NTNU – Trondheim
Norwegian University of
Science and Technology

A Study of Rotational Water Waves using Bifurcation Theory

Ailo Aasen

Master of Science in Physics and Mathematics
Submission date: June 2014
Supervisor: Mats Ehrnstrøm, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Sammendrag

Denne oppgaven omhandler vannbølgeproblemet. Vi bruker lokal bifurkasjonsteori til å etablere eksistens av reisende og periodiske løsninger av Eulerligningene med vortisitet og liten amplitude. Tilnæmingen vi bruker baserer seg på den til Ehrnström, Escher og Wahlén [6], der hovedforskjellen er at vi bruker nye bifurkasjonsparametere. Vi bifurkerer både fra en éndimensjonal og en todimensjonal kjerne, der den todimensjonale bifurkasjonen gir opphav til bølger med flere bølgetopper og -daler i hver minimale periode. Vi gir også et nytt og elementært bevis av Fredholm-egenskapen til den elliptiske differensialoperatoren assosiert med vannbølgeproblemet. Videre undersøker vi deriverte av bifurkasjonskurven, og gir et nytt resultat for det tilhørende lineære problemet.

Abstract

This thesis is concerned with the water wave problem. Using local bifurcation we establish small-amplitude steady and periodic solutions of the Euler equations with vorticity. Our approach is based on that of Ehrnström, Escher and Wahlén [6], the main difference being that we use new bifurcation parameters. The bifurcation is done both from a one-dimensional and a two-dimensional kernel, the latter bifurcation giving rise to waves having more than one crest in each minimal period. We also give a novel and rudimentary proof of a key lemma establishing the Fredholm property of the elliptic operator associated with the water wave problem. Furthermore, we investigate derivatives of the bifurcation curve, and present a new result for the corresponding linear problem.

Contents

1 Overview of sets and function spaces

$\kappa^{-1}\mathbb{S}$	The one-dimensional sphere of radius κ^{-1}.
$\mathbb{N}$	The set of positive integers.
$\mathbb{N}_0$	The set of nonnegative integers.
X_1	The function space $C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R})$.
X_2	The function space $C^{2,\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R})$.
X	The function space $X_1 \times X_2$.
Y_1	The function space $C^{1,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R})$.
Y_2	The function space $C^{0,\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R})$.
Y	The function space $Y_1 \times Y_2$.
$\tilde{X}_2$	The function space $\left\{\phi \in C^{2,\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R}) : \phi\|_{s=0} = 0\right\}$.
$\tilde{X}$	The function space $X_1 \times \tilde{X}_2$.
$\ker T$	For a function $T : X \to Y$, the set $\{x \in X : T(x) = 0\}$.
$\operatorname{ran} T$	The a function $T : X \to Y$, the set $\{T(x) : x \in X\}$.

2 Introduction

2.1 Previous work

The *steady water wave problem* concerns the flow of an inviscid fluid of uniform density, uniformly translating in the horizontal direction, subject to the external forces of gravity, surface tension, or both. This reduces to studying the Euler equations (see (2.1) below). In our case, we consider only pure gravity waves. The mathematical study of steady water waves has until recently focused on irrotational flow, i.e. flows with a curl-free velocity field. See Groves [14] for a general overview of this theory. There are however many naturally occurring situations in which it is necessary to take vorticity into account. Examples are running water with nonuniform velocity, and any region where the wind is blowing along a water surface.

Already in 1802 Gerstner [11] constructed an explicit example of a periodic traveling wave on water of infinite depth with a particular nonzero vorticity. In 1934 Dubreil-Jacotin [4], using a power series approach, constructed steady, periodic small-amplitude solutions with a general vorticity distribution. In 2004 Constantin and Strauss [2], using a modern functional-analytic framework, were the first to construct large-amplitude rotational waves. They applied global bifurcation theory to prove the existence of a connected solution set of exact pure gravity waves over a flat bottom, with vorticity encoded in a large class of admissible vorticity functions γ, depending only on the stream function of the flow. This breakthrough allows for the identification of new types of waves solving the Euler equations exactly, and paved the way for a new branch of research studying the properties of waves with nonzero vorticity.

One direction of study is to allow for waves with a non-uniform background flow, something that is excluded in the analysis in [2]. Some papers in this direction are [6, 19, 9, 8]. The paper [19] by Wahlén is concerned with the case where the vorticity function γ is a constant, while [6] considered the wider class of affine vorticity functions. As an effect of vorticity, these papers established the existence of waves with *critical layers*—a connected part of the fluid domain consisting only of closed streamlines—a feature not encountered in the earlier mathematical study of rotational flows, and never for irrotational flows.

2.2 The work at hand

This thesis is based on the paper [6] by Ehrnström, Escher and Wahlén, which reformulates the water wave problem in the setting of functional analysis, and then uses local bifurcation theory to prove existence of solutions. This technique exploits the fact that there are some obvious trivial solutions to the Euler equations, namely that of laminar flows which are uniform in the horizontal direction with a flat surface. Bifurcation theory allows us to study solutions which are sufficiently close (in a sense made precise later) to a trivial solution, and deduce various properties of these solutions.

The main novelty of our approach compared to that of [6] is that bifurcations are performed using a different parameter, which presents a new technical difficulty we call the orthogonality condition (Lemma 5.3). We also give detailed proofs of some claims made in [6], but never proved in that paper, and give a new proof of the Fredholm property of the linearized operator (Theorem 4.6) valid in a larger function class than previous work. In section 5.3 we present a situation where we can give a local classification of all solutions of the water wave problem in a given function space. In section 7 we present a new result for the linearized problem, showing that, with the right choice of parameters, the kernel of the linear problem can have any specified finite dimension. This parallels a result in [7]. In section 8 we compute the values of certain derivatives in the Lyapunov-Schmidt reduction, which elicit new information about the nature of our solution curves.

Any proposition which is taken directly from existing literature, and whose proof is either omitted, or for which our supplied proof does not contain any new ideas, has been clearly marked with the relevant reference. All other results are novel contributions by the author.

2.3 The governing equations

We now present the equations governing the propagation of pure gravity waves on water. Gravity is assumed to be the only external force acting on the water, and so we are neglecting effects of e.g. surface tension. We stipulate that the bottom of the domain is flat, and we choose the origin to lie at the bottom. We use a Cartesian coordinate system oriented so that the x-axis is horizontal, the y-axis points vertically upwards, and direct the z-axis so as to give a right-handed orthogonal coordinate system. The bottom is therefore given by the equation $y = 0$. In its undisturbed state with no waves, the equation of the flat surface is given by $y = d$ for some $d > 0$. In the presence of waves, we define the function $\eta_\star$ such that the free surface is given[1] by $y = d + \eta_\star(t, x, z)$.

In the mathematical treatment of steady waves, it is physically realistic to treat water as having constant density [16], and, at least for gravity waves, as being inviscid [3]. The governing equations for inviscid fluid motion are the Euler equations, and in the case of an incompressible fluid, they take the form

$$\begin{aligned} \nabla \cdot \mathbf{u}_\star &= 0, \\ \rho(\partial_t + \mathbf{u}_\star \cdot \nabla)\mathbf{u}_\star &= -\nabla P_\star + \mathbf{f}, \end{aligned} \tag{2.1}$$

expressing conservation of mass and conservation of momentum, respectively, in the domain $0 < y < d+\eta_\star(t, x, z)$ during some period of time. Here ρ is the density of the fluid, $\mathbf{u}_\star$ is the velocity field, $P_\star$ is the pressure and $\mathbf{f} = (0, -\rho g, 0)$ is the sum of external force densities. The functions $\mathbf{u}_\star$ and $P_\star$ are functions of t, x, y

[1]The subscript $\star$ is only a device for not mixing up function definitions at a later stage.

and z, and since $\eta_\star$ is a priori unknown, the domain (i.e. the form of the body of water) is also a priori unknown.

We now supply (2.1) with boundary conditions. First, we require that water does not penetrate the flat bed, which is the same as demanding that the flow at the bottom is tangential:

$$v_\star = 0 \qquad \text{on } y = 0. \tag{2.2}$$

Second, we require that a water particle with coordinates $(x(t), y(t), z(t))$ which is on the surface at time t, i.e. $y(t) = \eta_\star(t, x(t), y(t))$, will stay on the surface for all times. This is equivalent to demanding $\frac{d}{dt}y(t) = \frac{d}{dt}\eta_\star(t, x(t), z(t))$, giving rise to the boundary condition

$$v_\star = (\eta_\star)_t + (\eta_\star)_x u_\star + (\eta_\star)_z w_\star \qquad \text{on } y = d + \eta_\star(t, x, z). \tag{2.3}$$

Third, we impose the condition that the pressure at the surface is equal to the (constant) atmospheric pressure,

$$P = P_{atm} \qquad \text{on } y = d + \eta_\star(t, x, z). \tag{2.4}$$

As we will see, equation (2.4) has the effect of decoupling the motion of water from the motion of the overlying air.

To tackle (2.1)–(2.4), we need to make some simplifying assumptions. First, we reduce from three spatial dimensions to two: We assume that the motion is identical on any line parallel to the z-axis, and that the direction of wave propagation is along the x-axis. This is reasonable since the motion of most waves propagating on the surface of the sea or in a channel is close to identical in any direction parallel to the crest line. Second we restrict our search for solutions to that of steady waves, also called traveling waves, traveling with a given constant speed $c > 0$. This entails that all functions involved have a space-time dependence of the form $(x - ct, y)$, which allows us to eliminate time from the problem. We can therefore write $\mathbf{u}_\star(t, x, y, z) = (u(x - ct, y), v(x - ct, y), 0)$, where u and v are the horizontal and vertical velocity components. Similarly, we can write $\eta_\star(t, x, z) = \eta(x - ct)$ and $P_\star(t, x, y, z) = P(x - ct, y)$ for some functions η and P. With these simplifications, and by normalizing the density ρ to 1, which amounts to scaling P and g by the constant factor ρ, the equations (2.1) reduce to

$$u_x + v_y = 0, \tag{2.5}$$
$$(u - c)u_x + vu_y = -P_x, \tag{2.6}$$
$$(u - c)v_x + vv_y = -P_y - g, \tag{2.7}$$
$$v_x - u_y = \omega \tag{2.8}$$

which are to hold for all points in $\Omega_\eta = \{(x, y) \in \mathbb{R}^2 : 0 < y < d + \eta(x)\}$. Here we have also introduced the vorticity $\omega = v_x - u_y$, which we include as a separate equation since we will be looking for solutions where ω takes a specific form. The boundary conditions (2.2)–(2.4) become

$$v = 0 \qquad \text{on } y = 0, \tag{2.9}$$

$$v = (u - c)\eta_x \qquad \text{on } y = d + \eta(x), \tag{2.10}$$

$$P = P_{atm} \qquad \text{on } y = d + \eta(x). \tag{2.11}$$

The time t and the spatial dimension z is at this point eliminated from our analysis, while the interpretation of our new coordinates (x, y) is as being measured along a coordinate system traveling in the direction of propagation with speed c. The last simplification is that we will only be searching for periodic waves, and so we introduce the wave number $\kappa > 0$, and stipulate that all functions involved be $2\pi/\kappa$-periodic in the x-variable.

2.4 Stream function formulation

We now reformulate the water wave problem (2.5)–(2.11) in terms of a potential ψ, called the *relative stream function*. We will largely follow the approach of [5]. From mass conservation (2.5), stating that (u, v) is a divergence-free vector field, under the assumption that Ω_η is simply connected we know that there exists a function ψ having the properties

$$\psi_x = -v, \qquad \psi_y = u - c. \tag{2.12}$$

That Ω_η is simply connected is essentially equivalent to the assumption $\eta > -d$. This means that no waves are so large that the bottom $y = 0$ is exposed directly to the air. Equation (2.8) now becomes

$$\Delta\psi = -\omega.$$

Also observe that the condition (2.9) can be reformulated as $\psi_x = 0$ at $y = 0$, and so ψ is constant on the bottom. Furthermore, since

$$\begin{aligned} \frac{d}{dx}\psi(x, d + \eta(x)) &= \psi_x(x, d + \eta(x)) + \psi_y(x, d + \eta(x))\eta_x(x) \\ &= -v(x, d + \eta(x)) + (u(x, d + \eta(x)) - c)\eta_x(x), \end{aligned}$$

we see that surface condition (2.10) is precisely the statement that ψ is constant along the surface.

Observe that equations (2.6) and (2.7) can be written in the form

$$(\mathbf{v} \cdot \nabla) \cdot \mathbf{v} = -\nabla P - \nabla(gy),$$

where we have introduced $\mathbf{v} = (u - c, v, 0)$. Using the identity $(\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times (\nabla \times \mathbf{v})$, we get

$$\nabla P = -\nabla\left(\tfrac{1}{2}\mathbf{v} \cdot \mathbf{v} + gy\right) + \mathbf{v} \times (\nabla \times \mathbf{v}). \tag{2.13}$$

A small calculation yields

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = (u-c, v, 0) \times (0,0,\omega) = (v\omega, -(u-c)\omega, 0) = (\psi_x \Delta\psi, \psi_y \Delta\psi, 0). \tag{2.14}$$

If we take the curl of (2.13) we therefore get $(\psi_y \Delta\psi)_x - (\psi_x \Delta\psi)_y = 0$, which can be simplified to

$$\{\psi, \Delta\psi\} = 0,$$

where we have introduced the Poisson bracket $\{f, g\} = f_y g_x - f_x g_y$.

By differentiating the boundary condition $P(x, d+\eta(x)) = P_{atm}$, we get

$$\begin{aligned} & & P_x + \eta_x P_y = 0 \\ \iff & & (u-c)u_x + vu_y + \eta_x((u-c)v_x + vv_y + g) = 0 \\ \iff & & (u-c)u_x + vv_x + \eta_x((u-c)u_y + vv_y) + (-v + \eta_x(u-c))\omega = -g\eta_x \\ \iff & & \tfrac{1}{2}((u-c)^2 + v^2)_x + \tfrac{1}{2}\eta_x((u-c)^2 + v^2)_y = -g\eta_x \end{aligned} \tag{2.15}$$

which after integration is equivalent to the nonlinear surface condition (2.21).

$$\tfrac{1}{2}|\nabla\psi|^2 + g\eta = \text{constant} \qquad \text{on } y = d + \eta(x).$$

Observe finally that ψ is $2\pi/\kappa$-periodic in the horizontal variable. To see this, first recall that $\psi_x = -v$, and so

$$\psi(x_0 + 2\pi/\kappa, y) - \psi(x_0, y) = -\int_{x_0}^{x_0+2\pi/\kappa} v(s, y)\,\mathrm{d}s \tag{2.16}$$

The y-derivative of the integral in (2.16) is 0, which is a consequence of $v_y = -u_x$ coupled with the $2\pi/\kappa$-periodicity of u. Using that $v|_{y=0} = 0$, we conclude that $\psi(x_0 + 2\pi/\kappa, y) - \psi(x_0, y) = 0$.

Proposition 2.1. *[5] **(Stream function formulation)** Given $\eta \in C^3_{per}(\mathbb{R})$ and $u, v \in C^2_{per}(\overline{\Omega}_\eta)$ (where the subscript* per *denotes horizontal $2\pi/\kappa$-periodicity) the steady water-wave problem* (2.5)–(2.11) *is equivalent to the stream function formulation*

$$\Delta\psi = -\omega \qquad \text{in} \quad \Omega_\eta, \tag{2.17}$$

$$\{\psi, \Delta\psi\} = 0 \qquad \text{in} \quad \Omega_\eta, \tag{2.18}$$

$$\psi = m_0 \qquad \text{on} \quad y = 0, \tag{2.19}$$

$$\psi = m_1 \qquad \text{on} \quad y = d + \eta(x), \tag{2.20}$$

$$\tfrac{1}{2}|\nabla\psi|^2 + g\eta = Q \qquad \text{on} \quad y = d + \eta(x), \tag{2.21}$$

for $\psi \in C^3_{\mathrm{per}}(\overline{\Omega}_\eta)$, and some constants m_0, m_1 and Q.

Proof. We have already shown that the steady water wave problem implies the stream function formulation, and we now prove the converse. Given a $\psi \in C^3_{\mathrm{per}}(\overline{\Omega}_\eta)$

define $u - c = \psi_y$ and $v = -\psi_x$. Then $u, v \in C^3_{\text{per}}(\overline{\Omega}_\eta)$, and the calculation preceding this proposition makes it clear that mass conservation (2.5), the bottom condition (2.9) and the surface condition (2.10) all hold. What remains is how to define P, and show that conservation of momentum (2.6) and (2.7) as well as the pressure condition (2.11) hold.

Recall that (2.6) and (2.7) is equivalent to

$$\nabla P = -\nabla\left(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + gy\right) + \mathbf{v}\times(\nabla\times\mathbf{v}), \qquad \mathbf{v} = (\psi_y, -\psi_x, 0).$$

We can use this to *define* P (up to a constant) if the curl of the right hand side is $\mathbf{0}$, which as we have seen holds due to (2.18). Furthermore, the nonlinear surface condition (2.21) coupled with the computation (2.15) show that P is constant and at the surface, and since P up to is only defined up to a constant, we can arrange it so that $P = P_{atm}$ on $y = d + \eta(x)$. □

2.5 The vorticity function

We now show that under the condition $u < c$, there exists a function with the property that $\omega(x, y) = \gamma(\psi(x, y))$ throughout the fluid. γ is called the vorticity function, and is a measure of the strength of the vorticity. The assumption $u < c$ is supported by empirical evidence: For wave patterns not near the breaking state, the propagation speed c of the surface wave is considerably higher than the horizontal velocity u of individual water particles [16].

Lemma 2.2. *[2]* ***(Existence of a vorticity function.)*** *Suppose $u < c$. Then there exists a function γ such that $\omega(x, y) = \gamma(\psi(x, y))$ for all $(x, y) \in \Omega_\eta$.*

Proof. Since $\psi_y = u - c < 0$, we see that $\psi(x, y)$ is strictly decreasing as a function of y for fixed x. This implies the existence of a (bijective) coordinate transformation

$$q = x, \qquad p = -\psi(x, y).$$

In particular it is possible to write the vorticity as a function of q and p, i.e. $\omega = \omega(q, p)$. To prove the lemma, it suffices to show that $\partial_q\omega = 0$ throughout the fluid. First note that[2]

$$\omega_x = \omega_q + \omega_p p_x = \omega_q - \omega_p v,$$
$$\omega_y = \omega_p p_y = -\omega_p(u - c),$$

and therefore

$$\omega_q = \omega_x - \frac{v}{c - u}\omega_y, \tag{2.22}$$

where we used that $c - u$ is never 0 by assumption. Note that

$$\omega_q = 0 \qquad \Longleftrightarrow \qquad (u - c)\omega_x + v\omega_y = 0 \qquad \Longleftrightarrow \qquad \{\psi, \Delta\psi\} = 0,$$

and the latter equation is just (2.18). □

[2]We use the usual convention that partials indicate which coordinates we consider ω to depend on.

We will as announced assume γ to be affine; i.e. $\omega = \alpha\psi + \delta$, where $\alpha \neq 0$ and δ are real constants. In this case $\{\psi, \Delta\psi\} = 0$ is trivially satisfied. Making the shift $\psi - \delta/\alpha \mapsto \psi$ and the corresponding shifts of m_0 and m_1, we can assume that γ is linear: $\omega = \alpha\psi$. By also redefining $g\eta \mapsto \eta$, the stream function formulation reduces to

$$\begin{aligned} \psi_{xx} + \psi_{yy} &= \alpha\psi \quad \text{in} \quad \Omega_\eta, \\ \psi &= m_0 \quad \text{on} \quad y = 0, \\ \psi &= m_1 \quad \text{on} \quad y = d + \eta(x), \\ \tfrac{1}{2}|\nabla\psi|^2 + \eta &= Q \quad \text{on} \quad y = d + \eta(x), \end{aligned} \tag{2.23}$$

Seeing as solutions of (2.23) induces solutions of the water wave problem also when $\psi_y = u - c < 0$ is not satisfied, and we therefore drop this assumption. In fact, some of the solutions we find will satisfy $u \geq c$ at some points, and these will therefore not be covered by the general existence theory of rotational waves in Constantin and Strauss [2]. Furthermore, (2.23) makes sense also for less regular function spaces than specified in Proposition 2.1, and we will therefore allow for less regular (but still classical) solutions. See section 4.

3 Background material

We now record some mathematical machinery this thesis relies upon. The treatment will be minimalistic. In this chapter only, X and Y will denote arbitrary Banach spaces over $\mathbb{R}$, and not the special spaces given on the Notation page.

3.1 Calculus in Banach spaces

We now review the general facts about differential calculus in Banach spaces. Given two Banach spaces X and Y, by $L(X,Y)$ we mean the bounded linear operators from X to Y, which is a Banach space with norm

$$\|T\|_{L(X,Y)} = \sup_{\|x\|_X \leq 1} \|T(x)\|_Y .$$

Fréchet Derivatives. Given an open set $U \subset X$ and a function $F : U \to Y$, we say that F is Fréchet differentiable at $x_0 \in U$ if there exists $A \in L(X,Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x_0+h) - F(x_0) - Ah\|_Y}{\|h\|_X} = 0.$$

If such an operator exists, it is unique and is called the Fréchet derivative of F at x_0. We write $A = DF(x_0)$, and sometimes $A = DF[x_0]$ when we want to emphasize that the dependence on x_0 is not (necessarily) linear. Furthermore, we say that F is Fréchet differentiable on U if it is Fréchet differentiable at every point of U. Just as in the Euclidean case, Fréchet differentiability implies continuity.

It is readily seen from the definition that the Fréchet derivative of a sum of two functions is the sum of the Fréchet derivatives. A not so immediate consequence is the following.

Lemma 3.1. *[1] **(The chain rule)** Let X, Y, Z be Banach spaces, $U \subset X, V \subset Y$ open sets and suppose that $F : U \to Y$ and $G : V \to Z$ are such that $F(U) \subset V$, and that $DF[x_0]$ and $DG[F(x_0)]$ exist. Then $D(G \circ F)[x_0]$ exists and*

$$D(G \circ F)[x_0] = DG[F(x_0)] \circ DF[x_0].$$

We will often use the following corollary to the Chain Rule: If $T : Y \to Z$ and $H : W \to X$ are bounded linear operators, then $D(T \circ F)[x] = T \circ DF[x]$ and $D(F \circ H)[x] = DF[H(x)]$. A more general result is the following:

Lemma 3.2. *Let Λ be an open set in a Banach space W, and suppose that $\mathcal{T} : \Lambda \to L(X,Y)$ and $\mathcal{L} : \Lambda \to L(Y,Z)$ are differentiable. Then also $G : \Lambda \to L(X,Z)$ given by $G(\lambda) = \mathcal{L}(\lambda) \circ \mathcal{T}(\lambda)$ is differentiable, with*

$$DG[\lambda] = D_\lambda \mathcal{L}[\lambda] \circ \mathcal{T}(\lambda) + \mathcal{L}(\lambda) \circ D_\lambda \mathcal{T}[\lambda].$$

Proof. Rearranging terms and applying the triangle inequality, we obtain the estimate

$$\begin{aligned}
&\|\mathcal{L}(\lambda+h)\circ\mathcal{T}(\lambda+h)-\mathcal{L}(\lambda)\circ\mathcal{T}(\lambda)-(D_\lambda\mathcal{L}[\lambda]h)\circ\mathcal{T}(\lambda)-\mathcal{L}(\lambda)\circ(D_\lambda\mathcal{T}[\lambda)h)\|_{L(X,Z)}\\
&\le \|\mathcal{L}(\lambda+h)-\mathcal{L}(\lambda)-D_\lambda\mathcal{L}[\lambda]h\|_{L(Y,Z)}\|\mathcal{T}(\lambda)\|_{L(X,Y)}\\
&+\|\mathcal{L}(\lambda+h)\|_{L(Y,Z)}\|\mathcal{T}(\lambda+h)-\mathcal{T}(\lambda)-D_\lambda\mathcal{T}[\lambda]h\|_{L(X,Y)}\\
&+\|\mathcal{L}(\lambda+h)-\mathcal{L}(\lambda)\|_{L(Y,Z)}\|D_\lambda\mathcal{T}[\lambda]h\|_{L(X,Y)},
\end{aligned}$$

and so dividing by $\|h\|_W$ and letting $h\to 0$, the limit is 0, which proves Lemma 3.2. □

We will often be interested in finding the Fréchet derivative of an operator $\lambda\mapsto f(\lambda)T$, where f is some real-valued function and T is a bounded linear operator not depending on λ. In this special case, the above lemma gives the derivative $f'(\lambda)T$. See also Corollary 3.4.

The next lemma is an analogue of the mean value theorem in the Euclidean setting.

Lemma 3.3. *[1]* ***(The mean value theorem.)*** *Let X and Y be Banach spaces, $U\subset X$ a an open and convex set, and let $F: U\to Y$ be Fréchet differentiable at each point of U with*

$$\sup\{\|DF[x]\|_{L(X,Y)} : x\in U\} = M<\infty.$$

Then, for all $x_1, x_2\in U$,

$$\|F(x_1)-F(x_2)\|_Y\le M\|x_1-x_2\|_X.$$

Higher order derivatives. If F is Fréchet differentiable on an open set U, then higher order Fréchet derivatives can be defined in the obvious manner. Note that if $D^2F[x_0]$ exists it will be an element of $L(X,L(X,Y))$, and similarly $D^3F[x_0]$ will be an element of $L(X,L(X,L(X,Y)))$. It thus seems that $D^kF[x_0]$ is an element of a complicated space, but turns out that it has a simpler and more useful characterization, which we now discuss.

Let $n\in\mathbb{N}$. A mapping $m: X^n\to Y$ is said to be a multilinear operator, in this case n-linear, if it is linear in each variable separately, i.e. for all $k\in\{1,\dots,n\}$ and fixed $x_j\in X$, $j\ne k$, the map

$$x\mapsto m(x_1,\dots,x_{k-1},x,x_{k+1},\dots,x_n)\quad\text{is linear in } x.$$

It is called a bounded and symmetric multilinear operator if also

$$\|m\| = \sup\{\|m(x_1,x_2,\dots,x_n)\| : \|x_1\|,\dots\|x_n\|\le 1\}<\infty \tag{3.1}$$

and

$$m(x_1,\dots,x_n)=m(x_{\pi(1)},\dots,x_{\pi(n)})\quad\text{for all }\pi\in S_n.$$

where S_n is the symmetric group of order n. Let $\mathcal{M}^n_{\text{sym}}(X,Y)$ denote the set of all bounded and symmetric multilinear operators from X^n to Y, where we let $M^0_{\text{sym}}(X,Y) = Y$. With the norm (3.1), $\mathcal{M}^n_{\text{sym}}(X,Y)$ becomes a Banach space.

Suppose that $D^2F[x_0] \in L(X, L(X,Y))$ exists, and let $x_1, x_2 \in X$. Then $D^2F[x_0]x_1 \in L(X,Y)$ and $D^2F[x_0](x_1,x_2) \in Y$. We will also use the notation $D^2F[x_0]x_1x_2$ for $D^2F[x_0](x_1,x_2)$. It can be proven that the map $(x_1,x_2) \mapsto D^2F[x_0]x_1x_2$ is a bounded and symmetric 2-linear operator. With this identification, we more generally have that, if the k'th Fréchet derivative of F exists at x_0, then $D^kF[x_0] \in \mathcal{M}^k_{\text{sym}}(X,Y)$. The notation $\|D^kF[x_0]\|$ will always mean the $\mathcal{M}^k_{\text{sym}}(X,Y)$-norm.

If we combine the chain rule with Lemma 3.2, we get a formula for the second derivative of a composition:

Corollary 3.4. *[1] Let X and Y be Banach spaces, and $U \subset X$, $V \subset Y$ open sets. Suppose that $F : U \to Y$ and $G : V \to Z$ are twice differentiable and that $F(U) \subset V$. Then*

$$D^2(F \circ G)[x_0] = D^2F[G(x_0)](DG[x_0], DG[x_0]) + DF[G(x_0)]D^2G[x_0].$$

We also have the following generalization to a classical theorem from calculus.

Theorem 3.5. *[1]* ***(Taylor's Theorem.)*** *Let X be a Banach space with $U \subset X$ open and convex. Let $F \in C^{n+1}(U,Y)$, for some $n \in \mathbb{N}$, and $x, x_0 \in U$. Then if we define*

$$R_n(x,x_0) = F(x) - \sum_{k=0}^{n} \frac{1}{k!} D^kF[x_0](x-x_0)^k,$$

we have that

$$\|R_n(x,x_0)\| \leq \frac{\|x-x_0\|^{n+1}}{(n+1)!} \sup_{0\leq t\leq 1} \left\|D^{n+1}F[tx_0 + (1-t)x]\right\|.$$

Analytic operators. The operators we will be working with have Fréchet derivatives of all orders where they are defined. They will moreover by analytic, a concept we now define. We say that a function $F : X \to Y$ is analytic at $x_0 \in X$ if there exists a number $r > 0$ and a sequence $a_0, a_1, a_2, \ldots$ of bounded, symmetric multilinear operators $a_n \in \mathcal{M}^n_{\text{sym}}(X,Y)$, such that for all x with $\|x - x_0\|_X < r$, we have

$$\sum_{n=0}^{\infty} \|a_n\| \, \|x-x_0\|_X^n < \infty, \quad \text{and} \quad F(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n. \qquad (3.2)$$

In that case, it can be shown that F has Fréchet derivatives of all orders, that F is analytic at all points x with $\|x - x_0\| < r$, and that we in fact have $a_n = \frac{1}{n!}D^nF[x_0]$. In particular observe that if we are able to represent F by a sum of the form (3.2), we will be able to find $D^nF[x_0](x-x_0)^n$ for any n. This will be utilized in chapter 8.1.

