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Gauge Dependence of the Quantum Field Theory Effective Potential

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MASTER'S THESIS

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Gauge Dependence of the
Quantum Field Theory
Effective Potential

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Abstract

Using the absolute stability requirement of the Standard Model vacuum, we compute the Higgs mass bound for the 1-loop Standard Model effective potential with gauge dependence in the R_ξ gauges together with 3-loop beta functions, 3-loop anomalous dimension and 2-loop threshold corrections. We find that the bound changes by $+0.1\text{GeV}$ when we change the gauge parameter ξ from 0 to 50. We also report that the Higgs bound plateaus as we increase ξ beyond 100.

Preface

This thesis is the result of the course TFY4900 Physics, Master's Thesis at NTNU in the spring of 2013. The work has been done at the Jefferson Physical Laboratory at the Department of Physics, Harvard University. My adviser at Harvard has been Professor Matthew D. Schwartz and the work has been done in collaboration with him and graduate student Willam Frost at Harvard University. My adviser at NTNU has been Professor Kåre Olaussen.

The initial question about the gauge dependence of Higgs mass bound from absolute stability requirements comes from Prof. Schwartz, and the this thesis grew out of the work understanding the necessary background material, relevant literature and unanswered questions related to this gauge dependence.

This thesis contains a introduction to quantum field theory and effective potentials relevant to our work, and the main work has been done studying the gauge dependence of the Abelian Higgs model and the Standard model. When starting this work, I did not expect that we would finish the analysis finding the gauge dependence of the bound on the Higgs mass, which is the main finding in this thesis. There are however still many questions that needs to be answered, and we hope to answer them in our future work.

Acknowledgments

I would first like to thank Prof. Matthew Schwartz for letting me stay at Harvard and work with his group this year. I have really enjoyed all the time we have spent discussing physics together related to this thesis, and I appreciate that he always has time to talk to me when I need guidance.

Second, I would like to thank my good friend William Frost for the numerous discussions that has resulted in the work presented in this thesis. I have learned a lot from working with him, and this thesis would not be the same without him.

Third, I want to thank all my friends at Harvard for making the last two years the best time of my life. I want to especially thank my office mates Prahar Mitra, William Frost and Andrew Marantan for interesting discussion about physics and almost any other topic imaginable. I am also very grateful for their help in proofreading parts of this thesis.

Fourth, I would like to thank Prof. Kåre Olaussen for being by adviser at NTNU and for all the help in the finishing stages of writing this thesis.

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Chapter 1

Introduction

The Standard Model of particle physics has proven to be a very good model of the electromagnetic, weak and strong interactions between the subatomic particles we have seen in experiments. To date most of its predictions have been confirmed at the LHC, and they are now working hard to find new signs of physics beyond the Standard Model.

One approach to finding physics beyond the Standard Model is to see where the current model breaks down or make predictions not realized in nature. When the Standard Model fails, we have an indication of some new physics to fix the discrepancy.

In this thesis we will study the Higgs potential, and it will be valuable to study the conditions on this potential under which the Standard Model is stable. If the Standard Model turns out to be unstable, we know that there must be new physics beyond the Standard Model to stabilize the theory.

1.1 Stability of the Standard Model

This goal of this thesis is to study the stability of the Standard Model. This relates back to Sidney Coleman's seminal paper *The Fate of the False Vacuum* [1] describing the possibility of a metastable universe in a false vacuum decaying to a stable vacuum. It turns out that it is possible for the Standard Model to be in a false vacuum, and this possibility is strongly dependent on the Higgs boson and top quark masses. Cabbibio et al. [2] and Hung [3] described in 1979 how one can find bounds on the Higgs and top mass by requiring that the Standard Model should be stable, i.e. that we are in the true vacuum. The bound on the Higgs mass is a lower bound, and the top mass gets an upper bound.

The analysis of the stability of the Standard Model has been improved multiple times as new experimental data have constrained the size of the parameter

space. Lindner et al. [4] analyzed the stability in the context of the top search at Fermilab in 1989. Altarelli and Isidori [5] updated the Higgs mass bound in 1994, and they concluded that the Higgs would be too heavy to be found at LEP due to the large top mass. More recently, Degraasi et al. [6] put out another paper on the subject in 2012, and this was updated in July 2013 by Buttazzo et al. [7].

In 1989 Arnold [8] argued that it is possible to have a sensible theory of our universe located in a false vacuum as long as the time it takes to tunnel to the true vacuum is longer than the age of the universe. This is called meta-stability, and from now on we will use the term "absolute stability" to describe the scenario in which our vacuum is the true vacuum. Isidori et al. [9] gave an updated account of the metastability condition in 2001.

With the discovery of the Higgs boson by the ATLAS Collaboration [10] and CMS Collaboration [11], we are finally able to quantify the possibility that we live in the true or false vacuum. This is all assuming that our universe is described by just the Standard Model up to the Planck scale, which is the energy scale above which we can no longer safely ignore the contributions of gravity.

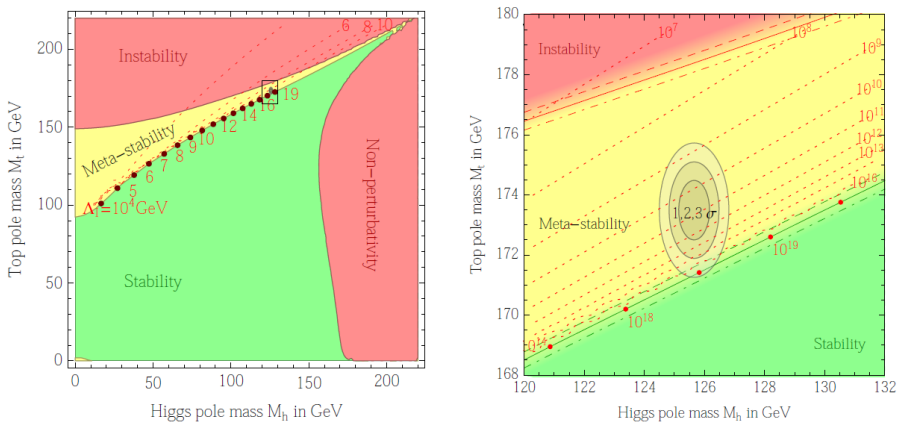


Figure 1.1: The left plot is the Standard Model phase diagram in terms of the Higgs and top pole masses. The plane is divided into four different regions: Instability, Meta-stability, Absolute Stability and Non-perturbativity of the Higgs quartic coupling.

The right plot shows the region of interest with the preferred experimental ranges indicated by rings corresponding to 1,2 or 3σ . The dotted lines indicate at which scale the instability occurs. The figure is taken from Buttazzo et al. [7].

With the most recent work by Buttazzo et al. [7], the calculations of the

effective potential of the Standard Model have been carried out with impressive precision. The Higgs potential and the top Yukawa coupling are computed with 2-loop NNLO (next-to-next-to-leading order) precision, and the Standard Model parameters are computed with full 3-loop NNLO RGE precision up to the Planck scale.

The result by Buttazzo et al. [7] can be found in figure 1.1. We see that we are living in a very interesting place at the border between absolute stability and meta-stability, but with experimental data favoring meta-stability. Meta-stability would mean that we live in a false vacuum, but with a lifetime greater than the age of our universe, so we have not yet tunneled to the true vacuum state with the lowest-energy configuration. It is also very interesting to note that the experimental values lie well within the range of parameters in which we can extrapolate the Standard Model up to the Planck scale without having to add any new physics to make the theory consistent. This provides us with a method to check the consistency of the Standard Model at energy scales far beyond the scale of which collider experiments can operate today.

Fixing the top pole mass to $m_t = 173.36\text{GeV}$, we find a condition on the Higgs pole mass m_H to be in the region of absolute stability. Buttazzo et al. [7] found the bound to be

$$m_H > (129.6 \pm 1.5)\text{GeV}. \quad (1.1)$$

With this result they concluded that absolute vacuum stability of the Standard Model up to the Planck scale is excluded at 2.5σ (99.3% confidence level one-sided) with the current measured value $m_H = 125.66 \pm 0.34\text{GeV}$ [12]. We find this result to be one of the most interesting consequences of the measurement of the Higgs boson mass.

We will reproduce the plots in figure 1.1 and the Higgs mass bound in eq. (1.1) in chapter 5. This serves as a check that we have understood all the details of this calculation, and it also provides a verification of the results found by Buttazzo et al. [7].

1.2 Gauge Dependence

All the stability calculations in the Standard Model and the Higgs mass bound have been computed using the Standard Model effective potential in the Landau gauge. However, Jackiw showed in 1974 [13] that the effective potential in a quantum field theory with a gauge symmetry will be gauge-dependent. In other words, if we gauge fix in the R_ξ gauges, the effective potential will depend on the gauge parameter ξ .

One would hope that physical predictions would always come out gauge independent in a proper quantum field theory analysis, but we believe an explicit

calculation with a free gauge parameter ξ is necessary to see whether this happens or not. Only after such an analysis can one safely conclude what the physical prediction really is.

Another important result by Jackiw, that was later extended to 2-loops by Kang [14], was the calculation of the scalar-to-vector mass ratio for massless scalar QED that is spontaneously broken by radiative corrections, as was first described by Coleman and Weinberg [15]. They found that if one considers $\lambda = \mathcal{O}(e^4)$ keeping only the terms to leading order in e , the physical prediction of the mass ratio comes out gauge-independent even though the effective potential is gauge-dependent. We will see how this works in detail in chapter 4.

We have noted that understanding the gauge dependence of the effective potential and the resulting predictions is still an open question in the literature; for example, Patel and Ramsey-Musolf [16] and Wainwright et al. [17, 18] are trying to tackle this issue. Due to these observations we think it is important to investigate how the gauge dependence of the Standard Model effective potential affects the calculation of the Higgs mass bound from absolute stability requirements. Buttazzo et al. [7] have performed all their calculations in the Landau gauge $\xi = 0$, and it is a priori unclear from their procedure if the value of the Higgs mass bound will depend on the choice of ξ .

1.3 Outline of Thesis

The structure of this thesis will be as follows. Chapter 2 begins with a derivation of the basic tools necessary for our analysis aimed at students relatively new to quantum field theory. This chapter works as a point of reference in defining notation and vocabulary and the experienced reader can skip this without much trouble.

Chapter 3 continues with an introduction to effective field theories, and in this chapter we derive and describe the specific tools used in our analysis in chapters 4 and 5, including how to compute the effective potential using the renormalization group equations and how to find the resummed effective potential. The last part of the chapter contains a discussion of the gauge dependence of the effective potential, the Nielsen Identity, stability and meta-stability, which we assume to be less familiar to most readers.

Since the gauge dependence of the effective potential is an unfamiliar topic, chapter 4 is devoted to studying the Abelian Higgs model, which is an easier theory with which to get used to the new calculations and concepts than the Standard Model. This chapter contains some more technical calculations, and we will rederive results of historical importance by Coleman and Weinberg [15] and Jackiw [13]. We complete the chapter by doing an analogous calculation of the resummed potential that will be important in chapter 5, and we use these results to look quantitatively on the gauge dependence of the effective potential

for massless scalar QED.

Chapter 5 contains our analysis of the Standard Model effective potential. Replicating the calculations from chapter 4, we find the effective potential with gauge dependence to 1-loop and we use the renormalization group equations to find the resummed effective potential. We reproduce the findings of Buttazzo et al. [7] for the Higgs mass bound, and we then do the same analysis with different choices of the gauge parameter ξ . We have found a dependence on the Higgs mass bound of order 0.1GeV between $\xi = 0$ and $\xi = 50$, and we have analyzed some interesting features of this dependence for larger values of ξ . We conclude with a discussion of our findings and an outlook for future work.

1.3. OUTLINE OF THESIS

Chapter 2

Preliminaries

This chapter contains the basic knowledge needed to read and understand the main work in this thesis. The thesis is written with the assumption that the reader has a knowledge equivalent to two semesters of Quantum Field Theory, but if the reader is unfamiliar with certain topics I recommend reading Schwartz [19] which has been my main source in writing this thesis. Other good sources are Peskin and Schroeder [20], Zee, [21], Weinberg, [22, 23] and Srednicki[24] listed in my preferred order. This chapter is mainly based on Schwartz [19], the source from which I learned QFT, and this is assumed to be the source if no other reference is listed.

The main point of this chapter is to include the background material relevant to my work, making the thesis as self-contained as possible. Every topic covered will be used in some sense in later chapters, and in addition I also include some extra details and comments where I find the standard textbook discussion confusing or insufficient.

We begin by defining some notation and reviewing the path integral formalism since much of the later chapters will be based on functional methods. We will then discuss symmetries (including a discussion of Noether's theorem which differs from most textbooks) with a focus on gauge symmetries, gauge fixing and BRST invariance. We also include a section on symmetry breaking including a brief summary of the Standard Model and electroweak symmetry breaking. We end this chapter by discussing renormalization and the renormalization group equations.

2.1 Notation

The metric used will be the well known Minkowski metric $\eta_{\mu\nu}$. We will never use curved space backgrounds, so we will use $g_{\mu\nu} = \eta_{\mu\nu}$, and we will be using the

2.1. NOTATION

so-called *west coast* metric signature

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (2.1)$$

where all the blank entries are zeros and are intentionally left out for simplicity.

We always work in natural units with $\hbar = c = 1$ so that every quantity in which we are interested is given in terms of its mass dimension. For example, the energy E , mass m , derivative operator ∂_μ and momentum p_μ have mass dimension one,

$$[E] = [m] = [p_\mu] = [\partial_\mu] = M^1. \quad (2.2)$$

Position (and time) x^μ have mass dimension -1 ,

$$[x] = [t] = M^{-1}, \quad (2.3)$$

and velocity v and the action S will always be dimensionless,

$$[v] = [S] = \left[\int d^d x \mathcal{L} \right] = 0. \quad (2.4)$$

It follows that in d dimensional space-time the Lagrangian \mathcal{L} has mass dimension

$$[\mathcal{L}] = M^d. \quad (2.5)$$

Converting from natural units back to SI units can simply be done using $1 = \hbar c = 197.327 \text{ MeV fm}$ to convert from one GeV to meters and $c = 1 = 2.998 \times 10^8 \text{ m/s}$ to convert meters to seconds. We can also use $1\text{eV} = 1.6 \times 10^{-19} \text{ Joule} = 1.6 \times 10^{-19} \text{ kgm}^2/\text{s}^2$ that allows us to convert from GeV to kilograms. The conversion factors are summarized in table 2.1.

Table 2.1: Conversion table between natural and SI units

Quantity	SI units	Natural units	Conversion
Length	m	M^{-1}	$1\text{GeV}^{-1} = 0.197 \times 10^{-15} \text{m}$
Energy	kgm^2/s^2	M	$1\text{GeV} = 1.6 \times 10^{-10} \text{kgm}^2/\text{s}^2$
Mass	kg	M	$1\text{GeV} = 1.782 \times 10^{-27} \text{kg}$
Time	s	M^{-1}	$1\text{GeV}^{-1} = 6.58 \times 10^{-25} \text{s}$
Momentum	kgm/s	M	$1\text{GeV} = 5.39 \times 10^{-19} \text{kgm/s}$
Velocity	m/s	M^0	$1 = 2.998 \times 10^8 \text{m/s}$

We will use the Einsteins summation conventions everywhere! If two indices are repeated in the same term with one lower and one upper it means that it is implicitly summed over. For example, the momenta p^μ and q^μ can be contracted, $p^\mu q_\mu = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3$. Sometimes it is easier to read an expression when all the indices are lowered, and we will adopt the convention used in Schwartz [19] and, when there is no ambiguity, simply write

$$p_\mu q^\mu = p_\mu q_\mu = p^\mu q^\mu. \quad (2.6)$$

2.2 Path Integral Formulation of Quantum Field Theory

We will now review the path integral formalism, which was developed by Richard Feynman [25]. For completeness, we start in quantum mechanics, and then carry the results over to field theory. We also give some examples of how to actually do calculations with the path integral in section 2.2.5, which will be very useful for readers unfamiliar with functional integration.

