), bearing in mind that the total number of runs is still 16, 1, 1, and 2 factors can be allocated to stage 1 for the respective cases. As for D2, one factor has to be taken out at stage 2 in case 2 in order for the subplots to be run as mirror image pairs.

Similar to the notation we introduced for two-stage split-plot designs, we will use MSP and MSPMIP to distinguish between whether runs on different stages are run as mirror image pairs or not. Some possible designs are given in Table 6 where for example $\mathrm{MSP}(N, f_1, f_2, \cdots, f_l)(n_2, \cdots, n_l)_3$ denotes a $P = 3$ multistage design with N runs and n_i runs in stage i, $i > 1$ and for which a maximum of f_i factors can be accommodated in stage i.

[Insert Table 6 here]

The procedure based on the Kronecker product representation can easily be applied to more than three stages. For example for a four-stage process, a design with 64 runs with 4 runs in stage 1, 4 runs in stage 2 for each run of stage 1, two runs in stage 3 for each run of stage 2 and 2 runs in stage 4 for each of stage 3 can be obtained using

$$\mathbf{2}^6 = \mathbf{2}^1 \otimes \mathbf{2}^1 \otimes \mathbf{2}^2 \otimes \mathbf{2}^2 \qquad (8)$$

We can also show that the maximum number of factors for stages 1 through 4 is 3, 12, 16 and 32 respectively yielding a $\mathrm{MSP}(64,3,12,16,32)(4,2,2)$ design. Similarly it is possible to construct a $\mathrm{MSPMIP}(64,2,4,8,16)(4,2,2)_3$ design, where subplots are run as mirror image pairs by allowing for 2, 4, 8 and 16 factors in stages 1 through 4 respectively.

6. SPLIT-PLOT MULTISTAGE EXPERIMENTS USING NONREGULAR DESIGNS

Important alternatives to the regular two-level designs are the nonregular designs which apparently exist for the number of runs $N = 4m,\ m \geq 3$. Besides their flexible run sizes, these are known to have far better projection properties than the regular two-level designs. In fact most of the nonregular designs are $P \geq 3$ designs in $N-1$ factors. Their alias structure is usually complex, but it often involves partial rather than full confounding. As a result it is normally possible, in contrast to the regular designs, to de-alias two-factor interactions from each other even for a design of projectivity $P = 3$. The best known nonregular designs are the Plackett and Burman (PB) designs with the number of runs $N \neq 2^k$.

Construction of MSPMIP designs for cases where for each whole plot, two subplots are run as mirror image pairs from nonregular designs is discussed in Tyssedal and Kulahci (2014). For instance a split-plot design with four subplots for each whole plot can be obtained through the operation $\mathbf{2}^2 \otimes \text{PB12}$ as

$$\begin{bmatrix} \text{PB12} & \text{-PB12} & \text{-PB12} & \text{PB12} \\ \text{PB12} & \text{PB12} & \text{-PB12} & \text{-PB12} \\ \text{PB12} & \text{-PB12} & \text{PB12} & \text{-PB12} \\ \text{PB12} & \text{PB12} & \text{PB12} & \text{PB12} \end{bmatrix} \qquad (9)$$

In (9), the column block $\begin{bmatrix} \text{PB12} \\ \text{PB12} \\ \text{PB12} \\ \text{PB12} \end{bmatrix}$ represents the whole plot factors and we get a

$\text{SP}(48,11,33)(4)_3$ or a $\text{SPMIP}(48,11,22)(4)_3$ design by removing the column block

$\begin{bmatrix} -\text{PB12} \\ \text{PB12} \\ -\text{PB12} \\ \text{PB12} \end{bmatrix}$. In a similar way a three stage $\text{MSPMIP}(48,11,11,22)(2,2)_3$ design is given using the Kronecker product representation $\mathbf{2}^1 \otimes \mathbf{2}^1 \otimes \text{PB12}$ to obtain

$$\begin{bmatrix} \text{PB12} & -\text{PB12} & -\begin{bmatrix} \text{PB12} & \text{PB12} \\ \text{PB12} & -\text{PB12} \end{bmatrix} \\ \text{PB12} & \text{PB12} & \\ \text{PB12} & -\text{PB12} & \begin{bmatrix} \text{PB12} & \text{PB12} \\ \text{PB12} & -\text{PB12} \end{bmatrix} \\ \text{PB12} & \text{PB12} & \end{bmatrix} \tag{10}$$

