Constraints to a justification of commutativity of multiplication

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In this paper we investigate a pre-service teacher's whole-class discussion at Grade 6, where she attempts to justify the general claim that multiplication is commutative in ℕ*. Our analysis points at two conditions that constrain the discussion in class: The first is that the diagram used to represent a multiplicative situation—and the way the diagram is used—is inadequate because it does not illustrate the meaning of multiplication. The second is that the diagram enables the students to respond "adequately", even if they may not have understood why multiplication is commutative. The latter is a constraint because it deprives the pre-service teacher of the opportunity to get feedback that might have let her understand that she would need to revise her intervention.*

Keywords: Multiplicative situation, commutativity, representation, justification, teacher education.

Introduction

Addition and multiplication in ℕ (or, more generally, in ℝ) are commutative and associative. Moreover, multiplication is distributive over addition. These basic properties of addition and multiplication play a crucial role in abstract algebra, but also in arithmetic and algebra in elementary school. The work on fluency of the multiplication table gets considerably easier if the students make use of the basic properties. In fact, all arithmetic strategies can be shown to have origin in the basic properties of commutativity, associativity and distributivity. Promoting these properties in elementary school is important for integrating arithmetic and algebra (Carpenter, Franke, & Levi, 2003). Involved in this is the activity of generalising and formalising relationships and constraints, which is one of three strands of algebra identified by Kaput (2008). Furthermore, the basic properties of addition and multiplication are also important in "transformational activities" (Kieran, 2004), which involve syntactically guided manipulation of symbols (e.g., simplifying expressions, exponentiation with polynomials, and solving equations).

At the background on the above, the question raises as to how it can be discussed with students that addition and multiplication are both commutative and associative, and that multiplication is distributive over addition? If we want these properties to emerge from students' mathematical reasoning in the classroom—and not from the teacher's just presenting them as rules—how could it be done, and what challenges may arise? In this paper we discuss this question with data from teacher education, where a pre-service teacher has designed and implemented a lesson, aimed at whole-class discussion with 12 year-old students on basic properties of multiplication. The research question set out to answer is: *What conditions constrain a pre-service teacher's whole-class discussion with Grade 6 students about commutativity of multiplication in* ℕ***?***

Theoretical framework

Multiplicative structures—modelling situations involving equal-sized groups

From a mathematical point of view, multiplication and division by natural and rational numbers may appear easy. However, from a psychological point of view, it is more complex. In a teaching situation, these operations are dealt with not only as abstract binary operations, but also in terms of how they model different situations. Vergnaud (1988) claims that "[m]athematical concepts are rooted in situations and problems" (p. 142). Because a single concept does not refer to only one type of situation, and a single situation cannot be studied with only one concept, Vergnaud proposes that researchers study "conceptual fields". This is defined as a set of situations, the mastery of which depends on mastery of a conceptual structure. For instance, the conceptual field of *multiplicative structures* consists of all situations that can be analysed as simple and multiple proportion problems, where the necessary operation is multiplication or division (Vergnaud, 1988).

According to Greer (1992), the most important types of situations where multiplication of integers is involved are:

- equivalent groups (e.g., 6 tables, each with 4 children)
- multiplicative comparison (e.g., 3 times as many girls as boys)
- rectangular arrays/areas (e.g., 4 rows of 7 students, or area of a rectangle)
- Cartesian product (e.g., the number of possible trousers-sweater pairs)

Fishbein, Deri, Nello and Marino (1985) investigated how 623 pupils enrolled in 13 Italian schools (Grades 5, 7 and 9) responded on 26 multiplication and division word problems. Their findings confirmed the impact of *repeated addition* as an intuitive model on multiplication, in which a number of groups of the same size are put together—that is, equivalent-groups situations.

In Norwegian schools, multiplication is usually introduced through situations with equivalent groups, where $4 \cdot 7$ means $7 + 7 + 7 + 7$, while $7 \cdot 4$ means $4 + 4 + 4 + 4 + 4 + 4 + 4$. Here, it is not obvious that multiplication is commutative. The first of the factors—the number of equivalent groups—is taken as the operator (termed *multiplicator*); the other factor—the size of each group—is taken as the operand (termed *multiplicand*). In this model, the multiplicand can be any positive quantity, but the multiplicator must be an integer (Fishbein et al., 1985).

