NTNU – Trondheim
Norwegian University of
Science and Technology

Modifications of the Kramers-Kronig Relations for the Magnetic Permeability

Tarjei Bondevik

Nanotechnology
Submission date: February 2014
Supervisor: Johannes Skaar, IET

Norwegian University of Science and Technology
Department of Electronics and Telecommunications

Problem description

The Kramers-Kronig (KK) relations provide a mathematical relation between the real and imaginary part of the magnetic susceptibility, $\mu - 1$. In some cases, however, the KK relations are not satisfied, due to the behavior of the magnetic permeability μ on very small and very large frequencies. In order to nevertheless obtain relation between the real and imaginary part of μ, this thesis will address the following two questions for a metamaterial:

- How can μ generally behave, for very small and very large frequencies?
- Given the behavior of μ, how can the KK relations be modified in order to provide a relation between the real and imaginary part of μ?

One sub-goal is also to specifically investigate how transmission line metamaterials behave with respect to the KK relations.

Abstract

The Kramers-Kronig (KK) relations provide a mathematical relation between the real and imaginary part of a complex function that is analytic in the upper half plane and tends to zero as the argument tends to infinity. The magnetic susceptibility, $\chi = \mu - 1$, may satisfy the KK relations, but not in all cases. In cases where χ does not satisfy the usual KK relations, the analogous Modified Kramers-Kronig (MKK) relations were defined, and we applied them on a function $\xi(\mu)$, where μ is the magnetic permeability. The MKK relations provided a connection between the real and imaginary part of μ. It is the behavior of μ for low and high frequencies that determines if the MKK relations are satisfied.

The behavior of the magnetic permeability as a function of the frequency was studied, for a general metamaterial. From causality, $\mu(\omega)$ was found to behave only as $O\left(\omega^{-2}\right)$, $O\left(\omega^{-1}\right)$ or $O\left(\omega^{0}\right)$ when $\omega \to 0$. Similarly, when $\omega \to \infty$, $\mu(\omega)$ was also found to behave only as $O\left(\omega^{-2}\right)$, $O\left(\omega^{-1}\right)$ or $O\left(\omega^{0}\right)$. This lead to $3^2 = 9$ different possibilities of how $\mu(\omega)$ could behave.

The nine different possibilities show that μ may behave in ways not characterized by the usual KK relations. Thus, causality was found to be less restrictive than the KK relations.

For some of the nine possibilities, the function $\xi = \mu$ would satisfy the MKK relations directly. However, for other possibilities, a slight modification of ξ was needed. These modifications were found, and based on the nine different possibilities, the following set of functions ξ were found to satisfy the MKK relations,

$$\xi = \left\{\mu\ ,\ \omega\mu\ ,\ \omega(\mu - \mu_1)\ ,\ \mu - \mu_1\ ,\ \frac{\mu\omega}{1+\omega^2}\ ,\ \frac{\mu}{\omega}\right\},$$

where $\mu_1 \equiv \mu(\omega = \infty)$. Note that these function does not satisfy the MKK relations simultaneously, but in different cases, depending on the behavior of μ.

Passive transmission line metamaterials were also studied specifically. Their magnetic permeability μ were found to generally satisfy the MKK relations, given that their circuit topology consisted of solely serial and parallel circuits with loss.

Sammendrag

Kramers-Kronig (KK)-relasjonene gir en matematisk sammenheng mellom den reelle og den imaginære delen til en kompleks funksjon som er analytisk i det øvre halvplan og går mot null når argumentet går mot uendelig. Den magnetiske suskeptibiliteten, $\chi = \mu - 1$, kan i noen tilfeller tilfredsstille KK-relasjonene, men ikke i alle. For de tilfellene der χ ikke tilfredsstiller de vanlige KK-relasjonene, definerte vi derfor de Modifiserte Kramers-Kronig (MKK)-relasjonene, og brukte dem på en funksjon $\xi(\mu)$, der μ er magnetiske permeabiliteten. MKK-relasjonene ville dermed kunne gi en sammenheng mellom den reelle og imaginære delen til μ. Det er oppførselen til μ for lave og høye frekvenser som bestemmer om MKK-relasjonene blir tilfredsstilt.

Oppførselen til den magnetiske permeabiliteten som funksjon av frekvensen ω ble undersøkt for et generelt metamateriale. Fra kausalitet ble $\mu(\omega)$ funnet til å bare kunne oppføre seg som $O\left(\omega^{-2}\right)$, $O\left(\omega^{-1}\right)$ eller $O\left(\omega^{0}\right)$, når $\omega \to 0$. Ved $\omega \to \infty$ ble $\mu(\omega)$ også funnet til å bare kunne oppføre seg som $O\left(\omega^{-2}\right)$, $O\left(\omega^{-1}\right)$ eller $O\left(\omega^{0}\right)$. Dette ga totalt $3^2 = 9$ ulike muligheter på hvordan $\mu(\omega)$ kunne oppføre seg.

De ni ulike mulighetene viser at μ kan oppføre seg på måter som ikke er karakterisert av de vanlige KK-relasjonene. Dermed var kausalitet funnet til å være mindre restriktivt enn KK-relasjonene.

For enkelte av de ni mulighetene tilfredsstilte funksjonen $\xi = \mu$ MKK-relasjonene direkte. Imidlertid var det også tilfeller der man måtte modifisere ξ for at MKK-relasjonene skulle bli tilfredsstilt. Disse modifikasjonene ble funnet, og basert på de ni ulike mulighetene, ble det følgende settet av funksjoner ξ funnet til å tilfredsstille MKK-relasjonene,

$$\xi = \left\{\mu \, , \, \omega\mu \, , \, \omega(\mu - \mu_1) \, , \, \mu - \mu_1 \, , \, \frac{\mu\omega}{1+\omega^2} \, , \, \frac{\mu}{\omega}\right\},$$

der $\mu_1 \equiv \mu(\omega = \infty)$. Merk at disse funksjonene ikke tilfredsstiller MKK-relasjonene samtidig, men i ulike tilfeller, basert på oppførselen til μ.

Passive transmisjonslinjemetamaterialer ble også studert. Deres magnetiske permeabilitet μ ble funnet til å generelt tilfredsstille MKK-relasjonene, gitt at kretstopologien bestod utelukkende av serie- og parallellkoplinger.

Contents

1 Introduction

1.1 Background

Metamaterials may be defined as artificial media structured on a size smaller than the wavelength of the external stimuli [1]. When the size of the unit cell is much shorter than the electromagnetic wavelength, the external electromagnetic influence can be considered to be constant across several neighboring unit cells. An example of a metamaterial structure is shown in figure 1.

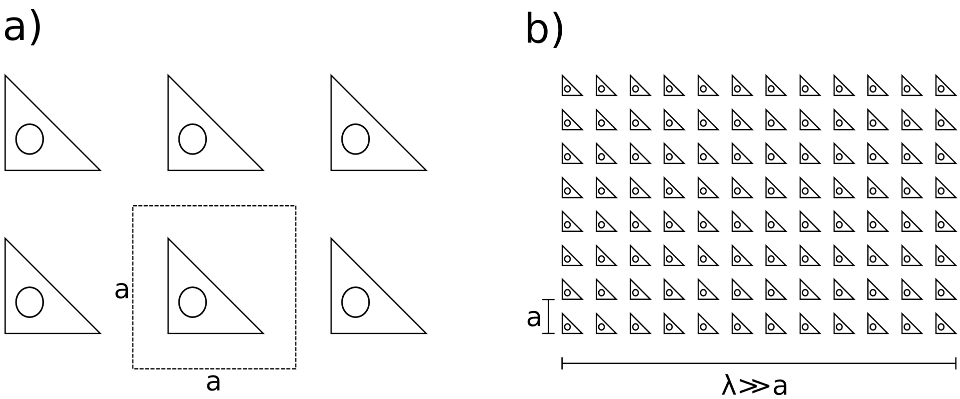

Figure 1: Part a) shows six unit cells. Each unit cell has side length a. Part b) shows several unit cells, illustrating the metamaterial condition $\lambda \gg a$.

Figure 1 a) shows a small part of a two-dimensional metamaterial, containing six unit cells. One of the unit cells is indicated with a dotted square, the length of its edges is a. Figure 1 b) shows a larger part of the metamaterial, containing many unit cells. The wavelength λ is much greater than the unit cell length a. $\lambda \gg a$ is a condition that needs to be satisfied for a metamaterial, ref. the above definition.

The electromagnetic characteristic of metamaterials does not primarily depend on the properties of its atoms and molecules, but rather the structure of the metamaterial [1]. A unit cell in a metamaterial play an analogous role as an atom do in a conventional material. Thus, by modifying its unit cells, it is possible to artificially create metamaterials with a wide range of properties. Metamaterials will have an impact across the entire range of technologies were electromagnetic radiation is used [1]. The negative refractive index is one example of an interesting property. By building a proper metamaterial structure, it is possible to fine-tune the permeability μ and permittivity ϵ, resulting in a negative refractive index. We will study this phenomenon in further detail in chapter 5.

If the wavelength of the external stimuli becomes sufficiently small – that is, the frequency becomes sufficiently large – the condition $\lambda \gg a$ no longer holds. This frequency, we define as $\omega_{\max}$, the metamaterial limit. For frequencies above $\omega_{\max}$, the external electromagnetic influence will not be considered constant, and the metamaterial ceases to be a metamaterial.

In this work we are, among other things, interested in investigating what happens with a metamaterial's magnetic permeability μ for large frequencies ω. μ is defined as the degree of magnetization a medium obtains in response to an applied magnetic field. For very large frequencies, we would expect $\mu \to 1$. This may be explained by physical intuition: As ω approaches a sufficiently large frequency, the direction of the applied magnetic field on the medium will switch extremely fast – faster than the medium's charges can respond. Consequently, the medium would not obtain any magnetization, yielding a relative permeability μ equal to 1.

Metamaterials where μ approaches unity for sufficiently large frequencies may satisfy the Kramers-Kronig (KK) relations – a set of mathematical relations that connects the real and imaginary part of μ [2]. However, in this work, we will explore the possibility of metamaterials with a permeability *not necessarily* approaching unity when the frequency gets sufficiently large. In this case, the usual KK relations would not be satisfied, and we would have to modify the relations to provide the connection between the real and imaginary part of μ. This leads us to two of the problems that will be addressed in this work:

- What different types of asymptotic behaviors $\mu \sim \omega^n$ are possible for sufficiently large frequencies?
- Given the different asymptotic behaviors $\mu \sim \omega^n$; how can we modify the KK relations, such that we obtain a connection between the real and imaginary part of μ?

In the above list, by "sufficiently large frequencies", we refer to a frequency that is large enough for μ to behaves asymptotically as $\sim \omega^n$, but not large enough for the material to cease to be a metamaterial. This gap is illustrated with the green area in figure 2. For frequencies $\omega < \omega_{\text{asymp}}$, μ does not behave asymptotically. For frequencies $\omega_{\text{asymp}} < \omega < \omega_{\max}$, the medium can still be considered as a metamaterial, while μ behaves asymptotically. We will, in this work, assume that $\omega_{\text{asymp}} < \omega_{\max}$ in general. Note, nonetheless, that this assumption is not necessarily always true.

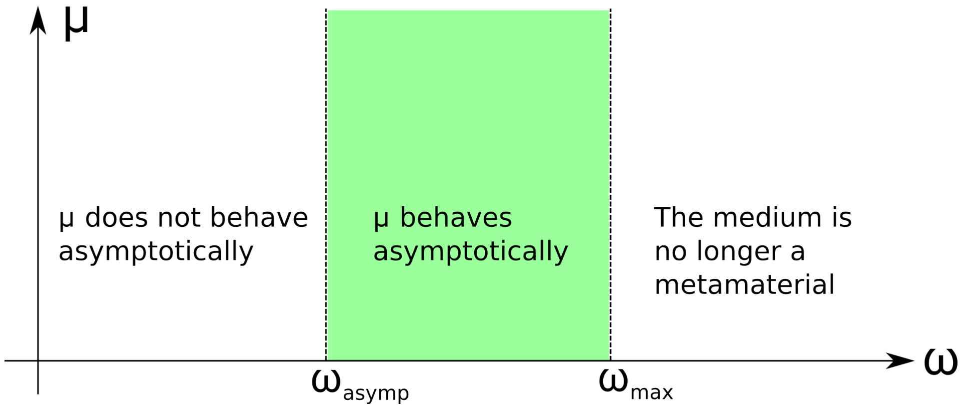

Figure 2: The behavior of μ as a function of the frequency ω. In the green area, where $\omega_{\text{asymp}} < \omega < \omega_{\text{max}}$, μ behaves asymptotically while the material is within the metamaterial limit. When $\omega > \omega_{\text{max}}$, the material ceases to be a metamaterial: The condition $\lambda \gg a$ is no longer satisfied. In this thesis, we will investigate only the regime where $\omega < \omega_{\text{max}}$.

Throughout the thesis, we will consider only the regime of frequencies lower than the metamaterial limit ω_{max}. When discussing the frequency $\omega \to \infty$ in upcoming chapters, we refer to a very large frequency *within the given regime*, not a frequency actually approaching infinity. The choice of restricting the notation $\omega \to \infty$ to within the metamaterial limit is in accordance with existing literature [2][3].

Every physical medium is causal, which means that the cause has to occur before the effect. For the magnetic permeability specifically, this means that the medium cannot generate a response before an external field is applied. Since μ describes a causal response, it follows that μ is analytic in the upper half complex plane [2]. Therefore, μ often satisfies the KK relations. The relations, which will be derived in chapter 3, can be written as follows [3]:

$$\chi'(\omega) = \mu'(\omega) - 1 = \frac{2}{\pi} P \int_0^\infty \mu''(z) \frac{z}{z^2 - \omega^2} dz \tag{1}$$

$$\chi''(\omega) = \mu''(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \mu'(z) \frac{1}{z^2 - \omega^2} dz, \tag{2}$$

where ’ stands for the real part, ” stands for the imaginary part, P is Cauchy’s principal value, ω is the frequency of the magnetic field and χ is the magnetic susceptibility. The KK relations are often considered to be generally valid for the magnetic permeability μ. Yet, the relations may provide results that contradicts with the physical reality. In this introduction we will give an example.

Let us consider a passive medium that experiences a constant magnetic field, which implies $\omega = 0$. Equation (1) then becomes

$$\mu'(0) - 1 = \frac{2}{\pi} P \int_0^\infty \frac{\mu''(z)}{z} dz > 0. \tag{3}$$

For a passive medium, $\mu''(z) > 0$ for $z > 0$ [3], which makes the integral term greater than zero in the above equation. Re-arranging terms, we obtain

$$\mu'(0) > 1. \tag{4}$$

Equation (4) says that *any* medium will obtain a magnetization in response to a constant, applied magnetic field. This is not correct according to physical reality, as it contradicts the existence of diamagnetic material ($\mu < 1$), which are proven to exist [4]. How can we avoid this absurd? Silveirinha suggests a modification of the KK relations, which enables the possibility of $\mu \leq 1$ at zero frequency [5]. We will get back to this example in section 4.1.

The idea of a modification would be central throughout the thesis. We already know that the magnetic susceptibility, $\chi = \mu - 1$, may satisfy the KK relations [2]. However, the above example shows that $\chi = \mu - 1$ does not always provide a reasonable result. For some behavior of μ, the usual KK relations would not be satisfied. How can we still obtain a mathematical relation between the real and the imaginary part of μ, which is what the KK relations provide? We will define a set of functions, labeled ξ. $\xi(\mu, \omega)$ is defined from the permeability, and are created to satisfy the Modified Kramers-Kronig (MKK) relations. The MKK relations, which will be derived in chapter 3, provides a relation between the real and imaginary part of ξ.

To explain the idea of modification, we provide a simple example. Let us, as a thought experiment assume that $\mu(\omega)$ has a second order pole in the origin. As we will see in chapter 3, $\chi = \mu - 1$ would not satisfy the KK relations if it has a second order pole in the origin. If we rather set $\xi = \omega\mu$, ξ will only have a first order pole in the origin, and the MKK relations may be satisfied. The extended set of functions ξ is illustrated in figure 3. Finding a set of functions ξ will increase our sample space in obtaining relations that connects the real and imaginary part of μ.