Here PB12 contains the 11 factor columns in a 12 run PB design. This design offers the practitioner greater flexibility in the number of factors in each stage than a 64 run MSPMIP design constructed from a regular two-level design if a $P \geq 3$ design is required. In general we can construct the factor columns for a multistage experiments from the operation $\mathbf{2}^{s_i} \otimes \mathbf{2}^{s_{i-1}} \otimes \cdots \otimes \mathbf{2}^{s_1} \otimes \text{NR}$ in the same way where NR stands for a nonregular design.

Tyssedal and Kulahci (2014) also propose using different columns of a nonregular design in each stage of a multistage design. For example for a three stage process, we can have MSPMIP $(48, f_1, f_2, f_3 \mid f_1 + f_2 + f_3 \leq 13)_3$. The design matrix is given below

$$\begin{matrix} \textit{Stage } 1 & \textit{Stage } 2 & \textit{Stage } 3 \end{matrix}$$

$$\begin{bmatrix} \mathbf{i} & \begin{bmatrix} \mathbf{S}1 \\ \mathbf{S}1 \\ \mathbf{S}1 \\ \mathbf{S}1 \end{bmatrix} & \begin{bmatrix} \mathbf{S}2 & \mathbf{i} \\ -\mathbf{S}2 & -\mathbf{i} \\ \mathbf{S}2 & \mathbf{i} \\ -\mathbf{S}2 & -\mathbf{i} \end{bmatrix} & \begin{bmatrix} \mathbf{S}3 & \mathbf{i} \\ \mathbf{S}3 & \mathbf{i} \\ -\mathbf{S}3 & -\mathbf{i} \\ -\mathbf{S}3 & -\mathbf{i} \end{bmatrix} \\ \mathbf{i} & & & \\ \mathbf{i} & & & \\ \mathbf{i} & & & \end{bmatrix} \tag{11}$$

where $[\mathbf{S}1 \quad \mathbf{S}2 \quad \mathbf{S}3] = \text{PB12}$ and $\mathbf{i}$ is a 12×1 vector of only 1's. The design can accommodate "only" thirteen factors in total, but has the advantage that no two-factor interactions are fully aliased.

All nonregular designs constructed from Hadamard matrices are known to be of strength at least two. This property offers a possibility to construct cost efficient multi-stage designs that can be used in situations with few whole plot factors. For example with two whole plot factors, we can for each of their level combination have three subplots, and thereby have the possibility to run a $\mathrm{SP}(12,2,9)(3)_3$ design. Similarly we can also construct an SP design with 1 whole plot factor, i.e. $\mathrm{SP}(12,1,10)(6)_3$ design.

[Insert Table 7 here]

An *l*-stage design can be constructed from a *l*-1 stage design. For instance the design given by $\mathbf{2}^1 \otimes \mathrm{SP}(12,2,9)(3)_3$ is a three stage $\mathrm{SP}(24,2,9,11)(3,2)_3$ design. By using different columns for each stage as in (11) a three stage design in 24 runs where no two-factor interaction is fully aliased can be obtained. With reference to Table 7, for each of the four level combinations of factors A and B, let $\mathrm{SP}i = \left[\mathrm{SP}i(f_2), \mathrm{SP}i(f_3)\right],\ i = 1,2,3,4$ constitute the *i*-th set of three runs for the last 9 columns. A $\mathrm{SP}(24,2,f_2,f_3 \mid f_2 + f_3 \leq 9)(3,2)_3$ design is given in Table 8 where $\mathbf{i}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

[Insert Table 8 here]

Another useful approach to generate designs with good projective properties is to use the fold-over technique. The factor columns in Table 9 are obtained by taking the fold-over of the PB 12 design and adding a column (column 4) which is the three-factor interaction column of the first 3 columns. If we omit this column, the design is of projectivity $P = 4$. From these factor columns it is possible to construct $\mathrm{SP}(24,3,9)(3)_4$, $\mathrm{SP}(24,2,9)(6)_4$ and $\mathrm{SP}(24,4,9)(3)_3$ designs as well as a $\mathrm{SP}(24,2,2,9)(2,3)_3$ design.

