Output feedback boundary control of 2×2 semilinear hyperbolic systems ⋆

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Abstract

We consider the control and state estimation of a class of 2×2 semilinear hyperbolic systems with actuation and sensing collocated at one boundary. Our approach exploits the dynamics on the characteristic lines of the hyperbolic system. The control method using full-state feedback can be used for both stabilization of an equilibrium and tracking at an arbitrary point in the domain. The control objective is achieved globally in minimum time. A Lyapunov function is constructed to prove exponential convergence in the spatial supremum norm. For linear systems, the control input can be written explicitly as the inner product of kernels with the state, and turns out to be equivalent to the control input obtained from previously known backstepping methods. The observer achieves exact state estimation also in minimum time and, combined with the state-feedback controller, solves the output feedback control problem. The performance is demonstrated in a numerical example.

Key words: Distributed-parameter systems, Boundary Control, Minimum-time control, Exponential Stability, Tracking, Estimation

1 Introduction

Many relevant systems are described by 2×2 hyperbolic systems, such as open water channels [15,7,9], heat exchangers [28], gas flow in pipelines [14], road traffic [3], and oil wells [1]. In this paper, we are concerned with the semilinear case. That is, the system involves source terms depending nonlinearly on the state, and a nonlinearity at the uncontrolled boundary, but the propagation speeds are assumed to be state-independent. In particular, we are motivated by disturbance rejection in oil well drilling [1]. The oil well is modeled as a hydraulic transmission line with nonlinear friction, and disturbances enter both at the boundary and within the domain, interacting nonlinearly with the state [25,26]. Therefore, the purpose of this paper is to present a constructive controller and observer design method for a very general class of semilinear hyperbolic systems for which there have been no constructive methods for global minimum-time feedback control yet.

One approach to the stabilization of hyperbolic systems is to design dissipative boundary conditions under which the system is stable. First developed for conservation laws, i.e. systems without source terms, stability is established either using the explicit evolution of the Riemann invariants [13,7], via a Lyapunov function [5], or by a frequency domain approach [20]. Stabilization of hyperbolic systems with source terms was considered by [9,21,8]. While their approach achieves closed-loop asymptotic stability, it does not achieve finite-time convergence.

Exact controllability of linear hyperbolic PDEs is well established, see [22] for an overview. For semilinear systems, controllability of a 1-d semilinear wave equation with coefficients that are constant in space was proven in [29]. More recently, [12] showed the exact controllability of semilinear hyperbolic systems with spatially varying coefficients in multiple dimensions. For quasilinear systems, only local controllability results exist, see [19] and references therein. In [18], the minimum time for exact one-sided boundary controllability is given. In these papers, the existence of a control input driving the state from the initial condition to a desired state within a given time, which needs to be larger than some minimally required time, is proven using non-constructive methods. Numerical algorithms to compute the actuation based on the discretized problem exist, see [10] for an overview. However, they can fail if the discretization is refined, and require computationally expensive iterations, limiting their real-time applicability. Moreover, these papers

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discuss only the existence of an open-loop control signal over a fixed interval for given initial and terminal states. They do not consider state feedback, or stability of the controlled system. Still, these results indicate that exact control is feasible for this class of systems.

For some special cases, exact control laws have been found. In [4], a simple feedforward controller was developed to exactly cancel boundary disturbances in linear conservation laws. In [15], an exact controller was designed for a particular nonlinear conservation law. The state was assumed to be in a subcritical range and the control designed to be sufficiently slow in order to avoid shocks.

Over the last years, backstepping has been established as a constructive method for the exact control of linear hyperbolic PDEs. First developed for parabolic PDEs [23], backstepping has been used for the exact control of first-order [17] and second-order [27] hyperbolic systems. In [6] local H^2-stability of a quasilinear system was shown using a backstepping controller designed for the linearized system. Exploiting results from [27], the disturbance rejection problem in the linear case was solved in [1]. More recently, the backstepping approach has been generalized to general heterodirectional systems [16]. However, exact control of hyperbolic PDEs by backstepping is (still) limited to linear systems.

The paper is organized as follows. In Section 2, we give preliminaries that are used in the remainder of the paper. The precise problem statement is given in Section 2.1. In Section 3, we design an state-feedback controller. The control method is used to drive the system to an equilibrium in Section 3.1, and a Lyapunov function to prove global exponential stability in the spatial supremum norm is constructed in Section 3.2. The same control approach is used for tracking, as well as rejecting predictable disturbances in Section 3.3. In Section 4, we design an observer using boundary measurements collocated with the actuation. We combine the controller with the observer to solve the output feedback control problem in Section 5. In Section 6, the controller performance is demonstrated in a numerical example, before concluding remarks are given in Section 7. We include technical proofs in the appendix.

2 Preliminaries

2.1 Problem statement

We consider the semilinear 2×2 hyperbolic system

$$u_t(x,t) = -\epsilon_u(x)u_x(x,t) + F_u((u,v)(x,t),x,t), \quad (1)$$
$$v_t(x,t) = \epsilon_v(x)v_x(x,t) + F_v((u,v)(x,t),x,t), \quad (2)$$
$$u(0,t) = f(v(0,t),t), \quad (3)$$
$$v(1,t) = U(t), \quad (4)$$
$$u(x,0) = u_0(x), \quad (5)$$
$$v(x,0) = v_0(x), \quad (6)$$

for $x \in [0,1]$ and $t \geq 0$. The subscripts $_t$ and $_x$ denote partial derivatives with respect to time and space, respectively. $U(t)$ is the control input. The notation $(u,v)(x,t)$ in F_u and F_v represents the state, (u,v), evaluated at (x,t), although we sometimes omit the (x,t) part for brevity. We assume there exist positive bounds ϵ_1 and ϵ_2 such that $\epsilon_1 \leq \epsilon_u(x) \leq \epsilon_2$ and $\epsilon_1 \leq \epsilon_v(x) \leq \epsilon_2$ for all $x \in [0,1]$, and that F_u, F_v and f are uniformly Lipschitz continuous in the state, i.e. there exist positive constants L_u, L_v, L_f such that

$$|F_u((u_1,v_1),x,t) - F_u((u_2,v_2),x,t)| \leq L_u(|u_1-u_2| + |v_1-v_2|), \quad (7)$$
$$|F_v((u_1,v_1),x,t) - F_v((u_2,v_2),x,t)| \leq L_v(|u_1-u_2| + |v_1-v_2|), \quad (8)$$
$$|f(v_1,t) - f(v_2,t)| \leq L_f|v_1-v_2|, \quad (9)$$

for all (u_1,v_1), (u_2,v_2), x, t. Note that this Lipschitz condition excludes finite-time blow up, hence ensuring global existence of a solution. We also assume that F_u and F_v are measurable in x and t along every one-dimensional curve in $[0,1] \times [0,\infty)$, and that ϵ_u and ϵ_v are measurable in x.

The functions F_u, F_v and f are allowed to be time-varying. Within the scope of this paper, we assume that at every time, F_u, F_v and f are exactly known d_u into the past and d_v into the future (see Equation (21) for definitions of d_u and d_v).

We let $\mathcal{X}[0,1]$ denote the state space of bounded functions on $[0,1]$, i.e.,

$$\mathcal{X}[0,1] = \{f : [0,1] \to \mathbb{R} : |f(x)| < \infty \, \forall x \in [0,1]\} \quad (10)$$

and we use $\|\cdot\|_\infty$ to denote the spatial supremum norm, i.e. $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$ for $f \in \mathcal{X}[0,1]$. The initial conditions u_0 and v_0 are assumed to be in $\mathcal{X}[0,1]$. If the input $U(t)$ is discontinuous in time, classical solutions of (1)-(6) cannot be expected. Therefore, throughout this paper we mean by solution of (1)-(6) functions satisfying the integral equations that are obtained by integrating (1)-(6) along its characteristic lines, as it is done in detail in Appendix A. The solutions u and v are absolutely continuous [1] along the characteristic lines of (1) and (2), respectively, but in general not continuous in different directions. Therefore, all differential equations in this paper are assumed to hold almost everywhere.

We consider two separate control problems:

(1) Drive the system to the origin $(u,v) \equiv (0,0)$ (Section 3.1) from an arbitrary state in minimum time under the additional assumptions

$$F_u((0,0),x,t) = 0 \quad \text{for all } x \in [0,1], t \geq 0, \quad (11)$$
$$F_v((0,0),x,t) = 0 \quad \text{for all } x \in [0,1], t \geq 0, \quad (12)$$
$$f(0,t) = 0 \quad \text{for all } t \geq 0, \quad (13)$$

and that f has a Lipschitz continuous inverse in the first argument, i.e. there exists a function f^{-1} and

[1] A function f is absolutely continuous if there exists a locally Lebesgue-integrable function g such that $f(x) = \int_0^x g(\xi)d\xi$.

constant $L_{f^{-1}} > 0$ such that

$$v^0 = f^{-1}(u^0, t) \quad \Leftrightarrow \quad u^0 = f(v^0, t), \tag{14}$$
$$|f^{-1}(u_1, t) - f^{-1}(u_2, t)| \le L_{f^{-1}} |u_1 - u_2|, \tag{15}$$

for all $t \ge 0$ and all $u_1, u_2 \in \mathbb{R}$. Moreover, the origin shall be an exponentially stable equilibrium of the closed-loop system.

(2) Tracking at $\bar{x} \in [0, 1]$ (Section 3.3) without assuming (11)-(13). The tracking objective is to achieve

$$v(\bar{x}, t) = g(u(\bar{x}, t), t) \tag{16}$$

in minimum time, where g is merely a function of t if $\bar{x} = 0$. Moreover, the states should remain bounded in $\mathcal{X}[0, 1]$. In this general form, F_u, F_v and f can include disturbance terms for which exact measurements (which need to be stored d_u into the past) and short term predictions (d_v into the future) are available.

Moreover, we consider the problem of estimating the infinite-dimensional state from the measurement $y(t) = u(1, t)$ (Section 4) and solve the output feedback problems corresponding to objectives 1 and 2 above (Section 5).

2.2 Characteristic lines

The method of characteristics is a popular tool for the analysis of hyperbolic systems. Since the propagation speeds ϵ_u and ϵ_v are state-independent, the characteristic lines are known a priori. They are illustrated in Figure 1. We parameterize the characteristic lines over

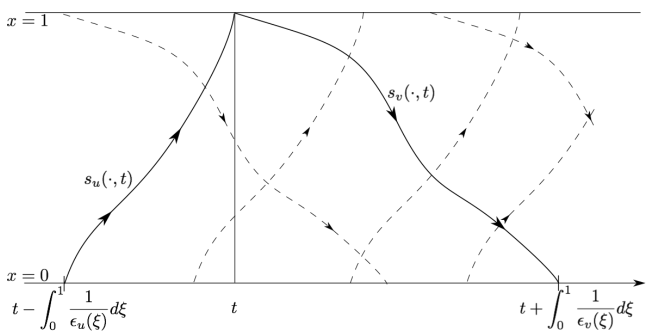

Fig. 1. Characteristic lines of u ("upwards") and v ("downwards").

space, x, and the time t at which they are at $x = 1$. The characteristic lines for v are given by

$$\tau_v(x, t) = t + \int_x^1 \frac{1}{\epsilon_v(\xi)} d\xi, \tag{17}$$
$$s_v(x, t) = (x, \tau_v(x, t)). \tag{18}$$

That is, the characteristic line originating from location $x = 1$ at time t is at location x at time $\tau_v(x, t)$. Analogously for u, we define

$$\tau_u(x, t) = t - \int_x^1 \frac{1}{\epsilon_u(\xi)} d\xi, \tag{19}$$
$$s_u(x, t) = (x, \tau_u(x, t)) \tag{20}$$

We also define the delay times

$$d_u = \int_0^1 \frac{1}{\epsilon_u(\xi)} d\xi, \qquad d_v = \int_0^1 \frac{1}{\epsilon_v(\xi)} d\xi. \tag{21}$$

2.3 Dynamics on the characteristic line s_v

The control input $U(t)$ propagates from $x = 1$ to $x = 0$ with finite speed ϵ_v along the characteristic line $s_v(\cdot, t)$, which is depicted as the wider line in Figure 1. Due to the finite propagation speed, the actuation cannot affect the state in the whole domain immediately. Loosely speaking, the control input $U(t)$ affects only the states "after" $s_v(\cdot, t)$ (including v but excluding u on s_v). That is, $U(t)$ affects the state $v(x, \cdot)$ at some location $x \in [0, 1]$ only at the future time $\tau_v(x, t)$. All states "before" $s_v(\cdot, t)$ (including u but excluding v on s_v) are entirely determined by the state at time t, i.e. they cannot be affected by $U(t)$. Therefore, we base our analysis on the dynamics on s_v.