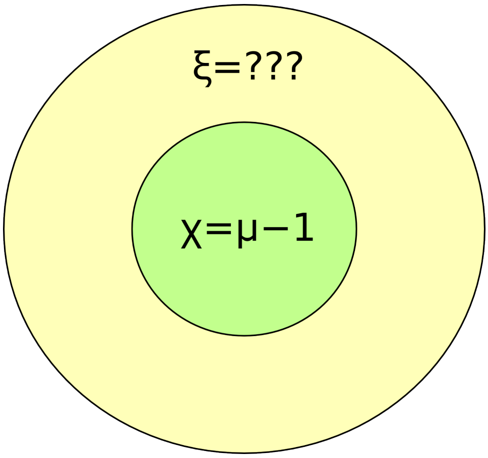

Figure 3: We already know that $\chi = \mu - 1$ may satisfy the KK relations [2]. In this thesis, our goal is to investigate the extended class of functions ξ that satisfies the MKK relations, and provide relations between the real and imaginary part of μ.

> We divide between the MKK relations and the KK relations to avoid confusion. When we write the KK relations, we refer to the relations connecting the imaginary and real part of the magnetic susceptibility $\chi = \mu - 1$. When we write the MKK relations, we refer to the relations connecting the imaginary and real part of the function ξ, where ξ is a function of μ and ω; $\xi = \xi(\mu, \omega)$.

The magnetic permeability is an interesting property of a metamaterial, since a metamaterial enables fine-tuning of μ such that $\mu < 0$ [6]. A metamaterial with simultaneous negative permeability and permittivity is known as a left-handed material [7]. *Science* magazine named the concept as one of the top ten scientific breakthroughs in 2003 [8]. We will examine two types of metamaterials more closely, and investigate if and how we can make them satisfy the MKK relations. The metamaterials to be examined are split ring cylinders (in section 4.2) and transmission line metamaterials (in chapter 5). The transmission line approach seems to be promising in order to create left-handed metamaterials with relatively low loss and large bandwidth [7].

Throughout the thesis, we will assume a medium that

- is linear, isotropic and passive
- has no spatial dispersion
- is time-invariant
- has a time dependency $e^{-i\omega t}$

When we chose a time dependency $e^{-i\omega t}$ rather than $e^{j\omega t}$, it implies, for instance, that the impedance of a capacitor would be $Z_C = \frac{i}{\omega C}$ rather than $Z_C = \frac{1}{j\omega C}$, and so on for the other circuit elements. Even though we do not consider the permittivity ϵ in detail in this thesis (except in chapter 5), we note that most of the relations discussed for μ applies for ϵ as well.

1.2 Outline

After this introduction, we provide some theoretical background, deriving the MKK relations. In the derivation we will point out the important requirements for a function ξ to satisfy the MKK relations. We will also discuss properties of passive media.

Further, we will examine an expression for the permeability of a split ring cylinder, and see how we have to modify the function ξ in order to satisfy the MKK relations. In the section about transmission line metamaterials, we will show that given a few valid assumptions, their permeability is found to generally satisfy the MKK relations.

Finally, we will examine the general behavior of a metamaterial's μ as a function of ω, for very small and very large frequencies. By finding how μ behaves, we can also find what types of modifications we must apply to ξ, in order to satisfy the MKK relations for different μ. Then we will obtain a new set of functions, in addition to $\chi = \mu - 1$, that yields a relation between the real and imaginary part of μ. This goal is illustrated in figure 3.

2 PASSIVE MEDIA

As mentioned in the introduction, we will consider only passive media in this thesis. Passive media has the important property, $\omega > 0 \Rightarrow \mu'' > 0$, which we will briefly explain in this chapter.

A passive medium is a medium that cannot amplify signals. We may have a transfer of energy from external electromagnetic fields to the medium, thus, the passive medium may consume energy. Resistors, capacitors and inductors are examples of passive electrical components, while a transistor, for instance, is an active component. For passive media, Landau and Lifshitz derived an expression for the electrical dissipation Q of a monochromatic electromagnetic field [3], shown in equation (5). We will examine the derivation of this expression more closely in section (6.1).

$$Q = \omega \left(\mathrm{Im}(\epsilon) \left\langle E^2 \right\rangle + \mathrm{Im}(\mu) \left\langle H^2 \right\rangle \right), \tag{5}$$

where ϵ is the permittivity, ω is the frequency, E is the electric field, H is the magnetic field and $\langle , \rangle$ means time averaging. Since we are considering a passive system, there is no external electromagnetic energy supply, thus we know that $Q \geq 0$. Furthermore, we know that in every physical system, some loss is inevitable, converting energy to heat [3]. With this is mind, we have for the electrical dissipation,

$$Q > 0. \tag{6}$$

By inserting this condition into equation (5), we see that when considering positive frequencies ($\omega > 0$), we require

$$\mathrm{Im}\, \mu = \mu'' > 0 \tag{7}$$

and

$$\mathrm{Im}\, \epsilon = \epsilon'' > 0. \tag{8}$$

However, the expression in equation (5) suggests that μ'' and ϵ'' do not necessarily need to be positive simultaneously, and there has been some controversy on this issue. As long as the positive term is largest, they might have different signs and still satisfy $Q > 0$, Koschny et al argued [9]. This lead to a discussion if μ'' and ϵ'' had to be positive simultaneously.

Efros argued against Koschny et al [10]. We will briefly explain his arguments here. Let us assume that we place a medium in a macroscopic capacitor, shown in figure 4. When we let the distance between the plates be very small, ($d \to 0$),

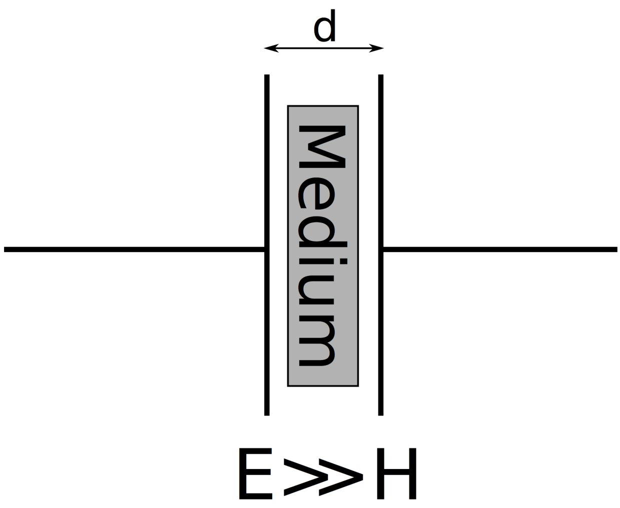

Figure 4: The medium is placed between the plates in a parallel plate capacitor. With a small distance d, the electric field will be strong relative to the magnetic field, yielding the condition $E \gg H$.

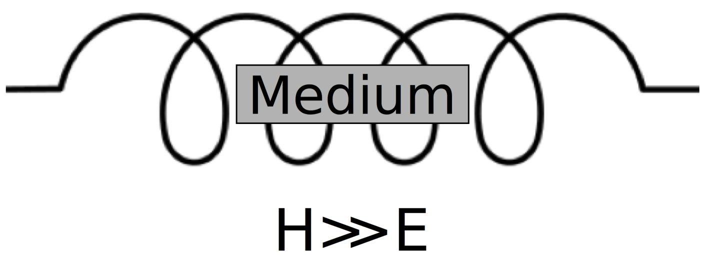

Figure 5: The medium is placed inside a solenoid, with length l. The number of wire windings N around the medium is much greater than showed in this illustration. With a high $\frac{N}{l}$ ratio, the magnetic field will be strong relative to the electric field, yielding the condition $H \gg E$.

we obtain a strong electric field, by the formula $E = \frac{V}{d}$. This yields the condition $E \gg H$, and from equation (5), we then require $\epsilon'' > 0$ in order to ensure $Q > 0$.

Let us now move the medium to a long and thin macroscopic solenoid with length l and N windings, shown in figure 5. Inside the solenoid, the magnetic field is $H = \frac{NI}{l}$ [11], and the electric field will be negligible compared to the magnetic field. Thus, in order to ensure $Q > 0$, we have from equation (5) that $\mu'' > 0$. The properties μ and ϵ of a medium are independent of the configuration of the electric and magnetic field in the medium [10], that is, they would not change when we move the medium from the capacitor to the solenoid. Thus, both ϵ'' and μ'' need to be positive, simultaneously. Now, there seem to be a scientific consensus for Efros' reasoning, so in this thesis we will assume that for all passive media, $\mu'' > 0$ and $\epsilon'' > 0$ simultaneously.

Note that this only applies for positive frequencies ($\omega > 0$). When $\omega < 0$, we see from equation (5) that we obtain the opposite case: $\mu'' < 0$ and $\epsilon'' < 0$, simultaneously.

3 Derivation of the Modified Kramers-Kronig relations

In this chapter, we will derive the MKK relations for a complex function ξ. We will have to assume ξ to be analytic in the upper half plane. Furthermore, ξ being a physical function, we require it to be causal. We start by stating Cauchy's Integral Theorem, as the theorem plays an important role in the derivation. Then we will show that μ is analytic in the upper half complex plane and derive the symmetry properties of μ, both being important in the following derivation. Finally, we will do the actual derivation of the relations. The derivation of the MKK relation is similar to the derivation of the KK relations provided in Jackson's *Classical Electrodynamics*[2].

Cauchy's Integral Theorem is as follows [12]: If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D, we have that

$$\oint_C f(z)dz = 0 \tag{9}$$

3.1 The magnetic permeability is analytic in the upper half complex plane

In this section, we will show that $\mu(\omega)$ is analytic in the upper half complex plane. The upper half complex plane is where $\text{Im}\,\omega > 0$, above the real axis.

We start by expressing the magnetic polarization in the time domain,

$$\vec{m}(t) = x(t) * \vec{h}(t) = \int_0^\infty x(\tau)\vec{h}(t-\tau)d\tau, \tag{10}$$

which has the Fourier transform

$$\vec{M}(\omega) = \mathscr{F}\left\{\vec{m}(t)\right\} = \chi(\omega)\vec{H}(\omega), \tag{11}$$

where $\chi(\omega)$ is the magnetic susceptibility, defined as $\chi(\omega) = \mathscr{F}\{x(t)\}$. $x(\tau)$ from equation (10) is a real function weighting how much the field $\vec{h}(t-\tau)$ contributes to the magnetic polarization at time t. Put another way, $x(\tau)$ is how much the magnetic field contributed to the polarization a time τ ago.

The polarization is causal, thus, $x(\tau) = 0$ for $\tau < 0$. This may be explained intuitively: Since $x(\tau)$ is how much the magnetic field contributed a time τ ago, a negative τ implies that we are considering the effect of a future magnetic field on a present polarization. Due to causality, it is impossible for a future magnetic field to influence the present polarization. Consequently, $x(\tau) = 0$ when $\tau < 0$.

For the magnetic susceptibility χ, which is the Fourier transform of x, we then obtain

$$\chi(\omega) = \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt = \int_{0}^{\infty} x(t) e^{i\omega t} dt. \tag{12}$$

From equation (12), we observe that in the upper half complex plane, where $\mathrm{Im}\,\omega > 0$, the exponential term will converge since the power in the exponent will be negative. Consequently, χ will be analytic in the upper half complex plane [2]. Noting that $\mu = \chi + 1$, we can conclude:

> Due to causality, the magnetic permeability, μ, is analytic in the upper half complex plane.

3.2 The symmetry properties of the magnetic permeability

Now we will show that μ' is an even, and μ'' an odd, function of ω. We recall that x from equation (10) is real. Complex conjugation of the magnetic susceptibility χ thus leads to

$$\chi^*(\omega) = \int_{0}^{\infty} x(t) e^{-i\omega t} dt = \chi(-\omega). \tag{13}$$

We split $\chi(\omega)$ into a real and an imaginary part, $\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$. Equation (13) provides the following relations:

$$\begin{aligned} \chi(-\omega) &= \chi^*(\omega) \\ \chi'(-\omega) + i\chi''(-\omega) &= \chi'(\omega) - i\chi''(\omega) \\ \\ \Rightarrow \chi'(-\omega) &= \chi'(\omega) \\ \chi''(-\omega) &= -\chi''(\omega) \end{aligned} \tag{14}$$

Since $\mu(\omega) = \chi + 1$, it will have the same symmetry properties. Thus, we conclude the following:

The real part of μ is even in ω, while its imaginary part is odd in ω. Stated algebraically,

$$\begin{aligned} \mu'(\omega) &= \mu'(-\omega) \\ \mu''(\omega) &= -\mu''(-\omega) \end{aligned} \tag{15}$$

The above symmetry relations will prove to be useful several times later in this work.

3.3 The MKK relations spanning all frequencies

In order to find the MKK relations, we integrate the function $\frac{\xi(z)}{z-\omega}$ along the closed path C, shown in figure 6. Note that we use z instead of ω as the variable of integration, even though we are integrating in the complex frequency plane. The reason is that we want to find the value of ξ at a specific frequency ω on the real axis, that is, $\xi(\omega)$. As we shall see, $\xi(\omega)$ depends on the value of ξ on the rest of the real axis. It is then very convenient to introduce a different variable of integration. Thus, in this context, the integration variable z is the frequency. The path of integration C is the same as Silveirinha chose in his derivation of the KK relations [5]. We have the following requirements for a function $\xi(z)$ to satisfy the MKK relations [5][2]:

- $\xi(z)$ must be analytic in the upper half complex plane, except in the origin: Here we allow for a pole of maximum order one.
- $\xi(z)$ must approach 0 as $|z| \to \infty$

In the following derivation of the MKK relations we assume that the above requirements are met. Since $\xi(z)$ is analytic in the upper half plane, the total value of the integral around C is zero, according to Cauchy's Integral Theorem (equation (9)). Hence,

$$\oint_C \frac{\xi(z)}{z-\omega} dz = 0. \tag{16}$$

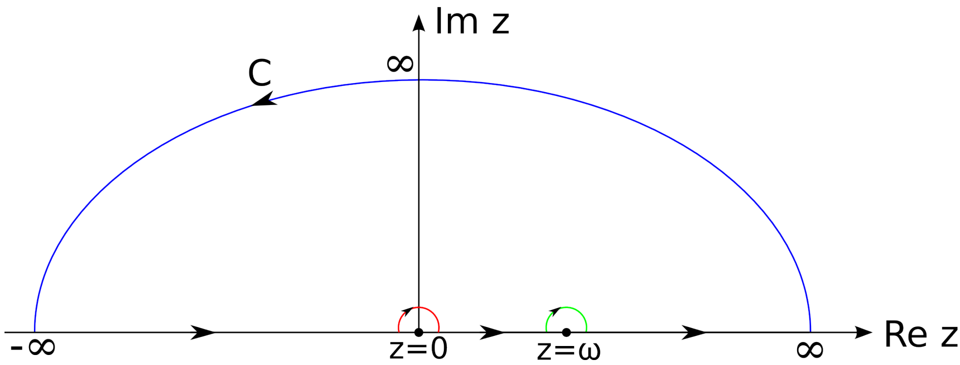

Figure 6: The integration path C, where $\frac{\xi(z)}{z-\omega}$ is to be integrated. The four parts are: the very large semi circle, the two infinitely small semi circles on the real axis and the rest of the real axis. The colors of the terms corresponds to the colors in equation (17). The integration path C avoids the poles; $\xi(z)$ is thus analytic in the area C encloses, and Cauchy's Integral Formula may be applied.

The integral can be split into four parts, shown in the following equation. The first term a clockwise integration around a possible pole in the origin. The second term is a clockwise integration around the pole at $z = \omega$, a pole that occurs since we are integrating with $z - \omega$ in the denominator. The third term is a counter-clockwise integration around the infinitely large semicircle, and finally, the fourth term is the integral along the z-axis, where P denotes Cauchy's principal value. For clarity, the colors of each term corresponds with the colors in figure 6.

$$\oint_C \frac{\xi(z)}{z-\omega} dz = \int_{\curvearrowright z=0} \frac{\xi(z)}{z-\omega} dz + \int_{\curvearrowright z=\omega} \frac{\xi(z)}{z-\omega} dz + \int_{\curvearrowleft |z|=\infty} \frac{\xi(z)}{z-\omega} dz + P\int_{-\infty}^{\infty} \frac{\xi(z)}{z-\omega} dz = 0 \tag{17}$$

The first term in equation (17) is a clockwise integration along an infinitely small half circle around a possible pole of ξ in the origin. In the following evaluation of the integral we use the substitution $z = re^{i\phi} \Rightarrow dz = ire^{i\phi} d\phi$. Also, we note that $\frac{\xi(z)z}{z-\omega}$ can be considered constant in the infinitely small integration region, thus simplifying the expression. A detailed sketch over the integration path is provided in figure 7; equation (18) provides the integration of the first term.