Justification in the elementary classroom

Algebraic thinking is a term used to describe particular ways of thinking applied when we are looking beyond quantities and operations on quantities. It includes analysing relationships between quantities, noticing structure, studying invariance and change, generalising, problem solving, modelling, conjecturing, justifying and proving (Cai et al., 2005). In this section we concentrate on justification. There are several ways of approaching justification and proof in school mathematics. Balacheff (1988) has identified four types of reasoning in 14-15 year-old students' practice of proving a conjecture that applies on infinitely many examples:

- *Naïve empiricism* is where students think that some examples (even one or two) are sufficient to justify a conjecture.
- The *crucial experiment* is where students think that the validity of a conjecture is accomplished by testing it on an instance that has some complexity—the reasoning being "if it works here, it will always work". The crucial experiment is different from naïve empiricism in that the generality at stake is explicitly articulated.
- The *generic example* involves making explicit the reasons for the truth of an assertion by means of operations on an object that is a representative of the class of elements considered. A generic example is an example *of* something—the validity of a hypothesis is argued for by the characteristic properties of this example.
- The *thought experiment* requires that the one who produces the proof distances him from the actions of solving the problem—he must give up the actual object for the class of objects on

which relations and operations are to be represented in formalised symbolic expressions. Proof by induction is an example of a thought experiment.

Proofs by naïve empiricism, the crucial experiment, and the generic example are based on actual actions and references to examples—these proofs are referred to as *pragmatic* proofs (Balacheff, 1988). The thought experiment is based on abstract formulations of properties and of relationships among properties—this proof is referred to as a *conceptual* proof. Balacheff emphasises that a proof by naïve empiricism or by a crucial experiment does not establish the truth of an assertion, and the reason why he refers to them as "proofs" is because they are recognised as such by the students who produce them. He asserts, further, that the generic example and the thought experiment are mathematically valid proofs. They involve a fundamental shift in the students' reasoning because the nature of the truth of a claim is established by giving reasons. When using a visual representation to justify a claim of generality, the representation needs to have some properties. Schifter (2009) has identified three criteria for a representation in elementary grades to be adequate: (1) the meaning of the operation(s) involved is represented in diagrams, manipulatives, possibly complemented by story contexts; (2) the representation is accessible for a class of instances; and, (3) the conclusion of the claim follows from the structure of the representation (p. 76).

Justification in the elementary classroom of the commutative property of multiplication can be done through a generic example based on a rectangular-area situation. In this situation, multiplication in $\mathbb{Q}^+$ is commutative because the area of a rectangle is the same regardless of the order in which its side lengths are multiplied. However, given the impact of the equivalent-groups situation, we consider it important also to be able to build on this intuitive interpretation in justifying that multiplication is commutative. Then of course, the asymmetry of this situation is a challenge. In the following we explain how a justification of commutativity of multiplication in $\mathbb{N}$ can be constructed, taking the situation of equivalent groups as a starting point. We discuss a justification first by a generic example, then by a thought experiment.

Let $4 \cdot 3$ be interpreted as the total number of discs when we have 4 equivalent groups of 3 discs, as shown in the upper row of Figure 1. The discs can be regrouped into 3 equivalent groups of 4 discs, which corresponds to $3 \cdot 4$ (illustrated in Figure 1[1]).

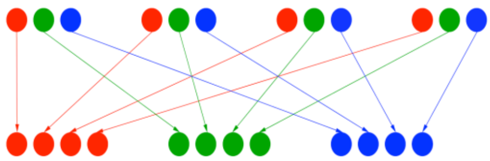

Figure 1. A generic example illustrating the symmetrisation of an asymmetric situation

The total number of discs is not changed, and consequently we have that $4 \cdot 3 = 3 \cdot 4$. This process of regrouping can be imagined with an arbitrary number of groups a, and an arbitrary number of discs b—that is, for any $a \cdot b$ where a and b are natural numbers. The number of discs in the equivalent groups in the original grouping transforms into the number of groups in the new grouping. The

[1] The discs are coloured to make the process clearer. The arrows signify the movement of the discs.

example with the transformation of $4 \cdot 3$ into $3 \cdot 4$ is thus used as a generic example in the justification.

Justification by a thought experiment can be rather similar to the generic example presented above, though the reasoning is done in general terms: Given natural numbers a and b, then $a \cdot b$ can be interpreted as the total number of discs when we have a groups with b discs in each group. Regrouping of discs by taking one by one disc from each group to make a new group gives b groups with a discs in each group—that is, a situation in which the total number of discs can be represented by $b \cdot a$. Since the number of discs is not changed in the process of regrouping, we can conclude that $a \cdot b = b \cdot a$ for all natural numbers a and b.