Definition 1 *We define the state on s_v as*

$$\tilde{u}(x, t) = u(x, \tau_v(x, t)), \tag{22}$$

$$\tilde{v}(x, t) = v(x, \tau_v(x, t)). \tag{23}$$

Theorem 2 *For every t, there exists a continuous operator $\Phi^t : \mathcal{X}[0, 1] \times \mathcal{X}[0, 1) \to \mathcal{X}(0, 1]$, independent of $U(t)$, such that*

$$\tilde{u}(\cdot, t) = \Phi^t(u(\cdot, t), v(\cdot, t)). \tag{24}$$

If (11)-(13) hold, there exists a constant c_Φ such that

$$\|\tilde{u}(\cdot, t)\|_\infty \le c_\Phi \|(u(\cdot, t), v(\cdot, t))\|_\infty. \tag{25}$$

Moreover, $\tilde{u}$ and $\tilde{v}$ satisfy the PDE-ODE system

$$\tilde{u}_t(x, t) = -\frac{\epsilon_u(x)\epsilon_v(x)}{\epsilon_u(x) + \epsilon_v(x)} \tilde{u}_x(x, t) + \frac{\epsilon_v(x)}{\epsilon_u(x) + \epsilon_v(x)} F_u((\tilde{u}, \tilde{v})(x, t), x, \tau_v(x, t)), \tag{26}$$
$$\tilde{v}_x(x, t) = -\frac{1}{\epsilon_v(x)} F_v((\tilde{u}, \tilde{v})(x, t), x, \tau_v(x, t)), \tag{27}$$
$$\tilde{u}(0, t) = f(\tilde{v}(0, t), \tau_v(0, t)), \tag{28}$$
$$\tilde{v}(1, t) = U(t). \tag{29}$$
$$\tilde{u}(\cdot, 0) = \Phi^0(u_0, v_0), \tag{30}$$

Proof. For the existence of Φ^t and c_Φ, see Appendix A. For the second statement, we use $\frac{d}{dt}$ and $\frac{d}{dx}$ to denote the total derivative with respect to t and x, while $_t$ and $_x$ are partial derivatives w.r.t. time and space. We get

$$\begin{aligned}\tilde{u}_t(x,t) &= \frac{d}{dt}u(x,\tau_v(x,t)) \\ &= u_t(x,\tau_v(x,t))\frac{d\tau_v(x,t)}{dt} = u_t(x,\tau_v(x,t)), \qquad (31) \\ \tilde{u}_x(x,t) &= \frac{d}{dx}u(x,\tau_v(x,t)) \\ &= u_x(x,\tau_v(x,t)) + u_t(x,\tau_v(x,t))\frac{d\tau_v(x,t)}{dx} \\ &= u_x(x,\tau_v(x,t)) - \frac{1}{\epsilon_v(x)}u_t(x,\tau_v(x,t)) \\ &= -\frac{1}{\epsilon_v(x)}u_t(x,\tau_v(x,t)) - \frac{1}{\epsilon_u(x)}\left[u_t(x,\tau_v(x,t))\right. \\ &\quad \left.-F_u((u,v)(x,\tau_v(x,t)),x,\tau_v(x,t))\right] \\ &= -\frac{\epsilon_u+\epsilon_v}{\epsilon_u\epsilon_v}u_t(x,\tau_v) + \frac{1}{\epsilon_u}F_u((\tilde{u},\tilde{v})(x,t),x,\tau_v). \qquad (32)\end{aligned}$$

Substituting $u_t(x,\tau_v(x,t))$ in the latter equation by (31) yields (26). For $\tilde{v}$, we get

$$\begin{aligned}\tilde{v}_x(x,t) &= \frac{d}{dx}v(x,\tau_v(x,t)) \\ &= v_x(x,\tau_v(x,t)) - \frac{1}{\epsilon_v(x)}v_t(x,\tau_v(x,t)) \\ &= -\frac{1}{\epsilon_v(x)}v_t(x,\tau_v(x,t)) + \frac{1}{\epsilon_v(x)}\left[v_t(x,\tau_v(x,t))\right. \\ &\quad \left.-F_v((u,v)(x,\tau_v(x,t)),x,\tau_v(x,t))\right] \\ &= -\frac{1}{\epsilon_v(x)}F_v((\tilde{u},\tilde{v})(x,t),x,\tau_v(x,t)). \qquad (33)\end{aligned}$$

Note that (31)-(33) must be interpreted in the appropriate weak sense as mentioned in Section 2.1. The initial and boundary conditions follow from $\tilde{u}(\cdot,0) = \Phi^0(u(\cdot,0),v(\cdot,0))$, $\tilde{u}(0,t) = u(0,\tau_v(0,t))$, $\tilde{v}(0,t) = v(0,\tau_v(0,t))$, and $\tilde{v}(1,t) = v(1,t)$. ■

2.4 Dynamics with virtual input

The control actuation $U(t)$ is located at $x = 1$. However, we see that in (26), information propagates from the inflow boundary condition (28) at $x = 0$ to $x = 1$. The tracking objective (16) is also located at some $\bar{x} \in [0,1]$, where in general $\bar{x} \neq 1$. Therefore, the idea is to control the system via a 'virtual control input' $U^*(t)$ located at $\bar{x} \in [0,1]$. Exploiting the fact that (27) is a simple ODE in space, we then construct $U(t)$ such that $\tilde{v}(\bar{x},t)$ becomes $U^*(t)$. That is, the boundary condition (29) can be replaced by $\tilde{v}(\bar{x},t) = U^*(t)$, as stated by the following theorem.

Theorem 3 *For $\bar{x} \in [0,1]$ and $t \geq 0$, $\phi \in \mathcal{X}[0,1]$ and $\varphi_0 \in \mathbb{R}$, consider the ODE*

$$\varphi_x(x) = -\frac{1}{\epsilon_v(x)}F_v((\phi,\varphi)(x),x,\tau_v(x,t)) \qquad (34)$$

on the interval $x \in [0,1]$ with initial condition $\varphi(\bar{x}) = \varphi_0$, and denote $\varphi_1 = \varphi(1)$. Consider the continuous mapping

$$\Psi^t_{\bar{x}}\colon \mathcal{X}[0,1]\times\mathbb{R}\to\mathbb{R}\colon (\phi,\varphi_0)\mapsto\varphi_1. \qquad (35)$$

The system consisting of (26)-(30) in closed loop with $U(t) = \Psi^t_{\bar{x}}(\tilde{u}(\cdot,t),U^*(t))$ satisfies

$$\begin{aligned}\bar{u}_t(x,t) &= -\frac{\epsilon_u(x)\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)}\bar{u}_x(x,t) \\ &\quad+\frac{\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)}F_u((\bar{u},\bar{v})(x,t),x,\tau_v(x,t)), \qquad (36) \\ \bar{v}_x(x,t) &= -\frac{1}{\epsilon_v(x)}F_v((\bar{u},\bar{v})(x,t),x,\tau_v(x,t)), \qquad (37) \\ \bar{u}(0,t) &= f(\bar{v}(0,t),\tau_v(0,t)), \qquad (38) \\ \bar{v}(\bar{x},t) &= U^*(t). \qquad (39) \\ \bar{u}(x,0) &= \Phi^0(u_0,v_0), \qquad (40)\end{aligned}$$

Proof. By the Carathéodory theorem, and due to the Lipschitz condition on F_v, the ODE (34) has a unique solution for every given initial condition, even for discontinuous right-hand sides [11]. Therefore, if two solutions φ and $\bar{\varphi}$ both satisfy (34), then

$$\bar{\varphi}(\bar{x}) = \varphi(\bar{x}) \Leftrightarrow \bar{\varphi}(1) = \varphi(1). \qquad (41)$$

Since (34) is a copy of (27) for $\phi = \tilde{u}(\cdot,t)$, this is equivalent to

$$U(t) = \Psi^t_{\bar{x}}(\tilde{u}(\cdot,t),U^*(t)) \Leftrightarrow \tilde{v}(\bar{x},t) = U^*(t). \qquad (42)$$

The rest is mere change of notation. Continuity of $\Psi^t_{\bar{x}}$ holds since the solution of (34) is Lipschitz continuous in φ_0 and ϕ. ■

Lemma 4 *The closed loop solution of (1) -(6) with $U(t) = \Psi^t_{\bar{x}}\left(\Phi^t(u(\cdot,t),v(\cdot,t),U^*(t))\right)$ satisfies*

$$u(x,t) = \bar{u}\left(x, t - \int_x^1 \frac{1}{\epsilon_v(\xi)}d\xi\right), \qquad (43)$$

$$v(x,t) = \bar{v}\left(x, t - \int_x^1 \frac{1}{\epsilon_v(\xi)}d\xi\right) \qquad (44)$$

for all $x \in [0,1]$ and $t \geq \int_x^1 \frac{1}{\epsilon_v(\xi)}d\xi$, where $\bar{u}$ and $\bar{v}$ are governed by (36)-(40).

Proof. This follows directly from Definition 1, the definition of τ_v (see (17)) and Theorems 2 and 3. ■

From now on, we perform the analysis on the closed loop system (36)-(40) and use (43)-(44) to make conclusions on the original system.

Remark 5 *Our proof of existence of Φ^t is constructive. Hence, Φ^t can be implemented by following Appendix A. That is, at time t, the PDE system (1)-(3) is solved in the*

domain $\left\{(x,\theta) : x \in [0, 1-\delta), \theta \in \left[t, t+\int_x^{1-\delta} \frac{1}{\epsilon_v(\xi)} d\xi\right]\right\}$ *for infinitesimally small* $\delta > 0$. *This can be done either via successive approximations as in Section A.2 or, more efficiently, by discretizing the PDEs in space and integrating in time. Then,* $\bar{u}(\cdot, t)$ *is obtained by taking the limit as in Section A.3.*
The operator Ψ^t *is implemented by solving the Cauchy problem (34) with* $\varphi(\bar{x}) = U^*(t)$.

Remark 6 *As a practical alternative to continuously evaluating* Φ^t, *one could evaluate* Φ^t *only after a certain time passed or the state changed more than some margin, and update the predicted state* $\bar{u}$ *via (36)-(39) in between these evaluations of* Φ^t. *However, this would introduce an additional error, and analyzing this strategy is beyond the scope of this paper.*

2.5 Dynamics on the characteristic line s_u

For estimation, we assume that the measurement $y(t) = u(1, t)$ is available. The measurement at time t evolved along the characteristic line $s_u(\cdot, t)$, which was defined in Equation (20) and is depicted in Figure 1. Due to the finite propagation speed, information from within the domain cannot be sensed at the boundary $x = 1$ immediately. Loosely speaking, the state at some location x and time $\theta > \tau_u(x, t)$ has no influence on $y(t)$. Only the past state on $s_u(\cdot, t)$ can be sensed via $y(t)$. Therefore, we base the observer design on the dynamics on s_u.

Definition 7 *We define the state on* s_u *as*

$$\check{u}(x,t) = u(x, \tau_u(x,t)), \tag{45}$$

$$\check{v}(x,t) = v(x, \tau_u(x,t)) \tag{46}$$

for $x \in [0, 1]$ *and* $t \geq \int_x^1 \frac{1}{\epsilon_u(\xi)} d\xi$.

