SHARP NORM ESTIMATES FOR COMPOSITION OPERATORS AND HILBERT-TYPE INEQUALITIES

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Abstract. Let $\mathscr{H}^2$ denote the Hardy space of Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ with square summable coefficients and suppose that φ is a symbol generating a composition operator on $\mathscr{H}^2$ by $\mathscr{C}_\varphi(f) = f\circ\varphi$. Let ζ denote the Riemann zeta function and $\alpha_0 = 1.48\ldots$ the unique positive solution of the equation $\alpha\zeta(1+\alpha) = 2$. We obtain sharp upper bounds for the norm of $\mathscr{C}_\varphi$ on $\mathscr{H}^2$ when $0 < \operatorname{Re}\varphi(+\infty) - 1/2 \leq \alpha_0$, by relating such sharp upper bounds to the best constant in a family of discrete Hilbert-type inequalities.

1. Introduction

Let $0 < \alpha < \infty$. The main object of study in the present paper is the family of discrete bilinear forms

$$B_\alpha(a,b) := \sum_{m=1}^\infty \sum_{n=1}^\infty a_m b_n \frac{(mn)^{\alpha-1/2}}{[\max(m,n)]^{2\alpha}}. \tag{1}$$

Let $\|B_\alpha\|$ denote the norm of the bilinear form (1) on $\ell^2\times\ell^2$, that is the smallest positive number C_α such that

$$|B_\alpha(a,b)| \leq C_\alpha \|a\|_{\ell^2}\|b\|_{\ell^2}$$

holds for every pair of sequences $a,b\in\ell^2$. Our interest in $\|B_\alpha\|$ stems from its connection to sharp norm estimates for composition operators. For the moment, we postpone the discussion of this connection and focus on (1). Let

$$\zeta(s) := \sum_{n=1}^\infty n^{-s}$$

denote the Riemann zeta function and let $\alpha_0 = 1.48\ldots$ denote the unique positive solution of the equation $\alpha\zeta(1+\alpha) = 2$. We have been unable to compute $\|B_\alpha\|$ for every $0<\alpha<\infty$, but we can prove the following result.

Theorem 1. *For $0<\alpha<\infty$,*

$$\max\left(\frac{2}{\alpha},\, \zeta(1+2\alpha)\right) \leq \|B_\alpha\| \leq \max\left(\frac{2}{\alpha},\, \zeta(1+\alpha)\right).$$

In particular, if $0<\alpha\leq\alpha_0 = 1.48\ldots$, then $\|B_\alpha\| = 2/\alpha$.

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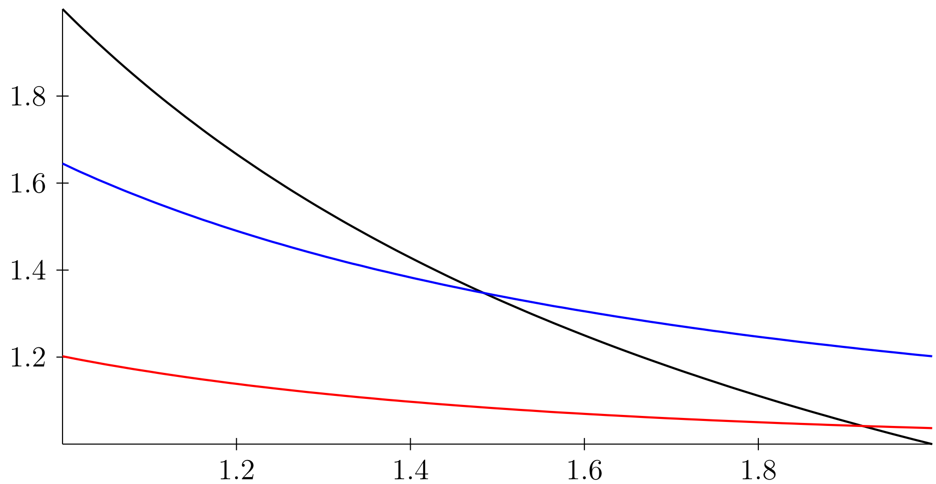

FIGURE 1. $2/\alpha$, $\zeta(1+\alpha)$ and $\zeta(1+2\alpha)$ for $1 \leq \alpha \leq 2$.

See Figure 1 for a graph of the functions $2/\alpha$, $\zeta(1+\alpha)$ and $\zeta(1+2\alpha)$ on the interval $[1,2]$, where both intersections occur. Theorem 1 is presented as a combination of several results. The upper bound comes from Lemma 8. The first part of the lower bound is Lemma 7, while the second is obtained by the point estimate (5) and Theorem 2.

Our approach to the bilinear form (1) and to Theorem 1 is classical. We will exploit that the positive and symmetric kernels

$$K_\alpha(x,y) = \frac{(xy)^{\alpha-1/2}}{[\max(x,y)]^{2\alpha}} \tag{2}$$

enjoy the homogeneity property $K_\alpha(\lambda x, \lambda y) = \lambda^{-1} K_\alpha(x,y)$ for $x, y, \lambda > 0$. Hence (1) is a Hilbert–type bilinear form as studied by Hardy, Littlewood and Pólya [4, Ch. IX]. In fact, $\alpha = 1/2$ in Theorem 1 is [4, Thm. 341]. Note also that the result in [2] covers the case $\alpha = 1$.

We first investigate the continuous version of the discrete bilinear form (1). As expected, it is easy to show that the norm of the continuous bilinear form is $2/\alpha$ for every $0 < \alpha < \infty$ (see Theorem 6). Inspired by [4], we aim to use discretization to obtain a sharp result for (1). This approach is successful when $0 < \alpha \leq 1$. In fact, the upper bound in Theorem 1 can be deduced directly from [4, Thm. 318] in this range.