2.2.1 Gaussian Integrals

As silly as it might seem, the only integrals we really know how to do in the path integral formalism are Gaussian integrals. We list the results here, and the proofs are given in great detail in Appendix A.

Let M_{ab} be a real symmetric $n \times n$ matrix, and let J_a and x_a be n -dimensional vectors. The Gaussian integral is

$$\int d^n x e^{-\frac{1}{2} x_a M_{ab} x_b + J_a x_a} = \sqrt{\frac{(2\pi)^n}{\det M}} e^{\frac{1}{2} J_a M_{ab}^{-1} J_b}. \quad (2.7)$$

If we are integrating over complex coordinates the Gaussian integral is

$$\int d^n z d^n z^* e^{-z_i^* H_{ij} z_j + J_i z_i^* + J_i^* z_i} = \frac{(2\pi)^n}{\det H} e^{J_k^* H_{ij}^{-1} J_j} \quad (2.8)$$

where H_{ij} is a Hermitian matrix and z_i , z_i^* and J_i are complex n -dimensional vectors.

For a Grassmann variable, the Gaussian integral will be

$$\int d^n \theta^* d^n \theta e^{-\theta_i^* A_{ij} \theta_j + \xi_i \theta_i^* + \eta_i \theta_i} = e^{\eta_i A_{ij}^{-1} \xi_j} \det A \quad (2.9)$$

where θ_i , θ_i^* , ξ_i and η_i are n dimensional Grassmann vectors, and A_{ij} is an antisymmetric $n \times n$ matrix.

2.2.2 The Path Integral in Quantum Mechanics

For simplicity, we will introduce the path integral in quantum mechanics, for which the framework is simpler than for quantum field theory. The theory is specified through a Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + V(\hat{x}, t), \quad (2.10)$$

where \hat{H} , \hat{p} and \hat{x} are operators acting on the Hilbert space, and t is just a number (sometimes called a c -number). By $|x_i\rangle$ we denote a state such that at the point x_i and time t_i the position operator \hat{x} acts as $\hat{x}|x_i\rangle = x_i|x_i\rangle$, and similarly for a state with momentum p_i we have $\hat{p}|p_i\rangle = p_i|p_i\rangle$.

Given an initial state $|i\rangle = |x_i\rangle$ and a final state $\langle f| = \langle x_f|$, we want to compute the transition amplitude $\langle f|i\rangle$. If \hat{H} were independent of t the result would simply come from the time evolution of the states,

$$\langle f|i\rangle = \langle x_f| e^{-i(t_f-t_i)\hat{H}} |x_i\rangle. \quad (2.11)$$

We want to be more general than that, and we will assume that $\hat{H}(t)$ is a smooth function of t . For each infinitesimal time interval $\delta t = \frac{t_f-t_i}{n}$ we can use the right hand side of eq. (2.11) and we find the amplitude to be

$$\begin{aligned} \langle f|i\rangle &= \int dx_n \cdots dx_1 \langle x_f| e^{-i\hat{H}(t_f)\delta t} |x_n\rangle \langle x_n| \cdots |x_2\rangle \times \\ &\quad \langle x_2| e^{-i\hat{H}(t_2)\delta t} |x_1\rangle \langle x_1| e^{-i\hat{H}(t_1)\delta t} |x_i\rangle. \end{aligned} \quad (2.12)$$

We can evaluate each matrix element separately by inserting a complete set of momentum eigenstates and using $\langle p|x\rangle = e^{-ipx}$

$$\begin{aligned} \langle x_{i+1}| e^{-i\hat{H}(t_{i+1})\delta t} |x_i\rangle &= \int \frac{dp}{2\pi} \langle x_{i+1}|p\rangle \langle p| e^{-i\left[\frac{p^2}{2m} + V(x_i, t_i)\right]\delta t} |x_i\rangle \\ &= e^{-iV(x_i, t_i)\delta t} \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t + ip(x_{i+1}-x_i)} \\ &= \sqrt{\frac{m}{2\pi i\delta t}} e^{i\left[\frac{1}{2}m\frac{(x_{i+1}-x_i)^2}{(\delta t)^2} - V(x_i, t_i)\right]\delta t} \\ &= \sqrt{\frac{m}{2\pi i\delta t}} e^{iL(x_i, \dot{x}_i)\delta t} \end{aligned} \quad (2.13)$$

where we used eq. (2.7) to do the Gaussian integral, and we have defined the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x, t). \quad (2.14)$$

Technically we must assume that δt in eq. (2.13) has a small negative imaginary part for the integral to converge. This corresponds to a determination of the time-ordering of the path integral.

We find

$$\langle f|i\rangle = \lim_{n \rightarrow \infty} \left(\sqrt{\frac{m}{2\pi i \delta t}} \right)^n \int dx_n \cdots dx_1 e^{i \int_{t_i}^{t_f} dt L(x, \dot{x})} = \int_{x_i, t_i}^{x_f, t_f} \mathcal{D}x e^{iS[x]}, \quad (2.15)$$

where we have defined the measure

$$\mathcal{D}x \equiv \lim_{n \rightarrow \infty} \left(\sqrt{\frac{m}{2\pi i \delta t}} \right)^n dx_n \cdots dx_1, \quad (2.16)$$

and the action is defined as

$$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}). \quad (2.17)$$

2.2.3 The Path Integral in Quantum Field Theory

In quantum field theory the derivation of the path integral is almost the same, but we have to be more careful about the intermediate states. For simplicity we'll start by considering just the vacuum matrix element $\langle 0; t_f | 0; t_i \rangle$. The position and momentum operators \hat{x} and \hat{p} in quantum mechanics are replaced by the Schrödinger-picture fields $\hat{\phi}(\vec{x})$ and $\hat{\pi}(\vec{x})$ that are defined as

$$\begin{aligned} \hat{\phi}(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}), \\ \hat{\pi}(\vec{x}) &= -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} (a_p e^{i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}), \end{aligned} \quad (2.18)$$

and satisfy

$$\left[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y}) \right] = i\delta^3(\vec{x} - \vec{y}). \quad (2.19)$$

The equivalent of $|x\rangle$ and $|p\rangle$ from quantum mechanics is the complete set of eigenstates

$$\begin{aligned} \hat{\phi}(\vec{x}) |\Phi\rangle &= \Phi(\vec{x}) |\Phi\rangle, \\ \hat{\pi}(\vec{x}) |\Pi\rangle &= \Pi(\vec{x}) |\Pi\rangle, \end{aligned} \quad (2.20)$$

that satisfies

$$\langle \Pi | \Phi \rangle = \exp \left(-i \int d^3x \Pi(\vec{x}) \Phi(\vec{x}) \right), \quad (2.21)$$

2.2. PATH INTEGRAL FORMULATION OF QUANTUM FIELD THEORY

in the same way as $\langle p|x\rangle = e^{-i\vec{p}\cdot\vec{x}}$. These operators and their commutation relation, together with the Hamiltonian $H(t) = \int d^3x \mathcal{H}$, where \mathcal{H} is the Hamiltonian density

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{\pi}^2 + \mathcal{V}(\hat{\phi}), \quad (2.22)$$

define a quantum field theory.

Computing the vacuum matrix element is done following the same logic as in quantum mechanics,

$$\begin{aligned} \langle 0; t_f | 0; t_i \rangle &= \int \mathcal{D}\Phi_1(x) \cdots \mathcal{D}\Phi_n(x) \langle 0 | e^{-i\hat{H}(t_f)} | \Phi_n \rangle \times \\ &\quad \langle \Phi_n | \cdots | \Phi_1 \rangle \langle \Phi_1 | e^{-i\hat{H}(t_i)} | 0 \rangle. \end{aligned} \quad (2.23)$$

As before, each intermediate piece can be evaluated by itself by inserting a complete set of momentum states $\int \mathcal{D}\Pi_i | \Pi_i \rangle \langle \Pi_i |$, applying eq. (2.21) and performing the Gaussian integral. The result is

$$\begin{aligned} \langle \Phi_{i+1} | e^{-i\hat{H}(t_{i+1})\delta t} | \Phi_i \rangle &= N e^{\left(i\delta t \int d^3x \left[\frac{1}{2} \left(\frac{\Phi_{i+1}(\vec{x}) - \Phi_i(\vec{x})}{\delta t} \right)^2 - \mathcal{V}(\Phi_i) \right] \right)} \\ &= N e^{\left(i\delta t \int d^3x \mathcal{L}[\Phi_i, \partial_t \Phi_i] \right)}, \end{aligned} \quad (2.24)$$

where

$$\mathcal{L}[\Phi_i, \partial_t \Phi_i] = \frac{1}{2}(\partial_t \Phi_i)^2 - \mathcal{V}[\Phi_i]. \quad (2.25)$$

Putting it all together in eq. (2.23), we find

$$\langle 0; t_f | 0; t_i \rangle = N \int \mathcal{D}\Phi(x, t) e^{i \int d^4x \mathcal{L}[\Phi]} = N \int \mathcal{D}\Phi(x, t) e^{iS[\phi]}, \quad (2.26)$$

where the time integral goes from t_i to t_f . When calculating S -matrix elements, we are taking $t_i = -\infty$ and $t_f = +\infty$, so the action $S[\Phi]$ is the integral of the Lagrangian density over all space-time. N is just some (infinite) constant that comes from the Gaussian integral. We will see soon that it drops out of all of our calculations.

2.2.4 Time-Ordered Products and the Generating Functional

So far we have only computed the vacuum matrix element using the QFT path integral. We will now see how we can compute more general matrix elements.

Start by considering the following integral

$$\mathcal{I} = \int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_i(x_i), \quad (2.27)$$

where $\Phi_i(x_i)$ means the field Φ at a time t_i , i.e. $\Phi_i(x_i) = \Phi(x_i, t_i)$. Going back to eq. (2.23) we find

$$\begin{aligned} \mathcal{I} = \int \mathcal{D}\Phi_1(x) \cdots \mathcal{D}\Phi_n(x) \langle 0 | e^{-i\hat{H}(t_f)} | \Phi_n \rangle \times \\ \langle \Phi_n | \cdots | \Phi_1 \rangle \langle \Phi_1 | e^{-i\hat{H}(t_1)} | 0 \rangle \Phi_i(x_i). \end{aligned} \quad (2.28)$$

This means that we can replace the field $\Phi_i(x_i)$ by the operator $\hat{\phi}$ acting on the complete set of fields at time t_i

$$\int \mathcal{D}\Phi_i e^{-i\hat{H}(t_i)\delta t} | \Phi_i \rangle \Phi_i(x_i) \langle \Phi_i | = \hat{\phi}(x_i) \int \mathcal{D}_i e^{-i\hat{H}(t_i)\delta t} | \Phi_i \rangle \langle \Phi_i |. \quad (2.29)$$

This means that our integral is nothing but

$$\mathcal{I} = \int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_i(x_i) = \langle 0 | \hat{\phi}(x_i) | 0 \rangle. \quad (2.30)$$

Now if we go through the same process with two fields inserted into the integral,

$$\mathcal{I} = \int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_i(x_i) \Phi_j(x_j), \quad (2.31)$$

we will get two operators out, $\hat{\phi}(x_i)$ and $\hat{\phi}(x_j)$, and the result will be the matrix element of these two operators. The problem is that these two operators do not necessarily commute. So what is their ordering, i.e. $\hat{\phi}(x_i)\hat{\phi}(x_j)$ or $\hat{\phi}(x_j)\hat{\phi}(x_i)$? In eq. (2.23) the intermediate complete set of states was ordered with the later times on the left and the earlier times on the right. We conclude that the operators $\hat{\phi}(x_i)$ and $\hat{\phi}(x_j)$ must come out time-ordered with the operator acting on later times on the left.

$$\int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_i(x_i) \Phi_j(x_j) = \langle 0 | T \left\{ \hat{\phi}(x_1) \hat{\phi}(x_2) \right\} | 0 \rangle, \quad (2.32)$$

and in general,

$$\int \mathcal{D}\Phi e^{iS[\Phi]} \Phi_1(x_1) \cdots \Phi_n(x_n) = \langle 0 | T \left\{ \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \right\} | 0 \rangle. \quad (2.33)$$

Notice that the time-ordering almost came for free in our derivation. In reality it is related to the how the path integral was derived when we assumed that the δt

had a small imaginary part necessary to make the integrals converge. The details of how to include the $i\varepsilon$ prescription in the path integral formalism can be found in Schwartz [19].

In an interacting theory, the vacuum $|\Omega\rangle$ is different from the free vacuum $|0\rangle$, and we normalize the interacting vacuum such that $\langle\Omega|\Omega\rangle = 1$. This fixes the overall normalization of the path integral, and the infinite constant N drops out,

$$\langle\Omega|T\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\}|\Omega\rangle = \frac{\int \mathcal{D}\Phi e^{iS[\Phi]}\Phi_1(x_1)\cdots\Phi_n(x_n)}{\int \mathcal{D}\Phi e^{iS[\Phi]}}. \quad (2.34)$$

A convenient way of calculating these correlation functions is using what's called the generating functional. We consider an action $S[\phi]$ in the presence of a classical current J

$$Z[J] = \int \mathcal{D}\phi \exp\left\{iS[\phi] + i \int d^4x J(x)\phi(x)\right\}, \quad (2.35)$$

which is defined such that $J = 0$ corresponds to the vacuum matrix element

$$Z[0] = \int \mathcal{D}\phi \exp\{iS[\phi]\}. \quad (2.36)$$

We will now show that by taking derivatives with respect to J , we can build up all the correlation functions. Taking n derivatives of $Z[J]$ and normalizing by dividing by $Z[J]$ we find

$$\begin{aligned} \frac{(-i)^n}{Z[J]} \frac{\partial^n Z[J]}{\partial J(x_1)\cdots\partial J(x_n)} &= \frac{\int \mathcal{D}\phi e^{iS[\phi]+i\int d^4x J(x)\phi(x)} (\phi(x_1)\cdots\phi(x_n))}{\int \mathcal{D}\phi e^{iS[\phi]+i\int d^4x J(x)\phi(x)}} \\ &\equiv \langle\Omega|T\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\}|\Omega\rangle_J, \end{aligned} \quad (2.37)$$

which is the vacuum matrix element with a current J present. Setting $J = 0$, we find that

$$\begin{aligned} \frac{(-i)^n}{Z[0]} \frac{\partial^n Z[J]}{\partial J(x_1)\cdots\partial J(x_n)} \Bigg|_{J=0} &= \frac{\int \mathcal{D}\phi e^{iS[\phi]} (\phi(x_1)\cdots\phi(x_n))}{\int \mathcal{D}\phi e^{iS[\phi]}} \\ &= \langle\Omega|T\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\}|\Omega\rangle \end{aligned} \quad (2.38)$$

reproducing eq. (2.34).

2.2.5 Path Integral Example

The path integral is very useful in many cases. First of all, it provides a more convenient framework for formal derivations and proofs than perturbative calculations, and it is also useful in quantifying non-perturbative effects like instantons

[26]. We will use the path integral to derive many results about effective field theories in chapter 3, and we will now give an example of how one computes the path integral in the case where the action is quadratic. This will be used several times in the calculation of the effective potential in section 3.3.