Theorem 8 $\check{u}$ *and* $\check{v}$ *satisfy the PDE-ODE system*

$$\check{u}_x(x,t) = \frac{1}{\epsilon_u(x)} F_u((\check{u},\check{v})(x,t), x, \tau_u(x,t)), \tag{47}$$

$$\check{v}_t(x,t) = \frac{\epsilon_u(x)\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)} \check{v}_x(x,t) + \frac{\epsilon_u(x)}{\epsilon_u(x)+\epsilon_v(x)} F_v((\check{u},\check{v})(x,t), x, \tau_u(x,t)), \tag{48}$$

$$\check{u}(0,t) = f(\check{v}(0,t), t), \tag{49}$$

$$\check{v}(1,t) = U(t) \tag{50}$$

$$\check{v}(x, -\tau_u(x,0)) = v_0(x), \tag{51}$$

for $x \in [0, 1]$ *and* $t \geq \int_x^1 \frac{1}{\epsilon_u(\xi)} d\xi$. *Moreover, for every* t, *there exists a continuous operator* $\Lambda^t : \mathcal{X}[0,1] \times \mathcal{X}[0,1) \to \mathcal{X}[0,1] \times \mathcal{X}[0,1)$, *independent of* $U(t)$, *such that*

$$(u(\cdot,t), v(\cdot,t)) = \Lambda^t(\check{u}(\cdot,t), \check{v}(\cdot,t)). \tag{52}$$

Proof.

$$\begin{aligned}
\check{u}_x(x,t) &= \frac{d}{dx} u(x, \tau_u(x,t)) \\
&= u_x(x, \tau_u(x,t)) + u_t(x, \tau_u(x,t)) \frac{d\tau_u(x,t)}{dx} \\
&= u_x(x, \tau_u(x,t)) + \frac{1}{\epsilon_u(x)} u_t(x, \tau_u(x,t)) \\
&= \frac{1}{\epsilon_u(x)} u_t(x, \tau_u(x,t)) - \frac{1}{\epsilon_u(x)} [u_t(x, \tau_u(x,t)) \\
&\quad - F_u((u,v)(x, \tau_u(x,t)), x, \tau_u(x,t))] \\
&= \frac{1}{\epsilon_u(x)} F_u((\check{u},\check{v})(x,t), x, \tau_u(x,t)).
\end{aligned} \tag{53}$$

For $\check{v}$, we get

$$\begin{aligned}
\check{v}_t(x,t) &= \frac{d}{dt} v(x, \tau_u(x,t)) \\
&= v_t(x, \tau_u(x,t)) \frac{d\tau_u(x,t)}{dt} = v_t(x, \tau_u(x,t)),
\end{aligned} \tag{54}$$

$$\begin{aligned}
\check{v}_x(x,t) &= \frac{d}{dx} v(x, \tau_u(x,t)) \\
&= v_x(x, \tau_u(x,t)) + v_t(x, \tau_u(x,t)) \frac{d\tau_u(x,t)}{dx} \\
&= v_x(x, \tau_u(x,t)) + \frac{1}{\epsilon_u(x)} v_t(x, \tau_u(x,t)) \\
&= \frac{1}{\epsilon_u(x)} v_t(x, \tau_u(x,t)) + \frac{1}{\epsilon_v(x)} [v_t(x, \tau_u(x,t)) \\
&\quad - F_v((u,v)(x, \tau_u(x,t)), x, \tau_u(x,t))] \\
&= \frac{\epsilon_u + \epsilon_v}{\epsilon_u \epsilon_v} v_t(x, \tau_u) - \frac{1}{\epsilon_v} F_v((\check{u},\check{v})(x,t), x, \tau_u).
\end{aligned} \tag{55}$$

Substituting $v_t(x, \tau_u(x,t))$ in the latter equation by (54) yields (48). The initial and boundary conditions follow directly from the definitions of $\check{u}$, $\check{v}$ and τ_u.
Regarding existence of Λ^t, the same methods as in Appendix A (i.e. transforming the differential equation in integral equations and applying a successive approximation argument) can be used to show that the PDE system (1)-(3) with "initial condition" $u(x, \tau_u(x,t)) = \check{u}(x,t)$ and $v(x, \tau_u(x,t)) = \check{v}(x,t)$ has a unique solution (u, v) in the domain $\mathcal{S} = \{(x,\theta) : x \in [0,1], \theta \in [\tau_u(x,t), t]\}$. This set includes $(u(\cdot,t), v(\cdot,t))$, which is the output of Λ^t. ∎

Remark 9 *The operator* Λ^t *can be implemented by solving the PDE system (1)-(3) in the domain* $\{(x,\theta) : x \in [0,1), \theta \in [\tau_u(x,t), t]\}$.

3 State feedback control

3.1 Stabilization

In this section, we design a control law such that the states converge to the origin in minimum time, and that the origin becomes an exponentially stable equilibrium of the closed-loop system, under the additional assumptions (11)-(13). Loosely speaking, the idea is to 'set' the inflow boundary condition at $x = 0$ to zero, and this zero 'propagates' towards $x = 1$.

Theorem 10 *If* $U(t) = \Psi_0^t(\tilde{u}(\cdot,t), 0)$, *i.e.* $\bar{x} = 0$ *and*

$U^*(t) = 0$, *the solution of (36)-(40) satisfies*

$$\bar{u}(x,t) = \bar{v}(x,t) = 0 \text{ for all } (x,t) \in \mathcal{A} \tag{56}$$

where $\mathcal{A} = \{(x,t) : x \in [0,1], t \geq \int_0^x \frac{1}{\epsilon_v(\xi)} + \frac{1}{\epsilon_u(\xi)} d\xi\}$.

Proof. We transform the PDEs into integral equations, as it is done in more detail for a similar system in Appendix A.1. Defining

$$\bar{\phi}(x) = \int_0^x \frac{\epsilon_u(\xi) + \epsilon_v(\xi)}{\epsilon_u(\xi)\epsilon_v(\xi)} d\xi, \tag{57}$$

$$\bar{\xi}(x,t,s) = \bar{\phi}^{-1}(\bar{\phi}(x) - t + s), \tag{58}$$

$$\bar{s}^0(x,t) = t - \bar{\phi}(x), \tag{59}$$

we integrate (36)-(40) along its characteristic lines to obtain, for $(x,t) \in \mathcal{A}$,

$$\bar{u}(x,t) = \bar{u}(0, \bar{s}^0(x,t)) + \int_{\bar{s}^0}^t \frac{\epsilon_v}{\epsilon_u + \epsilon_v} F_u\left((\bar{u},\bar{v})\left(\bar{\xi}(x,t,s),s\right), \bar{\xi}(x,t,s), \tau_v\left(\bar{\xi}(x,t,s),s\right)\right) ds, \tag{60}$$

$$\bar{v}(x,t) = \bar{v}(0,t) - \int_0^x \frac{1}{\epsilon_v(\xi)} F_v((\bar{u},\bar{v})(\xi,t), \xi, \tau_v(\xi,t)) d\xi. \tag{61}$$

Since $\bar{x} = 0$ and $U^*(t) = 0$, (39) gives that $\bar{v}(0,t) = 0$ and, due to (13), $\bar{u}(0,t) = 0$ for all $t \geq 0$. For all $(x,t) \in \mathcal{A}$, we have $\bar{s}^0(x,t) \geq 0$, hence $\bar{u}(0,\bar{s}^0(x,t)) = 0$. For $(x,t) \in \mathcal{A}$, we also have that $(\bar{\xi}(x,t,s),s) \in \mathcal{A}$ for all $s \in [\bar{s}^0(x,t), t]$, and $(\xi,t) \in \mathcal{A}$ for all $\xi \in [0,x]$. Inserting (56) into (60)-(61), we see that, using (11)-(12), the right-hand sides become zero. That is, (56) solves (60)-(61). Using the same methods as in Appendix A.2, it is also possible to show that the solution of (60)-(61) is unique. Therefore, we can reverse the statement to say that the solution of (60)-(61) satisfies (56), and the same holds for the solution of the original PDE system (36)-(40). ■

Combining Theorem 10 and Lemma 4, we conclude the following:

Theorem 11 *The system consisting of (1)-(6) in closed loop with* $U(t) = \Psi_0^t(\Phi^t(u(\cdot,t), v(\cdot,t)), 0)$ *satisfies*

$$u(x,t) = v(x,t) = 0 \tag{62}$$

for all $x \in [0,1]$ *and* $t \geq d_u + d_v$ *(see (21) for definitions of* d_u *and* d_v*).*

3.2 Lyapunov stability

We construct a Lyapunov function to prove exponential stability in the spatial supremum norm. First, we establish an inequality to bound $\bar{v}$ by $\bar{u}$.

Lemma 12 *For* $k > 0$ *and* $L = \frac{1}{\epsilon_1} L_v$, *where* ϵ_1 *is the lower bound on the transport speeds and* L_v *is the Lipschitz constant in (8), we have*

$$\|e^{-kx}\bar{v}(x,t)\|_\infty \leq \left(e^L - 1\right) \|e^{-kx}\bar{u}(x,t)\|_\infty. \tag{63}$$

Proof. Exploiting (11)-(12) and (8), we have

$$\begin{aligned} \frac{d}{dx}|e^{-kx}\bar{v}(x,t)| &\leq -k|e^{-kx}\bar{v}(x,t)| + |e^{-kx}\bar{v}_x(x,t)| \\ &\leq e^{-kx}|F_v((\bar{u},\bar{v})(x,t), x, \tau_v(x,t))| \\ &\leq Le^{-kx}\left(|\bar{v}(x,t)| + |\bar{u}(x,t)|\right) \end{aligned} \tag{64}$$

where for every t, the derivative $\bar{v}_x$ as given in (37) is well defined for almost all x. Applying a Gronwall-type inequality [2], we obtain the bound

$$|e^{-kx}\bar{v}(x,t)| \leq \int_0^x e^{L(x-\xi)} L |e^{-k\xi}\bar{u}(\xi,t)| d\xi. \tag{65}$$

Consequently,

$$\begin{aligned} \|e^{-kx}\bar{v}(x,t)\|_\infty &\leq \int_0^1 e^{L(1-\xi)} L \|e^{-kx}\bar{u}(x,t)\|_\infty d\xi \\ &= \left(e^L - 1\right)\|e^{-kx}\bar{u}(x,t)\|_\infty. \end{aligned} \tag{66}$$ ■

Next, we note that (36) is an advection equation with zero 'inflow' at $x = 0$. However, the coupling term F_u means that $\bar{u}$ can grow while propagating from $x = 0$ to $x = 1$. We compensate this effect by weighting the spatial supremum with a term decreasing in x. More precisely, we seek a Lyapunov function of the form

$$V(\bar{u}(\cdot,t)) = \sup_{x \in [0,1]} \left|e^{-kx}\bar{u}(x,t)\right|, \tag{67}$$

where the weighting coefficient k is to be designed. Therefore, we analyze the dynamics of $w(x,t) = e^{-kx}\bar{u}(x,t)$.

In order to bound the derivative of V, we first establish a bound on the derivative of w along its characteristic lines.

Lemma 13 *For* $\epsilon = \frac{\epsilon_u\epsilon_v}{\epsilon_u+\epsilon_v}$, $L^* = \sup_{x\in[0,1]} \frac{\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)} L_u$, $c = L^* e^L$ *and fixed* $x \in [0,1]$, $w(x,t) = e^{-kx}\bar{u}(x)$ *satisfies the differential inequality*

$$\frac{d}{ds}\, w\left(\bar{\xi}(x,t,s),s\right)\big|_{s=t} \leq -\epsilon(x)kw(x,t) + c\|w(\cdot,t)\|_\infty, \tag{68}$$

where $\bar{\xi}$ *was defined in (58), for almost all* $t \geq 0$.