A phase change occurs at $\alpha = 1$, and the discretization argument gives here an upper bound which (from its application) clearly cannot be sharp. The main point of Theorem 1 is therefore that the upper bound obtained by discretization for $0 < \alpha \leq 1$ extends beyond the phase change at $\alpha = 1$ to (at least) $\alpha_0 = 1.48\ldots$. Note that the upper bound $2/\alpha$ cannot hold when $\alpha \geq 2$, since it would contradict that $\|B_\alpha\| > 1$. In fact, we have verified that the upper bound $2/\alpha$ fails when $\alpha \geq 1.7$ (see Section 5).

To set the stage for the discussion of the relationship between B_α and composition operators, let $H^2(\mathbb{D})$ denote the Hardy space of the unit disc $\mathbb{D} := \{z\,:\,|z|<1\}$, consisting of analytic functions $F(z)=\sum_{k\geq 0} a_k z^k$ with square summable coefficients,

$$\|F\|_{H^2(\mathbb{D})} := \left(\sum_{k=0}^{\infty} |a_k|^2\right)^{1/2} = \lim_{r\to 1^-}\left(\int_0^{2\pi} |F(re^{i\theta})|^2\,\frac{d\theta}{2\pi}\right)^{\frac{1}{2}}.$$

For the following basic facts about Hardy spaces of the unit disc and their composition operators, we refer to [11, Ch. 11]. Suppose that $\phi\colon \mathbb{D}\to\mathbb{D}$ is analytic. Littlewood's subordination principle [6] gives the following upper bound for the norm of the composition operator defined by $\mathscr{C}_\phi(F) = F\circ\phi$:

$$\|\mathscr{C}_\phi\|_{H^2(\mathbb{D})\to H^2(\mathbb{D})} \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}. \tag{3}$$

From the functional of point evaluation, we obtain the following lower bound:

$$\|\mathscr{C}_\phi\|_{H^2(\mathbb{D})\to H^2(\mathbb{D})} \geq \sqrt{\frac{1}{1-|\phi(0)|^2}}. \tag{4}$$

Both (3) and (4) are sharp for any value $\phi(0)=w\in\mathbb{D}$. Indeed, for the lower bound, take $\phi(z)=w$, and for the upper bound, take the Möbius transform

$$\phi(z) = \frac{w-z}{1-\overline{w}z}.$$

In general, the computation of $\|\mathscr{C}_\phi\|_{H^2(\mathbb{D})\to H^2(\mathbb{D})}$ is difficult, but it follows from (3) and (4) that the composition operator $\mathscr{C}_\phi$ is a contraction on $H^2(\mathbb{D})$ if and only if $\phi(0)=0$.

We will use Theorem 1 to obtain sharp norm estimates for composition operators on $\mathscr{H}^2$, the Hardy space of Dirichlet series $f(s)=\sum_{n\geq 1} a_n n^{-s}$ with square summable coefficients,

$$\|f\|_{\mathscr{H}^2} := \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}.$$

The basic properties of $\mathscr{H}^2$ can be found in [5, 9]. Using the Cauchy–Schwarz inequality, it is easy to see that $\mathscr{H}^2$ is a space of absolutely convergent Dirichlet series in the half-plane $\mathbb{C}_{1/2}$, where

$$\mathbb{C}_\theta := \{s\,:\,\mathrm{Re}(s)>\theta\}.$$

To see that $\mathbb{C}_{1/2}$ is generally the largest domain of convergence for functions in $\mathscr{H}^2$, consider $f(s)=\zeta(1/2+\varepsilon+s)$ for $\varepsilon>0$.

The study of composition operators on $\mathscr{H}^2$ was initiated by Gordon and Hedenmalm in their pioneering paper [3] (see also [8, 10]), where they proved that an analytic function $\varphi\colon \mathbb{C}_{1/2}\to\mathbb{C}_{1/2}$ generates a composition operator on $\mathscr{H}^2$ if and only if it is a member of the following class.

Definition. The *Gordon–Hedenmalm* class, $\mathscr{G}$, consists of symbols of the form

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \varphi_0(s),$$

where c_0 is a non-negative integer. The series φ_0 converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and satisfies the following mapping properties:

(a) If $c_0 = 0$, then $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$ then $\varphi_0 \equiv 0$ or $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_0$.

Observe that the case (b), which is when $\varphi(+\infty) = +\infty$, corresponds to $\phi(0) = 0$ considered above. Indeed, it was shown in [3] that composition operators generated by symbols with $c_0 \geq 1$ are contractive. Considering the case (a), the functional of point evaluation for $\mathscr{H}^2$ (see [3, 5]) gives that

$$\|\mathscr{C}_\varphi\| \geq \sqrt{\zeta(2\operatorname{Re}(c_1))}, \tag{5}$$

since $c_1 = \varphi(+\infty)$. The lower bound (5) is sharp for $\varphi(s) = c_1$. Our initial motivation for studying B_α was to investigate sharp upper bounds analogous to (3) for composition operators on $\mathscr{H}^2$ for the case (a), something which has been unresolved since [3]. Here is the main result of the present paper.

Theorem 2. *Fix $w \in \mathbb{C}_{1/2}$ and let $\alpha = \operatorname{Re}(w) - 1/2$. Then*

$$\sup_{\substack{\varphi \in \mathscr{G} \\ \varphi(+\infty) = w}} \|\mathscr{C}_\varphi\| = \sqrt{\|B_\alpha\|}.$$

The proof of Theorem 2 is obtained by combining observations and ideas from [1, 2, 3]. Our desired sharp upper bound for composition operators is easily deduced from Theorem 1 and Theorem 2.