The implicit function theorem. Due to its central role in this thesis, we include a proof of the implicit function theorem in a Banach space setting. We prove a version which gives quantitative bounds on the domain of existence for the implicit function. The approach is based on notes by C. Liverani [17]. Given an element $x_0 \in X$ and a $\delta > 0$, we will use the standard notation

$$B_\delta(x_0) = \{x \in X : \|x - x_0\| < \delta\}, \qquad \overline{B}_\delta(x_0) = \{x \in X : \|x - x_0\| \leq \delta\}.$$

First recall the following elementary result.

Lemma 3.6. *[1] **(Neumann series)** Let $T \in L(X, X)$, where X is a Banach space. If $\|I - T\| < 1$, then T^{-1} exists and is given by the Neumann series*

$$T^{-1} = \sum_{j=0}^{\infty} (I - T)^j.$$

Remark 3.7. *Lemma 3.6 in particular shows that $T \mapsto T^{-1}$, as an operator from the set of invertible bounded linear operators into itself, is analytic at I.*

Theorem 3.8. ***(Quantitative implicit function theorem).*** *Let X, Y and Z be Banach spaces, $U \times V \subset X \times Y$ an open subset, and $(x_0, y_0) \in U \times V$. Let also $F \in C^k(U \times V, Z)$ for some $k \in \mathbb{N}$, such that $F(x_0, y_0) = 0$ and the partial derivative $D_x F[x_0, y_0] \in L(X, Z)$ a homeomorphism[1]. Choose $0 < \delta < 1$ such that*

$$\sup_{(x,y) \in V_\delta} \left\| I - (D_x F[x_0, y_0])^{-1} D_x F[x, y] \right\| < \frac{1}{2},$$

where $V_\delta = \overline{B}_\delta(x_0) \times \overline{B}_\delta(y_0) \subset U \times V$. (Such a δ exists due to the continuity of $D_x F$.) Define

$$K_\delta = \sup_{(x,y) \in V_\delta} \|D_y F[x, y]\|, \qquad M = \left\|(D_x F[x_0, y_0])^{-1}\right\|,$$

and set $\delta_1 = \min(\delta, (2MK_\delta)^{-1}\delta)$. Then there exists a $\phi \in C^k(B_{\delta_1}(y_0), X)$ such that all solutions of the equation $F(x, y) = 0$ in the open set $S = B_\delta(x_0) \times B_{\delta_1}(y_0)$ are given by $\{(\phi(y), y) : y \in B_{\delta_1}(y_0)\}$.

Proof. Define $\Theta : \overline{B}_\delta(x_0) \times \overline{B}_{\delta_1}(y_0) \to X$ by

$$\Theta(x, y) = x - (D_x F[x_0, y_0])^{-1} F(x, y),$$

and observe that $\Theta(x, y) = x$ if and only if $F(x, y) = 0$. For all $(x, y), (x', y') \in \overline{B}_\delta(x_0) \times \overline{B}_{\delta_1}(y_0)$, we have the estimate

$$\begin{aligned} \|\Theta(x, y) - \Theta(x', y')\| &\leq \|\Theta(x, y) - \Theta(x', y)\| + \|\Theta(x', y) - \Theta(x', y')\| \\ &\leq \sup_{(x,y) \in V_\delta} \left\| I - (D_x F[x_0, y_0])^{-1} D_x F[x, y] \right\| \|x - x'\| \\ &\quad + M \sup_{(x,y) \in V_\delta} \|D_y F[x, y]\| \|y - y'\| \\ &< \tfrac{1}{2} \|x - x'\| + M K_\delta \|y - y'\|. \end{aligned} \tag{3.3}$$

[1]See also Remark 3.9.

From this estimate, we conclude the following:

i. By letting $(x', y') = (x_0, y_0)$ and using that $MK_\delta \delta_1 \leq \delta/2$, we see that $\|\Theta(x,y) - x_0\| < \delta$.

ii. By letting $y = y'$, we find that $\Theta(\cdot, y)$ is a contraction: $\|\Theta(x,y) - \Theta(x',y)\| < \frac{1}{2}\|x - x'\|$.

It follows that $\Theta(\cdot, y)$ is a contraction mapping from $\overline{B}_\delta(x_0)$ into itself. Appealing to the contraction mapping theorem, we find that there exists a unique function $\phi : \overline{B}_\delta(x_0) \to \overline{B}_{\delta_1}(y_0)$ such that $\Theta(\phi(y), y) = \phi(y)$, equivalently $F(\phi(y), y) = 0$. Since the inequality $\|\Theta(x,y) - x_0\| < \delta$ is strict, we find that ϕ actually maps into the open ball $B_\delta(x_0)$, and thus by restricting ϕ to the open ball $B_{\delta_1}(y_0)$, we have established everything in the theorem except the regularity of ϕ.

First observe that ϕ is Lipschitz continuous, since by setting $x = \phi(y)$ and $x' = \phi(y')$ in (3.3) we get

$$\|\phi(y) - \phi(y')\| \leq \tfrac{1}{2}\|\phi(y) - \phi(y')\| + MK_\delta \|y - y'\|,$$

Consequently, using the differentiability of F we must have

$$F(\phi(y+h), y+h) - F(\phi(y), y) - D_x F[\phi(y), y](\phi(y+h) - \phi(y)) - D_y F[\phi(y), y]h = o(h),$$

and since $F(\phi(y+h), y+h) = F(\phi(y), y) = 0$, we can multiply both sides with $(D_x F[\phi(y), y])^{-1}$—which exists by Lemma 3.6—to get

$$\phi(y+h) - \phi(y) + (D_x F[\phi(y), y])^{-1} D_y F[\phi(y), y]h = o(h).$$

Hence ϕ is differentiable, with

$$D\phi(y) = -(D_x F[\phi(y), y])^{-1} D_y F[\phi(y), y]. \tag{3.4}$$

Due to the formula (3.4), it is a consequence of the chain rule that ϕ is of class C^k if F is of class C^k. □

Remark 3.9. *By the bounded inverse theorem, if $D_x F[x_0, y_0] \in L(X, Z)$ is bijective then it is also a homeomorphism.*

3.2 Hölder spaces over compact sets

Let X and Y be Banach spaces. Let Ω be an open subset of X which is precompact, i.e. the closure $\overline{\Omega}$ is compact. Given a nonnegative integer k, let $C^k(\overline{\Omega}, Y)$ be the set of functions u which are k times continuously differentiable on $\overline{\Omega}$, i.e. $D^j u : \Omega \to \mathcal{M}^j_{\mathrm{sym}}(\Omega, Y)$ can be continuously extended to $\overline{\Omega}$, $j = 0, 1, \ldots k$. $C^k(\overline{\Omega}, Y)$ is a Banach space with norm

$$\|u\|_{C^k(\overline{\Omega}, Y)} = \sum_{j=0}^{k} \sup_{x \in \Omega} \left\| D^j u[x] \right\|.$$

If one identifies $D^j u$ with its continuous extension to $\overline{\Omega}$, then we of course have $\sup_{x\in\Omega} \|D^j u\| = \sup_{x\in\overline{\Omega}} \|D^j u\|$. With this identification, we also see that $D^j u : \overline{\Omega} \to Y$ is uniformly continuous, $j = 0, 1, \dots k$. This follows from the general fact that continuous functions with domain a compact set are uniformly continuous. We moreover define $C^\infty(\overline{\Omega}, Y) = \cap_{k=1}^\infty C^k(\overline{\Omega}, Y)$.

Given a nonnegative integer k and a $\beta \in (0, 1)$, we define the Hölder space $C^{k,\beta}(\overline{\Omega}, Y)$ as the set

$$C^{k,\beta}(\overline{\Omega}, Y) = \left\{ u \in C^k(\overline{\Omega}, Y) : [D^k u]_{\beta;\Omega} < \infty \right\},$$

where

$$[D^k u]_{\beta;\Omega} = \sup_{x,y\in\Omega,\, x\neq y} \frac{\|D^k u[x] - D^k u[y]\|}{\|x - y\|^\beta}, \tag{3.5}$$

$C^{k,\beta}(\overline{\Omega}, Y)$ is becomes a Banach spaces when equipped with the norm

$$\|u\|_{C^{k,\beta}(\overline{\Omega},Y)} = \|u\|_{C^k(\overline{\Omega},Y)} + [D^k u]_{\beta;\Omega}.$$

If one identifies $D^k u$ with its continuous extension to $\overline{\Omega}$, then we can let the supremum in (3.5) range over $x, y \in \overline{\Omega}$, $x \neq y$.

Lemma 3.10. *Let X be a Banach space, and suppose that $\Omega \subset X$ is open, precompact and convex. If $u \in C^1(\overline{\Omega}, Y)$, then $u \in C^{0,\beta}(\overline{\Omega})$ for any $\beta \in (0, 1)$.*

Proof. By the mean value theorem, we have

$$[u]_{\beta;\Omega} = \sup_{x,y\in\Omega,\, x\neq y} \frac{\|u(x) - u(y)\|_X}{\|x - y\|^\beta} \leq \left(\sup_{x\in\Omega} \|Du[x]\| \right) \operatorname{diam}(\Omega)^{1-\beta} < \infty.$$

□

Lemma 3.11. *Let X, Y and Z be Banach spaces. Let $\Omega_X \subset X$, $\Omega_Y \subset Y$ be open, precompact and convex. If $g \in C^{2,\beta}(\overline{\Omega}_X, Y)$, $f \in C^{2,\beta}(\overline{\Omega}_Y, Z)$ and $g(\overline{\Omega}_X) \subset \overline{\Omega}_Y$, then $f \circ g \in C^{2,\beta}(\overline{\Omega}_X, Z)$.*

Proof. We have that

$$D^2(f \circ g)[x] = D^2 f[g(x)](Dg[x], Dg[x]) + Df[g(x)]D^2 g[x],$$

and by rearranging terms and applying the triangle inequality, we find

$$\begin{aligned}
&\|D^2(f \circ g)[x_1] - D^2(f \circ g)[x_2]\| \\
&\leq \|(D^2 f[g(x_1)] - D^2 f[g(x_2)])(Dg[x_1], Dg[x_1])\| && (I) \\
&+ \|D^2 f[g(x_2)]((Dg[x_1], Dg[x_1]) - (Dg[x_2], Dg[x_2]))\| && (II) \\
&+ \|Df[g(x_1)](D^2 g[x_1] - D^2 g[x_2])\| && (III) \\
&+ \|(Df[g(x_1)] - Df[g(x_2)])D^2 g[x_2]\| && (IV)
\end{aligned}$$

We now estimate the terms (I)-(IV). Using the notation $\left\|D^j u\right\|_\infty = \sup_x \left\|D^j u(x)\right\|$, and using the mean value theorem we find that

$$(I) \leq \left\|D^2 f[g(x_1)] - D^2 f[g(x_2)]\right\| \left\|Dg[x_1]\right\|^2 \leq [D^2 f]_\beta \left\|Dg\right\|_\infty^{2+\beta} \left\|x_1 - x_2\right\|^\beta ,$$

$$\begin{aligned}(II) &= \left\|D^2 f[g(x_2)]((Dg[x_1] - Dg[x_2], Dg[x_1]) + (Dg[x_2], Dg[x_1] - Dg[x_2]))\right\| \\ &\leq 2\left\|D^2 f\right\|_\infty \left\|Dg\right\|_\infty \left\|Dg[x_1] - Dg[x_2]\right\|_\infty \leq 2\left\|D^2 f\right\|_\infty \left\|Dg\right\|_\infty \left\|D^2 g\right\|_\infty \left\|x_1 - x_2\right\| ,\end{aligned}$$

$$(III) \leq \left\|Df\right\|_\infty [D^2 g]_\beta \left\|x_1 - x_2\right\|^\beta ,$$

$$(IV) \leq \left\|D^2 f\right\|_\infty \left\|D^2 g\right\|_\infty \left\|Dg\right\|_\infty \left\|x_1 - x_2\right\| .$$

Since all terms are bounded by a constant times $\left\|x_1 - x_2\right\|^\beta$, the proof is complete. □

3.3 Fredholm operators and the Lyapunov-Schmidt reduction

Given two Banach spaces X and Y, a bounded linear operator $T : X \to Y$ is called a Fredholm operator if it satifies

i) $\dim(\ker T) < \infty$,

ii) $\operatorname{codim}(\operatorname{ran} T) < \infty$,

iii) $\operatorname{ran} T$ is closed in Y.

Here $\operatorname{codim}(\operatorname{ran} T) = \dim(\operatorname{coker} T)$, where $\operatorname{coker}(T)$ is the quotient space $Y/\operatorname{ran}(T)$. The integer $\dim(\ker T) - \operatorname{codim}(\operatorname{ran} T)$ is called the *Fredholm index* of T. The condition *iii)* in the definition of a Fredholm operator is customary to include, but is redundant as it follows from conditions *i)* and *ii)*. Recalling that any finite-dimensional subspace of a Banach space is closed, the redundancy of condition *iii)* follows from condition *ii)* coupled with the following proposition.

Proposition 3.12. *Let X and Y be Banach spaces, and $T \in L(X, Y)$ an operator such that $Y = \operatorname{ran} T \oplus C$ for some closed subspace $C \subset Y$. Then $\operatorname{ran} T$ is closed.*

Proof. Since $\ker T$ is closed also $X/\ker T$ is Banach space, and so by replacing T with the induced map from this quotient space, we can without loss of generality assume that T is injective. Introduce the operator $S : X \oplus C \to Y$ defined by $S(x, c) = T(x) + c$. It is clear that S is linear, injective and surjective, and also bounded since $S(x, c) \leq \|T\| \|x\| + \|c\| \leq \max(1, \|T\|)(\|x\| + \|c\|)$. By the bounded inverse theorem, we have that T is a homeomorphism. Since $X \oplus \{0\}$ is closed in X, it follows that $\operatorname{ran} T = S(X \oplus \{0\})$ is closed in Y. □

We will also need the following two results.

Proposition 3.13. *Let $X = C_1 \oplus C_2$ where X is a Banach space and C_1, C_2 are closed subspaces. Then the projection $\Pi_1 : X \to C_1$ is bounded.*

Proof. $C_1 \times C_2$ is a Banach space with the norm $\|(x_1, x_2)\|_{C_1 \times C_2} = \|x_1\|_X + \|x_2\|_X$. Define $S : C_1 \times C_2 \to X$ by $S(x_1, x_2) = x_1 + x_2$. S is linear, bijective, and by the triangle inequality also bounded. Applying the bounded inverse theorem we conclude that, for all $(x_1, x_2) \in C_1 \times C_2$, we have $\|x_1\| + \|x_2\| \leq M \|x_1 + x_2\|$ for some $M > 0$, and so in particular $\|x_1\| \leq M \|x_1 + x_2\|$. Thus Π_1 is bounded. □

Proposition 3.14 (Folland [10])**.** *For any finite dimensional subspace F of a Banach X, there exists a closed subspace $C \subset X$ such that $X = F \oplus C$.*

Let us now return to the setting of the implicit function theorem:

$$\begin{aligned} X, Y, Z \text{ are Banach spaces}, \quad \tilde{U} \times \tilde{V} \subset X \times Y \text{ is open}, \\ F \in C^k(\tilde{U} \times \tilde{V}, Z), \quad F(x_0, y_0) = 0, \end{aligned} \tag{3.6}$$

While we in the implicit function theorem assumed that $D_x F(x_0, y_0)$ is a homeomorphism, the Lyapunov-Schmidt reduction can be seen as a generalization of the implicit function theorem to the case where $D_x F[x_0, y_0]$ is a (not necessarily invertible) Fredholm operator. If $D_x F[x_0, y_0]$ is Fredholm, we can invoke propositions 3.13 and 3.14 to define the closed subspaces $X_0 \subset X$ and $Z_0 \subset Z$ such that

$$X = N \oplus X_0, \qquad Y = R \oplus Z_0, \tag{3.7}$$

where $N = \ker D_x F[x_0, y_0]$ and $R = \operatorname{ran} D_x F[x_0, y_0]$, with corresponding bounded projections

$$\begin{aligned} &\Pi_N : X \to N \text{ parallel to } X_0, && N = \ker D_x F[x_0, y_0], \\ &\Pi_{Z_0} : Z \to Z_0 \text{ parallel to } R, && R = \operatorname{ran} D_x F[x_0, y_0], \end{aligned} \tag{3.8}$$

Note that both Π_N and Π_{Z_0} project onto finite-dimensional subspaces.

Theorem 3.15 (Kielhöfer [15])**.** ***(Lyapunov-Schmidt reduction)*** *Assume* (3.6) *and that $D_x F[x_0, y_0]$ is a Fredholm operator. Define the spaces N, X_0, R, Z_0 and projections Π_N, Π_{Z_0} according to* (3.7) *and* (3.8)*. There is a neighborhood $U \times V \subset \tilde{U} \times \tilde{V}$ of (x_0, y_0) such that the problem*

$$F(x, y) = 0 \qquad \textit{for} \qquad (x, y) \in U \times V$$

is equivalent to the problem

$$\Phi(u_\star, y) = 0 \qquad \textit{for} \qquad (u_\star, y) \in U_\star \times V, \tag{3.9}$$

where $U_\star$ is an open subset of $U \cap N$ (which is finite-dimensional), and $\Phi \in C^k(U_\star \times V, Z_0)$. A formula for Φ is given in (3.12) *below.*

Proof. Note that the problem $F(x, y) = 0$ can be written

$$\Pi_{Z_0} F\left(\Pi_N x + (I - \Pi_N)x, y\right) = 0, \tag{3.10}$$
$$(I - \Pi_{Z_0}) F\left(\Pi_N x + (I - \Pi_N)x, y\right) = 0. \tag{3.11}$$

Choose an open neighborhood $U_\star \subset N$ containing $\Pi_N x_0$ and an open neighborhood $W \subset X_0$ containing $(I - \Pi_N)x_0$ such that $U_\star + W \subset \tilde{U}$. Choose also any neighborhood $V \subset \tilde{V}$ of y_0. Define the function $G : U_\star \times W \times V \to R$ given by

$$G(u_\star, w, y) = (I - \Pi_{Z_0})F(u_\star + w, y).$$

Because $F(x_0, y_0) = 0$ we also have $G(\Pi_N x_0, (I - \Pi_N)x_0, y_0) = 0$. Furthermore, by our choice of the spaces we see that $D_w G(\Pi_N x_0, (I - \Pi_N)x_0, y_0) = (I - \Pi_{Z_0}) D_x F\left(x_0, y_0\right) : X_0 \to R$ is bijective. Applying the implicit function theorem then yields

$$\begin{aligned} & G(u_\star, w, y) = 0 \text{ for } (u_\star, w, y) \in U_\star \times W \times V \\ \text{if and only if} \quad & w = \psi(u_\star, y), \end{aligned}$$

for some $\psi \in C^k(U_\star \times V, W)$. (According to the implicit function theorem, we might need to shrink $U_\star$, W and V, but we retain the notation.)

We have now solved (3.11) locally. Define $U = U_\star + W$, and $\Phi : U_\star \times V \to Z$ by

$$\Phi(u_\star, y) := \Pi_{Z_0} F\left(u_\star + \psi(u_\star, y), y\right).$$

Then

$$F(x, y) = 0 \qquad \text{for} \qquad (x, y) \in U \times V$$

is by (3.10) and (3.11) equivalent to there being some $u_\star \in U_\star$ such that

$$x = u_\star + \psi(u_\star, y) \quad \text{and} \quad \Phi(u_\star, y) := \Pi_{Z_0} F\left(u_\star + \psi(u_\star, y), y\right) = 0. \tag{3.12}$$

□

Remark 3.16. *If Y is finite dimensional, then* (3.9) *is a finite dimensional problem, and is equivalent to a set of* $\dim Z_0$ *(nonlinear) equations in* $\dim N + \dim Y$ *real variables. The main difficulty with solving the system $\Phi(u_\star, y) = 0$ is that ψ is only known implicitly, as the unique function satisfying $(I - \Pi_{Z_0})F(u_\star + \psi(u_\star, y), y) = 0$.*

Corollary 3.17 (Kielhöfer [15]). *In the terminology of the proof of Theorem 3.15, we have that*

$$\psi(\Pi_N x_0, y_0) = (I - \Pi_N)x_0, \tag{3.13}$$
$$D_{u_\star}\psi(\Pi_N x_0, y_0) = 0 \in L(N, X_0), \quad D_{u_\star}\Phi(\Pi_N x_0, y_0) = 0 \in L(N, Z_0). \tag{3.14}$$

Proof. The property (3.13) is immediate from the definition of ψ.

Differentiating the identity $(I - \Pi_{Z_0})F(u_\star + \psi(u_\star, y), y) = 0$ with respect to $u_\star$ gives, for all $(u_\star, y) \in U_\star \times V$,

$$(I - \Pi_{Z_0})D_x F[u_\star + \psi(u_\star, y), y](I_N + D_{u_\star}\psi(u_\star, y)) = 0$$

Evaluating at $(u_\star, y) = (\Pi_N x_0, y_0)$ and using the fact that N is the kernel of $D_x F[x_0, y_0]$, we get

$$(I - \Pi_{Z_0})D_x F[x_0, y_0]D_{u_\star}\psi(\Pi_N x_0, y_0) = 0.$$

Because $D_{u_\star}\psi(\Pi_N x_0, y_0)$ maps into X_0, which is complementary to N, and $(I - \Pi_{Z_0})D_x F[x_0, y_0]|_{X_0}$ is a bijection, we get that $D_{u_\star}\psi(\Pi_N x_0, y_0) = 0$. From the formula $\Phi(u_\star, y) = \Pi_{Z_0} F(u_\star + \psi(u_\star, y), y)$ it then follows that

$$D_{u_\star}\Phi(\Pi_N x_0, y_0) = \Pi_{Z_0} D_x F[x_0, y_0](I_N + D_{u_\star}\psi(\Pi_N x_0, y_0)) = 0,$$

because $R = \operatorname{ran} D_x F[x_0, y_0]$ is in the kernel of Π_{Z_0}. □

4 The functional-analytic setting

We recall the problem of interest

$$\psi_{xx} + \psi_{yy} = \alpha\psi \qquad \text{in } \Omega_\eta, \tag{4.1a}$$
$$\tfrac{1}{2}|\nabla\psi|^2 + \eta = Q \qquad \text{on } S, \tag{4.1b}$$
$$\psi = m_0 \qquad \text{on } B, \tag{4.1c}$$
$$\psi = m_1 \qquad \text{on } S, \tag{4.1d}$$

where the interior Ω_η, the bottom B and the surface S are subsets of $\mathbb{R}^2$ given by

$$\Omega_\eta = \{(x, y) : 0 < y < 1 + \eta(x)\},$$
$$B = \{(x, y) : y = 0\}, \quad S = \{(x, y) : y = 1 + \eta(x)\}.$$

We let α in (4.1a) be an arbitrary negative constant[1]. We will be looking for solutions $\eta \in C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S}, \mathbb{R})$ and $\psi \in C^{2,\beta}_{\text{per, even}}(\overline{\Omega}_\eta, \mathbb{R})$ of (4.1), where the subscripts signify that we are in the subspace[2] of functions which are $2\pi/\kappa$-periodic and even in the horizontal variable, and $2\pi/\kappa$-periodic functions are identified with functions on the scaled unit circle $\kappa^{-1}\mathbb{S}$. The Hölder exponent $\beta \in (0, 1)$ and the wave number $\kappa > 0$ are fixed, but arbitrary constants.

Remark 4.1. *All steady water waves known to exist are horizontally symmetric. See e.g. [7], which shows that the converse always holds: symmetric waves are steady waves. Our goal is not to prove the existence of nonsymmetric waves, and so in this thesis we impose symmetry from the start by assuming that waves are even in the horizontal variable.*

4.1 Trivial solutions and the flattening transform

Trivial solutions. The solutions of problem (4.1) we construct are perturbations of laminar flows in which the velocity field is horizontal but depth-dependent. These are the trivial solutions of (4.1) in the sense that $\eta = 0$ and $\psi(x, y)$ is independent of x. By integrating equation (4.1a), we see that the trivial solutions take the form

$$\psi_0(y; \mu, \alpha, \lambda) = \mu \cos\left(|\alpha|^{1/2}(y - 1) + \lambda\right), \qquad \mu, \lambda \in \mathbb{R}, \tag{4.2}$$

with corresponding $Q = Q(\mu, \alpha, \lambda)$, $m_0 = m_0(\mu, \alpha, \lambda)$ and $m_1 = m_1(\mu, \alpha, \lambda)$ determined from equations (4.1b), (4.1c) and (4.1d) as

$$Q(\mu, \alpha, \lambda) = \frac{\mu^2|\alpha|\sin^2(\lambda)}{2}, \quad m_0(\mu, \alpha, \lambda) = \mu\cos(\lambda - |\alpha|^{1/2}), \quad m_1(\mu, \lambda) = \mu\cos(\lambda).$$

[1] If $\alpha > 0$ then two-dimensional bifurcation is not possible, see [6].

[2] This is a closed subspace, and is therefore itself a Banach space.

We will find nontrivial solutions of (4.1) for certain combinations of μ, α and λ, corresponding to some particular values of Q, m_0 and m_1. For technical reasons which we will elucidate later (see Appendix A), it is assumed that

$$(\psi_0)_y(1) = -\mu|\alpha|^{1/2}\sin(\lambda) \neq 0. \tag{4.3}$$

As in (4.3), we will often omit the dependence on μ, α and λ in the notation for ψ, and write $\psi_0(y)$ instead of $\psi_0(y, \mu, \alpha, \lambda)$.

The flattening transform. The main difficulty with the system (4.1) is that it is a free-boundary problem, which entails that the domain is a priori unknown. This can, however, be remedied by the change of variables

$$G : (x, y) \mapsto \left(x, \frac{y}{1+\eta(x)}\right), \tag{4.4}$$

giving a bijection from the sets $\overline{\Omega}_\eta$, B and S onto

$$\overline{\hat{\Omega}} = \{(x, s) : s \in [0, 1]\}, \qquad \hat{B} = \{(x, s) : s = 0\}, \qquad \hat{S} = \{(x, s) : s = 1\},$$

respectively, where we have introduced $\hat{\Omega} = \{(x, s) : s \in (0, 1)\}$. The map G is well-defined if $\eta > -1$. Under this assumption, and recalling that $\eta \in C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S}, \mathbb{R})$, both G and its inverse $G^{-1}(x, s) = (x, (1 + \eta(x))s)$ are of class $C^{2,\beta}$. If we let $\hat{\psi}(x, s) = \psi(x, y)$, where $s = y/(1 + \eta(x))$, straightforward differentiation gives that

$$\psi_x = \hat{\psi}_x - \frac{s\eta_x\hat{\psi}_s}{1+\eta}, \qquad \psi_{xx} = \left(\hat{\psi}_x - \frac{s\eta_x\hat{\psi}_s}{1+\eta}\right)_x - \frac{s\eta_x}{1+\eta}\left(\hat{\psi}_x - \frac{s\eta_x\hat{\psi}_s}{1+\eta}\right)_s,$$

and

$$\psi_y = \frac{\hat{\psi}_s}{1+\eta}, \qquad \psi_{yy} = \frac{\hat{\psi}_{ss}}{(1+\eta)^2},$$

where ψ and its derivatives are evaluated at (x, y), η is evaluated at x, and $\hat{\psi}$ and its derivatives are evaluated at $(x, y/(1 + \eta(x)))$. Inserting these expressions into equations (4.1a) and (4.1b), we get

$$\begin{cases} \dfrac{1}{2}\left(\hat{\psi}_x - \dfrac{s\eta_x\hat{\psi}_s}{1+\eta}\right)^2 + \dfrac{1}{2}\left(\dfrac{\hat{\psi}_s}{1+\eta}\right)^2 + \eta - Q = 0 & \text{on } \hat{S}, \\ \left(\hat{\psi}_x - \dfrac{s\eta_x\hat{\psi}_s}{1+\eta}\right)_x - \dfrac{s\eta_x}{1+\eta}\left(\hat{\psi}_x - \dfrac{s\eta_x\hat{\psi}_s}{1+\eta}\right)_s + \dfrac{\hat{\psi}_{ss}}{(1+\eta)^2} - \alpha\hat{\psi} = 0 & \text{in } \hat{\Omega}. \end{cases} \tag{4.5}$$

Let $\mathcal{E}((\eta, \hat{\psi}), \Lambda)$ denote the vector consisting of the left hand sides of equations (4.5). We then have

Lemma 4.2 ([6]). *When* $\min \eta > -1$ *the steady water-wave problem* (4.1) *in the given function classes, is equivalent to the transformed problem*

$$\mathcal{E}((\eta,\hat{\psi}),\Lambda) = 0, \quad (\eta,\hat{\psi}) \in C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R}) \times \left\{\hat{\Omega} \in C^{2,\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R}) : \hat{\psi}|_{s=0} = m_0, \hat{\psi}|_{s=1} = m_1\right\},$$

and $\mathcal{E}((0,\hat{\psi}),\Lambda) = 0$ *for solutions* $\hat{\psi} = \hat{\psi}(s)$ *independent of the horizontal variable if and only if* $\hat{\psi}(s) = \psi_0(s)$.