Proof. At $s = t$ we have

$$\begin{aligned} \frac{d}{ds}w\left(\bar{\xi}(x,t,s),s\right) &= w_t(x,t) + w_x(x,t)\frac{1}{\bar{\phi}'(x)} \\ &= w_t(x,t) + \epsilon(x)w_x(x,t). \end{aligned} \tag{69}$$

[2] i.e. bounding $|e^{-kx}\bar{v}(x,t)|$ by the solution of the differential *equation* $\frac{d}{dx}|e^{-kx}\bar{v}(x,t)| = Le^{-kx}\left(|\bar{v}(x,t)| + |\bar{u}(x,t)|\right)$ with $|\bar{v}(0,t)| = 0$, which can be solved analytically. Sets of measure zero where $\frac{d}{dx}|e^{-kx}\bar{v}(x,t)|$ is not well defined do not affect this bound.

Differentiating $w(x,t)$ with respect to t and x, and abbreviating $F = \frac{\epsilon_v}{\epsilon_u+\epsilon_v} F_u$, gives

$$
\begin{aligned}
w_t(x,t) &= e^{-kx}\bar{u}_t(x,t), && (70)\\
\epsilon w_x(x,t) &= -\epsilon k e^{-kx}\bar{u}(x,t) + e^{-kx}\epsilon\bar{u}_x(x,t)\\
&= -\epsilon k e^{-kx}\bar{u} + e^{-kx}(-\bar{u}_t + F((\bar{u},\bar{v}),x,\tau_v)). && (71)
\end{aligned}
$$

Inserting (70) and (71) into (69) yields at $s=t$

$$
\begin{aligned}
\frac{d}{ds} w\left(\bar{\xi}(x,t,s),s\right) &= -\epsilon(x)kw(x,t)\\
&\quad + e^{-kx}F((\bar{u},\bar{v})(x,t),x,\tau_v(x,t)).
\end{aligned} \tag{72}
$$

The measurability assumptions on F_u implies that for fixed x, the right-hand side of (72) is well-defined for almost all t. Next, we bound the F-term. Exploiting the Lipschitz condition on F, (63) and the definition of w, we obtain

$$
\begin{aligned}
&|e^{-kx}F((\bar{u},\bar{v})(x,t),x,\tau_v(x,t))|\\
&\quad\le e^{-kx}L^*(|\bar{u}(x,t)| + |\bar{v}(x,t)|)\\
&\quad= L^*(|e^{-kx}\bar{u}(x,t)| + |e^{-kx}\bar{v}(x,t)|)\\
&\quad\le L^*\left(\|e^{-kx}\bar{u}(\cdot,t)\|_\infty + \|e^{-kx}\bar{v}(\cdot,t)\|_\infty\right)\\
&\quad\le L^*\left(1+\left(e^L-1\right)\right)\|e^{-kx}\bar{u}(\cdot,t)\|_\infty\\
&\quad\le L^* e^L \|\bar{w}(\cdot,t)\|_\infty.
\end{aligned} \tag{73}
$$

■

For $\varepsilon > 0$ and $w(\cdot,t) \in \mathcal{X}[0,1]$ given, we define the set

$$
\mathcal{D}_\varepsilon = \{x \in [0,1] : |w(x,t)| > (1-\varepsilon)\|w(\cdot,t)\|_\infty\}. \tag{74}
$$

Lemma 14 *If $\mathcal{D}_\varepsilon \neq \{1\}$ for all $\varepsilon > 0$, then $\dot{V} = \frac{d}{dt}V(\bar{u}(\cdot,t))$ exists for almost all $t \ge 0$ and satisfies $\dot{V} \le -\bar{c}(k)V$ for $\bar{c}(k) = 0.5\,\epsilon_1 k - c$ and $k > \frac{2c}{\epsilon_1}$.*

Proof. We fix $\varepsilon = 0.5$. For $x^* \in \mathcal{D}_\varepsilon$, we assume without loss of generality that $w(x^*,t) > 0$, otherwise we multiply by (-1). Hence, by Lemma 13,

$$
\begin{aligned}
\frac{d}{ds}\, w\left(\bar{\xi}(x^*,t,s),s\right)\big|_{s=t} &\le -\epsilon k w(x^*,t) + c\|w(\cdot,t)\|_\infty\\
&\le (-\epsilon k(1-\varepsilon)+c)\|w(\cdot,t)\|_\infty = -\bar{c}(k)\|w(\cdot,t)\|_\infty
\end{aligned} \tag{75}
$$

for almost all $t \ge 0$. Next, we bound the derivative of V by analyzing how the supremum of $|w|$ changes along its characteristic lines. For given t and s, we define the set

$$
\Xi^{s,t} = \{x \in [0,1] : \bar{\xi}(x,t,s) \in [0,1]\}. \tag{76}
$$

Note that $\Xi^{s,t}$ is non empty for $s \in [t, t+\bar{\phi}(1)]$. Moreover, for $s \in [t, t+\bar{\phi}(1)]$,

$$
[0,1] = \left\{\bar{\xi}(x,t,s) : x \in \Xi^{s,t}\right\} \cup \left\{\bar{\xi}(0,\theta,s) : \theta \in [t,s]\right\}, \tag{77}
$$

i.e. every point in $[0,1]$ lies on a characteristic line of w going through either (x,t) for some $x \in [0,1]$, or $(0,\theta)$ for some $\theta \in [t,s]$. For all (x,t), $w\left(\bar{\xi}(x,t,s),s\right)$ is continuous in s. Since $w(0,\theta) = 0$ for all θ, there exists a $\delta > 0$ such that

$$
|w(\bar{\xi}(0,\theta,s),s)| \le 0.5 \sup_{x\in[0,1]} |w(x,s)| \tag{78}
$$

for all $s \in [t, t+\delta]$ and all $\theta \in [t,s]$. Due to (77), this implies

$$
\sup_{x\in[0,1]} |w(x,s)| = \sup_{x\in\Xi^{s,t}} |w(\bar{\xi}(x,t,s),s)|. \tag{79}
$$

At time t, we have

$$
\sup_{x\in\mathcal{D}_\varepsilon} |w(\bar{\xi}(x,t,t),t)| = \sup_{x\in\mathcal{D}_\varepsilon} |w(x,t)| = \|w(\cdot,s)\|_\infty. \tag{80}
$$

Again due to continuity of $w\left(\bar{\xi}(x,t,s),s\right)$ in s, we can decrease δ further if necessary such that

$$
\sup_{x\in\mathcal{D}_\varepsilon\cap\Xi^{s,t}} |w(\bar{\xi}(x,t,s),s)| = \|w(\cdot,s)\|_\infty \tag{81}
$$

for all $s \in [t,t+\delta]$. The set $\mathcal{D}_\varepsilon \cap \Xi^{s,t}$ is non empty for small enough δ due to the assumption $\mathcal{D}_\varepsilon \neq \{1\}$. Thus, we can bound the derivative of V by

$$
\dot{V} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\sup_{x\in[0,1]}|w(x,t+\Delta t)| - \sup_{x\in[0,1]}|w(x,t)|\right) \tag{82}
$$

$$
\begin{aligned}
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Bigg(\sup_{x\in\mathcal{D}_\varepsilon\cap\Xi^{t+\Delta t,t}} |w(\bar{\xi}(x,t,t+\Delta t),t+\Delta t)|\\
&\qquad\qquad - \sup_{x\in\mathcal{D}_\varepsilon}|w(x,t)|\Bigg)\\
&\le \lim_{\Delta t\to 0}\frac{1}{\Delta t}\sup_{x\in\mathcal{D}_\varepsilon\cap\Xi^{t+\Delta t,t}}\big(|w(\bar{\xi}(x,t,t+\Delta t),t+\Delta t)|\\
&\qquad\qquad - |w(x,t)|\big)\\
&= \sup_{x\in\mathcal{D}_\varepsilon\cap\Xi^{t+\Delta t,t}}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\big(|w(\bar{\xi}(x,t,t+\Delta t),t+\Delta t)|\\
&\qquad\qquad -|w(x,t)|\big) \le -\bar{c}(k)V,
\end{aligned} \tag{83}
$$

where the limit is taken over $\Delta t \in [0,\delta]$, and we applied (75) in the very last step. ■

Lemma 15 *If $\mathcal{D}_\varepsilon = \{1\}$ for some $\varepsilon > 0$, then $V(\bar{u}(\cdot,t))$ is discontinuous in t and decreases by a jump.*

Proof. If $\mathcal{D}_\varepsilon = \{1\}$ for some $\varepsilon > 0$, $|w|$ has a unique maximum at $x = 1$ and there exists a $\bar{\varepsilon}$ such that $|w(x,t)| < |w(1,t)| - \bar{\varepsilon}$ for all $x \neq 1$. We have $\bar{\xi}(1,t,s) \notin [0,1]$ for all $s > t$. Note that for every x_1, t and s, there exists a x_2 such that $x_1 = \bar{\xi}(x_2,t,s)$. Hence, by continuity in s, $|w(x_1,s)| = |w(\bar{\xi}(x_2,t,s),s)| < |w(1,t)| - \bar{\varepsilon}$ for all $x \neq 1$ and $s \in [t,t+\bar{\delta}]$ for some $\bar{\delta} > 0$. ■

We are now in position to formulate the main theorem on exponential stability using the composition $V \circ \Phi^t$ as a Lyapunov function.

Theorem 16 *For every $\gamma > 0$ there exists a $M > 0$ such that for all $t \geq 0$ and all initial conditions in $\mathcal{X}[0,1]$*

$$\|(u(\cdot,t), v(\cdot,t))\|_\infty \leq M e^{-\gamma t} \|(u(\cdot,0), v(\cdot,0))\|_\infty. \quad (84)$$

Proof. We first establish exponential decay for V. Let $\{t_1, t_2, \ldots\}$ denote the times at which there exists an $\varepsilon > 0$ such that $\mathcal{D}_\varepsilon = \{1\}$ and $t_0 = 0$, and abbreviate $V(t) = V(\bar{u}(\cdot,t))$. We establish the bound

$$V(t) \leq V(0) e^{-\bar{c}(k)t} \quad (85)$$

for all $t = t_i$ and all $t \in (t_i, t_{i+1})$, $i = 0, 1, \ldots$, by induction. For $i = 0$, clearly $V(t_0) = V(0)$. Assume (85) holds for $t = t_i$. For $t \in (t_i, t_{i+1})$,

$$\begin{aligned} V(t) &= V(t_i) + \int_{t_i}^{t} \dot{V}(s) ds \\ &\leq V(t_i) + \int_{t_i}^{t} -\bar{c}(k) V(s) ds = V(t_i) e^{-\bar{c}(k)(t-t_i)} \\ &\leq V(0) e^{-\bar{c}(k)t_i} e^{-\bar{c}(k)(t-t_i)} = V(0) e^{-\bar{c}(k)t}, \end{aligned} \quad (86)$$

where we applied Lemma 14 and used the fact that sets of measure zero do not affect the integral. Then, at $t = t_{i+1}$ we can apply Lemma 15 to conclude that

$$\begin{aligned} V(t_{i+1}) &\leq \lim_{t \nearrow t_{i+1}} V(t) \leq \lim_{t \nearrow t_{i+1}} V(0) e^{-\bar{c}(k)t} \\ &= V(0) e^{-\bar{c}(k)t_{i+1}}. \end{aligned} \quad (87)$$

By the definition of V (see (67)),

$$\begin{aligned} \|\bar{u}(\cdot,t)\|_\infty &\leq e^k V(t) \leq e^k V(0) e^{-\bar{c}(k)t} \\ &\leq e^k \|\bar{u}(\cdot,0)\|_\infty e^{-\bar{c}(k)t} \end{aligned} \quad (88)$$

for all t. Finally, (25) and (B.13) imply

$$\begin{aligned} \|(u(\cdot,t), v(\cdot,t))\|_\infty &\leq c_{\Phi^{-1}} \|\bar{u}(\cdot,t)\|_\infty \\ &\leq c_{\Phi^{-1}} e^k e^{-\bar{c}(k)t} \|\bar{u}(\cdot,0)\|_\infty \\ &\leq c_\Phi c_{\Phi^{-1}} e^k e^{-\bar{c}(k)t} \|(u(\cdot,0), v(\cdot,0))\|_\infty. \quad \blacksquare \end{aligned} \quad (89)$$

3.3 Tracking

The tracking problem at $\bar{x} \in [0,1]$ is solved by simply setting $U^*(t)$ as desired by the tracking objective (16) using the prediction $\bar{u}(\bar{x},t)$ obtained by evaluating Φ^t, i.e.,

$$\bar{u}(\cdot,t) = \Phi^t(u(\cdot,t), v(\cdot,t)), \quad (90)$$
$$U^*(t) = g(\bar{u}(\bar{x},t), \tau_v(\bar{x},t)). \quad (91)$$

For tracking, we do not require the additional assumptions (11)-(13). In this generality, F_u, F_v and f can include the effect of disturbances for which, at time t, exact predictions are available in the interval $[t, \tau_v(\bar{x},t)]$. The effect of such disturbances is considered, implicitly, when evaluating Φ^t and Ψ^t.