Corollary 3. *Let $\alpha_0 = 1.48\ldots$ denote the unique positive solution to the equation $\alpha\zeta(1+\alpha) = 2$. Suppose that φ is in $\mathscr{G}$ with $c_0 = 0$ and that $0 < \operatorname{Re}(c_1) - 1/2 \leq \alpha_0$. Then*

$$\|\mathscr{C}_\varphi\| \leq \sqrt{\frac{2}{\operatorname{Re}(c_1) - 1/2}}, \tag{6}$$

Moreover, for every $0 < \operatorname{Re}(c_1) - 1/2 \leq \alpha_0$, there are φ in $\mathscr{G}$ attaining (6).

The present paper is organized as follows. The proof of Theorem 2 is presented in Section 2. The following two sections are devoted to the bilinear form (1) and Theorem 1. In Section 3 we follow [4] and investigate the continuous version of (1). Here we also obtain the lower bound in Theorem 1 for all $0 < \alpha < \infty$ and the upper bound when $0 < \alpha \leq 1$ and $\alpha \geq 3$. Section 4 contains the proof of the upper bound in Theorem 1 in the most intricate cases $1 < \alpha < 2$ and $2 \leq \alpha < 3$. Finally, Section 5 contains a few remarks pertaining to the relationship between composition operators on $H^2(\mathbb{D})$ and $\mathscr{H}^2$. Also found in Section 5 are some observations regarding Theorem 1 for $\alpha > \alpha_0$ and two interesting or appealing special cases of (1).

2. Proof of Theorem 2

For fixed $0 < \alpha < \infty$, the conformal map

$$\mathcal{T}_\alpha(z) := \alpha \frac{1-z}{1+z} \tag{7}$$

sends $\mathbb{D}$ to $\mathbb{C}_0$. Let $\mathcal{S}_\theta(s) = s + \theta$, and define $H^2_{\mathrm{i}}(\mathbb{C}_\theta, \alpha)$ as the space of analytic functions in $\mathbb{C}_\theta$ such that $f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha$ is in $H^2(\mathbb{D})$, and set

$$\|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_\theta,\,\alpha)} := \|f \circ \mathcal{S}_\theta \circ \mathcal{T}_\alpha\|_{H^2(\mathbb{D})} = \left(\frac{\alpha}{\pi}\int_{-\infty}^{\infty} |f(\theta+it)|^2 \frac{dt}{\alpha^2+t^2}\right)^{1/2}. \tag{8}$$

For fixed θ and varying α, the norms (8) are equivalent, but not equal. This means that the space $H^2_{\mathrm{i}}(\mathbb{C}_\theta,\,\alpha)$ does not depend on the parameter α.

We are now ready to begin with the proof of Theorem 2. Note that the first statement of the following lemma can be found in [3], but we include a short proof for the reader's benefit.

Lemma 4. *Let $\varphi \in \mathscr{G}$ with $c_0 = 0$ and $\varphi(+\infty) = c_1 > 1/2$. If $\alpha = c_1 - 1/2$ and $f \in \mathscr{H}^2$, then*

$$\|\mathscr{C}_\varphi f\|_{\mathscr{H}^2} \le \|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)}. \tag{9}$$

Moreover, for every $c_1 > 1/2$ there are $\varphi \in \mathscr{G}$ with $\varphi(+\infty) = c_1$ such that equality in (9) is attained simultaneously for all $f \in \mathscr{H}^2$.

Proof. A computation (or [8, Thm. 2.31]) shows that if $g \in \mathscr{H}^2$ converges uniformly in $\mathbb{C}_0$, then

$$\|g\|_{\mathscr{H}^2} = \lim_{\beta\to\infty} \|g\|_{H^2_{\mathrm{i}}(\mathbb{C}_0,\,\beta)}.$$

In particular, if f is a Dirichlet polynomial and φ is in $\mathscr{G}$ with $c_0 = 0$, then by (8) we get that

$$\|\mathscr{C}_\varphi f\|_{\mathscr{H}^2} = \lim_{\beta\to\infty} \|f\circ\varphi\|_{H^2_{\mathrm{i}}(\mathbb{C}_0,\,\beta)} = \lim_{\beta\to\infty} \|f\circ\varphi\circ\mathcal{T}_\beta\|_{H^2(\mathbb{D})}. \tag{10}$$

Define $F \in H^2(\mathbb{D})$ and $\phi\colon \mathbb{D} \to \mathbb{D}$ by

$$\begin{aligned} F &:= f \circ \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha \\ \phi &:= \mathcal{T}_\alpha^{-1} \circ \mathcal{S}_{1/2}^{-1} \circ \varphi \circ \mathcal{T}_\beta \end{aligned}$$

for some $0 < \alpha < \infty$ to be decided later. It now follows from (3) that

$$\|f\circ\varphi\circ\mathcal{T}_\beta\|_{H^2(\mathbb{D})} = \|F\circ\phi\|_{H^2(\mathbb{D})} \le \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}\|F\|_{H^2(\mathbb{D})}. \tag{11}$$

We compute

$$\lim_{\beta\to\infty} \phi(0) = \lim_{\beta\to\infty} \mathcal{T}_\alpha^{-1}(\varphi(\beta) - 1/2) = \mathcal{T}_\alpha^{-1}(c_1 - 1/2) = 0,$$

where the final equality is obtained by choosing $\alpha = c_1 - 1/2$. Combining (10) and (11) we find that

$$\|\mathscr{C}_\varphi f\|_{\mathscr{H}^2} \leq \|F\|_{H^2(\mathbb{D})} = \|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)}.$$

For a general $f \in \mathscr{H}^2$, we prove (9) by approximating f by a sequence of Dirichlet polynomials. The convergence on the right hand side is then justified by [5, Thm. 4.11].