Proof. Under the assumption that $\eta > -1$, we have as mentioned that G given by (4.4) is a diffeomorphism of class $C^{2,\beta}$. Since compositions of maps of class $C^{2,\beta}$ are again of class $C^{2,\beta}$, we see that ψ and $\hat{\psi} = \psi \circ G^{-1}$ are both of class $C^{2,\beta}$ if one of them is. Moreover, from the form of G we also see that both ψ and $\hat{\psi}$ are even and $2\pi/\kappa$-periodic in the horizontal variable if one of them is.

The equivalence of the problem (4.1) for (η,ψ) and the equation $\mathcal{E}((\eta,\hat{\psi}),\Lambda) = 0$ for $(\eta,\hat{\psi})$ thus follows from the calculations preceding this lemma, and the fact that we in the transformed problem have incorporated the boundary conditions (4.1c) and (4.1d) directly into the function space. When we insert $\eta = 0$ and $\hat{\psi} = \hat{\psi}(s)$ into the second equation of (4.5), we end up with the same equation that determined ψ_0 in the non-transformed problem. Thus the trivial, x-independent solutions of $\mathcal{E}((0,\hat{\phi}),\Lambda) = 0$ are the same for the original problem and the transformed problem, namely $s \mapsto \psi_0(s)$. □

4.2 The linearized problem

We want to linearize the problem around a trivial solution ψ_0. We therefore introduce $\hat{\psi} = \psi_0 + \hat{\phi}$, and the spaces

$$X = X_1 \times X_2 = C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R}) \times \left\{\hat{\phi} \in C^{2,\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R}) : \hat{\phi}|_{s=0} = \hat{\phi}|_{s=1} = 0\right\}$$

and

$$Y = Y_1 \times Y_2 = C^{1,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R}) \times C^{\beta}_{\text{per,even}}(\overline{\hat{\Omega}},\mathbb{R}).$$

To capture our assumptions, it is convenient to define the subsets

$$\mathcal{O} = \{(\eta,\hat{\phi}) \in X : \min \eta > -1\},$$

and, to enforce the assumptions $\alpha < 0$ and (4.3), the set

$$\mathcal{U} = \{(\mu,\alpha,\lambda) \in \mathbb{R}^3 : \mu \neq 0, \alpha < 0, \sin(\lambda) \neq 0\}.$$

From now on we will use the abbreviation $w = (\eta,\hat{\phi})$ for elements of X, and elements of $\mathcal{U}$ will be written $\Lambda = (\mu,\alpha,\lambda)$. We now define the operator $\mathcal{F} : \mathcal{O} \times \mathcal{U} \to Y$ by

$$\mathcal{F}(w,\Lambda) = \mathcal{E}\left((\eta,\psi_0(s;\Lambda) + \hat{\phi}),\Lambda\right),$$

or, written out component-wise (and remembering that $\psi_{0x} = 0$),

$$\mathcal{F}_1(w,\Lambda) = \frac{1}{2}\left[\left(\hat{\phi}_x - \frac{s\eta_x\left(\psi_{0s}+\hat{\phi}_s\right)}{1+\eta}\right)^2 + \frac{\left(\psi_{0s}+\hat{\phi}_s\right)^2}{(1+\eta)^2}\right]_{s=1} + \eta - Q(\Lambda), \quad (4.6)$$

$$\begin{aligned}\mathcal{F}_2(w,\Lambda) = &\left(\hat{\phi}_x - \frac{s\eta_x(\psi_{0s}+\hat{\phi}_s)}{1+\eta}\right)_x - \frac{s\eta_x}{1+\eta}\left(\hat{\phi}_x - \frac{s\eta_x(\psi_{0s}+\hat{\phi}_s)}{1+\eta}\right)_s \\ &+ \frac{\psi_{0ss}+\hat{\phi}_{ss}}{(1+\eta)^2} - \alpha(\psi_{0ss}+\hat{\phi}),\end{aligned} \quad (4.7)$$

where $\psi_0 = \psi_0(\cdot\,;\Lambda)$. Note that the codomains of $\mathcal{F}_1$ and $\mathcal{F}_2$ are Y_1 and Y_2, respectively. It is not immediately obvious that $\mathcal{F}$ given by (4.6) and (4.7) really does map $\mathcal{O}\times\mathcal{U}$ into Y, but now we justify this. Since the order of the highest derivative of η and ϕ appearing in $\mathcal{F}_1$ and $\mathcal{F}_2$ is one and two, respectively, we must have that $\mathcal{F}(\mathcal{O}\times\mathcal{U}) \subset C^{1,\beta}(\kappa^{-1}\mathbb{S},\mathbb{R}) \times C^{\beta}(\overline{\hat{\Omega}},\mathbb{R})$. Moreover, due to the way the derivatives appear in (4.6) and (4.7), we see that $\mathcal{F}(\cdot,\Lambda)$ preserves horizontal periodicity and evenness; in other words, if η and ϕ are T-periodic and even in the horizontal variable, then this is also true for $\mathcal{F}_1((\eta,\phi),\Lambda)$ and $\mathcal{F}_2((\eta,\phi),\Lambda)$.

Lemma 4.3 ([6])**.** *The problem $\mathcal{F}(w,\Lambda) = 0$, $(w,\Lambda) \in \mathcal{O}\times\mathcal{U}$ is equivalent to problem* (4.1)*, and $\mathcal{F}((0,\hat{\phi}),\Lambda) = 0$ for $\hat{\phi} = \hat{\phi}(s)$ if and only if $\hat{\phi} = 0$. Moreover $\mathcal{F} \in C^\infty(\mathcal{O}\times\mathcal{U},Y)$.*

Proof. That $\mathcal{F}(\eta,\hat{\phi}) = 0$, $(\eta,\hat{\phi}) \in \mathcal{O}$ is equivalent to problem (4.1) follows from the definition of $\mathcal{F}$ and Lemma 4.2. That $\mathcal{F} \in C^\infty(\mathcal{O}\times\mathcal{U},Y)$ follows from the fact that compositions of smooth maps are smooth, using that $\psi_0 \in C^\infty(\mathbb{R}\times\mathcal{U},\mathbb{R})$, that $\mathcal{F}$ depends polynomially on η and $\hat{\phi}$ and its partial derivatives, and that $\mathcal{F}$ is a rational function in $1+\eta > 0$. ☐

Linearization. We obtain the linearized problem by taking the Fréchet derivative of $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$ at the point $w = 0$. As shown in section 8.1, we have that

$$D_w\mathcal{F}_1(0,\Lambda)w = \left[\psi_{0s}\hat{\phi}_s - (\psi_{0s})^2\eta + \eta\right]_{s=1}, \quad (4.8)$$

$$D_w\mathcal{F}_2(0,\Lambda)w = (\partial_x^2 + \partial_s^2 - \alpha)\hat{\phi} - s\psi_{0s}\eta_{xx} - 2\psi_{0ss}\eta. \quad (4.9)$$

Define

$$\tilde{X}_2 = \left\{\phi \in C^{2,\beta}_{\mathrm{per,even}}(\overline{\hat{\Omega}},\mathbb{R}) : \phi|_{s=0} = 0\right\}, \qquad \tilde{X} = \left\{(\eta,\phi) \in X_1 \times \tilde{X}_2\right\}.$$

Then we have the set inclusions $X_2 \subset \tilde{X}_2$ and $X \subset \tilde{X} \subset Y$. We will typically use the symbol ϕ to denote an element in $\tilde{X}_2$. From this point onward we start relying on the assumption (4.3) that $\psi_{0s}(1) \neq 0$.

Proposition 4.4 ([6]). ***(The $\mathcal{T}$ isomorphism.)*** *The bounded linear operator $\mathcal{T}(\Lambda) : \tilde{X}_2 \to X$ given by*

$$\mathcal{T}(\Lambda)\phi = \left(-\frac{\phi|_{s=1}}{\psi_{0s}(1)}, \phi - \frac{s\psi_{0s}}{\psi_{0s}(1)}\phi|_{s=1} \right) \tag{4.10}$$

is an isomorphism of normed spaces, i.e. it is bijective and has a bounded inverse. Define $\mathcal{L}(\Lambda) = D_w\mathcal{F}(0,\Lambda)\mathcal{T}(\Lambda) : \tilde{X}_2 \to Y$. Then

$$\mathcal{L}(\Lambda)\phi = \left(\left[\psi_{0s}\phi_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}} \right) \phi \right]_{s=1}, (\partial_x^2 + \partial_s^2 - \alpha)\phi \right). \tag{4.11}$$

Proof. From the definition (4.10) of $\mathcal{T}(\Lambda)$, it is clear that it preserves horizontal evenness and $2\pi/\kappa$-peridocity. That $\mathcal{T}(\Lambda)$ is bounded follows readily from the definition of the $C^{n,\beta}$-norms, and linearity is clear. One can verify that its inverse is given by

$$(\mathcal{T}(\Lambda))^{-1}(\eta, \hat{\phi}) = \hat{\phi} - s\psi_{0s}\eta, \tag{4.12}$$

which also is seen to be linear and bounded.

Let us now show (4.11). To prove the equality in the first component, we use (4.8) and (4.10) to calculate

$$\begin{aligned}
&D_w\mathcal{F}_1(0,\Lambda)\mathcal{T}(\Lambda)\phi \\
&= \left[(\psi_{0s}) \left(\phi - \frac{s\psi_{0s}\phi|_{s=1}}{\psi_{0s}(1)} \right)_s - (\psi_{0s})^2 \left(-\frac{\phi|_{s=1}}{\psi_{0s}(1)} \right) + \left(-\frac{\phi|_{s=1}}{\psi_{0s}(1)} \right) \right]_{s=1} \\
&= \left[\psi_{0s}\phi_s - (\psi_{0s} + s\psi_{0ss})\phi|_{s=1} + \psi_{0s}\phi|_{s=1} - \frac{\phi|_{s=1}}{\psi_{0s}(1)} \right]_{s=1} \\
&= \left[\psi_{0s}\phi_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}} \right) \phi \right]_{s=1}
\end{aligned}$$

To prove the equality in the second component of (4.11), we recall that $\psi_0(s;\Lambda) = \mu\cos\left(|\alpha|^{1/2}(s-1) + \lambda\right)$ and $\alpha < 0$, and compute

$$\begin{aligned}
&(\partial_x^2 + \partial_s^2 - \alpha)\,(\mathcal{T}(\Lambda))^{-1}(\eta, \hat{\phi}) \\
&= (\partial_x^2 + \partial_s^2 - \alpha)(\hat{\phi} - s\psi_{0s}\eta) \\
&= (\partial_x^2 + \partial_s^2 - \alpha)\hat{\phi} - (s\psi_{0s}\eta_{xx}) - (2\psi_{0ss}\eta - s\psi_{0sss}\eta) - \alpha\,(s\psi_{0s}\eta) \\
&= D_w\mathcal{F}_2(0,\Lambda)(\eta, \hat{\phi}).
\end{aligned}$$

Introducing the notation $\mathcal{L}_j(\Lambda) = D_w\mathcal{F}_j(0,\Lambda) \circ \mathcal{T}$, $j = 1, 2$, we thus find that $\mathcal{L}_2(\Lambda) = (\partial_x^2 + \partial_s^2 - \alpha)$. This concludes the proof. □

We introduce the complex parameter

$$\theta_k = \sqrt{\alpha + k^2} = \begin{cases} \sqrt{\alpha + k^2}, & \alpha + k^2 \geq 0, \\ i\sqrt{|\alpha + k^2|} & \alpha + k^2 < 0. \end{cases}$$

We will soon be dealing with the functions $\cosh(\theta_k s)$ and $\sinh(\theta_k s)/\theta_k$ of s, which are real-valued functions of s:

$$\cosh(\theta_k s) = \begin{cases} \cosh(\theta_k s), & \alpha + k^2 \geq 0, \\ \cos(|\theta_k|), & \alpha + k^2 < 0, \end{cases}$$
$$\frac{\sinh(\theta_k s)}{\theta_k} = \begin{cases} \sinh(\theta_k s)/\theta_k, & \alpha + k^2 \geq 0, \\ \sin(|\theta_k| s)/|\theta_k|, & \alpha + k^2 < 0. \end{cases} \tag{4.13}$$

In the case $\theta_k = 0$ we interpret $\cosh(\theta_k s)$ as 1 and $\sinh(\theta_k s)/\theta_k$ as s.

The following lemma is stated, but not proved, in [6].

Lemma 4.5 ([6])**.** *Let* $\Lambda \in \mathcal{U}$*. A basis for* $\ker \mathcal{L}(\Lambda)$ *is given by* $\{\phi_k\}_{k\in M}$*, where*

$$\phi_k(x,s) = \begin{cases} \cos(kx)\sinh(\theta_k s)/\theta_k, & \theta_k \neq 0, \\ \cos(kx)s, & \theta_k = 0, \end{cases}$$

and M *is the set of all* $k \in \kappa\mathbb{N}$ *for which*

$$\theta_k \coth(\theta_k) = \frac{1}{\mu^2\theta_0^2 \sin^2(\lambda)} + \theta_0 \cot(\lambda). \tag{4.14}$$

Furthermore, M *is finite.*

Proof. Suppose that $\phi \in \tilde{X}_2$ is such that $\mathcal{L}(\Lambda)\phi = 0$. Since for each $s \in [0,1]$, the function $\phi(\cdot, s)$ is $2\pi/\kappa$-periodic, even and of class $C^{2,\beta}$, we know that we have the Fourier expansion

$$\phi(x,s) = \frac{1}{2}a_0(s) + \sum_{n=1}^{\infty} a_n(s)\cos(n\kappa x), \quad a_n(s) = \frac{2}{2\pi/\kappa}\int_0^{2\pi/\kappa} \phi(x,s)\cos(n\kappa x)\,\mathrm{d}x, \tag{4.15}$$

and that for each fixed s, the series is uniformly convergent in x. The fact that $\mathcal{L}_2(\Lambda)\phi = 0$ implies $\int_0^{2\pi/\kappa}(\phi_{xx} + \phi_{ss} - \alpha\phi)(x,s)\cos(n\kappa x)\,\mathrm{d}x = 0$, and using that $\phi \in \tilde{X}_2$ we deduce that

$$a_{nss}(s) - (\alpha + (n\kappa)^2)a_n(s) = 0, \quad s \in (0,1), \qquad n \in \mathbb{N}_0. \tag{4.16}$$

Using the fact that $\phi(\cdot, 0) = 0$ (by definition of $\tilde{X}_2$), and the fact that $\mathcal{L}_1(\Lambda)\phi = 0$, we get the boundary conditions

$$a_n(0) = 0, \qquad \psi_{0s}(1)a_{ns}(1) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)a_n(1) = 0, \qquad n \in \mathbb{N}_0. \tag{4.17}$$

i.e. a Dirichlet condition at the bottom, and a Robin condition at the top. Now, the general solution of (4.16) is

$$a_n(s) = A_n \cosh(\theta_{n\kappa}s) + B_n \frac{\sinh(\theta_{n\kappa}s)}{\theta_{n\kappa}}, \qquad A_n, B_n \in \mathbb{R}, \qquad n \in \mathbb{N}_0.$$

Note that $\{\cosh(\theta_{n\kappa}s), \sinh(\theta_{n\kappa}s)/\theta_{n\kappa}\}$ is a solution basis of (4.16) even in the case $\alpha + (n\kappa)^2 = 0$.

Incorporating the Dirichlet boundary condition yields that $A_n = 0$, and the Robin boundary condition then reduces to

$$\left(\psi_{0s}(1)\cosh(\theta_{n\kappa}) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)\frac{\sinh(\theta_{n\kappa})}{\theta_{n\kappa}}\right) B_n = 0. \tag{4.18}$$

Observe that if the coefficient in front of B_n in (4.18) is nonzero, then we must have $B_n = 0$ and therefore $a_n = 0$. If the coefficient is zero, then we can only conclude that $a_n(s) = B_n \sinh(\theta_{n\kappa}s)/\theta_{n\kappa}$ for some $B_n \in \mathbb{R}$. Suppose now that the coefficient is zero, i.e. that

$$\psi_{0s}(1)\cosh(\theta_{n\kappa}) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)\frac{\sinh(\theta_{n\kappa})}{\theta_{n\kappa}} = 0. \tag{4.19}$$

Note that if (4.19) holds, then $\sinh(\theta_{n\kappa})/\theta_{n\kappa} \neq 0$; otherwise (4.19) would imply that $\cosh(\theta_{n\kappa}) = \sinh(\theta_{n\kappa}) = 0$, which is not possible. Thus (4.19) is equivalent to

$$\theta_{n\kappa}\coth(\theta_{n\kappa}) = \frac{\psi_{0ss}(1)}{\psi_{0s}(1)} + \frac{1}{\psi_{0s}(1)^2}. \tag{4.20}$$

Recalling that $\psi_0(s;\Lambda) = \mu\cos(\theta_0(s-1)+\lambda)$, we find

$$\psi_{0s}(1;\Lambda) = -\mu\theta_0\sin(\lambda), \qquad \psi_{0ss}(1;\Lambda) = -\mu\theta_0^2\cos(\lambda),$$

and thus (4.20) is equivalent to

$$\theta_{n\kappa}\coth(\theta_{n\kappa}) = \frac{1}{\mu^2\theta_0^2\sin^2(\lambda)} + \theta_0\cot(\lambda), \tag{4.21}$$

where $\theta_{n\kappa}\coth(\theta_{n\kappa})$ is naturally interpreted as 1 in the case $\theta_{n\kappa} = 0$.

Define now the set M be the set of $k \in \kappa\mathbb{N}$ satisfying (4.21). We claim that M is finite. This is a consequence of the following three facts: that the right hand side of (4.21) is independent of n, that there are only finitely many $n \in \mathbb{N}$ for which $\alpha + (n\kappa)^2 < 0$, and that for $\alpha + (n\kappa)^2 \geq 0$ the function $n \mapsto \theta_{n\kappa}\coth(\theta_{n\kappa})$ is strictly increasing.

We have thus deduced that if $\phi \in \ker\mathcal{L}(\Lambda)$, then ϕ can be represented by the finite sum

$$\phi(x,s) = \sum_{n\kappa\in M} B_n \frac{\sinh(\theta_{n\kappa}s)}{\theta_{n\kappa}}\cos(kx). \tag{4.22}$$

Since also each function $\phi_k(x,s) = \frac{\sinh(\theta_k s)}{\theta_k}\cos(kx)$ is easily verified to be in $\ker\mathcal{L}(\Lambda)$ for $k \in M$, we have proved Lemma 4.5. □

The following result is stated, but not proved, in [6].

Theorem 4.6 ([6])**.** *For each $\Lambda \in \mathcal{U}$, the operator $\mathcal{L}(\Lambda)$ is Fredholm with index 0. Let $Z = \{(\eta_\phi, \phi) : \phi \in \ker \mathcal{L}(\Lambda)\} \subset \tilde{X} \subset Y$. The range of $\mathcal{L}(\Lambda)$ is the orthogonal complement of Z in Y with respect to the inner product*

$$\Big\langle (\eta_1, \hat{\phi}_1),\, (\eta_2, \hat{\phi}_2) \Big\rangle_Y = \int_0^{2\pi/\kappa} \eta_1 \eta_2 \,\mathrm{d}x + \int_0^1 \int_0^{2\pi/\kappa} \hat{\phi}_1 \hat{\phi}_2 \,\mathrm{d}x\,\mathrm{d}s, \qquad (\eta_j, \hat{\phi}_j) \in Y.$$

Let $\tilde{w}_k = (\eta_{\phi_k}, \phi_k)$ for $k \in M$, where ϕ_k and M are as in Lemma 4.5. The projection $\Pi_Z : Y \to Z$ parallel to $\operatorname{ran} \mathcal{L}(\Lambda)$ *is then given by*

$$\Pi_Z w = \sum_{k \in M} \frac{\langle w,\, \tilde{w}_k \rangle_Y}{\|\tilde{w}_k\|_Y^2} \tilde{w}_k. \tag{4.23}$$

Proof. That $(Y, \langle \cdot,\, \cdot \rangle_Y)$ is an inner product space follows from the fact that $L^2(\kappa^{-1}\mathbb{S}, \mathbb{R})$ and $L^2(\hat{\overline{\Omega}}, \mathbb{R})$ are inner product spaces. Observe that we can decompose Y into a direct sum in two ways, namely

$$Y = \operatorname{ran} \mathcal{L}(\Lambda) \oplus \operatorname{coker} \mathcal{L}(\Lambda) = \operatorname{ran} \mathcal{L}(\Lambda) \oplus \operatorname{ran} \mathcal{L}(\Lambda)^\perp.$$

If we are able to show that $Z = \{(\eta_\phi, \phi) : \phi \in \ker \mathcal{L}(\Lambda)\}$ is the orthogonal complement of $\operatorname{ran} \mathcal{L}(\Lambda)$ in $(Y, \langle \cdot,\, \cdot \rangle_Y)$, elementary considerations allows us to conclude

$$\dim(\operatorname{coker} \mathcal{L}(\Lambda)) = \dim\left(\operatorname{ran} \mathcal{L}(\Lambda)^\perp\right) = \dim(Z) = \dim(\ker \mathcal{L}(\Lambda)),$$

proving that $\mathcal{L}(\Lambda)$ is a Fredholm operator of index 0.

To show that $Z = \operatorname{ran} \mathcal{L}(\Lambda)^\perp$ in $(Y, \langle \cdot,\, \cdot \rangle_Y)$, let (η_0, ϕ_0) be an arbitrary but fixed element of $\operatorname{ran} \mathcal{L}(\Lambda)^\perp \subset Y$. Recalling the definition of $\mathcal{L}(\Lambda)$,

$$\mathcal{L}(\Lambda)\phi = \left(\left[\psi_{0s}\phi_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}} \right) \phi \right]_{s=1}, (\partial_x^2 + \partial_s^2 - \alpha)\phi \right).$$

we see that (η_0, ϕ_0) has to satisfy the following identity for all $\phi \in \tilde{X}_2$:

$$\begin{aligned} 0 = &\int_0^{2\pi/\kappa} \left[\psi_{0s}\phi_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}} \right) \phi \right]_{s=1} \eta_0 \,\mathrm{d}x \\ &+ \int_0^1 \int_0^{2\pi/\kappa} \phi_0 (\phi_{xx} + \phi_{ss} - \alpha\phi) \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{4.24}$$

In particular, for all $\phi \in \tilde{X}_2$ that has support not intersecting the boundary lines $s = 0$ and $s = 1$, we have that

$$\int_0^1 \int_0^{2\pi/\kappa} \phi_0 (\phi_{xx} + \phi_{ss} - \alpha\phi) \,\mathrm{d}x\,\mathrm{d}s = 0, \tag{4.25}$$

and we will use this to deduce properties of ϕ_0.

First, we claim that (4.25) also holds for an arbitrary $\phi \in C_c^\infty(\mathbb{R} \times (0,1), \mathbb{R})$. To see this, we first show that (4.25) holds if ϕ is any element of $C_c^\infty((0, 2\pi/\kappa) \times (0,1)$.

Indeed, defining $\tilde{\phi}(x,s) = \frac{1}{2}(\phi(\pi/\kappa - x, s) + \phi(\pi/\kappa + x, s))$, we can use partial integration and the symmetry-properties of ϕ_0 to deduce that

$$\int_0^1 \int_0^{2\pi/\kappa} \phi_0(\phi_{xx} + \phi_{ss} - \alpha\phi)\,\mathrm{d}x\,\mathrm{d}s = \int_0^1 \int_0^{2\pi/\kappa} \phi_0(\tilde{\phi}_{xx} + \tilde{\phi}_{ss} - \alpha\tilde{\phi})\,\mathrm{d}x\,\mathrm{d}s,$$

but since $\tilde{\phi}$ can be periodically extended to an element of $\tilde{X}_2$, both of these integrals have to be zero. A similar symmetry-trick allows us extend this to $\phi \in C_c^\infty(\mathbb{R} \times (0,1), \mathbb{R})$, and we therefore conclude that ϕ_0 solves $(\partial_x^2 + \partial_s^2 - \alpha)\phi_0 = 0$ in $\mathbb{R} \times (0,1)$ in the sense of distribution. By interior elliptic regularity (Folland [10, p. 327]) we conclude that $\phi_0 \in Y_2 \cap C^\infty(\hat{\Omega})$, and moreover

$$\mathcal{L}_2(\Lambda)\phi_0 = (\partial_x^2 + \partial_s^2 - \alpha)\phi_0 = 0 \qquad \text{in } \hat{\Omega}. \tag{4.26}$$

For $s \in [0,1]$ and $x \in \mathbb{R}$, we have the Fourier expansion

$$\phi_0(x,s) = \frac{1}{2}a_0(s) + \sum_{n=1}^\infty a_n(s)\cos(n\kappa x), \quad a_n(s) = \frac{2}{2\pi/\kappa}\int_0^{2\pi/\kappa} \phi_0(x,s)\cos(n\kappa x)\,\mathrm{d}x,$$

where for each s the series converges in $L^2(\kappa^{-1}\mathbb{S}, \mathbb{R})$ (Hölder continuity even implies pointwise convergence, but we will not need this fact). In the same manner as the proof of Lemma 4.5, we find

$$a_{nss}(s) - (\alpha + (n\kappa)^2)a_n(s) = 0 \quad \text{for} \quad s \in (0,1), \quad a_n(0) = 0$$

for all $n \in \mathbb{N}_0$. This entails that

$$a_n(s) = B_n \frac{\sinh(\theta_{n\kappa} s)}{\theta_{n\kappa}}, \qquad s \in [0,1], \tag{4.27}$$

where $B_n \in \mathbb{R}$. Define now the family of sequences

$$A_l^m(s) = \left(\left(\frac{1}{2}\right)^{\delta_{j0}} (n\kappa)^l a_n^{(m)}(s) \right)_{n=0}^\infty, \qquad s \in [0,1], \quad l,m \in \mathbb{N}_0, \tag{4.28}$$

where δ is the Kronecker delta. For $s \in (0,1)$, $A_l^m(s)$ is the sequence of Fourier coefficients of $\partial_x^l \partial_s^m \phi_0(\cdot, s)$, modulo a sign. In particular, from Parseval's theorem we have

$$\left\|\partial_x^l \partial_s^m \phi_0(\cdot,s)\right\|_{L^2(\kappa^{-1}\mathbb{S},\mathbb{R})} = \|A_l^m(s)\|_{L^2(\mathbb{N}_0,\mathbb{R})}, \qquad s \in (0,1).$$

Observe now from (4.27) that for all m, $s \mapsto |a_j^{(m)}(s)|$ is strictly increasing for $s \geq 0$, and therefore we have $A_l^m(0) \in L^2(\mathbb{N}_0, \mathbb{R})$ and $\lim_{s\downarrow 0} A_l^m(s) = A_l^m(0)$ for all nonnegative integers l and m. Using dominated convergence (in $L^2(\mathbb{N}_0, \mathbb{R})$) and Parseval's theorem, we see in particular that $\lim_{s\downarrow 0}\phi_0(\cdot, s)$ and $\lim_{s\downarrow 0}\phi_{0s}(\cdot, s)$ exist in $L^2(\kappa^{-1}\mathbb{S}, \mathbb{R})$, with the former limit being 0. This will now be utilized.

