On Interface Transfer Terms in Two-Fluid Models

Tore Flåtten[a,*], Alexandre Morin[b]

[a]*SINTEF Energy Research, P.O. Box 4761 Sluppen, NO-7465 Trondheim, Norway.*
[b]*Department of Energy and Process Engineering, Norwegian University of Science and Technology (NTNU), Kolbjørn Hejes vei 1B, NO-7491 Trondheim, Norway.*

Abstract

In this note we consider two-fluid models based on the usual formulations for conservation of mass, total momentum and total energy. We present some potentially useful general relationships between the interface exchange terms and the evolution of the mechanical variables. In particular, we discuss the possibility of obtaining in this framework a model that is both thermodynamically reversible and possesses real eigenvalues. We formally prove that such a model must include terms associated with the virtual mass force.

We then address a technical issue regarding the modelling of interface transfer terms in the energy equations. In particular, we demonstrate how the formulation of the non-conservative products in these equations determine whether the interface exchange terms represent heat or energy transfer.

Keywords: two-fluid model, heat transfer, hyperbolic
2010 MSC: 76T10, 35L60

1. Introduction

We are interested in the classical compressible model for two-phase flow assuming mechanical equilibrium between the phases, and a separate velocity field for each phase (Bendiksen et al., 1991; Bestion, 1990; Stewart and Wendroff, 1984). With the standard closure assumptions, this model possesses complex eigenvalues (Stewart and Wendroff, 1984; Toumi, 1996; Toumi and Kumbaro, 1996). The mathematical and physical implications of this fact have been extensively discussed during the past decades (Keyfitz et al., 2003, 2004; Lax, 1957, 1980; Sever, 2005, 2008; Stewart and Wendroff, 1984).

In particular, this model is generally *ill-posed* in the sense that smooth solutions are expected to be absolutely unstable under perturbations (Sever, 2005). Obviously, this calls into question the usefulness of these equations for modelling and simulation. A common practice is to introduce regularizing terms to render the eigenvalues real. These terms typically take the form of interface momentum exchange terms, and may be classified in two main categories:

- *interface pressure corrections* (Bestion, 1990; Munkejord and Papin, 2007; Stuhmiller, 1977), involving spatial derivatives in the volume fraction;

---

*Corresponding author.
Email addresses: `Tore.Flatten@sintef.no` (Tore Flåtten), `alexandre.morin@ntnu.no` (Alexandre Morin)

- *virtual mass force terms* (Bestion, 1990; Lahey, 1991; Städtke, 2006; Toumi, 1996), involving spatial derivatives in the velocities.

A very general analysis including both these effects was performed by Jones and Prosperetti (1985). Considering the incompressible limit, the authors here showed that hyperbolicity is a necessary condition for stability of steady uniform flows, even in the presence of algebraic momentum source terms.

It is known (Saurel et al., 2003), but not widely discussed in the literature, that such differential regularizing terms tend to introduce a fundamental problem on the physical level; the model ceases to satisfy the second law of thermodynamics. In fact, several issues regarding the modelling of interface transfer terms seem to be only implicitly discussed in the recent literature. The aim of this paper is to clarify some of these issues. In this respect, we provide what seems to us some explicit original calculations, although the topics we address are highly classical and our conclusions should not be surprising.

In particular, we aim to shed light on the following two modelling issues:

1. The apparent incompatibility between thermodynamic reversibility and wellposedness for our two-fluid models. Assuming thermal equilibrium, we here provide a general explicit condition on the interface momentum exchange term that is necessary and sufficient for global entropy to be conserved for smooth solutions. This condition is rather strict and excludes a large class of models from being simultaneously well-posed and reversible.

2. The interpretation of the interface transfer terms in the energy balance equations. We argue that in the standard formulation, these terms should be interpreted as heat transfer terms rather than energy transfer terms, and we make this interpretation mathematically precise.

Our paper is organized as follows. In Section 2, we describe the general framework for the two-fluid models we will consider. In Section 3, we derive a relationship between interphasic heat transfer and the evolution of the pressure and volume fraction. In Section 4, we present a similar result for the momentum exchange term. A main result of our paper is the equation (24), which gives a simple general relationship between heat and momentum transfer in our two-fluid models.

In Section 5, we apply these results by considering the special case of thermal equilibrium between the phases. In particular, we are able to derive the general, and rather restrictive, explicit condition (36) that must be satisfied by the momentum exchange term in order for the model to be thermodynamically reversible. This result allows us to prove that such a model can be well-posed with real eigenvalues only if this term includes spatial derivatives in the velocities. The purpose of this analysis is not to advocate the use of such a model; rather, the main insight gained is that models in our framework that *do not* satisfy this condition are unquestionably fundamentally unphysical.