2.2.5.1 Functional determinants

Consider an action that is of a quadratic form

$$S[\phi] = \int d^4x d^4y \phi_a(x) M_{ab}(x-y) \phi_b(x). \quad (2.39)$$

Two examples are a free scalar field theory with $\Phi = \phi$ and $M(x-y) = \delta^4(x-y)(\square + m^2)$, and a massive spin-1 field with $\Phi = A_\mu$ and $M_{\mu\nu}(x-y) = \delta^4(x-y)(\square g_{\mu\nu} - \partial_\mu \partial_\nu + m^2)$. One way to look at this expression is as though we are summing over four indices: a, b, x and y . The first two run over $a, b = 1, 2, \dots, N$ and x, y go from $-\infty$ to ∞ . The action $S[\phi]$ is a real number, so M_{ab} must be real, and we see that it's symmetric, $M_{ab}(x-y) = M_{ba}(y-x)$. If we want to perform the path integral

$$\int \mathcal{D}\phi e^{iS[\phi]} = \int \mathcal{D}\phi e^{i \int d^4x d^4y \phi_a(x) M_{ab}(x-y) \phi_b(y)}, \quad (2.40)$$

we apply eq. (2.7) and find

$$\int \mathcal{D}\phi e^{iS[\phi]} = N \frac{1}{\sqrt{\text{Det}M}}, \quad (2.41)$$

where $\text{Det}M$ is the functional determinant of M and N is some unimportant constant. If you have a hard time understanding how this *really* works, you should read Appendix A.6. In the functional determinant we are not only taking the determinant over the matrix indices, but we are also including the position coordinates. We will now show how to evaluate this functional determinant.

The easiest way of finding the determinant is by first taking the logarithm. Since the determinant is just the product of the eigenvalues, it follows that

$$\ln \text{Det}M = \text{Tr} \ln M, \quad (2.42)$$

where Tr is the functional trace and \ln is the functional natural logarithm, meaning that we trace and take the determinant over both matrix indices and position coordinates. By Fourier transforming,

$$\ln M(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \ln M(p), \quad (2.43)$$

we can now take the functional trace. The trace over the spatial coordinates means setting $x = y$ and summing over them, i.e. integrating over d^4x , and we also have to take the trace over $M(p)$.

$$\begin{aligned}
 \ln \text{Det}M(x - y) &= \text{Tr} \ln M(x - y) \\
 &= \text{Tr} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \ln M(p) \\
 &= \int d^4x \int \frac{d^4p}{(2\pi)^4} \text{tr} \ln M(p) \\
 &= \mathcal{V}_4 \int \frac{d^4p}{(2\pi)^4} \ln \det M(p)
 \end{aligned} \tag{2.44}$$

where \det is the normal matrix determinant, \ln is the standard natural logarithm and \mathcal{V}_4 is the volume of space-time.

In conclusion, we have found that

$$\int \mathcal{D}\Phi e^{iS[\Phi]} = N \frac{1}{\sqrt{\text{Det}M}} = \exp \left(-\frac{1}{2} \mathcal{V}_4 \int \frac{d^4p}{(2\pi)^4} \ln \det M(p) \right). \tag{2.45}$$

2.2.6 Fermionic path integral

The fermionic path integral is really no different than the normal path integral discussed so far. The only difference is that we must use eq. (2.9) instead of eq. (2.7). Given a Lagrangian $\mathcal{L} = \bar{\psi}M\psi$, the result is

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[\psi, \bar{\psi}]} = N \text{Det}M \tag{2.46}$$

and the functional determinant is evaluated in exactly the same way as described before.

2.3 Symmetries

Symmetries play a very important role in quantum field theory. Prior knowledge of the symmetry often simplifies calculations and gives us a deeper understanding of the theory. Apriori knowledge of the symmetries of the theory is not crucial, and the symmetry will still be there even if we are not aware of it. The famous example is Maxwell's equations which are Lorentz invariant, but they were written down in an early form in 1865 [27] long before Lorentz invariance [28] and Einstein's special relativity [29].

In this section we will give a description of the Euler-Lagrange equations and the classical symmetries of the action $S[\phi]$ leading us to Noether's theorem. Then we will focus on the description of gauge symmetries which play an important

role in the quantum field theory of the real world. Finally we will discuss the concept of symmetry breaking.

2.3.1 Euler-Lagrange Equations

Consider a set of scalar fields $\phi_i(x)$ and a Lagrangian $\mathcal{L} = \mathcal{L}[\phi_i(x), \partial_\mu \phi_i(x)]$. The principle of least action tells us that the solution $\phi_i^{(0)}$ minimizes the action $S = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x))$. Equivalently, an infinitesimal variation away from the solution $\phi_i = \phi_i^{(0)} + \delta\phi_i$ gives $\delta S = 0$ to linear order in $\delta\phi_i$.

As we will show below, the principal of least action implies a differential equation for $\phi_i^{(0)}$. The solution is then fixed by imposing either a set of initial conditions or boundary conditions on ϕ_i^0 . In QFT, we choose to set the following boundary conditions

- $\lim_{|\vec{x}| \rightarrow \infty} \phi_i^{(0)}(t, \vec{x}) = 0$
- $\phi_i^{(0)}(\pm\infty, \vec{x}) = \phi_i^{(0)}(\vec{x})$

Note that since we are looking for a particular solution with fixed boundary conditions, one only allows variations that respect these boundary conditions. Thus, we must have $\delta\phi_i = 0$ at spatial and temporal infinity.

We find

$$\begin{aligned}
 S[\phi_i^{(0)}] &\rightarrow S[\phi_i^{(0)} + \delta\phi_i] = \int d^4x \mathcal{L}[\phi_i^{(0)} + \delta\phi_i, \partial_\mu \phi_i^{(0)} + \delta\partial_\mu \phi_i] \\
 &= \int d^4x \left[\mathcal{L} + \delta\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \delta(\partial_\mu \phi_i) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] \\
 &= S[\phi_i] + \int d^4x \left[\delta\phi_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \right) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right) \right].
 \end{aligned} \tag{2.47}$$

The last term is a total derivative, and after integrating over space-time we are left with $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i$ evaluated at the boundary. Since $\delta\phi_i$ vanishes at the boundary, this term is zero. Thus, for the action to be invariant under an arbitrary $\delta\phi_i$ we conclude

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0, \tag{2.48}$$

which are the Euler-Lagrange equations.

2.3.2 Noether's Theorem

In this section we will review Noether's theorem [30] in a slightly different way than most textbooks [31]. Noether's theorem states that for every continuous

global symmetry there exists a conserved current and a corresponding conserved quantity, and in the context of quantum field theory it's important to remember that this is only true when the Euler-Lagrange equations are satisfied.

Specifically, a symmetry means that we can do a field transformation $\phi_i \rightarrow \phi'_i$ along with a coordinate transformation $x^\mu \rightarrow x'^\mu$ that leaves the action $S[\phi_i]$ invariant. A continuous symmetry means that it can be written in an infinitesimal form $\phi_i(x) \rightarrow \phi_i(x) + \epsilon \delta \phi_i(x)$, where ϵ is small. Note that a symmetry transformation relates to a solution $\phi_i^{(0)}$ to another solution $\phi_i^{\prime(0)}$. In general, the two solutions are described by two different boundary conditions. Thus, we cannot assume the variation $\epsilon \delta \phi_i$ to vanish at the boundary.

There are two ways of doing an infinitesimal transformation on our action

- Infinitesimal field redefinition: $\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \delta \phi_i(x)$
- Infinitesimal change in coordinates: $x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu(x)$. For a scalar field we get $\phi_i(x) = \phi'_i(x') \Rightarrow \phi'_i(x) = \phi_i(x) - \delta x^\mu(x) \partial_\mu \phi_i(x)$.

So the most general infinitesimal transformation we can have is

$$\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \delta \phi_i(x) - \delta x^\mu(x) \partial_\mu \phi_i(x) \equiv \phi_i(x) + \Delta \phi_i(x), \quad (2.49)$$

where we have defined $\Delta \phi_i(x) \equiv \delta \phi_i(x) - \delta x^\mu(x) \partial_\mu \phi_i(x)$. For this transformation the action transforms as

$$\begin{aligned} S \rightarrow S' &= \int d^4x' \mathcal{L}[\phi'_i(x'), \partial_\mu \phi'_i(x')] \\ &= \int d^4x \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| (\mathcal{L}[\phi'_i(x), \partial_\mu \phi'_i(x)] + \delta x^\mu \partial_\mu \mathcal{L}) \\ &= \int d^4x (1 + \partial_\mu \delta x^\mu) \left(\mathcal{L} + \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \right] \Delta \phi_i \right. \\ &\quad \left. + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i \right) + \delta x^\mu \partial_\mu \mathcal{L} \right) \\ &= \int d^4x \left(\mathcal{L} + \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \right] \Delta \phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i \right) \right. \\ &\quad \left. + \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \delta x^\mu \mathcal{L} \right) \quad (2.51) \\ &= \int d^4x \left(\mathcal{L} + \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \right] \Delta \phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i + \delta x^\mu \mathcal{L} \right) \right) \end{aligned}$$

$$\Rightarrow \delta S = \int d^4x \left(\left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \right] \Delta \phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i + \delta x^\mu \mathcal{L} \right) \right).$$

Note that the total derivative does not necessarily vanish since $\Delta \phi$ and δx^μ need not vanish at the boundary as described above. For this to be a symmetry we require $\delta S = 0$. We therefore have

$$\left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \right] \Delta \phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \Delta \phi_i + \delta x^\mu \mathcal{L} \right) = 0. \quad (2.52)$$

If $\phi_i(x)$ satisfies the equations of motion given in eq. (2.48), the first term is zero, and we find by using the definition of $\Delta \phi_i(x)$ that there exists a conserved current $\partial_\mu J^\mu = 0$ with

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \delta \phi_i - \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta_\nu^\mu \mathcal{L} \right] \delta x^\nu. \quad (2.53)$$

We see that for every continuous global symmetry we find a corresponding conserved current. For this conserved current we define a charge $Q(t) = \int d^3x J^0(t, \vec{x})$ which is conserved

$$\frac{dQ}{dt} = \int d^3x \partial_t J^0(t, x) = \int d^3x \vec{\nabla} \cdot \vec{J}(t, x) = 0, \quad (2.54)$$

which is zero since the current vanishes by assumption at the boundary.

With this general formula, consider a simple example with a space-time translation symmetry $x^\mu \rightarrow x^\mu + a^\mu$ for an arbitrary constant a^μ without transforming the scalar field, i.e. $\delta \phi_i = 0$. We find

$$J^\mu = - \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta_\nu^\mu \mathcal{L} \right] a^\nu \equiv -T^{\mu\nu} a_\nu, \quad (2.55)$$

and we have found the conserved Energy-Momentum tensor with $\partial_\mu T^{\mu\nu} = 0$ since $\partial_\mu J^\mu = 0$. For a free scalar field theory with $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2$ we find

$$T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - g_{\mu\nu} \frac{1}{2} \phi(\square + m^2)\phi. \quad (2.56)$$

The above discussion was about classical field theory, and it is worth mentioning what happens in the quantum case. There are cases where there exist a classical symmetry in the action but that is not a symmetry in the quantum theory. When this happens the symmetry is said to be anomalous. One example is the chiral anomaly. Here we have an action that has a chiral symmetry, but it turns out that the measure in the path integral is not invariant, and the charge corresponding to the chiral symmetry will not be conserved as it would be classically.

Gauge symmetries, which will be discussed in the next section, cannot have any anomalies. The reason for this is that if the current corresponding to the gauge symmetry is not conserved, it would be possible to produce unphysical longitudinal polarizations. Technically, the Ward identity would be violated. It turns out that this is a strong requirement for consistency of a quantum field theory, and in the Standard Model, it forces the electric charge to be quantized, and it relates the quark and lepton charges. A thorough discussion can be found in Schwartz [19].

2.3.3 Gauge Symmetries

The general idea behind gauge symmetries is that we start with a theory with a continuous global symmetry, and we will make the symmetry local. This sounds like a fairly simple idea, but it turns out that this is a very powerful concept. Gauge symmetries play an extremely important role in quantum field theory.

The Standard Model is a $SU(3) \times SU(2) \times U(1)$ gauge theory. We start with a theory with quarks, leptons and a scalar (Higgs), and by gauging it we get the strong force through the gluons, the weak force through the W^\pm and Z , and the electromagnetic force through the photon A . Add in spontaneous symmetry breaking discussed in section 2.3.8, and you have the main ingredients of the Standard Model. It's quite remarkable that so much comes out from this one concept.

2.3.3.1 $U(1)$ Gauge symmetry

To illustrate how gauge symmetries work we will start by considering a simple theory with one complex scalar field $\mathcal{L} = -\phi^*(\square + m^2)\phi$. This theory is invariant under a global $U(1)$ symmetry $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ for $\alpha \in \mathbb{R}$ and $\alpha \sim \alpha + 2\pi$. We will now make the symmetry local, i.e. $\alpha \rightarrow \alpha(x)$.

The first problem that arises in making the theory local (besides the fact that the Lagrangian is obviously not invariant) is that we can no longer compare the value of the fields at two different points x^μ and y^μ . To see this, consider $|\phi(y) - \phi(x)|$. For a global symmetry this transforms to $|e^{i\alpha}(\phi(y) - \phi(x))| = |\phi(y) - \phi(x)|$. But for a gauge symmetry we get under a transformation

$$\phi(y) - \phi(x) \rightarrow e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x), \quad (2.57)$$

and we see that we don't know how to compare fields at different points. Thus, we also do not know how to compute derivatives since the derivative is essentially a difference between two points x and $x + \delta x$.

To deal with this problem we introduce a new field $W(x, y)$ called a Wilson line [32] that transforms as

$$W(x, y) \rightarrow e^{i\alpha(x)}W(x, y)e^{-i\alpha(y)}. \quad (2.58)$$

We now see that

$$\begin{aligned} W(x, y)\phi(y) - \phi(x) &\rightarrow e^{i\alpha(x)}W(x, y)e^{-i\alpha(y)}e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) \\ &= e^{i\alpha(x)} [W(x, y)\phi(y) - \phi(x)]. \end{aligned} \quad (2.59)$$

We see that $|W(x, y)\phi(y) - \phi(x)|$ is independent of $\alpha(x)$, and we will now use this to define a new derivative called the covariant derivative

$$D_\mu\phi(x) \equiv \lim_{\delta x \rightarrow 0} \frac{W(x, x + \delta x)\phi(x + \delta x) - \phi(x)}{\delta x^\mu}, \quad (2.60)$$

which by definition transforms as

$$D_\mu\phi(x) \rightarrow e^{i\alpha(x)}D_\mu\phi(x). \quad (2.61)$$

With the definition of the covariant derivative we must have $W(x, x) = 1$, and for small δx we can expand

$$W(x, x + \delta x) = 1 - ie\delta x^\mu A_\mu(x) + \mathcal{O}(\delta x^2), \quad (2.62)$$

where the constant e is arbitrary and $A_\mu(x)$ is our gauge field introduced as a connection. Applying this expansion to eq. (2.58) we find that our gauge field $A_\mu(x)$ must transform as

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \quad (2.63)$$

We also find an explicit form of the covariant derivative

$$D_\mu\phi(x) = \partial_\mu\phi(x) - ieA_\mu(x)\phi(x). \quad (2.64)$$

Since $D_\mu\phi(x)$ transforms just as $\phi(x)$, we must also have that $D_\mu D_\nu\phi(x)$ transforms as $\phi(x)$. Hence

$$[D_\mu, D_\nu]\phi(x) \rightarrow e^{i\alpha(x)}[D_\mu, D_\nu]\phi(x). \quad (2.65)$$

It turns out that this commutator is actually not an operator, but just a function

$$[D_\mu, D_\nu]\phi(x) = -ie(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))\phi(x) \equiv -ieF_{\mu\nu}\phi(x), \quad (2.66)$$

where we have defined the field strength $F_{\mu\nu} \equiv \frac{i}{e}[D_\mu, D_\nu]$ which is invariant under the gauge transformation. Since it respects the gauge symmetry we can add it to the Lagrangian to study the dynamics of the gauge field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 - m^2|\phi|^2, \quad (2.67)$$

where the $-\frac{1}{4}$ is just convention and we have replaced $\partial_\mu \rightarrow D_\mu$ compared to the original Lagrangian. This is the Lagrangian for scalar QED and it has a $U(1)$ gauge symmetry. It is the simplest example of a gauge theory since $U(1)$ is Abelian.