To see that φ can be chosen to attain equality in (9) for every $f \in \mathscr{H}^2$, we follow an observation from [1] (which in turn was inspired by the transference principle from [10]) and consider the symbol defined by

$$\varphi_\alpha(s) := \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha(2^{-s}) = \frac{1}{2} + \alpha\frac{1-2^{-s}}{1+2^{-s}}.$$

Clearly, $c_1 = 1/2 + \alpha$ as required. Observe now that the subspace $\mathscr{X}$ of $\mathscr{H}^2$ consisting of Dirichlet series of the form

$$f(s) = \sum_{k=0}^\infty a_{2^k} 2^{-ks}$$

is isometrically isometric to $H^2(\mathbb{D})$, through the map $2^{-s} \mapsto z$. In particular, since $\mathscr{C}_{\varphi_\alpha}$ maps $\mathscr{H}^2$ into $\mathscr{X}$, we find that

$$\|\mathscr{C}_{\varphi_\alpha} f\|_{\mathscr{H}^2} = \|f \circ \varphi_\alpha\|_{\mathscr{X}} = \|f \circ \mathcal{T}_\alpha\|_{H^2(\mathbb{D})} = \|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)},$$

so equality in (9) is attained for φ_α. □

We are now ready for the second part of the proof of Theorem 2, which relies on an idea from [2].

Lemma 5. *For $0 < \alpha < \infty$, let C_α denote the optimal constant in the embedding*

$$\|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)} \leq C_\alpha \|f\|_{\mathscr{H}^2}.$$

Then $C_\alpha = \sqrt{\|B_\alpha\|}$ where B_α is the bilinear form (1).

Proof. Let $x > 0$. As in [2], we begin by computing the integral

$$I_\alpha(x) := \frac{\alpha}{\pi}\int_{-\infty}^\infty x^{it}\,\frac{dt}{\alpha^2+t^2} = \frac{1}{[\max(x,1/x)]^\alpha}.$$

We insert a Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ in (8) with $\theta = 1/2$ and compute

$$\|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)} = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m\overline{a_n}}{\sqrt{mn}}\,I_\alpha(n/m) = \sum_{m=1}^\infty \sum_{n=1}^\infty a_m\overline{a_n}\frac{(mn)^{\alpha-1/2}}{[\max(m,n)]^{2\alpha}}.$$

Since the matrix associated to the bilinear form (1) is real and symmetric, it is self-adjoint. This means that the norm is attained by considering only $b = \overline{a}$. Hence we have obtained the sharp estimate

$$\|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)} \leq \|B_\alpha\| \|f\|^2_{\mathscr{H}^2},$$

as desired. □

Final part in the proof of Theorem 2. Fix $w \in \mathbb{C}_{1/2}$. The norm of $\mathscr{H}^2$ is invariant under vertical translations, so we get from Lemma 4 that

$$\sup_{\substack{\varphi\in\mathscr{G}\\ \varphi(+\infty)=w}} \|\mathscr{C}_\varphi\| = \sup_{\substack{\varphi\in\mathscr{G}\\ \varphi(+\infty)=\mathrm{Re}(w)}} \|\mathscr{C}_\varphi\| = \sup_{\|f\|_{\mathscr{H}^2}=1} \|f\|_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2},\,\alpha)}$$

with $\alpha = \mathrm{Re}(w) - 1/2$. We complete the proof by using Lemma 5. $\square$

3. Continuous bilinear forms and Riemann sums

As explained in the introduction, we will initiate our study of (1) by investigating the continuous version. Let K_α be as in (2) and consider

$$H_\alpha(f,g) := \int_0^\infty \int_0^\infty K_\alpha(x,y) f(x) g(y)\, dydx.$$

We have the following result about the norm of H_α on $L^2(0,\infty)$.

Theorem 6. *Let $\alpha > 0$. Then $\|H_\alpha\| = 2/\alpha$, that is the sharp estimate*

$$|H_\alpha(f,g)| \leq \frac{2}{\alpha}\|f\|_{L^2}\|g\|_{L^2}$$

holds for every pair of functions $f, g \in L^2(0,\infty)$.

Proof. Following [4, Ch. IX], we apply the Cauchy–Schwarz inequality with weights $\sqrt{x/y}$ and $\sqrt{y/x}$ to find that

$$|H_\alpha(f,g)| \leq \left(\int_0^\infty |f(x)|^2\sqrt{x}\int_0^\infty K_\alpha(x,y)\,\frac{dy}{\sqrt{y}}\,dx\right)^{1/2} \times \left(\int_0^\infty |g(y)|^2\sqrt{y}\int_0^\infty K_\alpha(x,y)\,\frac{dx}{\sqrt{x}}\,dy\right)^{1/2}.$$

We then use a substitution and the homogeneity property to conclude that an upper bound for $\|H_\alpha\|$ is

$$\sqrt{x}\int_0^\infty K_\alpha(x,y)\frac{dy}{\sqrt{y}} = \sqrt{x}\int_0^\infty K_\alpha(x,xy)\frac{x\,dy}{\sqrt{xy}} = \int_0^\infty K_\alpha(1,y)\,\frac{dy}{\sqrt{y}} =: C_\alpha.$$

We then easily compute

$$C_\alpha = \int_0^1 y^{\alpha-1}\,dy + \int_1^\infty y^{-\alpha-1}\,dy = \frac{1}{\alpha} + \frac{1}{\alpha} = \frac{2}{\alpha}. \tag{12}$$

To prove optimality, let $0 < \varepsilon < \alpha$ and set

$$f(t) = g(t) = \begin{cases} 0, & t \in (0,1), \\ t^{-1/2-\varepsilon}, & t \in (1,\infty). \end{cases}$$

A direct computation gives that

$$H_\alpha(f,g) = \int_1^\infty \int_1^\infty \frac{(xy)^{\alpha-1-\varepsilon}}{[\max(x,y)]^{2\alpha}}\,dydx = \left(\frac{1}{\alpha-\varepsilon} + \frac{1}{\alpha+\varepsilon}\right)\|f\|_{L^2}^2 + O(1).$$

Clearly $\|f\|_{L^2} \to \infty$ as $\varepsilon \to 0^+$, so we find that

$$\|H_\alpha\| \geq \lim_{\varepsilon\to 0^+} \left(\frac{1}{\alpha-\varepsilon} + \frac{1}{\alpha+\varepsilon}\right) = \frac{2}{\alpha}. \qquad \square$$

We proceed by showing that the lower bound from the continuous setting carries across to $\|B_\alpha\|$.