Given any $\varepsilon \in (0,1/2)$, we have the equalities

$$\begin{aligned}
&\int_\varepsilon^{1-\varepsilon}\int_0^{2\pi/\kappa} \phi_0(\phi_{xx}+\phi_{ss}-\alpha\phi)\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int_\varepsilon^{1-\varepsilon}\int_0^{2\pi/\kappa} (\phi_{0x}\phi_x+\phi_{0s}\phi_s-\alpha\phi_0\phi)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad+\int_0^{2\pi/\kappa}\left([\phi_0\phi_s]_{s=1-\varepsilon}-[\phi_0\phi_s]_{s=\varepsilon}\right)\,\mathrm{d}x \\
&= \int_\varepsilon^{1-\varepsilon}\int_0^{2\pi/\kappa} (\phi_{0xx}+\phi_{0ss}-\alpha\phi_0)\phi\,\mathrm{d}x\,\mathrm{d}s \\
&\quad+\int_0^{2\pi/\kappa}\left([\phi_0\phi_s-\phi_{0s}\phi]_{s=1-\varepsilon}-[\phi_0\phi_s-\phi_{0s}\phi]_{s=\varepsilon}\right)\,\mathrm{d}x \\
&= \int_0^{2\pi/\kappa}\left([\phi_0\phi_s-\phi_{0s}\phi]_{s=1-\varepsilon}-[\phi_0\phi_s-\phi_{0s}\phi]_{s=\varepsilon}\right)\,\mathrm{d}x
\end{aligned}$$

where in the last line we used (4.26). Taking the limit $\varepsilon\downarrow 0$, we get the equality

$$\int_0^1\int_0^{2\pi/\kappa} \phi_0(\phi_{xx}+\phi_{ss}-\alpha\phi)\,\mathrm{d}x\,\mathrm{d}s = \int_0^{2\pi/\kappa}[\phi_0\phi_s]_{s=1}\,\mathrm{d}x-\lim_{\varepsilon\downarrow 0}\int_0^{2\pi/\kappa}[\phi_{0s}\phi]_{s=1-\varepsilon}\,\mathrm{d}x,$$

where we used that $\lim_{s\downarrow 0}\phi(\cdot,s)=\lim_{s\downarrow 0}\phi_0(\cdot,s)=0$, and that $\lim_{s\downarrow 0}\phi_{0s}(\cdot,s)$ and $\lim_{s\downarrow 0}\phi(\cdot,s)$ exist, the limits being taken in $L^2(\kappa^{-1}\mathbb{S},\mathbb{R})$. We have therefore shown that (4.24) is equivalent to

$$\begin{aligned}
\lim_{\sigma\uparrow 1}\int_0^{2\pi/\kappa}[\phi_{0s}\phi]_{s=\sigma}\,\mathrm{d}x &= \int_0^{2\pi/\kappa}[(\psi_0\eta_0+\phi_0)\phi_s]_{s=1}\,\mathrm{d}x \\
&\quad-\int_0^{2\pi/\kappa}\left[\left(\psi_{0ss}+\frac{1}{\psi_{0s}}\right)\phi\right]_{s=1}\eta_0\,\mathrm{d}x
\end{aligned}\tag{4.29}$$

for all $\phi\in\tilde{X}_2$. In the special case $\phi(x,s)=g(s)\hat{\phi}(x)$ for functions $x\mapsto\hat{\phi}(x)$ in $C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R})$ and $s\mapsto g(s)$ in $C^\infty([0,1],\mathbb{R})$ with $g(0)=0$, (4.29) reduces to

$$\begin{aligned}
g(1)\lim_{\sigma\uparrow 1}\int_0^{2\pi/\kappa}[\phi_{0s}\hat{\phi}]_{s=\sigma}\,\mathrm{d}x &= g'(1)\int_0^{2\pi/\kappa}[(\psi_0\eta_0+\phi_0)\hat{\phi}]_{s=1}\,\mathrm{d}x \\
&\quad-g(1)\int_0^{2\pi/\kappa}\left[\left(\psi_{0ss}+\frac{1}{\psi_{0s}}\right)\hat{\phi}\right]_{s=1}\eta_0\,\mathrm{d}x.
\end{aligned}\tag{4.30}$$

Here we used that $\lim_{\sigma\uparrow 1}\int_0^{2\pi/\kappa}[\phi_{0s}\hat{\phi}]_{s=\sigma}\,\mathrm{d}x$ exists[3] for all $\hat{\phi}\in C^{2,\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R})$, which is needed to pull out $g(1)$ from the limit on the left hand side of (4.30).

[3]This can be established by choosing g such that $g(s)=1$ for $s>1/2$, and appealing to (4.29).

Letting $g(s) = s(s-1)$ in (4.30), and using the fact that $g(1) = 0$ and $g'(1) = 1$ while $\hat{\phi}$ is arbitrary, we conclude that $\int_0^{2\pi/\kappa}[(\psi_0\eta_0 + \phi_0)\hat{\phi}]_{s=1}\,\mathrm{d}x = 0$, which due to the continuity of $(\psi_0\eta_0 + \phi_0)$ and the arbitrariness of $\hat{\phi}$ implies

$$\phi_0(x, 1) = -\frac{\eta_0(x)}{\psi_{0s}(1)}. \tag{4.31}$$

In particular, this shows that $\phi_0|_{s=1}$ is of class $C^{1,\beta}$. Letting $g(s) = s$, (4.30) reduces to

$$\lim_{\sigma\uparrow 1}\int_0^{2\pi/\kappa}[\phi_{0s}]_{s=\sigma}\hat{\phi}\,\mathrm{d}x = -\int_0^{2\pi/\kappa}\left[\psi_{0ss} + \frac{1}{\psi_{0s}}\right]_{s=1}\eta_0\hat{\phi}\,\mathrm{d}x. \tag{4.32}$$

showing that, as $\sigma \uparrow 1$, $\phi_{0s}(\cdot, \sigma)$ converges to $-\left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)\eta_0(\cdot)$ in the sense of distributions.

We now show that $\lim_{\sigma\uparrow 1}\phi_{0s}(\cdot, \sigma)$ actually converges in $L^2(\kappa^{-1}\mathbb{S}, \mathbb{R})$, which is equivalent to the limit $\lim_{\sigma\uparrow 1} A_0^1(\sigma)$ existing in $L^2(\mathbb{N}_0, \mathbb{R})$. First we show that $A_0^1(1) = (B_j\cosh(\theta_j))_{j=0}^{\infty} \in L^2(\mathbb{N}_0, \mathbb{R})$. To see why this is true, first choose N so that $n \geq N$ implies $\theta_{n\kappa} \in \mathbb{R}$; it is sufficient to show that

$$\sum_{n=N}^{\infty} B_n^2 e^{2\theta_{n\kappa}} < \infty. \tag{4.33}$$

To this purpose, we will use the fact that $\phi_0|_{s=1}$ is of class $C^{1,\beta}$, implying that $\phi_{0x}|_{s=1} \in L^2(\kappa^{-1}\mathbb{S}, \mathbb{R})$. Applying Parseval's theorem again we find that

$$\sum_{n=N}\frac{(\kappa n)^2}{\theta_{n\kappa}^2}B_n^2\sinh^2(\theta_{n\kappa}) < \infty,$$

and therefore also $\sum_{n=N}^{\infty}\frac{(\kappa n)^2}{\theta_{n\kappa}^2}B_n^2 e^{2\theta_{n\kappa}} < \infty$. Noting that

$$\frac{(\kappa n)^2}{\theta_{n\kappa}^2} = \frac{1}{1 + \alpha(\kappa n)^{-2}} = 1 + O(n^{-2}) \qquad \text{as } n \to \infty,$$

we conclude that (4.33) is true, and consequently that $A_0^1(1) \in L^2(\mathbb{N}_0, \mathbb{R})$. Due to the monotonicity of $s \mapsto |a_j'(s)|$ we find that $\left\|A_0^1(s)\right\|_{L^2(\mathbb{N}_0,\mathbb{R})} \leq \left\|A_0^1(1)\right\|_{L^2(\mathbb{N}_0,\mathbb{R})}$ for $s \in (0, 1)$, and since $\lim_{s\uparrow 1} A_0^1(s) = A_0^1(1)$ pointwise (i.e. component-wise), dominated convergence ensures that also $\lim_{s\uparrow 1} A_0^1(s) = A_0^1(1)$ in $L^2(\mathbb{N}_0, \mathbb{R})$, and applying Parseval we therefore also get

$$\lim_{\sigma\uparrow 1}\phi_{0s}(x, \sigma) = \frac{1}{2}a_0'(1) + \sum_{n=1}^{\infty} a_n'(1)\cos(n\kappa x) \qquad \text{in } L^2(\kappa^{-1}\mathbb{S}, \mathbb{R}). \tag{4.34}$$

Having established $L^2(\kappa^{-1}\mathbb{S},\mathbb{R})$-convergence of $\lim_{\sigma\uparrow 1}\phi_{0s}(\cdot,\sigma)$, (4.32) and (4.31) show that

$$\begin{aligned}\lim_{\sigma\uparrow 1}\phi_{0s}(x,\sigma) &= -\left(\psi_{0ss}(1)+\frac{1}{\psi_{0s}(1)}\right)\eta_0(x)\\ &= \left(\frac{\psi_{0ss}(1)}{\psi_{0s}(1)}+\frac{1}{\psi_{0s}(1)^2}\right)\phi_0(x,1)\\ &= \left(\frac{\psi_{0ss}(1)}{\psi_{0s}(1)}+\frac{1}{\psi_{0s}(1)^2}\right)\left(\frac{1}{2}a_0(1)+\sum_{n=1}^{\infty}a_n(1)\cos(n\kappa x)\right),\end{aligned} \tag{4.35}$$

equalities also here meant as between elements of $L^2(\kappa^{-1}\mathbb{S},\mathbb{R})$. Combining (4.34) and (4.35), and using that elements of $L^2(\kappa^{-1}\mathbb{S},\mathbb{R})$ are equal precisely when their Fourier coefficients are equal, we once again end up with the condition

$$a_n'(1) = \left(\frac{\psi_{0ss}(1)}{\psi_{0s}(1)}+\frac{1}{\psi_{0s}(1)^2}\right)a_n(1), \qquad n\in\mathbb{N}_0.$$

Repeating the argument in the proof of Lemma 4.5, we find that only finitely many functions $s\mapsto a_n(s)$ are nonzero, with ϕ_0 given by the representation (4.22). Thus we have the regularity $\phi_0\in\tilde{X}_2$, and moreover that $\phi_0\in\ker\mathcal{L}(\Lambda)$. Furthermore, (4.31) shows that $\eta_0=\mathcal{T}_2(\Lambda)\phi_0$. Thus

$$(\eta_0,\phi_0)\in Z,$$

showing that $\operatorname{ran}\mathcal{L}(\Lambda)^\perp\subset Z$. Conversely, we alo have the inclusion $Z\subset\operatorname{ran}\mathcal{L}(\Lambda)^\perp$: Given $(\eta_0,\phi_0)\in Z$, ϕ_0 is smooth and therefore the statement $(\eta_0,\phi_0)\in\ker\mathcal{L}(\Lambda)^\perp$ is equivalent to (4.29), which by direct substitution is easily seen to hold for each basis function $(\eta_{\phi_k},\phi_k)\in Z$. Thus we indeed have the equality

$$Z=\operatorname{ran}\mathcal{L}(\Lambda)^\perp.$$

Lastly, we prove the formula for the projection Π_Z. Clearly $\{\phi_k\}_{k\in M}$ is a basis for $\ker\mathcal{L}(\Lambda)$, and we need only to show that it is orthogonal, i.e. that

$$\left\langle\tilde{w}_{k_i},\tilde{w}_{k_j}\right\rangle_Y=0 \qquad \text{for } k_i\neq k_j.$$

Recalling the formula

$$\eta_{\phi_k}(x) = -\frac{\phi_k(x,1)}{\psi_{0s}(1)} = -\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\cos(kx),$$

we get, for $k_i \neq k_j$,

$$
\begin{aligned}
\langle \tilde{w}_{k_i}, \tilde{w}_{k_j} \rangle_Y &= \int_0^{2\pi/\kappa} \eta_{k_i}\eta_{k_j}\,\mathrm{d}x + \int_0^1 \int_0^{2\pi/\kappa} \hat{\phi}_{k_i}\hat{\phi}_{k_j}\,\mathrm{d}x\,\mathrm{d}s \\
&= \left(\frac{\sinh(\theta_{k_i})\sinh(\theta_{k_j})}{\theta_{k_i}\theta_{k_j}\psi_{0s}(1)^2}\right)^2 \int_0^{2\pi/\kappa} \cos(k_i x)\cos(k_j x)\,\mathrm{d}x \\
&+ \int_0^1 \frac{\sinh(\theta_{k_i} s)\sinh(\theta_{k_j} s)}{\theta_{k_i}\theta_{k_j}} \int_0^{2\pi/\kappa} \cos(k_i x)\cos(k_j x)\,\mathrm{d}x\,\mathrm{d}s \\
&= 0.
\end{aligned}
$$

□

Remark 4.7. *Our proof of Theorem 4.6 is valid even in the case* $\beta = 0$*, where we identify* $C^{2,0}(\overline{\Omega}) = C^2(\overline{\Omega})$*. The reason for having* $\beta > 0$ *is that traditional proofs of the Fredholm property of the elliptic operator associated with the water wave problem rely on so-called Schauder estimates (see e.g. [12, 2, 18]), which are invalid in the* $C^{2,0}$*-setting.*

Corollary 4.8. *Given* $\Lambda \in \mathcal{U}$*, the operator* $D_w\mathcal{F}(0,\Lambda)$ *is a Fredholm operator of index* 0*.*

Proof. Since $\mathcal{L}(\Lambda) = D_w\mathcal{F}(0,\Lambda)\mathcal{T}(\Lambda)$ and $\mathcal{T}(\Lambda)$ is an isomorphism, we get

$$\dim\left(\ker D_w\mathcal{F}(0,\Lambda)\right) = \dim\left(\ker \mathcal{L}(\Lambda)\right)$$

and

$$\dim\left(\operatorname{ran} D_w\mathcal{F}(0,\Lambda)\right) = \dim\left(\operatorname{ran} \mathcal{L}(\Lambda)\right).$$

□

Remark 4.9. *In appendix A we show that if* $\psi_{0s}(1;\Lambda) = 0$*, i.e. if* $\Lambda \notin \mathcal{U}$*, then* $D_w\mathcal{F}(0,\Lambda)$ *is not a Fredholm operator.*

The bifurcation results in chapters 5 and 6 are valid under the assumption that $\ker \mathcal{L}(\Lambda)$ is respectively one- and two-dimensional. This corresponds to the cardinality of the set M in Lemma 4.5 being one or two. We now record some results on the kernel equation (4.14), which in particular show that it is indeed possible to choose $\Lambda \in \mathcal{U}$ such that $\dim(\ker \mathcal{L}(\Lambda))$ is one or two.

Lemma 4.10 ([6])**.** *(Bifurcation kernels.) Let* $k_1, k_2 \in \kappa\mathbb{N}$ *and* $\Lambda \in \mathcal{U}$*.*

(i) For every α *and any* k_1 *there are* μ *and* λ *such that the kernel equation* (4.14) *holds for at least* $k = k_1$*. The points* α *for which there exist* λ *and* μ *such that equation* (4.14) *holds for more than one* $k \in \kappa\mathbb{N}$ *are isolated.*

(ii) For any λ *with* $\cot(\lambda) \leq 0$*, and any pair* k_1, k_2 *with* $k_2^2 \geq k_1^2 + 9\pi^2/4$*, there are* $\alpha \in (-k_2^2, -k_1^2 - \pi^2)$ *and* μ *such that* (4.14) *holds for* $k = k_1, k_2$ *and for no other* $k > k_1$*.*

(iii) *For any k_1, k_2 such that $k_2^2 > k_1^2 + 3\pi^2$ and $k_2^2 - k_1^2 \notin (2\mathbb{Z}+1)\pi^2$, there are $\alpha < -k_2^2$, μ and λ such that the kernel equation* (4.14) *holds at least for $k = k_1, k_2$.*

Remark 4.11. *It follows from (i) that we can get a kernel which is one-dimensional. Moreover, if we choose $k_1 = \kappa$ in (ii), we get a kernel which is two-dimensional.*

Proof. Let $r(\mu, \alpha, \lambda) = 1/\left(\mu^2\theta_0^2 \sin^2(\lambda)\right) + \theta_0 \cot(\lambda)$ denote the right hand side of the kernel equation. For fixed α and λ such that $\cot(\lambda) < 0$, observe that we have

$$\operatorname*{ran}_{\mu \neq 0} r(\mu, \alpha, \lambda) = (\theta_0 \cot(\lambda), \infty) \supset \mathbb{R}_0^+,$$

Introducing the positive parameter $t = |\alpha|$, we study the function

$$l(t,k) = \begin{cases} \sqrt{t-k^2}\cot\left(\sqrt{t-k^2}\right), & t > k^2 \quad \text{(the trigonometric regime)},\\ \sqrt{k^2-t}\coth\left(\sqrt{k^2-t}\right), & k^2 > t \quad \text{(the hyperbolic regime)},\\ 1, & t = k^2, \end{cases}$$

i.e. the left hand side of the kernel equation. Letting $l_{tri}(x) = x\cot(x)$ and $l_{hyp}(x) = x\coth(x)$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\coth(x)\right) = \frac{\cosh(x)\sinh(x) - x}{\sinh^2(x)} > 0, \qquad 0 < x,$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\cot(x)\right) = \frac{\cos(x)\sin(x) - x}{\sin^2(x)} < 0, \qquad 0 < x \notin \pi\mathbb{N}.$$

It follows that $l(t,k)$ is strictly increasing in k and strictly decreasing in t, which of course does not apply across the singularities $\sqrt{t-k^2} \in \pi\mathbb{N}$ in the trigonometric regime. Moreover, we have that

$$\lim_{x\to a} l_{tri}(x) = \infty \qquad \text{for } a \in \{\pi^+, (2\pi)^+, (3\pi)^+, \ldots\},$$
$$\lim_{x\to a} l_{tri}(x) = -\infty \qquad \text{for } a \in \{\pi^-, (2\pi)^-, (3\pi)^-, \ldots\},$$
$$\lim_{x\to\infty} l_{hyp}(x) = \infty.$$

The functions l_{tri} and l_{hyp} are plotted in Figure 4.1.

Having established these elementary properties of l, we now prove (*i*). Given any $\alpha < 0$ and any $k_1 \in \kappa\mathbb{N}$ one can choose μ and λ such that k_1 solves the kernel equation: first choose λ such that $\theta_0\cot(\lambda) < l(|\alpha|, k_1)$, and then decrease μ such that the right hand side $r(\mu, \alpha, \lambda)$ equals $l(\alpha, k_1)$.

Next we show the second part of (*i*), namely that the values of α for which the kernel equation holds simultaneously for two different values of k are isolated. To this end, we first establish the following weaker, preliminary result: given two wave numbers k_1 and k_2, the values of α for which the kernel equation is satisfied for both k_1 and k_2 are isolated. We therefore fix some $t_0 \in \mathbb{R}$, and suppose that

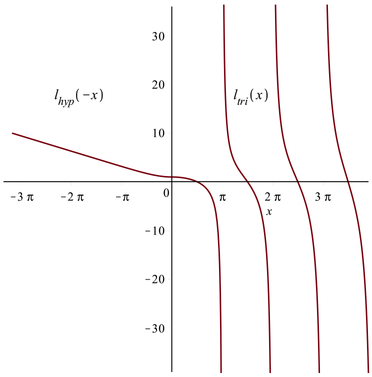

Figure 4.1: A plot showing $l_{tri}(x)$ for $x > 0$, and $l_{hyp}(-x)$ for $x < 0$.

$l(t_0, k_1) = l(t_0, k_2)$ for some $k_1 < k_2$. Note that due to the monotonicity of $l(t, k)$ for $k^2 \geq t$, we must have that $t_0 > k_1^2$. To prove the preliminary result, it suffices to show that there is an open interval containing t_0, in which the only zero of the function $f(t) = l(t, k_1) - l(t, k_2)$ is t_0. We split the analysis into three cases:

Case I: $k_2^2 < t_0$. Then locally about t_0 we have $f(t) = \sqrt{t - k_1^2} \cot\left(\sqrt{t - k_1^2}\right) - \sqrt{t - k_2^2} \cot\left(\sqrt{t - k_2^2}\right)$. Since f is an analytic real-valued function of a real variable, either f has isolated zeros, or f is identically zero[4]. To show that f is not identically zero, we compute

$$
\begin{aligned}
f'(t) &= \frac{\cot\left(\sqrt{t - k_1^2}\right)}{2\sqrt{t - k_1^2}} - \frac{1}{2}\left(1 + \cot^2\left(\sqrt{t - k_2^2}\right)\right) \\
&\quad - \frac{\cot\left(\sqrt{t - k_2^2}\right)}{2\sqrt{t - k_2^2}} + \frac{1}{2}\left(1 + \cot^2\left(\sqrt{t - k_2^2)}\right)\right) \\
&= \frac{(t - k_2^2)\sqrt{t - k_1^2} \cot\left(\sqrt{t - k_1^2}\right) - (t - k_1^2)\sqrt{t - k_2^2} \cot\left(\sqrt{t - k_2^2}\right)}{2(t - k_2^2)(t - k_2^2)} \\
&\quad + \frac{(t - k_1^2)(t - k_2^2)\left(\cot^2\left(\sqrt{t - k_2^2}\right) - \cot^2\left(\sqrt{t - k_1^2}\right)\right)}{2(t - k_2^2)(t - k_2^2)}.
\end{aligned}
$$

[4]Suppose namely that there was a strictly monotone sequence $(\tau_n)_n$ converging to t_0 with $f(\tau_n) = 0$ for all n. By applying the mean value theorem repeatedly, we conclude that there is a strictly monotone sequence $(\tau_n')_n$ also converging to t_0 with $f'(\tau_n') = 0$ for all n, and by continuity $f'(t_0) = 0$. By induction, we conclude that $f^{(m)}(t_0) = 0$ for all m, so by analyticity f is zero in a whole neighborhood about t_0.

If $f(t) = 0$, the expression for the derivative can be simplified to

$$f'(t) = \frac{(k_2^2 - k_1^2) l_{tri}\left(\sqrt{t - k_1^2}\right)\left(l_{tri}\left(\sqrt{t - k_1^2}\right) - 1\right)}{2(t - k_1^2)(t - k_2^2)}, \tag{4.36}$$

Since $l_{tri}\left(\sqrt{\cdot - k_1^2}\right)$ is locally strictly decreasing, $f'(t)$ can not be zero for all t in a neighborhood of t_0. Thus f is not locally identically zero, and so by analyticity its zeros are isolated.

Case II: $k_1^2 < t_0 < k_2^2$. We now let

$$f(t) = \sqrt{t - k_1^2} \cot\left(\sqrt{t - k_1^2}\right) - \sqrt{k_2^2 - t} \coth\left(\sqrt{k_2^2 - t}\right),$$

for which

$$\begin{aligned} f'(t) &= \frac{\cot\left(\sqrt{t - k_1^2}\right)}{2\sqrt{t - k_1^2}} - \frac{1}{2}\left(1 + \cot^2\left(\sqrt{t - k_1^2}\right)\right) \\ &+ \frac{\coth\left(\sqrt{k_2^2 - t}\right)}{2\sqrt{k_2^2 - t}} - \frac{1}{2}\left(1 + \coth^2\left(\sqrt{k_2^2 - t}\right)\right) \\ &= \frac{(k_2^2 - t)\sqrt{t - k_1^2}\cot(\sqrt{t - k_1^2}) + (t - k_1^2)\sqrt{k_2^2 - t}\coth(\sqrt{k_2^2 - t})}{2(k_2^2 - t)(t - k_2^2)} \\ &- \frac{(t - k_1^2)(k_2^2 - t)\left(\coth^2(\sqrt{k_2^2 - t}) + \cot^2(\sqrt{t - k_1^2})\right)}{2(k_2^2 - t)(t - k_2^2)}. \end{aligned}$$

If $f(t) = 0$, the expression for the derivative can be simplified to

$$f'(t) = \frac{(k_2^2 - k_1^2) l_{tri}\left(\sqrt{t - k_1^2}\right)\left(1 - l_{tri}\left(\sqrt{t - k_1^2}\right)\right)}{2(t - k_2^2)(t - k_1^2)}.$$

We can now repeat the argument following (4.36) in Case I, and conclude that f has isolated zeros.

Case III: $t_0 = k_2^2$. In this case $f(t) = l(t, k_1) - l(t, k_2)$ must be given by a piecewise defined formula near t_0. However, a straightforward computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} l(t, k_1)|_{t=t_0} = -\frac{1}{2},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} l(t, k_2)|_{t=t_0} = -\frac{1}{3},$$

and thus $f'(t_0) = -1/6$ in this case, showing that t_0 has to be an isolated zero of f.

We now define, for an arbitrary $t_0 \in \mathbb{R}$, the finite set $S_{tri}(t_0) = \{k \in \kappa\mathbb{N} : k^2 - t_0 < 0\}$, i.e. the set of all k's such that (t_0, k) is in the trigonometric regime.

We know that if $l(t_0, k_i) = l(t_0, k_j)$, then the smaller of k_i and k_j has to be in $S_{tri}(t_0)$. Due to the continuity of l, the monotonicity of l in the hyperbolic domain, and the preliminary result just proved, we know that for each $k_i \in S_{tri}(t_0)$ there is an $\varepsilon_i > 0$ such that

$$0 < |t - t_0| < \varepsilon_i \quad \Longrightarrow \quad l(t,k) \neq l(t,k_i) \quad \forall \quad k \in \kappa\mathbb{N}.$$

Letting

$$\varepsilon = \min_{k_i \in S_{tri}(t_0)} \varepsilon_i$$

we find that, for all t with $0 < |t - t_0| < \varepsilon$, we have $l(t,k_i) \neq l(t,k_j)$ for all distinct $k_i, k_j \in \kappa\mathbb{N}$. This concludes the proof of (i).

To prove (ii), choose λ such that $\cot(\lambda) < 0$ and any pair $k_1, k_2 \in \kappa\mathbb{N}$ with $k_2^2 \geq k_1^2 + 9\pi^2/4$, and consider the set I of $t = |\alpha|$ such that

$$\pi^2 < t - k_1^2 < \left(\frac{3\pi}{2}\right)^2 \quad \text{and} \quad \operatorname{ran}_{t \in I} l(t,k_1) = (1, \infty). \tag{4.37}$$

Due to the monotonicity of l with respect to k, the only possible $k > k_1$ such that $l(t,k) = l(t,k_1)$, must belong to the hyperbolic regime $t - k^2 < 0$, and there is at most one such k for a given t. Figure 1 should help make this clearer. Note now that if t fulfills (4.37), then since $k_2^2 \geq k_1^2 + 9\pi^2/4$ we have $k_2^2 - t > 0$. Since, in that case,

$$1 < l(t,k_2) = \sqrt{k_2^2 - t}\coth\left(\sqrt{k_2^2 - t}\right) < k_2 \coth(k_2),$$

while $l(t,k_1)$ spans $(1,\infty)$ as t spans I, there must by the intermediate value theorem exist $t^* \in I$ such that $l(t^*, k_1) = l(t^*, k_2)$. We then choose λ and μ such that $r(\mu, -t^*, \lambda) = l(t^*, k_1)$. This proves (ii).

We now prove (iii). Choose k_1, k_2 such that $k_2 > k_1^2 + 3\pi^2$ and $k_2^2 - k_1^2 \notin (2\mathbb{Z}+1)\pi^2$. Since we are interested in $\alpha < -k_2^2$, we study $t > k_2^2$, i.e. (t,k) in the hyperbolic regime. By assumption, there exists a unique $n \in \mathbb{N}$ such that

$$(2n+1)\pi^2 < k_2^2 - k_1^2 < (2n+3)\pi^2.$$

This implies that

$$(n+1)^2\pi^2 < k_2^2 - k_1^2 + n^2\pi^2 < k_2^2 - k_1^2 + (n+1)^2\pi^2 < (n+2)^2\pi^2. \tag{4.38}$$

On the interval $I = (k_2^2 + n^2\pi^2, k_2^2 + (n+1)^2\pi^2)$, the function $h(\cdot, k_2)$ spans $\mathbb{R}$. Inequalites (4.38) show that, over I, $h(\cdot, k_1)$ spans a bounded set, namely the interval (a,b) where

$$a = \sqrt{k_2^2 - k_1^2 + n^2\pi^2}\coth(k_2^2 - k_1^2 + n^2\pi^2)$$
$$b = \sqrt{k_2^2 - k_1^2 + (n+1)^2\pi^2}\coth(\sqrt{k_2^2 - k_1^2 + (n+1)^2\pi^2}).$$

By the intermediate value theorem, we must therefore have that $l(t_0, k_1) = l(t_0, k_2)$ for some $t_0 \in (k_2^2+n^2\pi^2, k_2^2+(n+1)^2\pi^2)$. Since $t_0 > k_2^2$, we get by setting $\alpha = -t_0$ that $\alpha < -k_2^2$, and thus the proof of part *(iii)* is finished by choosing μ and λ such that $l(|\alpha|, k_1) = r(\mu, \alpha, \lambda)$. □

4.3 The Lyapunov-Schmidt reduction

Recall that $\mathcal{L}(\Lambda)$ maps from $\tilde{X}_2$ to Y. Let Λ^* denote a triple $(\mu^*, \alpha^*, \lambda^*)$ such that $\mathcal{L}(\Lambda^*)$ holds with nontrivial kernel, say with basis $\{\phi_1^*, \ldots, \phi_n^*\}$. Let $X \ni w_j^* = \mathcal{T}(\Lambda^*)\phi_j^*$, $j = 1, \ldots, n$. Since $D_w\mathcal{F}(0, \Lambda^*)$ is Fredholm, we know from section 3.3 that there exists a closed subspace $X_0 \subset X$, and a decomposition of X into the direct sum $X = \ker D_w\mathcal{F}(0, \Lambda^*) \oplus X_0$. In our particular setting, we can even give an explicit characterization of X_0, which we now do. First, recall from Proposition 4.6 that we can decompose Y into the direct sum

$$Y = Z \oplus \operatorname{ran} \mathcal{L}(\Lambda^*),$$

where $Z = \operatorname{span}\{(\eta_{\phi_j^*}, \phi_j^*)\}_{j=1}^n$. Since $\tilde{X} \subset Y$, the decomposition of Y induces the decomposition

$$\tilde{X} = Z \oplus \left(\operatorname{ran} \mathcal{L}(\Lambda^*) \cap \tilde{X}\right).$$

Define now the map $\mathcal{R} : \tilde{X} \to X$ by

$$\mathcal{R}(\eta, \phi) = \left(\eta, \phi - \frac{s\psi_{0s}\phi|_{s=1}}{\psi_{0s}(1)}\right).$$

Note that $\mathcal{R}$ is a surjection, being the composition of the bijection

$$(\mathrm{id}, \mathcal{T}(\Lambda^*)) : X_1 \times \tilde{X}_2 \to X_1 \times X, \qquad (\eta, \phi) \mapsto (\eta, \mathcal{T}(\Lambda^*)\phi)$$

and the surjection

$$(\mathrm{id}, \pi_2) : X_1 \times X \to X_1 \times X_2, \qquad (\eta_1, (\eta_2, \phi)) \mapsto (\eta_1, \phi).$$

Observe furthermore that $\mathcal{R}(\eta, \phi) \in \ker D_w\mathcal{F}(0, \Lambda^*)$ if and only if $(\eta, \phi) \in Z$. Letting $X_0 = \mathcal{R}\left(\operatorname{ran} \mathcal{L}(\Lambda^*) \cap \tilde{X}\right)$, it follows that we can write X a sum of subspaces, namely $X = \ker D_w\mathcal{F}(0, \Lambda^*) + X_0$. This sum is also direct, because if $w \in \ker D_w\mathcal{F}(0, \Lambda^*) \cap X_0$, then $\mathcal{R}^{-1}(w) \subset Z \cap \operatorname{ran} \mathcal{L}(\Lambda^*) = \{0\}$.

$$X = \ker D_w\mathcal{F}(0, \Lambda^*) \oplus X_0.$$

Moreover, as a consequence Proposition 3.12, X_0 is closed.