Using the Euler-Lagrange equation for A_μ given in eq. (2.48), we find $\partial_\mu F^{\mu\nu} = 0$. This reproduces Maxwell's equation in covariant form

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= 0, \\ \partial_{[\mu} F_{\nu\lambda]} &= 0,\end{aligned}\tag{2.68}$$

where the brackets in the second equation means that we antisymmetrize all the indices. This is automatically satisfied with our definition of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

2.3.4 $SU(N)$ Gauge Symmetry

Now we will consider $SU(N)$ which is a more complicated example of a gauge symmetry since it is a non-Abelian group, but is of huge importance in the Standard Model. Consider the Lagrangian with N complex scalar fields with a global $SU(N)$ symmetry

$$\mathcal{L} = -\Phi_i^*(\square + m^2)\Phi_i.\tag{2.69}$$

The field Φ_i transforms as

$$\Phi_i \rightarrow \left[e^{i\alpha^a T^a} \right]_{ij} \Phi_j,\tag{2.70}$$

where T^a are the $SU(N)$ generators in the fundamental representation. I will not go into details about the representation of these groups, but if the reader is unfamiliar with the group theory used here, I recommend reading Howard Georgi's book on Lie Algebras in Particle Physics [33].

Making the symmetry local, $\alpha^a = \alpha^a(x)$, we must as before find a way of comparing fields at different points in space-time. We introduce a Wilson line that transforms as

$$W(x, y) \rightarrow U(x)W(x, y)U^\dagger(y) = e^{i\alpha^a(x)T^a} W(x, y)e^{-i\alpha^a(y)T^a},\tag{2.71}$$

where we have defined $U(x) = \exp(i\alpha^a(x)T^a)$ and used $T^{a\dagger} = T^a$, which simply follows from $U^\dagger(x) = U^{-1}(x)$. For $y^\mu = x^\mu + \delta x^\mu$ with δx^μ being infinitesimal, we expand and find

$$W(x, x + \delta x) = 1 - igA_\mu^a(x)T^a\delta x^\mu + \mathcal{O}(\delta x^2),\tag{2.72}$$

where we have defined the gauge field $A_\mu^a(x)$. It follows that the covariant derivative will be defined in a similar way as for the abelian case,

$$D_\mu \Phi_i(x) = \partial_\mu \Phi_i(x) - igA_\mu^a(x)T_{ij}^a \Phi_j(x).\tag{2.73}$$

To see how the gauge field transforms it's easiest to start with the fact that $D_\mu \Phi_i(x) \rightarrow U(x)D_\mu \Phi_i(x)$. Let $\mathbf{A}_\mu \equiv A_\mu^a T^a$ denote the Lie-Algebra valued field, and we find

$$U(\partial_\mu - ig\mathbf{A}_\mu)\Phi = (\partial_\mu - ig\mathbf{A}'_\mu)U\Phi, \quad (2.74)$$

where we have suppressed some of the indicies and denoted the transformed gauge field \mathbf{A}'_μ . Solving for \mathbf{A}'_μ we find

$$\mathbf{A}'_\mu = U\mathbf{A}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \quad (2.75)$$

or written in an infinitesimal form

$$A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a(x) - f^{abc}\alpha^b A^c_\mu(x) + \mathcal{O}(\alpha^2), \quad (2.76)$$

where f^{abc} are the $SU(N)$ structure constants.

The field strength is again defined as

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= F^a_{\mu\nu} T^a = \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu) - ig[\mathbf{A}_\mu, \mathbf{A}_\nu] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)T^a, \end{aligned} \quad (2.77)$$

and we our new Lagrangian which is invariant under a local $SU(N)$ transformation is

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + |D_\mu \Phi_i|^2 - m^2|\Phi_i|^2. \quad (2.78)$$

2.3.5 Quantization and Faddeev-Popov Gauge Fixing

In the previous section we introduced the gauge field A^a_μ , a massless spin-1 field with two degrees of freedom. Note that the vector A^a_μ in general has four degrees of freedom, and we must remove the extra degrees of freedom before we can quantize the theory. The missing ingredient is what we call gauge fixing.

To get some intuition, we start with a physical picture that we are familiar with. For a massive spin-1 particle we know that we can go to the particle's rest frame, and there are 3 possible polarizations by rotational symmetry. In the massless case we cannot go to the particles rest frame since it's moving with the speed of light. In this case, the only two polarizations are transverse to the direction of motion of the particle. Hence we see that the gauge field's two degrees of freedom are the two physical transverse polarizations. The problem is that A^a_μ , with the four degrees of freedom, can have a unphysical longitudinal polarization.

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We will now see that this unphysical polarization is very much related to the gauge symmetry where this all started. Consider a state A_μ^a with a given transverse physical polarization. Now recall that the infinitesimal gauge transformation for A_μ^a is

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a(x) - f^{abc} \alpha^b A_\mu^c(x). \quad (2.79)$$

Notice that in momentum space the $\partial_\mu \rightarrow \pm i p_\mu$, so we are effectively shifting A_μ^a with something proportional to its momentum p_μ . Since the longitudinal polarization that is proportional to the momentum is unphysical, we conclude that shifting A_μ^a by $\frac{1}{g} \partial_\mu \alpha^a(x)$ will not change the physical polarization. The last term is just mixing the different gauge fields in the non-Abelian case, so if we started out with a set of fields with physical polarization, we have now gone to another linear combination of these physical polarizations. We usually say that we have a redundant description of the gauge field in A_μ^a since multiple vectors can represent the same physical state.

Consider a manifold containing all different configurations of the gauge field, and start by considering a point A_μ^a . Using the transformation given above, we can trace out a curve called a gauge orbit. Every point along this curve corresponds to the same two physical polarizations, but different unphysical polarizations. To quantize in the canonical way, we must get rid of the redundancy by restricting ourselves to only one point on each gauge orbit. This can be done by choosing another constraint called the gauge fixing. Typically we choose the Lorenz gauge $\partial_\mu A^\mu = 0$ since it is manifestly Lorentz invariant, but other choices like the axial gauge with $A_0 = 0$ are sometimes used.

We would like to understand how this works in the path integral formalism, and we will follow the approach known as the Fadeev-Popov gauge fixing procedure [34]. Before we dig into the details, let's look at the general idea. Start with the path integral

$$\mathcal{I} = \int \mathcal{D}A e^{iS[A]}. \quad (2.80)$$

If we naïvely think of this integration as integrating over all values of the field A_μ^a we are overcounting due to the redundancy described above. Hence we want to restrict the integral to be over only the field values where the field satisfy some constraint $F[A_\mu^a] = 0$, where $F[A_\mu^a]$ is what we call the gauge fixing function. This can simply be done by inserting a delta function

$$\mathcal{I} \sim \int \mathcal{D}A \delta(F[A_\mu^a]) e^{iS[A]}. \quad (2.81)$$

In the following we will show how we can achieve this rigorously.

We start by choosing a gauge, i.e. choose one element of the equivalence class of gauge fields, and call it \hat{A}_μ^a , and choose a constraint F such that $F[\hat{A}_\mu^a] = 0$. Due to our gauge symmetry we can write any other element of the equivalent class of gauge fields as $A_\mu^a = \hat{A}_\mu^a + \frac{1}{g}D_\mu\alpha^a$. The gauge fixing condition can be written as $F[A_\mu^a - \frac{1}{g}D_\mu\alpha^a] = 0$

Now observe that

$$1 = \int dF\delta(F) = \int \mathcal{D}\alpha^b \det\left(\frac{\delta F}{\delta\alpha^b}\right) \delta(F), \quad (2.82)$$

where $F = F[A_\mu^a - \frac{1}{g}D_\mu\alpha^a]$ and we have done a change in integration variables. Starting with a Lagrangian $\mathcal{L} = \mathcal{L}[A^a, \phi_i]$ with a gauge symmetry, we multiply the generating functional $Z[0]$ by the 1 in eq. (2.82) and find

$$\begin{aligned} Z[0] &= \int \mathcal{D}A\mathcal{D}\phi_i e^{iS} \\ &= \int \mathcal{D}A^a\mathcal{D}\phi_i\mathcal{D}\alpha^b \det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu\alpha^a]}{\delta\alpha^b}\right) \delta\left(F[A_\mu^a - \frac{1}{g}D_\mu\alpha^a]\right) e^{iS}. \end{aligned} \quad (2.83)$$

We now perform a shift corresponding to a gauge transformation on $A_\mu^a \rightarrow A_\mu^a + \frac{1}{g}D_\mu\beta^a$ and a corresponding gauge transformation in the scalar fields. The action and measure are invariant under this transformation, and the delta function becomes

$$\begin{aligned} &\det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu(\alpha^a + \beta^a)]}{\delta\alpha^b}\right) \delta\left(F[A_\mu^a - \frac{1}{g}D_\mu(\alpha^a + \beta^a)]\right) \\ &= \det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu(\alpha^a + \beta^a)]}{\delta(\alpha^c + \beta^c)}\right) \det\left(\frac{\delta(\alpha^c + \beta^c)}{\delta\alpha^b}\right) \\ &\quad \times \delta\left(F[A_\mu^a - \frac{1}{g}D_\mu(\alpha^a + \beta^a)]\right). \end{aligned} \quad (2.84)$$

We keep α^a fixed when we are doing the integral over $\mathcal{D}A$, and we are allowed to make the choice $\beta^a = -\alpha^a$. Taking this limit we take the determinant and evaluate it at zero

$$\begin{aligned} \det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu(\alpha^a + \beta^a)]}{\delta(\alpha^c + \beta^c)}\right) \Big|_{\beta^a = -\alpha^a} &\xrightarrow{\beta^a = -\alpha^a} \det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu\alpha^a]}{\delta\alpha^c}\right) \Big|_{\alpha=0} \\ &\equiv \det\left(\frac{\delta F}{\delta\alpha}\right), \end{aligned} \quad (2.85)$$

where we on the second line have defined a short hand notation for this determinant. Note that the determinant is independent of α after taking the limit, so we

can factor out the integral over $\mathcal{D}\alpha$ which is just the volume of the gauge group. We find

$$\begin{aligned} Z[0] &= \int \mathcal{D}A \mathcal{D}\phi_i e^{iS} \\ &= \left(\int \mathcal{D}\alpha^b \right) \int \mathcal{D}\phi_i \mathcal{D}A^a \det \left(\frac{\delta F}{\delta \alpha} \right) \delta(F[A_\mu^a]) e^{iS}. \end{aligned} \quad (2.86)$$

The volume of the gauge group is just some constant independent of the fields, and it will in general be dropped from our calculation.

Using the path integral in eq. (2.9) we introduce the non-physical ghost fields c and \bar{c} and write

$$\det \left(\frac{\delta F}{\delta \alpha} \right) = \int \mathcal{D}c \mathcal{D}\bar{c} \exp \left(i \int d^4x \bar{c} \left[-\frac{\delta F}{\delta \alpha} \right] c \right). \quad (2.87)$$

We also note that shifting F by a constant doesn't change the value of the determinant due to the differentiation, so we can shift F in the delta function in eq. (2.86) by a constant χ . The value of the constant is irrelevant, and we choose to average over a Gaussian-weighted selection

$$\int \mathcal{D}\chi \exp \left\{ -i \int d^4x \frac{\chi^2}{2\xi} \delta(F[A_\mu^a - \chi]) \right\} = \exp \left\{ -i \int d^4x \frac{F[A_\mu^a]^2}{2\xi} \right\}. \quad (2.88)$$

In conclusion we find

$$Z[0] = \int \mathcal{D}A^a \mathcal{D}\phi_i \mathcal{D}c \mathcal{D}\bar{c} \exp \left\{ i \int d^4x \left[\mathcal{L} - \bar{c} \frac{\delta F}{\delta \alpha} c - \frac{F^2}{2\xi} \right] \right\}. \quad (2.89)$$

We will denote the gauge fixed action by $I = \mathcal{L} - \bar{c} \frac{\delta F}{\delta \alpha} c - \frac{F^2}{2\xi}$, and this gauge fixing is sometimes called the R_ξ gauges which is parametrized by ξ .

2.3.5.1 U(1) Gauge theory

Let's apply the Faddeev-Popov Gauge fixing to scalar QED, a U(1) gauge theory. The original action is

$$S = \int d^4x \mathcal{L} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - m^2 |\phi|^2 \right), \quad (2.90)$$

and we choose the gauge fixing condition

$$F[A_\mu] = \partial_\mu A_\mu. \quad (2.91)$$

For the U(1) theory the ghost term is

$$\begin{aligned} \det\left(\frac{\delta F}{\delta\alpha}\right) &= \det\left(\frac{\delta F[A_\mu - \frac{1}{e}\partial_\mu\alpha]}{\delta\alpha}\right)\Big|_{\alpha\rightarrow 0} \\ &= \det\left(\frac{\delta(\partial_\mu A_\mu - \frac{1}{e}\square\alpha)}{\delta\alpha}\right)\Big|_{\alpha\rightarrow 0} = \det\left(-\frac{\square}{e}\right). \end{aligned} \quad (2.92)$$

The gauge fixed action becomes

$$I = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 - m^2|\phi|^2 - \bar{c}\square c - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 \right), \quad (2.93)$$

where we have rescaled the ghost fields to remove the factor of e .

Note that in scalar QED with this choice of F the ghost fields do not interact with the other fields. We say that they decouple from the theory, and they can without loss of generality be neglected when computing Green's functions as in eq. (2.34) since the term from the numerator cancel the term in the denominator. Note that the ghost term will in general be important if you are computing something else than Green's functions, e.g. the free energy.

2.3.5.2 SU(N) Gauge theory

Now we follow the same steps as in the U(1) case, but for the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + |D_\mu\Phi_i|^2 - m^2|\Phi_i|^2, \quad (2.94)$$

where the gauge transformation for A_μ^a is

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{g}D_\mu^{ab}\alpha^b, \quad (2.95)$$

and we choose the gauge fixing condition

$$F[A_\mu^a] = \partial_\mu A_\mu^a. \quad (2.96)$$

Evaluating the determinant we find in the SU(N) case

$$\begin{aligned} \det\left(\frac{\delta F}{\delta\alpha^c}\right) &= \det\left(\frac{\delta F[A_\mu^a - \frac{1}{g}D_\mu^{ab}\alpha^b]}{\delta\alpha^c}\right)\Big|_{\alpha=0} \\ &= \det\left(\frac{\delta(\partial_\mu A_\mu^a - \frac{1}{g}\partial_\mu D_\mu^{ab}\alpha^b)}{\delta\alpha^c}\right)\Big|_{\alpha=0} \\ &= \det\left(-\frac{1}{g}\partial_\mu D_\mu^{ac}\right). \end{aligned} \quad (2.97)$$

The gauge fixed action becomes

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + |D_\mu\Phi_i|^2 - m^2|\Phi_i|^2 - \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 - \bar{c}^a\partial_\mu D_\mu^{ab}c^b, \quad (2.98)$$

where we again have rescaled the ghost fields to get rid of the charge g .