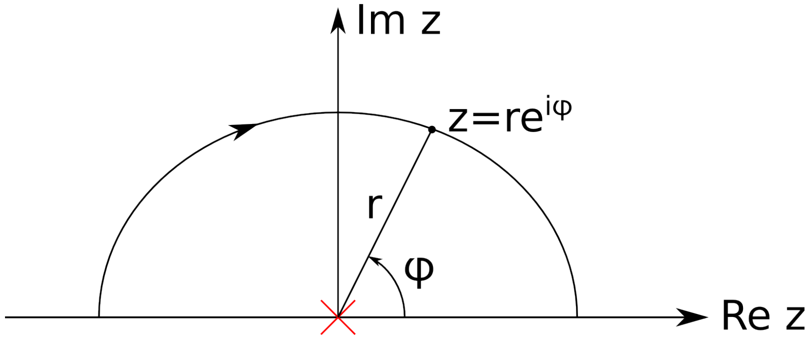

Figure 7: The integration path around a possible pole in the origin. The size of r is magnified in this figure; in fact, r is infinitely small. The origin is marked with a red cross, and the integration goes clockwise from $\phi = \pi$ to $\phi = 0$.

$$\int\limits_{\curvearrowright\, z=0} \frac{\xi(z)}{z-\omega} dz = \left.\frac{\xi(z)z}{z-\omega}\right|_{z=0} \cdot \int\limits_{\curvearrowright\, z=0} \frac{dz}{z} = \left.\frac{\xi(z)z}{z-\omega}\right|_{z=0} \cdot \int_{\pi}^{0} \frac{ire^{i\phi} d\phi}{re^{i\phi}} = \frac{i\pi}{\omega}\Big[\xi(z)z\Big]_{z=0} \tag{18}$$

For the second term in equation (17), we use the same substitution $z = re^{i\phi} \Rightarrow dz = ire^{i\phi} d\phi$. Assuming ξ to be constant in the infinitely small semi circle, we have

$$\int\limits_{\curvearrowright\, z=\omega} \frac{\xi(z)}{z-\omega} dz = \xi(\omega) \int\limits_{\curvearrowright\, z=\omega} \frac{dz}{z-\omega} = \xi(\omega) \int_{\pi}^{0} \frac{ire^{i\phi} d\phi}{re^{i\phi}} = -i\pi\xi(\omega). \tag{19}$$

For the third term in equation (17), we apply the inequalities

$$\begin{aligned} \int\limits_{\curvearrowleft\, |z|=\infty} \frac{\xi(z)}{z-\omega} dz &\le \left| \int\limits_{\curvearrowleft\, |z|=\infty} \frac{\xi(z)}{z-\omega} dz \right| \\ &\le \int\limits_{\curvearrowleft\, |z|=\infty} \left| \frac{\xi(z)}{z-\omega} \right| dz \le \lim_{|z|\to\infty} \max\left(\frac{\xi(z)}{z-\omega}\right) \pi |z|. \end{aligned} \tag{20}$$

Noting that we have already assumed $|z| \to \infty \Rightarrow \xi(z) \to 0$ (this was one of the requirements for achieving the MKK relations in the first place), the above limit becomes

$$\lim_{|z|\to\infty} \max\left(\frac{\xi(z)}{z-\omega}\right)\pi|z| = \xi(z)\frac{|z|}{|z|}\pi = 0\cdot\pi = 0 \quad \Rightarrow \int\limits_{\curvearrowleft\, |z|=\infty} \frac{\xi(z)}{z-\omega}dz = 0. \tag{21}$$

Consequently, the third term vanishes. Leaving the fourth term as it is, inserting the values of the three first terms back into equation (17) and re-organizing leads to

$$\xi(\omega) = \left.\frac{\xi(z)z}{\omega}\right|_{z=0} + \frac{1}{\pi i}P\int\limits_{-\infty}^{\infty}\frac{\xi(z)}{z-\omega}dz \tag{22}$$

By expanding ξ to a real and an imaginary part, $\xi(\omega) = \xi'(\omega) + i\xi''(\omega)$, we obtain

$$\xi'(\omega) + i\xi''(\omega) = \left.\frac{\xi'(z)z}{\omega}\right|_{z=0} + \left.\frac{i\xi''(z)z}{\omega}\right|_{z=0} + \frac{-i}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\xi'(z)}{z-\omega}dz + \frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\xi''(z)}{z-\omega}dz$$

$$\Rightarrow \xi'(\omega) = \left.\frac{\xi'(z)z}{\omega}\right|_{z=0} + \frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\xi''(z)}{z-\omega}dz \tag{23}$$

$$\xi''(\omega) = \left.\frac{\xi''(z)z}{\omega}\right|_{z=0} - \frac{1}{\pi}P\int\limits_{-\infty}^{\infty}\frac{\xi'(z)}{z-\omega}dz \tag{24}$$

These are the Modified Kramers-Kronig relations, providing a mathematical relation between the real and imaginary part of ξ for a given frequency ω. We may simplify the relations if ξ has a symmetry property (which often is the case). If $\xi'(z)$ is even, for instance, the first term on the right side in equation (23) will be odd and thus equal to zero. Similarly, if $\xi''(z)$ is even, the first term on the right side in equation (24) will vanish.

3.4 The MKK relations spanning only positive frequencies

For many functions $\xi(\mu,\omega)$, we can take advantage of the symmetry properties of μ. For instance, when ξ equals the magnetic permeability, $\xi = \mu$, equation (15) yields: $\xi'(-z) = \xi'(z)$ and $\xi''(-z) = -\xi''(z)$. With this, we can transform the MKK integral term to span only positive frequencies. First we re-write the integral term for the real part in equation (23):

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\xi''(z)}{z-\omega} dz &= \int_{-\infty}^{0} \frac{\xi''(z)}{z-\omega} dz + \int_{0}^{\infty} \frac{\xi''(z)}{z-\omega} dz = \int_{\infty}^{0} \frac{\xi''(-z)}{-z-\omega}(-dz) + \int_{0}^{\infty} \frac{\xi''(z)}{z-\omega} dz \\
&= \int_{\infty}^{0} \frac{-\xi''(z)}{-z-\omega}(-dz) + \int_{0}^{\infty} \frac{\xi''(z)}{z-\omega} dz = \int_{0}^{\infty} \frac{-\xi''(z)}{-z-\omega} dz + \int_{0}^{\infty} \frac{\xi''(z)}{z-\omega} dz \\
&= \int_{0}^{\infty} \xi''(z) \left(\frac{1}{z+\omega} + \frac{1}{z-\omega} \right) dz = \int_{0}^{\infty} \xi''(z) \frac{2z}{z^2-\omega^2} dz \qquad (25)
\end{aligned}$$

Inserting the integral from equation (25) into equation (23) and noting that $\left.\frac{\xi'(z)z}{\omega}\right|_{z=0} = 0$ due to symmetry, we get a simplified equation which spans only positive frequencies:

$$\xi'(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} \xi''(z) \frac{z}{z^2-\omega^2} dz \qquad (26)$$

We can carry out the same procedure in re-writing the integral term for the imaginary part, from equation (24):

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\xi'(z)}{z-\omega} dz &= \int_{-\infty}^{0} \frac{\xi'(z)}{z-\omega} dz + \int_{0}^{\infty} \frac{\xi'(z)}{z-\omega} dz = \int_{\infty}^{0} \frac{\xi'(-z)}{-z-\omega}(-dz) + \int_{0}^{\infty} \frac{\xi'(z)}{z-\omega} dz \\
&= \int_{\infty}^{0} \frac{\xi'(z)}{z+\omega} dz + \int_{0}^{\infty} \frac{\xi'(z)}{z-\omega} dz = \int_{0}^{\infty} \xi'(z) \left(\frac{-1}{z+\omega} + \frac{1}{z-\omega} \right) dz \\
&= \int_{0}^{\infty} \xi'(z) \frac{2\omega}{z^2-\omega^2} dz \qquad (27)
\end{aligned}$$

Inserting the integral from equation (27) into equation (24), we obtain, just as with the real part, a simplified equation for the imaginary part:

$$\xi''(\omega) = \left.\frac{\xi''(z)z}{\omega}\right|_{z=0} - \frac{2\omega}{\pi} P \int_{0}^{\infty} \xi'(z) \frac{1}{z^2-\omega^2} dz \qquad (28)$$

The equations (26) and (28) are representations of the MKK relations for a function $\xi(\omega)$ which span only positive frequencies, which is applicable in many cases. Note that we require $\xi(-\omega) = \xi^*(\omega)$ for these representations.

4 Examples of the Kramers-Kronig relations not providing reasonable results

As mentioned in the introduction, the KK relations sometimes lead to results that contradict with our physical intuition. In this chapter, we will provide two examples where a modification is needed. The first example is a more detailed description of the absurd presented in the introduction, where the KK relations prohibited the existence of diamagnetic materials. In the second example, we use an expression for μ for a split ring cylinder, derived by Pendry et al [13], and show how a modification does the trick.

4.1 The magnetic susceptibility at zero frequency

The magnetic susceptibility, $\mu - 1$, is a commonly used quantity. Investigating its properties regarding the KK relations would therefore be of interest. When $\frac{\mu(z)-1}{z-\omega}$ is integrated around the path shown in figure 6, it satisfies the KK relation when assuming that

- $\mu(z)$ has no pole outside the origin, and at most a simple pole in the origin
- $\mu(z) \to 1$ as $|z| \to \infty$; then it follows that $\big(\mu(z) - 1\big) \to 0$

For the real part of the magnetic susceptibility, equation (26) yields

$$\mu'(\omega) - 1 = \frac{2}{\pi} P \int_0^\infty \mu''(z) \frac{z}{z^2 - \omega^2} dz, \tag{29}$$

where we have used the fact that $\mathrm{Im}[\mu(\omega) - 1] = \mu''(\omega)$. We consider a passive material. Then we know that $\mu''(z) > 0$ for $z > 0$ [10]. Noting that $\mu''(z) > 0$ inside the integral and evaluating at $\omega = 0$, equation (29) becomes

$$\begin{aligned} \mu'(0) - 1 &= \frac{2}{\pi} P \int_0^\infty \frac{\mu''(z)}{z} dz > 0 \\ &\Rightarrow \mu'(0) > 1 \end{aligned} \tag{30}$$

This result implies that when considering a constant external magnetic field, that is, $\omega = 0$, the passive material will be magnetizable, regardless of the set of elements the material consist of.

Physically, this result is counterintuitive. Let us, as a thought experiment, assume that we have a passive metamaterial with resonance, consisting of aluminum. We apply a constant external magnetic field on our material. According to the result in equation (30), the piece of aluminum would be magnetizable: If we tried to put the aluminum on a permanent magnet (e.g. a piece of iron), they would stick together. This contradicts our physical intuition; aluminum is not magnetizable, so this should not happen.We note that equation (30) also prohibits the existence of diamagnetic materials ($\mu < 1$) under these conditions, which is another absurd.

To avoid these issues, we may apply a modification. Instead of setting $\chi = \mu - 1$ and apply the KK relations, we set $\xi = \mu - \mu_2$, where $\mu_2 \equiv \mu(\omega_2)$, and apply the MKK relations. We assume ω_2 to be sufficiently large, such that the permeability is constant and real-valued on the semi circumference $|\omega| = \omega_2$ [5]. We require $\xi = 0$ at the counter-clockwise integration around the large semi circle in order to satisfy the MKK relations. To meet this requirement, we integrate around $|z| = \omega_2$ rather than $|z| \to \infty$, since $\mu(\omega_2) - \mu_2 = 0$, by definition. See figure 8.

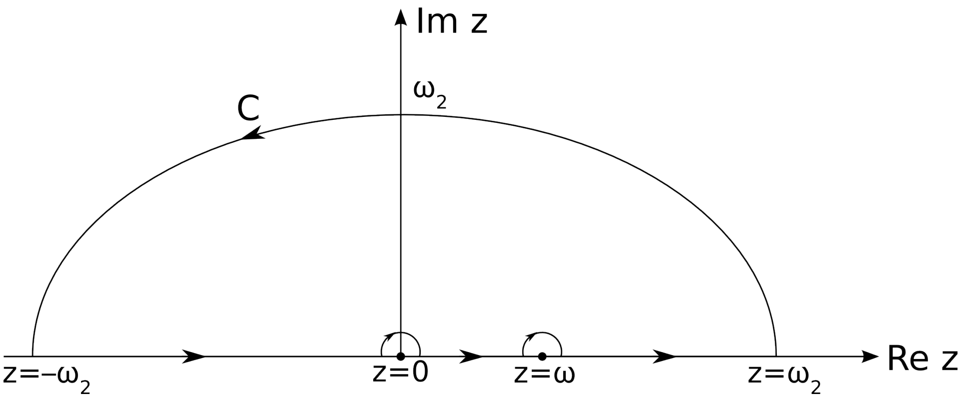

Figure 8: The integration path we use when finding the MKK relations for $\xi = \mu - \mu_2$. In order to ensure $\xi \to 0$ on the large semi circle, we have to integrate at $|z| = \omega_2$ rather than $|z| \to \infty$.

With $\xi = \mu - \mu_2$ evaluated at zero frequency, equation (26) becomes,

$$\mu'(0) - \mu_2 = \frac{2}{\pi} P \int_0^{\omega_2} \frac{\mu''(z)}{z} dz > 0 \Rightarrow \mu'(0) > \mu_2. \tag{31}$$

Since μ_2 can be an arbitrary real number [5], we no longer obtain the problematic result $\mu(0) > 1$.

This example shows that at zero frequency, the KK relations are not valid for $\chi = \mu - 1$. A quick an incorrect conclusion could have been that it is not possible to obtain a valid relation between the real and imaginary part of $\mu(0)$. This is, however, possible; we have to set $\xi = \mu - \mu_2$ and rather apply the MKK relations.

4.2 Pendry's expression for μ does not satisfy the KK relations

We will now provide an example of a structure whose magnetic permeability does not satisfy the KK relations for $\chi = \mu - 1$. A sketch of the structure is provided in Figure 9.

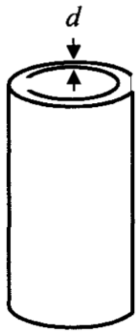

Figure 9: A split-ring cylinder, with distance d between the sheets. Each unit cell contains one cylinder. Reproduced from [13].

Pendry et al [13] calculated the permeability for a set of split-ring cylinders, with one cylinder within each unit cell. The set of cylinders had a metamaterial length scale, satisfying the metamaterial limit condition

$$a \ll \lambda \tag{32}$$

where a is the characteristic size of the unit cell and λ is the electromagnetic wavelength. This condition prevents internal diffraction within the unit cell, and we can assume the external electromagnetic influence to be equal throughout the unit cell. The expression Pendry et al found for the split-ring cylinders was [13]:

$$\mu = 1 - \frac{F}{1 + \frac{2\sigma i}{\omega r \mu_0} - \frac{3}{\pi^2 \mu_0 \omega^2 C r^3}}, \tag{33}$$

where F is the volume fraction of the unit cell covered by the cylinder, σ is the resistance of the cylinder per unit area, ω is the frequency of the external magnetic field, r is the radius of the cylinder and C is the capacitance per unit area between the two sheets.

Let us investigate what happens when we try to meet the KK requirements with the expression in equation (33). We observe that

$$\begin{aligned}\omega \to \infty \quad &\Rightarrow \quad \mu \to (1-F) \\ &\Rightarrow \quad \big(\mu - 1\big) \to -F \neq 0.\end{aligned} \tag{34}$$

Inserting $\mu - 1$ into the third term in equation (17) (the integral along the infinitely large semi circle) yields

$$\int_{\curvearrowleft\,|z|=\infty} \frac{\mu(z)-1}{z-\omega} dz = \int_{\curvearrowleft\,|z|=\infty} \frac{-F}{z-\omega} dz \neq 0. \tag{35}$$

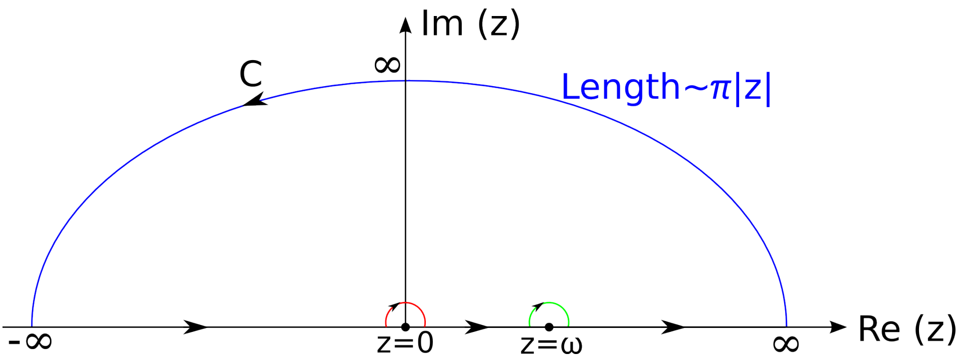

Figure 10: The path of integration in the derivation of the KK relations. The blue line is the path around the infinitely large semi circle, where the length $\pi|z|_{z\to\infty}$ is shown.