Applying the Lyapunov-Schmidt reduction, Theorem 3.15 and Corollary 3.17, we obtain the following lemma.

Lemma 4.12. *There exist open neighborhoods* $\mathcal{N}$ *of* 0 *in* $\ker D_w F(0, \Lambda^*)$*,* $\mathcal{M}$ *of* 0 *in* X_0*, and* $\mathcal{U}'$ *of* Λ^* *in* $\mathbb{R}^3$*, and a unique function* $\psi : \mathcal{N} \times \mathcal{U}' \to \mathcal{M}$ *such that*

$$\mathcal{F}(w, \Lambda) = 0 \qquad \text{for } w \in \mathcal{N} + \mathcal{M},\ \Lambda \in \mathcal{U}',$$

if and only if $w = w^* + \psi(w^*, \Lambda)$ *and* $w^* = t_1 w_1^* + \cdots + t_n w_n^* \in \mathcal{N}$ *solves the finite-dimensional problem*

$$\Phi(t, \Lambda) = 0 \qquad \text{for } t \in \mathcal{V},\ \Lambda \in \mathcal{U}', \tag{4.39}$$

in which

$$\Phi(t, \Lambda) = \Pi_Z F(w, \Lambda) \quad \text{and} \quad \mathcal{V} = \{t \in \mathbb{R}^n : t_1 w_1^* + \cdots + t_n w_n^* \in \mathcal{N}\}$$

(Since $\mathcal{N}$ *is open,* $\mathcal{V}$ *contains a small ball in* $\mathbb{R}^n$ *centered at* 0*.) The function* ψ *has the following properties:* $\psi \in C^\infty(\mathcal{N} \times \mathcal{U}', \mathcal{M})$*,* $\psi(0, \Lambda) = 0$ *for all* $\Lambda \in \mathcal{U}'$*, and* $D_w \psi(0, \Lambda^*) = 0$*.*

5 One-dimensional bifurcation

We show that a curve of nontrivial solutions bifurcates from a point $(0, \Lambda^*) \in X \times \mathcal{U}$ where the kernel of $D_w\mathcal{F}(0, \Lambda^*)$ is one-dimensional, given that Λ^* satisfies an additional technical condition. We also we use the one-dimensional bifurcation result to give a local classification of all solutions near a trivial solution having an associated one-dimensional kernel.

5.1 Preliminaries

Lemma 5.1. *Suppose that $\phi_j \in \ker \mathcal{L}(\Lambda)$, where $\phi_j = \phi_{k_j}$ is the basis function given by the formula (4.5). Then if $\tilde{w}_j = (\eta_{\phi_j}, \phi_j)$ is the corresponding basis function of Z, we have*

$$\langle D_\lambda \mathcal{L}(\Lambda)\phi_j, \, \tilde{w}_j \rangle_Y = A \left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right)^2, \tag{5.1}$$

where

$$A = \frac{\pi}{\psi_{0s}(1)} \left[\mu\theta_0 \cos(\lambda)\theta_{k_j} \coth(\theta_{k_j}) + \mu\theta_0^2 \sin(\lambda) + \frac{\cos(\lambda)}{\mu\theta_0 \sin^2(\lambda)} \right] \neq 0.$$

Moreover,

$$\langle D_\lambda \mathcal{L}(\Lambda)\phi_j, \, \tilde{w}_j \rangle_Y = 0 \quad \iff \quad \cot(\lambda) = -\frac{\mu^2 |\alpha|^{3/2}}{2}.$$

Remark 5.2. *Observe that, for a given Λ, the quantity A is independent of k_j, in the sense that it has the same value for all k_j such that (k_j, Λ) solves the kernel equation. This obervation will be important in chapter 6, when we invoke this lemma in the proof of the two-dimensional bifurcation result.*

Proof. First we list formulas for some derivatives of $\psi_0(s, \Lambda) = \mu \cos(\theta_0(s-1)+\lambda)$, namely

$$\begin{aligned} \psi_{0s}(s, \Lambda) &= -\mu\theta_0 \sin(\theta_0(s-1) + \lambda), \\ \psi_{0s\lambda}(s, \Lambda) &= -\mu\theta_0 \cos(\theta_0(s-1) + \lambda), \\ \psi_{0ss\lambda}(s, \Lambda) &= \mu\theta_0^2 \sin(\theta_0(s-1) + \lambda). \end{aligned} \tag{5.2}$$

Recall that $\mathcal{L}(\Lambda) = \left(\left[\psi_{0s}\partial_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}} \right) \mathrm{id} \right]_{s=1}, \partial_x^2 + \partial_s^2 - \alpha \, \mathrm{id} \right)$. Using the remark following Lemma 3.2, we know that the Frechet derivative $D_\lambda \mathcal{L}(\Lambda)$ can be computed by formal differentiation with respect to λ, i.e.

$$D_\lambda \mathcal{L}(\Lambda) = \left(\left[\psi_{0s\lambda}\partial_s - \left(\psi_{0ss\lambda} - \frac{\psi_{0s\lambda}}{\psi_{0s}^2} \right) \mathrm{id} \right]_{s=1}, 0 \right). \tag{5.3}$$

Thus, using that $\phi_j(x,s) = \cos(k_j x)\sinh(\theta_{k_j} s)/\theta_{k_j}$ and the formulas (5.2), we find

$$
\begin{aligned}
&D_\lambda \mathcal{L}(\Lambda)\phi_j(x,s) \\
&= \left(\left[-\mu\theta_0 \cos(\lambda)\cosh(\theta_{k_j}) - \left(\mu\theta_0^2 \sin(\lambda) + \frac{\cos(\lambda)}{\mu\theta_0 \sin^2(\lambda)} \right) \frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right] \cos(k_j x), 0 \right) \\
&= \left(\left[- \left(\mu\theta_0 \cos(\lambda)\theta_{k_j} \coth(\theta_{k_j}) + \mu\theta_0^2 \sin(\lambda) + \frac{\cos(\lambda)}{\mu\theta_0 \sin^2(\lambda)} \right) \frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right] \cos(k_j x), 0 \right) \\
&= \left(-\frac{A\psi_{0s}(1)}{\pi} \frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \cos(k_j x), 0 \right),
\end{aligned}
$$

where we from the first to the second line divided by $\sinh(\theta_{k_j})/\theta_{k_j}$, which is nonzero since (k_j, Λ) satisfies the kernel equation (4.14). Since $\tilde{w}_j = (\eta_{\phi_j}, \phi_j)$, we also need to recall that

$$
\eta_{\phi_j}(x) = -\frac{\phi_j(x,1)}{\psi_{0s}(1)} = -\frac{\cos(k_j x)}{\psi_{0s}(1)} \frac{\sinh(\theta_{k_j})}{\theta_{k_j}}.
$$

Now,

$$
\begin{aligned}
\langle D_\lambda \mathcal{L}(\Lambda)\phi_j,\, \tilde{w}_j \rangle_Y &= \int_0^{2\pi/\kappa} (D_\lambda \mathcal{L}(\Lambda)\phi_j)_1\, \eta_{\phi_j}\, \mathrm{d}x + \int_0^1 \int_0^{2\pi/\kappa} (D_\lambda \mathcal{L}(\Lambda)\phi_j)_2\, \phi_j\, \mathrm{d}x\, \mathrm{d}s \\
&= A \left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right)^2 \int_0^{2\pi/\kappa} \frac{\cos^2(k_j x)}{\pi}\, \mathrm{d}x \\
&= A \left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right)^2.
\end{aligned}
$$

The inner product is zero if and only if $A = 0$, i.e. if

$$
0 = \mu\theta_0 \cos(\lambda)\theta_k \coth(\theta_k) + \mu\theta_0^2 \sin(\lambda) + \frac{\cos(\lambda)}{\mu\theta_0 \sin^2(\lambda)} \tag{5.4}
$$

$$
\iff \quad \theta_k \coth(\theta_k) = -\theta_0 \tan(\lambda) - \frac{1}{\mu^2\theta_0 \sin^2(\lambda)}. \tag{5.5}
$$

From (5.4) to (5.5) we divided with μ, θ_0 and $\cos(\lambda)$. That μ and θ_0 are nonzero follows from the assumption $\Lambda \in \mathcal{U}$. Moreover, note that $\cos(\lambda) = 0$ will by (5.4) imply that $\mu\theta_0 \sin(\lambda) = 0$, which is not possible again due to the fact that $\Lambda \in \mathcal{U}$. Thus all divisions are justified. We can now combine the kernel equation with (5.5) to deduce that

$$
\begin{aligned}
&\theta_0 \cot(\lambda) + \frac{1}{\mu^2\theta_0^2 \sin^2(\lambda)} = -\theta_0 \tan(\lambda) - \frac{1}{\mu\theta_0^2 \sin^2(\lambda)} \\
\iff \quad &\tan(\lambda) + \cot(\lambda) = -\frac{2}{\mu^2\theta_0^3 \sin^2(\lambda)} \\
\iff \quad &1 = -\frac{2\cos(\lambda)}{\mu^2|\alpha|^{3/2} \sin(\lambda)} \\
\iff \quad &\cot(\lambda) = -\frac{\mu|\alpha|^{3/2}}{2}.
\end{aligned}
$$

□

Let us from now on call

$$\cot(\lambda) = -\mu^2|\alpha|^{3/2}/2 \tag{5.6}$$

the orthogonality condition. We now show that if the orthogonality condition holds for some $\Lambda \in \mathcal{U}$, then it is possible to rechoose μ and λ such that the right hand side $r(\mu,\alpha,\lambda)$ of the kernel equation remains unchanged, while the orthogonality condition fails. Recall the definition

$$r(\mu,\alpha,\lambda) = \frac{1}{\mu^2\theta_0^2\sin^2(\lambda)} + \theta_0\cot(\lambda).$$

Lemma 5.3. *Suppose that $\Lambda = (\mu,\alpha,\lambda) \in \mathcal{U}$ is such that the orthogonality condition* (5.6) *holds. Then there are $\tilde{\mu},\tilde{\alpha} \in \mathbb{R}$ with $\tilde{\Lambda} = (\tilde{\mu},\alpha,\tilde{\lambda}) \in \mathcal{U}$ such that*

$$r(\tilde{\mu},\alpha,\tilde{\lambda}) = r(\mu,\alpha,\lambda) \quad \textit{and} \quad \cot(\tilde{\lambda}) \neq -\frac{\tilde{\mu}^2|\alpha|^{3/2}}{2}.$$

Proof. First observe that since

$$\partial_\mu r(\mu,\alpha,\lambda) = \frac{-2}{\mu^3\theta_0^2\sin^2(\lambda)} \neq 0,$$

there is an open rectangle $I_1 \times I_2$ centered in (μ^*,λ^*) and a unique function $\mu : I_2 \to I_1$ such that $r(\mu(\lambda),\alpha^*,\lambda) = r(\mu^*,\alpha^*,\lambda^*)$ for all $\lambda \in I_2$. We will show that there exists $\lambda \in I_2$ such that

$$\cot(\lambda) \neq -\mu(\lambda)^2\theta_0^3/2.$$

Assuming the contrary, we get the explicit expression $\mu(\lambda) = \mathrm{sgn}(\mu^*)\sqrt{2|\cot(\lambda)|/\theta_0^3}$. Using $\cot(\lambda) = -\mu(\lambda)^2\theta_0^3/2$ to eliminate λ from the right hand side of the kernel equation, we end up with

$$r(\mu(\lambda),\alpha^*,\lambda) = \frac{4+\mu(\lambda)^4\theta_0^6}{4\theta_0^2\mu(\lambda)^2} - \frac{1}{2}\theta_0^3\mu(\lambda)^2, \tag{5.7}$$

which by our assumptions and the definition of $\lambda \mapsto \mu(\lambda)$ should be the constant $r(\mu^*,\alpha^*,\lambda^*)$ for all $\lambda \in I_2$. However, note that for fixed θ_0 the expression (5.7) is not a constant function over any interval for $\mu(\lambda)$. For if $(4+x^4\theta_0^6)/(4\theta_0^2x^2) - \theta_0^3x^2/2 = C$, then also $4+x^4\theta_0^6 - 4C\theta_0^2x^2 - 2\theta_0^5x^4 = 0$, which by the fundamental theorem of algebra can not hold over a whole interval in x for fixed θ_0. Since $\mu(\lambda) = \mathrm{sgn}(\mu^*)\sqrt{2|\cot(\lambda)|/\theta_0^3}$ obviously is continuous and not the constant function μ^*, we finally arrive at a contradiction. Thus we see that, it is in fact possible to choose (μ,λ) arbitrarily close to (μ^*,λ^*) such that $r(\mu,\alpha^*,\lambda) = r(\mu^*,\alpha^*,\lambda^*)$, while $\cot(\lambda) \neq -\mu(\lambda)^2\theta_0^3/2$. □

5.2 The one-dimensional bifurcation result

The following result is essentially an application of the famous Crandall-Rabinowitz bifurcation theorem (see e.g. Kielhöfer [15]). To clarify the proof of the two-dimensional bifurcation in the next section, we will nonetheless spell out the details of the proof.

Theorem 5.4 ([6])**.** *Suppose that $\Lambda^* \in \mathcal{U}$ is such that $\ker \mathcal{L}(\Lambda^*) = 1$, so that*

$$\ker D_w \mathcal{F}(0, \Lambda^*) = \operatorname{span}\{w^*\},$$

where $w^* = \mathcal{T}(\Lambda^*)\phi_k$ for the unique $k \in \kappa\mathbb{N}$ such that (Λ^*, k) satisfies the kernel equation (4.14)*. Suppose furthermore that we have the non-orthogonality condition*

$$\cot(\lambda^*) \neq \frac{(\mu^*)^2|\alpha^*|^{3/2}}{2}. \tag{5.8}$$

Then there exists a C^∞-curve of small-amplitude nontrivial solutions $\{(\overline{w}(t), \overline{\lambda}(t)) : 0 < |t| < \varepsilon\}$ of

$$\mathcal{F}(w, \mu^*, \alpha^*, \lambda) = 0,$$

in $\mathcal{O} \times \mathbb{R}$, passing through $(\overline{w}(0), \overline{\lambda}(0)) = (0, \lambda^*)$, with

$$\overline{w}(t) = tw^* + O(t^2) \qquad \text{in } \mathcal{O} \text{ as } t \to 0. \tag{5.9}$$

In a neighborhood of $(0, \lambda^*)$ in $\mathcal{O} \times \mathbb{R}$, these are the only nontrivial solutions of (5.9)*. For small enough $|t| > 0$, the surface profile of $\overline{w}(t)$ has minimal period $2\pi/k$, has one crest and one trough per minimal period, and is strictly monotone between crest and trough.*

Proof. Using the terminology from section 4.3, we know that there exists a neighborhood of $(0, \lambda^*) \in \mathcal{O} \times \mathbb{R}$ for which the equation $\mathcal{F}(w, \mu^*, \alpha^*, \lambda) = 0$ is equivalent to $\Phi(t, \mu^*, \alpha^*, \lambda) = 0$, where $t \in \mathbb{R}$. Since the range of Φ, Z, in our case is one-dimensional, we find

$$\Phi(t, \Lambda) = \Phi_1(t, \Lambda)\tilde{w}^*,$$

where $\tilde{w}^* = (\eta_{\phi_k}, \phi_k) \in Z \subset Y$, and Φ_1 is a smooth real-valued function. From Lemma 4.12 we know that we have the identity $\Phi(0, \Lambda) = 0$, and therefore also $\Phi_1(0, \Lambda) = 0$. Hence we can write

$$\begin{aligned}
\Phi_1(t, \Lambda) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}z}(\Phi_1(tz, \Lambda))\,\mathrm{d}z \\
&= t\int_0^1 (\Phi_{1t})(tz, \Lambda)\,\mathrm{d}z \\
&= t\Psi(t, \Lambda),
\end{aligned}$$

where we have defined $\Psi(t, \Lambda) = \int_0^1 \Phi_{1t}(tz, \Lambda)\,\mathrm{d}z$. For $t \neq 0$ the equations $\Phi_1 = 0$ and $\Psi = 0$ are equivalent, whence to prove the theorem we need only concern ourselves with zeros of Ψ.

We want to apply the implicit function theorem to Ψ. We start by showing that $\Psi_\lambda(0,\Lambda^*) \neq 0$. Now, by definition of Ψ we find

$$\Psi_\lambda(0,\Lambda^*) = \Phi_{1t\lambda}(0,\Lambda^*), \tag{5.10}$$

and from the formula $\Phi(t,\Lambda) = \Phi_1(t,\Lambda)\tilde{w}^*$, we get

$$\Phi_{1t\lambda}(0,\Lambda^*)\tilde{w}^* = \Phi_{t\lambda}(0,\Lambda^*), \tag{5.11}$$

By definition of Φ we see that

$$\begin{aligned}\Phi_t(t,\Lambda) &= \partial_t\left(\Pi_Z \mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)\right)\\ &= \Pi_Z D_w\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)(w^* + \psi_w(tw^*,\Lambda)w^*)\end{aligned}$$

and so by evaluating in $t = 0$, and using the properties of ψ listed in Lemma 4.12, we find

$$\Phi_t(0,\Lambda) = \Pi_Z D_w\mathcal{F}(0,\Lambda)(w^* + \psi_w(0,\Lambda)w^*). \tag{5.12}$$

Furthermore we have

$$\begin{aligned}\Phi_{t\lambda}(0,\Lambda) &= \partial_\lambda\left(\Pi_Z D_w\mathcal{F}(0,\Lambda)(w^* + \psi_w(0,\Lambda)w^*)\right)\\ &= \Pi_Z D^2_{w\lambda}\mathcal{F}(0,\Lambda)(w^* + \psi_w(tw^*,\Lambda)) + D_w\mathcal{F}(0,\Lambda)\psi_{w\lambda}(w^*,\Lambda),\end{aligned}$$

and so by evaluating in $\Lambda = \Lambda^*$, and using that Z is the orthogonal complement of $\operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$, we get

$$\begin{aligned}\Phi_{t\lambda}(0,\Lambda^*) &= \Pi_Z D^2_{w\lambda}\mathcal{F}(0,\Lambda)w^*\\ &= \frac{\left\langle D^2_{w\lambda}\mathcal{F}(0,\Lambda^*)w^*,\, \tilde{w}^*\right\rangle_Y}{\|\tilde{w}^*\|^2_Y}\tilde{w}^*,\end{aligned} \tag{5.13}$$

where we in the second line used the formula (4.23) for the projection Π_Z. Thus we see from (5.11) that $\Phi_{1t\lambda}(0,\Lambda^*)$ equals the coefficient of $\tilde{w}^*$ in (5.13). Going back to (5.10) we therefore see that $\Psi_\lambda(0,\Lambda^*) \neq 0$ if and only if

$$\left\langle D^2_{w\lambda}\mathcal{F}(0,\Lambda^*)w^*,\, \tilde{w}^*\right\rangle_Y \neq 0.$$

Now, by definition of $\mathcal{L}$ we have $D_w\mathcal{F}(0,\Lambda)w^* = \mathcal{L}(\Lambda)\circ\mathcal{T}(\Lambda)^{-1}w^*$, so using Lemma 3.2 we find

$$D^2_{w\lambda}\mathcal{F}(0,\Lambda^*)w^* = D_\lambda\mathcal{L}(\Lambda^*)\phi_k + \mathcal{L}(\Lambda^*)\circ D_\lambda\left(\mathcal{T}(\Lambda^*)^{-1}\right)w^*.$$

The second term is in $\operatorname{ran}\mathcal{L}(\Lambda^*)$, which is the orthogonal complement of Z. Thus $\Psi_\lambda(0,\Lambda^*) \neq 0$ if and only if $\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_k,\, \tilde{w}^*\rangle_Y \neq 0$, and the latter holds true due to Lemma 5.1 and the non-orthogonality assumption (5.8).

We now show that $\Psi(0,\Lambda^*) = 0$. To see this, observe first that $\Psi(0,\Lambda) = \partial_t\Phi_1(0,\Lambda)$, and since $\Phi(t,\Lambda) = \Phi_1(t,\Lambda)\tilde{w}^*$, it suffices to show that $\partial_t\Phi(0,\Lambda^*) = 0$. This follows from (5.12) evaluated in $\Lambda = \Lambda^*$, and once again using that Z is the orthogonal complement of $\operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$.

Finally, since $\Psi(0,\Lambda^*) = 0$ and $\partial_\lambda \Psi(0,\Lambda^*) \neq 0$, we can invoke the implicit function theorem, and deduce that there exists an $\varepsilon > 0$, a C^∞-function $\overline{\lambda} : (-\varepsilon, \varepsilon)$ with $\overline{\lambda}(0) = \lambda^*$ such that $\Psi(t, \mu^*, \alpha^*, \lambda(t)) \equiv 0$, and since $\Phi(t,\Lambda) = t\Psi(t,\Lambda)w^*$, we have

$$\mathcal{F}(tw^* + \psi[tw^*, \mu^*, \alpha^*, \overline{\lambda}(t)], \mu^*, \alpha^*, \overline{\lambda}(t)) = 0 \quad \text{for all } 0 < |t| < \varepsilon,$$

or in the notation of Theorem 5.4, $\overline{w}(t) = tw^* + \psi(tw^*, \mu^*, \alpha^*, \overline{\lambda}(t))$. It follows that

$$\dot{\overline{w}}(t) = w^* + \psi_w(tw^*, \mu^*, \alpha^*, \overline{\lambda}(t))w^* + \psi_\lambda(tw^*, \mu^*, \alpha^*, \overline{\lambda}(t))\dot{\overline{\lambda}}(t),$$

and we can conclude, using the properties of ψ given in Lemma 4.12, that $\overline{w}(0) = 0$ and $\dot{\overline{w}}(0) = w^*$. Consequently,

$$\overline{w}(t) = tw^* + O(t^2).$$

Letting $\overline{w}(t) = (\overline{\eta}(t), \overline{\phi}(t))$ we now prove that, for small enough $|t| > 0$, the surface profiles $\overline{\eta}(t)$ have one crest and one trough per minimal period, and are strictly monotone between crest and trough. First recall that $w^* = (\mathcal{T}_1(\Lambda^*)\phi_k, \mathcal{T}_2(\Lambda^*)\phi_k)$, where

$$\mathcal{T}_1(\Lambda^*)\phi_k = \eta_{\phi_k}(x) = -\frac{\phi_k(x,1)}{\psi_{0s}(1)} = -\frac{\sinh(\theta_k)}{\theta_k \psi_{0s}(1)} \cos(kx).$$

We know that

$$\|\overline{\eta}(t) - t\eta_{\phi_k}\|_{C^2} \leq Kt^2$$

for small enough $|t|$. In the following we consider $t > 0$; the analysis for $t < 0$ is similar. For a given $\varepsilon > 0$, we define the open sets

$$I_1(\varepsilon) = \{x \in \mathbb{R} : |\eta'_{\phi_k}(x)| > \varepsilon\}, \qquad I_2(\varepsilon) = \{x \in \mathbb{R} : |\eta''_{\phi_k}(x)| > \varepsilon\}.$$

Since η_{ϕ_k} is simply a scaled cosine, it is possible to choose ε_0 such that $I_1(\varepsilon_0) \cup I_2(\varepsilon_0) = \mathbb{R}$. Then for $t < \varepsilon_0/K$ we have

$$\operatorname{sgn}(\overline{\eta}(t)'(x)) = \operatorname{sgn}\left(\eta'_{\phi_k}(x)\right), \qquad \text{for } x \in I_1(\varepsilon_0),$$

and

$$\operatorname{sgn}(\overline{\eta}(t)''(x)) = \operatorname{sgn}\left(\eta''_{\phi_k}(x)\right), \qquad \text{for } x \in I_2(\varepsilon_0).$$

It follows that $\overline{\eta}(t)'$ will have alternating signs on each neighboring connected component of I_1, while $\overline{\eta}(t)'$ will have alternating signs on each connected component of I_2. It follows that the minimal period of $\overline{\eta}(t)$ is the same as for $x \mapsto \cos(kx)$, i.e. $2\pi/k$. Moreover, over the course of one minimal period each of $\overline{\eta}(t)'$ and $\overline{\eta}(t)'$ have exactly two zeros, with all four zeros distinct. In other words, $\overline{\eta}(t)$ has one local minimum and one local maximum per period whenever $0 < t < \varepsilon_0/K$. □

Remark 5.5. *As shown in Theorem 4.6 in [6], there is an analogous version of Theorem 5.4 where one instead of λ uses μ as the bifurcation parameter, but without any non-orthogonality restriction. We have not been able to determine whether the two resulting bifurcation curves consist of the same solutions of $F(w,\Lambda) = 0$.*

5.3 Local classification of solutions

Everything in this section is original.

Using Theorem 5.4, we now classify all solutions in a neighborhood of $(0, \Lambda^*) \in X \times \mathcal{U}$ for any Λ^* for which $\ker D_w\mathcal{F}(0, \Lambda^*)$ is one-dimensional and the non-orthogonality condition $\cot(\lambda^*) \neq -(\mu^*)^2|\alpha^*|^{3/2}/2$ is fulfilled. This can be achieved by several careful applications of the implicit function theorem. The details are as follows. Suppose that we have a one-dimensional kernel with wavenumber k, i.e. $\ker D_w\mathcal{F}(0, \Lambda^*) = \operatorname{span}\{\phi_k\}$. For this fixed k, define

$$K(\mu, \alpha, \lambda) = \theta_k \coth(\theta_k) - \frac{1}{\mu^2\theta_0^2 \sin^2(\lambda)} - \theta_0 \cot(\lambda).$$

We then have $K(\mu^*, \alpha^*, \lambda^*) = 0$, and

$$\partial_\lambda K(\mu, \alpha, \lambda) = \frac{\sin(2\lambda)}{\mu^2\theta_0^2 \sin^4(\lambda)} + \frac{\theta_0}{\sin^2(\lambda)}, \tag{5.14}$$

which is zero if and only if

$$\cot(\lambda) = -\frac{\mu^2\theta_0^3}{2},$$

which is not true at $\Lambda = \Lambda^*$ by assumption.

By applying the Lyapunov-Schmidt reduction we obtain a neighborhood $B(\Lambda^*)\times \mathcal{U}_0(\Lambda^*)$ of $(0, \Lambda^*)$ in which the equation $\mathcal{F}(w, \Lambda)$ is equivalent to a finite-dimensional equation $\Phi(t, \Lambda) = 0$. Since we are assuming that (5.14) is nonzero at $\Lambda = \Lambda^*$, the implicit function theorem guarantees that there is a neighborhood of Λ^* and a unique function $(\mu, \alpha) \mapsto \lambda(\mu, \alpha)$ satisfying

$$K(\mu, \alpha, \lambda(\mu, \alpha)) = 0,$$

for (μ, α) close to (μ^*, α^*), and moreover $(\mu, \alpha) \mapsto \lambda(\mu, \alpha)$ is of class C^∞. Let $\mathcal{C} \subset \mathbb{R}^3$ denote the graph of $(\mu, \alpha) \mapsto \lambda(\mu, \alpha)$. Using the fact that the values of α for which $\dim\ker D_w\mathcal{F}(0, \Lambda) > 1$ form an isolated set, we can assume, by possibly shrinking $\mathcal{U}(\Lambda^*)$, that all $\Lambda \in \mathcal{C} \cap \mathcal{U}(\Lambda^*)$ are such that $\dim\ker \mathcal{L}(\Lambda) = 1$, implying that in fact $\ker \mathcal{L}(\Lambda) = \operatorname{span}\{\phi_k\}$. By possibly shrinking $\mathcal{U}(\Lambda^*)$ further, we can assume that the non-orthogonality condition is satisfied for each $\Lambda \in \mathcal{U}(\Lambda^*)$, which therefore allows us to invoke Theorem 5.4 at each $\Lambda \in \mathcal{C} \cap \mathcal{U}(\Lambda^*)$. In other words, for all $\Lambda \in \mathcal{C} \cap \mathcal{U}(\Lambda^*)$ we get a solution curve bifurcating out of the point. As Λ varies, we therefore get a smooth two-dimensional sheet $\mathcal{S}$ of bifurcating solution curves.