In Section 6, we discuss the interpretation of the interface transfer terms in the standard formulation of the energy equations. In particular, we provide two mathematically equivalent formulations of the energy equations; in one formulation, the right-hand side terms will represent *heat transfer*; in the alternative formulation, these terms will represent *internal energy* transfer. The relationship between these two kinds of source terms is made explicit, and gives us an expression for the amount of heat transferred that will be converted to mechanical work.

Finally, in Section 7, the results of this paper are summarized.

2. The Two-Fluid Model

We consider here the highly classical two-fluid model presented for instance by Stewart and Wendroff (1984), based on fundamental conservation principles. For simplicity, we will limit our discussion to the formulation in one space dimension.

Physically, it is commonly recognized that such a formulation is most sensibly interpreted as an *averaging* of a local description of separate flow fields (Ishii, 1975; Stewart and Wendroff, 1984). In (Ishii, 1975) the focus is on time averaging, but spatial and ensemble averaging are alternative viable approaches (Drew and Passman, 1999; Stewart and Wendroff, 1984). In this framework, we focus on the model derived from the following basic assumptions of conservation of masses, momentum and total energy:

A1: Mass is conserved for each phase:

$$\frac{\partial}{\partial t}\left(\rho_g \alpha_g\right) + \frac{\partial}{\partial x}\left(\rho_g \alpha_g v_g\right) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\rho_\ell \alpha_\ell\right) + \frac{\partial}{\partial x}\left(\rho_\ell \alpha_\ell v_\ell\right) = 0. \tag{2}$$

A2: Total momentum is conserved in the form:

$$\frac{\partial}{\partial t}\left(\rho_g \alpha_g v_g + \rho_\ell \alpha_\ell v_\ell\right) + \frac{\partial}{\partial x}\left(\rho_g \alpha_g v_g^2 + \rho_\ell \alpha_\ell v_\ell^2 + p\right) = 0. \tag{3}$$

A3: Total energy is conserved in the form:

$$\frac{\partial}{\partial t}\left(E_g + E_\ell\right) + \frac{\partial}{\partial x}\left((E_g + \alpha_g p)v_g + (E_\ell + \alpha_\ell p)v_\ell\right) = 0. \tag{4}$$

Herein, external and dissipative forces have been neglected, and we have assumed the following notation for the phase $k \in \{g, \ell\}$:

ρ_k - density,
v_k - velocity,
α_k - volume fraction,
E_k - energy,
p - pressure common to both phases.

Here the volume fractions satisfy

$$\alpha_g + \alpha_\ell = 1, \tag{5}$$

and the phasic energies are given by

$$E_k = \rho_k \alpha_k \left(e_k + \frac{1}{2}v_k^2\right), \tag{6}$$

where e_k is the specific internal energy.

Within the context of averaging, (1)–(2) can be taken as the definition of the velocities v_k. That these velocities appear in unmodified form also for the momentum and energy equations (3) and (4) is here an assumption, although common, that is mainly based on the desire to avoid excessive complexities in the model (Stewart and Wendroff, 1984). In this respect, we remark

3

that alternative formulations of (3) and (4) may, and perhaps should, be considered (Song and Ishii, 2001; Stewart and Wendroff, 1984).

For the thermodynamic closure, we assume that each phase may be equipped with a free energy $G_k(p, T_k)$, and that the fundamental thermodynamic differential

$$\mathrm{d}e_k = T_k \,\mathrm{d}s_k + \frac{p}{\rho_k^2} \,\mathrm{d}\rho_k \tag{7}$$

is valid. Herein, T_k and s_k are the temperatures and specific entropies of the phases. It should be noted that (7) is in itself a rather strong assumption; given that the entropies, energies and densities are to be interpreted in an *averaged* sense, relating them through a unique equation of state is a simplification motivated mainly by convenience (Stewart and Wendroff, 1984).

2.1. Well-posedness and reversibility

In addition to these basic conservation principles, we want our model to satisfy the second law of thermodynamics. In particular, we here insist that the model should be purely fluid-mechanical, i. e. thermodynamically *reversible* for smooth solutions.

In particular, we assume that some model is given that is locally fully defined, including a complete set of constitutive relations. Then reversibility should hold whenever this model is applied to a physical region with no exchange of mass, energy or heat with the surroundings. Mathematically, we represent such a region as a closed loop in space (periodic boundary conditions), and we exclude any terms representing interactions with the environment.

Given these considerations, we impose the following requirement:

A4: Global entropy is conserved for smooth solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_R (\rho_g \alpha_g s_g + \rho_\ell \alpha_\ell s_\ell) \,\mathrm{d}x = 0, \tag{8}$$

where the integral is taken over over any *closed* region R, i.e. we have

$$R = [x_1, x_2),$$

with periodic boundary conditions.

We also want our model to be globally linearizable, and the velocity of information propagation should be finite:

A5: The model can be written in quasilinear form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}(\boldsymbol{U}) \frac{\partial \boldsymbol{U}}{\partial x} = 0, \tag{9}$$

where $\boldsymbol{U}$ is the vector of evolved variables and $\boldsymbol{A}(\boldsymbol{U})$ is a smooth function.