The integration is shown as the blue path in figure 10. At first sight, it might be tempting to conclude that the integral in equation (35) in fact is zero; after all, the denominator $(z-\omega) \to 0$ when $|z| \to \infty$. Nonetheless, the length of the integration path is $z\pi|_{z\to\infty}$, which means that the numerator and denominator go to infinity in the same order. Consequently, we cannot set the integral equal to zero. Since the value of the integral is unknown and not necessarily zero, the term will not vanish in the next step of the derivation (equation (22)), and we will not obtain the KK relations. This example shows why $\big(\mu(z)-1\big) \to 0$ as $z \to \infty$ is an absolute requirement for the KK relations.

Can we modify the function slightly, in order to satisfy the MKK relations? Instead of using the function $\chi = \mu - 1$, we rather use $\xi = \mu - \mu_1$, where $\mu_1 \equiv \mu(\infty)$, and investigate if the MKK relations are satisfied. Now, the integral along the infinitely large semi circle (third term, equation (17)) becomes

$$\int_{\curvearrowleft |z|=\infty} \frac{\xi(z)}{z-\omega} dz = \int_{\curvearrowleft |z|=\infty} \frac{\mu(z)-\mu(\infty)}{z-\omega} dz = \int_{\curvearrowleft z=\infty} \frac{0}{z-\omega} dz = 0. \tag{36}$$

Thus, the term vanishes. We also observe that $\xi(\omega) = \mu(\omega) - \mu_1$ is analytic in the upper half plane, which means that the MKK relations are satisfied. For the real part, we get from equation (26) the following relation,

$$\mu'(\omega) - \mu_1 = \frac{2}{\pi} P \int_0^\infty \mu''(z) \frac{z}{z^2 - \omega^2} dz. \tag{37}$$

With the MKK relations satisfied, let us investigate if they provide a meaningful result. We apply a constant external magnetic field, that is, $\omega = 0$. Noting that

$$\mu_1 = \lim_{|z| \to \infty} \mu = 1 - F \tag{38}$$

and $\mu(0) = 1$, insertion into equation (37) yields for the left side,

$$\mu'(0) - \mu_1 = 1 - (1 - F) = F, \tag{39}$$

leading to

$$F = \frac{2}{\pi} P \int_0^\infty \frac{\mu''(z)}{z} dz > 0. \tag{40}$$

Based on the assumptions that we are considering a passive material ($\mu'' > 0$), the integral in equation (40) has a positive value, which means that the right side of the equation has to be larger than zero. This does not contradict the left side of the equation, since $F > 0$ by definition. We can not confirm that the MKK relations are correct with this modified function, but at least we do not observe a clear contradiction, as we did when we applied the KK relations for $\chi = \mu - 1$ in section 4.1.

The examples in chapter 4 show that the magnetic susceptibility $\chi = \mu - 1$ does not necessarily provide a meaningful result when applied in the KK relations – nor does it always satisfy requirements of the KK relations. Still, with simple modifications, we have shown that these issues might be solved. Later in this thesis, we will provide a general overview of what other types of modifications we can apply in order to satisfy the MKK relations, given the limitations on the behavior of μ.

5 TRANSMISSION LINE METAMATERIALS

Transmission line metamaterials is another type of materials whose susceptibility does not necessarily satisfy the KK relations. Transmission metamaterials are particularly suitable for obtaining negative refraction; this chapter will start with discussing that concept. Further, we will derive expressions for the effective permeability and effective permittivity in a transmission line, illustrating the flexibility regarding the sign of the refractive index. Finally, we will show how the MKK relations may be applied for these types of materials.

5.1 THE CONCEPT OF NEGATIVE REFRACTION INDEX

Recently, the interest in creating metamaterials with simultaneously negative permeability (μ) and permittivity (ϵ) has increased. Although the concept was already theorized in the late 60's by Veselago [14], the experimental verification did not occur until the turn of the millennium [7]. One astonishing property occurs when $\mu < 0$ and $\epsilon < 0$: For the refractive index, we will now show that $\mu, \epsilon < 0$ simultaneously leads to a negative real value of n.

The refractive index is defined as $n = \pm\sqrt{\mu\epsilon}$ [6], where μ and ϵ are complex quantities, thus, the sign of n is, by definition, ambiguous. Let us consider a passive medium, which requires $\epsilon'' > 0$ and $\mu'' > 0$ simultaneously, as explained in chapter 2. We also consider a medium with negative real values of μ and ϵ, thus,

$$\epsilon = -\epsilon' + i\epsilon'', \tag{41}$$

and

$$\mu = -\mu' + i\mu'', \tag{42}$$

where $\epsilon' > 0$ and $\mu' > 0$. For the square of the refractive index, n^2, we obtain

$$n^2 = \mu\epsilon = (-\mu' + i\mu'')(-\epsilon' + i\epsilon'') = \mu'\epsilon' - \epsilon''\mu'' - i(\epsilon'\mu'' + \epsilon''\mu'). \tag{43}$$

Now, we want to find the square root of the complex number n^2, we apply the formula

$$n = \sqrt{n^2} = \pm\left(\sqrt{\frac{|n^2| + \mathrm{Re}[n^2]}{2}} + i\ \mathrm{sgn}(\mathrm{Im}[n^2])\sqrt{\frac{|n^2| - \mathrm{Re}[n^2]}{2}}\right) \tag{44}$$

From equation (43) we have that $\text{Im}[n^2] = -\epsilon'\mu'' - \epsilon''\mu' < 0$, that is, a *negative sign*. Equation (44) then becomes

$$n = \pm\left(\sqrt{\frac{|n^2| + \text{Re}[n^2]}{2}} - i\sqrt{\frac{|n^2| - \text{Re}[n^2]}{2}}\right). \tag{45}$$

The sign of the complex square root is ambiguous, however, we will now introduce a constraint which will determine the sign. For passive media, we have that Im $n > 0$. This is because the wave propagates along the z-axis as $\sim e^{iknz}$, and the wave converges only if Im $n > 0$. We note that the right square root in equation (45) has to be positive, since

$$\text{Im}\ n > 0 \Rightarrow |n^2| - \text{Re}[n^2] > 0. \tag{46}$$

Thus, in order to satisfy the constraint Im $n > 0$, the $\pm$ sign in equation (45) has to be negative, hence

$$n = -\sqrt{\frac{|n^2| + \text{Re}[n^2]}{2}} + i\sqrt{\frac{|n^2| - \text{Re}[n^2]}{2}}. \tag{47}$$

From equation (47), we can conclude that for passive media, with negative real values for ϵ and μ, we have a negative value for the real part of the refractive index.

Now, we will briefly discuss the consequences of a negative refractive index. Let us consider a boundary between vacuum (with $n_1 = 1$) and a medium with negative refractive index, say, $n_2 = -1$. Inserted into Snell's law, we obtain

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \Rightarrow \sin\theta_1 = -1 \cdot \sin\theta_2 \Rightarrow \theta_1 = -\theta_2, \tag{48}$$

where θ_1 and θ_2 are the angles of incidence and refraction, respectively. From equation (48), we see that θ_1 and θ_2 have different signs, and this is counter intuitive with our normal understanding of refraction. A sketch that illustrates the difference is provided in figure 11. In part a), we see an example of negative refraction, where $\theta_1 = -\theta_2$, and the ray partly refracts back towards its source. In b), the refraction behaves more "normally" according to elements that exist naturally.

Another interesting property with materials where $n < 0$ is the different sign of the phase velocity and the group velocity [6]. As an incident wave crosses a boundary where n becomes negative, the phase velocity switches sign. Yet, the group velocity (and thus, the propagation of energy) maintains its sign; we then

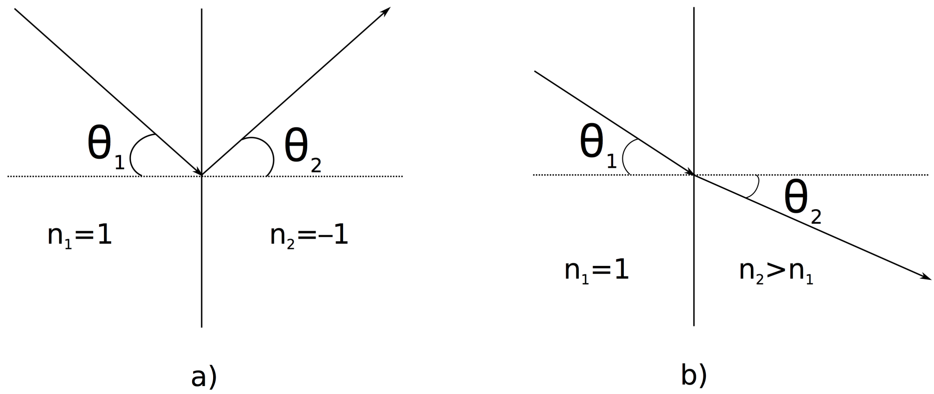

Figure 11: Part a) shows negative refraction of an incident electromagnetic wave, while part b) shows "normal" refraction that occurs naturally.

get different directed phase and group velocities.

The first experimental verifications with negative refractive index was with copper split ring resonators and copper wire strips [15]. Soon, however, transmission line materials were considered as a more convenient alternative for microwave applications, due to wider bandwidth and lower loss [7]. Thus, the properties of transmission line metamaterials are highly interesting, and we will discuss their MKK relations later in this section. First we will derive expressions for their permeability and permittivity. The derivation follows the same procedure as Eleftheriades et al [16], although, instead of using a two-dimensional unit cell, we will here use a one-dimensional unit cell.

5.2 Derivation of expressions for μ and ϵ

Let us consider a unit cell of length a, shown in figure 12. In reality, the leakage between the transmission lines occurs in a continuous – and not discrete – way, so a discrete unit cell representation is not perfectly accurate, but sufficient for our purpose.

The small change in current from the left unit cell to the right unit cell equals

$$I_1 - I_2 = -\frac{dI}{dz}a \tag{49}$$

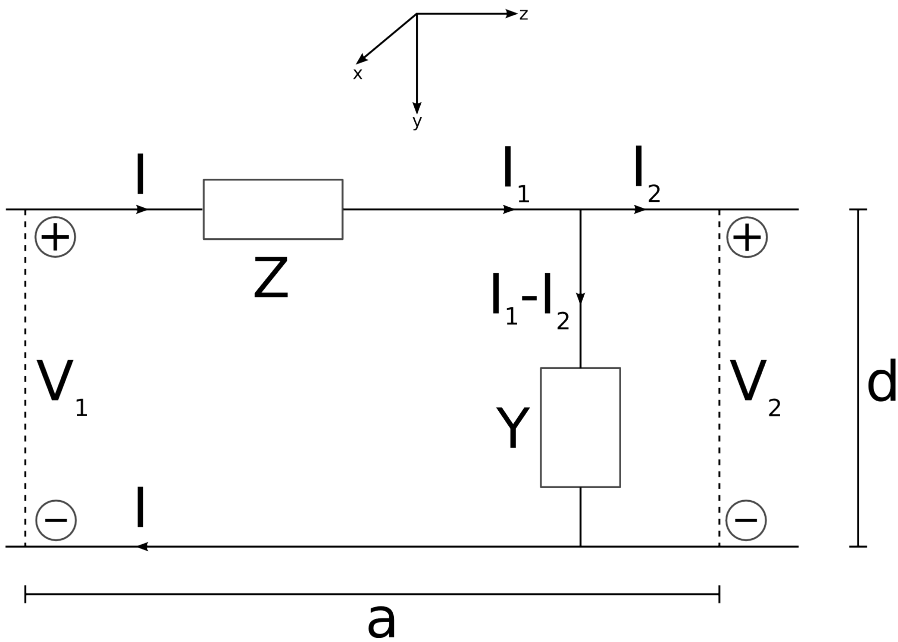

Figure 12: Transmission line unit cell. a is the length of the unit cell, I is the current through the transmission line, Z is the series impedance and Y is the shunt admittance. The small current $I_1 - I_2$ is added for clarity when using Kirchhoff's Current Law.

Assuming a negligible voltage drop through the impedance Z, we can approximate $V_1 \approx V_2 \approx V$. From figure 12, we have that

$$I_1 - I_2 = VY. \tag{50}$$

Combining equations (49-50), we obtain

$$\frac{dI}{dz} = -\frac{1}{a}VY \tag{51}$$

For the (small) voltage drop, we have that

$$V_1 - V_2 = -\frac{dV}{dz}a \tag{52}$$

Assuming that $I_1 \approx I_2 \approx I$, we get the following expression for the voltage drop across the series impedance:

$$V_1 - V_2 = IZ \tag{53}$$

Combining equations (52-53) yields

$$\frac{dV}{dz} = -\frac{1}{a} IZ \tag{54}$$

Equations (51) and (54) are the so-called telegrapher's equations. From equation (54) we have that

$$I = -\frac{a}{Z}\frac{dV}{dz}. \tag{55}$$

Insertion into equation (51) yields

$$\begin{aligned} \frac{dI}{dz} &= -\frac{d^2V}{dz^2}\frac{a}{Z} = -\frac{1}{a} YV \\ \Rightarrow &\quad \frac{d^2V}{dz^2} - \frac{ZY}{a^2} V = 0 \end{aligned} \tag{56}$$

We want to obtain a relation for μ and ϵ. Our strategy will be to relate the current to the magnetic field by applying Ampere's law, and the potential to the electric field by applying the definition of potential difference. For the electric field, we see from figure 12 that

$$\vec{E} = E_y e^{-i\omega t}\hat{y} = \frac{V}{d} e^{-i\omega t}\hat{y} \Rightarrow V = E_y d, \tag{57}$$

where E_y is the y-component of the electric field and $e^{-i\omega t}$ is the time dependence.