Theorem 17 *The system (1)-(6) in closed loop with $U(t) = \Psi^t_{\bar{x}}(\Phi^t(u(\cdot,t), v(\cdot,t)), U^*(t))$, with $U^*(t)$ as in (90)-(91), satisfies the tracking objective (16) for all $t \geq \tau_v(\bar{x},0)$. Moreover, the trajectories of the closed-loop system remain bounded if there exist positive constants c_u and c_v such that, for all $x \in [0,1]$ and $t \geq 0$,*

$$|F_u((0,0),x,t)| \leq c_u, \quad |F_v((0,0),x,t)| \leq c_v, \quad (92)$$

and the following additional conditions are satisfied. In case of tracking at $\bar{x} = 0$, there exist positive c_u^0 and c_v^0 such that for all $t \geq 0$

$$U^*(t) \leq c_v^0, \qquad f(U^*(t),t) \leq c_u^0. \quad (93)$$

In case of tracking at $\bar{x} \neq 0$, there exist positive L_g and c_g such that for all $\bar{u} \in \mathbb{R}$ and $t \geq 0$

$$|g(\bar{u},t)| \leq L_g |\bar{u}| + c_g, \quad (94)$$

and $\hat{L} = e^{L\bar{x}}(L_g + 1) - 1$ satisfies $L_f \hat{L} < 1$, where $L = \frac{1}{\epsilon_1} L_v$.

Proof. The design of $U^*(t)$ ensures that the closed loop system (36)-(40) satisfies the control objective for all $t \geq 0$. Thus, the first claim follows from Lemma 4.
To prove boundedness, we proceed as in Section 3.2. First, we consider the case $\bar{x} = 0$. For $k \geq 0$, we have

$$\frac{d}{dx}|e^{-kx}\bar{v}(x,t)| \leq \frac{e^{-kx}}{\epsilon_1}\left(L_v(|\bar{v}(x,t)| + |\bar{u}(x,t)|) + c_v\right) \quad (95)$$

for fixed t and almost all x, and $|\bar{v}(0,t)| \leq c_v^0$. Like in (65), a variant of the Gronwall inequality gives

$$\begin{aligned} |e^{-kx}\bar{v}(x,t)| \leq e^{Lx} c_v^0 + \int_0^x & e^{L(x-\xi)} \epsilon_1^{-1} \\ & \left(L_v |e^{-k\xi}\bar{u}(\xi,t)| + c_v\right) d\xi, \end{aligned} \quad (96)$$

where $L = \epsilon_1^{-1} L_v$. Consequently,

$$\begin{aligned} \|e^{-kx}\bar{v}(x,t)\|_\infty \leq{} & e^L c_v^0 \\ & + \frac{e^L - 1}{\epsilon_1 L}\left(L_v \|e^{-kx}\bar{u}(x,t)\|_\infty + c_v\right). \end{aligned} \quad (97)$$

As in the proof of Lemma 13, Equation (73), we can bound the term $e^{-kx}F$ by

$$\begin{aligned} &|e^{-kx}F((\bar{u},\bar{v})(x,t),x,\tau_v(x,t))| \\ &\quad\le \epsilon^* \left(L_u \left(\|e^{-kx}\bar{u}(\cdot,t)\|_\infty + \|e^{-kx}\bar{v}(\cdot,t)\|_\infty\right) + c_u\right) \\ &\quad\le \epsilon^* L_u e^L \|e^{-kx}\bar{u}(\cdot,t)\|_\infty + d \end{aligned} \tag{98}$$

where we abbreviated $\epsilon^* = \sup_{x\in[0,1]} \frac{\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)}$ and

$$d = \epsilon^* \left(L_u \left(e^L c_v^0 + \frac{e^L - 1}{\epsilon_1 L} c_v \right) + c_u \right). \tag{99}$$

Thus, $w(x,t) = e^{-kx}\bar{u}(x,t)$ satisfies at $s = t$

$$\frac{d}{ds} w\left(\bar{\xi}(x^*,t,s),s\right) \le -\epsilon(x)kw(x,t) + c\|w(\cdot,t)\|_\infty + d \tag{100}$$

for every fixed x^* and almost all t, where ϵ and c are as in Lemma 13. Moreover, $|w(0,t)| \le c_u^0$. We can then choose $x^* \in \mathcal{D}_\varepsilon$ for $\varepsilon < 0.5$, and assume without loss of generality that $w(x^*,t) > 0$, so that w satisfies

$$\frac{d}{ds}\, w\left(\bar{\xi}(x^*,t,s),s\right)\big|_{s=t} \le -\bar{c}(k)\|w(\cdot,t)\|_\infty + d, \tag{101}$$

where $\bar{c}(k)$ becomes positive for k large enough. In order to ensure that (81) holds, i.e. the maximum is not attained at $x = 0$, we further assume that $\|w(\cdot,t)\|_\infty > 2c_u^0$. Thus, we conclude as in Section 3.2 that V decreases exponentially if $\|w(\cdot,t)\|_\infty > 2\min\left\{d/c(k), c_u^0\right\}$.
Second, if $\bar{x} \ne 0$, the assumptions on the data imply

$$\begin{aligned} &|e^{-k\bar{x}}U^*(t)| \le e^{-k\bar{x}}\left(L_g|\bar{u}(\bar{x},t)| + c_g\right) \\ &\quad\le L_g|e^{-k\bar{x}}\bar{u}(\bar{x},t)| + c_g \le L_g\|w(\cdot,t)\|_\infty + c_g. \end{aligned} \tag{102}$$

Similarly to above, the Gronwall inequality can be applied to derive an inequality of the same structure as (98) but with different constants, and conclude a differential inequality like (101). For $\bar{x} \ne 0$, there is no a priori bound on $|\bar{v}(0,t)|$ like (93), i.e. we need to additionally design a condition to ensure that $\|w(\cdot,t)\|_\infty$ does not increase via the boundary condition at $x = 0$. By the Gronwall inequality,

$$\begin{aligned} &|e^{-kx}\bar{v}(x,t)| \le e^{-L(x-\bar{x})}|e^{-k\bar{x}}U^*(t)| \\ &\quad + \int_{\bar{x}}^{x} e^{-L(x-\xi)}\epsilon_1^{-1}\left(L_v|e^{-k\xi}\bar{u}(\xi,t)| + c_v\right)d\xi \end{aligned} \tag{103}$$

for $x < \bar{x}$, thus

$$\begin{aligned} |\bar{v}(0,t)| \le\, & e^{L\bar{x}}|e^{-k\bar{x}}U^*(t)| \\ & + \frac{e^{L\bar{x}} - 1}{\epsilon_1 L}\left(L_v\|w(\cdot,t)\|_\infty + c_v\right). \end{aligned} \tag{104}$$

Inserting (102) into (104) gives

$$|\bar{v}(0,t)| \le \hat{L}\|w(\cdot,t)\|_\infty + \hat{d}, \tag{105}$$

where $\hat{d} = e^{L\bar{x}}c_g + \frac{e^{L\bar{x}}-1}{\epsilon_1 L}c_v$. Finally,

$$\begin{aligned} |w(0,t)| =& |\bar{u}(0,t)| = |f(\bar{v}(0,t),t)| \le L_f|\bar{v}(0,t)| + d_f \\ &\le L_f\hat{L}\|w(\cdot,t)\|_\infty + L_f\hat{d} + d_f. \end{aligned} \tag{106}$$

Since we assume $L_f\hat{L} < 1$, $\|w(\cdot,t)\|_\infty > \frac{L_f\hat{d}+d_f}{1-L_f\hat{L}}$ implies $|w(0,t)| < \|w(\cdot,t)\|_\infty$, i.e. (81) holds for some $\varepsilon > 0$. ■

Example 18 *In [1], a linear system with boundary condition*

$$u(0,t) = qv(0,t) + d(t) \tag{107}$$

was considered, where $q \ne 0$ and d is a disturbance term. Predictability of d was assumed, implicitly, via the linear model

$$d(t) = CX(t), \qquad \dot{X}(t) = AX(t). \tag{108}$$

The tracking objective was

$$u(0,t) = rv(0,t) \tag{109}$$

for $r \ne q$. Thus, g can be constructed by inserting (107) into (109) and solving for $v(0,t)$, yielding $g(*,t) = \frac{1}{r-q}d(t)$. Hence, setting

$$U^*(t) = \frac{1}{r-q}d(t+d_v) \tag{110}$$

solves the disturbance rejection problem from [1] for all $t \ge d_v$ if an exact prediction of d in $[t, t+d_v]$ is available. Disturbance rejection at $\bar{x} \in (0,1)$ for the same linear system was considered in [2]. The tracking objective was $g(u(\bar{x},t),t) = \frac{1}{r}u(\bar{x},t)$ for $r \ne 0$, which is achieved for all $t \ge \tau_v(\bar{x},0)$ by $U^*(t) = \frac{1}{r}\bar{u}(\bar{x},t)$.
In [24], disturbances entering also inside the domain were considered, again for a linear model. The source terms were written as

$$F_u((u,v)(x,t),x,t) = c_1(x)v(x,t) + d_1(x,t), \tag{111}$$
$$F_v((u,v)(x,t),x,t) = c_2(x)v(x,t) + d_2(x,t). \tag{112}$$

In our approach, the effect of the in-domain disturbances d_1 and d_2 is handled, implicitly, in the Φ^t and Ψ^t operators, and $U^*(t)$ can be constructed as above.

3.4 *Explicit state-feedback law for linear systems*

For the special case of linear systems of the form

$$u_t = -\epsilon_1(x)u_x + c_1(x)v, \tag{113}$$
$$v_t = \epsilon_2(x)v_x + c_2(x)u, \tag{114}$$
$$v(1,t) = U(t), \tag{115}$$

$$u(0,t) = qv(0,t), \tag{116}$$

with the same smoothness requirements as in [27], it is possible to make the feedback law explicit in the state, i.e. without solving the PDEs when evaluating Φ^t and Ψ^t. The explicit control law can be derived by the ansatz

$$\begin{aligned}\bar{v}(x,t) = \int_0^x & K^u(x,\xi)u(\xi,\tau_v(x,t)) \\ & + K^v(x,\xi)v(\xi,\tau_v(x,t))d\xi,\end{aligned} \tag{117}$$

with $U(t) = \bar{v}(1,t)$. Differentiating the right-hand side of (117) with respect to x, inserting the dynamics (113)-(116), integrating by parts and equating the result with $\bar{v}_x(x,t)$ as given in (37), it turns out that the kernels K^u and K^v must satisfy

$$\begin{aligned}&\epsilon_2(x)K^u_x(x,\xi) - K^u_\xi(x,\xi)\epsilon_1(\xi) \\ &\qquad - K^u(x,\xi)\epsilon_1'(\xi) - K^v(x,\xi)c_2(\xi) = 0, \quad (118)\\ &\epsilon_2(x)K^v_x(x,\xi) + K^v_\xi(x,\xi)\epsilon_2(\xi) \\ &\qquad + K^v(x,\xi)\epsilon_2'(\xi) - K^u(x,\xi)c_1(\xi) = 0 \quad (119)\end{aligned}$$

for all $x \in [0,1]$ and $\xi \in [0,x]$, and

$$K^u(x,x)\left(\epsilon_1(x) + \epsilon_2(x)\right) = -c_2(x), \tag{120}$$
$$q\epsilon_1(0)K^u(x,0) = \epsilon_2(0)K^v(x,0) \tag{121}$$

for $x \in [0,1]$. These are precisely two of the kernel PDEs from [27] (they are denoted K^{vu} and K^{vv} in [27]). Thus, the two control laws are equivalent.
For linear systems with disturbance terms, equivalence of the state-feedback controller in this paper and the backstepping-based results in [1,24] can be shown by the same steps sketched above.