From the 20 and 24 run nonregular designs we can in the same way construct split-plot designs with 5 and 6 subplots for each whole plot. The generalization to nonregular designs with $N = 4m$ is obvious.

Finally another possibility is to use H_2, H_3 and H_4, three of the five different Hadamard matrices for $N = 16$, as defined in Box and Tyssedal (2001). The design in Table 10 is constructed from H_2. In this design it is possible to allocate up to 12 factors and construct a $P = 3$ design. We can then have $\mathrm{SPMIP}(16,4,8)(2)_3$, $\mathrm{SP}(16,2,10)(4)_3$, $\mathrm{SPMIP}(16,2,8)(4)_3$ and $\mathrm{MSPMIP}(16,2,2,8)(2,2)_3$ designs among others.

[Insert Table 9 here]

[Insert Table 10 here]

7. CONCLUDING REMARKS

A new methodology for constructing two-level split-plot designs and designs for multistage processes by means of Kronecker product representation is introduced. The methodology can be used for any number of stages and for various numbers of runs in each stage using both regular and nonregular designs as base designs. Using the proposed methodology it is easy to find the maximum allowable number of factors in each stage and to obtain designs with good projection properties where runs at different stages can be run as mirror image pairs. Taking these properties into account the proposed designs are divided into four classes that provide flexible starting points for search algorithms based on a chosen design criterion such as maximum number of clear two factor interactions among factors from different stages to obtain the best suitable design for the needs and requirements of the experimental circumstances. Furthermore we believe the methodology provided in this paper is flexible enough to accommodate situations such as blocking in multi-stage experimentation, mixed level multi-stage experimentation and even response surface methodology studies. We are currently working on some of these issues but there are still many other possibilities for the use of this methodology for research and practical applications.

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APPENDIX

Kronecker Product

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be $m \times n$ and $p \times q$ matrices, respectively. Then the Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \left(a_{ij}\mathbf{B}\right)$$

is an $mp \times nq$ matrix expressible as a partitioned matrix with $a_{ij}\mathbf{B}$ as the $(i, j)^{\text{th}}$ partition, $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

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D1						
1	2	3	4	5	6	7
-	-	+	-	+	+	-
	-	+	+	-	-	+
	+	-	-	+	-	+
	+	-	+	-	+	-
+	-	-	-	-	+	+
	-	-	+	+	-	-
	+	+	-	-	-	-
	+	+	+	+	+	+

D2				
1	4	5	6	7
-	-	+	+	-
	+	-	-	+
	-	+	-	+
	+	-	+	-
+	-	-	+	+
	+	+	-	-
	-	-	-	-
	+	+	+	+

Figure 1. Two 8 run split-plot designs with 1 whole plot and 6 (D1) and 4 (D2) subplot factors respectively. The low and the high level of a factor is represented with – and + respectively.

Table 1. The Alias Matrix for D1

		ws						ss															
		12	13	14	15	16	17	23	24	25	26	27	34	35	36	37	45	46	47	56	57	67
w	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1
s	2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
	3	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
	4	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
	5	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
	6	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
	7	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0

Table 2. The Alias Matrix for D2

		ws				ss					
		14	15	16	17	45	46	47	56	57	67
w	1	0	0	0	0	**1**	0	0	0	0	**1**
s	4	0	**1**	0	0	0	0	0	0	0	0
	5	**1**	0	0	0	0	0	0	0	0	0
	6	0	0	0	**1**	0	0	0	0	0	0
	7	0	0	**1**	0	0	0	0	0	0	0

Table 3. The maximum allowable number of whole plot and subplot factors in split plot designs with the total number of runs equal to 8, 16, 32 and 64. SP(n) and SPMIP(n) stand for split plot designs with n subplots and split-plot designs with n subplots run as mirror image pairs respectively. Similarly SP(n)$_P$ or SPMIP(n)$_P$ denote the corresponding designs of projectivity P