We now argue that, locally, the sheet $\mathcal{S}$ comprises all solutions. For each $\Lambda \in \mathcal{C} \cap \mathcal{U}(\Lambda^*)$, applying Theorem 5.4 yields a uniqueness-domain

$$U(\Lambda) = \{(w', \mu, \alpha, \lambda') \in X \times \mathcal{U}(\Lambda^*) : \|w'\| < r(\Lambda), |\lambda' - \lambda| < r(\Lambda)\},$$

such that the only nontrivial solutions $(w', \lambda') \in X \times \mathbb{R}$ of $\mathcal{F}(w', \mu, \alpha, \lambda') = 0$ lying in the set $U(\Lambda)$ are given by the solution curve $\{(\overline{w}(t; \Lambda), \overline{\lambda}(t; \Lambda)),\ 0 < |t| < \varepsilon(\Lambda)\}$

from Theorem 5.4. Using the quantitative version of the implicit function theorem, we can assume that $r(\Lambda)$ is continuous, and subsequently—by possibly shrinking $\mathcal{U}(\Lambda^*)$—that $r(\Lambda)$ is equal to a constant $r_0 > 0$ on $\mathcal{U}(\Lambda^*)$. Consider now the set $A \subset X \times \mathcal{U}$ defined by

$$A = \{(w, \mu, \alpha, h) : (\mu, \alpha, \lambda) \in \mathcal{C} \cap \mathcal{U}(\Lambda^*), |h - \lambda| < r_0 \text{ and } \|w\| < r_0\}.$$

From its definition it is clear that A is an open set, and that all solutions of $\mathcal{F}(w, \Lambda) = 0$ in A are given by the two-dimensional sheet of solutions curves obtained by letting $\Lambda \in \mathcal{C} \cap \mathcal{U}(\Lambda^*)$ and applying Theorem 5.4. See also Figure 5.1.

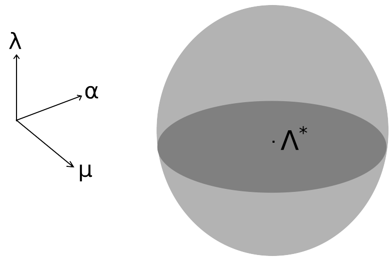

Figure 5.1: In this illustration, $\mathcal{U}(\Lambda^*) \subset \mathbb{R}^3$ is a ball centered in Λ^* (light grey), and $\mathcal{C}$ is a portion of a plane (dark grey). Since we can assume that the uniqueness-domain has uniform size for all $\Lambda \in \mathcal{C}$, all solutions of $\mathcal{F}(w, \Lambda) = 0$ near $(0, \Lambda^*)$ are obtained by applying Theorem 5.4 at each dark grey point.

6 Two-dimensional bifurcation

6.1 Preliminaries

In this chapter we will in addition to λ also be using α as a bifurcation parameter, and so we are eventually going to need the analogue of Lemma 5.1 for the parameter α. To avoid lengthy calculations, we will prove a version giving us only what is needed for carrying out the proof of the two-dimensional bifurcation result.

Lemma 6.1. *Suppose that $\phi_j \in \ker \mathcal{L}(\Lambda)$, where $\phi_j = \phi_{k_j}$ is the basis function given by the formula (4.5). Then if $\tilde{w}_j = (\eta_{\phi_j}, \phi_j)$ is the corresponding basis function of Z, we have*

$$\langle D_\alpha \mathcal{L}(\Lambda)\phi_j, \tilde{w}_j\rangle_Y = B\left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}}\right)^2 + f(k_j),$$

where

$$B = \frac{-\pi}{\psi_{0s}(1)}\left[\psi_{0s\alpha}(1)\theta_{k_j}\cot(\theta_{k_j}) - \psi_{0ss\alpha}(1) + \frac{\psi_{0s\alpha}(1)}{\psi_{0s}(1)^2}\right]\left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}}\right)^2,$$

$$f(k_j) = \begin{cases} \frac{\pi}{2}\frac{\theta_{k_j} - \cosh(\theta_{k_j})\sinh(\theta_{k_j})}{\theta_{k_j}^3}, & \theta_{k_j} \neq 0, \\ -\frac{\pi}{3} & \theta_{k_j} = 0. \end{cases}$$

Remark: For a given Λ, we see that the quantity B is independent of k_j.

Proof. Recall that $\mathcal{L}(\Lambda) = \left(\left[\psi_{0s}\partial_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}}\right)\mathrm{id}\right]_{s=1}, \partial_x^2 + \partial_s^2 - \alpha\,\mathrm{id}\right)$. Using the remark following Lemma 3.2, we know that the Fréchet derivative $D_\alpha\mathcal{L}(\Lambda)$ can be computed by formal differentiation with respect to α, giving

$$D_\alpha \mathcal{L}(\Lambda) = \left(\left[\psi_{0s\alpha}\partial_s - \left(\psi_{0ss\alpha} - \frac{\psi_{0s\alpha}}{\psi_{0s}^2}\right)\mathrm{id}\right]_{s=1}, -\mathrm{id}\right).$$

Thus, using that $\phi_j(x,s) = \cos(k_j x)\sinh(\theta_{k_j}s)/\theta_{k_j}$, we find

$$\begin{aligned} &D_\alpha \mathcal{L}(\Lambda)\phi_j(x,s) \\ &= \left(\psi_{0s\alpha}(1)\cos(\theta_{k_j}) - \left(\psi_{0ss\alpha}(1) - \frac{\psi_{0s\alpha}(1)}{\psi_{0s}(1)^2}\right)\frac{\sinh(\theta_{k_j})}{\theta_{k_j}}, -\frac{\sinh(\theta_{k_j}s)}{\theta_{k_j}}\right)\cos(k_j x). \end{aligned}$$

Recalling that

$$\eta_{\phi_j}(x) = -\frac{\sinh(\theta_k)}{\psi_{0s}(1)\theta_{k_j}}\cos(k_j x),$$

we find

$$\begin{aligned}
&\int_0^{2\pi/\kappa} (D_\alpha \mathcal{L}(\Lambda)\phi_j)_1 \, \eta_{\phi_j} \, \mathrm{d}x \\
&= \frac{-\pi}{\psi_{0s}(1)} \left[\psi_{0s\alpha}(1)\theta_{k_j} \coth(\theta_{k_j}) - \psi_{0ss\alpha}(1) + \frac{\psi_{0s\alpha}(1)}{\psi_{0s}(1)^2} \right] \left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right)^2 \\
&= B \left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}} \right)^2 .
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi/\kappa} (D_\alpha \mathcal{L}(\Lambda)\phi_j)_2 \, \phi_j \, \mathrm{d}x \, \mathrm{d}s \\
&= - \int_0^1 \int_0^{2\pi/\kappa} (\phi_j)^2 \, \mathrm{d}x \, \mathrm{d}s \\
&= - \int_0^{2\pi/\kappa} \cos^2(k_j x) \, \mathrm{d}x \cdot \int_0^1 \left(\frac{\sinh(\theta_k s)}{\theta_k} \right)^2 \mathrm{d}x \\
&= -\pi \int_0^1 \left(\frac{\sinh(\theta_k s)}{\theta_k} \right)^2 \mathrm{d}x \\
&= \begin{cases} \frac{\pi}{2\theta_k^2} - \frac{\pi \sinh(2\theta_k)}{4\theta_k^3} = \frac{\pi}{2} \frac{\theta_{k_j} - \cosh(\theta_{k_j}) \sinh(\theta_{k_j})}{\theta_{k_j}^3}, & \theta_{k_j} \neq 0 \\ -\frac{1}{3}, & \theta_{k_j} = 0, \end{cases}
\end{aligned}$$

and thus $\int_0^1 \int_0^{2\pi/\kappa} (D_\alpha \mathcal{L}(\Lambda)\phi_j)_2 \, \phi_j \, \mathrm{d}x \, \mathrm{d}s = f(k_j)$. This concludes the proof.

□

Lemma 6.2. *Let $\kappa > 0$, and let $k_1, k_2 \in \kappa\mathbb{N}$ with $k_2/k_1 \notin \mathbb{N}$. Then given a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous, $2\pi/k_1$-periodic and even, we have that $\int_0^{2\pi/\kappa} f(x) \cos(k_2 x) \, \mathrm{d}x = 0$.*

Proof. The Fourier series of f converges to f in L^2, and thus it suffices to show that $\int_0^{2\pi/\kappa} \cos(nk_1 x) \cos(k_2 x) \, \mathrm{d}x = 0$ for all $n \in \mathbb{N}_0$. Using exponential notation, we find

$$\begin{aligned}
&\int_0^{2\pi/\kappa} \cos(nk_1 x) \cos(k_2 x) \, \mathrm{d}x \\
=&\frac{1}{4} \int_0^{2\pi/\kappa} \left(e^{i(nk_1 + k_2)x} + e^{i(-nk_1 + k_2)x} + e^{i(nk_1 - k_2)x} + e^{-i(nk_1 + k_2)x} \right) \mathrm{d}x \\
=&0.
\end{aligned}$$

□

Suppose that we have a two-dimensional kernel, $\ker \mathcal{L}(\Lambda^*) = \operatorname{span}\{\phi_{k_1}, \phi_{k_2}\}$, with ϕ_{k_j} given by formula (4.5). Again, we want to find solutions of $\mathcal{F}(w, \Lambda) = 0$ in the vicinity of $(0, \Lambda^*)$. In the analysis of this problem, we will be able to utilize the one-dimensional bifurcation result, coupled with the fact that our discussion so far is valid for arbitrary wave numbers κ. From now on, we let $X^{(k_1)}$ denote the subset of X which consists of functions which are $2\pi/k_1$-periodic in the horizontal variable, and we will in the same manner superscript with (k_j) all spaces and operators which one gets by instead of operating with the wave number κ, one uses the wave number k_j ($\in \kappa\mathbb{N}$). Let $\mathcal{O}^{(k_1)}$ and $\mathcal{O}^{(k_2)}$ be the subspaces of $\mathcal{O}$ consisting of functions which are respectively $2\pi/k_1$ and $2\pi/k_2$-periodic in the horizontal variable. Then $\mathcal{F}^{(k_2)}$ ($= \mathcal{F}|_{\mathcal{O}^{(k_2)}}$) is well-defined, and the kernel of $D_w\mathcal{F}^{(k_2)}(0, \Lambda)$ is one-dimensional, spanned by ϕ_{k_2}. Using the classification from section 5.3, we obtain the set $\mathcal{S}^{(k_2)}$ of all solutions of $\mathcal{F}(w, \Lambda) = 0$ in a neighborhood of $(0, \Lambda^*)$ in $X^{(k_2)} \times \mathcal{U}$. Similarly, under the condition[1] $k_2/k_1 \notin \mathbb{N}$, the kernel of $\mathcal{F}^{(k_1)}$ is spanned by ϕ_{k_1}, and we obtain the set $\mathcal{S}^{(k_1)}$ of all solutions of $\mathcal{F}(w, \Lambda) = 0$ in a neighborhood of $(0, \Lambda^*)$ in $X^{(k_1)} \times \mathcal{U}$. In this section we prove a two-dimensional bifurcation result, which shows that there are solutions of $\mathcal{F}(w, \Lambda) = 0$ near $(0, \Lambda^*)$ which are neither in $X^{(k_1)} \times \mathcal{U}$ nor $X^{(k_2)} \times \mathcal{U}$.

In the one-dimensional bifurcation result, Theorem 5.4, we fixed $\mu = \mu^*$ and $\alpha = \alpha^*$, and used λ as a bifurcation parameter. When the kernel is two-dimensional we need two bifurcation parameters, which we will take to be α and λ, and hence we will be looking for solutions of the equation

$$\mathcal{F}(w, \mu^*, \alpha, \lambda) = 0. \tag{6.1}$$

Let, for $j = 1, 2$, $\mathcal{S}^{(k_j)}_{\mu^*}$ denote all $(w, \mu, \alpha, \lambda) \in \mathcal{S}^{(k_j)}$ where $\mu = \mu^*$, i.e. the near-zero solutions of equation (6.1) which are horizontally $2\pi/k_j$-periodic.

6.2 The two-dimensional bifurcation result

Theorem 6.3 ([6])**.** *Suppose that we have a two-dimensional kernel*

$$\ker D_w\mathcal{F}(0, \Lambda^*) = \operatorname{span}\{\mathcal{T}(\Lambda^*)\phi_{k_1}, \mathcal{T}(\Lambda^*)\phi_{k_2}\}, \quad k_1, k_1 \in \kappa\mathbb{N}, \quad k_1 < k_2,$$

that the non-orthogonality condition

$$\cot(\lambda^*) \neq -\frac{(\mu^*)^2|\alpha^*|^{3/2}}{2} \tag{6.2}$$

is fulfilled, and define

$$a = \theta_{k_1}\coth(\theta_{k_1}) = \theta_{k_2}\coth(\theta_{k_2}), \tag{6.3}$$

i.e. the left hand side of the kernel equation. Furthermore, assume either that $a \notin \{0, 1\}$, or that $\theta_{k_2} = 0$ (in which case $a = 1$).

[1]Unless this condition is fulfilled, any $2\pi/k_2$-periodic function will be $2\pi/k_1$-periodic as well.

(i) If $k_2/k_1 \notin \mathbb{N}$, there exists a smooth sheet of small-amplitude nontrivial solutions,

$$\mathcal{S}^{mixed} = \{\big(\overline{w}(t_1,t_2), \overline{\alpha}(t_1,t_2), \overline{\lambda}(t_1,t_2)\big) : 0 < |t_1|, |t_2| < \varepsilon\}$$

of (6.1) *in $\mathcal{O} \times \mathbb{R} \times \mathbb{R}$, passing through $\big(\overline{w}(0,0), \overline{\alpha}(0,0), \overline{\lambda}(0,0)\big) = (0, \alpha^*, \lambda^*)$, with*

$$\overline{w}(t_1,t_2) = t_1 w_1^* + t_2 w_2^* + O(t_1^2 + t_2^2). \tag{6.4}$$

In a neighborhood of $(0, \alpha^*, \lambda^*)$ in $\mathcal{O} \times \mathbb{R} \times \mathbb{R}$ the set $\mathcal{S}^{mixed} \cup \mathcal{S}_{\mu^*}^{(k_1)} \cup \mathcal{S}_{\mu^*}^{(k_2)}$ contains all nontrivial solutions of (6.1)*.*

(ii) Let $\delta > 0$. If $k_2/k_1 \in \mathbb{N}$, there exists a smooth sheet of small-amplitude nontrivial solutions

$$\mathcal{S}_\delta = \{\big(\overline{w}(r,v), \overline{\alpha}(r,v), \overline{\lambda}(r,v)\big) : 0 < r < \varepsilon, \delta < |v| < \pi - \delta\}$$

of (6.1) *in $\mathcal{O} \times \mathbb{R}^2$, passing through $\big(\overline{w}(0,v), \overline{\alpha}(0,v), \overline{\lambda}(0,v)\big) = (0, \alpha^*, \lambda^*)$, with*

$$\overline{w}(r,v) = r\cos(v) w_1^* + r\sin(v) w_2^* + O(r^2). \tag{6.5}$$

In a neighborhood of $(0, \alpha^*, \lambda^*)$ in $\mathcal{O} \times \mathbb{R}^2$ the union $\mathcal{S}_\delta \cup \mathcal{S}_{\mu^*}^{(k_2)}$ contains all nontrivial solutions of (6.1) *such that $\delta < |v| < \pi - \delta$, where $r\cos(v) w_1^* + r\sin(v) w_2^*$ is the projection of $\overline{w}(r,v)$ on $\ker D_w\mathcal{F}(0,\Lambda^*)$ parallel to X_0.*

Proof. The start is the same as in the one-dimensional bifurcation: Define $\tilde{w}_j^* = (\eta_{\phi_{k_j}}, \phi_{k_j})$, and recall that $Z = \operatorname{span}\{\tilde{w}_1^*, \tilde{w}_2^*\}$ and $X = \ker D_w\mathcal{F}(0,\Lambda^*) \oplus X_0$. Applying the Lyapunov-Schmidt reduction, we get the existence of a smooth ψ defined in a neighborhood of $(0,\Lambda^*)$ in $\ker D_w\mathcal{F}(0,\Lambda^*) \times \mathcal{U}$ with codomain X_0, and a neighborhood of $(0,\Lambda^*)$ in $X \times \mathcal{U}$ in which the equation $\mathcal{F}(w,\Lambda) = 0$ is equivalent to $\Phi(t_1,t_2,\Lambda) = 0$, where

$$\Phi(t_1,t_2,\Lambda) = \Pi_Z \mathcal{F}(t_1 w_1^* + t_2 w_2^* + \psi(t_1 w_1^* + t_2 w_2^*, \Lambda), \Lambda). \tag{6.6}$$

Let Π_1 and Π_2 denote the projection onto $\operatorname{span}\{\tilde{w}_1^*\}$ and $\operatorname{span}\{\tilde{w}_2^*\}$, respectively. Then $\Pi_Z = \Pi_1 + \Pi_2$, and we have

$$\Pi_1\Phi(t_1,t_2,\Lambda) = \Phi_1(t_1,t_2,\Lambda)\tilde{w}_1^* \quad \text{and} \quad \Pi_2\Phi(t_1,t_2,\Lambda) = \Phi_2(t_1,t_2,\Lambda)\tilde{w}_2^* \tag{6.7}$$

for some smooth, real-valued functions Φ_1 and Φ_2. The equation $\Phi(t_1,t_2,\Lambda) = 0$ can be rewritten as

$$\begin{aligned} \Phi_1(t_1,t_2,\Lambda) &= 0, \\ \Phi_2(t_1,t_2,\Lambda) &= 0. \end{aligned} \tag{6.8}$$

This is a system of two equations in five unknowns, all of which are real numbers. Note that we have the trivial solution $(0,0,\Lambda)$ for all $\Lambda \in \mathcal{U}'$; this follows from the fact that $\psi(0,\Lambda) = 0$ and $\mathcal{F}(0,\Lambda) = 0$ for all $\Lambda \in \mathcal{U}'$.

Let us first consider case (i), where $k_2/k_1 \notin \mathbb{Z}$. We then claim that

$$\begin{aligned} \Phi_1(0, t_2, \Lambda) &= 0, \qquad \text{for all } t_2, \Lambda, \\ \Phi_2(t_1, 0, \Lambda) &= 0, \qquad \text{for all } t_1, \Lambda. \end{aligned} \tag{6.9}$$

We now show that $\Phi_1(0, t_2, \Lambda) = 0$. Using that $\ker D_w \mathcal{F}^{(k_2)}(0, \Lambda^*) = \operatorname{span}\{w_2^*\}$, an application of the Lyapunov-Schmidt reduction in $X^{(k_2)}$ yields a unique function $\psi_\star$ with codomain $X_0^{(k_2)}$ satisfying

$$(I - \Pi_2)\mathcal{F}(tw_2^* + \psi_\star(tw_2^*, \Lambda), \Lambda) = 0, \tag{6.10}$$

for all t in a neighborhood of $0 \in \mathbb{R}$ (recall that $\psi_\star$ is defined as the unique function satisfying (6.10)). Due to the $2\pi/k_2$-periodicity of $\mathcal{F}(tw_2^* + \psi_\star(tw_2^*, \Lambda), \Lambda)$, coupled with the formula for the projection Π_1, which involves integrating against $\cos(k_1 x)$, we can use Lemma 6.2 to conclude that

$$\Pi_1 \mathcal{F}(tw_2^* + \psi_\star(tw_2^*, \Lambda), \Lambda) = 0 \tag{6.11}$$

as well. Thus (6.10) holds with $I - \Pi_2$ replaced with $I - \Pi_Z$, and so by uniqueness of ψ, we must have $\psi(\cdot, \Lambda)|_{X^{(k_2)}} = \psi_\star(\cdot, \Lambda)$ (where in this equality we might have to shrink the domain of one of the functions). Recalling the definition of Φ_1, we thus see that formula (6.11) is equivalent to the identity $\Phi_1(0, t_2, \Lambda) = 0$. The second identity in (6.9) is proven by a similar argument.

Using (6.9) we find that

$$\Phi_1(t_1, t_2, \Lambda) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}z} \Phi_1(zt_1, t_2, \Lambda)\, \mathrm{d}z = t_1 \int_0^1 \partial_{t_1} \Phi_1(zt_1, t_2, \Lambda)\, \mathrm{d}z,$$

and a similar identity holds for Φ_2. Defining

$$\begin{aligned} \Psi_1(t_1, t_2, \Lambda) &= \int_0^1 (\partial_{t_1} \Phi_1)(zt_1, t_2, \Lambda)\, \mathrm{d}z, \\ \Psi_2(t_1, t_2, \Lambda) &= \int_0^1 (\partial_{t_2} \Phi_2)(t_1, zt_2, \Lambda)\, \mathrm{d}z, \end{aligned} \tag{6.12}$$

we therefore have $\Phi_j = t_j \Psi_j$, and thus for $t_j \neq 0$, $j = 1, 2$, $\Phi_j(t_1, t_2, \Lambda) = 0$ is equivalent to $\Psi_j(t_1, t_2, \Lambda) = 0$.

There are four possibilities. The case $t_1, t_2 = 0$ reduces to trivial solutions. When $t_1 = 0$ but $t_2 \neq 0$ the system reduces to $\Phi_2(0, t_2, \Lambda) = 0$, the solutions of which are given by[2] $\mathcal{S}_{\mu^*}^{(k_2)}$; when $t_2 = 0$ but $t_1 \neq 0$ we get solutions which lie in $\mathcal{S}_{\mu^*}^{(k_1)}$. The remaining case is where we allow both t_1 and t_2 to be nonzero, which amounts to investigating the solutions of $\Psi_1(t_1, t_2, \Lambda) = \Psi_2(t_1, t_2, \Lambda) = 0$ in a neighborhood of $(0, 0, \Lambda^*)$.

[2]This amounts to saying that $t_2 w_2^* + \psi(t_2 w_2^*, \Lambda)$ is in $X^{(k_2)}$, which is true since the argument following (6.10) shows that $\psi(t_2 w_2^*, \Lambda) \in X^{(k_2)}$.

We claim that $\Psi_1(0,0,\Lambda^*) = \Psi_2(0,0,\Lambda^*) = 0$. First note that $\Psi_j(0,0,\Lambda^*) = \partial_{t_j}\Phi(0,0,\Lambda^*)$, $j = 1,2$. Furthermore, observe that from the definition (6.6) of Φ, we get

$$\begin{aligned}&\partial_{t_j}\Phi(t_1,t_2,\Lambda^*)\\&= \Pi_Z D_w\mathcal{F}(t_1w_1^* + t_2w_2^* + \psi(t_1w_1^* + t_2w_2^*,\Lambda^*),\Lambda^*)(w_j^* + \psi_w(t_1w_1^* + t_2w_2^*,\Lambda^*)w_j^*),\end{aligned}$$

and so evaluating in $(t_1,t_2) = (0,0)$, and using that $\Pi_Z D_w\mathcal{F}(0,\Lambda^*) = 0$, we find $\partial_{t_j}\Phi(t_1,t_2,\Lambda^*) = 0$.

If we can in addition prove that the matrix

$$\begin{pmatrix}\partial_\lambda\Psi_1(0,0,\Lambda^*) & \partial_\lambda\Psi_2(0,0,\Lambda^*)\\ \partial_\alpha\Psi_1(0,0,\Lambda^*) & \partial_\alpha\Psi_2(0,0,\Lambda^*)\end{pmatrix}$$

is invertible, we can apply the implicit function theorem, which will prove (i) (except for the asymptotic formula (6.4) for small (t_1,t_2), which follows by a first order Taylor expansion in the same way as in the proof of the one-dimensional bifurcation). To this purpose, note that (6.12) yields

$$\partial_\lambda\Psi_1(0,0,\Lambda) = \partial_{t_1}\partial_\lambda\Phi_1(0,0,\Lambda),$$

and, by doing a similar computation as in the proof of the one-dimensional bifurcation, we get

$$\partial_{t_1}\partial_\lambda\Phi_1(0,0,\Lambda^*) = \Pi_1 D^2_{w\lambda}\mathcal{F}(0,\Lambda^*)w_1^*.$$

By using the formula for the projection, we have

$$\Pi_1 D^2_{w\lambda}\mathcal{F}(0,0,\Lambda^*)w_1^* = \frac{\langle D^2_{w\lambda}\mathcal{F}(0,\Lambda^*)w_1^*,\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2}.$$

As we have already seen, using that by definition $D_w\mathcal{F}(0,\Lambda) = \mathcal{L}(\Lambda)\mathcal{T}(\Lambda)^{-1}$ we have

$$D^2_{w\lambda}\mathcal{F}(0,0,\Lambda^*)w_1^* = D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1} + \mathcal{L}(\Lambda^*)D_\lambda\mathcal{T}(\Lambda^*)^{-1}w_1^*.$$

Since the last term is in $\operatorname{ran}\mathcal{L}(\Lambda^*) \subset \ker\Pi_1$, we conclude that

$$\partial_\lambda\Psi_1(0,0,\Lambda^*) = \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2}.$$

Similar arguments show that

$$\partial_\lambda\Psi_2(0,0,\Lambda^*) = \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2},$$

and

$$\partial_\alpha\Psi_j(0,0,\Lambda^*) = \frac{\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_j},\, \tilde{w}_j^*\rangle_Y}{\|\tilde{w}_j\|_Y^2}, \qquad j = 1,2.$$

It follows that

$$
\begin{aligned}
&\det\begin{pmatrix}\partial_\lambda\Psi_1(0,0,\Lambda^*) & \partial_\lambda\Psi_2(0,0,\Lambda^*)\\ \partial_\alpha\Psi_1(0,0,\Lambda^*) & \partial_\alpha\Psi_2(0,0,\Lambda^*)\end{pmatrix}\\
&= C\det\begin{pmatrix}\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\,\tilde w_1^*\rangle_Y & \langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_2},\,\tilde w_2^*\rangle_Y\\ \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\,\tilde w_1^*\rangle_Y & \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_2},\,\tilde w_2^*\rangle_Y\end{pmatrix}
\end{aligned}
$$

where $C = \|\tilde w_1^*\|_Y^{-2}\,\|\tilde w_2^*\|_Y^{-2} \neq 0$. The inner products appearing in the determinant have already been computed in Lemmas 5.1 and 6.1, namely

$$
\begin{aligned}
\big\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_j},\,\tilde w_j^*\big\rangle_Y &= A\left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}}\right)^2,\\
\big\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_j},\,\tilde w_j^*\big\rangle_Y &= B\left(\frac{\sinh(\theta_{k_j})}{\theta_{k_j}}\right)^2 + f(k_j),
\end{aligned}
$$

where A and B do not depend on j, $A \neq 0$ due to our explicit assumption (6.2), and

$$
f(k_j) = \begin{cases} \frac{\pi}{2}\frac{1-a^{-1}\cosh^2(\theta_{k_j})}{\theta_{k_j}^2}, & \theta_{k_j}\neq 0,\\ -\frac{\pi}{3} & \theta_{k_j} = 0.\end{cases} \tag{6.13}
$$

Here we have rewritten the expression for $f(k_j)$ given in Lemma 6.1, using the quantity $a \notin \{0,1\}$ introduced in (6.3). Using elementary properties of determinants, we can thus make the following calculation.

$$
\begin{aligned}
&\det\begin{pmatrix}\langle D_\mu\mathcal{L}(\Lambda^*)\phi_{k_1},\,\tilde w_1^*\rangle_Y & \langle D_\mu\mathcal{L}(\Lambda^*)\phi_{k_2},\,\tilde w_2^*\rangle_Y\\ \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\,\tilde w_1^*\rangle_Y & \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_2},\,\tilde w_2^*\rangle_Y\end{pmatrix}\\
&= \det\begin{pmatrix} A\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 & A\left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2\\ B\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 + f(k_1) & B\left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2 + f(k_2)\end{pmatrix}\\
&= A\det\begin{pmatrix}\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 & \left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2\\ f(k_1) & f(k_2)\end{pmatrix}\\
&= A\left(\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 f(k_2) - \left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2 f(k_1)\right).
\end{aligned}
$$

To show that the quantity in the parentheses is nonzero[3], we split the analysis up into cases, and refer to (6.13).

[3] *If* $a = 0$, we have $k_j^2 + \alpha < 0$, $\cosh(\theta_{k_j}) = 0$ and $f(k_j) = \pi/(2\theta_{k_j}^2)$, $j = 1, 2$. This implies that the quantity in the parantheses is zero.