4 Observer design

We assume we can measure $y(t) = u(1,t)$ and design an observer to estimate the distributed state. In (48), information propagates from $x = 1$ to $x = 0$. Moreover, both boundary values at $x = 1$, $\check{u}(1,t) = y(t)$ and $\check{v}(1,t) = U(t)$, are known. Therefore, we design the observer as a copy of (47)-(51), where we replace the boundary condition (49) by the measurement at $x = 1$:

$$\hat{u}_x(x,t) = \frac{1}{\epsilon_u(x)}F_u(\hat{u},\hat{v})(x,t),x,\tau_u(x,t)), \tag{122}$$
$$\begin{aligned}\hat{v}_t(x,t) = &\frac{\epsilon_u(x)\epsilon_v(x)}{\epsilon_u(x)+\epsilon_v(x)}\hat{v}_x(x,t) \\ &+ \frac{\epsilon_u(x)}{\epsilon_u(x)+\epsilon_v(x)}F_v((\hat{u},\hat{v})(x,t),x,\tau_u(x,t)),\end{aligned} \tag{123}$$
$$\hat{u}(1,t) = y(t), \tag{124}$$
$$\hat{v}(1,t) = U(t), \tag{125}$$
$$\hat{v}(x,0) = \hat{v}_0(x), \tag{126}$$

where $\hat{v}_0$ is some initial guess.

Theorem 19 *The observer errors $e^u = \hat{u} - \check{u}$ and $e^v = \hat{v} - \check{v}$ satisfy*

$$e^u(x,t) = e^v(x,t) = 0 \text{ for all } (x,t) \in \mathcal{B}, \tag{127}$$

where $\mathcal{B} = \{(x,t) : x \in [0,1], t \geq \int_x^1 \frac{1}{\epsilon_u(\xi)} + \frac{1}{\epsilon_v(\xi)} d\xi\}$.

Proof. Subtracting (47)-(51), with (49) replaced by $\check{u}(1,t) = y(t)$, from (122)-(126) yields

$$e^u_x(x,t) = E_u(\hat{u},\hat{v},\check{u},\check{v},x,t), \tag{128}$$
$$e^v_t(x,t) = \frac{\epsilon_u\epsilon_v}{\epsilon_u+\epsilon_v}e^v_x(x,t) + E_v(\hat{u},\hat{v},\check{u},\check{v},x,t), \tag{129}$$
$$e^u(1,t) = 0, \tag{130}$$
$$e^v(1,t) = 0, \tag{131}$$

where

$$\begin{aligned}E_u(\hat{u},\hat{v},\check{u},\check{v},x,t) = &\frac{1}{\epsilon_u(x)}(F_u((\hat{u},\hat{v})(x,t),x,\tau_u(x,t)) \\ &- F_u((\check{u},\check{v})(x,t),x,\tau_u(x,t))),\end{aligned} \tag{132}$$
$$\begin{aligned}E_v(\hat{u},\hat{v},\check{u},\check{v},x,t) = &\frac{\epsilon_u(x)}{\epsilon_u(x)+\epsilon_v(x)}(F_v((\hat{u},\hat{v})(x,t), \\ &x,\tau_u(x,t)) - F_v((\check{u},\check{v}),x,\tau_u(x,t))).\end{aligned} \tag{133}$$

This PDE-ODE system is well defined in the domain $\{(x,t) : x \in [0,1], t \geq \int_x^1 \frac{1}{\epsilon_u(\xi)}d\xi\}$, which contains $\mathcal{B}$. Defining

$$\hat{\phi}(x) = \int_x^1 \frac{\epsilon_u(\xi)+\epsilon_v(\xi)}{\epsilon_u(\xi)\epsilon_v(\xi)}d\xi, \tag{134}$$
$$\hat{\xi}(x,t,s) = \hat{\phi}^{-1}(\hat{\phi}(x) - t + s), \tag{135}$$
$$\hat{s}^0(x,t) = t - \hat{\phi}(x), \tag{136}$$

we integrate (128)-(131) along its characteristic lines to obtain, for $(x,t) \in \mathcal{B}$,

$$e^u(x,t) = e^v(1,t) + \int_1^x E_u(\hat{u},\hat{v},\check{u},\check{v},\xi,t)d\xi, \tag{137}$$
$$\begin{aligned}e^v(x,t) = &e^v(1,\hat{s}^0(x,t)) \\ &+ \int_{\hat{s}^0(x,t)}^t E_v(\hat{u},\hat{v},\check{u},\check{v},\hat{\xi}(x,t,s),s)ds.\end{aligned} \tag{138}$$

Next, we exploit the fact that $e^u(x,t) = e^v(x,t) = 0$, i.e. $\check{u}(x,t) = \hat{u}(x,t)$ and $\check{v}(x,t) = \hat{v}(x,t)$, implies

$$E_u(\hat{u},\hat{v},\check{u},\check{v},x,t) = E_v(\hat{u},\hat{v},\check{u},\check{v},x,t) = 0. \tag{139}$$

Inserting (127) into (137)-(138), we therefore see that the right-hand sides become zero. Thus, (127) solves (137)-(138). Since the solution is unique, we can reverse the statement to say that the solution of (137)-(138) satisfies (127), and the same holds for the error system (128)-(131). ■

Thus, we can determine the current state from the observer state $(\hat{u}, \hat{v})$ by evaluating Λ^t:
Theorem 20 *The observer (122)-(126) achieves*

$$(u(\cdot,t), v(\cdot,t)) = \Lambda^t(\hat{u}(\cdot,t), \hat{v}(\cdot,t)) \qquad (140)$$

for all $t \geq d_u + d_v$

Proof. This follows directly from the definition of Λ^t (Theorem 8) and Theorem 19. ■

Remark 21 *Making a coordinate change from* x *to* $z = 1-x$*, the estimation error system (128)-(131) has exactly the same structure as (36)-(40) for* $\bar{x} = 0$ *and* $U^*(t) = 0$*. Thus, we can proceed exactly as in Section 3.2 to show that*

$$V^e(e^v(\cdot,t)) = \sup_{x\in[0,1]} \left| e^{-k(1-x)} e^v(x,t) \right| \qquad (141)$$

for sufficiently large k *is a Lyapunov function for* (e^u, e^v)*, proving exponential decay of the estimation error.*
Remark 22 *The question arises whether the observer is the same as the one in [27] for linear systems. They are not the same, although both deliver exact estimates in minimum time. The observer (122)-(126) has zero error at* $x = 1$*, and this zero error propagates towards* $x = 0$*. The observer in [27] feeds the potentially non-zero error* $u(1,t) - \hat{u}(1,t)$ *back into the estimated state at all* $x \in [0, 1]$*.*

5 Output feedback control

Combining Theorems 11 and 17, respectively, with Theorem 20, we have solved the output feedback control problem.
Theorem 23 *Consider the output feedback controller consisting of the the observer (122)-(126) and the control law*

$$U(t) = \Psi^t_{\bar{x}}(\Phi^t(\Lambda^t(\hat{v}(\cdot,t), \hat{v}(\cdot,t)), U^*(t)). \qquad (142)$$

Assuming (11)-(13) and using $U^*(t) = 0$*, the closed loop system becomes zero within* $2(d_u + d_v)$*. Using (90)-(91) for* $U^*(t)$*, the closed loop system satisfies the tracking objective (16) within* $d_u + 2d_v$*.*

6 Example

We illustrate the performance of the controller in an example with

$$\epsilon_u(x) = \begin{cases} 0.2 & \text{if } x < 0.5, \\ 2 - x & \text{if } x \geq 0.5, \end{cases}, \qquad (143)$$

$$\epsilon_v(x) = 1 + 0.5\,x, \qquad (144)$$

$$F_u((u,v),x,t) = \frac{1}{3-x}\sin(u+v), \qquad (145)$$

$$F_v((u,v),x,t) = \sin(v-u), \qquad (146)$$

$$f(v^0,t) = -v^0, \qquad (147)$$

and initial condition $u_0 = v_0 = 1$. With these propagation speeds, the delay times are $d_u \approx 2.9$ and $d_v \approx 0.8$. First, we consider stabilization of the origin using output feedback. The initial condition of the observer is set to zero. The operators Φ^t, Ψ^t and Λ^t are implemented as sketched in Remarks 5 and 9, respectively. The PDEs are discretized in space by finite differences, and Matlab's ode45 is used for time-integration. Using 40 spatial discretization elements, evaluating Φ^t and Λ^t takes approximately 40 ms and 120 ms, respectively, on a Mac OS X 10.10.5 with a 2.2 GHz Intel Core i7 with 16 GB 1600 MHz DDR3 memory and Matlab 2014b. In order to illustrate the system behavior in open loop, the controller is switched on only after $t = 10$. For $t < 10$, the input is set to $U(t) = 0$. The resulting trajectories of the state, and the error between the true state and the estimated state $(u_{est}, v_{est}) = \Lambda(\hat{u}, \hat{v})$, are depicted in Figure 2. The error between true and estimated state becomes zero within $d_u + d_v \approx 3.7$, up to numerical errors. Once the controller is switched on, the states also become zero within $d_u + d_v$.
Second, we consider a tracking problem using output feedback. The tracking objective is defined by $\bar{x} = 0$ and $g_0(u^0, v^0, t) = v^0 - v_{ref}(t)$ for $v_{ref}(t) = \sin(0.5\,t) + \cos(0.7\,t)$, i.e. $v(0,t)$ shall track the reference signal v_{ref}. The trajectory of v, as well as $v(0,t)$, $v_{ref}(t)$, and the actuation $U(t)$, are depicted in the right column of Figure 2. The tracking objective is achieved for $t > 4.5$, up to numerical errors.

7 Conclusions

We have solved the boundary control and estimation problem for a class of 2×2 semilinear hyperbolic PDEs. The control scheme can be used for stabilization of the origin, as well as tracking at an arbitrary location in the presence of predictable disturbances. Evaluation of the controller involves solving a PDE and an ODE. For linear system, the control input is the same as the one obtained from previously known backstepping methods. This equivalence bears the potential to analyze properties, such as robustness, of backstepping controllers by the approach presented in this paper, and vice versa. We also designed an observer using measurements collocated with the actuation, and, combining the observer with the the state-feedback controller, solved the output feedback problem. The controller can be applied to several physical systems, such as transmission lines with nonlinear friction. In order to make the evaluation of the control input less computationally demanding, the scheme mentioned in Remark 6 can be investigated in future work. Moreover, it would be interesting to see if the methods in this paper can be extended to $(n+m) \times (n+m)$ systems, interconnected hyperbolic systems, or two-sided boundary control.

A Existence of Φ^t

We prove the existence of Φ^t in Theorem 2. Without loss of generality, we assume Φ^t is evaluated at $t = 0$, otherwise we can shift time.