Finally, the initial value problem should be well posed:

A6: All eigenvalues of $\boldsymbol{A}(\boldsymbol{U})$ are real for all $\boldsymbol{U}$ in some physically relevant domain $\mathcal{D}$.

Additional relations are needed to close the model. Although a multitude of such closures have so far been proposed in the literature, we are not aware of any full model that satisfies all conditions A1-A6 in any general sense.

In this paper, we will derive a potentially useful condition that such a model must satisfy. We first derive some basic mathematical relationships.

3. Entropy Exchange Terms

With no loss of generality, we may write entropy evolution equations for each phase in the form

$$\frac{\partial}{\partial t}\left(\rho_k \alpha_k s_k\right) + \frac{\partial}{\partial x}\left(\rho_k \alpha_k s_k v_k\right) = \sigma_k, \tag{10}$$

where the local entropy modification term σ_k is so far unknown. However, we may state the following general result.

Proposition 1. *If the mass conservation assumption A1 holds, then in the context of* (10) *we have*

$$T_k \sigma_k = \frac{\alpha_k}{\Gamma_k}\left(\frac{\partial p}{\partial t} + v_k \frac{\partial p}{\partial x}\right) + \frac{\rho_k c_k^2}{\Gamma_k}\left(\frac{\partial \alpha_k}{\partial t} + \frac{\partial}{\partial x}(\alpha_k v_k)\right), \tag{11}$$

where

$$c_k^2 = \left(\frac{\partial p}{\partial \rho_k}\right)_{s_k} \tag{12}$$

represents the phasic sound velocity and

$$\Gamma_k = \frac{1}{\rho_k T_k}\left(\frac{\partial p}{\partial s_k}\right)_{\rho_k} \tag{13}$$

is the Grüneisen coefficient.

Proof. By assumption A1 we obtain

$$\sigma_k = \rho_k \alpha_k \left(\frac{\partial s_k}{\partial t} + v_k \frac{\partial s_k}{\partial x}\right) \tag{14}$$

and

$$\alpha_k \left(\frac{\partial \rho_k}{\partial t} + v_k \frac{\partial \rho_k}{\partial x}\right) = -\rho_k \left(\frac{\partial \alpha_k}{\partial t} + \frac{\partial}{\partial x}(\alpha_k v_k)\right). \tag{15}$$

The result then follows from the differential

$$T_k \, \mathrm{d}s_k = \frac{1}{\Gamma_k \rho_k}\left(\mathrm{d}p - c_k^2 \, \mathrm{d}\rho_k\right). \tag{16}$$

$\square$

3.1. Relation to Internal Energy

Through the fundamental differential (7), this can be recast in terms of internal energy evolution as follows:

$$\frac{\partial}{\partial t}\left(\rho_k \alpha_k e_k\right) + \frac{\partial}{\partial x}\left(\rho_k \alpha_k e_k v_k\right) = \frac{\alpha_k}{\Gamma_k}\left(\frac{\partial p}{\partial t} + v_k \frac{\partial p}{\partial x}\right) + \left(\frac{\rho_k c_k^2}{\Gamma_k} - p\right)\left(\frac{\partial \alpha_k}{\partial t} + \frac{\partial}{\partial x}(\alpha_k v_k)\right), \tag{17}$$

where we have used A1:

$$\frac{\partial}{\partial t}\left(\rho_k \alpha_k e_k\right) \frac{\partial}{\partial x}\left(\rho_k \alpha_k e_k v_k\right) = \rho_k \alpha_k \left(\frac{\partial e_k}{\partial t} + v_k \frac{\partial e_k}{\partial x}\right). \tag{18}$$

5

4. Momentum Exchange Terms

With no loss of generality, the momentum conservation assumption A2 can be rewritten as

$$\frac{\partial}{\partial t}\left(\rho_g \alpha_g v_g\right) + \frac{\partial}{\partial x}\left(\rho_g \alpha_g v_g^2\right) + \alpha_g \frac{\partial p}{\partial x} + \mathcal{M} = 0, \tag{19}$$

$$\frac{\partial}{\partial t}\left(\rho_\ell \alpha_\ell v_\ell\right) + \frac{\partial}{\partial x}\left(\rho_\ell \alpha_\ell v_\ell^2\right) + \alpha_\ell \frac{\partial p}{\partial x} - \mathcal{M} = 0, \tag{20}$$

where the interface momentum exchange term $\mathcal{M}$ is determined from the closure relations. Using assumption A1, we can then derive kinetic energy evolution equations:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_g \alpha_g v_g^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho_g \alpha_g v_g^3\right) = -v_g\left(\mathcal{M} + \alpha_g \frac{\partial p}{\partial x}\right), \tag{21}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_\ell \alpha_\ell v_\ell^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho_\ell \alpha_\ell v_\ell^3\right) = v_\ell\left(\mathcal{M} - \alpha_\ell \frac{\partial p}{\partial x}\right). \tag{22}$$

We then obtain the following potentially useful proposition.