Lemma 7. *For $0 < \alpha < \infty$, we have that $\|B_\alpha\| \geq 2/\alpha$.*

Proof. Set $a_m = m^{-1/2-\varepsilon}$ and $b_n = n^{-1/2-\varepsilon}$ for some $0 < \varepsilon < \alpha$. We get

$$\|B_\alpha\| \geq \frac{1}{\zeta(1+2\varepsilon)} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(mn)^{\alpha-1-\varepsilon}}{[\max(m,n)]^{2\alpha}} =: \frac{D_\alpha}{\zeta(1+2\varepsilon)}. \tag{13}$$

Now,

$$D_\alpha = \sum_{m=1}^\infty m^{-\alpha-1-\varepsilon} \sum_{n=1}^m n^{\alpha-1-\varepsilon} + m^{\alpha-1-\varepsilon} \sum_{n=m+1}^\infty n^{-\alpha-1-\varepsilon}.$$

Standard computations shows that

$$\sum_{n=1}^m n^{\alpha-1-\varepsilon} = \frac{n^{\alpha-\varepsilon}}{\alpha-\varepsilon} + O(n^{\alpha-1}),$$
$$\sum_{n=m+1}^\infty n^{-\alpha-1-\varepsilon} = \frac{n^{-\alpha-\varepsilon}}{\alpha+\varepsilon} + O(n^{-\alpha-1}),$$

so we get that

$$D_\alpha = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(mn)^{\alpha-1-\varepsilon}}{[\max(m,n)]^{2\alpha}} = \zeta(1+2\varepsilon)\left(\frac{1}{\alpha-\varepsilon} + \frac{1}{\alpha+\varepsilon}\right) + O(1).$$

We insert this into (13) and let $\varepsilon \to 0^+$ to complete the proof. $\square$

To investigate upper bounds for $\|B_\alpha\|$, we use the same weighted Cauchy–Schwarz inequality as in the proof of Theorem 6 to get

$$|B_\alpha(a,b)| \leq \left(\sum_{m=1}^\infty |a_m|^2 \sqrt{m} \sum_{n=1}^\infty \frac{(mn)^{\alpha-1/2}}{[\max(m,n)]^{2\alpha}} \sqrt{\frac{1}{n}}\right)^{1/2} \times \left(\sum_{n=1}^\infty |b_n|^2 \sqrt{n} \sum_{m=1}^\infty \frac{(mn)^{\alpha-1/2}}{[\max(m,n)]^{2\alpha}} \sqrt{\frac{1}{m}}\right)^{1/2}.$$

By symmetry, we obtain the upper bound $\|B_\alpha\| \leq \sup_m S_\alpha(m)$, where

$$S_\alpha(m) := \sqrt{m} \sum_{n=1}^\infty \frac{(mn)^{\alpha-1/2}}{[\max(m,n)]^{2\alpha}} \sqrt{\frac{1}{n}}. \tag{14}$$

Our next goal is to prove the following result, which by the preceding discussion constitutes the upper bound in Theorem 1.

Lemma 8. *Let* $0 < \alpha < \infty$ *and let* S_α *be as in* (14)*. Then*

$$\sup_m S_\alpha(m) = \max\left(\frac{2}{\alpha}\,,\,\zeta(1+\alpha)\right).$$

The first step towards the proof of Lemma 8 is to rewrite (14) as

$$\begin{aligned} S_\alpha(m) &= m^{-\alpha}\sum_{n=1}^{m} n^{\alpha-1} + m^{\alpha}\sum_{n=m+1}^{\infty} n^{-\alpha-1} \\ &= \frac{1}{m}\sum_{n=1}^{m}\left(0+\frac{n}{m}\right)^{\alpha-1} + \sum_{j=1}^{\infty}\frac{1}{m}\sum_{n=1}^{m}\left(j+\frac{n}{m}\right)^{-\alpha-1}, \end{aligned} \tag{15}$$

to see that $S_\alpha(m)$ is a Riemann sum of the two integrals in (12) with step length m^{-1}, taking the value at the upper endpoint for each interval. Hence we conclude that

$$\lim_{m\to\infty} S_\alpha(m) = \frac{2}{\alpha}. \tag{16}$$

Note also that $S_\alpha(1) = \zeta(1+\alpha)$. Hence Lemma 8 states that the first or the "last" element of the sequence S_α is always the biggest. The fact that $S_\alpha(m)$ are Riemann sums of (12) directly gives the following simple proof.

Proof of Lemma 8: $0 < \alpha \leq 1$. If $0 < \alpha \leq 1$, then $y \mapsto y^{\alpha-1}$ and $y \mapsto y^{-\alpha-1}$ are decreasing functions on $(0,1)$ and $(1,\infty)$, respectively. This means that both sums in (15) are increasing sequences, and the limit (16) is also the supremum of the combined sequence. □

Note that when $1 < \alpha < \infty$, the function $y \mapsto y^{\alpha-1}$ is increasing on $(0,1)$ so we can no longer take the limit and obtain the supremum. This is the phase change mentioned in the introduction. Nevertheless, when α is large enough, we may conclude by rather savage estimates.