Methodological approach

The pre-service teacher (henceforth PST) participating in the research reported here was in her first year of a 4-year undergraduate teacher education programme for Grades 1-7 in Norway. The investigation has been done within a compulsory mathematics course in the programme, which involves an integration of mathematics and didactics. The data were collected at the end of the second semester. The main content of the mathematics course previous to data collection was multiplicative thinking, and the emphasis was on different strategies for, reasoning with, and properties of, multiplication and division. The second author, together with a colleague, taught the mathematics course and carried out the data collection.

During the mathematics course, the PSTs in the class worked on several assignments that involved practice of teaching in school (Grades 4-7), all concerned with strategies for and properties of multiplication. In the fourth assignment, from which the data analysed in this paper emerged, the PSTs were asked to plan and carry out a discussion with students concerning a given strategy or property of multiplication or division. The PSTs video-recorded and transcribed their discussions with students. The transcript analysed here is from one of 25 classroom discussions that were carried out and analysed.

Our research question concerns basic properties of multiplication, and we are interested in PSTs´ handling of general justifications—that is, justifications for an infinite number of cases. In most episodes where the PSTs discussed properties of multiplication with the students, there was no attempt to generalise and justify. There were discussions of particular examples, usually succeeded by a conclusion along the lines of "this will apply for all numbers". In this paper, we present an analysis of one of the discussions, the case of Janet (a PST). This case is chosen because Janet actually tried to discuss with students *why* a property of multiplication applies in general (in $\mathbb{N}$) and it demonstrates challenges thereof, which are also traced in some of the other transcripts (not analysed here). We will explain what conditions that prevent the case of Janet from being successful in the sense of including a valid argument for the claim of generality.

Results

Establishing a situation to interpret a multiplication problem

Janet and the students have discussed the products $12 \cdot 10$ and $10 \cdot 12$, and the students have come to the recognition that the products are the same. David says it is because “the numbers have simply changed places” (turn 10). Then Janet provides another example:[2]

11. Janet: We will get the same answer, it’s just the calculation that is reversed… If we take another arithmetic problem, will that be similar, too? $13 \cdot 17$ and $17 \cdot 13$ [writes on the blackboard]. [Pause 7 sec.]. Will this be the same, or are they different? [Pause 11 sec.]. What do you think? Do you think it will be the same answer, or are they different?
12. Brian: I think it will be the same, because you have just exchanged the numbers.
13. Janet: You think it will be the same? Mary, do you think it will be like Brian said?
14. Mary: Yes.
15. Janet: I don’t know whether you have done this before, made a story or a drawing. Is there anyone who would try to make a story for $13 \cdot 17$, if we just concentrate on this [product]? Can someone make a story or drawing that might explain $13 \cdot 17$? Is there anyone who dares to do that? [Pause 5 sec.].
16. Janet: What does it mean? Could it mean that we have 13 of something that we shall have 17 times? If we imagine having a baking tray with muffins for instance. If we imagine having a baking tray [draws on the blackboard]. Can someone try to figure out how the drawing will be, if we have a baking tray with $13 \cdot 17$ muffins? [Pause 3 sec.].
17. Janet: Where should we place 13 for instance? Should we just draw them all over the place, or should we place them across or down? Anyone who dares to try? Trying is allowed. Remember, no answer is silly. [Pause 5 sec.].
18. Janet: Nobody dares to try? Well, OK. If we imagine that we have 13 muffins across here [draws on the blackboard], and we have 17 down. We fill out the whole tray, but I don’t bother to draw them all. You understand that we have 13 across and 17 down. If we were to calculate this instead of counting all the muffins, how could we do that? You may want to take $13 \cdot 17$. Then we can think that we have 13 across and 17 down [points at the blackboard]. However, if we had $17 \cdot 13$ [points at the blackboard], can someone figure out what the drawing would look like? Carl?
19. Carl: It will be 17 across and 13 down.
20. Janet: Uh-huh. Anne, can you repeat what Carl said?
21. Anne: It will be 17 across and 13 down.
22. Janet: Yes, we would have had 17 here and 13 down [points at the blackboard and explains]. Have we changed how many muffins we have on the tray? [Tim shakes his head].

