Topic Study Group No. 11: Teaching and Learning of Algebra

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The Programme

TSG-11 on Teaching and Learning of Algebra had a small number of presentations in the main session, leaving enough space for discussions and dialogue. The TSG planned to cover the salient themes and ideas in algebra education, including early algebra, algebraic thinking, conjecturing, proving and generalizing and algebra instruction. Each of the sessions had two presentations, one of which was an invited talk by an eminent scholar in the field, focusing on one or more of the themes that were identified in the TSG and another one selected from the papers submitted to the group. The TSG was able to bring forth significant ideas for discussion within the group. The presentations gave theoretical, methodological and empirical insights into students' construction of algebraic knowledge. Below, we give the programme details and brief summary of the sessions.

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Day	Speaker	Title
Tuesday, July 26, 2016	Kaye Stacey	Algebra research to guide teaching
	Andrew Izsák, Sybilla Beckmann, Eun Jung, Ibrahim Burak Ölmez	Connecting multiplication, unit fractions, and equations
Wednesday, July 27, 2016	Maria Blanton, Barbara M. Brizuela and Ana C. Stephens	Children's understanding and use of variable notation
	Jan Block	Flexible algebraic action: Solving of algebraic equations
Friday, July 29, 2016	Jinfa Cai	Early algebra learning: Answered and unanswered questions
	Thomas Janßen	Developing algebraic structure sense in linear equations as tuning into a new activity
Saturday, July 30, 2016	Heidi Strømskag	Evolution of the *milieu* for a particular piece of mathematical knowledge
	Erik Tillema and Andrew Gatza	A quantitative approach to establishing cubic identities

Kaye Stacey's talk introduced the participants to the ideas that have emerged through decades of research in the area of algebra education and how they can guide teaching. In the process, she introduced us to the content and structure of the new book The Teaching and Learning of Algebra: Ideas, Insights and Activities, she has co-authored with Abraham Arcavi and Paul Drijvers. It is a useful resource for teachers and researchers. The talk highlighted the knowledge generated about the aims of algebra and its use and the ideas that as students and teachers, one has to deal with. It reiterated the key ideas of algebra, that is, generalizing, exploring properties and relationships, problems solving, and proving theorems. Students' difficulties and challenges with algebra (like, coming to terms with the letter, use it for representing, making sense of it in different contexts and work with symbolic expressions with meaning) and the insights they offer for teaching algebra were discussed. Technology has the potential in supporting teaching and the possibilities need more exploration.

Andrew Izsak's presentation argued for a continuity between arithmetic and algebra and also going beyond the whole numbers to fractions. The presentation talked about an intervention study with pre-service teachers at the middle school, which aimed at helping them connect fractions, proportional relationships and linear equations. The study took a quantitative meaning of multiplication to see fractions as multipliers of unknowns. For instance, in the equation $M \cdot N = P$, the letter M refers to the number of equal-size groups, N refers to the number of units in each of those groups, and P refers to the number of units in M groups. The data from the one-on-one clinical interviews with six pre-service teacher participants was presented. The study revealed that the understanding of unit fraction (1/b) as the number of groups (in $1/b \bullet X$) requires coordination of multiple knowledge elements—unit fraction as a result of splitting a whole, unit fraction as partitioning a group of X units, meanings for multiplication and definitions of fraction and interpreting symbolic expressions.

It was essential for this group also to engage with the early algebra thoughts and literature because it makes us aware of multiple possibilities for introducing algebraic thinking in early years. Maria Blanton presented her group's work on children's understanding and use of variable notation across elementary grades. The talk discussed an intervention study in early algebra for children in grades 3–5 which dealt with generalized arithmetic, equivalence, expressions, equations and inequalities and functional thinking. Children in the intervention group (here only grades 3 and 4 were reported) performed significantly better in the post-test compared to a similar control group, in tasks which required them to make an expression and an equation using the variable and a representation for a functional relationship. It showed the readiness and preference of young children to use the variable notation. She also reported from other intervention studies with children of grades of K-1 showing their capacity to use variable notation for modeling situations as well as in functional relations.