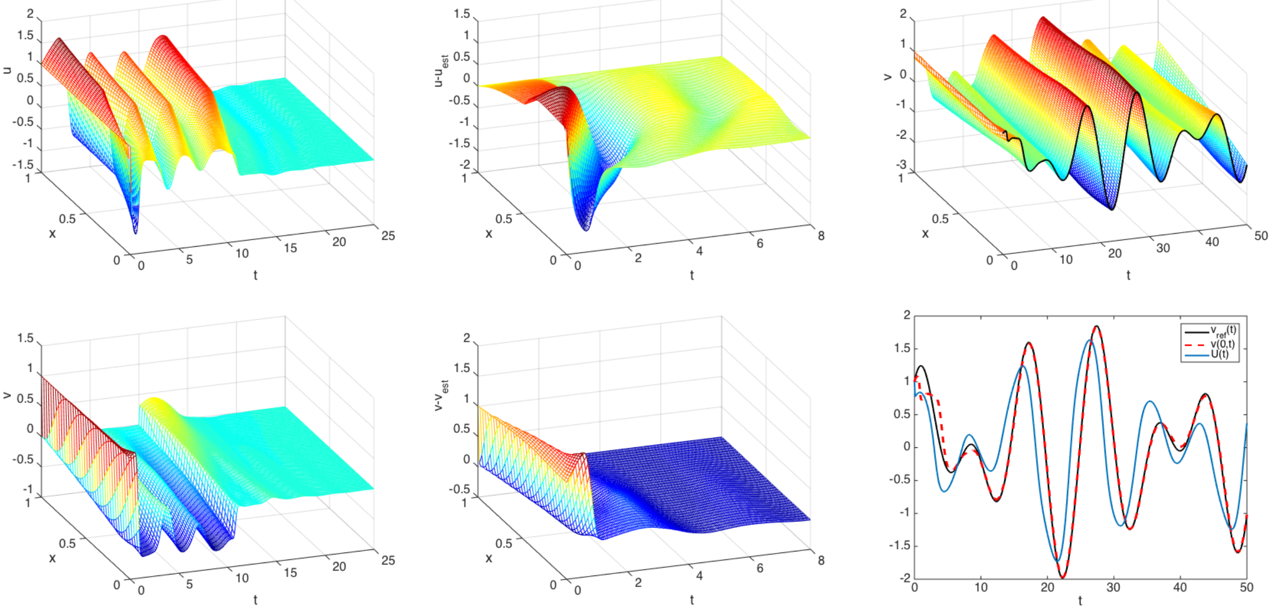

Fig. 2. System state and estimation error trajectories. Left and centre: stabilization example. Right: tracking example.

In the first step, we prove the existence of a unique solution of the system

$$u_t(x,t) = -\epsilon_u u_x(x,t) + g_u(x,t) + \tilde{F}_u(u,v,x,t), \quad \text{(A.1)}$$
$$v_t(x,t) = \epsilon_v v_x(x,t) + g_v(x,t) + \tilde{F}_v(u,v,x,t), \quad \text{(A.2)}$$
$$u(0,t) = h(t) + \tilde{f}(v,t), \quad \text{(A.3)}$$
$$u(x,0) = u_0, \quad \text{(A.4)}$$
$$v(x,0) = v_0, \quad \text{(A.5)}$$

where

$$\tilde{F}_u(u,v,x,t) = F_u((u,v)(x,t),x,t) - F_u((0,0),x,t), \quad \text{(A.6)}$$
$$\tilde{F}_v(u,v,x,t) = F_v((u,v)(x,t),x,t) - F_v((0,0),x,t), \quad \text{(A.7)}$$
$$g_u(x,t) = F_u((0,0),x,t), \quad \text{(A.8)}$$
$$g_v(x,t) = F_v((0,0),x,t), \quad \text{(A.9)}$$
$$\tilde{f}(v^0,t) = f(v^0,t) - f(0,t), \quad \text{(A.10)}$$
$$h(t) = f(0,t), \quad \text{(A.11)}$$

in the domain

$$\mathcal{T}^\delta = \left\{ (x,t) : x \in [0, 1-\delta], t \in \left[0, \int_x^{1-\delta} \frac{1}{\epsilon_v(\xi)} d\xi\right]\right\} \quad \text{(A.12)}$$

for $\delta > 0$, see also Figure A.1. $\mathcal{T}^\delta$ is designed to cut off the boundary condition at $x = 1$, which corresponds to the input U and is not known yet when Φ^t is evaluated. Therefore, the values on the inflow boundary are fully specified, and thus the system (A.1)-(A.5) is well posed.

Moreover, the solution depends continuously on the input data (u_0, v_0) and, if (11)-(13) hold, is sublinear in (u_0, v_0). The idea of the proof follows Appendix A in [6], but we have to consider the nonlinearity and have a different domain. That is, we transform the PDEs into integral equations (Section A.1) and prove convergence of a successive approximation series (Section A.2).
Second, we show that $u(x,t)$ for $(x,t) \in s_v = \{(x,t) : x \in [0,1), t = \int_x^1 \frac{1}{\epsilon_v(\xi)} d\xi\}$, where $s_v \not\subseteq \mathcal{T}^\delta$ for any $\delta > 0$, equals the limit of u evaluated at points inside $\mathcal{T}^\delta$ for $\delta \to 0$ (Section A.3).
Finally, u on s_v is the actual output of the operator Φ^t for the input (u_0, v_0).

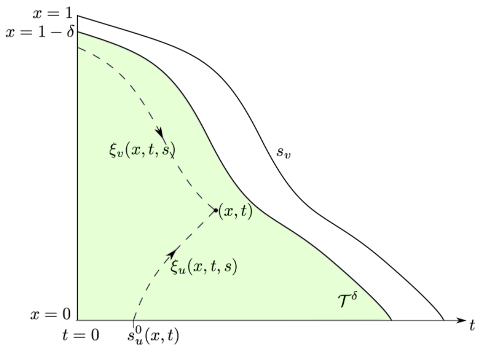

Fig. A.1. Illustration of the domain $\mathcal{T}^\delta$ (green) and the characteristic lines.

A.1 *Transformation into integral equations*

We denote the characteristic lines passing through (x,t) along which (A.1)-(A.2) evolve by $(\xi_u(x,t,s), s)$

and $(\xi_v(x,t,s),s)$, respectively, where the argument s is used for parameterization in time. First, we define

$$\phi_u(x) = \int_0^x \frac{1}{\epsilon_u(\xi)} d\xi, \qquad \phi_v(x) = \int_x^1 \frac{1}{\epsilon_v(\xi)} d\xi. \tag{A.13}$$

Since ϵ_u and ϵ_v are positive, ϕ_u and ϕ_v are well defined and monotonically increasing, and thus invertible. Then, ξ_u and ξ_v are given by

$$\xi_u(x,t,s) = \phi_u^{-1}(\phi_u(x) - t + s), \tag{A.14}$$
$$\xi_v(x,t,s) = \phi_v^{-1}(\phi_v(x) - t + s). \tag{A.15}$$

The following Lemma holds.

Lemma 24 *If $(x,t) \in \mathcal{T}^\delta$, then $(\xi_u(x,t,s),s) \in \mathcal{T}^\delta$ for all $s \in [s_u^0(x,t), t]$, where $s_u^0(x,t) := \max\{0, t - \phi_u(x)\}$, and $(\xi_v(x,t,s),s) \in \mathcal{T}^\delta$ for all $s \in [0,t]$*

Next, (A.1)-(A.2) are integrated along their characteristic lines to obtain

$$\begin{aligned} u(x,t) = u(\xi_u(x,t,s_u^0(x,t)), s_u^0(x,t)) \\ + G_u(x,t) + I_u[u,v](x,t), \end{aligned} \tag{A.16}$$
$$v(x,t) = v(\xi_v(x,t,0),0) + G_v(x,t) + I_v[u,v](x,t), \tag{A.17}$$

where

$$G_u(x,t) = \int_{s_u^0(x,t)}^{t} g_u(\xi_u(x,t,s),s)ds, \tag{A.18}$$
$$I_u[u,v](x,t) = \int_{s_u^0(x,t)}^{t} \tilde{F}_u(u,v,\xi_u(x,t,s),s)ds, \tag{A.19}$$
$$G_v(x,t) = \int_0^t g_v(\xi_v(x,t,s),s)ds, \tag{A.20}$$
$$I_v[u,v](x,t) = \int_0^t \tilde{F}_v(u,v,\xi_v(x,t,s),s)ds. \tag{A.21}$$

Inserting the boundary conditions (A.3)-(A.4) into (A.16) and (A.5) into (A.17), we get

$$\begin{aligned} u(x,t) = H_u(x,t) + G_u(x,t) + P(x,t) \\ + Q[u,v](x,t) + I_u[u,v](x,t), \end{aligned} \tag{A.22}$$
$$v(x,t) = H_v(x,t) + G_v(x,t) + I_v[u,v](x,t), \tag{A.23}$$

where $H_v(x,t) = v_0(\xi_v(x,t,0))$, and

$$H_u(x,t) = u_0(\xi_u(x,t,0)), \tag{A.24}$$
$$Q[u,v](x,t) = P(x,t) = 0. \tag{A.25}$$

if $t - \phi_u(x) < 0$, and

$$H_u(x,t) = h(t - \phi_u(x)), \tag{A.26}$$
$$P(x,t) = f(H_v(0, t-\phi_u(x)), t - \phi_u(x)), \tag{A.27}$$
$$Q[u,v](x,t) = f(I_v[u,v](0, t-\phi_u(x)), t-\phi_u(x)) \tag{A.28}$$

if $t - \phi_u(x) \geq 0$.

A.2 Solution of integral equations via successive approximation

For $w = (u,v)^T$, define the successive approximation sequence

$$w^0 = \begin{pmatrix} H_u + P + G_u \\ H_v + G_v \end{pmatrix}, \ \Omega[w] = \begin{pmatrix} I_u[u,v] + Q[u,v] \\ I_v[u,v] \end{pmatrix}, \tag{A.29}$$

and $w^{n+1} = \Omega[w^n] + w^0$. The limit $w(x,t) = \lim_{n\to\infty} w^n(x,t)$, if convergent, solves the integral equations (A.22)-(A.23), and thus the PDEs (A.1)-(A.5). The idea is to prove existence of the limit via convergence of the series

$$w = \lim_{n\to\infty} w^n = \sum_{n=0}^{\infty} \Delta w^n, \tag{A.30}$$

where $\Delta w^n = w^n - w^{n-1}$ and $\Delta w^0 = w^0$ by definition. Next, we establish a contraction property for Ω. We denote the 1-norm in $\mathbb{R}^2$ by $\|\cdot\|_1$, i.e. $\|w\|_1 = |w_1| + |w_2|$ for $w = (w_1, w_2)^T$.

Lemma 25 *Assume that $\|w_1(x,t) - w_2(x,t)\|_1(x,t) \leq c\,C^m \frac{t^m}{m!}$ for $C = L_u + L_v(1 + L_f)$, $m \geq 0$, some constant $c \geq 0$, and all $(x,t) \in \mathcal{T}^\delta$. Then, for $(x,t) \in \mathcal{T}^\delta$,*

$$\|\Omega[w_1](x,t) - \Omega[w_2](x,t)\|_1 \leq c\,C^{m+1} \frac{t^{m+1}}{(m+1)!}. \tag{A.31}$$

Proof. Denoting $w_1 = (u_1, v_1)^T$ and $w_2 = (u_2, v_2)^T$, we have

$$\begin{aligned} &|I_u[u_1,v_1](x,t) - I_u[u_2,v_2](x,t)| \\ &\quad = \left| \int_{s_u^0(x,t)}^t \tilde{F}_u(u_1,v_1,\xi_u(x,t,s),s)ds \right. \\ &\qquad \left. - \int_{s_u^0(x,t)}^t \tilde{F}_u(u_2,v_2,\xi_u(x,t,s),s)ds \right| \\ &\quad \leq \int_{s_u^0(x,t)}^t |(\tilde{F}_u(u_1,v_1,\xi_u(x,t,s),s) \\ &\qquad - \tilde{F}_u(u_2,v_2,\xi_u(x,t,s),s)|ds \\ &\quad \leq L_u \int_{s_u^0(x,t)}^t \|w_1(\xi_u(x,t,s),s) \\ &\qquad - w_2(\xi_u(x,t,s),s)\|_1 ds \\ &\quad \leq cL_uC^m \frac{1}{m!} \int_{s_u^0}^t s^m ds \leq cL_uC^m \frac{t^{m+1}}{(m+1)!}. \end{aligned} \tag{A.32}$$

The same steps for I_v yield

$$|I_v[u_1,v_1](x,t) - I_v[u_2,v_2](x,t)| \leq cL_vC^m \frac{t^{m+1}}{(m+1)!}. \tag{A.33}$$

Similarly, for Q_u if $t \geq \phi_u(x)$,

$$\begin{aligned} &|Q_u[u_1,v_1](x,t) - Q_u[u_2,v_2](x,t)| \\ &\qquad = |f(I_v[u_1,v_1](0,t-\phi_u(x))) \\ &\qquad\quad - f(I_v[u_2,v_2](0,t-\phi_u(x)))| \\ &\qquad \leq L_f|I_v[u_1,v_1](0,t-\phi_u(x)) \\ &\qquad\quad - I_v[u_2,v_2](0,t-\phi_u(x)))| \\ &\qquad \leq cL_vL_fC^m\frac{t^{m+1}}{(m+1)!}. \end{aligned} \tag{A.34}$$

The claim follows from

$$\begin{aligned} &\|\Omega[w_1](x,t) - \Omega[w_2](x,t)\|_1 \\ &\quad \leq |I_u[u_1,v_1](x,t) - I_u[u_2,v_2](x,t)| \\ &\quad + |Q_u[u_1,v_1](x,t) - Q_u[u_2,v_2](x,t)| \\ &\quad + |I_v[u_1,v_1](x,t) - I_v[u_2,v_2](x,t)|. \end{aligned} \tag{A.35}$$ ■

The following proposition discusses convergence of the successive approximation series. We define the constants

$$\bar{w} = \sup_{(x,t)\in\mathcal{T}^\delta} \|w^0(x,t)\|_1, \quad \tilde{w} = \bar{w}(1+Cd_v), \tag{A.36}$$

where d_v is defined in (21) and $\|\cdot\|_1$ is the 1-norm in $\mathbb{R}^2$.