Proof of Lemma 8: $3 \leq \alpha < \infty$. Consider the sums in (15). We know that if $\alpha > 1$, then the first sum is decreasing and the second sum is increasing. In particular, if $m \geq 2$, then

$$S_\alpha(m) \leq 2^{-\alpha}\left(1+2^{\alpha-1}\right) + \frac{1}{\alpha} = 2^{-\alpha} + 1/2 + \frac{1}{\alpha}.$$

Furthermore, $S_\alpha(1) = \zeta(1+\alpha) \geq 1 + 2^{-\alpha-1}$, so we get that $S_\alpha(1) \geq S_\alpha(m)$ whenever

$$1 + 2^{-\alpha-1} \geq 2^{-\alpha} + 1/2 + \frac{1}{\alpha} \qquad \Longleftrightarrow \qquad 1/2 - \frac{1}{\alpha} - 2^{-\alpha-1} \geq 0.$$

This final expression is clearly increasing in α and positive for $\alpha = 3$, so we conclude that $\sup_m S_\alpha(m) = \zeta(1+\alpha)$ when $\alpha \geq 3$. □

4. Proof of Lemma 8: $1 \leq \alpha \leq 3$

For fixed $\alpha > 1$, we do not know if the sequence $S_\alpha(m)$ is increasing or decreasing, since it is the sum of one increasing and one decreasing sequence. Our general approach is therefore to obtain decreasing upper bounds for $S_\alpha(m)$ when $m \geq 2$, which we then compare with $2/\alpha$ and $\zeta(1+\alpha)$.

To obtain these estimates, we will apply the Euler–Maclaurin summation formula (see [7, Ch. B]). In preparation, let us recall a few properties of Bernoulli polynomials, denoted $B_k(x)$. We will only have use of the first five polynomials, which are

$$B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \qquad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$$

To analyse the remainder terms in the Euler–Maclaurin summation formula, we make use of the following simple result.

Lemma 9. *If g is positive, continuous and decreasing on $[0,1]$, then*

$$\operatorname{sign}\left(\int_0^1 g(x)B_{2k+1}(x)\,dx\right) = (-1)^{k-1}.$$

Proof. The following well-known facts can be found in [7, Thm. B.1]:

$$\begin{aligned} B_{2k+1}(x) &= -B_{2k+1}(1-x) && (0 < x < 1),\\ \operatorname{sign}\big(B_{2k+1}(x)\big) &= (-1)^{k-1} && (0 < x < 1/2). \end{aligned}$$

The statement now follows from a symmetry consideration. □

The Bernoulli numbers are defined by $B_k := B_k(0)$. We will only use that $B_2 = 1/6$ and $B_4 = -1/30$. Let $\{x\}$ denote the fractional part of x. Suppose that $f \in C^{2k+1}\big([m,\infty)\big)$. For $m \geq 1$ and $k \geq 0$ we have that

$$\sum_{n=m+1}^{\infty} f(n) = \int_m^\infty f(x)\,dx - \frac{f(m)}{2} - \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(m) + R_k(m). \tag{17}$$

The remainder term in (17) is given by

$$R_k(m) = \frac{1}{(2k+1)!}\int_m^\infty f^{(2k+1)}(x)B_{2k+1}(\{x\})\,dx.$$

Similarly to (17), if $f \in C^{(2k+1)}\big([1,m]\big)$ it holds for $m \geq 1$ and $k \geq 0$ that

$$\begin{aligned} \sum_{n=1}^{m} f(n) &= \int_1^m f(x)\,dx + \frac{f(m)+f(1)}{2} \\ &\quad + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(m) - f^{(2j-1)}(1)\right) + \widetilde{R}_k(m). \end{aligned} \tag{18}$$

The remainder term in (18) is given by

$$\widetilde{R}_k(m) = \frac{1}{(2k+1)!}\int_1^m f^{(2k+1)}(x)B_{2k+1}(\{x\})\,dx.$$

We are now ready to obtain our estimates.

Lemma 10. *Let $0 < \alpha < \infty$. Then*

$$m^\alpha \sum_{n=m+1}^{\infty} n^{-\alpha-1} \le \frac{1}{\alpha} - \frac{1}{2m} + \frac{(\alpha+1)}{12m^2}, \tag{19}$$

$$\zeta(1+\alpha) \ge \frac{1}{\alpha} + \frac{1}{2} + \frac{(\alpha+1)}{12} - \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{720}. \tag{20}$$

If $1 \le \alpha \le 2$, then

$$\frac{1}{m^\alpha}\sum_{n=1}^{m} n^{\alpha-1} \le \frac{1}{\alpha} + \frac{1}{2m} + \frac{(\alpha-1)}{12m^2} - \frac{(\alpha-3)(\alpha-4)}{12\alpha}\frac{1}{m^\alpha}, \tag{21}$$

and if $2 \le \alpha \le 3$, then

$$\frac{1}{m^\alpha}\sum_{n=1}^{m} n^{\alpha-1} \le \frac{1}{\alpha} + \frac{1}{2m} + \frac{(\alpha-1)}{12m^2}. \tag{22}$$

Proof. To get (19) and (20) we apply (17) to $f(x) = x^{-\alpha-1}$ with $k = 1$ and $k = 2$, respectively. Note that for (20) we take $m = 1$. To verify the sign of the remainder term, we appeal to Lemma 9 for $g(x) = -f^{(2k+1)}(x)$.