Proposition 2. *If the assumptions A1-A3 and the differential* (7) *hold, the momentum exchange term* $\mathcal{M}$ *satisfies*

$$\left(v_g - v_\ell\right)\mathcal{M} = \left(\frac{\rho_g c_g^2}{\Gamma_g} - \frac{\rho_\ell c_\ell^2}{\Gamma_\ell}\right)\frac{\partial \alpha_g}{\partial t} + \frac{\rho_g c_g^2}{\Gamma_g}\frac{\partial}{\partial x}(\alpha_g v_g) + \frac{\rho_\ell c_\ell^2}{\Gamma_\ell}\frac{\partial}{\partial x}(\alpha_\ell v_\ell)$$
$$+ \left(\frac{\alpha_g}{\Gamma_g} + \frac{\alpha_\ell}{\Gamma_\ell}\right)\frac{\partial p}{\partial t} + \left(\frac{\alpha_g v_g}{\Gamma_g} + \frac{\alpha_\ell v_\ell}{\Gamma_\ell}\right)\frac{\partial p}{\partial x}. \tag{23}$$

Proof. Add (21) and (22) to (17) and compare with (4). $\qquad\square$

Note the general validity of (23), which at first sight may look like a *definition* of $\mathcal{M}$. However, on the contrary, this equation merely provides us with information about how the interface momentum term affects the evolution of the pressure and volume fraction.

In particular, we have the following simple relation:

$$T_g \sigma_g + T_\ell \sigma_\ell = \left(v_g - v_\ell\right)\mathcal{M}, \tag{24}$$

which follows from (11) and (23). From this we immediately see that if entropy is conserved along the flow in each phase, i. e.

$$\sigma_k \equiv 0, \tag{25}$$

then our only choice of $\mathcal{M}$ that conserves total energy is

$$\mathcal{M} \equiv 0, \tag{26}$$

which is the standard non-hyperbolic model (Stewart and Wendroff, 1984) for which real-valued eigenvalues occur only for $v_g = v_\ell$. This indicates a fundamental incompatibility between well-posedness and reversibility for models satisfying the assumptions A1-A3. We will now investigate this issue further by relaxing the requirement that $\sigma_k = 0$.

6

5. Thermal Equilibrium

For simplicity, we now limit ourselves to the special case that the phases are in thermal equilibrium, i. e.

$$T = T_{\mathrm{g}} = T_{\ell}. \tag{27}$$

This simplification is justified by the fact that any valid *general* model must also be valid for the equilibrium states (27). Furthermore, the equilibrium condition (27) may also be imposed as a closure relation for the model, as was done for instance by Martínez Ferrer et al. (2012).

We now introduce the local total entropy variation $\mathcal{S}$:

$$\mathcal{S} = \sigma_{\mathrm{g}} + \sigma_{\ell}, \tag{28}$$

so that the relation (24) simplifies to

$$T\mathcal{S} = (v_{\mathrm{g}} - v_{\ell})\mathcal{M}. \tag{29}$$

We may then write the reversibility condition A4 as:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_R (\rho_{\mathrm{g}}\alpha_{\mathrm{g}}s_{\mathrm{g}} + \rho_{\ell}\alpha_{\ell}s_{\ell})\,\mathrm{d}x = \oint_R \left(\mathcal{S} - \frac{\partial}{\partial x}\left(\rho_{\mathrm{g}}\alpha_{\mathrm{g}}s_{\mathrm{g}}v_{\mathrm{g}} + \rho_{\ell}\alpha_{\ell}s_{\ell}v_{\ell}\right)\right)\mathrm{d}x = \oint_R \mathcal{S}\,\mathrm{d}x = 0, \tag{30}$$

for any distribution $U(x)$ on the closed region R. Hence if $\mathcal{S}$ is a function of U, it becomes an algebraic entropy source term, and (30) can only be generally satisfied if

$$\mathcal{S}(U) \equiv 0. \tag{31}$$

However, our condition A5 allows $\mathcal{S}$ to be a function of the spatial derivative of U:

$$\mathcal{S} = \mathcal{S}(\partial_x U) = \sum_i B_i(U)\frac{\partial U_i}{\partial x}, \tag{32}$$

and the condition (30) may still be non-trivially satisfied. Now, the classical *gradient theorem* states that a line integral over an arbitrary closed path in a vector field is identically zero if and only if the integrand is a gradient of some potential function. We now recall that the condition A4 states that reversibility must hold for *all* smooth solutions; hence the entropy integral (30) must be zero for *any* spatial distribution $U(x)$.