For the magnetic field, $\vec{H}$, we need to sketch a larger part of the structure, before applying Ampere's Law. An expanded sketch of the transmission line metamaterial is provided in figure 13. We apply Ampere's law on the path C in figure 13. By using the right hand rule, we observe that the magnetic field above the conductors is directed in the $+\hat{x}$-direction; under the conductors it is directed in the $-\hat{x}$-direction. By Ampere's law, the current becomes

$$\begin{aligned} \oint_C \vec{H}\cdot d\vec{l} &= \int_S \vec{J}\cdot d\vec{S} \Rightarrow H_x \cdot 4b \cdot 2 = 4I \Rightarrow H_x = \frac{I}{2b} \\ \Rightarrow &\quad I = H_x 2b. \end{aligned} \tag{58}$$

With a relation between the magnetic field and the current, and a relation between the electric field and the voltage, we next apply Maxwell's equations to

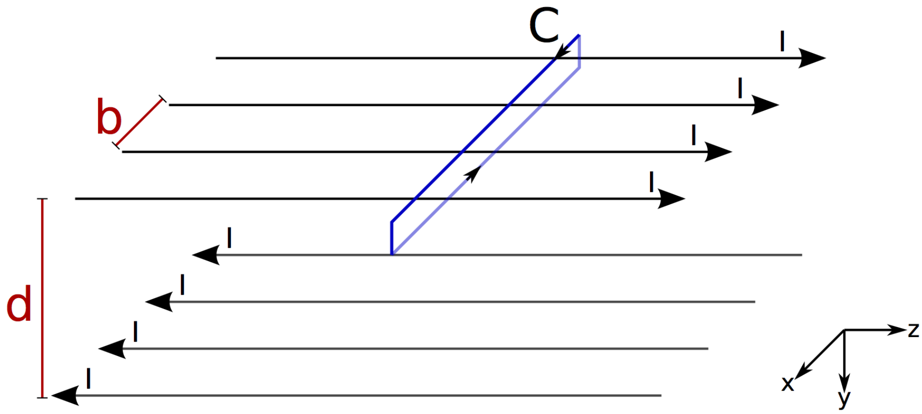

Figure 13: Expanded sketch of the transmission line metamaterial. The distance d between the transmission lines is the same distance as the one referred to in figure 12. The figure indicates a set of discrete conductors where current is flowing, but in reality, since the metamaterial scale is small, we assume a constant surface current density as a function of x.

relate these quantities to μ and ϵ. We use Maxwell's equations in the following form:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{59}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{60}$$

We observe from figure 12 that between the transmission lines, the magnetic field is $\vec{H} = -H_x \hat{x} e^{-i\omega t}$, where $e^{-i\omega t}$ is the time dependence. Maxwell's equations then provides the relation

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{i} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -H_x & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial z} H_x \hat{y} + \frac{\partial}{\partial y} H_x \hat{z} = -\epsilon i \omega E_y \hat{y}$$

$$\Rightarrow \frac{\partial}{\partial z} H_x = \epsilon i \omega E_y, \tag{61}$$

where the time dependence $e^{-i\omega t}$ on both sides vanishes. By insertion of $H_x = \frac{I}{2b}$, $E_y = \frac{V}{d}$ and equation (51) into equation (61), we arrive at

$$
\begin{aligned}
\frac{\partial}{\partial z}\frac{I}{2b} &= \epsilon i\omega\frac{V}{d} \\
&\Rightarrow \quad \frac{\partial I}{\partial z} = \frac{\epsilon i\omega 2b}{d}V = -\frac{1}{a}VY \\
&\Rightarrow \quad \epsilon = -\frac{d}{2ba}\frac{Y}{i\omega}.
\end{aligned} \tag{62}
$$

We observe that ϵ depends on easily adjustable parameters, such as the size of the unit cell and the admittance of the metamaterial. To find an expression for μ, we also apply Maxwell's equations:

$$
\nabla \times \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial}{\partial z}E_y\hat{x} + \frac{\partial}{\partial x}E_y\hat{z} = -(-\mu i\omega H_x\hat{x}), \tag{63}
$$

where the time dependence $e^{-i\omega t}$ on both sides vanishes. Insertion of equations (54), (57) and (58) into equation (63) yields

$$
\begin{aligned}
-\frac{\partial}{\partial z}\frac{V}{d} &= -\mu i\omega\frac{I}{2b} \\
&\Rightarrow \quad \frac{\partial V}{\partial z} = \frac{\mu i\omega d}{2b}I = -\frac{1}{a}IZ \\
&\Rightarrow \quad \mu = -\frac{2b}{da}\frac{Z}{i\omega}
\end{aligned} \tag{64}
$$

Since we are free to adjust the parameters of the metamaterials, we can set $\frac{2b}{d} = \frac{d}{2b} = 1$, for convenience. Then the equations for ϵ and μ becomes

$$
\epsilon = -\frac{1}{a}\frac{Y}{i\omega} \tag{65}
$$

$$
\mu = -\frac{1}{a}\frac{Z}{i\omega}. \tag{66}
$$

Recall that the leakage between the transmission lines occurs in a continuous and not a discrete matter. Consequently, it is reasonable to study the impedance per length, $Z' = \frac{Z}{a}$, and the admittance per length, $Y' = \frac{Y}{a}$. Instead of applying the terms Z' and Y', however, we re-define Z and Y to mean impedance per length and admittance per length, respectively, for the rest of this chapter. This is to ease the notation and terminology, and to focus on the central topic, which is the MKK relations – not the difference between Z and Z'. With these new definitions, we obtain for the permeability and permettivity,

$$\epsilon = -\frac{Y}{i\omega} \tag{67}$$

$$\mu = -\frac{Z}{i\omega}. \tag{68}$$

From the above equations, we observe a remarkable flexibility for the permittivity and permeability. By adjusting the impedance and admittance of the metamaterial, it is possible to achieve a large variety of values for μ and ϵ, including negative values [16]. In the following section, we will discuss a couple of the possibilites and investigate if they satisfy the MKK relations.

5.3 Investigation if μ and ϵ satisfy the MKK relations

Now we want to investigate how the above relations vary as a function of the frequency ω, and if they satisfy the MKK relations. Let us first consider a transmission line circuit where there is no loss, i.e. the resistance is zero. A sketch of the circuit is provided in figure 14. Here, the series impedance Z will be equal to the impedance of an inductor: $Z = Z_L = -i\omega L$. For the case of free space, where the constraint $\mu = \mu_0$ applies, equation (68) becomes

$$-i\omega\mu = -i\omega\mu_0 = -i\omega L \Rightarrow L = \mu_0. \tag{69}$$

With no resistance, the shutter admittance is solely determined by the capacitor, $Y = \frac{1}{Z_C} = \frac{1}{\frac{1}{-i\omega C}} = -i\omega C$. Inserted in equation (67), we similarly obtain $C = \epsilon_0$ in free space. Thus, the vacuum constants for the permittivity and permeability are constraints for L and C in this system.

The use of the relations in equations (67-68) is, however, not restricted to free space [16]. We are, of course, free to adjust L and C. We may choose negative inductance and capacitance, yielding a dielectric with negative permittivity and

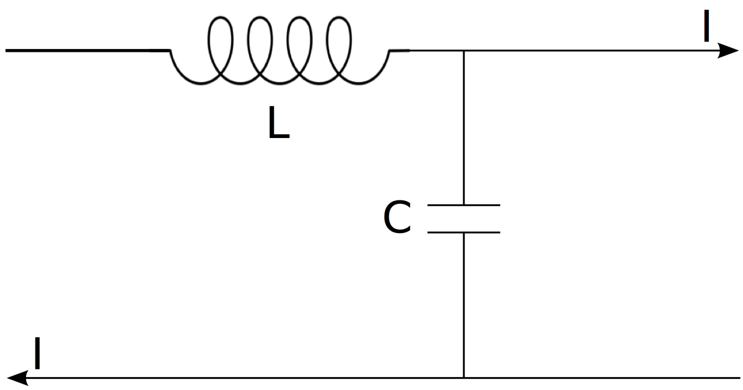

Figure 14: Transmission line unit cell. Idealized situation with no loss.

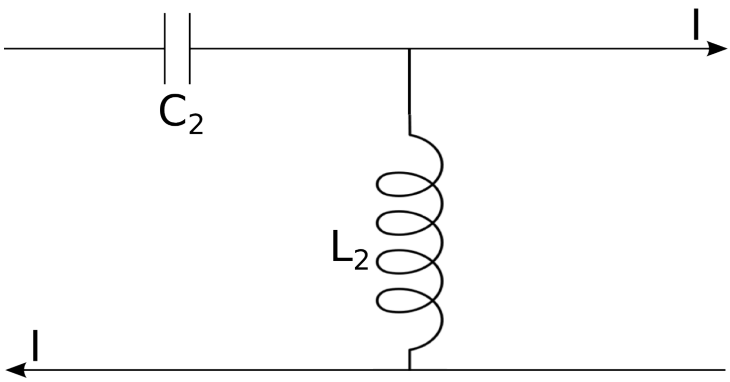

Figure 15: Transmission line unit cell. Due to the case of simultaneous negative permeability and permittivity, the roles of the capacitance and inductance are exchanged compared to the previous figure. Idealized situation with no loss.

permeability [16]. From the impedance and admittance's perspective, this will exchange the role of L and C, since $Z_L = -i\omega L$ and $Z_C = \frac{1}{-i\omega C}$. Figure 15 provides a sketch of this system, where the inductance and capacitance are labeled L_2 and C_2, respectively. Inserting L_2 and C_2 into equations (67-68), we obtain [16]

$$-i\omega\mu = Z = \frac{1}{-i\omega C_2} \quad \Rightarrow \quad \mu = -\frac{1}{\omega^2 C_2} \tag{70}$$

$$-i\omega\epsilon = Y = \frac{1}{-i\omega L_2} \quad \Rightarrow \quad \epsilon = -\frac{1}{\omega^2 L_2} \tag{71}$$

Note that with negative ϵ and μ, this is an example of negative refractive index [6]. By adjusting simple parameters like the inductance and the capacitance, it is thus possible to obtain $n < 0$ for a transmission line metamaterial.

Now, let us investigate if these relations satisfy the MKK relations. We observe immediately that both expressions have a pole of second order in the origin, which means that the relations are not satisfied, according to the requirements we set in section 3. Here we will provide the details in the calculations, explaining why the relations will not be satisfied. The following calculations are performed on μ but account also for ϵ, since they behave similarly, ref. equations (70-71).

As with the derivation of the MKK relations in section 3, we want to integrate $\xi = \mu = -\frac{1}{z^2 C_2}$ along a path C, sketched in figure 16.

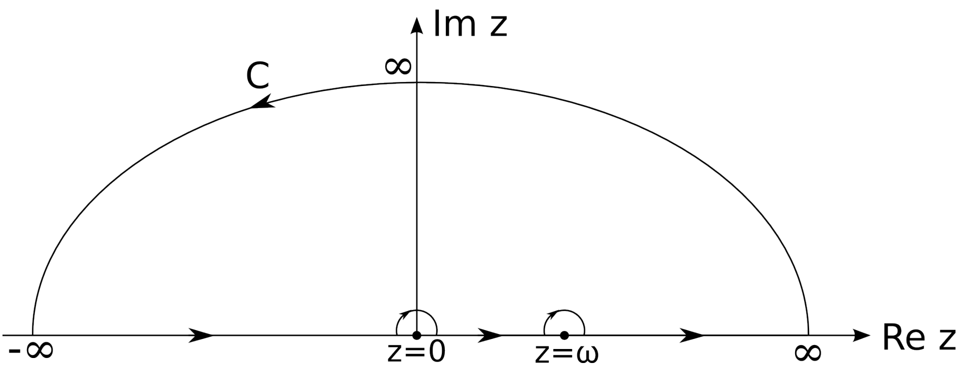

Figure 16: Integration path around the curve C. In order to preserve analyticity within the domain enclosed by C, which satisfies Cauchy's Integral Theorem, the integration path avoids the pole in the origin.

By using Cauchy's Integral Theorem, we obtain the equation

$$\oint_C \frac{-1}{z^2 C_2} \frac{1}{z-\omega} dz = \int_{\curvearrowright z=0} \frac{-1}{z^2 C_2} \frac{1}{z-\omega} dz + \int_{\curvearrowright z=\omega} \frac{-1}{z^2 C_2} \frac{1}{z-\omega} dz + \int_{\curvearrowleft |z|=\infty} \frac{-1}{z^2 C_2} \frac{1}{z-\omega} dz + P \int_{-\infty}^{\infty} \frac{-1}{z^2 C_2} \frac{1}{z-\omega} dz = 0. \quad (72)$$

The three last terms are similar to what we have already calculated in chapter 3, and we know they would not cause any problem. The first term in equation (72), though, is an integral halfway around a second order pole, and we suspect it to be infinity. By using the substitution $z = re^{i\phi}$, we get for this term

$$
\begin{aligned}
\int\limits_{\curvearrowright z=0} \frac{-1}{z^2C_2}\frac{1}{z-\omega}dz &= \frac{-1}{C_2(z-\omega)}\int\limits_{\pi}^{0}\frac{ire^{i\phi}}{re^{i\phi}re^{i\phi}}d\phi = \frac{-i}{rC_2(z-\omega)}\int\limits_{\pi}^{0}e^{-i\phi}d\phi \\
&= \frac{-i}{rC_2(z-\omega)}\frac{1}{-i}\left[e^{-i\phi}\right]_{\pi}^{0} = \frac{1}{rC_2(z-\omega)}\left(\frac{1}{e^0}-\frac{1}{e^{-i\pi}}\right) \\
&= \frac{2}{rC_2(z-\omega)} \to \infty, \qquad (73)
\end{aligned}
$$

since $r \to 0$ on the infinitely small semi circle. If we insert this term back into equation (72), we have that

$$
P\int\limits_{-\infty}^{\infty}\frac{-1}{z^2C_2}dz = -\infty \qquad (74)
$$

which is not very interesting, and certainly not a MKK relation. Thus, these calculations prove that with a second order pole in the origin, the MKK relations are not satisfied. Furthermore, it shows that lossless transmission metamaterials with negative refractive index do not satisfy the MKK relations.

Can we modify μ to make it satisfy the MKK relations? One option is to remove the assumption that we have no loss. In any real physical system, it would be impossible to eliminate the loss completely, so the adjustment of adding resistance would make the system's response more realistic. A one dimensional sketch of the unit cell with resistance included is provided in figure 17. A leakage resistance is added in parallel with the capacitor, and another resistance is added in series with the inductor. To simplify the notation, we have chosen to set both resistors to the same value R. This is, however, not a requirement, and the choice does not effect the upcoming result.

Let us now find expressions for μ and ϵ, when taking the resistance into account. From figure 17, we observe that the series impedance on the transmission line is

$$
\begin{aligned}
\frac{1}{Z} &= -i\omega C + \frac{1}{R} = \frac{1-Ri\omega C}{R} \\
&\Rightarrow \quad Z = \frac{R}{1-i\omega CR} \qquad (75)
\end{aligned}
$$

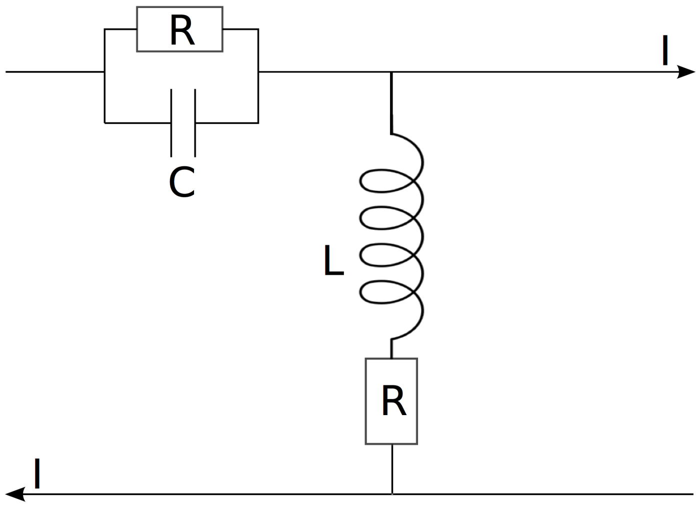

Figure 17: Transmission line unit cell with loss. Some loss in circuits are inevitable, this unit cell is then more realistic than the unit cell provided in figure 15.

By insertion into equation (68), we obtain

$$-i\omega\mu = \frac{R}{1-i\omega CR} \Rightarrow \mu = -\frac{R}{i\omega(1-i\omega CR)} \tag{76}$$

In equation (76), we observe a first order pole at $\omega = 0$ and another first order pole at $\omega = \frac{-i}{CR}$. In addition, $\mu \to 0$ as $\omega \to \infty$, so a meaningful MKK relation is possible. What about ϵ? From figure 17, we find the shutter impedance across the transmission line to be

$$Z_{\text{across}} = -i\omega L + R \Rightarrow Y = \frac{1}{R-i\omega L}, \tag{77}$$

and when inserting into equation (67), we obtain the permittivity

$$-i\omega\epsilon = \frac{1}{R-i\omega L} \Rightarrow \epsilon = -\frac{1}{i\omega(R-i\omega L)} \tag{78}$$

By inspection of the above equation, we conclude that ϵ has two first order poles: one at $\omega = 0$ and one at $\omega = \frac{-iR}{L}$. Also, we observe that $\epsilon \to 0$ as $\omega \to \infty$. This implies that if we use the same integration path as the one sketched in figure 16, the MKK relations would be satisfied. Consequently, when we assume loss in the system, it enables us to derive the MKK relations for both ϵ and μ.

5.4 Proof of no second order pole in upper half plane

In section 5.3, we added loss to our system and the requirements for applying the MKK relations were then met. What if this applies for every transmission line metamaterial? That is, would the occurrence of loss prevent second order poles on the real axis for a transmission line metamaterial, in general? In this section we prove that, given a few simple assumptions, this is indeed true.

First, we restate the general relations for μ and ϵ (equations (67) and (68)):

$$\epsilon = \frac{Y}{-i\omega} \tag{79}$$

$$\mu = \frac{Z}{-i\omega} \tag{80}$$

From the above equations, we see that in order to avoid a second order pole for μ and ϵ in the origin, we have to prove that Y and Z have no first order pole in the origin. In addition, to satisfy the MKK relations, we require analyticity. Thus, we would also have to show that a pole would not occur elsewhere in the upper half plane, nor the real axis.

5.4.1 The impedance and admittance in the upper half plane

We start with the upper half plane, leaving the real axis for later. The impedance function Z for a passive system is a so-called *postive* function [17]. A *positive* function can be defined the following way [18]:

> Z is considered to be a *positive* function if its real part is greater than zero in the upper half plane, and greater than or equal to zero on the real axis. Expressed mathematically,
>
> $$\text{Im}\,\omega > 0 \Rightarrow \text{Re}\,Z > 0\,,\ \text{Im}\,\omega = 0 \Rightarrow \text{Re}\,Z \geq 0. \tag{81}$$

We list up some relevant properties of a *positive* function Z [18]:

- $Y = \frac{1}{Z}$ is also a *positive* function.
- Its zeros and poles lie in the lower half plane or on the real axis.
- The possible zeros or poles on the real axis must be simple.