2.3.6 BRST Invariance

In the previous section we showed how we could break the gauge symmetry by adding a gauge fixing term, but there is in fact a residual global symmetry left. The Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + |D_\mu\Phi_i|^2 - m^2|\Phi_i|^2 - \frac{1}{2\xi}(\partial_\mu A_\mu^a)^2 - \bar{c}^a\partial_\mu D_\mu^{ab}c^b \quad (2.99)$$

is actually still invariant under the transformation [19]

$$\begin{aligned} \Phi_i &\rightarrow \Phi_i + i\theta c^a T_{ij}^a \Phi_j, \\ A_\mu^a &\rightarrow A_\mu^a + \frac{1}{g}\theta D_\mu^{ab}c^b, \\ \bar{c}^a &\rightarrow \bar{c}^a - \frac{1}{g}\theta\frac{1}{\xi}\partial_\mu A_\mu^a, \\ c^a &\rightarrow c^a - \frac{1}{2}\theta f^{abc}c^b c^c, \end{aligned} \quad (2.100)$$

where θ is a Grassmann number. This symmetry is called BRST after Becchi, Rouet, Stora and Tyutin [35, 36]. Note that the transformation for Φ_i and A_μ^a is just the normal gauge transformation with $\alpha^a = \theta c^a$, and it follows that after gauge fixing anything that is gauge invariant also is BRST invariant.

BRST invariance is a global symmetry, and it has proven very useful in different contexts. We will use it to derive the Nielsen Identity in section 3.3.3.1, and it is also an important ingredient in section 2.3.7 where we discuss how to show that S -matrix elements and Green's functions of gauge invariant operators are in fact gauge independent. Since BRST is an exact symmetry of the Lagrangian, it is preserved even when computing loops. This can be used to prove that non-Abelian theories are renormalizable.

2.3.7 Gauge Invariant Quantities

After having gone through the process in the previous sections, we should now stop and think about what we have done. We started with a theory with a classical symmetry that we made local. This new gauge symmetry caused a problem with a redundant description of our gauge fields, and we had to gauge fix

using the Faddeev-Popov procedure which introduced the gauge fixing parameter ξ . The redundancy just means that we did not have a unique description for our physical state, so in the way that everything has been set up, we should expect any physical quantity to be independent of what value I choose for ξ .

If we have done everything correctly there should be no gauge dependence in any physical prediction, but as a double check we should be able to prove that this in fact is the case. As it turns out, the BRST symmetry is exactly what gives us this confirmation. The proof is rather involved, and we will now just summarize the key steps that goes into the proof [37].

Slavnov [38] and Taylor [39] showed that having BRST invariance implies a generalized set of Ward-Takahashi identities sometimes called the Slavnov-Taylor identities. Consider a generating functional $Z_\xi(J)$ which depends on the gauge fixing parameter and an external current J . It can be shown [40] that an infinitesimal change $\xi \rightarrow \xi + \delta\xi$ and using the Slavnov-Taylor identities leaves the S -matrix invariant meaning that the S -matrix must be independent of the value of ξ . One can also show that Green's functions of gauge invariant operators are independent of ξ [41].

Knowing that the S -matrix and Green's functions of gauge invariant operators are in fact gauge independent, we can safely just choose a value for ξ that is the most convenient for us to simplify the calculation. If you choose to leave ξ in there, you will see that the ξ dependence will drop out when adding up all the diagrams order by order in perturbation theory.

2.3.8 Symmetry Breaking

In this section we will give a brief discussion about symmetry breaking which is an important concept in Quantum Field Theory. In the previous section we saw how the gauge fixing term broke the gauge symmetry of our Lagrangian, and this was needed for us to quantize the theory. This was an example of explicit symmetry breaking. Another form of symmetry breaking is spontaneous symmetry breaking (SSB). In this case we have a symmetry that is not broken at the level of our Lagrangian, but the ground state of the theory is not invariant under the symmetry. To illustrate the point, we will start with an example in the linear sigma model before we discuss the more general case.

2.3.8.1 The Linear Sigma Model

Consider the following Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2 - \frac{\lambda}{3!} |\phi|^4. \quad (2.101)$$

This Lagrangian has a global $U(1)$ symmetry $\phi \rightarrow e^{i\alpha} \phi$. The general form of the potential $V = -m^2 |\phi|^2 + \frac{\lambda}{3!} |\phi|^4$ is shown in figure 2.1 with $m^2 > 0$. From the

figure it is obvious that the ground state is not going to be $\phi = 0$. The value that minimizes the potential is $|\phi|^2 = \frac{3m^2}{\lambda}$. Due to the U(1) symmetry we now have an infinite number of equivalent vacua $|\Omega_\theta\rangle$ with $\langle\Omega_\theta|\phi|\Omega_\theta\rangle = \sqrt{\frac{3m^2}{\lambda}}e^{i\theta} \equiv \frac{v}{\sqrt{2}}$ for any $\theta \in \mathbb{R}$. All the vacua are equivalent, and we will normally choose $\theta = 0$ making the vacuum expectation value (vev) real. Expanding around the vev and writing the Lagrangian in terms of real fields $\phi = \frac{v+\phi_1+i\phi_2}{\sqrt{2}}$ we get a new Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{m^2}{4}\phi_1^2 + \frac{m^2}{4}\phi_2^2 - \frac{1}{4!}(\phi_1^2 + \phi_2^2)^2 \\ & + \frac{9m^4}{8\lambda} + \frac{1}{2}m^2v\phi_1 - \frac{m^2}{2v}\phi_1^3 - \frac{m^2}{2v}\phi_1\phi_2^2, \end{aligned} \quad (2.102)$$

which now have the right sign mass term for the ϕ_1 field. Expanding around ϕ_1 in this way is not the most natural choice since ϕ_2 still has the wrong sign mass term. Considering the U(1) symmetry it's more natural to define

$$\phi(x) = \left(\frac{v + \sigma(x)}{\sqrt{2}}\right) e^{i\frac{\pi(x)}{F_\pi}} = \left(\sqrt{\frac{3m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma(x)\right) e^{i\frac{\pi(x)}{F_\pi}}, \quad (2.103)$$

for two real fields $\pi(x)$ and $\sigma(x)$ and $F_\pi \in \mathbb{R}$ is a constant. Substituting this into the Lagrangian in eq. (2.101) we find

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu\sigma)^2 + \left(\sqrt{\frac{3m^2}{\lambda}} + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F_\pi^2}(\partial_\mu\pi)^2 \\ & + \frac{3m^4}{2\lambda} - m^2\sigma^2 - \sqrt{\frac{\lambda m^2}{6}}\sigma^3 - \frac{\lambda}{4!}\sigma^4. \end{aligned} \quad (2.104)$$

There are a few things to note about the field $\pi(x)$ in this expression. First we see that with the expansion in eq. (2.103) the interacting part of the Lagrangian is automatically independent of $\pi(x)$. We also see that if we make the choice $F_\pi = v$ the field $\pi(x)$ is canonically normalized. A third thing to notice, is that $\pi(x)$ is a massless. This massless field is called a Nambu-Goldstone boson [42, 43, 44] and it follows from the very general Goldstone theorem which says that we will have one massless particle for each broken continuous symmetry. The theory described by eq. (2.104) is called the linear sigma model.

2.3.8.2 Comment on the expansion around the vev

In the previous section we saw that we started with a theory which was not expanding around the vacuum of the theory, but it was instead expanded around $\phi = 0$. The question you should ask yourself is why we have to expand the theory

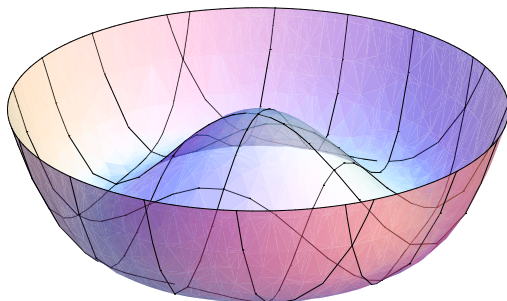


Figure 2.1: The potential in eq. (2.101) with $m^2 > 0$ is called the Mexican hat potential, and it illustrates that the ground state of the system is not $\phi = 0$ and it is not invariant under the $U(1)$ symmetry.

around the true vacuum. Is it not possible to do the calculations around $\phi = 0$, or is it maybe possible to start by doing the calculations around $\phi = 0$ and later do some sort of expansion $\phi \rightarrow \phi + v$? To understand this question, let us start by reviewing a crucial assumption about perturbation theory [31].

Consider a theory described by a free Hamiltonian H_0 where the particle states are eigenstates of H_0 . Now we introduce interactions to our theory by adding an interaction operator V , and our new Hamiltonian is $H = H_0 + V$. Using the S-matrix to do perturbation theory there are two assumptions. The first assumption is that the Hilbert space of H describes particles, i.e. there are eigenstates of H similar to the free-particle eigenstates of some free Hamiltonian. The other assumption is that V is small in the sense that it does not change the particle spectrum of H_0 . In other words, the spectrum of H is close to the spectrum of H_0 .

In the case of a spontaneously broken theory we will have different particle spectrum, i.e. there will be states in H that are not in H_0 . These states are the massless Goldstone bosons, one for each broken symmetry. To cope with this, we must redefine our free Hamiltonian. As you might have guessed, this is done by expanding our Lagrangian around the vacuum expectation value, the real (classical) ground state of our theory. We see that expanding our theory around the right value is crucial to not violate the the assumptions of S-matrix theory.

2.3.9 Higgs Mechanism

So far we have studied spontaneous symmetry breaking of bosons in a scalar theory. Now we will study the case where we have a gauge boson associated with the broken symmetry. The end result is the famous Higgs mechanism, which is the name of the process where the Goldstone boson disappears from the spectrum and the gauge boson acquires a mass. The Higgs mechanism is named after Peter Higgs, but was originally proposed by Anderson [45] in 1962 and later developed into a relativistic theory in 1964 by Brout and Englert [46]; Guralnik, Hagen and Kibble [47]; and Higgs [48].

2.3.9.1 Abelian Higgs Model

We will now consider a gauged version of the theory described in section 2.3.8.1. The theory is called the Abelian Higgs model, and the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 + m^2|\phi|^2 - \frac{\lambda}{3!}|\phi|^4, \quad (2.105)$$

where $D_\mu = \partial_\mu + ieA_\mu$. For $m^2 > 0$ we will have spontaneous symmetry breaking with a minimum at $|\phi|^2 = \frac{3m^2}{\lambda} \equiv \frac{v^2}{2}$. Expanding

$$\phi(x) = \left(\frac{v + \sigma(x)}{\sqrt{2}} \right) e^{i\frac{\pi(x)}{F_\pi}}, \quad (2.106)$$

the Lagrangian is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{(v + \sigma)^2}{2} \left(eA_\mu + \frac{\partial_\mu\pi}{F_\pi} \right)^2 \\ &\quad + \frac{3m^4}{2\lambda} - m^2\sigma^2 - \sqrt{\frac{\lambda m^2}{6}}\sigma^3 - \frac{\lambda}{4!}\sigma^4 \\ &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2 A_\mu^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \mathcal{L}_{\text{int}}, \end{aligned} \quad (2.107)$$

where $m_A = ev$ and $m_\sigma = \sqrt{2}m$ are the masses of A_μ and σ , respectively, and \mathcal{L}_{int} is the interacting part of the Lagrangian (also including the overall constant $\frac{3m^4}{2\lambda}$ for simplicity). Also note that the field $\pi(x)$ is massless, and we have used $F_\pi = v$ to make $\pi(x)$ canonically normalized. The field σ is better known as the Higgs boson.

A difficulty of this Lagrangian is that there is a term $\frac{1}{2}m_A^2 \frac{1}{eF_\pi} A_\mu \partial_\mu \pi$. This cross term causes what's called *kinetic mixing* between A_μ and π , and it complicates the process of interpreting the physical spectrum of the theory. We will see much more of kinetic mixing in chapter 4, and how one can deal with it. We will here quickly see that it can be removed through choosing a gauge.

The usual gauge symmetry $\phi(x) \rightarrow e^{-i\alpha(x)}\phi(x)$ is now replaced by

$$\begin{aligned}\pi(x) &\rightarrow \pi(x) - F_\pi(x)\alpha(x), \\ A_\mu(x) &\rightarrow A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x),\end{aligned}\tag{2.108}$$

where the transformation for A_μ is the same as before. We see that it is possible to make a choice of $F_\pi\alpha(x)$ such that $\pi(x) = 0$. This gauge is called the Unitary gauge. Another choice is the Lorenz gauge $\partial_\mu A_\mu = 0$. In this gauge the cross term vanishes after integration by part $A_\mu(\partial_\mu\pi) \rightarrow -(\partial_\mu A_\mu)\pi = 0$. In the Lorenz gauge we have a massive gauge boson with 3 degrees of freedom that has to satisfy one constraint ($\partial_\mu A_\mu = 0$), and the Goldstone boson has one degree of freedom. In the unitary gauge we have just the massive gauge field and no constraints, i.e. the number of degrees of freedom are the same, as it should be. In the unitary gauge the Goldstone boson has disappeared, and one typically says that the gauge boson eats the Goldstone boson through the Higgs mechanism.

We will use the Abelian Higgs model many times later since it the simplest model to investigate the main topic of this thesis, i.e. the gauge dependence of effective potentials. All of this will be described in chapter 4. However this model is also of interest for other purposes. The Abelian Higgs model is used in the Ginzburg-Landau model of superconductivity [49] to describe superconductors near the critical temperature. I will not go into any details, but it can be used to simply describe the Meissner effect, see Schwartz [19] and Weinberg [23] for a discussion on the subject.

As an end note I would like to mention that this model has an interesting feature as was first studied by Coleman and E. Weinberg [15]. In the case where we set the mass to zero, it seems like there will be no symmetry breaking since the potential is just $V = \frac{\lambda}{4!}\phi^4$. But it turns out that quantum corrections will spontaneously break the symmetry. We will study this more in section 4.2.3.

2.3.10 Electroweak symmetry breaking

Now we will consider a slightly more complicated example than the Abelian Higgs model, the Electroweak symmetry breaking. The theory is very similar to the model described in the previous section as we start with the Lagrangian

$$\mathcal{L} = (\partial_\mu H)^\dagger + m^2 H^\dagger H - \lambda (H^\dagger H)^2,\tag{2.109}$$

where H is a complex doublet. The Lagrangian is invariant under a global SU(2) and U(1) transformation. By gauging the U(1) symmetry we add the B_μ hypercharge gauge boson, just as we added A_μ in the Abelian Higgs model starting from the linear sigma model. Gauging the SU(2) symmetry, we also add W_μ^a which are the SU(2) gauge bosons. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}(W_{\mu\nu}^a)^2 - \frac{1}{4}(B_{\mu\nu})^2 + (D_\mu H)^\dagger(D_\mu H) + m^2 H^\dagger H - \lambda(H^\dagger H)^2,\tag{2.110}$$

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where $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2 f^{abc} W_\mu^b W_\nu^c$, f^{abc} are the SU(2) structure constants, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, the covarian derivative is

$$D_\mu H = \partial_\mu H - ig_2 W_\mu^a \tau^a H - \frac{1}{2} ig_1 B_\mu H, \quad (2.111)$$

where $\tau^a = \frac{1}{2}\sigma^a$ and σ^a are the Pauli matrices, g_2 is the SU(2) coupling and g_1 is the U(1) coupling. The factor of $\frac{1}{2}$ in the covariant derivative comes from the fact that we define the Higgs doublet to have hypercharge $Y = \frac{1}{2}$.