To prove (21), we use (18) with $f(x) = x^{\alpha-1}$ and $k = 2$. To see that the remainder term is negative, we note that $g(x) = f^{(5)}(x)$ is positive and decreasing when $1 < \alpha < 2$ and use Lemma 9. Hence we get that

$$\begin{aligned} m^{-\alpha}\sum_{n=1}^{m} n^{\alpha-1} &\le \frac{1}{\alpha} + \frac{1}{2m} + \frac{(\alpha-1)}{12}\frac{1}{m^2} - \frac{(\alpha-1)(\alpha-2)}{720}\left(\frac{1}{m^4} - \frac{1}{m^\alpha}\right) \\ &\qquad + \left(-\frac{1}{\alpha} + \frac{1}{2} - \frac{(\alpha-1)}{12}\right)\frac{1}{m^\alpha} \\ &\le \frac{1}{\alpha} + \frac{1}{2m} + \frac{(\alpha-1)}{12}\frac{1}{m^2} - \frac{(\alpha-3)(\alpha-4)}{12\alpha}\frac{1}{m^\alpha}, \end{aligned}$$

where we in the second inequality used (twice) that $1 \le \alpha \le 2$ to conclude that the fourth term is negative.

Finally, for (22), we again use (18) with $f(x) = x^{\alpha-1}$ and $k = 1$. The remainder term is negative by Lemma 9, since $g(x) = -f^{(3)}(x)$ is positive and decreasing when $2 < \alpha < 3$. Hence

$$\frac{1}{m^\alpha}\sum_{n=1}^{m} \le \frac{1}{\alpha} + \frac{1}{2m} + \frac{(\alpha-1)}{12}\frac{1}{m^2} + \left(-\frac{1}{\alpha} + \frac{1}{2} - \frac{(\alpha-1)}{12}\right)\frac{1}{m^\alpha}.$$

The factor in front of $m^{-\alpha}$ is negative when $2 < \alpha < 3$. □

Proof of Lemma 8: $2 \leq \alpha \leq 3$. We want to prove that $S_\alpha(1) \geq S_\alpha(m)$. We combine (19) and (22) to get that if $m \geq 2$, then

$$S_\alpha(m) \leq \frac{2}{\alpha} + \frac{\alpha}{6m^2} \leq \frac{2}{\alpha} + \frac{\alpha}{24}.$$

Recall that $S_\alpha(1) = \zeta(1+\alpha)$. By (20) and the fact that $2 \leq \alpha \leq 3$, we get

$$\zeta(1+\alpha) \geq \frac{1}{\alpha} + \frac{1}{2} + \frac{(\alpha+1)}{12} - \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{720} \geq \frac{1}{\alpha} + \frac{2}{3}.$$

We complete the proof by checking that

$$\frac{2}{\alpha} + \frac{\alpha}{24} < \frac{1}{\alpha} + \frac{2}{3}$$

for $2 \leq \alpha \leq 3$. □

The proof of the following lemma is a straightforward calculus argument, but it is very tedious and therefore omitted.

Lemma 11. *For $1 \leq \alpha \leq 2$, consider the following functions*

$$h_1(\alpha) := \frac{(\alpha-3)(\alpha-4)}{12\alpha}\frac{1}{2^\alpha} - \frac{\alpha}{24}, \tag{23}$$

$$h_2(\alpha) := \frac{1}{2} + \frac{(\alpha+1)}{12} - \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{720} - \frac{1}{\alpha} + h_1(\alpha). \tag{24}$$

Then

- h_1 *is strictly decreasing on* $[1,2]$ *and the equation* $h_1(\alpha) = 0$ *has the unique solution* $\alpha_1 = 1.553\ldots$,
- h_2 *is strictly increasing on* $[1,2]$ *and the equation* $h_2(\alpha) = 0$ *has the unique solution* $\alpha_2 = 1.507\ldots$,

and in particular, $\alpha_1 > \alpha_2$.

Proof of Lemma 8: $1 \leq \alpha \leq 2$. We combine (19) and (21) to obtain

$$S_\alpha(m) \leq \frac{2}{\alpha} + \frac{\alpha}{6}\frac{1}{m^2} - \frac{(\alpha-3)(\alpha-4)}{12\alpha}\frac{1}{m^\alpha}.$$

We check that the right hand side is decreasing in $m \geq 2$ for $1 \leq \alpha \leq 2$ to conclude that

$$S_\alpha(m) \leq \frac{2}{\alpha} + \frac{\alpha}{24} - \frac{(\alpha-3)(\alpha-4)}{12\alpha}\frac{1}{2^\alpha}. \tag{25}$$

We then use (25) and (20) to obtain

$$\frac{2}{\alpha} - S_\alpha(m) \geq h_1(\alpha),$$
$$\zeta(1+\alpha) - S_\alpha(m) \geq h_2(\alpha),$$

where h_1 and h_2 are the functions from (23) and (24), respectively. By Lemma 11 we can therefore conclude that

$$\max\left(\frac{2}{\alpha}\,,\,\zeta(1+\alpha)\right) \geq S_\alpha(m).$$ □

5. Concluding remarks

5.1. Our first remarks concern the relationship between the upper bound for composition operators on $H^2(\mathbb{D})$ from (3), Theorem 2 and Corollary 3. We first observe that the behaviour of the upper bound (6) as $\varphi(+\infty)$ approaches the boundary of $\mathbb{C}_{1/2}$ is identical to the behaviour of the upper bound (3) as $\phi(0)$ approaches the boundary of $\mathbb{D}$, since $1+|\phi(0)| \to 2$.

Let us next discuss the transference principle from [10]. Suppose that ϕ is the symbol of a composition operator on $H^2(\mathbb{D})$. The composition operator $\mathscr{C}_\phi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ can be transferred to an operator on $\mathscr{H}^2$ with symbol

$$\varphi_\alpha := \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha \circ \phi \circ \mathcal{I}, \tag{26}$$

where $\mathcal{T}_\alpha$ is as in (7) and $\mathcal{I}(s) := 2^{-s}$. In the second part of the proof of Lemma 4, we essentially transfer $\phi(z) = z$ in this way. Note that in [10] only $\alpha = 1$ is considered, but that particular choice of α is not important for their considerations. We can obtain the following result about the norm of the transferred composition operator.