Considering these properties of *positive* functions, we know that the impedance Z has no zeros or poles in the upper half plane. Furthermore, we know that this accounts also for the admittance Y. However, we do not yet know if the real axis has no poles. Figure 18 provides an illustration of the sign of Z and Y in different regions of the complex plane, given that Z is a *positive* function.

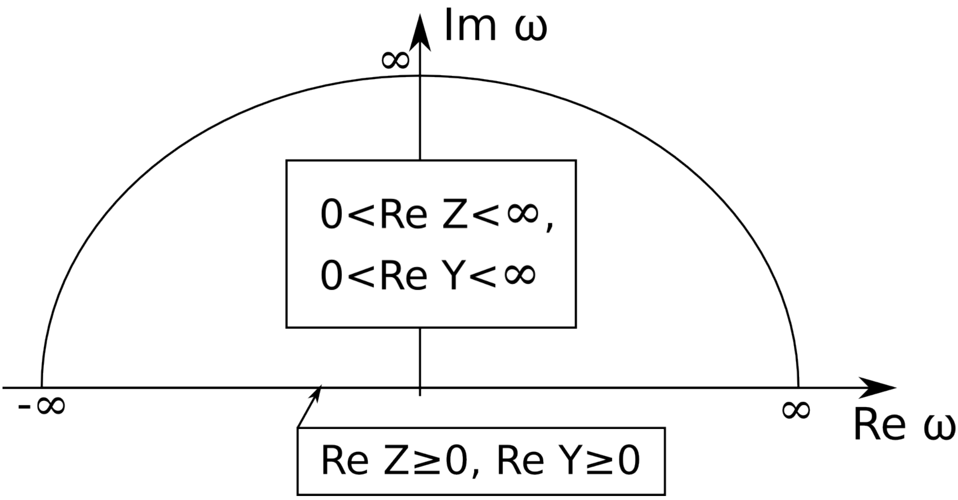

Figure 18: The impedance Z and the admittance Y in the upper half plane. There are no poles or zeros in the upper half plane, but the real axis are yet to be investigated more thoroughly.

Next, we examine the real axis, showing that Z and Y have no poles in the origin.

5.4.2 The impedance and admittance on the real axis

Let us consider a general transmission line metamaterial with a series impedance Z and a shutter admittance Y. The unit cell is shown in figure 19.

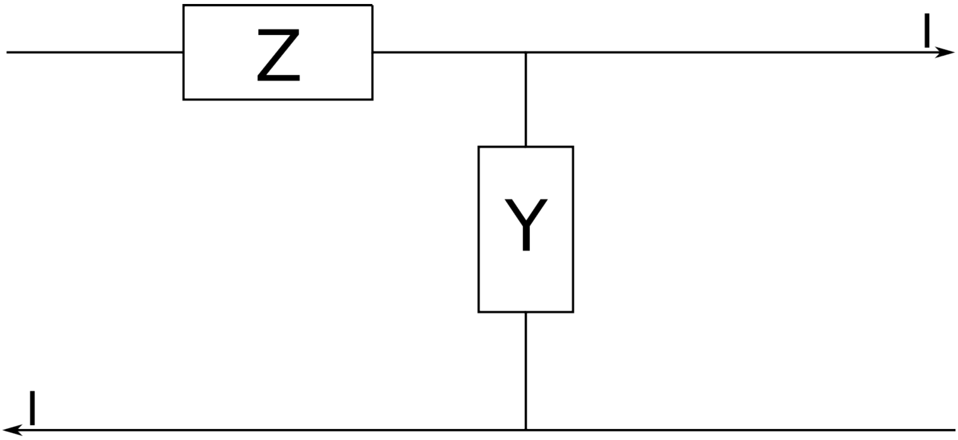

Figure 19: A simple, general transmission line unit cell, with a series impedance Z and a shutter admittance Y.

For the impedance Z we have the following well-known relations,

$$\begin{aligned} Z_{\text{ser}} &= \sum_n Z_n \\ \frac{1}{Z_{\text{par}}} &= \sum_n \frac{1}{Z_n} = \sum_n Y_n, \end{aligned} \tag{82}$$

for the serial and parallel connection, respectively. Similarly, for the admittance we have

$$\begin{aligned} Y_{\text{par}} &= \sum_n Y_n \\ \frac{1}{Y_{\text{ser}}} &= \sum_n \frac{1}{Y_n} = \sum_n Z_n, \end{aligned} \tag{83}$$

for the parallel and serial connection, respectively. We have to emphasize that we assume that we only have parallel and serial connections for Z and Y. Thus, the relations that are to be shown do not apply if the circuit has a topology that cannot be created with the combinations of serial and parallel connections.

From the above relations, we see that Z and Y are calculated *solely* by adding the individual terms Z_n and Y_n. Consequently, as long as the individual terms Z_n and Y_n have no poles, Z and Y would not have any poles, either.

Thus, we have to calculate the impedance and admittance for the individual elements in the transmission line circuit. The two possible elements are a capacitor and an inductor, both with a leakage resistance. The elements are sketched in figures 20 and 21. A series resistance is added to both elements, so that we are dealing with a circuit that takes loss into consideration, which every physical circuit will do to some degree. Note, once again, that all the resistors are set to the same value R, to simplify the notation.

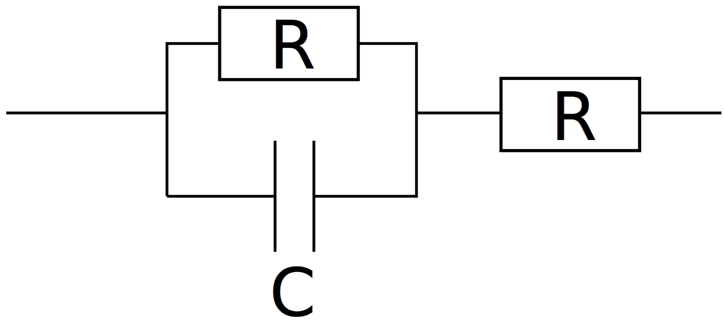

Figure 20: A capacitor with a leakage resistance connected in parallel. A resistance is added in series, since we are considering loss in the circuit.

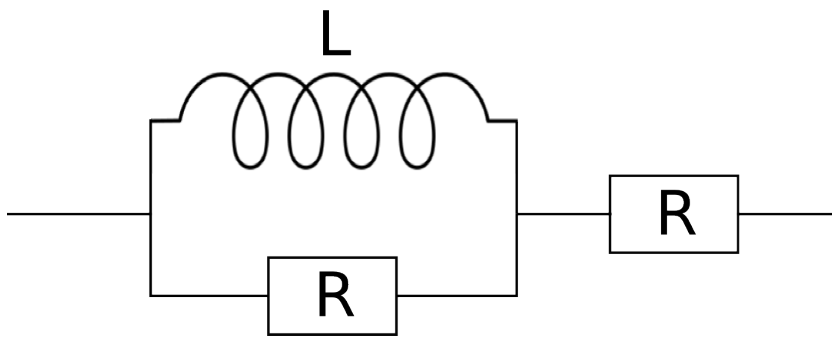

Figure 21: An inductor with a leakage resistance connected in parallel. A resistance added in series, since we are considering a circuit with loss.

For the parallel connection to the left in figure 20, we have for the impedance

$$\frac{1}{Z_{\text{left}}} = \frac{1}{\frac{1}{-i\omega C}} + \frac{1}{R} = \frac{1 - i\omega RC}{R} \Rightarrow Z_{\text{left}} = \frac{R}{1 - i\omega RC}, \tag{84}$$

which leads to the following impedance and admittance for the capacitor:

$$Z_C = Z_{\text{left}} + R = \frac{R}{1 - i\omega RC} + R = \frac{R(2 - i\omega RC)}{1 - i\omega RC} \tag{85}$$

$$\Rightarrow Y_C = \frac{1 - i\omega RC}{R(2 - i\omega RC)} \tag{86}$$

Similarly, for the parallel connection to the left in figure 21, we have the impedance

$$\frac{1}{Z_{\text{left}}} = \frac{1}{-i\omega L} + \frac{1}{R} = \frac{R - i\omega L}{-Ri\omega L} \Rightarrow Z_{\text{left}} = \frac{-Ri\omega L}{R - i\omega L}, \tag{87}$$

and thus for the whole circuit element in figure 21,

$$Z_L = Z_{\text{left}} + R = \frac{-Ri\omega L}{R - i\omega L} + R = \frac{R(R - 2i\omega L)}{R - i\omega L} \tag{88}$$

$$\Rightarrow Y_L = \frac{R - i\omega L}{R(R - 2i\omega L)} \tag{89}$$

Now we want to investigate if equations (85)-(89) have a pole on the real axis. In order to determine this, we have to state the sign of R. Since we are considering passive materials, we have that $R > 0$ (the circuit is incapable of power gain). Also, we assume that $L > 0$ and $C > 0$. Then we get the results summarized in table 1.

Table 1: The impedances and admittances of all the four possible circuit elements (from equations (85)-(89)), and their corresponding poles.

Element	**Pole**
$Z_C = \frac{R(2-i\omega RC)}{1-i\omega RC}$	$\omega = \frac{-i}{RC}$
$Y_C = \frac{1-i\omega RC}{R(2-i\omega RC)}$	$\omega = \frac{-2i}{RC}$
$Z_L = \frac{R(R-2i\omega L)}{R-i\omega L}$	$\omega = \frac{-iR}{L}$
$Y_L = \frac{R-i\omega L}{R(R-2i\omega L)}$	$\omega = \frac{-iR}{2L}$

From table 1 we observe that none of the elements have a pole on the real axis; all the poles are in the lower half plane. Since the individual terms have no poles on the real axis, we have from equations (82)-(83) that Z and Y have no poles, as Z and Y are obtained *solely* by summarizing the above individual terms. To sum up our findings, we conclude:

For a passive transmission line metamaterial, we assume

- a circuit consisting of *only* serial and parallel connections, or the combination of them,
- $L > 0$, $C > 0$ for the inductance and the capacitance.

Then the impedance Z and the admittance Y has no zeros or poles in the upper half plane, including the real axis. Consequently, ϵ and μ,

$$\epsilon = \frac{Y}{-i\omega} \tag{90}$$

$$\mu = \frac{Z}{-i\omega}, \tag{91}$$

has no second order pole in the origin. Since, in addition, $\epsilon, \mu \to 0$ as $\omega \to \infty$, we can conclude that ϵ and μ satisfy the MKK relations.

This is a remarkable result. When we assume no loss, the transmission line metamaterials would not satisfy the MKK relations. Yet, when loss is taken into consideration, they generally do.

For the transmission line metamaterials, the behavior $\mu \sim \frac{1}{\omega^2}$ caused problems when applying the MKK relations. In the next chapter we will, for a general metamaterial, examine the validity of the behavior $\mu \sim \omega^N$ for different N, and investigate how we can modify $\xi(\mu, \omega)$ to satisfy the MKK relations.

6 THE LIMITATIONS ON THE BEHAVIOR OF μ

In this chapter, we will find the limitations on the behavior of μ in two cases: When $\omega \to 0$ and when $\omega \to \infty$. Specifically, we are interested in the order of μ as a function of ω. The central question is this: For the asymptotic behavior $\mu \sim \omega^N$, what integers N are possible?

To determine the possible integers, we need a constraint. For the upper half complex plane, we require $\operatorname{Im} \omega\mu > 0$. This constraint will be derived in the following section, and it uses the same procedure as Landau & Lifshitz used when deriving $\omega > 0 \Rightarrow \mu'' > 0$ [3], which we briefly explained in chapter 2. We will then use the constraint $\operatorname{Im} \omega\mu > 0$ to investigate the behavior of a function $F \equiv \omega\mu$ when $\omega \to 0$ and $\omega \to \infty$. Finally, knowing how F behaves, we may also conclude how $\mu = \frac{F}{\omega}$ behaves. Then, in the next chapter, we can – based on knowledge of the behavior of μ – discuss how μ may satisfy the MKK relations.

6.1 IM $\omega\mu > 0$ IN THE UPPER HALF PLANE

When deriving the constraint $\operatorname{Im} \omega\mu > 0$, we start with defining the Poynting vector [2],

$$\vec{S} = \vec{E} \times \vec{H}, \tag{92}$$

which is a measure of the rate of energy transfer per unit area of an electromagnetic field. Then we have for average heat evolved pr unit time and volume [3],

$$Q = -\nabla \cdot \vec{S}\Big|_{\text{avg}} \tag{93}$$

where the avg index denotes the time-averaged value of $-\nabla \cdot \vec{S}$. In a physical process, some heat will always be produced, due to the fundamental thermodynamics that implies the inevitable increase of entropy. Thus, we have the constraint

$$Q > 0, \tag{94}$$

which will be important in this derivation. Vector calculus yields

$$-\nabla \cdot \vec{S} = -\nabla \cdot (\vec{E} \times \vec{H}) = -\left(\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H}\right) = \vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t}, \tag{95}$$

where two of Maxwell's equations, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$, have been used. Assuming the fields to be complex, and a time dependency $e^{-i\omega t}$, we can substitute

$$
\begin{aligned}
\vec{E} &= \frac{1}{2}(\vec{E} + \vec{E}^*) && (96) \\
\vec{H} &= \frac{1}{2}(\vec{H} + \vec{H}^*) && (97) \\
\frac{\partial \vec{D}}{\partial t} &= \frac{1}{2}(-i\omega\epsilon\vec{E} + i\omega^*\epsilon^*\vec{E}^*) && (98) \\
\frac{\partial \vec{B}}{\partial t} &= \frac{1}{2}(-i\omega\epsilon\vec{H} + i\omega^*\epsilon^*\vec{H}^*) && (99)
\end{aligned}
$$

Since we want to average $-\nabla \cdot \vec{S}$ with respect to time, we can set $\vec{E} \cdot \vec{E} = 0$, $\vec{E}^* \cdot \vec{E}^* = 0$ as they include a rapidly oscillating term $e^{\mp 2i\omega t}$. The same arguments apply for $\vec{H}$. Applying the above substitutions, we obtain from equation (95)

$$
\begin{aligned}
Q &= \left[\frac{1}{2}(\vec{H} + \vec{H}^*)\frac{1}{2}(-i\omega\mu\vec{H} + i\omega^*\mu^*\vec{H}^*) + \frac{1}{2}(\vec{E} + \vec{E}^*)\frac{1}{2}(-i\omega\epsilon\vec{E} + i\omega^*\epsilon^*\vec{E}^*)\right]_{\text{avg}} \\
&= \frac{1}{4}\left(|\vec{H}|^2(i\omega^*\mu^* - i\omega\mu) + |\vec{E}|^2(i\omega^*\epsilon^* - i\omega\epsilon)\right) > 0 \\
&\Rightarrow \quad i\omega^*\mu^* - i\omega\mu > 0 \quad \text{and} \quad i\omega^*\epsilon^* - i\omega\epsilon > 0 && (100)
\end{aligned}
$$

We use the first part of equation (100) to determine an important constraint for μ:

$$
\begin{aligned}
i\omega^*\mu^* - i\omega\mu &= i(\omega' - i\omega'')(\mu' - i\mu'') - i(\omega' + i\omega'')(\mu' + i\mu'') \\
&= i\left(\omega'\mu' - \omega''\mu'' - i(\omega''\mu' + \omega'\mu'')\right) - i\left(\omega'\mu' - \omega''\mu'' + i(\omega''\mu' + \omega'\mu'')\right) \\
&= 2(\omega''\mu' + \omega'\mu'') > 0. && (101)
\end{aligned}
$$

Recognizing that $\text{Im}\,\omega\mu = \omega''\mu' + \omega'\mu''$, equation (101) provides the constraint

$$\text{Im}\,\omega\mu > 0 \tag{102}$$

for the upper half plane.

Figure 22 provides a sketch of the constraint. In the next section, we will define a function $F \equiv \omega\mu$, and investigate how the requirement $\text{Im}\,F > 0$ will restrict F. Figure 22 provides a visualization of the requirement.