From the Higgs potential $V(H) = -m^2 H^\dagger H + \lambda(H^\dagger H)^2$ we see that the minimum is $|H|^2 = \frac{m^2}{2\lambda} \equiv \frac{v^2}{2}$. As in the Abelian Higgs model we expand

$$H = \exp\left(2i\frac{\pi^a \tau^a}{v}\right) \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}. \quad (2.112)$$

Doing this expansion is a rather messy process, so we will choose the unitary gauge, $\pi^a = 0$, and break the Lagrangian up to smaller pieces. First some comments about notation. It turns out that the gauge fields in the Lagrangian can be written in a more natural way

$$\begin{aligned} W_\mu^+ &= \frac{1}{\sqrt{2}} (W_\mu^1 - W_\mu^2), \\ W_\mu^- &= \frac{1}{\sqrt{2}} (W_\mu^1 + W_\mu^2), \\ Z_\mu &= \cos\theta_w W_\mu^3 - \sin\theta_w B_\mu, \\ A_\mu &= \sin\theta_w W_\mu^3 + \cos\theta_w B_\mu, \end{aligned} \quad (2.113)$$

with θ_w defined as

$$\tan\theta_w = \frac{g_1}{g_2}, \quad (2.114)$$

and the strength of the electromagnetic force is

$$e = g_1 \cos\theta_w = g_2 \sin\theta_w. \quad (2.115)$$

The gauge part of the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} Z_{\mu\nu}^2 + \frac{1}{2} m_Z^2 Z_\mu^2 + m_W^2 W^+ W^- \\ &\quad - \frac{1}{2} (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+) (\partial_\mu W_\nu^- - \partial_\nu W_\mu^-) + \mathcal{L}_{\text{gauge interactions}} \end{aligned} \quad (2.116)$$

with the masses

$$m_Z = \frac{g_2 v}{2 \cos\theta_w}, \quad m_A = 0, \quad m_W = \frac{g_2 v}{2}. \quad (2.117)$$

We don't write out the terms in $\mathcal{L}_{\text{gauge interactions}}$ because it's rather messy. The details and thorough discussion can be found in Schwartz [19]. The Higgs Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = & -\frac{1}{2}(\square + m_h^2)h - \frac{g_2 m_h^2}{4m_W} h^3 - \frac{g_2^2 m_h^2}{32m_W^2} h^4 \\ & + 2\frac{h}{v} \left(m_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_Z^2 Z_\mu^2 \right) + \left(\frac{h}{v} \right)^2 \left(m_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_Z^2 Z_\mu^2 \right) \end{aligned} \quad (2.118)$$

where $m_h = \sqrt{2}m = \sqrt{2\lambda}v$. Using some experimental values given by Schwartz [19] $\alpha(m_e) = \frac{e^2}{4\pi} = \frac{1}{137.036}$, $m_Z = 91.2\text{GeV}$, $m_W = 80.399\text{GeV}$ and $m_h = 126\text{GeV}$ gives

$$e = 0.303, \quad \sin^2 \theta_w = 0.223, \quad g_2 = \frac{e}{\sin \theta_w} = 0.64, \quad g_1 = \frac{e}{\cos \theta_w} = 0.34. \quad (2.119)$$

To summarize, we have seen how we started with a theory with massless gauge bosons in eq. (2.110). Due to the shape of the Higgs potential, the theory is spontaneously broken, and out comes our massless photon A and the massive W^\pm and Z bosons in addition to the Higgs boson which recently has been observed at the LHC [10, 11].

We will study this Lagrangian more in chapter 5 where we will compute the effective potential for the Higgs.

2.3.11 Fermions in the Standard Model

So far we have only discussed the electroweak gauge bosons in the Standard Model. In this section we will briefly summarize the fermion sector of the Standard Model, and discuss how the fermions and bosons couple to each other.

The Standard Model has 3 generations of SU(2) doublet pairs of quarks and leptons

$$\begin{aligned} L^i &= \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \\ Q^i &= \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \end{aligned} \quad (2.120)$$

where $i = 1, 2, 3$ labels the generations and the subscript L indicates that the quarks and leptons are left-handed Weyl spinors, transforming in the $(\frac{1}{2}, 0)$ representation of the Lorentz group. The corresponding right-handed quarks and leptons are

$$\begin{aligned} e_R^i &= \{e_R, \mu_R, \tau_R\}, & \nu_R^i &= \{\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\} \\ u_R^i &= \{u_R, c_R, t_R\}, & d_R^i &= \{d_R, s_R, b_R\} \end{aligned} \quad (2.121)$$

which are all SU(2) singlets transforming in the $(0, \frac{1}{2})$ representation of the Lorentz group. Note that the right-handed neutrinos has not been observed in nature, but is included here for completeness in case they do exist.

All the quarks and leptons have hypercharge and hence couple to the hypercharge gauge boson, but only the left-handed fermions couple to the SU(2) gauge bosons. The gauge interactions are

$$\begin{aligned}
 \mathcal{L} = & i\bar{L}_i \left(\not{\partial} - ig_2 \mathcal{W}^a \tau^a + i\frac{g_1}{2} \not{B} \right) L_i + i\bar{Q}_i \left(\not{\partial} - ig_2 \mathcal{W}^a \tau^a - i\frac{g_1}{6} \not{B} \right) Q_i \\
 & + i\bar{e}_R^i \left(\not{\partial} + ig_1 \not{B} \right) e_R^i + i\bar{u}_R^i \left(\not{\partial} - ig_1 \frac{2}{3} \not{B} \right) u_R^i \\
 & + i\bar{d}_R^i \left(\not{\partial} + ig_1 \frac{1}{3} \not{B} \right) d_R^i + i\bar{\nu}_R^i \not{\partial} \nu_R^i
 \end{aligned} \tag{2.122}$$

We will not do much with the fermions in the Standard Model, but in chapter 5 we will include the top quark as this is the biggest contribution to the Standard Model effective potential.

2.3.11.1 Fermion masses

The fermions in the Standard Model get their mass when H gets a vacuum expectation value. We will call the terms that will produce the fermion masses the Yukawa terms $\mathcal{L}_{\text{Yukawa}}$, and since they have to respect the full Standard Model symmetry, we they have to be of a very specific form. We will only need the top quark mass in chapter 5, so we will focus on the details relevant to the top. The rest of the details can be found in Schwartz [19]. The top mass will be generated from the Yukawa term

$$\mathcal{L}_{\text{Yukawa}} = -y_{ij} \bar{Q}^i \tilde{H} u_R^j + h.c. \tag{2.123}$$

where $\tilde{H} = i\sigma_2 H^*$ and y_{ij} is the Yukawa matrix. When the symmetry is broken as in eq. (2.112) we find that the top mass is

$$\mathcal{L}_{\text{top mass}} = -\frac{y_t}{\sqrt{2}} v (\bar{t}_R t_L + \bar{t}_L t_R) \equiv -m_t \bar{\psi}_t \psi_t, \tag{2.124}$$

where we have introduced the Dirac spinor for the top ψ_t , $m_t = \frac{y_t}{\sqrt{2}v}$ and y_t is the top Yukawa coupling.

2.4 Renormalization and the RGE

A big part of learning how to do calculations in quantum field theory is learning how to deal with all the divergent loop integrals. To illustrate renormalization and how couplings run with scale, we will look at an example in QED from [19] and derive the renormalization group equations (RGE).

2.4.1 Renormalization

The QED Lagrangian in $d = 4 - \varepsilon$ dimensions is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}^0(i\not{\partial} - e^0\gamma^\mu A_\mu^0 - m^0)\psi^0, \quad (2.125)$$

where $F_{\mu\nu} = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0$ and the subscript A^0 , m^0 , etc. indicate that these are bare quantities (i.e. not renormalized). Being in d dimensional space-time we can find the dimensions of the fields from the Lagrangian

$$[A_\mu^0] = M^{\frac{d-2}{2}}, \quad [\psi^0] = [\bar{\psi}^0] = M^{\frac{d-1}{2}}, \quad [e^0] = M^{\frac{4-d}{2}}. \quad (2.126)$$

We now define our renormalized mass, charge and fields

$$A_\mu = \frac{1}{\sqrt{Z_3}}A_\mu^0, \quad \psi = \frac{1}{\sqrt{Z_2}}\psi^0, \quad m_R = \frac{1}{Z_m}m^0, \quad e_R = \frac{1}{Z_e}\mu^{\frac{d-4}{2}}e^0, \quad (2.127)$$

where we have included a factor of $\mu^{\frac{d-4}{2}}$ to make the renormalized charge dimensionless. Expanding $Z_x = 1 + \delta_x$ where δ_x formally starts at order $\mathcal{O}(e_R^2)$ we find

$$\begin{aligned} \mathcal{L}_{QED} = & -\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}\not{\partial}\psi - m_R\bar{\psi}\psi - e_R\mu^{\frac{4-d}{2}}\bar{\psi}A\psi \\ & -\frac{1}{4}\delta_3F_{\mu\nu}^2 + i\delta_2\bar{\psi}\not{\partial}\psi - (\delta_2 + \delta_m)m_R\bar{\psi}\psi - e_R\mu^{\frac{4-d}{2}}\delta_1\bar{\psi}A\psi. \end{aligned} \quad (2.128)$$

We will use these counterterms to absorb the infinities from the loop diagrams.

We have here introduced a subscript R on the renormalized coupling and mass. This just means that it's a value set by a renormalization condition at some given scale. If we care about at which point we are evaluating the coupling at, we will explicitly write the scale dependence instead of the subscript R , for example e.g. $e_R = e(\mu_0)$ for some scale μ_0 .

Consider the vacuum polarization and counterterm diagram at 1-loop which are computed in Schwartz[19]



$$\text{Diagram 1} + \text{Diagram 2} \quad (2.129)$$

$$= -i(p^2 g^{\mu\nu} - p^\mu p^\nu) (e_R^2 \Pi_2(p^2) + \delta_3),$$

where

$$\Pi_2(p^2) = \frac{1}{2\pi^2} \int_0^1 dx x(1-x) \left[\frac{2}{\varepsilon} + \ln \left(\frac{\tilde{\mu}^2}{m_R^2 - p^2 x(1-x)} \right) \right] \quad (2.130)$$

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and we have used dimensional regularization with $d = 4 - \varepsilon$ and $\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$. In Fourier space, the Coulomb potential can now be written

$$V(p^2) = e_R^2 \frac{1 - e_R^2 \Pi_2(p^2)}{p^2}. \quad (2.131)$$

At this point we want to impose a renormalization condition. Thinking about this physically, it is natural to define e_R as the value we measure from the potential at a scale p_0 , i.e. $V(p_0) \equiv \frac{e_R^2}{p_0^2}$. This will fix the counter term δ_3 , and in the limit when $p^2 \gg m^2$ we find

$$V(p^2) = \frac{e_R^2}{p^2} \left(1 + \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{p_0^2} \right). \quad (2.132)$$

An alternative approach called the $\overline{\text{MS}}$ subtraction scheme. Using $\overline{\text{MS}}$ we define the counterterm to subtract the $\frac{2}{\varepsilon}$ and the $4\pi e^{-\gamma_E}$ in the log. We find

$$V(p^2) = \frac{e_R^2}{p^2} \left(1 + \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2} \right). \quad (2.133)$$

We see that in this case we actually get almost the same result, and hence we often refer to μ as the renormalization scale, i.e. the physical scale at which the theory is renormalized.

In eq. (2.133) we notice that we have a problem if p^2 gets large. We get a large logarithm, and we can no longer trust our perturbative expansion. We will now look at two different ways of improving this result.

The first approach will be to add up more diagrams. We have computed the one loop correction, but we can put together multiple loops and add them all together

$$\begin{array}{c} \text{---} \\ p \end{array} + \begin{array}{c} \text{---} \\ p \end{array} \text{---} \text{---} \text{---} \text{---} \begin{array}{c} \text{---} \\ p \end{array} + \begin{array}{c} \text{---} \\ p \end{array} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \begin{array}{c} \text{---} \\ p \end{array} + \dots \quad (2.134)$$

The Coulomb potential becomes

$$\begin{aligned} V(p^2) &= \frac{e_R^2}{p^2} \left[1 + \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2} + \left(\frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2} \right)^2 + \dots \right] \\ &= \frac{1}{p^2} \left[\frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2}} \right] \equiv \frac{e_{\text{eff}}^2(p^2)}{p^2}, \end{aligned} \quad (2.135)$$

where we have defined

$$e_{\text{eff}}^2(p^2) \equiv \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2}}, \quad (2.136)$$

which is the effective charge that runs as we change p^2 . This is called the leading log resummation since it reproduces the leading logarithmic terms. This means that if we did the full two loop calculation we would find the $\left(\frac{e_R^2}{12\pi^2}\right)^2 \left(\ln \frac{p^2}{\mu^2}\right)^2$ term, but there could also be terms like $\left(\frac{e_R^2}{12\pi^2}\right)^2 \left(\ln \frac{p^2}{\mu^2}\right)$. Note that the latter term is not reproduced in our resummation, so the resummed effective charge is only an approximate solution to how the electric charge runs.

Another way of approaching the resummation is by realizing that the scale μ at which we renormalize the theory is arbitrary. If we chose a different scale, the physics should still be the same. The Coulomb potential $V(p^2)$ has both explicit and implicit μ^2 dependence. We see the explicit dependence in eq. (2.135), and the implicit dependence is in $e_R^2 = e_{\text{eff}}^2(\mu^2)$. Independence of μ can be expressed as

$$\mu \frac{d}{d\mu} V(p^2) = \mu \frac{d}{d\mu} \left\{ \frac{e_{\text{eff}}^2(\mu^2)}{p^2} \left[1 + \frac{e_{\text{eff}}^2(\mu^2)}{12\pi^2} \ln \frac{p^2}{\mu^2} + \dots \right] \right\} = 0. \quad (2.137)$$

Solving this equation to leading order gives us

$$\beta_e \equiv \mu \frac{de_{\text{eff}}(\mu)}{d\mu} = \frac{e_{\text{eff}}^3(\mu)}{12\pi^2}. \quad (2.138)$$

This is the beta function for the running coupling, and this equation is referred to as the renormalization group equation (RGE).

Eq. (2.138) is just a differential equation, and given the initial condition $e_{\text{eff}}(\mu) = e_R$ we find

$$e_{\text{eff}}^2(p^2) \equiv \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{p^2}{\mu^2}}, \quad (2.139)$$

which is the same result as we got in eq. (2.136). Note that there was no need to add up any diagrams with this approach. Solving the RGE resums the leading logarithmic terms for us. In the rest of this thesis we will use the beta function method to resum the large logarithms.

2.4.2 RGE

In this section we will see how we can find the beta function without thinking about Coulomb potentials or similar quantities. Looking back at the bare Lagrangian in eq. (2.125) all the bare fields and couplings are independent of μ . For the bare electric charge e^0 we find using eq. (2.127)

$$\begin{aligned} 0 &= \mu \frac{d}{d\mu} e^0 = \mu \frac{d}{d\mu} \left[Z_e \mu^{\frac{4-d}{2}} e_R \right] \\ &= \mu^{\frac{\epsilon}{2}} e_R Z_e \left[\frac{\epsilon}{2} + \frac{1}{Z_e} \mu \frac{dZ_e}{d\mu} + \frac{1}{e_R} \mu \frac{de_R}{d\mu} \right], \end{aligned} \quad (2.140)$$

where we have used $d = 4 - \varepsilon$.

In $\overline{\text{MS}}$ the counterterm is [19] $Z_e = 1 + \frac{e_R^2}{12\pi^2}$ to 1-loop order. Solving for β_e to order e_R^3 we find

$$\beta_e = \mu \frac{de_R}{d\mu} = -\frac{\varepsilon}{2}e_R + \frac{e_R^3}{12\pi^2}, \quad (2.141)$$

which reduces to eq. (2.138) in the limit as $\varepsilon \rightarrow 0$.