We may consequently apply the gradient theorem to the space of physically admissible states $\mathcal{D}$ to which U belongs, i. e. we consider arbitrary curves in $\mathcal{D}$ parametrized by the variable $x \in [x_1, x_2) = R$, representing the possible initial conditions $U(x)$. By this, the condition that (30) must hold for *any* distribution $U(x)$ implies the existence of a potential function $Z(U)$ such that

$$\oint_R \mathcal{S}\,\mathrm{d}x = \oint_R \nabla_U Z(U) \cdot \frac{\partial U}{\partial x}\,\mathrm{d}x = 0. \tag{33}$$

Hence $\mathcal{S}\,\mathrm{d}x$ must be an *exact* differential, i. e. we have

$$B_i = \frac{\partial Z}{\partial U_i}, \tag{34}$$

and in particular

$$\mathcal{S} = \frac{\partial}{\partial x}Z(U). \tag{35}$$

In other words, $\mathcal{S}$ can be interpreted as an *entropy flux*. This gives us a main result of this paper.

Proposition 3. *Consider a subdomain $\mathcal{D}$ of the admissible thermal equilibrium states. Consider a two-fluid model satisfying the assumptions A1-A3, A5 and the fundamental differential (7) for all $\boldsymbol{U} \in \mathcal{D}$. Then, for all $\boldsymbol{U} \in \mathcal{D}$, the reversibility condition A4 is satisfied if and only if there exists a function $\boldsymbol{W}(\boldsymbol{U})$ such that the interface momentum exchange term can be written as*

$$\mathcal{M} = T(v_\mathrm{g} - v_\ell)\frac{\partial W}{\partial x} + 2WT\frac{\partial}{\partial x}(v_\mathrm{g} - v_\ell) \tag{36}$$

for all $\boldsymbol{U} \in \mathcal{D}$.

Proof. With no loss of generality, we may write Z as

$$Z(\boldsymbol{U}) = W(\boldsymbol{U})(v_\mathrm{g} - v_\ell)^2, \tag{37}$$

where $W(\boldsymbol{U})$ is some function. Now substituting (37) in (29) and cancelling terms, we obtain (36). $\qquad\square$

This result opens for the possibility that some appropriate $W(\boldsymbol{U})$ may be found, making the reversible model at least conditionally hyperbolic. This question will not be pursued in the current paper.

However, we may use (36) as a convenient tool for testing the thermodynamic consistency of various established models. In particular, we have the following proposition.

Proposition 4. *Any model satisfying the assumptions A1-A5 with $\mathcal{M} \neq 0$ must involve terms of the form $\partial_x v_k$ in $\mathcal{M}$.*

Proof. If this does not hold, it follows from (36) that $W(\boldsymbol{U})$ would have to satisfy

$$v_\mathrm{s}\frac{\partial W}{\partial v_\mathrm{s}} + 2W = 0, \tag{38}$$

where

$$v_\mathrm{s} = v_\mathrm{g} - v_\ell. \tag{39}$$

Now (38) can be integrated to yield

$$W(\boldsymbol{U}) = Cv_\mathrm{s}^{-2}, \tag{40}$$

where C is independent of v_s. By substituting this result into (37), it follows that $Z(\boldsymbol{U})$, and hence $\mathcal{S}$, must be independent of v_s. However, it follows from (29) and the smoothness of $\mathcal{M}$ that $\mathcal{S}$ must disappear when $v_\mathrm{s} = 0$. Hence we must have $\mathcal{S} \equiv 0$, giving $\mathcal{M} \equiv 0$. $\qquad\square$

We remark that $\mathcal{M} = 0$ corresponds to the standard non-hyperbolic formulation of the model, violating the condition A6 for all $v_\mathrm{s} \neq 0$. Hence Proposition 4 may be restated as follows:

Any model satisfying the assumptions A1-A6 in any general sense must involve spatial derivatives of the velocities in the momentum exchange term. Physically, such velocity derivatives are most naturally interpreted as being associated with the *virtual mass force* (Jones and Prosperetti, 1985; Lahey, 1991).

In particular, this result immediately rules out all models based solely on interface pressure corrections in the framework A1-A3 (Bestion, 1990; Martínez Ferrer et al., 2012; Stuhmiller, 1977). We remark that hydrostatic pressure corrections, used to simulate surface waves and regime transitions (De Henau and Raithby, 1995; Holmås et al., 2008), typically operate with separate pressures in each phase and hence do not fit into our framework A2.

8

We emphasize that the converse of Proposition 4 does not necessarily hold. The standard formulations of the virtual mass force terms do not in general satisfy (36) – and hence the resulting model is not thermodynamically reversible.

It should also be noted that the standard formulations of the virtual mass force (Jones and Prosperetti, 1985; Lahey, 1991) involve not only spatial, but also temporal, derivatives of the velocities. However, our current framework applies also in this case; the temporal derivatives can always be equivalently reformulated in terms of spatial derivatives through a mathematical transformation. This point will be demonstrated in the next section, where such a transformation is performed on the energy equations.