	Total Number of Runs			
Design	8	16	32	64
SP (2)	(3,4)	(7,8)	(15,16)	(31,32)
SPMIP(2)	(3,4)	(7,8)	(15,16)	(31,32)
$SP(2)_3$	(2,2)	(4,4)	(8,8)	(16,16)
$SPMIP(2)_3$	(2,2)	(4,4)	(8,8)	(16,16)
SP (4)	(1,6)	(3,12)	(7,24)	(15,48)
SPMIP(4)	(1,4)	(3,8)	(7,16)	(15,32)
$SP(4)_3$	(1,3)	(2,6)	(4,12)	(8,24)
$SPMIP(4)_3$	(1,2)	(2,4)	(4,8)	(8,16)
SP (8)		(1,14)	(3,28)	(7,56)
SPMIP(8)		(1,8)	(3,16)	(7,32)
$SP(8)_3$		(1,7)	(2,14)	(4,28)
$SPMIP(8)_3$		(1,4)	(2,8)	(4,16)

Table 4. Factor assignments in split-plot designs with n subplots run as mirror image pairs for (SPMIP$(n)_P$) of projectivity $P = 3$

		Total Number of Runs			
Design		8	16	32	64
$\text{SPMIP}(2)_3$	Whole-plot	1,2	1,2, 3,123	1,2,3,123,4,124, 134,234	1,2,3,123,4,124,134,234, 5,125,135,145,234, 235,345,12345
	Sub-plot	3,123	4,124, 134,234	5,125,135,145,234, 235,345,12345	6, 126,136,146,156,236,246,256, 346,356,456,12346,12356,12456, 13456,23456
$\text{SPMIP}(4)_3$	Whole-plot	1	1,2	1,2,3,123	1,2,3,123,4,124,134,234,
	Sub-plot	3,123	4,124, 134,234	5,125,135,145,234, 235,345,12345	6, 126,136,146,156,236,246,256, 346,356,456,12346,12356,12456, 13456,23456
$\text{SPMIP}(8)_3$	Whole-plot		1	1,2	1,2,3,123
	Subplot		4,124, 134,234	5,125,135,145,234, 235,345,12345	6, 126,136,146,156,236,246,256, 346,356,456,12346,12356,12456, 13456,23456

Table 5. Factor assignments in SP and SPMIP designs of projectivity $P > 3$ with total number of runs N

$N =$	Designs	Whole-plot columns	Sub-plot columns
16	$\text{SPMIP}(16,1,4)(4/8)_4$	1	4,24,34,1234
	$\text{SP}(16,2,3)(4)_4$	1,2	3,4,1234
	$\text{SPMIP}(16,2,2)(4)_4$	1,2	4,1234
32	$\text{SPMIP}(32,1,5)(4/8)_4$	1	5,25,35,45,12345
	$\text{SPMIP}(32,2,4)(4/8)_5$	1,2	5,35,45,12345
	$\text{SPMIP}(32,3,3)(4)_4$	1,2,3	5,45,12345
64	$\text{SPMIP}(64,1,7)(4/8)_4$	1	6,26,36,46,56,1236,13456
	$\text{SPMIP}(64,1,6)(4/8)_6$	1	6,26,36,46,56,123456
	$\text{SPMIP}(64,2,6)(4/8)_4$	1,2	6,36,46,56,1346,23456
	$\text{SPMIP}(64,2,5)(4/8)_5$	1,2	6,36,46,56,12346
	$\text{SPMIP}(64,3,5)(4/8)_4$	1,2,3	6,46,56,1456,23456
	$\text{SPMIP}(64,3,4)(4/8)_6$	1,2,3	6,46,56,123456
	$\text{SPMIP}(64,4,4)(4)_4$	1,2,3,4	6,56,1236,23456
	$\text{SPMIP}(64,4,3)(4)_5$	1,2,3,4	6,56,12346
	$\text{SPMIP}(64,5,3)(4)_4$	1,2,3,4,5	6,1236,3456
	$\text{SPMIP}(64,5,2)(4)_6$	1,2,3,4,5	6,123456

Table 6. Examples of SP and SPMIP designs for a 3-stage process with a total of 16 runs

Case1	Case 2	Case 3
$\mathrm{MSP}(16,1,2,12)(2,4)$	$\mathrm{MSP}(16,1,6,8)(4,2)$	$\mathrm{MSP}(16,3,4,8)(2,2)$
$\mathrm{MSPMIP}(16,1,2,8)(2,4)$	$\mathrm{MSPMIP}(16,1,4,8)(4,2)$	$\mathrm{M\ SPMIP}(16,3,4,8)(2,2)$
$\mathrm{MSP}(16,1,1,6)(2,4)_3$	$\mathrm{MSP}(16,1,3,4)(4,2)_3$	$\mathrm{MSP}(16,2,2,4)(2,2)_3$
$\mathrm{MSPMIP}(16,1,1,4)(2,4)_3$	$\mathrm{MSPMIP}(16,1,2,4)(4,2)_3$	$\mathrm{MSPMIP}(16,2,2,4)(2,2)_3$