In a more general theory with other couplings, we can find the beta function for the couplings in the same way if we know what the counterterms are. The beta function for a coupling g is always defined as

$$\beta_g = \mu \frac{dg}{d\mu}. \quad (2.142)$$

The mass parameters in our theory can also run¹, and using the same approach as above we find

$$\begin{aligned} 0 &= \mu \frac{d}{d\mu} m^0 = \mu \frac{d}{d\mu} [Z_m m_R] \\ &= Z_m m_R \left[\frac{\mu}{m_R} \frac{dm_R}{d\mu} + \frac{\mu}{Z_m} \frac{dZ_m}{d\mu} \right]. \end{aligned} \quad (2.143)$$

We define the anomalous dimension γ_M as

$$\gamma_m \equiv \frac{\mu}{m_R} \frac{dm_R}{d\mu} = -\frac{\mu}{Z_m} \frac{dZ_m}{d\mu}, \quad (2.144)$$

which tells us how m_R changes with the scale μ . Note the extra minus sign here. In the literature there exists conventions where they have the opposite convention for γ_m than used in eq. (2.144).

2.4.2.1 RGE for Green's functions

Consider the following Green's function in QED with bare fields (n photons and m fermions)

$$G_{n,m}^{(0)} = \langle \Omega | T \{ A_{\mu_1}^0 \cdots A_{\mu_n}^0 \psi_1^0 \cdots \psi_m^0 | \Omega \} \rangle. \quad (2.145)$$

Since there are only bare fields here, this Green's function must be independent of μ

$$\mu \frac{d}{d\mu} G_{n,m}^{(0)} = 0. \quad (2.146)$$

¹Note that the pole masses does not run with any scale. Typically we will talk about the running of the $\overline{\text{MS}}$ mass.

Expressing the Green's function in terms of renormalized fields we find

$$G_{n,m}^{(0)} = Z_3^{\frac{n}{2}} Z_2^{\frac{m}{2}} G_{n,m}. \quad (2.147)$$

The Green's function $G_{n,m}$ can in general be a function of all the different momenta, the coupling e_R , the mass m_R and the renormalization scale μ . Evaluating eq. (2.146) using the chain rule and defining

$$\gamma_2 = \frac{\mu}{Z_2} \frac{dZ_2}{d\mu}, \quad \gamma_3 = \frac{\mu}{Z_3} \frac{dZ_3}{d\mu}, \quad (2.148)$$

we find

$$\left[\mu \frac{\partial}{\partial \mu} + \frac{m}{2} \gamma_2 + \frac{n}{2} \gamma_3 + \beta_e \frac{\partial}{\partial e_R} + \gamma_m m_R \frac{\partial}{\partial m_R} \right] G_{n,m} = 0. \quad (2.149)$$

We call this the RGE equation for Green's functions. Depending on the subtraction scheme² used people also call have other specific names for this equation [50]. It's most often called the Callan-Symanzik [51, 52] equation, but this technically assumes that we are using the on-shell physical renormalization scheme. Using $\overline{\text{MS}}$ it is sometimes referred to as the 't Hooft-Weinberg equation [53, 54]. We will use a similar equation for the effective potential in section 3.3.2.

²Note that this process including computing the beta functions is a subtraction scheme dependent process. We will use $\overline{\text{MS}}$ since it is simple and also since it is the one that is mostly used in the relevant literature.

Chapter 3

Effective Field Theory

An effective field theory is a theory whose tree level correlation functions and S -matrix elements are the full quantum correlation functions of another field theory, which will be referred to as the full theory. The effective theory and full theory are defined through the actions Γ and S , respectively. S will typically have more degrees of freedom than Γ , usually meaning fewer fields, but by focusing on only the relevant degrees of freedom the calculation will be much simpler. Specifically we want to talk about the 1PI effective action where tree level calculations using Γ reproduces all the quantum effects from loop calculations using S . Fields that are in the full theory but not the effective action are said to be integrated out. Since this sounds too good to be true, it must come at a cost: we typically only find the effective action for a subset of the fields, so we cannot compute the scattering with this subset as external fields.

There are different ways of calculating the effective action, and we will mention two of the most common approaches. We will start by explaining the effective action defined through matching and then we will go through the effective action defined using functional methods coming from the Feynman path integrals. The latter will be our preferred method that will be used in future calculations in chapter 4 and 5.

After defining the effective action we go on to define the effective potential, which will be of great interest in the rest of this thesis. We will describe different ways of computing the effective potential. We start by discussing how to compute the effective action using the original action using diagrams in section 3.3.1.1 and using diagrams with background fields in section 3.3.1.2. We then discuss a more unfamiliar method to find the effective potential using tadpole diagrams in section 3.3.1.3, and the last method we will describe is the functional method in section 3.3.1.4. These are all fixed order calculations, and we will use the RGE in section 3.3.2 to find the resummed effective potential. We also include a short discussion of the gauge dependence of the effective potential and we derive the

3.1.2 Functional Methods

Another equivalent approach, which we will follow throughout this thesis, is to identify the effective action as a Legendre transform of the generating functional of connected diagrams $W[J]$, where $W[J]$ is related to the generating functional through $Z[J] = e^{iW[J]}$. This will be derived in this section.

3.1.2.1 Deriving the Effective Action

Given a scalar field $\phi(x)$ and the action $S[\phi]$ ¹, the generating functional eq. (2.35) that is used to compute vacuum amplitudes with sources is

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \exp \left[iS[\phi] + i \int d^4x \phi(x) J(x) \right], \quad (3.7)$$

where $W[J]$ will be defined shortly. Remember from section 2.2.4 that $Z[J]$ generates Green's functions

$$(-i)^n \frac{1}{Z[J]} \frac{\partial^n Z[J]}{\partial J(x_1) \cdots \partial J(x_n)} = \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle_J \quad (3.8)$$

including both connected and disconnected diagrams. Usually we set $J = 0$ to compute vacuum matrix elements, but here (with $J \neq 0$) $Z[J]$ generates the Green's functions for ϕ with a classical background current J .

We also define a new functional $W[J] \equiv -i \ln Z[J]$, and $W[J]$ is the generator of all connected diagrams

$$(-i)^n \frac{\partial^n W[J]}{\partial J(x_1) \cdots \partial J(x_n)} = \langle \Omega | T \{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle_J^{\text{connected}}. \quad (3.9)$$

Let's do an example to illustrate this. Take $n = 2$, and we find

$$\begin{aligned} (-i)^2 \frac{\partial^2 W}{\partial J_1 \partial J_2} &= (-i)^3 \frac{\partial}{\partial J_1} \left(\frac{1}{Z} \frac{\partial Z}{\partial J_2} \right) \\ &= (-i)^3 \frac{1}{Z} \frac{\partial^2 Z}{\partial J_1 \partial J_2} - (-i)^3 \left(\frac{1}{Z} \frac{\partial Z}{\partial J_1} \right) \left(\frac{1}{Z} \frac{\partial Z}{\partial J_2} \right) \\ &= (-i) \left[\langle \Omega | T \{ \phi_1 \phi_2 \} | \Omega \rangle_J - \langle \Omega | \phi_1 | \Omega \rangle_J \langle \Omega | \phi_2 | \Omega \rangle_J \right] \end{aligned} \quad (3.10)$$

where $J_i = J(x_i)$, $\phi_i = \phi(x_i)$, $Z = Z[J]$, and $W = W[J]$. The first term is the full Green's function including connected and disconnected diagrams. The second term are exactly the disconnected terms, which are subtracted off, leaving only the connected diagrams.

¹Although we are only writing this in terms of a scalar field, it is straightforward to generalize this to cases where ϕ is a field with Lorentz indices or is a fermionic field.

We will now define the action $\Gamma[\phi]$ in terms of $W[J]$, and after studying this definition we will show that this is in fact the 1PI effective action. We define $\Gamma[\phi]$ as a Legendre transform of the functional $W[J]$

$$\Gamma[\phi] = W[J_\phi] - \int d^4x J_\phi(x) \phi(x) \quad (3.11)$$

where J_ϕ is an implicit functional of ϕ defined as the solution to

$$\left. \frac{\partial W[J]}{\partial J(x)} \right|_{J=J_\phi} = \phi(x). \quad (3.12)$$

All we know at this point is that $W[J]$ generates all the connected Feynman diagrams for the full theory, and $\Gamma[\phi]$ is whatever it is satisfying this equation. By varying eq. (3.11) with respect to ϕ

$$\frac{\partial \Gamma[\phi]}{\partial \phi(x)} = \int d^4y \left[\frac{\partial J_\phi(y)}{\partial \phi(x)} \frac{\partial W[J_\phi]}{\partial J_\phi(y)} - \frac{\partial J_\phi(y)}{\partial \phi(x)} \phi(y) \right] - J_\phi(x) = -J_\phi(x). \quad (3.13)$$

We can also write down the inverse Legendre transform

$$W[J] = \Gamma[\phi_J] + \int d^4x J(x) \phi_J(x) \quad (3.14)$$

where ϕ_J is an implicit functional of J that satisfies

$$\left. \frac{\partial \Gamma[\phi]}{\partial \phi(x)} \right|_{\phi=\phi_J} = -J(x). \quad (3.15)$$

Varying eq. (3.14) with respect to J we find

$$\frac{\partial W[J]}{\partial J(x)} = \int d^4y \left[\frac{\partial \phi_J(y)}{\partial J(x)} \frac{\partial \Gamma[\phi_J]}{\partial \phi_J(y)} + J(y) \frac{\partial \phi_J(y)}{\partial J(x)} \phi(y) \right] + \phi_J(x) = \phi_J(x). \quad (3.16)$$

Notice that this is just the one point function

$$\frac{\partial W[J]}{\partial J(x)} = -i \frac{1}{Z[J]} \frac{\partial Z[J]}{\partial J(x)} = \langle \Omega | \phi(x) | \Omega \rangle_J = \phi_J. \quad (3.17)$$

This gives us a physical interpretation of the Legendre transform in eq. (3.11). Given a classical field configuration $\phi_c(x)$, the current $J_{\phi_c}(x)$ is the current that has to be present to give the expectation value of ϕ to be $\langle \Omega | \phi | \Omega \rangle_{J_{\phi_c}} = \phi_c$. From the other point of view, we have from eq. (3.17) that $\phi_J(x) = \frac{\partial W[J]}{\partial J(x)} = \langle \Omega | \phi(x) | \Omega \rangle_J$. Thus, we see that $\phi_{J_\phi} = \phi$ and $J_{\phi_J} = J$, and we see that our two expressions are self-consistent.

3.1. 1PI EFFECTIVE ACTION

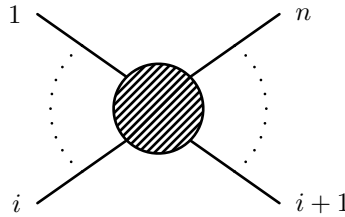
3.1.2.2 Powers of \hbar

Before we go any further, I want to make one point clear that is independent of this discussion about effective field theory. We will use this observation very soon. Start with a theory given by the action $S[\phi]$ and restore the factors of \hbar which normally are set to one. The generating functional $Z[J]$ is

$$Z[J] = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar} \left\{ S[\phi] + \int d^4x \phi(x) J(x) \right\} \right]. \quad (3.18)$$

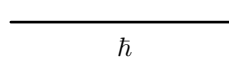
When computing Feynman diagrams in this theory we see that all vertices come with a factor of $\frac{1}{\hbar}$ because it multiplies the coupling constants and all propagators comes with \hbar since the propagator is the inverse of the kinetic term.

Consider the following n -point amplitude



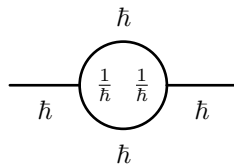
$$(3.19)$$

Assume that we start with a diagram with zero loops. There are two ways that we can include a loop. The first way is to add a loop on one of the propagators in the n -point amplitude. The propagator is to begin with



$$\sim \hbar \quad (3.20)$$

Adding a loop we get three more propagators but only two more vertices



$$\sim \hbar^2 \quad (3.21)$$

We see that adding in one loop increased the power of \hbar by one.

The other way we can add in a loop is by adding a propagator between two

different lines in the n -point amplitude. We illustrate this by two lines

$$\begin{array}{c} \hline \hbar \\ \hline \hbar \\ \hline \end{array} \sim \hbar^2 \tag{3.22}$$

Adding in one propagator gives also in this case three propagators and two vertices.

$$\begin{array}{c} \hbar \quad \frac{1}{\hbar} \quad \hbar \\ \hline \hbar \quad \hbar \\ \hline \end{array} \sim \hbar^3 \tag{3.23}$$

We see that in both cases we get one $\frac{1}{\hbar}$ more factor of \hbar by adding in a loop.

From quantum mechanics we know that $\hbar \rightarrow 0$ restores the classical limit. Taking this limit in our quantum field theory, we see that the tree-level diagrams will be leading contribution. All the loops will be higher order in \hbar and they vanish in this limit. This is why we say that the tree-level diagrams are classical and loops give quantum effects.

3.1.2.3 The Emergence of the 1PI Effective Action

Returning to our original calculation, we now want to rewrite eq. (3.14) using the method of stationary phases. Consider the following equation

$$\lim_{\hbar \rightarrow 0} (-i\hbar) \ln \left[\int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \left[\Gamma[\phi] + \int d^4x J(x)\phi(x) \right] \right\} \right]. \tag{3.24}$$

The method of stationary phases says that in the limit of $\hbar \rightarrow 0$ the integral is equal to the field configuration that extremize the action. We denote this field configuration ϕ_J since the extremum condition is

$$\left. \frac{\partial \Gamma[\phi]}{\partial \phi(x)} \right|_{\phi=\phi_J} = -J(x) \tag{3.25}$$

as in eq. (3.15). Hence, we can write


$$\begin{aligned} W[J] &= \left[\Gamma[\phi_J] + \int d^4x J(x)\phi_J(x) \right] \Bigg|_{\frac{\partial \Gamma[\phi_J]}{\partial \phi_J} + J=0} \\ &= \lim_{\hbar \rightarrow 0} (-i\hbar) \ln \left[\int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} \left[\Gamma[\phi] + \int d^4x J(x)\phi(x) \right] \right\} \right] \end{aligned} \tag{3.26}$$

that can be written as



$$(3.31)$$

there will be a corresponding current diagram such that



$$(3.32)$$

The set of diagrams that *cannot* be written as eq. (3.31) are the 1PI diagrams. Hence we may write eq. (3.29) as

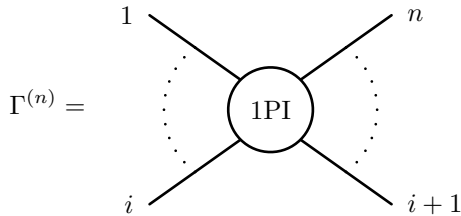
$$e^{i\Gamma[\phi_0]} = \int_{\text{1PI}} \mathcal{D}\phi e^{iS[\phi+\phi_0]}, \quad (3.33)$$

where the subscript 1PI is just a label indicating that we only integrate over what corresponds to 1PI diagrams. The right hand side of eq. (3.33) will involve both the connected and disconnected diagrams, but analogous to the previous discussion relating $Z[J]$ and $W[J]$ we see that $\Gamma[\phi_0]$ will be the effective action only including the connected diagrams. We will use this equation much in chapter 4 and 5. The value ϕ_0 that we shifted ϕ by is usually called the background field, and we will usually denote the background field with a hat, $\hat{\phi}$. The background field method is the topic of section 3.2, which will give the details of how one calculates the effective action using eq. (3.33).

The effective action can be expanded in terms powers of ϕ ,

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \cdots, x_n) \phi(x_1) \cdots \phi(x_n), \quad (3.34)$$

where $\Gamma^{(n)}$ in momentum space are the 1PI n -point diagram



$$(3.35)$$

Alternatively we can do an expansion around $\phi = \text{constant}$ which effectively is a power expansion of the powers of derivatives [55]. We can equivalently think of

3.2. BACKGROUND FIELDS

this as expanding around the point where all the external momenta vanish. We find

$$\Gamma[\phi] = \int d^4x \left[-V_{\text{eff}}[\phi] + \frac{1}{2}Z[\phi](\partial_\mu\phi)^2 + \dots \right], \quad (3.36)$$

where $V_{\text{eff}}[\phi]$ and $Z[\phi]$ are ordinary functions of ϕ . $V_{\text{eff}}[\phi]$ is called the effective potential, and this will be the topic of section 3.3.