Proposition 26 *The successive approximation series converges to a unique, bounded limit $w = \lim_{n\to\infty} w^n$. Moreover, w depends continuously on the input data (u_0, v_0) and, if (11)-(13) holds, is sublinear in (u_0, v_0) in the sense that there exists a constant $c_\Phi > 0$ such that*

$$\sup_{(x,t)\in\mathcal{T}^\delta} \|w(x,t)\|_1 \leq c_\Phi \|(u_0,v_0)\|_\infty. \tag{A.37}$$

Both continuity and (A.37) hold uniformly with respect to $(x,t) \in \mathcal{T}^\delta$ and $\delta > 0$.

Proof. For $n = 0$, we have $\|\Delta w^0(x,t)\|_1 \leq \bar{w}$. Next, we establish the bound

$$\|\Delta w^n(x,t)\|_1 \leq \tilde{w}C^{n-1}\frac{t^{n-1}}{(n-1)!} \tag{A.38}$$

for all $n \geq 1$ by induction. For $n = 1$, we exploit $\Omega[0] = 0$ and Lemma 25 for $m = 0$ to get

$$\begin{aligned} \|w^1(x,t)\|_1 &= \|\Omega[w^0](x,t)\|_1 \\ &= \|\Omega[w^0](x,t) - \Omega[0](x,t)\|_1 \leq \bar{w}Cd_v. \end{aligned} \tag{A.39}$$

Thus, $\|\Delta w^1\|_1 \leq \|w^1\|_1 + \|w^0\|_1 \leq \bar{w}(1+Cd_v)$. In the induction step for $n \geq 2$, we assume the inequality holds for $\Delta w^n = w^n - w^{n-1}$ and utilize Lemma 25 to get

$$\begin{aligned} \|\Delta w^{n+1}(x,t)\|_1 &= \|\Omega[w^n](x,t) \\ &\quad - \Omega[w^{n-1}](x,t)\|_1 \leq \tilde{w}C^n\frac{t^n}{n!}. \end{aligned} \tag{A.40}$$

We can now prove absolute convergence of the successive approximation series,

$$\begin{aligned} \|w(x,t)\|_1 &\leq \sum_{n=0}^{\infty} \|\Delta w^n(x,t)\|_1 \\ &= \sum_{n=1}^{\infty} \|\Delta w^n(x,t)\|_1 + \|\Delta w^0(x,t)\|_1 \\ &\leq \sum_{n=1}^{\infty} \tilde{w}C^{n-1}\frac{t^{n-1}}{(n-1)!} + \bar{w} \\ &\leq \bar{w}\left(1 + (1+Cd_v)e^{Ct}\right). \end{aligned} \tag{A.41}$$

Thus, the limit of the successive approximation series exists and is bounded. Regarding uniqueness, consider two solutions w and w'. The difference, $\tilde{w} = w - w'$, satisfies a system of the same structure as (A.1)-(A.5) with $g_u = 0$, $g_v = 0$, $h = 0$ and initial condition $\tilde{w}_0 = 0$. Thus, the difference satisfies (A.41) with $\bar{w} = 0$, implying $\tilde{w} = 0$. To prove continuity in the initial data, choose $\varepsilon > 0$ and consider again two solutions w and w' with $\|w_0 - w_0'\|_\infty < \delta'$. We choose $\delta' = \epsilon e^{-Cd_v}$. The difference $\tilde{w} = w - w'$ again satisfies a system of the same structure as (A.1)-(A.5) with $g_u = 0$, $g_v = 0$, and $h = 0$, hence $\bar{w} \leq \delta'$. Thus, (A.41) implies $\|\tilde{w}(x,t)\|_1 \leq \varepsilon \; \forall (x,t) \in \mathcal{T}^\delta$ and all $\delta > 0$. (A.37) follows from the fact that (11)-(13) imply $g_u = 0$, $g_v = 0$, and $h = 0$, i.e. $\bar{w} = \|(u_0,v_0)\|_\infty$. ■

A.3 u on s_v

Proposition 26 states the existence of a solution (u,v) of (A.1)-(A.5) in $\mathcal{T}^\delta$ for all $\delta > 0$. However, we need u on the characteristic line s_v (see (18)), which is not contained in any $\mathcal{T}^\delta$.

Lemma 27 *For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(x,t) \in s_v$ with $t - \varepsilon \geq s_u^0(x,t)$, $(\xi_u(x,t,t-\varepsilon), t-\varepsilon) \in \mathcal{T}^\delta$.*

Proof. From the definition of ξ_u (see (A.14)), we see that $\xi_u(x,t,t-\varepsilon) < x$. The shape of $\mathcal{T}^\delta$ implies that if $(x',t') \in \mathcal{T}^\delta$, then $(y,t') \in \mathcal{T}^\delta$ for all $y \leq x'$. Thus, choosing δ such that $(x,t-\varepsilon) \in \mathcal{T}^\delta$ ensures $(\xi_u(x,t,t-\varepsilon), t-\varepsilon) \in \mathcal{T}^\delta$.
If $(x,t) \in s_v$, we have $t = \int_x^1 \frac{1}{\epsilon_v(\xi)}d\xi$. We also have that $(x,t') \in \mathcal{T}^\delta$ for all $t' \leq \int_x^{1-\delta} \frac{1}{\epsilon_v(\xi)}d\xi$. Thus, the claim follows if we establish

$$t - \varepsilon = \int_x^1 \frac{1}{\epsilon_v(\xi)}d\xi - \varepsilon < \int_x^{1-\delta} \frac{1}{\epsilon_v(\xi)}d\xi, \tag{A.42}$$

which is equivalent to

$$\int_{1-\delta}^{1} \frac{1}{\epsilon_v(\xi)} d\xi < \varepsilon. \quad \text{(A.43)}$$

This is achieved by choosing $\delta < \varepsilon \inf_{\xi\in[0,1]} \epsilon_v(\xi)$. The extra constraint $t - \varepsilon \geq s_u^0(x,t)$ ensures that $\xi_u(x,t,t-\varepsilon) \geq 0$ and $t - \varepsilon \geq 0$. ■

Since u satisfies an ODE along its characteristic lines, the function $u(\xi_u(x,t,s),s)$ is uniformly continuous in s. Thus, for $(x,t) \in s_v$ with $x \in (0,1)$, the limit

$$u(x,t) = \lim_{\substack{\varepsilon\to 0 \\ \varepsilon \leq t - s_u^0(x,t)}} u(\xi_u(x,t,t-\varepsilon), t-\varepsilon) \quad \text{(A.44)}$$

exists and is attained uniformly. Since both continuity in the input data and (A.37) in Proposition 26 hold uniformly with respect to $(x,t) \in \mathcal{T}^\delta$ and $\delta > 0$, u on s_v is also continuous in (u_0, v_0) and, if (11)-(13) hold, can be bounded by $c_\Phi \|u_0, v_0\|_\infty$. For $x = 1$, we trivially have $u(1,0) = u_0(1,0)$ and $u(0, \phi_v(0))$ is determined by $U^*(t)$.

B Inverse of Φ^t

We show that the Φ^t operator has a bounded inverse under the assumptions of Section 3.2 by showing that the system

$$u_t(x,t) = -\epsilon_u u_x(x,t) + \tilde{F}_u(u,v,x,t), \quad \text{(B.1)}$$
$$v_t(x,t) = \epsilon_v v_x(x,t) + \tilde{F}_v(u,v,x,t), \quad \text{(B.2)}$$
$$u(x, \phi_v(x)) = \bar{u}(x), \quad \text{(B.3)}$$
$$v(0, \phi_v(0)) = 0, \quad \text{(B.4)}$$
$$v(0,t) = f^{-1}(u(0,t),t), \quad \text{(B.5)}$$

has a solution in the domain

$$\mathcal{T} = \{(x,t) : x \in [0,1], t \in [0, \phi_v(x)]\}. \quad \text{(B.6)}$$

The proof follows the steps in Sections A.1-A.2, but here we integrate backwards in time along the characteristic lines. Since we assume (11)-(13), some terms in the integral equations vanish:

$$u(x,t) = u(\xi_u(x,t,s_u^F(x,t)), s_u^F(x,t)) + \hat{I}_u[u,v](x,t), \quad \text{(B.7)}$$
$$v(x,t) = v(0, s_v^F(x,t)) + \hat{I}_v[u,v](x,t), \quad \text{(B.8)}$$

where

$$\hat{I}_u[u,v](x,t) = \int_{s_u^F(x,t)}^{t} \tilde{F}_u(u,v,\xi_u(x,t,s),s) ds, \quad \text{(B.9)}$$

$$\hat{I}_v[u,v](x,t) = \int_{s_v^F(x,t)}^{t} \tilde{F}_v(u,v,\xi_v(x,t,s),s) ds, \quad \text{(B.10)}$$

$$s_v^F(x,t) = t - \phi_v(x) + \phi_v(0). \quad \text{(B.11)}$$

$s_u^F(x,t)$ is defined by the intersection of $\xi_u(x,t,\cdot)$ with

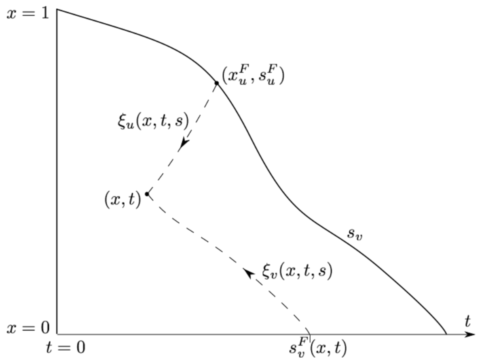

Fig. B.1. Integration paths for the inverse of Φ^t.

s_v. Denoting the intersecting point by (x_u^F, s_u^F), we have $x_u^F = \xi_u(x,t,s_u^F(x,t))$ and $\phi_v(x_u^F) = s_u^F$. Thus, s_u^F is given implicitly by the solution of

$$\phi_v^{-1}(s_u^F) = \phi_u^{-1}\left(\phi_u(\phi_v^{-1}(s_u^F)) - t + s_u^F\right). \quad \text{(B.12)}$$

Then, we can proceed as in Sections A.1-A.2, i.e. insert the boundary conditions, define the successive approximation series, and prove convergence. Since there are no terms g_u, g_v, and h as in (A.1)-(A.5), there exists a constant $c_{\Phi^{-1}}$ such that for all $(x,t) \in \mathcal{T}$, analogously to Proposition 26,

$$\|(u,v)(x,t)\| \leq c_{\Phi^{-1}} \|\bar{u}\|_\infty. \quad \text{(B.13)}$$

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