6. Energy Transfer Terms

We now turn our attention to the modelling of interface energy exchange terms. With no loss of generality, we may write the assumption A3 in the standard form (Martínez Ferrer et al., 2012; Munkejord et al., 2009; Paillère et al., 2003; Stewart and Wendroff, 1984):

$$\frac{\partial E_g}{\partial t} + \frac{\partial}{\partial x}\left(v_g(E_g + \alpha_g p)\right) + p\frac{\partial \alpha_g}{\partial t} + Q = 0, \tag{41}$$

$$\frac{\partial E_\ell}{\partial t} + \frac{\partial}{\partial x}\left(v_\ell(E_\ell + \alpha_\ell p)\right) + p\frac{\partial \alpha_\ell}{\partial t} - Q = 0. \tag{42}$$

Herein, the interpretation of the interface exchange term Q deserves some attention. Given that (41)–(42) balances total energy, one may be tempted to interpret Q as representing the amount of *energy* being transferred between the phases. However, the presence of the term $p\partial_t\alpha$ complicates this picture somewhat. In fact, in the form (41)–(42), the equations are not evolution equations for the energies; strictly speaking, they are evolution equations for the differential dJ_k given by

$$dJ_k = dE_k + p\,d\alpha_k. \tag{43}$$

In other words, the source term Q will modify the volume fractions as well as the energies of each phase. This means, in the context of (41)–(42), it makes more sense to interpret Q as *heat and kinetic energy* transfer terms rather than *energy transfer* terms. In the following, we will make this notion more precise, and present an alternative formulation of the energy balance equations where the source terms are truly energy transfer terms.

This may be considered an advantage from a purely heuristic point of view, although the resulting formulation is more involved than (41)–(42). Nevertheless, this alternative formulation has previously proven fruitful in devising numerical schemes (Martínez Ferrer et al., 2012; Munkejord et al., 2009).

In this respect, the main purpose of this section is to make the following point: *the modelling of the interface energy exchange terms is sensitive to the choice of formulation of the non-conservative terms representing mechanical work exchanged between the phases.*

6.1. Internal and Kinetic Energy

Using assumption A1 and the fundamental differential (7), we may rewrite the entropy equations (10) as follows:

$$\frac{\partial}{\partial t}\left(\rho_k\alpha_k e_k\right) + \frac{\partial}{\partial x}\left(\rho_k\alpha_k e_k v_k\right) + p\left(\frac{\partial \alpha_k}{\partial t} + \frac{\partial}{\partial x}(\alpha_k v_k)\right) = T_k\sigma_k. \tag{44}$$

Now adding (21)–(22) to (44) and comparing to (41)–(42), we obtain

$$Q = \mathcal{M}v_{\mathrm{g}} - T_{\mathrm{g}}\sigma_{\mathrm{g}} = \mathcal{M}v_{\ell} + T_{\ell}\sigma_{\ell}. \tag{45}$$

In other words, Q represents the sum of the interface heat transfer and kinetic energy transfer terms, as may be expected; the mechanical work the phases perform on each other is encoded in the term $p\partial_t\alpha_k$.

6.2. Energy Evolution Equations

As was done in (Martínez Ferrer et al., 2012; Munkejord et al., 2009), we now aim to reformulate (41)–(42) to replace the $p\partial_t\alpha$-term with spatial derivatives. We may rewrite (11) as an evolution equation for the volume fraction:

$$\beta\frac{\partial\alpha_{\mathrm{g}}}{\partial t} + \rho_{\mathrm{g}}\alpha_{\ell}c_{\mathrm{g}}^2\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}v_{\mathrm{g}}\right) - \rho_{\ell}\alpha_{\mathrm{g}}c_{\ell}^2\frac{\partial}{\partial x}\left(\alpha_{\ell}v_{\ell}\right) + \alpha_{\mathrm{g}}\alpha_{\ell}\left(v_{\mathrm{g}} - v_{\ell}\right)\frac{\partial p}{\partial x} = \alpha_{\ell}\Gamma_{\mathrm{g}}T_{\mathrm{g}}\sigma_{\mathrm{g}} - \alpha_{\mathrm{g}}\Gamma_{\ell}T_{\ell}\sigma_{\ell}, \tag{46}$$

where

$$\beta = \rho_{\mathrm{g}}\alpha_{\ell}c_{\mathrm{g}}^2 + \rho_{\ell}\alpha_{\mathrm{g}}c_{\ell}^2. \tag{47}$$