Case I. Assume that $k_1^2+\alpha<0$ and $k_2^2+\alpha<0$. Recalling the definition (6.3) of a and the representations (4.13), we find

$$\begin{aligned}
&\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 f(k_2)-\left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2 f(k_1)\\
&=\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\frac{\pi}{2}(-1)\frac{1-a^{-1}\cos^2(|\theta_{k_2}|)}{|\theta_{k_2}|^2}-\frac{\sin^2(|\theta_{k_2}|)}{|\theta_{k_2}|^2}\frac{\pi}{2}(-1)\frac{1-a^{-1}\cos^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\\
&=\frac{\pi}{2}\frac{a^{-1}}{|\theta_{k_1}|^2|\theta_{k_2}|^2}\left[\sin^2(|\theta_{k_2}|)(a-\cos^2(|\theta_{k_1}|)-\sin^2(|\theta_{k_1}|)(a-\sin^2(|\theta_{k_2}|))\right]\\
&=\frac{\pi}{2}\frac{a^{-1}(a-1)}{|\theta_{k_1}|^2|\theta_{k_2}|^2}\left(\sin^2(|\theta_{k_2}|)-\sin^2(|\theta_{k_1}|)\right)\neq 0.
\end{aligned}\tag{6.14}$$

To see that the quantity $\sin^2(|\theta_{k_2}|)-\sin^2(|\theta_{k_1}|)$ is nonzero, first note that $\sin(|\theta_{k_j}|)\neq 0$ follows from the fact that $\theta_{k_j}\neq 0$ and the kernel equation. We can then deduce the contradiction

$$\begin{aligned}
\sin^2(|\theta_{k_2}|)-\sin^2(|\theta_{k_1}|)=0 \quad &\Longrightarrow \quad \cot^2(|\theta_{k_1}|)=\cot^2(|\theta_{k_2}|)\\
&\Longrightarrow \quad |\theta_{k_1}|^2\cot^2(|\theta_{k_1}|)\neq|\theta_{k_2}|^2\cot^2(|\theta_{k_2}|).
\end{aligned}$$

Case II. Assume that $k_1^2+\alpha<0$ and $k_2^2+\alpha=0$. Then $a=1$, $\sinh(\theta_{k_2})/\theta_{k_2}=1$ and we compute

$$\begin{aligned}
&\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 f(k_2)-\left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2 f(k_1)\\
&=\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\left(-\frac{\pi}{3}\right)-(1)^2\frac{\pi}{2}(-1)\frac{1-1^{-1}\cos^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\\
&=\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\left(-\frac{\pi}{3}\right)+\frac{\pi}{2|\theta_{k_1}|^2}\sin^2(|\theta_{k_1}|)\\
&=\frac{\pi}{6}\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\neq 0.
\end{aligned}$$

Case III. Assume that $k_1^2+\alpha<0$ and $k_2^2+\alpha>0$. Then we have

$$\begin{aligned}
&\left(\frac{\sinh(\theta_{k_1})}{\theta_{k_1}}\right)^2 f(k_2)-\left(\frac{\sinh(\theta_{k_2})}{\theta_{k_2}}\right)^2 f(k_1)\\
&=\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\frac{\pi}{2}\frac{1-a^{-1}\cosh^2(|\theta_{k_2}|)}{|\theta_{k_2}|^2}-\frac{\sinh^2(|\theta_{k_2}|)}{|\theta_{k_2}|^2}\frac{\pi}{2}(-1)\frac{1-a^{-1}\cos^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2}\\
&=\frac{\pi}{2}\frac{\sin^2(|\theta_{k_1}|)}{|\theta_{k_1}|^2|\theta_{k_2}|^2}a^{-1}\left(a-1-\sinh^2(|\theta_{k_2}|)\right)+\frac{\pi}{2}\frac{\sinh^2(|\theta_{k_2}|)}{|\theta_{k_1}|^2|\theta_{k_2}|^2}a^{-1}\left(a-1+\sin^2(|\theta_{k_1}|)\right)\\
&=\frac{\pi}{2}a^{-1}(a-1)\left(\sin^2(|\theta_{k_1}|)+\sinh^2(|\theta_{k_2}|)\right)\neq 0.
\end{aligned}$$

This concludes the proof of part (i).

We now prove part (ii), dealing with the case $k_2/k_1 \in \mathbb{N}$. Defining Φ_1 and Φ_2 in the same way as before by (6.7), we still have

$$\Phi_1(0, t_2, \Lambda) = 0, \qquad \text{for all } t_2, \Lambda.$$

We introduce Ψ_1 with the same definition as before, but now using polar coordinates[4]:

$$\Psi_1(r, v, \Lambda) = \int_0^1 \partial_{t_1}\Phi_1(zr\cos(v), r\sin(v), \Lambda)\,\mathrm{d}z.$$

As before, $\Phi_1 = 0$ is equivalent to $\Psi_1 = 0$ for $t_1 \neq 0$. As for Φ_2, we can still use the identity $\Phi_2(0, 0, \Lambda) = 0$, and we redefine Ψ_2 as

$$\begin{aligned}&\Psi_2(r, v, \Lambda)\\ &= \int_0^1 [\partial_{t_1}\Phi_2(zr\cos(v), zr\sin(v), \Lambda)\cos(v) + \partial_{t_2}\Phi_2(zr\cos(v), zr\sin(v), \Lambda)\sin(v)]\,\mathrm{d}z.\end{aligned} \tag{6.15}$$

We then have that $\Phi_2(r, v, \Lambda) = r\Psi_2(r, v, \Lambda)$, as a consequence of the Fundamental Theorem of Calculus and the fact that the integrand in (6.15) equals

$$(1/r)\partial_z\Phi_2(zr\cos(v), zr\sin(v), \Lambda).$$

The solutions of $\Phi(0, t_2, \Lambda) = 0$ near $(0, \Lambda^*)$ all lie in $S_{\mu^*}^{(k_2)}$. When $t_1 \neq 0$, also $r \neq 0$, and we have that $\Phi(t_1, t_2, \Lambda) = 0$ is equivalent to the problem $(\Psi_1(r, v, \Lambda), \Psi_2(r, v, \Lambda)) = 0$, which we now consider. We have that $\Psi_1(0, v, \Lambda) = \Psi_2(0, v, \Lambda) = 0$ for all Λ. As before, we find that

$$\partial_\lambda\Psi_1(0, v, \Lambda^*) = \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2},$$

$$\partial_\alpha\Psi_1(0, v, \Lambda^*) = \frac{\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2}.$$

To find the derivatives of Ψ_2, first note that

$$\partial_\lambda\Psi_2(0, v, \Lambda^*) = \partial_{t_1}\partial_\lambda\Phi_2(0, 0, \Lambda^*)\cos(v) + \partial_{t_2}\partial_\lambda\Phi_2(0, 0, \Lambda^*)\sin(v).$$

Using the formula

$$\partial_{t_j}\partial_\lambda\Phi_2(0, 0, \Lambda^*) = \Pi_2 D^2_{w\lambda}\mathcal{F}(0, \Lambda^*)w_j^*, \quad j = 1, 2,$$

and the formula for the projection, we get that

$$\Pi_2 D^2_{w\lambda}\mathcal{F}(0, 0, \Lambda^*)w_j^* = \frac{\left\langle D^2_{w\lambda}\mathcal{F}(0, \Lambda^*)w_j^*,\, \tilde{w}_2^*\right\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}, \qquad j = 1, 2.$$

[4]To clarify the notation: here $\partial_{t_1}\Phi_1(zr\cos(v), zr\sin(v), \Lambda)$ means the partial derivative of Φ_1 with respect to the first variable, evaluated at $(zr\cos(v), zr\sin(v), \Lambda)$.

As we have already seen, using that by definition $D_w\mathcal{F}(0,\Lambda) = \mathcal{L}(\Lambda)\mathcal{T}(\Lambda)^{-1}$ we find

$$D^2_{w\lambda}\mathcal{F}(0,0,\Lambda^*)w_j^* = D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_j} + \mathcal{L}(\Lambda^*)D_\lambda\mathcal{T}(\Lambda^*)^{-1}w_j^*, \qquad j=1,2.$$

Since the last term is in $\operatorname{ran}\mathcal{L}(\Lambda^*) \subset \ker\Pi_Z$, we conclude that

$$\begin{aligned}\partial_\lambda\Psi_2(0,v,\Lambda^*) &= \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\cos(v) + \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\sin(v)\\ &= \frac{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\sin(v).\end{aligned}$$

Here we used that $\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_2^*\rangle_Y = 0$ due to the formula (5.3) for $D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1}$, showing that its x-dependence is $\cos(k_1x)$, together with the fact that $\tilde{w}_2^* = (\eta_{\phi_{k_j}}, \phi_{k_j})$ has x-dependence $\cos(k_2x)$, and trigonometric orthogonality in L^2.

Similary, we find

$$\begin{aligned}\partial_\alpha\Psi_2(0,v,\Lambda^*) &= \frac{\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\cos(v) + \frac{\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\sin(v)\\ &= \frac{\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\sin(v),\end{aligned}$$

where the inner product $\langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_2^*\rangle_Y$ is 0 once again due to trigonometric orthogonality in L^2.

Thus

$$\begin{aligned}&\det\begin{pmatrix}\partial_\lambda\Psi_1(0,v,\Lambda^*) & \partial_\lambda\Psi_2(0,v,\Lambda^*)\\ \partial_\alpha\Psi_1(0,v,\Lambda^*) & \partial_\alpha\Psi_2(0,v,\Lambda^*)\end{pmatrix}\\ &= C\sin(v)\det\begin{pmatrix}\langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_1^*\rangle_Y & \langle D_\lambda\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y\\ \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_1},\, \tilde{w}_1^*\rangle_Y & \langle D_\alpha\mathcal{L}(\Lambda^*)\phi_{k_2},\, \tilde{w}_2^*\rangle_Y\,,\end{pmatrix},\end{aligned}$$

where $C = \|\tilde{w}_1^*\|_Y^{-2}\,\|\tilde{w}_2^*\|_Y^{-2}$. We can therefore apply the implicit function theorem under the assumptions of Theorem 6.3 as long as $\sin(v) \neq 0$.

Since $(t_1,t_2) = (r\cos(v), r\sin(v))$, the asymptotic formula (6.5) for small r follows in the same way as in the one-dimensional bifurcation.

□

Remark 6.4. *In case (i) the wave profiles of $\overline{w}(t_1,t_2)$ have multiple crests and troughs per minimal period. This follows from the fact that $\overline{w}(t_1,t_2) = t_1w_1^* + t_2w_2^* + O(t^2)$, coupled with an analysis of the wave profiles similar as in the proof one-dimensional bifurcation result; the point being that $\overline{w}(t_1,t_2)$ is approximately a sum of the form $A_1\cos(k_1x) + A_2\cos(k_2x)$.*

7 Kernels of arbitrary dimension

We now address a question raised in [6]: does there exist values of the parameters Λ such that $\ker D_w\mathcal{F}(0,\Lambda)$ is at least three-dimensional? By also letting the wave number vary, this question was answered affirmatively in [9], which showed that there exist wave numbers κ and parameters Λ such that $\ker D_w\mathcal{F}(0,\Lambda)$ is at least three-dimensional, and also that parameters can be chosen so as to make it exactly three-dimensional. We here take a different approach. We will show that if $\kappa = \pi/2$, then for any integer $N \geq 1$ one can choose Λ such that $\ker D_w\mathcal{F}(0,\Lambda)$ is N-dimensional. Moreover, we find a dense subset of K of $\mathbb{R}^+$ with the property that if $\kappa \in K$, one can choose Λ such that $\ker D_w\mathcal{F}(0,\Lambda)$ has arbitrarily high dimension. Although at this in one sense is a stronger result than the one in [9], our method has other limitations, which we will also discuss. Everything in chapter 7 is original.

Let us formulate the problem. We want to determine $\kappa > 0$, $t > 0$ and $a \in \mathbb{R}$ such that the equation

$$\theta_k \coth(\theta_k) = a \tag{7.1}$$

has several solutions $k \in \kappa\mathbb{N}$. For any $a \in \mathbb{R}$ and $\alpha < 0$, we can always choose μ, λ so that the right hand side $r(\mu,\alpha,\lambda)$ of the kernel equation equals a. We look for solutions with $a = 0$, and thus search for $\kappa > 0$ and $t > 0$ such that

$$\sqrt{t-k^2} \in \left(\mathbb{N} - \frac{1}{2}\right)\pi.$$

for $k \in \kappa\mathbb{N}$. Writing $k = m\kappa$, this can be reformulated as determining $\kappa, t > 0$ such that the equation

$$\sqrt{t-m^2\kappa^2} = \left(n - \frac{1}{2}\right)\pi \tag{7.2}$$

holds for several $k \in \kappa\mathbb{N}$. We now rewrite (7.2) in the form

$$m^2 + \left(n-\frac{1}{2}\right)^2 \frac{\pi^2}{\kappa^2} = \frac{t}{\kappa^2}, \qquad (m,n) \in \mathbb{N}^2. \tag{7.3}$$

We first consider the case $\kappa = \pi/2$.

Lemma 7.1. *For $\kappa = \pi/2$ and any $N \in \mathbb{N}$, there exists $\Lambda \in \mathcal{U}$ such that $\dim\ker D_w\mathcal{F}(0,\Lambda) = N$.*

Proof. Letting $\kappa = \pi/2$ in equation (7.3), we get the problem

$$m^2 + (2n-1)^2 = \frac{4t}{\pi^2}, \qquad (m,n) \in \mathbb{N}^2.$$

By choosing t so that $H = 4t/\pi^2$ is an integer, we have reduced the problem to showing that there exists $H \in \mathbb{N}$ so that

$$m^2 + (2n-1)^2 = H$$

has exactly N solutions $(m,n) \in \mathbb{N}^2$. This is shown in e.g. [13]. More precisely, we have that given any $r \in \mathbb{N}$ and prime $p \in 4\mathbb{N}+1$, the number of ways to write p^{2r-1} as a sum of two squares of positive integers equals r. (In each of these representations, exactly one of the squares will be an odd number.)

□

The result for $\kappa = \pi/2$ can be partially generalized in the following way.

Proposition 7.2. *Let*

$$K = \left\{ \frac{\pi r}{2s} : r, s \in \mathbb{N},\ r \text{ odd} \right\}.$$

For $\kappa \in K$ and $N \in \mathbb{N}$, there exists $\Lambda \in \mathcal{U}$ such that $\dim \ker D_w \mathcal{F}(0,\Lambda) \geq N$.

Proof. Letting $\kappa = \frac{\pi}{2}\frac{r}{s}$ in (7.3), we get

$$r^2 m^2 + s^2(2n-1)^2 = \frac{4s^2}{\pi^2} t, \qquad (m,n) \in \mathbb{N}^2.$$

By Lemma 7.1, we know that we can choose $H \in \mathbb{N}$ such that the equation $a^2+b^2 = H$ has N solutions $(a_1,b_1),\dots,(a_N,b_N) \in \mathbb{N}\times\mathbb{N}$, where we can assume that the b_j's are odd. Defining t, $\tilde{m}_j$ and $\tilde{n}_j$ by

$$\frac{4s^2}{\pi^2} t = r^2 s^2 H, \qquad \text{and} \qquad \tilde{m}_j = s a_j, \quad 2\tilde{n}_j - 1 = r b_j, \qquad j = 1,\dots N,$$

we see that $(\tilde{m}_j, \tilde{n}_j)$, $j = 1,\dots N$ all solve

$$r^2 m^2 + s^2(2n-1)^2 = \frac{4s^2}{\pi^2} t, \qquad (m,n) \in \mathbb{N}.$$

□

Let us illustrate this and give an example of a four-dimensional kernel. If we let $\kappa = \pi/2$ and choose $H = 5125$ (i.e. $t = 5125\pi^2/4$), then since

$$5125 = 30^2 + 65^2 = 34^2 + 63^2 = 54^2 + 47^2 = 70^2 + 15^2,$$

we have obtained a four-dimensional kernel. In other words, letting

$$\kappa = \pi/2, \qquad t = \frac{5125\pi^2}{4},$$

we have

$$\cot(\sqrt{t-(m\kappa)^2}) = 0 \qquad \text{for } m = 30, 34, 54, 70.$$

(Note that this method gives explicit values of κ, t, and the wavenumbers $k = m\kappa$.)

Having found values of κ and Λ for which the linearized problem $D_w\mathcal{F}(0,\Lambda) = 0$ has solution space of arbitrarily high dimension, it is natural to ask whether these give rise to solutions of the full nonlinear problem $\mathcal{F}(w,\Lambda) = 0$. The two bifurcation results we have established, Theorem 5.4 and Theorem 6.3, deal with the cases where the kernel is precisely one- and two-dimensional, respectively. However, by the trick described in section 6.1, we may be able to choose a higher wave-number κ' such that, after restriction to $X^{(\kappa')}$, the kernel is one- or two-dimensional. Since $a = 0$ for our solutions, we can in any case not use the two-dimensional bifurcation result, and this also excludes the possibility bifurcation from kernels of higher dimensions. But the one-dimensional bifurcation result can indeed by restricting to the wave-numbers k_i in the kernel for which there are no other k_j such that $k_j | k_i$, as in our example of a four-dimensional kernel.

In contrast, the paper [9] establishes a three-dimensional kernel where $a > 1$, and subsequently proves a three-dimensional bifurcation result, using κ, μ and α as bifurcation parameters. We also remark that an obstacle for high-dimensional bifurcation results is that there are only four parameters to the problem, namely μ, α, λ and κ. Since for each dimension of the kernel one more parameter is required, one needs to be able to vary other parameters than just the four at our disposal. This may be remedied by e.g. including effects of surface tension.

8 Further properties of the solution curve

In this section we investigate derivatives of the bifurcation curve. Everything in section 8 is original, although the proofs of Propositions 8.1 and 8.5 are inspired by similar calculations in the book by Kielhöfer [15].

8.1 Analytic series expansion of $\mathcal{F}$

For a given $\Lambda \in \mathcal{U}$, we will now show that $\mathcal{F}(w,\Lambda)$ is analytic at $w=0$, and find a corresponding analytic series expansion. We use this to find explicit expressions for the derivatives $D_w^n \mathcal{F}(0,\Lambda)w^n$, $n=0,1,2,3$.

First, we perform the differentiations in the definition (4.6) of $\mathcal{F}_1$ and the definition (4.7) of $\mathcal{F}_2$, which yields

$$\begin{aligned}\mathcal{F}_1(w,\Lambda) &= \frac{1}{2}\left[\left(\hat{\phi}_x - \frac{s\eta_x\left(\psi_{0s}+\hat{\phi}_s\right)}{1+\eta}\right)^2 + \frac{\left(\psi_{0s}+\hat{\phi}_s\right)^2}{(1+\eta)^2}\right] + \eta - Q(\Lambda)\\ &= \frac{1}{2}\hat{\phi}_x^2 - \frac{s\hat{\phi}_x\eta_x\left(\psi_{0s}+\hat{\phi}_s\right)}{1+\eta} + \frac{(1+s^2\eta_x^2)\left(\psi_{0s}^2+2\psi_{0s}\hat{\phi}_s+\hat{\phi}_s^2\right)}{2(1+\eta)^2} + \eta - Q(\Lambda),\end{aligned}$$

(where we use the convention that in the expression for $\mathcal{F}_1$, all functions of s are tacitly assumed to be evaluated at $s=1$) and

$$\begin{aligned}\mathcal{F}_2(w,\Lambda) = &\left(\hat{\phi}_x - \frac{s\eta_x(\psi_{0s}+\hat{\phi}_s)}{1+\eta}\right)_x - \frac{s\eta_x}{1+\eta}\left(\hat{\phi}_x - \frac{s\eta_x(\psi_{0s}+\hat{\phi}_s)}{1+\eta}\right)_s\\ &+ \frac{\psi_{0ss}+\hat{\phi}_{ss}}{(1+\eta)^2} - \alpha(\psi_{0ss}+\hat{\phi})\\ =&\hat{\phi}_{xx} - s\frac{\eta_{xx}(\psi_{0s}+\hat{\phi}_s)+\eta_x\hat{\phi}_{sx}}{(1+\eta)} + s\frac{\eta_x^2(\psi_{0s}+\hat{\phi}_s)}{(1+\eta)^2} - \frac{s\eta_x}{1+\eta}\hat{\phi}_{xs}\\ &+\frac{s\eta_x^2(\psi_{0s}+\hat{\phi}_s)+s^2\eta_x^2(\psi_{0ss}+\hat{\phi}_{ss})}{(1+\eta)^2} + \frac{\psi_{0ss}+\hat{\phi}_{ss}}{(1+\eta)^2} - \alpha(\psi_{0ss}+\hat{\phi})\\ =&\hat{\phi}_{xx} - s\frac{\eta_{xx}(\psi_{0s}+\hat{\phi}_s)+2\eta_x\hat{\phi}_{sx}}{(1+\eta)}\\ &+\frac{2s\eta_x^2(\psi_{0s}+\hat{\phi}_s)+(s^2\eta_x^2+1)(\psi_{0ss}+\hat{\phi}_{ss})}{(1+\eta)^2} - \alpha(\psi_{0ss}+\hat{\phi}).\end{aligned}$$

For $\|\eta\|_X < 1$, we have that

$$\frac{1}{1+\eta} = 1 - \eta + \eta^2 - \eta^3 + \cdots = \sum_{j=0}^{\infty}(-1)^j\eta^j,$$
$$\frac{1}{(1+\eta)^2} = 1 - 2\eta + 3\eta^2 - 4\eta^3 + \cdots = \sum_{j=0}^{\infty} j(-1)^j\eta^j,$$

where the sums converge absolutely norm-wise in X. Inserting these expansions into the definitions of $\mathcal{F}_1(\cdot,\Lambda)$ and $\mathcal{F}_2(\cdot,\Lambda)$, we see—by referring to the definition of analyticity at a point given in section 3.1, and the subsequent remarks—that $\mathcal{F}(\cdot,\Lambda)$ is analytic at 0, and moreover that we can deduce explicit expressions for $D_w^n\mathcal{F}(0,\Lambda)w^n$. The derivatives to the third order can be read off the following Taylor expansions:

$$\begin{aligned}
&\mathcal{F}_1(w,\Lambda)\\
=&\frac{1}{2}\hat{\phi}_x^2 - s\hat{\phi}_x\eta_x(\psi_{0s}+\hat{\phi}_s)\left(1-\eta+\eta^2+\eta^3+O(\|\eta\|^4)\right)\\
&+\frac{1}{2}(1+s^2\eta_x^2)(\psi_{0s}^2+2\psi_{0s}\hat{\phi}_s+\hat{\phi}_s^2)\left(1-2\eta+3\eta^2-4\eta^3+O(\|\eta\|^4)\right)+\eta-Q(\Lambda)\\
=&-\psi_{0s}^2\eta+\psi_{0s}\hat{\phi}_s+\eta\\
&+\frac{1}{2}\hat{\phi}_x^2 - s\psi_{0s}\hat{\phi}_x\eta_x+\frac{3}{2}\psi_{0s}^2\eta^2+\frac{1}{2}\hat{\phi}_s^2+\frac{1}{2}s^2\psi_{0s}^2\eta_x^2-2\psi_{0s}\hat{\phi}_s\eta\\
&+s\psi_{0s}\hat{\phi}_x\eta_x\eta - s\hat{\phi}_x\eta_x\hat{\phi}_s-2\psi_{0s}^2\eta^3+3\psi_{0s}\hat{\phi}_s\eta^2-\hat{\phi}_s^2\eta-s^2\psi_{0s}^2\eta_x^2\eta+s^2\psi_{0s}^2\eta_x^2\hat{\phi}_s\\
&+O(\|w\|^4).
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_2(w,\Lambda)\\
=&\hat{\phi}_{xx}-s\left(\eta_{xx}(\psi_{0s}+\hat{\phi}_s)+2\eta_x\hat{\phi}_{sx}\right)\left(1-\eta+\eta^2+\eta^3+O(\|\eta\|^4)\right)\\
&+\left(2s\eta_x^2(\psi_{0s}+\hat{\phi}_s)+(s^2\eta_x^2+1)(\psi_{0ss}+\hat{\phi}_{ss})\right)\left(1-2\eta+3\eta^2-4\eta^3+O(\|\eta\|^4)\right)\\
&-\alpha(\psi_{0ss}+\hat{\phi})\\
=&(\partial_x^2+\partial_s^2-\alpha)\hat{\phi}-s\psi_{0s}\eta_{xx}-2\psi_{0ss}\eta\\
&+s\psi_{0s}\eta_{xx}\eta-s\eta_{xx}\hat{\phi}_s-2s\eta_x\hat{\phi}_{sx}+(2s\psi_{0s}+s^2\psi_{0ss})\eta_x^2+3\psi_{0ss}\eta^2-2\hat{\phi}_{ss}\eta\\
&-s\psi_{0s}\eta_{xx}\eta^2+s\eta_{xx}\hat{\phi}_s\eta+2s\eta_x\hat{\phi}_{sx}\eta-4s\psi_{0s}\eta_x^2\eta+2s\eta_x^2\hat{\phi}_s-2s^2\psi_{0ss}\eta_x^2\eta+s^2\eta_x^2\hat{\phi}_{ss}\\
&-4\psi_{0ss}\eta^3 3+\hat{\phi}_{ss}\eta^2\\
&+O(\|w\|^4).
\end{aligned}$$

8.2 The one-dimensional bifurcation curve

Recall Theorem 5.4, where we established the existence of a solution curve $t \mapsto (\overline{w}(t), \lambda(t))$ such that $\mathcal{F}(w(t), \mu^*, \alpha^*, \lambda(t)) = 0$. We already know that $\dot{\overline{w}}(0) = w^*$, but one may ask if it is possible to determine the higher order behavior of $\overline{w}$ at zero, say to the second order, which amounts to finding $\ddot{\overline{w}}(t)$. To consider this question, let us return to the expression for $\dot{\overline{w}}$ we found in the one-dimensional bifurcation result Theorem 5.4, namely

$$\dot{\overline{w}}(t) = w^* + \psi_w w^* + \psi_\lambda \dot{\lambda}(t),$$

where ψ_w and ψ_λ are evaluated at $(tw^*, \mu^*, \alpha^*, \lambda(t))$, and $\ker D_w\mathcal{F}(0, \Lambda^*) = \operatorname{span}\{w^*\}$. Another differentiation yields

$$\ddot{\overline{w}}(t) = \psi_{ww}(w^*)^2 + \psi_{w\lambda} w^* \dot{\lambda}(t) + \psi_{\lambda w}\dot{\lambda}(t) + \psi_{\lambda\lambda}\dot{\lambda}(t)^2 + \psi_\lambda \ddot{\lambda}(t).$$

This expression can be simplified. In view of Lemma 4.12, we have that $\psi(0, \Lambda) = 0$ for all Λ in an open set about Λ^*, and thus $\psi_\lambda(0, \Lambda^*)$ and $\psi_{\lambda\lambda}(0, \Lambda^*)$ are zero. We furthermore claim that $\dot{\lambda}(0) = 0$, so that, in fact, $\ddot{\overline{w}}(0) = \psi_{ww}(0, \Lambda^*)(w^*)^2$.