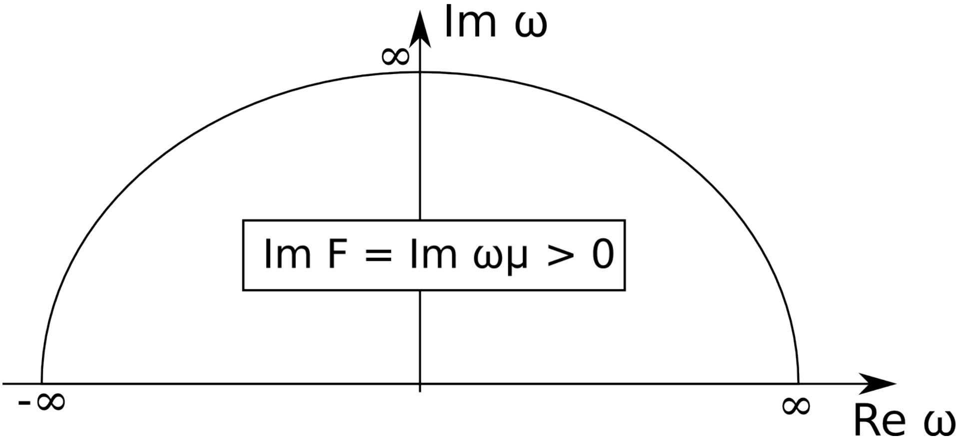

Figure 22: We require $\text{Im}(\omega\mu) > 0$ in the upper half complex plane.

6.2 THE BEHAVIOR OF F AS A FUNCTION OF ω

We require μ to be a rational function. Then $F = \omega\mu$ is also a rational function, and we may write

$$F(\omega) = \frac{P_n(\omega)}{Q_m(\omega)} = \frac{a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_0}{b_m\omega^m + b_{m-1}\omega^{m-1} + \cdots + b_0}, \tag{103}$$

where $P_n(\omega)$ and $Q_m(\omega)$ are polynomials of n-th and m-th order, respectively, and a_n and b_m are assumed to be nonzero. For large ω, we obtain

$$\lim_{\omega\to\infty} F(\omega) = \frac{a_n}{b_m}\omega^{n-m}. \tag{104}$$

For small ω, we obtain

$$\lim_{\omega \to 0} F(\omega) = \frac{a_k}{b_l} \omega^{k-l}, \tag{105}$$

where a_k and b_l are the lowest order non-zero coefficients of P_n and Q_m, respectively. Figure 23 provides a visualization on how F behaves for large and small ω. In the upcoming analysis, we will determine the limitations of $|k-l|$, that is, find an integer N yielding the constraint $|k-l| \leq N$. Then we will show that the same limitation applies also for $|n-m|$.

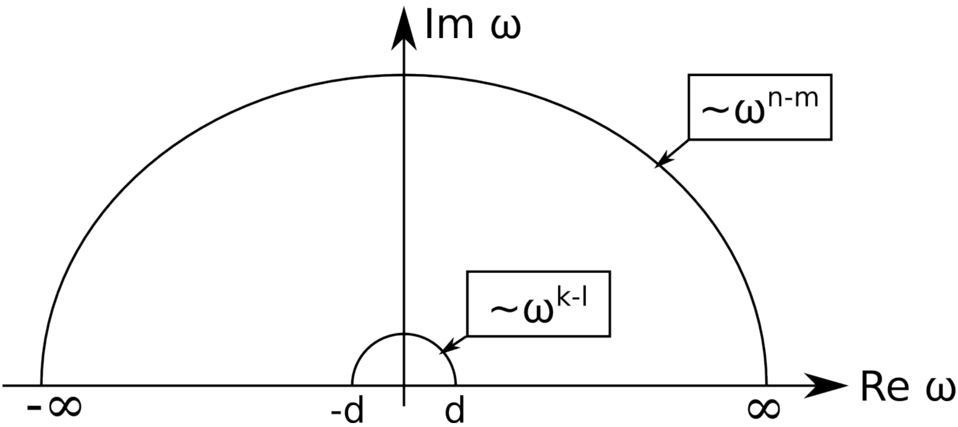

Figure 23: When $\omega \to 0$, the behavior of F depends on the difference of the coefficients k and l. When $\omega \to \infty$, however, the behavior of F depends on the difference of the coefficients n and m. Note that the quantity d is greatly magnified; in fact, $d \to 0$.

To investigate which constraints the requirement Im $F > 0$ puts on F as $\omega \to 0$, we will study how F behaves as a function of ω. Specifically, we will vary the integers $k-l$, and examine which integers that satisfy the requirement Im $F > 0$. We start by setting $k-l = 1$, applying Euler's formula

$$\lim_{\omega \to 0} F = \lim_{\omega \to 0} \frac{a_k}{b_l} \omega = \lim_{\omega \to 0} \frac{a_k}{b_l} |\omega| e^{i\theta} = \lim_{\omega \to 0} \frac{a_k}{b_l} |\omega| (\cos\theta + i \sin\theta). \tag{106}$$

Figure 24 shows the definition of θ in the complex plane. Note that in the upper half plane, $0 < \theta < \pi$.

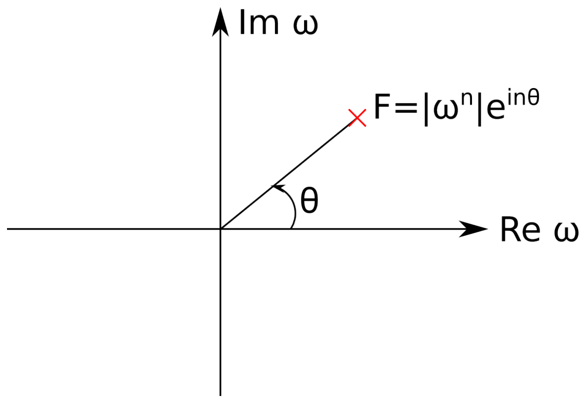

Figure 24: The complex quantity F shown on exponential form. For the upper half plane, the figure shows that $0 < \theta < \pi$.

For the imaginary part of $\lim_{\omega\to 0} F = \frac{a_k}{b_l}\omega$, we obtain

$$\lim_{\omega\to 0} \operatorname{Im} F = \lim_{\omega\to 0} \frac{a_k}{b_l}|\omega| \sin\theta \begin{cases} > 0 & \text{when } 0 < \theta < \pi \\ < 0 & \text{when } \pi < \theta < 2\pi, \end{cases} \tag{107}$$

which satisfies the requirement Im $F > 0$ in the upper half plane. In the above equation, we have chosen $\frac{a_k}{b_l} > 0$. Thus, $k - l = 1$ satisfies our requirements.

Next, we set $k - l = 0$. Towards the origin, F then becomes

$$\lim_{\omega\to 0} F = \lim_{\omega\to 0} \frac{a_k}{b_l}\omega^0 = \frac{a_k}{b_l}, \tag{108}$$

hence, for the imaginary part,

$$\lim_{\omega\to 0} \operatorname{Im} F = \operatorname{Im} \frac{a_k}{b_l} = 0, \tag{109}$$

which, at first sight, may seem to violate the condition Im $F > 0$. However, it does not. Im F *approaching* zero does not necessarily mean that Im F *equals* zero, as $\omega \to 0$. Thus, the condition Im $F > 0$ is not violated, and $k - l = 0$ satisfies our requirements.

Now, we set $k-l=-1$ and choose $\frac{a_k}{b_l}<0$, leading to

$$\begin{aligned}\lim_{\omega\to 0} F &= \lim_{\omega\to 0}\frac{a_k}{b_l}|\omega^{-1}|(\cos\theta - i\sin\theta)\\ \Rightarrow \lim_{\omega\to 0}\operatorname{Im} F &= \frac{a_k}{b_l}|\omega^{-1}|\sin(-\theta)\begin{cases}>0 & \text{when } 0<\theta<\pi\\ <0 & \text{when } \pi<\theta<2\pi,\end{cases}\end{aligned} \tag{110}$$

which satisfies the requirement of $\operatorname{Im} F>0$ in the upper half plane. The fact that we may choose the sign of the coefficient $\frac{a_k}{b_l}$ enables both $k-l=1$ and $k-l=-1$ to satisfy the requirement.

Next, we set $k-l=2$ and $\frac{a_k}{b_l}>0$, obtaining

$$\begin{aligned}\lim_{\omega\to 0} F &= \lim_{\omega\to 0}\frac{a_k}{b_l}|\omega^{2}|(\cos(2\theta) + i\sin(2\theta))\\ \Rightarrow \lim_{\omega\to 0}\operatorname{Im} F &= \lim_{\omega\to 0}\frac{a_k}{b_l}|\omega^{2}|\sin(2\theta) = \begin{cases}>0 & \text{when } 0<\theta<\frac{\pi}{2}\\ <0 & \text{when } \frac{\pi}{2}<\theta<\pi,\end{cases}\end{aligned} \tag{111}$$

which does not satisfy the requirement. Im F is greater than zero in the first quadrant, but less than zero in the second quadrant, as indicated in figure 25.

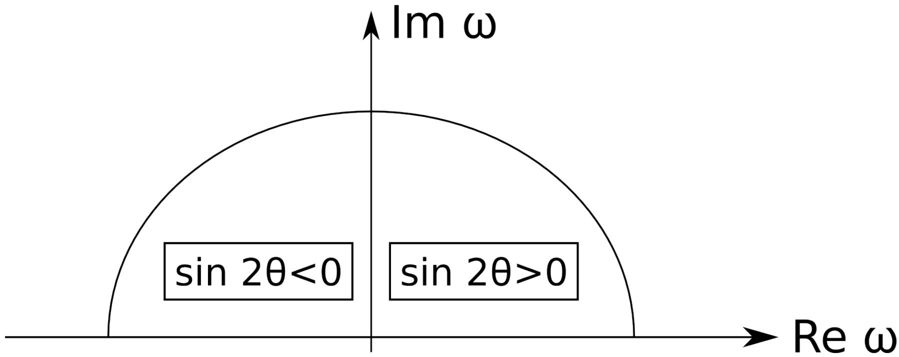

Figure 25: $\sin 2\theta$ is greater than zero when $0<\theta<\frac{\pi}{2}$, and less than zero when $\frac{\pi}{2}<\theta<\pi$. The sign of $\sin 2\theta$ switches every $\pi/2$ radians.

$\sin 2\theta$ has a period of π, which means it will change sign every $\frac{\pi}{2}$ radians – regardless of the coefficient in front of the sine term. This of course also applies for $\sin(-2\theta)$ and so on for all $\sin(|k-l|\theta)$ where $|k-l|\geq 2$.

Thus, we conclude that for $\lim_{\omega\to 0} F$, we require $|k-l| \le 1$. What about $\lim_{\omega\to\infty} F$?

We can apply the same procedure for F when $\omega \to \infty$. We observe that since

$$\lim_{\omega\to\infty} F = \frac{a_n}{b_m}\omega^{n-m}, \tag{112}$$

F has the same form when $\omega \to \infty$ and when $\omega \to 0$. Following the same procedure as above would therefore give an analogous result: We require $|n-m| \le 1$ in order to satisfy $\text{Im } F > 0$.

The above analysis demands $|k-l| < 1$ and $|n-m| < 1$ simultaneously. Yet, it does *not* demand $k-l = n-m$. Recall that F being a rational function, we can write

$$F(\omega) = \frac{P_n(\omega)}{Q_m(\omega)} = \frac{a_n\omega^n + a_{n-1}\omega^{n-1} + \cdots + a_0}{b_m\omega^m + b_{m-1}\omega^{m-1} + \cdots + b_0}. \tag{113}$$

As $\omega \to 0$, it is the lowest non-zero coefficients, a_k and b_l, that determine the order of F. However, as $\omega \to \infty$, it is the coefficients n and m that determines the order of F; a_k and b_l are not necessarily dependent of n and m. Thus, F might have a different order at zero and infinity. We summarize:

For the function $F = \omega\mu$, in the origin and at infinity, the degree of the numerator can differ with the degree of the denominator by at most one. Stated algebraically,

$$\begin{aligned} \omega \to 0 &\Rightarrow F = O(\omega) \text{ or } F = O(\omega^0) \text{ or } F = O(\omega^{-1}) \\ \omega \to \infty &\Rightarrow F = O(\omega) \text{ or } F = O(\omega^0) \text{ or } F = O(\omega^{-1}) \end{aligned} \tag{114}$$

However, the order of F in the origin does not necessarily equal its order at infinity.

6.3 The order of μ

In the previous section, we found that the function $F = \omega\mu$ was restricted to behave as $\sim\omega^{-1}$, $\sim\omega^0$ or $\sim\omega^1$. What restrictions does this put on the order of μ?

Since $\mu = \frac{F}{\omega}$, we obtain the same restrictions for μ, only with one lower degree. Thus, $\mu(\omega)$ is restricted to behave as $\sim\omega^{-2}$, $\sim\omega^{-1}$ or $\sim\omega^0$. Put formally:

For the magnetic permeability $\mu(\omega)$, in the origin and at infinity, the degree of the numerator can only be two, one or zero degrees lower than the degree of the denominator. Stated algebraically,

$$\begin{aligned} \omega \to 0 \ &\Rightarrow \mu = O(\omega^0) \ \text{ or } \ \mu = O(\omega^{-1}) \ \text{ or } \mu = O(\omega^{-2}) \\ \omega \to \infty \ &\Rightarrow \mu = O(\omega^0) \ \text{ or } \ \mu = O(\omega^{-1}) \ \text{ or } \mu = O(\omega^{-2}) \end{aligned} \tag{115}$$

However, the order of μ in the origin does not necessarily equal its order at the infinity.

This is a strong result. It implies clear restrictions on the physical quantity μ. Or, put differently, it enables opportunities on how μ may behave. If, for instance, $\mu \sim \omega^{-2}$ in the proximity of the origin, will we be able to satisfy the MKK relations for μ? Clearly the function $\xi = \mu$ would not satisfy the MKK relations due to a second order pole in the origin. But can we modify ξ to make it satisfy the MKK relations? We will discuss these types of issues in the next section.

The asymptotic behavior of μ shows that we may have several different types of μ that does not satisfy the usual KK relations. This leads to an important consequence:

Causality is less restrictive than the usual KK relations.

By this, we mean that from causality, we obtain the asymptotic behavior of μ. This asymptotic behavior allows both a second order pole in the origin, and that μ does not approach unity as $\omega \to \infty$. Yet, the usual KK relations prohibit μ to have a second order pole in the origin, and furthermore, they require $\mu \to 1$ as $\omega \to \infty$. Thus, causality is less restrictive than the KK relations. Put another way, this implies that the possible metamaterial responses are not so much restricted as the KK relations suggest.

6.4 An extended set of Kramers-Kronig relations for μ

As we have seen in the previous chapters, several functions ξ satisfies the MKK relations. With the behavior of μ for small and large ω derived in the previous section, we can now provide a general overview on which functions $\xi(\mu,\omega)$ that satisfy the MKK relations. Since the order of μ might have three different values, both in the origin and at infinity, we have $3^2 = 9$ possible combinations, listed in table 6.4.

We recall the requirements for a function $\xi(\omega)$ to satisfy the MKK relations:

- $\xi(\omega)$ is analytic in the upper half plane, except in the origin, where a pole of order one is possible.
- $\xi(\omega) \to 0$ as $\omega \to \infty$.

Table 6.4 lists the different possible combinations of μ. Also, the table includes a function ξ that will satisfy the MKK relations, given the behavior of μ. Note that the function ξ provided in each row in the table is not necessarily the *only* function satisfying the MKK relations – it is just *a* function satisfying the relations. To explain how the suggested ξ satisfies the MKK relations, we will examine each of the nine combinations more closely.

Table 2: The nine different combinations of the behavior of μ in the origin and at infinity. The numbering in the left column is for convenient reference; the right column contains suggestions for ξ satisfying the KK relations. Note that $\mu_1 \equiv \mu(\omega = \infty)$.

Case	$\boldsymbol{\mu}$ **as** $\boldsymbol{\omega \to 0}$	$\boldsymbol{\mu}$ **as** $\boldsymbol{\omega \to \infty}$	$\boldsymbol{\xi(\omega)}$
1	$\sim \omega^{-2}$	$\sim \omega^{-2}$	$\omega\mu$
2	$\sim \omega^{-1}$	$\sim \omega^{-2}$	μ
3	$\sim \omega^{0}$	$\sim \omega^{-2}$	μ
4	$\sim \omega^{-2}$	$\sim \omega^{-1}$	$\omega(\mu - \mu_1)$
5	$\sim \omega^{-1}$	$\sim \omega^{-1}$	μ
6	$\sim \omega^{0}$	$\sim \omega^{-1}$	μ
7	$\sim \omega^{-2}$	$\sim \omega^{0}$	$\mu \frac{\omega}{1+\omega^2}$
8	$\sim \omega^{-1}$	$\sim \omega^{0}$	$\mu - \mu_1$
9	$\sim \omega^{0}$	$\sim \omega^{0}$	$\frac{\mu}{\omega}$

CASE 1: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-2}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-2}$

Here, μ has a second order pole in the origin. In order to satisfy the MKK relations, we recall that a possible pole in the origin could at most have order one. To reduce the pole to a first order pole, we simply set $\xi = \omega\mu$, making $\xi \to\sim \omega^{-1}$ in the origin. Also, for infinity, we note that since $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-2}$, $\xi = \omega\mu \to\sim \omega^{-1} \to 0$. Thus, the requirement at infinity is also satisfied, and the MKK relations are hence satisfied.