The presentation by Jan Block explored the idea of flexible algebraic action among grade 9 and 10 students in the context of solving quadratic equations. Building on the theory of didactical-cut, it tried to explore what features of a quadratic equation are perceived by students and how they use this information and whether it facilitates or hinders flexible algebraic action. It discussed the wide use of the quadratic formula (pq-formula) and the trial-and-error method to solve equations among these students together with high error rate. It concluded by stating that teaching different strategies for solving quadratic equations is not going to lead to flexible algebraic activity. Rather one has to engage in meta-tasks to identify features that make certain strategies relevant for solving it.

In his talk, Jinfa Cai used statistics from the US National Educational Longitudinal Study (NELS) of 1988 to illustrate why algebra is important. NELS showed that students who take Algebra 1 in high school are much more likely to go to college than those who do not: 83% of students who take Algebra I go to college, whereas 36% of students without Algebra 1 do. Further, Cai showed that students who pass Algebra 2 in high school were 4.15 times more likely to graduate from college than students who have not. Then the LieCal project was presented, which longitudinally investigates the effects of the *Standards*-based Connected Mathematics Program (CMP) curriculum on students' learning of algebra to the effects of more traditional middle-school mathematics curricula. Cai presented example problems from the two curricula: the CMP curriculum represented a functional approach, with emphasis on change, variation, and relationships between variables; and, traditional curricula represented a structural approach, with emphasis on procedures and abstract work with symbols.

Thomas Janßen's presentation was about algebraic structure sense for linear equations, and how it can be developed from structure-seeing. The discussion was based on transcripts and drawings from video-recorded classroom observations of four Grade 8 students working on linear equations. The structure of linear equations had been introduced through a puzzle: On each side of the equal sign there was the same number of matches, some of them in matchboxes, with the same amount of matches in each box. The task was to find a way to determine the number of matches in each box. Janßen showed that the development of algebraic structure sense can be understood as happening in moments of *tuning*—where tuning is a form of social interaction characterized by a common interest and a common understanding of the situation and the goals of the activity, and further, by a common understanding of what actions are necessary to achieve the goals.

In her talk, Heidi Strømskag presented a semiotic analysis of three students teachers' engagement with a generalization task in geometry. She explained how an evolution of the milieu (in Brousseau's sense) enabled the student teachers to create manipulatives (plane geometrical figures) that were instrumental in the generalization process aiming at a relationship between percentage growth of length and area when looking at the enlargement of a square. It was shown how use of different notation systems constrained the interaction among the participants, and how transformation of percentage and fractional notation into geometrical figures—that belong to a different semiotic register—enabled the target mathematical knowledge to be expressed in algebraic notation. Strømskag made a general point about design of milieus for algebraic generalization: the adidactical potential of a situation depends upon a coordination between the particular values that students are asked to work on and the semiotic register(s) expected to the used.

Erik Tillema presented an interview study of eight Grade 10–12 students' generalizations made in the context of solving combinatorics problems about cubic relationships. Students' generalizing actions were characterized by schemes, where a scheme has three parts: an assimilatory mechanism; an activity; and, a result. Tillema showed how two schemes were pre-requisites to establishing the formula that $(x + 1)^3 = x^3 + 3 \cdot (x^2 \cdot 1) + 3 \cdot (x \cdot 1^2) + 1^3$. The first was a scheme to quantify the total number of three card hands using multiplication that was coordinated with a systematic way to list all possible outcomes, and the second was a scheme that enabled students to spatially structure 3-D arrays. Further, he showed that images based on quantitative relationships supported student generalizations. Tillema explained that the formula (above) that one student created was a formal statement of generalization (a reflection generalization) that was based on an abstraction in which she connected the activity of her scheme with the results of her schemes (reflective abstraction).