In turn 15, Janet invites the students to give an interpretation of the product $13 \cdot 17$. Nobody responds, after which Janet (turn 16) introduces multiplication in terms of equivalent groups: $13 \cdot 17$ is explained as the number of objects we will get when “we have 13 of something that we shall have 17 times”. This is a non-commutative situation, where 17 is the multiplicator and 13 is the multiplicand.

Then there is a shift to a rectangular-array situation, when Janet introduces a context of muffins on a baking tray to interpret the product $13 \cdot 17$ (turns 16-17). In turn 18 she explains how the product can be placed on the tray: 13 muffins across and 17 down. She says that the whole tray should be filled out, but draws only the first row and first column. The resulting diagram is reproduced in Figure 2, and we will refer to it as a “degenerated” array. With a proper (13x17)-array, it would have been possible to interpret the multiplication problem as an equivalent-groups situation in correspondence

[2] The transcript has been translated into English by the authors. Names are pseudonyms.

with Janet's initial explanation of multiplication: 13 muffins in a row could be interpreted as a group, and 17 rows could be interpreted as equivalent groups of 13 muffins. But the degenerated array and Janet's use of spontaneous concept ("across" and "down") instead of the scientific concepts "row" and "column", makes it unclear how the presented situation should be interpreted as the product $13 \cdot 17$.

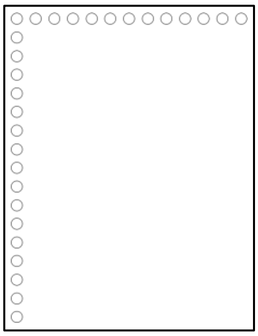

Figure 2. The diagram used by Janet to illustrate the product $13 \cdot 17$

The operation aimed at is just declared by Janet (turn 18): "You may want to take $13 \cdot 17$". When she asks how it would be if they had $17 \cdot 13$, Carl gives the desired answer (turn 19, repeated by Anne in turn 21). Nevertheless, this does not imply that Carl has understood what $17 \cdot 13$ means—it indicates only that he is able to substitute the numbers used by Janet. The diagram in Figure 2 does in fact represent two numbers (one across and another down), but the diagram does not represent the operation of *multiplying* these numbers.

Justifying that multiplication is commutative in $\mathbb{N}$

After having indicated how the products $13 \cdot 17$ and $17 \cdot 13$ should be interpreted as (degenerated) array situations in terms of muffins on a tray (as presented above), Janet sets out to justify that multiplication is commutative for all numbers in $\mathbb{N}$:

23. Janet: When we have $17 \cdot 13$, the tray would look like this, and if we have $13 \cdot 17$, we can just imagine that we rotate the tray. Then the arithmetic problem will be different. We may also think that we have a sheet of paper. If we imagine having $13 \cdot 17$ like this, and $17 \cdot 13$ like this [demonstrates on the sheet]. Then we can see that these arithmetic problems will be the same.

Janet uses the products $13 \cdot 17$ and $17 \cdot 13$—represented as drawings of muffins on a tray—to exemplify that multiplication is commutative. In turn 23, she refers to these products as being different arithmetic problems. The imagined rotation of the tray (possibly 90 degrees) is used to show that the arithmetic problems have the same result, the reasoning being that the rotation of the tray does not change the total number of muffins—hence $13 \cdot 17 = 17 \cdot 13$. We interpret the sheet mentioned in turn 23 as a representation enabling Janet to actually show the rotation and its effect on the arrangement of the muffins (a feature not afforded by the representation on the blackboard).

Having established that $13 \cdot 17 = 17 \cdot 13$, Janet then asks whether this property applies for all numbers:

29. Janet: How do you think it will be? Does it apply only for these numbers, or does it apply for all numbers? When we multiply two things… [Pause 5 sec.]. Mary?

[Mary says that she thinks that it applies for all numbers, and exemplifies by $1 \cdot 2$ and $2 \cdot 1$]

33. Janet: Uh-huh. Do you think it applies for all numbers, all whole numbers? [Pause 4 sec.]. Or are there numbers for which it doesn't apply? [Pause 5 sec.].

Several students respond that they think it applies for all numbers, and Janet asks why they think so.