Proposition 8.1. *In the one-dimensional bifurcation result Theorem 5.4, we have $\dot{\lambda}(0) = 0$.*

Proof. We use the terminology from the proof of Theorem 5.4. Recall the definitions

$$\Phi(t, \Lambda) = \Pi_Z \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda), \qquad \Phi(t, \Lambda) = \Phi_1(t, \Lambda)w^*,$$
$$\Psi(t, \Lambda) = \int_0^1 (\partial_t \Phi_1)(tz, \Lambda)\,\mathrm{d}z.$$

We showed that $\Psi_\lambda(0, \Lambda^*) \neq 0$, and that $\Psi(t, \mu^*, \alpha^*, \lambda(t)) = 0$ for $t \in (-\varepsilon, \varepsilon)$. Differentiating $\Psi(t, \mu^*, \alpha^*, \lambda(t)) = 0$ and evaluating at $t = 0$ yields

$$\Psi_t(0, \Lambda^*) + \Psi_\lambda(0, \Lambda^*)\dot{\lambda}(0) = 0, \quad \text{i.e.} \quad \dot{\lambda}(0) = -\frac{\Psi_t(0, \Lambda^*)}{\Psi_\lambda(0, \Lambda^*)}.$$

We are thus faced with the task of showing that $\Psi_t(0, \Lambda^*) = 0$. Now,

$$\Psi_t(t, \Lambda) = \int_0^1 (\partial_t^2 \Phi_1)(tz, \Lambda) z\,\mathrm{d}z, \quad \text{and so} \quad \Psi_t(0, \Lambda^*) = \frac{1}{2}(\partial_t^2 \Phi_1)(0, \Lambda^*).$$

Moreover, $\partial_t^2 \Phi_1(t, \Lambda)w^* = \partial_t^2 \Phi(t, \Lambda)$, and we find that

$$\begin{aligned}
\partial_t \Phi(t, \Lambda) =& \Pi_Z D_w \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda)(w^* + \psi_w(tw^*, \Lambda)w^*),\\
\partial_t^2 \Phi(t, \Lambda) =& \Pi_Z D_{ww} \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda)(w^* + \psi_w(tw^*, \Lambda)w^*)^2\\
&+ \Pi_Z D_w \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda)(\psi_{ww}(tw^*, \Lambda)(w^*)^2).
\end{aligned}$$

Since $\ker \Pi_Z = \mathrm{ran} D_w \mathcal{F}(0,\Lambda^*)$ we get $\Pi_Z D_w \mathcal{F}(0,\Lambda^*) = 0$, and using the properties of ψ given in Lemma 4.12 thus yields that $\partial_t^2 \Phi(t,\Lambda^*) = \Pi_Z D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2$. Recalling the formula for the projection Π_Z, we find

$$\Psi_t(0,\Lambda^*) = 0 \qquad \Longleftrightarrow \qquad \frac{\left\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2,\, \tilde{w}^* \right\rangle_Y}{\|\tilde{w}^*\|_Y^2} = 0.$$

Here $\tilde{w}^* = (\eta_{\phi_k}, \phi_k)$, where ϕ_k is as the unique function satisfying $w^* = \mathcal{T}(\Lambda^*)\phi_k$, where $\mathcal{T}(\Lambda) : \tilde{X}_2 \to X$ is the isomorphism from Proposition 4.4. We thus have to show that the inner product in the numerator is zero. Recalling that

$$\phi_k(x,s) = \cos(kx)\sin(\theta_k s)/\theta_k,$$

we get

$$\begin{aligned} \mathcal{T}(\Lambda)\phi_k &= \left(-\frac{\phi_k|_{s=1}}{\psi_{0s}(1)}, \phi_k - \frac{s\psi_{0s}\phi_k|_{s=1}}{\psi_{0s}(1)} \right) \\ &= \cos(kx)\left(-\frac{\sinh(\theta_k)}{\psi_{0s}(1)\theta_k}, \frac{\sinh(\theta_k s)}{\theta_k} - \frac{s\psi_{0s}(s)\sinh(\theta_k)}{\psi_{0s}(1)\theta_k} \right), \end{aligned} \tag{8.1}$$

Letting $w^* = (\eta, \hat{\phi})$, we see from (8.1) that both η and $\hat{\phi}$ are separable functions, with x-dependence $\cos(kx)$. We have

$$\begin{aligned} D_{ww}\mathcal{F}_1(0,\Lambda^*)(w^*)^2 &= \hat{\phi}_x^2 - 2s\psi_{0s}\hat{\phi}_x\eta_x + 3\psi_{0s}^2\eta^2 + \hat{\phi}_s^2 + s^2\psi_{0s}^2\eta_x^2 - 4\psi_{0s}\hat{\phi}_s\eta, \\ D_{ww}\mathcal{F}_2(0,\Lambda^*)(w^*)^2 &= 2s\psi_{0s}\eta_{xx}\eta - 2s\eta_{xx}\hat{\phi}_s - 4s\eta_x\hat{\phi}_{sx} + (4s\psi_{0s} + 2s^2\psi_{0ss})\eta_x^2 \\ &\quad + 6\psi_{0ss}\eta^2 - 4\hat{\phi}_{ss}\eta. \end{aligned}$$

Since each term is bilinear in w^*, each term in both of the above expressions will have an x-dependence of the form $\sin^a(kx)\cos^b(kx)$ where $a+b=2$. Since $\tilde{w} = (\eta_{\phi_k}, \phi_k)$, we see that when computing the inner product

$$\begin{aligned} \left\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2,\, \tilde{w}^* \right\rangle_Y = &\int_0^{2\pi/\kappa} \left(D_{ww}\mathcal{F}_1(0,\Lambda)(w^*)^2 \right) \eta_{\phi_k}\, \mathrm{d}x \\ &+ \int_0^1 \int_0^{2\pi/\kappa} \left(D_{ww}\mathcal{F}_2(0,\Lambda)(w^*)^2 \right) \phi_k\, \mathrm{d}x\, \mathrm{d}s, \end{aligned}$$

we will be integrating terms whose x-dependence is $\sin^a(kx)\cos^b(kx)$ where $a+b=3$. But since $\int_0^{2\pi/\kappa} \sin^a(kx)\cos^b(kx)\,\mathrm{d}x = 0$ whenever $a+b$ is odd, we conclude that

$$\left\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2,\, \tilde{w}^* \right\rangle_Y = 0. \tag{8.2}$$

□

Remark 8.2. *Since the proof only depended on the fact that the inner product $\left\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2,\, \tilde{w}^* \right\rangle_Y$ is 0, we conclude that in the version of the one-dimensional bifurcation result when one uses μ as the bifurcation parameter, we also have $\dot{\mu}(0) = 0$.*

Corollary 8.3. *In the one-dimensional bifurcation curve from Theorem 5.4, we have*

$$\dddot{w}(0) = \psi_{ww}(0,\Lambda^*)(w^*)^2.$$

We now derive a more explicit expression for $\psi_{ww}(0,\Lambda^*)(w^*)^2$. To do this, we will use the defining identity for ψ, namely that it is the unique function defined in a neighborhood of $(0,\Lambda^*) \in X \times \mathcal{U}$ mapping into X_0 satisfying

$$(I - \Pi_Z)\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda) = 0.$$

Differentiating this identity with respect to t gives

$$(I - \Pi_Z)D_w\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)(w^* + \psi_w(tw^*,\Lambda)w^*) = 0,$$

and taking yet another t-derivative yields

$$\begin{aligned}(I - \Pi_Z)&D_{ww}\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)(w^* + \psi_w(tw^*,\Lambda)w^*)^2 \\ +(I - \Pi_Z)&D_w\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)\psi_{ww}(tw^*,\Lambda)(w^*)^2 = 0.\end{aligned}$$

From this it follows that

$$D_w\mathcal{F}(0,\Lambda^*)\psi_{ww}(0,\Lambda^*)(w^*)^2 = -D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2, \tag{8.3}$$

where we can remove the projections $I - \Pi_Z$ since equation (8.2) establishes that $D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2 \in \operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$ (recall that $Z = \operatorname{span}\{\tilde{w}^*\}$ is the orthogonal complement of $\operatorname{ran}\mathcal{L}(\Lambda^*)$), in addition to the fact that $\ker \Pi_Z = \operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$. Note that since $D_w\mathcal{F}(0,\Lambda^*)|_{X_0}$ is invertible and $\psi_{ww}(0,\Lambda^*)(w^*)^2 \in X_0$, equation (8.3) determines $\psi_{ww}(0,\Lambda^*)(w^*)^2$ uniquely.

To solve for $\psi_{ww}(0,\Lambda)(w^*)^2$ using (8.3), we first give an explicit expression for $D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2$. Recalling that

$$\begin{aligned} w^* = (\eta,\hat{\phi}) &= \left(-\frac{\phi_k|_{s=1}}{\psi_{0s}(1)}, \phi_k - \frac{s\psi_{0s}\phi_k|_{s=1}}{\psi_{0s}(1)}\right) \\ &= \cos(kx)\left(-\frac{\sinh(\theta_k)}{\psi_{0s}(1)\theta_k}, \frac{\sinh(\theta_k s)}{\theta_k} - \frac{s\psi_{0s}(s)\sinh(\theta_k)}{\psi_{0s}(1)\theta_k}\right), \end{aligned} \tag{8.4}$$

we find

$$\begin{aligned} &D^2_{ww}\mathcal{F}_1(0,\Lambda^*)(w^*)^2 \\ &= \left[(\hat{\phi}_x)^2 - 2s\psi_{0s}\eta_x\hat{\phi}_x + (\hat{\phi}_s)^2 - 4\psi_{0s}\hat{\phi}_s\eta + 3\psi_{0s}^2\eta^2\right]_{s=1} \\ &= 0 - 0 + \left(\cosh(\theta_k) - \frac{(\psi_{0s}(1) + \psi_{0ss}(1))\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)^2\cos^2(kx) \\ &- 4\psi_{0s}(1)\left(\cosh(\theta_k) - \frac{(\psi_{0s}(1) + \psi_{0ss}(1))\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\cos^2(kx) \\ &+ 3\left(\frac{\sinh(\theta_k)}{\theta_k}\right)^2\cos^2(kx) \end{aligned}$$

and

$$
\begin{aligned}
&D_{ww}\mathcal{F}_2(0,\Lambda^*)(w^*)^2\\
=&2s\psi_{0s}\eta_{xx}\eta - 2s\eta_{xx}\hat{\phi}_s - 4s\eta_x\hat{\phi}_{sx} + (4s\psi_{0s} + 2s^2\psi_{0ss})\eta_x^2 + 6\psi_{0ss}\eta^2 - 4\hat{\phi}_{ss}\eta\\
=&2s\psi_{0s}(s)\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)^2(-k^2)\cos^2(kx)\\
&-2s\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\left(\cosh(\theta_k s) - \frac{(\psi_{0s}(s) + s\psi_{0ss}(s))\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)(-k^2)\cos^2(kx)\\
&-4s\left(-\frac{k\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\left(\cosh(\theta_k s) - \frac{(\psi_{0s}(s) + s\psi_{0ss}(s))\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)k^2\sin^2(kx)\\
&+(4s\psi_{0s}(s) + 2s^2\psi_{0ss}(s))\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)^2 k^2\sin^2(kx)\\
&+6\psi_{0ss}(s)\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)^2\cos^2(kx)\\
&-4\left(\theta_k\sinh(\theta_k s) - \frac{(2\psi_{0ss}(s) + s\psi_{0sss}(s))\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\left(-\frac{\sinh(\theta_k)}{\theta_k\psi_{0s}(1)}\right)\cos^2(kx).
\end{aligned}
$$

Note that because of the identities

$$
\sin^2(kx) = \frac{1}{2} - \frac{1}{2}\cos(2kx), \qquad \cos^2(kx) = \frac{1}{2} + \frac{1}{2}\cos(2kx),
$$

we can write $D_{ww}\mathcal{F}_2(0,\Lambda)(w^*)^2 = f_0(s) + f_2(s)\cos(2kx)$. Introduce the function $\zeta \in \tilde{X}_2$ by $\zeta = \mathcal{T}(\Lambda^*)^{-1}\psi_{ww}(0,\Lambda^*)(w^*)^2$. Then the equation (8.3) can be written

$$
\mathcal{L}(\Lambda^*)\zeta = -D_{ww}\mathcal{F}(0,\Lambda^*)(w^*)^2. \tag{8.5}
$$

Furthermore,

$$
\mathcal{L}(\Lambda^*)\zeta = \left(\left[\psi_{0s}\zeta_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}}\right)\zeta\right]_{s=1}, (\partial_x^2 + \partial_s^2 - \alpha^*)\zeta\right),
$$

and we will therefore consider the inhomogeneous Helmholtz equation

$$
(\partial_x^2 + \partial_s^2 - \alpha^*)\zeta = -D_{ww}\mathcal{F}_2(0,\Lambda^*)(w^*)^2.
$$

The function ζ must take the form

$$
\zeta(x,s) = a_0(s) + a_2(s)\cos(2kx),
$$

and so we have that

$$
\begin{aligned}
(\partial_x^2 + \partial_s^2 - \alpha)\left(a_0(s) + a_2(s)\cos(2kx)\right) = &\left[a_0''(s) - \alpha a_0(s)\right]\\
&+ \left[a_2''(s) - (4k^2 + \alpha)a_0(s)\right]\cos(2kx).
\end{aligned} \tag{8.6}
$$

We are thus left with two ordinary, linear differential equations of second order for a_0 and a_2, for which we already have identified the corresponding homogeneous solution basis. We can therefore obtain the inhomogeneous solution using variation of parameters. Let

$$u_1^m(s) = \cosh(\theta_{mk} s) \qquad u_2^m(s) = \sinh(\theta_{mk} s)/\theta_{mk}, \qquad m = 0, 2, \tag{8.7}$$

with the usual interpretations in the case $\theta_{mk} = 0$. Then $\{u_1^m, u_2^m\}$ is a solution basis for the homogeneous equation $a_m''(s) - (mk^2 + \alpha)a_m(s) = 0$. Since the Wronskian of the basis (8.7) equals 1, the well-known variation of parameters-formula in this case becomes,

$$a_0(s) = A_I(s)u_1(s) + B_I(s)u_2(s), \qquad a_2(s) = A_{II}(s)u_1(s) + B_{II}(s)u_2(s),$$

where

$$A_I(s) = -\int u_2(s) f_0(s)\,\mathrm{d}s, \qquad B_I(s) = \int u_1(s) f_0(s)\,\mathrm{d}s,$$
$$A_{II}(s) = -\int u_2(s) f_2(s)\,\mathrm{d}s, \qquad B_{II}(s) = \int u_1(s) f_2(s)\,\mathrm{d}s.$$

and the two constants of integration are determined by the conditions $\zeta|_{s=0} = 0$ (giving $a_0(0) = a_2(0) = 0$) and $\mathcal{L}_1(\Lambda^*)\zeta = -D_{ww}\mathcal{F}_1(0, \Lambda^*)(w^*)^2$. These integrals are possible to solve by elementary—but arduous—antidifferentiation. We will content ourselves with the following result:

Proposition 8.4. *We have that* $\dddot{w}(0) = \mathcal{T}(\Lambda)\zeta$*, where*

$$\zeta(x, s) = a_0(s) + a_2(s)\cos(2kx).$$

Furthermore $a_0(0) = a_2(0) = 0$*, and* $a_0(s)$ *and* $a_2(s)$ *are both nonzero polynomials of degree at most four in the expressions* s, $\cos(\theta_0(s-1)+\lambda)$, $\sin(\theta_0(s-1)+\lambda)$, $\sin(\theta_k s)$, $\cos(\theta_k s)$*, where in each term the power of* s *is at most two, the powers of* $\cos(\theta_0(s-1)+\lambda)$, $\sin(\theta_0(s-1)+\lambda)$ *sum to at most one, and the powers of* $\sin(\theta_k s)$, $\cos(\theta_k s)$ *sum to at most two.*

Proof. Everything has been proved except the description of a_0 and a_2. This follows from the fact that trigonometric and hyperbolic function can be written in exponential form, together with the fact that $\int s^j e^{as}\,\mathrm{d}s = p(s)e^{as} + C$, where p is a complex polynomial of degree j. □

8.3 The two-dimensional bifurcation curve

Proposition 8.5. *In case (i) of the two-dimensional bifurcation result Theorem 6.3, we have*

$$\nabla\alpha(0,0) = (0,0), \qquad \nabla\lambda(0,0) = (0,0). \tag{8.8}$$

Proof. We use the terminology in the proof of Theorem 6.3. Recall the definitions

$$\Phi(t_1,t_2,\Lambda) = \Pi_Z \mathcal{F}(t_1 w_1^* + t_2 w_2^* + \psi(t_1 w_1^* + t_2 w_2^*, \Lambda), \Lambda),$$

$$\Pi_1 \Phi(t_1,t_2,\Lambda) = \Phi_1(t_1,t_2,\Lambda), \quad \Pi_2 \Phi(t_1,t_2,\Lambda) = \Phi_2(t_1,t_2,\Lambda),$$

$$\Psi_1(t_1,t_2,\Lambda) = \int_0^1 (\partial_{t_1}\Phi_1)(zt_1,t_2,\Lambda)\,\mathrm{d}z, \qquad \Psi_2(t_1,t_2,\Lambda) = \int_0^1 (\partial_{t_2}\Phi_2)(t_1,zt_2,\Lambda)\,\mathrm{d}z.$$

We showed the local identities

$$\Psi_1(t_1,t_2,\mu^*,\alpha(t_1,t_2),\lambda(t_1,t_2)) = 0, \qquad \Psi_2(t_1,t_2,\mu^*,\alpha(t_1,t_2),\lambda(t_1,t_2)) = 0,$$

which, by differentiating with respect to t_j, $j=1,2$, and evaluating at $(t_1,t_2) = (0,0)$, yield the equalities

$$\begin{aligned} \partial_{t_1}\Psi_1(0,0,\Lambda^*) + \partial_\alpha \Psi_1(0,0,\Lambda^*)\alpha_{t_1}(0,0) + \partial_\lambda \Psi_1(0,0,\Lambda^*)\lambda_{t_1}(0,0) = 0,\\ \partial_{t_1}\Psi_2(0,0,\Lambda^*) + \partial_\alpha \Psi_2(0,0,\Lambda^*)\alpha_{t_1}(0,0) + \partial_\lambda \Psi_2(0,0,\Lambda^*)\lambda_{t_1}(0,0) = 0, \end{aligned} \tag{8.9}$$

and

$$\begin{aligned} \partial_{t_2}\Psi_1(0,0,\Lambda^*) + \partial_\alpha \Psi_1(0,0,\Lambda^*)\alpha_{t_2}(0,0) + \partial_\lambda \Psi_1(0,0,\Lambda^*)\lambda_{t_2}(0,0) = 0,\\ \partial_{t_2}\Psi_2(0,0,\Lambda^*) + \partial_\alpha \Psi_2(0,0,\Lambda^*)\alpha_{t_2}(0,0) + \partial_\lambda \Psi_2(0,0,\Lambda^*)\lambda_{t_2}(0,0) = 0. \end{aligned} \tag{8.10}$$

Equations (8.9) and (8.10) form a system of linear equations for the four quantites contained in the two vectors $\nabla\alpha(0,0)$ and $\nabla\lambda(0,0)$, and because we have also shown that

$$\det \begin{pmatrix} \partial_\lambda \Psi_1(0,0,\Lambda^*) & \partial_\lambda \Psi_2(0,0,\Lambda^*) \\ \partial_\alpha \Psi_1(0,0,\Lambda^*) & \partial_\alpha \Psi_2(0,0,\Lambda^*) \end{pmatrix} \neq 0,$$

it suffices—for the purpose of proving (8.8)—to show that the first order partial derivatives of $\Psi_1(\cdot,\cdot,\Lambda^*)$ and $\Psi_2(\cdot,\cdot,\Lambda^*)$ vanish at $(0,0)$. We will focus on the function Ψ_1, as the calculations for Ψ_2 are the same. We have that

$$\partial_{t_1}\Psi_1(0,0,\Lambda^*) = \frac{1}{2}(\partial_{t_1}\Phi_1)(0,0,\Lambda^*), \qquad \partial_{t_2}\Psi_1(0,0,\Lambda^*) = (\partial_{t_1}\partial_{t_2}\Phi_1)(0,0,\Lambda^*),$$

and furthermore

$$\begin{aligned} \partial_{t_1}\Phi_1(t_1,t_2,\Lambda) =& \Pi_1 D_w \mathcal{F}(tw_1^* + tw_2^* + \psi(tw_1^* + tw_2^*,\Lambda),\Lambda)(w_1^* + \psi_w(tw_1^* + tw_2^*,\Lambda)w_1^*),\\ \partial_{t_1}^2 \Phi_1(0,\Lambda^*) =& \Pi_1 D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^*)^2,\\ \partial_{t_1}\partial_{t_2}\Phi_1(0,\Lambda^*) =& \Pi_1 D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^* w_2^*). \end{aligned}$$

Using the formula for the projection, we find

$$\Psi_{1t_1t_1}(0,\Lambda^*) = \Pi_1 D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^*)^2 = \frac{\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^*)^2,\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2},$$

$$\Psi_{1t_1t_2}(0,\Lambda^*) = \Pi_1 D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^*)(w_2^*) = \frac{\langle D_{ww}\mathcal{F}(0,\Lambda^*)(w_1^*)(w_2^*),\, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2}.$$

By the same argument we employed in the proof of (8.2) we find that the first inner product is zero. The second inner product can be reduced to a linear combination of integrals of the form $\int_0^{2\pi/\kappa} \exp\left(i(\delta_1 k_1 + \delta_2 k_2 + \delta_3 k_1)x\right) \mathrm{d}x$ where $\delta_j \in \{-1, 1\}$, $j = 1, 2, 3$, which are all zero because $k_2/k_1 \neq 2$. □

Remark 8.6. *In the version of the two dimensional bifurcation result using α and μ as bifurcation parameters [6], one can prove the analogue of Proposition 8.5.*

In the two-dimensional bifurcation result Theorem 6.3 we have the solution curve

$$\overline{w}(t_1, t_2) = t_1 w_1^* + t_2 w_2^* + \psi(t_1 w_1^* + t_2 w_2^*, \mu^*, \overline{\alpha}(t_1, t_2), \overline{\lambda}(t_1, t_2)).$$

This implies that

$$\overline{w}_{t_1}(t_1, t_2) = w_1^* + \psi_w w_1^* + \psi_\alpha \overline{\alpha}_{t_1} + \psi_\lambda \overline{\lambda}_{t_1},$$

and

$$\begin{aligned}\overline{w}_{t_1 t_2}(t_1, t_2) = {} & \psi_{ww} w_1^* w_2^* + \psi_{w\alpha} w_1^* \overline{\alpha}_{t_2} + \psi_{w\lambda} w_1^* \overline{\lambda}_{t_2} \\ & + \psi_{\alpha w} \overline{\alpha}_{t_1} w_2^* + \psi_{\alpha\alpha} \overline{\alpha}_{t_1} \overline{\alpha}_{t_2} + \psi_{\alpha\lambda} \overline{\alpha}_{t_1} \overline{\lambda}_{t_2} + \psi_\alpha \overline{\alpha}_{t_1 t_2} \\ & + \psi_{\lambda w} \overline{\lambda}_{t_1} w_2^* + \psi_{\lambda\alpha} \overline{\lambda}_{t_1} \overline{\alpha}_{t_2} + \psi_{\lambda\lambda} \overline{\lambda}_{t_1} \overline{\lambda}_{t_2} + \psi_\lambda \overline{\lambda}_{t_1 t_2}.\end{aligned}$$

Using Proposition 8.5 and the fact that $\psi(0, \Lambda) = 0$ for all Λ near Λ^*, we conclude that $\overline{w}_{t_1 t_2}(0, 0) = \psi_{ww}(0, \Lambda^*) w_1^* w_2^*$. Doing similar derivations for the two other second derivatives, we find that

$$\overline{w}_{t_1 t_1}(0, 0) = \psi_{ww}(0, \Lambda^*)(w_1^*)^2, \quad \overline{w}_{t_2 t_2}(0, 0) = \psi_{ww}(0, \Lambda^*)(w_2^*)^2,$$
$$\overline{w}_{t_1 t_2}(0, 0) = \psi_{ww}(0, \Lambda^*) w_1^* w_2^*.$$

In the same way we proved (8.3), we can show that

$$D_w \mathcal{F}(0, \Lambda^*) \psi_{ww}(0, \Lambda^*) w_i^* w_j^* = -D_{ww} \mathcal{F}(0, \Lambda^*) w_i^* w_j^*, \qquad i, j \in \{1, 2\}.$$

We have not pursued an explicit expression for $\psi_{ww}(0, \Lambda^*) w_i^* w_j^*$, although we suspect that an argument along the lines of section 8.2 is possible.

A On the Fredholm property of the linearized problem

In Corollary 4.8 we established that if $\Lambda \in \mathcal{U}$, then $D_w\mathcal{F}(0,\Lambda)$ is a Fredholm operator of index 0; we now show that $D_w\mathcal{F}(0,\Lambda)$ is not Fredholm if $\Lambda \neq \mathcal{U}$, i.e. if $\psi_{0s}(1;\Lambda) = 0$. This is stated, but not proved, in [6]. Referring to (4.8) and (4.9), we see that when $\psi_{0s}(1;\Lambda) = 0$, $D_w\mathcal{F}(0,\Lambda) : X \to Y$ is given by

$$\begin{aligned} D_w\mathcal{F}_1(0,\Lambda)(\eta,\hat{\phi}) &= \eta \\ D_w\mathcal{F}_2(0,\Lambda)(\eta,\hat{\phi}) &= (\partial_x^2 + \partial_s^2 - \alpha)\hat{\phi} - s\psi_{0s}\eta_{xx} - 2\psi_{0ss}\eta. \end{aligned}$$

In this case however, the codimension of $\operatorname{ran} D_w\mathcal{F}(0,\Lambda)$ is not finite, and therefore $D_w\mathcal{F}(0,\Lambda)$ is not Fredholm. To see this, first observe that $\operatorname{ran} D_w\mathcal{F}_1(0,\Lambda) = Y_1 \supset X_1$. Recalling that

$$X_1 = C^{2+\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R}), \qquad Y_1 = C^{1+\beta}_{\text{even}}(\kappa^{-1}\mathbb{S},\mathbb{R}), \qquad \beta \in (0,1),$$

we see that it suffices to find an infinite family of linearly independent functions $\{f_n\}$ in $Y_1 \setminus X_1$. We give a constructive proof of this fact.

Proposition A.1. *Let $\beta \in (0,1)$. For $n = 1, 2, 3, \ldots$, define $f_n : \mathbb{R} \to \mathbb{R}$ by*

$$f_n(x) = \frac{1}{(n\kappa)(1+\beta)} \cos(n\kappa x)|\cos(n\kappa x)|^\beta.$$

Then $f_n \in X_1 \setminus Y_1$ and $\{f_n\}_{n\in\mathbb{N}}$ is linearly independent.

Proof. We readily find that f_n is differentiable, with

$$f_n'(x) = \sin(n\kappa x)|\cos(n\kappa x)|^\beta \operatorname{sgn}(\cos(n\kappa x)).$$

To see that f_n' is Hölder continuous of order β, note that f_n' is smooth away from the zeros of $x \mapsto \cos(n\kappa x)$. Moreover, if $\cos(n\kappa x_0) = 0$, then by the mean-value theorem we have

$$|f_n'(x_0+h) - f_n'(x_0)| = |f_n'(x_0+h)| = |\sin(n\kappa(x_0+h))||\sin(n\kappa\tilde{x}_0)|^\beta|h|^\beta$$

for some $\tilde{x}_0$ between x_0 and $x_0 + h$. Thus f_n' is Hölder continuous of order β, but not for any higher index $\beta' > \beta$. Consequently, $f_n \in Y_1 \setminus X_1$.

To show linear independence, it suffices to show that for $n > 1$, f_n is not a linear combination of $f_1, f_2, \ldots, f_{n-1}$. Lemma 6.2 shows that

$$\int_0^{2\pi/\kappa} f_m(x)\cos(n\kappa x)\,\mathrm{d}x = 0, \qquad m = 1, 2, \ldots n-1,$$

and so if f_m is a linear combination of $f_1, f_2, \ldots, f_{n-1}$ then we must also have

$$\int_0^{2\pi/\kappa} f_n(x)\cos(n\kappa x)\,\mathrm{d}x = \frac{1}{(n\kappa)(1+\beta)}\int_0^{2\pi/\kappa} |\cos(n\kappa x)|^{2+\beta}\,\mathrm{d}x = 0,$$

which is clearly not the case. Thus $\{f_n\}_{n\in\mathbb{N}}$ is linearly independent. □

References

[1] B. Buffoni and J. Toland. *Analytic theory of global bifurcation.* Princeton University Press, 2003.

[2] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Communications on Pure and Applied Mathematics*, Vol. 57 (2004), pp. 481–527

[3] G. D. Crapper. *Introduction to water waves.* Ellis Horwood Series: Mathematics and Its Applications. Ellis Horwood, Chichester; Halsted Press, New York, 1984.

[4] M.-L. Dubreil-Jacotin. Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finie. *Journal de Mathématiques Pures et Appliquées*, Vol. 13 (1934), 217-291.

[5] M. Ehrnström, J. Escher and G. Villari. Steady water waves with multiple critical layers: Interior Dynamics. *Journal of Mathematical Fluid Mechanics*, Vol. 14 (2012), 407–419.

[6] M. Ehrnström, J. Escher and E. Wahlén. Steady water waves with multiple critial layers. *SIAM Journal on Mathematical Analysis*, Vol. 43 (2011), pp. 1436–1456.

[7] M. Ehrnström, H. Holden and X. Raynaud. Symmetric waves are traveling waves. *International Mathematics Research Notices*, 2009, pp. 4578–4596,

[8] M. Ehrnström and G. Villari. Linear water waves with vorticity: Rotational features and particle paths. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis,* Vol. 14 (2007).

[9] M. Ehrnström and E. Wahlén. Trimodal steady waves. *arXiv:1311.0036.*

[10] G. B. Folland, *Real analysis: Modern techniques and their applications* (second edition). John Wiley and Sons Inc., New York 1999.

[11] F. Gerstner. Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile. *Annals of Physics*, Vol. 2 (1809), pp. 412–445.

[12] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order* (corrected 3rd printing of the 1998 edition). Springer 2013.

[13] E. Grosswald. *Representations of integers as sums of squares.* Springer-Verlag, New York, 1985.

[14] M. D. Groves. Steady water waves. *Journal of Nonlinear Mathematical Physics*, Vol. 43 (2004), pp. 435–460.

[15] H. Kielhöfer.*Bifurcation theory: An introdution with applications to partial differential equations* (second edition). Springer 2011.

[16] J. Lighthill. *Waves in Fluids*. Cambridge University Press, Cambridge 1978.

[17] C. Liverani. *Course notes in dynamical systems.* Retrieved from `http://www.mat.uniroma2.it/~liverani/SysDyn12/sd.html` (accessed 26/7-2014).

[18] E. Wahlén. Steady periodic capillary-gravity waves with vorticity. *SIAM Journal on Mathematical Analysis*, Vol. 38 (2006), pp. 921-943.

[19] E. Wahlén. Steady water waves with a critical layer. *Journal of Differential Equations*, Vol. 246 (2009), pp. 2468–2483.