Theorem 12. *Fix $0 < \alpha < \infty$. Suppose that $\phi \colon \mathbb{D} \to \mathbb{D}$ is analytic, and set $r = |\phi(0)| < 1$. Let*

$$\alpha_r := \alpha \frac{1-r}{1+r}.$$

Define φ_α as in (26). The upper bound $\|\mathscr{C}_{\varphi_\alpha}\| \leq \sqrt{\|B_{\alpha_r}\|}$ is sharp.

Proof. The proof is similar to that of Theorem 2. We begin by noting that

$$\|\mathscr{C}_{\varphi_\alpha}\|_{\mathscr{H}^2} = \|f \circ \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha \circ \phi\|_{H^2(\mathbb{D})}.$$

After a rotation, we may assume that $\phi(0) = r$. Let $\phi_r(z) := (r-z)/(1-rz)$. It now follows from (3) that

$$\|f \circ \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha \circ \phi_r \circ \phi_r^{-1} \circ \phi\|_{H^2(\mathbb{D})} \leq \|f \circ \mathcal{S}_{1/2} \circ \mathcal{T}_\alpha \circ \phi_r\|_{H^2(\mathbb{D})}, \tag{27}$$

since $(\phi_r^{-1} \circ \phi)(0) = 0$. The proof is completed by computing $\mathcal{T}_\alpha \circ \phi_r = \mathcal{T}_{\alpha_r}$, then using (8) and Lemma 5. Equality in (27) is attained for $\phi = \phi_r$. □

Note that when $0 < \alpha \leq \alpha_0$, we can combine Theorem 1 and (3) to restate the sharp upper bound $\|\mathscr{C}_{\varphi_\alpha}\| \leq \sqrt{\|B_{\alpha_r}\|}$ as

$$\|\mathscr{C}_{\varphi_\alpha}\|_{\mathscr{H}^2 \to \mathscr{H}^2} \leq \sqrt{\|B_\alpha\|} \cdot \|C_\varphi\|_{H^2(\mathbb{D}) \to H^2(\mathbb{D})}. \tag{28}$$

The upper bound (28) in fact holds for every $\alpha > 0$. This can be deduced from the arguments in [10, Sec. 9] and Theorem 2. However, (28) cannot be sharp for every $\alpha > \alpha_0$ unless $r = 0$. To see this, consider ϕ_r for some fixed $0 < r < 1$, then appeal to Theorem 1 and choose a large α such that

$$\frac{\zeta(1+\alpha_r)}{\zeta(1+2\alpha)} < \frac{1+r}{1-r}.$$

5.2. Let us now discuss Theorem 1 for $\alpha > \alpha_0$. We began our analysis of B_α with the application of the Cauchy–Schwarz inequality to obtain (14). To obtain an upper bound for $\|B_\alpha\|$, we computed the supremum of the sequence $S_\alpha(m)$. Note that $a_m = m^{-1/2-\varepsilon}$ and $b_n = n^{-1/2-\varepsilon}$ which gives the lower bound $2/\alpha$ in Lemma 7 is chosen to attain equality in the weighted Cauchy–Schwarz inequality. We get that $S_\alpha(+\infty) = 2/\alpha$ is also an upper bound when $\alpha \leq \alpha_0$, since the "tail estimate" dominates $S_\alpha(m)$ for all m.

We cannot expect the upper bound $S_\alpha(1) = \zeta(1+\alpha)$ to be attained in the same way, since the supremum is attained in the first summand of (14). Hence we conjecture that $\|B_\alpha\| \leq \zeta(1+\alpha)$ is not sharp for all $\alpha > \alpha_0$.

Our main effort has been directed at the upper bound in Theorem 1. It is easy to improve the lower bound coming from the point estimate (5). We offer only the following example result in this direction. If $\alpha > 1$, then

$$\|B_\alpha\| \geq 2 - \frac{\zeta(2\alpha)}{\zeta(2\alpha-1)}. \tag{29}$$

To prove (29), set $a_m = m^{-\alpha+1/2}$ and $b_n = n^{-\alpha+1/2}$. The estimate follows at once from the computation

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{[\max(m,n)]^{2\alpha}} = 2\zeta(2\alpha-1) - \zeta(2\alpha).$$

Comparing with (29), we find that the upper bound $\|B_\alpha\| \leq 2/\alpha$ cannot hold for $\alpha \geq 1.7$. Note also that the difference between the lower bounds from Theorem 1 and (29) is irrelevant for large α, compared to $\zeta(1+\alpha)$. In combination, these observations lead to the following questions.

(a) For which $1.48\ldots = \alpha_0 \leq \alpha < 1.7$ does the upper bound $\|B_\alpha\| \leq 2/\alpha$ cease to hold?
(b) What is the asymptotic decay of $\|B_\alpha\| - 1$ as $\alpha \to \infty$?

For question (a), we suggest investigating the attractive special case $\alpha = 3/2$, which can be formulated as follows. Let $a = (a_1, a_2, \ldots)$ be a non-negative sequence. Find the best constant C in the inequality

$$\sum_{m=1}^\infty \sum_{n=1}^\infty a_m a_n \frac{mn}{[\max(m,n)]^3} \leq C \sum_{m=1}^\infty a_m^2.$$

We know from Theorem 1 that $1.33\ldots = 4/3 \leq C \leq \zeta(5/2) = 1.34\ldots$.

To make question (b) precise, we deduce from Theorem 1 that there are positive constants C_1 and C_2 such that for $\alpha \geq 2$, we have

$$C_1 4^{-\alpha} \leq \|B_\alpha\| - 1 \leq C_2 2^{-\alpha}.$$

Note that (29) only changes the constant C_1. It would be interesting to decide which (if either) of these bounds is of the correct order. In analogy with (3) and (4), one might conjecture that $4^{-\alpha}$ is correct.

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