37. Mary: I think it has to apply for all numbers. Because it's about the same [pair of] numbers.
38. Brian: It can be a little demanding when you have very large numbers, like 1 million times 2 millions. It will be challenging to draw.
39. Janet: Uh-huh. Well, it will indeed be much to draw if we were to draw a million. But if we imagine that we take away all the muffins. If we imagine that we have only one sheet of paper [erases the muffins on the blackboard drawing]. We can imagine that we have 1 million times 2 millions, then we can place it like this [points at an array-model on the blackboard]. So, does anyone dare to formulate a rule for multiplying two numbers. When we use what we have just seen, which applies on those [points at the blackboard drawing]. [Pause 10 sec.]
40. Mary: It will be the same if we swap the numbers.
41. Brian: It is possible also to check out with this tray in case one is insecure.
42. Janet: Uh-huh. A rule can be that, when we do multiplication problems, the order does not matter. Whether we take $13 \cdot 17$ or $17 \cdot 13$ does not matter. We can see this [property] if we make such a drawing. If we rotate the drawing, the [total] number has not been changed, we just rotate the drawing.

The multiplication problem $13 \cdot 17$ is used as a generic example in the dialogue to justify the commutative property of multiplication in $\mathbb{N}$. The property that the factors in a multiplication problem commute is based on the idea of rotating a tray (or sheet) with muffins arranged in a rectangular array—this is Janet's intention, even if the diagram used is not a proper array. The generic properties of the example are, however, vaguely expressed: Janet suggests that the total number of muffins on the tray is not changed by a rotation (turns 23 and 42), but she does not express in clear text what the commutative property means in the actual situation (i.e., exchanging row and columns). When Brian (turn 38) provides an example that involves the product 1 million times 2 millions, it can be considered a crucial experiment (supplemented by Janet in turn 39): the validity of the conjecture of commutativity is accomplished by testing it on an instance that is quite complex (and impossible to draw). In turn 42, Janet utilizes the generic example of $13 \cdot 17$ when she articulates the conclusion of the claim—an important, last step in a justification process.

Discussion

The decision not to draw all the muffins (possibly because it would take too long) prevents the diagram in Figure 2 from representing the *meaning of the operation* at stake (even if Janet says that the whole tray should be filled out). Hence, Schifter's (2009) first criterion for a representation to be adequate is not met. It can be noticed that the other representation used, the sheet, does neither illustrate the meaning of multiplication, but it affords the rotation to be demonstrated physically. For the meaning of multiplication to be represented in a diagram, the total number of objects—the result of the operation—needs to be displayed. This entails that the numbers involved must be of manageable size, thus enabling them to be represented in diagrams or manipulatives. That Janet failed to draw the complete array indicates that the numbers she used in the generic example (13 and 17) were too big, as she possibly conceived of it.

It is possible to represent any pair of natural numbers in the diagram used by Janet, and hence, it seems as if Schifter's (2009) second criterion is met. Yet, this is irrelevant since the meaning of the operation is not represented in the diagram. Further, since the diagram does not represent multiplication at the outset, it is useless to check if Schifter's third criterion is met (i.e., whether the conclusion of the claim follows from the structure of the diagram).

The discussion in class (based on Figure 2) enables the students to evidence possession of some knowledge. This knowledge is, however, different from the knowledge aimed at by Janet: The students were able to say that the result—in the general case—would be the same even if the numbers in the multiplication problem were reversed. Yet, there is no indication that the result they refer to is the *product* of the two numbers. It is likely that the students imagine a diagram with objects in a formation similar to the one in Figure 2, and that they see that rotation does not change the total number of objects in the diagram. This is basically an aspect of the principle of number conservation, and it is doubtful whether the students have understood why multiplication is commutative for any pair of natural numbers, which was the aim of the lesson.

In conclusion, there are two conditions that constrain Janet's discussion with the students about commutativity of multiplication in $\mathbb{N}$: The first is that the diagram, as used by Janet, is inadequate because it does not illustrate the meaning of multiplication. The second is the matter of fact that the diagram (and the way it is used) enables the students to respond "adequately" (i.e., as expected by Janet), even if they may not have understood why the commutative property applies for multiplication. The latter is a constraint because it deprives Janet of the opportunity to get feedback that might have let her understand that she would need to change her approach.

The case of Janet can be used in teacher education to discuss with pre-service teachers criteria for, and impact of, generic examples (or representation-based proofs) used to justify general claims about properties of arithmetic operations. It is relevant to extend the research reported here by analysing written material from students' justification of properties of arithmetic operations.

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