Substituting (46) into (41)–(42) we obtain

$$\frac{\partial E_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}\left(E_{\mathrm{g}}v_{\mathrm{g}}\right) + \left(\alpha_{\mathrm{g}}v_{\mathrm{g}} - \eta\alpha_{\mathrm{g}}\alpha_{\ell}\left(v_{\mathrm{g}} - v_{\ell}\right)\right)\frac{\partial p}{\partial x} + \eta\rho_{\ell}\alpha_{\mathrm{g}}c_{\ell}^2\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}v_{\mathrm{g}} + \alpha_{\ell}v_{\ell}\right)$$
$$= \eta\left(\alpha_{\mathrm{g}}\Gamma_{\ell}T_{\ell}\sigma_{\ell} - \alpha_{\ell}\Gamma_{\mathrm{g}}T_{\mathrm{g}}\sigma_{\mathrm{g}}\right) - Q, \tag{48}$$

$$\frac{\partial E_{\ell}}{\partial t} + \frac{\partial}{\partial x}\left(E_{\ell}v_{\ell}\right) + \left(\alpha_{\ell}v_{\ell} + \eta\alpha_{\mathrm{g}}\alpha_{\ell}\left(v_{\mathrm{g}} - v_{\ell}\right)\right)\frac{\partial p}{\partial x} + \eta\rho_{\mathrm{g}}\alpha_{\ell}c_{\mathrm{g}}^2\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}v_{\mathrm{g}} + \alpha_{\ell}v_{\ell}\right)$$
$$= Q - \eta\left(\alpha_{\mathrm{g}}\Gamma_{\ell}T_{\ell}\sigma_{\ell} - \alpha_{\ell}\Gamma_{\mathrm{g}}T_{\mathrm{g}}\sigma_{\mathrm{g}}\right), \tag{49}$$

where

$$\eta = \frac{p}{\beta}. \tag{50}$$

6.2.1. Interpretation of Source Terms

To recapitulate, we may now write the energy equations in the two equivalent forms:

- Standard formulation:

$$\frac{\partial E_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}\left(v_{\mathrm{g}}(E_{\mathrm{g}} + \alpha_{\mathrm{g}}p)\right) + p\frac{\partial\alpha_{\mathrm{g}}}{\partial t} = \mathcal{H}_{\mathrm{g}} - \mathcal{M}v_{\mathrm{g}}, \tag{51}$$

$$\frac{\partial E_{\ell}}{\partial t} + \frac{\partial}{\partial x}\left(v_{\ell}(E_{\ell} + \alpha_{\ell}p)\right) + p\frac{\partial\alpha_{\ell}}{\partial t} = \mathcal{H}_{\ell} + \mathcal{M}v_{\ell}. \tag{52}$$

- Formulation with spatial derivatives:

$$\frac{\partial E_{\mathrm{g}}}{\partial t} + \frac{\partial}{\partial x}\left(E_{\mathrm{g}}v_{\mathrm{g}}\right) + \left(\alpha_{\mathrm{g}}v_{\mathrm{g}} - \eta\alpha_{\mathrm{g}}\alpha_{\ell}\left(v_{\mathrm{g}} - v_{\ell}\right)\right)\frac{\partial p}{\partial x} + \eta\rho_{\ell}\alpha_{\mathrm{g}}c_{\ell}^2\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}v_{\mathrm{g}} + \alpha_{\ell}v_{\ell}\right) = \mathcal{E}_{\mathrm{g}} - \mathcal{M}v_{\mathrm{g}}, \tag{53}$$

$$\frac{\partial E_{\ell}}{\partial t} + \frac{\partial}{\partial x}\left(E_{\ell}v_{\ell}\right) + \left(\alpha_{\ell}v_{\ell} + \eta\alpha_{\mathrm{g}}\alpha_{\ell}\left(v_{\mathrm{g}} - v_{\ell}\right)\right)\frac{\partial p}{\partial x} + \eta\rho_{\mathrm{g}}\alpha_{\ell}c_{\mathrm{g}}^2\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}v_{\mathrm{g}} + \alpha_{\ell}v_{\ell}\right) = \mathcal{E}_{\ell} + \mathcal{M}v_{\ell}. \tag{54}$$

Herein:

- $\mathcal{M}v_k$ are *kinetic energy* transfer terms;

- $\mathcal{H}_k = T_k\sigma_k$ are *heat* transfer terms;

- $\mathcal{E}_k$ are *internal energy* transfer terms.

We observe that the following relations hold between the heat and energy transfer terms:

$$\mathcal{E}_g = \mathcal{H}_g + \eta\left(\alpha_g\Gamma_\ell\mathcal{H}_\ell - \alpha_\ell\Gamma_g\mathcal{H}_g\right), \tag{55}$$

$$\mathcal{E}_\ell = \mathcal{H}_\ell - \eta\left(\alpha_g\Gamma_\ell\mathcal{H}_\ell - \alpha_\ell\Gamma_g\mathcal{H}_g\right). \tag{56}$$

In particular, the term

$$\mathcal{W} = \eta\left(\alpha_g\Gamma_\ell\mathcal{H}_\ell - \alpha_\ell\Gamma_g\mathcal{H}_g\right) \tag{57}$$

represents the mechanical work the phases perform on each other as a result of energy being transferred.