Table 7. The factor columns in a 12 run PB design rearranged such that the two first columns have the level combinations in a 2^2 design replicated three times

	Factors										
Run	A	B	C	D	E	F	G	H	I	J	K
1	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	+	+	+	+	+	+
3	-	-	+	+	+	-	-	-	+	+	+
4	-	+	-	+	+	-	+	+	-	-	+
5	-	+	+	-	+	+	-	+	-	+	-
6	-	+	+	+	-	+	+	-	+	-	-
7	+	-	+	+	-	-	+	+	-	+	-
8	+	-	+	-	+	+	+	-	-	-	+
9	+	-	-	+	+	+	-	+	+	-	-
10	+	+	+	-	-	-	-	+	+	-	+
11	+	+	-	+	-	+	-	-	-	+	+
12	+	+	-	-	+	-	+	-	+	+	-

Table 8. Factor columns in a three stage MSPMIP design constructed from the 12 run PB design

Stage 1		Stage 2	Stage 3
$-\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathrm{SP1}(f_2)$	$\mathrm{SP1}(f_3)$
$-\mathbf{i}_3$	$\mathbf{i}_3$	$\mathrm{SP2}(f_2)$	$\mathrm{SP2}(f_3)$
$\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathrm{SP3}(f_2)$	$\mathrm{SP3}(f_3)$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathrm{SP4}(f_2)$	$\mathrm{SP4}(f_3)$
$-\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathrm{SP1}(f_2)$	$-\mathrm{SP1}(f_3)$
$-\mathbf{i}_3$	$\mathbf{i}_3$	$\mathrm{SP2}(f_2)$	$-\mathrm{SP2}(f_3)$
$\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathrm{SP3}(f_2)$	$-\mathrm{SP3}(f_3)$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathrm{SP4}(f_2)$	$-\mathrm{SP4}(f_3)$

Table 9. Factor columns in a two stage 24 run $P = 3$ split-plot design with up to 4 factors at stage 1 constructed from the 12 run PB design

Stage 1				Stage 2
$\mathbf{i}_3$	$-\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathbf{i}_3$	SP1
$\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathbf{i}_3$	$-\mathbf{i}_3$	SP2
$\mathbf{i}_3$	$\mathbf{i}_3$	$-\mathbf{i}_3$	$-\mathbf{i}_3$	SP3
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_3$	SP4
$-\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_3$	$-\mathbf{i}_3$	-SP1
$-\mathbf{i}_3$	$\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathbf{i}_3$	-SP2
$-\mathbf{i}_3$	$-\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_3$	-SP3
$-\mathbf{i}_3$	$-\mathbf{i}_3$	$-\mathbf{i}_3$	$-\mathbf{i}_3$	-SP4

Table 10. Factor columns in a 16 run $P = 3$ split-plot design with 8 sub-plot factors.

Whole-plot Factors				Subplot factors							
A	B	C	D	P	Q	R	S	T	U	V	W
-	-	-	+	+	+	-	-	-	-	+	+
+	-	-	-	-	-	-	-	-	-	-	-
-	+	-	-	+	-	-	-	+	+	+	-
+	+	-	+	+	-	+	-	-	+	-	+
-	-	+	-	-	+	-	-	+	+	-	+
+	-	+	+	-	+	+	-	-	+	+	-
-	+	+	+	+	+	+	-	+	-	-	-
+	+	+	-	-	-	+	-	+	-	+	+
-	-	-	+	-	-	+	+	+	+	-	-
+	-	-	-	+	+	+	+	+	+	+	+
-	+	-	-	-	+	+	+	-	-	-	+
+	+	-	+	-	+	-	+	+	-	+	-
-	-	+	-	+	-	+	+	-	-	+	-
+	-	+	+	+	-	-	+	+	-	-	+
-	+	+	+	-	-	-	+	-	+	+	+
+	+	+	-	+	+	-	+	-	+	-	-