CASE 2: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-1}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-2}$

μ has a first order pole in the origin, which does not violate the MKK requirements. Also, $\omega \to \infty \Rightarrow \mu \to 0$, in accordance with the MKK requirements. Thus, setting $\xi = \mu$ will satisfy the MKK relations. Note, still, that the magnetic susceptibility $\chi = \mu - 1$ would not satisfy the KK relations, since in that case, $\omega \to \infty \Rightarrow \xi \to -1 \neq 0$.

CASE 3: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{0}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-2}$

This case is similar to Case 2: μ has no pole in the origin, and approaches 0 at infinity, thus, $\xi = \mu$ will satisfy the MKK relations, while $\chi = \mu - 1$ will not satisfy the KK relations, with the same arguments as above.

CASE 4: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-2}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-1}$

Since μ has a second order pole, we start by multiplying with ω to make it a first order pole. Yet, the function $\omega\mu$ will not satisfy the MKK relations at infinity: Since $\mu \to\sim \omega^{-1}$ at infinity, it follows that $\omega\mu \to\sim \omega^{0}$, which is not necessarily equal to zero. To rather set $\xi = \omega(\mu - \mu_1)$, where $\mu_1 \equiv \mu(\omega = \infty)$, does the trick. Since μ is a rational function, we may write

$$\mu = c_1\omega^{-1} + c_2\omega^{-2} + c_3\omega^{-3} + \cdots$$

$$\begin{aligned} \Rightarrow \mu - \mu_1 &= c_1\omega^{-1} + c_2\omega^{-2} + c_3\omega^{-3} + \cdots - c_1\omega^{-1} \\ &= c_2\omega^{-2} + c_3\omega^{-3} + c_4\omega^{-4} + \cdots \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad \omega(\mu - \mu_1) = O(\omega^{-1}) \text{ as } \omega \to \infty \\ &\Rightarrow \quad \lim_{\omega\to\infty} \omega(\mu - \mu_1) = 0. \end{aligned} \tag{116}$$

Thus, setting $\xi = \omega(\mu - \mu_1)$ satisfies the MKK relations.

CASE 5: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-1}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-1}$

This is similar to Case 2 and 3: μ has a first order pole in the origin and $\mu \to 0$ at infinity, which means that $\xi = \mu$ satisfies the MKK relations. As with case 2 and 3, $\chi = \mu - 1$ would not satisfy the KK relations.

CASE 6: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{0}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{-1}$

This is similar to Case 2, 3 and 5, where μ satisfies the MKK relations both in the origin and at infinity, thus, we may set $\xi = \mu$.

CASE 7: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-2}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{0}$

Since μ has a second order pole, we want to increase the order in the origin by one. We can achieve this by multiplying with ω. Nonetheless, since $\mu \to\sim \omega^0$ at infinity, we also want to reduce the order at infinity by one, and multiplying with ω only makes matters worse. Is it possible to satisfy the MKK relations in this case?

Multiplying with the function $g = \frac{\omega}{1+\omega^2}$ will do the trick. We observe that

$$g = O(\omega) \text{ as } \omega \to 0 \tag{117}$$

$$g = O(\omega^{-1}) \text{ as } \omega \to \infty. \tag{118}$$

Consequently, we obtain by setting $\xi = \mu\frac{\omega}{1+\omega^2}$,

$$\xi = \mu\frac{\omega}{1+\omega^2} = O(\omega^{-1}) \text{ as } \omega \to 0 \tag{119}$$

$$\xi = \mu\frac{\omega}{1+\omega^2} = O(\omega^{-1}) \text{ as } \omega \to \infty \tag{120}$$

With a first order pole in the origin, and $\xi \to 0$ as $\omega \to \infty$, the MKK relations are thus satisfied.

CASE 8: $\omega \to 0 \Rightarrow \mu \to\sim \omega^{-1}$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^{0}$

μ has a first order pole in the origin, which does not violate the MKK requirements. At infinity, however, $\mu \to\sim \omega^0$, which is not necessarily equal to zero. The function $\mu - \mu_1$ will do the trick:

$$\begin{aligned}
\mu - \mu_1 &= c_0\omega^0 + c_1\omega^{-1} + c_2\omega^{-2} + \cdots - c_0\omega^0 \\
&= c_1\omega^{-1} + c_2\omega^{-2} + c_3\omega^{-3} \\
&\Rightarrow \lim_{\omega\to\infty}(\mu - \mu_1) = O(\omega^{-1}) = 0.
\end{aligned} \tag{121}$$

Note that in Case 8, the function $\mu - 1$, that is, the magnetic susceptibility, also will satisfy the KK relations in the special case of $\mu \to 1$ as $\omega \to \infty$. But $\xi = \mu - \mu_1$ will satisfy the MKK relations in general.

CASE 9: $\omega \to 0 \Rightarrow \mu \to\sim \omega^0$, $\omega \to \infty \Rightarrow \mu \to\sim \omega^0$

We quickly observe that the conditions in Case 9 are similar to those in Case 8; thus, the function $\mu - \mu_1$ would satisfy the MKK relations, and $\chi = \mu - 1$ would in the special case of $\mu \to 1$ as $\omega \to \infty$ satisfy the KK relations. However, the absence of a pole in the origin also enables another possibility, which we will provide here, as an example of the variety of functions satisfying the MKK relations. Setting $\xi = \frac{\mu}{\omega}$ yields a first order pole in the origin, which is in accordance with the MKK requirements. At infinity,

$$\begin{aligned}
\frac{\mu}{\omega} &= O(\omega^{-1}) \\
&\Rightarrow \lim_{\omega\to\infty}\frac{\mu}{\omega} = 0.
\end{aligned} \tag{122}$$

Consequently, the MKK relations are satisfied for $\xi = \frac{\mu}{\omega}$.

Table 6.4 shows examples of the different functions satisfying the MKK relations. In addition to the magnetic susceptibility $\chi = \mu - 1$ satisfying the KK relations, the following set of functions ξ are found to satisfy the MKK relations:

$$\xi = \left\{\mu\,,\, \omega\mu\,,\, \omega(\mu - \mu_1)\,,\, \mu - \mu_1\,,\, \frac{\mu\omega}{1+\omega^2}\,,\, \frac{\mu}{\omega}\right\} \tag{123}$$

It is important to note, as pointed out in table 6.4: These functions does not all satisfy the MKK simultaneously, but in different cases, depending on the behavior of μ. Thus, we have obtained an extended class of functions where we are able to obtain a connection between the real part and the imaginary part. This is illustrated in figure 26.

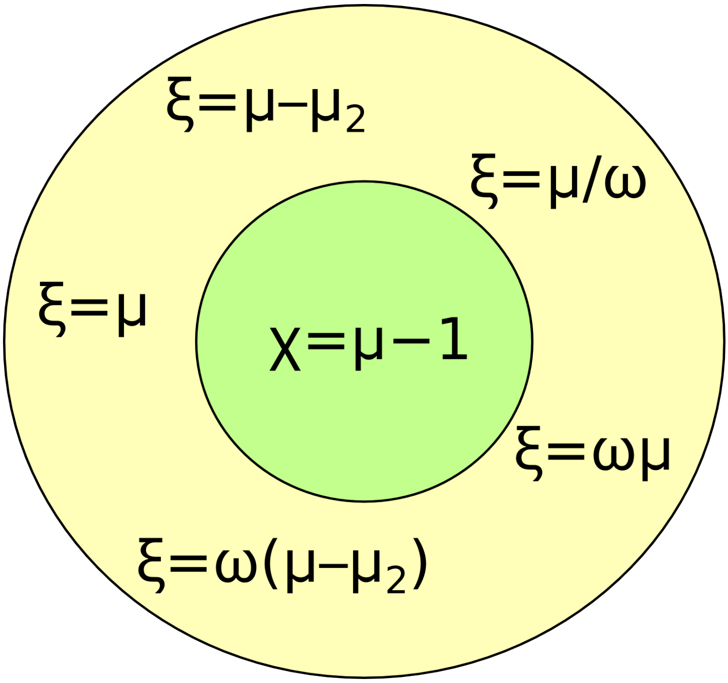

Figure 26: A visualization of the set of functions ξ satisfying the MKK relations for μ. As the figure illustrates, the magnetic susceptibility $\chi = \mu - 1$ is only one of several functions that provides a connection between the real part and the imaginary part of μ.

Yet, only in Case 8 and Case 9, there is a possibility that $\chi = \mu - 1$ will satisfy the KK relations. In the seven other possible behaviors of μ, we need to use different types of modifications to satisfy the MKK relations. This implies that, for Case 1-7, even though $\mu - 1$ will not satisfy the KK relations, μ will still be under an analogous mathematical constraint after ξ has been modified properly.

7 Conclusion and further work

The aim of this work has been to investigate how we may apply the MKK relations on the magnetic permeability, μ, for passive metamaterials. An important question was this: Would μ behave such that it satisfied the MKK relations, and in cases where μ did not, how could we modify the input function ξ to make μ satisfy the relations? Specifically, we have also studied the special case of transmission line metamaterials, to investigate whether they satisfy the MKK relations.

Transmission line metamaterials were initially found to have a second order pole in the origin, which is not in accordance with satisfying the MKK relations. However, when we included resistors in the materials, and assumed a circuit topology consisting solely of parallel and serial connections, the second order poles in the origin were reduced to first order poles. Consequently, under these assumptions, transmission line metamaterials satisfy the MKK relations in general.

For the behavior of μ for a metamaterial, we found that

$$\text{when } \omega \to 0\,;\ \mu = O(\omega^0)\ ,\ \mu = O(\omega^{-1}) \text{ or } \mu = O(\omega^{-2}) \tag{124}$$

$$\text{when } \omega \to \infty\,;\ \mu = O(\omega^0)\ ,\ \mu = O(\omega^{-1}) \text{ or } \mu = O(\omega^{-2}). \tag{125}$$

This lead to $3^2 = 9$ different combinations of how μ could behave. These combinations are a consequence of causality, and imply that μ is less restricted than the usual KK relations suggest: They allow both a second order pole in the origin, and that μ does not need to approach unity at infinity. This means that causality is less restrictive than the KK relations.

For the nine different behaviors of μ around the origin and at infinity, the following set of functions ξ were found to satisfy the MKK relations:

$$\xi = \left\{\mu\ ,\ \omega\mu\ ,\ \omega(\mu-\mu_1)\ ,\ \mu-\mu_1\ ,\ \frac{\mu\omega}{1+\omega^2}\ ,\ \frac{\mu}{\omega}\right\}. \tag{126}$$

Note that the above set of functions does not satisfy the MKK relations simultaneously, but in different cases, depending on the behavior of μ. The set of functions (126) can be seen as an extended sample space of functions satisfying the MKK relations. Instead of considering the KK relations for μ solely through the magnetic susceptibility $\chi = \mu - 1$, the set (126) shows that we – through MKK – have more possibilities. Figure 26 illustrates the extended set of functions ξ.

By modifying the input function ξ properly, any possible μ will satisfy the MKK relations. This might be seen both as a limitation and a possibility: A limitation, since μ thus is bound to a mathematical constraint, and a possibility, where the real part of μ might be used to find its imaginary part, and vice versa. For further work, two relevant quantities to investigate are the loss μ'' and the dispersion μ' for a given bandwidth. The extended set of functions (126) connecting μ' with μ'' might provide more possibilities and further insight to this investigation.

Recall from the introduction, we assumed the permeability to behave asymptoticly at frequencies within the metamaterial limit, ref. figure 2. This assumption might not always hold; $\omega_{\text{asymp}} > \omega_{\text{max}}$ might be the case. Then, there will be no frequency range where μ behaves asymptotically within the metamaterial limit. Then the MKK would not work, since we cannot guarantee μ to be well-behaved while integrating on the large semi circle. Further work may be to investigate the validity of this assumption in greater detail.

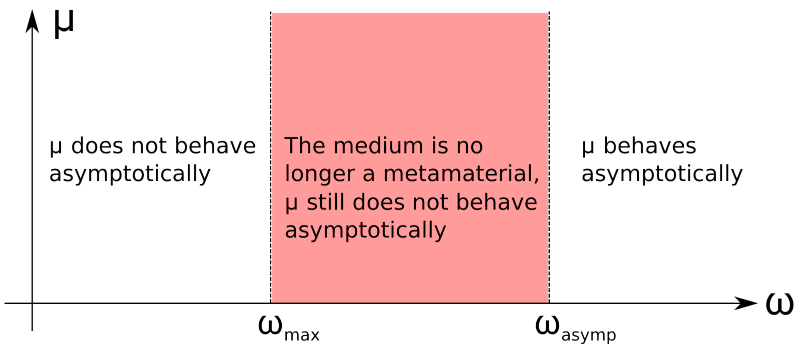

Figure 27: The behavior of μ for a metamaterial as a function of the frequency of the external stimuli. In the middle zone, the frequency exceeds the metamaterial limit, meaning that the condition $\lambda \gg a$ no longer holds. Then the material cease being a metamaterial. Yet, μ still does not behave asymptotically, and performing the MKK relations in this zone is thus not possible, symbolized by the red color. In the right zone, μ behaves asymptotically, but as we no longer consider a metamaterial, the MKK relations from this frequency range might not be relevant.

REFERENCES

[1] N. I. Zheludev, "The Road Ahead for Metamaterials", *Science*, 328, 2010.

[2] J. D. Jackson, "Classical Electrodynamics", *John Wiley and Sons*, 1962.

[3] L. D. Landau, E. M. Lifshitz, "Electrodynamics of Continuous Media", *Pergamon Press*, 1962.

[4] H. D. Young, F. W. Sears, M. Zemansky, "University Physics - 7th edition", *Addison Wesley*, 1992.

[5] M. G. Silveirinha, "Examining the validity of Kramers-Kronig relations for the magnetic permeability", *Physical Review*, 83, 2011.

[6] J. Skaar, "Negative refraction: A tutorial", Norwegian University of Science and Technology, 2010.

[7] A. Lai, C. Caloz, T. Itoh, "Composite Right/Left-Handed Transmission Line Metamaterials", *IEEE Microwave Magazine*, 2004.

[8] “Breakthrough of the year: The runners-up”, *Science*, 302, 2003.

[9] T. Koschny, P. Markos, D. R. Smith, C. M. Soukoulis, "Resonant and antiresonant frequency dependence of the effective parameters of metamaterials", *Physical Review*, 68, 2003.

[10] A. L. Efros, "Comment II on ’Resonant and antiresonant frequency dependence of the effective parameters of metamaterials’" *Physical Review*, 70, 2004.

[11] J. Skaar, "Elektromagnetisme", 2013.

[12] E. Kreyszig, "Advanced Engineering Mathematics, 9th Edition", *John Wiley and Sons*, 2006.

[13] J. B. Pendry, A. J. Holden, D. J. Robbins, W. J. Stewart, "Magnetism from Conductors and Enhanced Nonlinear Phenomena", *IEEE Transactions on Microwave Theory and Techniques*, 47, 1999.

[14] V. G. Veselago, "Electrodynamics of substances with simultaneously negative values of sigma and mu", *Soviet Physics Uspekhi-USSR*, 10, 1968.

[15] R. A. Shelby, D. R. Smith, S. Schultz, "Experimental Verification of a Negative Index of Refraction ", *Science*, 292, 2001.

[16] G. V. Eleftheriades, A. K. Iyer, P. C. Kremer, "Planar Negative Refractive Index Media Using Periodically L–C Loaded Transmission Lines", *IEEE Transactions on Microwave Theory and Techniques*, 50, 2002.

[17] E. Colgate, N. Hogan, "An analysis of contact instability in terms of passive physical equivalents", *IEEE International conference on Robotics and Automation*, 404, 1989.

[18] O. Brune, "Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency", PhD thesis, Massachusetts Institute of Technology, 1931.