7. Summary

We have addressed some technical issues regarding the modelling of interface transfer terms in a class of two-fluid models commonly studied in the literature. In particular, we have discussed the compatibility between thermodynamic reversibility and well-posedness in two-fluid models based on simple formulations for conservation of masses, energy and momentum. We have derived an explicit condition on the interface momentum exchange term for these models to be reversible. In particular, this condition states that any such well-posed, reversible model must include virtual mass force terms; more precisely, the momentum exchange term must include spatial derivatives in the velocities.

Furthermore, we have showed that in the standard formulation of the energy balance equations, interface exchange terms play the role of heat transfer terms. We have discussed an alternative formulation where the interface terms transfer energy. We have also provided an explicit relationship between the amount of heat transferred and the mechanical work exchanged between the phases.

References

Bendiksen, K. H., Malnes, D., Moe, R., Nuland, S., 1991. The dynamic two-fluid model OLGA: Theory and application. SPE Prod. Eng. 6, 171–180.

Bestion, D., 1990. The physical closure laws in the CATHARE code. Nucl. Eng. Des. 124, 229–245.

De Henau, V., Raithby, G. D., 1995. A transient two-fluid model for the simulation of slug flow in pipelines–I. Theory. Int. J. Multiphase Flow 21, 335–349.

Drew, D. A., Passman, S. L., 1999. Theory of Multicomponent Fluids. Vol. 135 of Applied Mathematical Sciences. Springer-Verlag, New York.

Holmås, H., Sira, T., Langtangen, H. P., 2008. Analysis of a 1D incompressible two-fluid model including artificial diffusion. IMA J. Appl. Math. 73, 651–667.

Ishii, M., 1975. Thermo-Fluid Dynamic Theory of Two-Phase Flow. Eyrolles, Paris.

Jones, A. V., Prosperetti, A., 1985. On the suitability of first-order differential models for two-phase flow prediction. Int. J. Multiphase Flow 11, 133–148.

Keyfitz, B. L., Sanders, R., Sever, M., 2003. Lack of hyperbolicity in the two-fluid model for two-phase incompressible flow. Discrete Cont. Dyn.-B 3, 541–563.

Keyfitz, B. L., Sever, M., Zhang, F., 2004. Viscous singular shock structure for a nonhyperbolic two-fluid model. Nonlinearity 17, 1731–1747.

Lahey, R. T., 1991. Void wave propagation phenomena in two-phase flow (Kern Award Lecture). AIChE J. 37, 123–135.

Lax, P. D., 1957. Asymptotic solutions of oscillatory initial value problems. Duke Math. J. 24, 627–646.

Lax, P. D., 1980. On the notion of hyperbolicity. Comm. Pure Appl. Math. 33, 395–397.

Martínez Ferrer, P. J., Flåtten, T., Munkejord, S. T., 2012. On the effect of temperature and velocity relaxation in two-phase flow models. ESAIM-Math. Model. Num. 46, 411–442.

Munkejord, S. T., Evje, S., Flåtten, T., 2009. A MUSTA scheme for a nonconservative two-fluid model. SIAM J. Sci. Comput. 31, 2587–2622.

Munkejord, S. T., Papin, M., 2007. The effect of interfacial pressure in the discrete-equation multiphase model. Comput. Fluids 36, 742–757.

Paillère, H., Corre, C., Carcía Gascales, J. R., 2003. On the extension of the AUSM+ scheme to compressible two-fluid models. Comput. Fluids 32, 891–916.

Saurel, S., Gavrilyuk, S., Renaud, F., 2003. A multiphase model with internal degrees of freedom: application to shock-bubble interaction. J. Fluid Mech. 495, 283–321.

Sever, M., 2005. Solutions of a nonhyperbolic pair of balance laws. ESAIM: M2AN Math. Model. Numer. Anal. 39, 37–58.

Sever, M., 2008. A model of discontinuous, incompressible two-phase flow. J. Math. Fluid Mech. 10, 203–223.

Song, J., Ishii, M., 2001. The one-dimensional two-fluid model with momentum flux parameters. Nucl. Eng. Des. 205, 145–158.

Städtke, H., 2006. Gasdynamic Aspects of Two-Phase Flow. Wiley-VCH Verlag GmbH & CO. KGaA, Weinheim, Germany.

Stewart, H. B., Wendroff, B., 1984. Review article; two-phase flow: Models and methods. J. Comput. Phys. 56, 363–409.

Stuhmiller, J. H., 1977. The influence of interfacial pressure forces on the character of two-phase flow model equations. Int. J. Multiphase Flow 3, 551–560.

Toumi, I., 1996. An upwind numerical method for two-fluid two-phase flow models. Nucl. Sci. Eng. 123, 147–168.

Toumi, I., Kumbaro, A., 1996. An approximate linearized riemann solver for a two-fluid model. J. Comput. Phys. 124, 286–300.