3.2 Background Fields

There are generally two ways of computing the effective action. You can use a diagrammatic approach computing Feynman diagrams or you can use functional methods with the path integral. We saw in eq. (3.33) how the background field came into the picture using the path integral, but you can also use the background fields to do calculations in terms of diagrams. We will see in section 3.3.1.2 how this is used to compute the effective potential. We will now give a brief discussion on the background field method based on Schwartz [19].

Given $S[\phi]$ we shift $\phi \rightarrow \phi + \hat{\phi}$ for an arbitrary non-dynamical background field configuration $\hat{\phi}(x)$. Non-dynamical just means that we won't integrate over it in the path integral. We denote the new action $S_b[\hat{\phi}, \phi] = S[\phi + \hat{\phi}]$ and the corresponding effective action $\Gamma_b[\hat{\phi}, \phi]$. With the new action $S_b[\hat{\phi}, \phi]$ we define the generating functional of the connected diagrams $W_b[J]$ as before

$$\exp\left(iW_b[\hat{\phi}, J]\right) = \int \mathcal{D}\phi \exp\left\{iS_b[\hat{\phi}, \phi] + i \int d^4x J(x)\phi(x)\right\}, \quad (3.37)$$

and the analogue of eq. (3.17) will in this case be $\frac{\partial W_b[\hat{\phi}, J]}{\partial J(x)} = \phi_{J,b}(x)$. Shifting $\phi \rightarrow \phi - \hat{\phi}$ in eq. (3.37) we find that

$$W_b[\hat{\phi}, J] = W[J] - \int d^4x J(x)\hat{\phi}(x), \quad (3.38)$$

which again implies

$$\phi_{J,b} = \phi_J - \hat{\phi} \quad (3.39)$$

which can easily be seen from differentiating with respect to J and using eq. (3.17). All this is saying is that when we shift the field value $\phi \rightarrow \phi + \hat{\phi}$, the expectation value will be shifted by $-\hat{\phi}$, which is what we expected.

In eq. (3.11) we defined the effective action through the Legendre transform of the generating functional $W[J]$. We now define the corresponding effective for

$W_b[\hat{\phi}, J]$,

$$\begin{aligned}\Gamma_b[\hat{\phi}, \phi] &= W_b[\hat{\phi}, J_{\phi,b}] - \int d^4x J_{\phi,b}(x)\phi(x) \\ &= W[J_{\phi,b}] - \int d^4x J_{\phi,b}(x)(\phi(x) + \hat{\phi}(x)),\end{aligned}\tag{3.40}$$

where we have used eq. (3.38).

Now we choose $\phi = \phi_{J,b}$ and use eq. (3.39) together with the fact that $J_{\phi_J} = J$, and we find

$$\begin{aligned}\Gamma_b[\hat{\phi}, \phi_{J,b}] &= W[J] - \int d^4x J(x)\phi_J(x) = \Gamma[\phi_J] \\ &= \Gamma[\hat{\phi} + \phi_{J,b}].\end{aligned}\tag{3.41}$$

This holds for any value of J , so we find

$$\Gamma_b[\hat{\phi}, \phi] = \Gamma[\hat{\phi} + \phi].\tag{3.42}$$

Specifically we find $\Gamma[\hat{\phi}] = \Gamma_b[\hat{\phi}, 0]$. This means that we can compute the functional form of $\Gamma[\phi]$ by computing $\Gamma_b[\hat{\phi}, 0]$. Finding $\Gamma_b[\hat{\phi}, 0]$ basically means that we compute the action with no external ϕ fields, but we include all the internal ϕ loops. This is basically the same statement as eq. (3.33) where we found that

$$e^{i\Gamma[\hat{\phi}]} = \int_{\text{1PI}} \mathcal{D}\phi e^{iS[\phi+\hat{\phi}]}.\tag{3.43}$$

Once the functional form of $\Gamma[\hat{\phi}]$ is determined, we can put the original field back in as the argument, $\Gamma[\phi]$. At this stage after finding the effective action, we will use the original field or the background field as arguments interchangeably.

Also in this case we can write $\Gamma_b[\hat{\phi}, 0]$ as a power expansion in $\hat{\phi}$,

$$\Gamma_b[\hat{\phi}, 0] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma_b^{(n)}(x_1, \cdots, x_n) \hat{\phi}(x_1) \cdots \hat{\phi}(x_n),\tag{3.44}$$

where $\Gamma_b^{(n)}$ are the n -point amplitude with only $\hat{\phi}$ as external lines, and only ϕ in the loops. In section 3.3 we will see how to do an explicit calculation of the effective potential to 1-loop order.

3.3 The Effective Potential

In this section we will study the effective potential $V_{\text{eff}}[\phi]$ that was defined in eq. (3.36), so we will treat ϕ as a constant field in space-time. The effective potential is of great interest to us because the location of the minimum will tell us if the theory is spontaneously broken or not which we will use in chapter 4 and 5.

3.3.1 Fixed Order Effective Potential

Computing the complete effective potential would involve summing up an infinite number of potentially complicated graphs, and it's clear that we need to do some approximation. As mentioned by Sher [55], it is very convenient to organize the calculation in what is called the loop expansion.

Start by introducing a parameter α that rescales the Lagrangian, $\mathcal{L} \rightarrow \alpha^{-1}\mathcal{L}$. Every vertex gets a factor of α^{-1} and every propagator gets a factor of α . You can convince yourself that the number of loops in a diagram is equal to the number of internal lines minus the number of vertices plus one. From this we see that the power of α in a diagram always will be one more than the number of loops. The good thing about this is that we see that the loop expansion corresponds to an expansion in a parameter that multiplying the whole Lagrangian, so it will be unaffected when we shift fields and if we split up the Lagrangian into free and interacting parts.

We will generally refer to a calculation done to a fixed loop order *a fixed order calculation* of the effective potential. In section 3.3.2 we will discuss the resummed potential.

We will now consider different ways of computing the effective potential. To make the whole process more explicit, we will take scalar ϕ^4 theory as an example and show how one goes about computing the effective potential with the different methods. The Lagrangian we will use for massless scalar ϕ^4 theory is

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4. \quad (3.45)$$

3.3.1.1 Diagrams Using the Original Action $S[\phi]$

We will now show how we can compute the effective action to 1-loop using the full theory that we started with given the action $S[\phi]$. We start with eq. (3.34) and realize that all we must do is to compute $\Gamma^{(n)} = \Gamma^{(n)}(p_i = 0)$ to 1-loop, where the $p_i = 0$ is indicating that there is no incoming momenta at the external legs since we are treating the field as constant. Ignoring the vacuum term that is independent of ϕ , we find that eq. (3.34) becomes

$$\Gamma[\phi] = \mathcal{V}_4 \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^{(n)}(p_i = 0) \phi^n. \quad (3.46)$$

To get this, we have used that eq. (3.34) is a local action in the way it is expanded. Hence there must be $n - 1$ delta-functions in $\Gamma^{(n)}(x_1, \dots, x_n)$, leaving only one integral when the fields are constant. This integral is equal to \mathcal{V}_4 . We will see that these n -point functions are divergent, but it turns out that when summing up all of them, we end up with one simple expression that we can handle with renormalization.

Now, let's do an example. We will do the calculation in massless scalar ϕ^4 theory given by the Lagrangian in eq. (3.45). The first diagram with two external ϕ fields is

$$\frac{i}{2!}\Gamma^{(2)} = \text{Diagram} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\frac{\lambda}{2}}{k^2 + i\varepsilon}. \quad (3.47)$$

The factors of $\frac{1}{2}$ are symmetry factors coming from the fact that there are two equivalent ways of doing the contraction for the internal fields and for the external fields. In general there would also be a term from the tree level term in the Lagrangian, but since the fields are constant and the mass is set to zero, this term does not contribute here. The amplitude with $2n$ external ϕ fields become

$$\frac{i}{(2n)!}\Gamma^{(2n)} = \text{Diagram} = \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{\lambda}{2}}{k^2 + i\varepsilon} \right)^n. \quad (3.48)$$

Here we have used that there are $(2n-1)!$ ways of ordering the internal vertices, so we are left with an overall $\frac{1}{2n}$. Using power counting we can see that for $n=1$ the diagram is UV divergent, $n=2$ is UV and IR divergent, and for $n>3$ they are IR divergent. Note that for the case $n=4$ there is also a tree level diagram that contributes

$$\frac{1}{4!}\Gamma^{(4)} = \text{Diagram} + \text{Diagram} = \frac{1}{4!}\lambda + \frac{1}{8} \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{\lambda}{2}}{k^2 + i\varepsilon} \right)^4 \quad (3.49)$$

Substituting this back into eq. (3.46), we find in terms of the effective potential $V_{\text{eff}}[\phi] = -\frac{\Gamma[\phi]}{\mathcal{V}_4}$ to be

$$\begin{aligned} V_{\text{eff}}[\phi] &= -\mathcal{L}[\phi] + i \sum_{n=1}^{\infty} \text{Diagram} \times \phi^{2n} \\ &= \frac{\lambda}{4!}\phi^4 + i \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{\lambda}{2}\phi^2}{k^2 + i\varepsilon} \right)^n \\ &= \frac{\lambda}{4!}\phi^4 - i \frac{1}{2} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \ln \left(1 - \frac{\lambda\phi^2}{2(k^2 + i\varepsilon)} \right) \\ &= \frac{\lambda}{4!}\phi^4 + \frac{2\pi^{\frac{d}{2}}}{2\Gamma(\frac{d}{2})} \mu^{4-d} \int dk_E k_E^{d-1} \ln \left(1 + \frac{\lambda\phi^2}{2k_E^2} \right) \\ &= \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2} \left[-\frac{2}{\epsilon} + \ln \left(\frac{\frac{\lambda}{2}\phi^2}{\tilde{\mu}^2} \right) - \frac{3}{2} \right] \end{aligned} \quad (3.50)$$

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Adding in the counterterm $\lambda = \lambda_R + \delta\lambda$, where $\delta\lambda$ is of order $\mathcal{O}(\lambda_R^2)$ we can renormalize the effective potential. The renormalization condition we will use is

$$V''''[\phi_R] = \lambda_R, \quad (3.51)$$

for some reference scale² ϕ_R . This gives

$$V_{\text{eff}}[\phi] = \frac{\lambda_R}{4!} \phi^4 \left[1 + \frac{3\lambda_R}{32\pi^2} \left(\ln \frac{\phi^2}{\phi_R^2} - \frac{25}{6} \right) \right] \quad (3.52)$$

which is the final form for the renormalized effective potential for massless scalar ϕ^4 theory.

3.3.1.2 Diagrams with Background Fields

In this section we will look at an alternative way of computing the effective potential using the action that is expanded around some background field $S[\phi + \hat{\phi}]$. Using eq. (3.44) for a constant background field $\hat{\phi}$, we want to calculate the $\Gamma_b^{(n)}$ to a given loop order. This means that we include all diagrams where ϕ can propagate in internal loops and we have only $\hat{\phi}$ as external fields.

We will again look at an example with massless ϕ^4 theory, and we will calculate the effective potential to 1-loop order. We start with the Lagrangian given in eq. (3.45) and replace $\phi \rightarrow \phi + \hat{\phi}$ where $\hat{\phi}$ is assumed to be constant. The new Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!} \left(\phi^4 + 4\phi^3\hat{\phi} + 6\phi^2\hat{\phi}^2 + 4\phi\hat{\phi}^3 + \hat{\phi}^4 \right). \quad (3.53)$$

Computing the 1PI vertices of $\Gamma_b^{(n)}$ to 1-loop we realize that the ϕ^4 , $\phi^3\hat{\phi}$ and $\phi\hat{\phi}^3$ terms will not be relevant since there is no way to draw diagrams with these vertices with only $\hat{\phi}$ as external fields and ϕ as internal propagating fields. Hence, the relevant Lagrangian to 1-loop is

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4}\phi^2\hat{\phi}^2 - \frac{\lambda}{4!}\hat{\phi}^4. \quad (3.54)$$

The diagram with $2n$ external $\hat{\phi}$ fields is

$$\frac{i}{(2n)!} \Gamma_b^{(2n)} = \text{Diagram} = \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{\lambda}{2}}{k^2 + i\varepsilon} \right)^n, \quad (3.55)$$

²The reason we evaluate ϕ at ϕ_R is that the potential is singular at $\phi = 0$.

where the dotted lines with small spacing are background fields and the solid line is ϕ . This result looks exactly the same as the $2n$ -point amplitude computed in eq. (3.48). The effective potential becomes

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}] &= -\mathcal{L}[\hat{\phi}] + i \sum_{n=1}^{\infty} \left(\text{Diagram} \right) \times \hat{\phi}^{2n} \\
 &= \frac{\lambda}{4!} \hat{\phi}^4 + i \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4k}{(2\pi)^4} \left(\frac{\frac{\lambda}{2} \hat{\phi}^2}{k^2 + i\varepsilon} \right)^n,
 \end{aligned} \tag{3.56}$$

and we see that there is really no need to continue this calculation. Following the same steps as in section 3.3.1.1 we will reproduce the result for the renormalized effective potential given in eq. (3.52).

3.3.1.3 The Effective Potential from Tadpole Diagrams

A problem with the above calculations of the diagrams for n -point amplitudes is that it gets very complicated and inefficient if we want to go beyond the 1-loop order. We will now look at another method that was first proposed by Lee and Sciacaluga [56] and also described by Sher [55].

First consider the case where we are given an action $S[\phi]$, and we can compute the corresponding effective action $\Gamma[\phi]$. For constant field values we can find the effective potential defined as

$$V_{\text{eff}}[\phi] = - \sum_n \frac{1}{n!} \Gamma^{(n)}(p_i = 0) \phi^n. \tag{3.57}$$

Now consider the same action S , but where the argument is the sum of two fields $S[\phi + \hat{\phi}]$. We saw in section 3.2 that we get an effective action $\Gamma_b[\hat{\phi}, \phi]$ and for constant fields we find an effective potential

$$V_{b;\text{eff}}[\hat{\phi}, \phi] = - \sum_n \frac{1}{n!} \Gamma^{(n)}(p_i = 0, \hat{\phi}) \phi^n, \tag{3.58}$$

where $\Gamma^{(n)}(p_i = 0, \hat{\phi})$ in general can depend on all powers of $\hat{\phi}$. We also saw in section 3.2 that $\Gamma[\hat{\phi} + \phi] = \Gamma_b[\hat{\phi}, \phi]$, and it follows that $V_{\text{eff}}[\hat{\phi} + \phi] = V_{b;\text{eff}}[\hat{\phi}, \phi]$.

Now let us consider the effective potential $V_{\text{eff}}[\phi]$ and we choose to rewrite it as

$$V_{\text{eff}}[\phi] = V_{b;\text{eff}}[\hat{\phi}, \phi - \hat{\phi}]. \tag{3.59}$$

Taking taking a derivative of this potential we can write

$$\frac{dV_{\text{eff}}[\hat{\phi}]}{d\hat{\phi}} = \left. \frac{dV_{\text{eff}}[\phi]}{d\phi} \right|_{\phi=\hat{\phi}} = \left. \frac{dV_{b;\text{eff}}[\hat{\phi}, \phi - \hat{\phi}]}{d\phi} \right|_{\phi=\hat{\phi}} = \left. \frac{dV_{b;\text{eff}}[\hat{\phi}, \psi]}{d\psi} \right|_{\psi=0}. \tag{3.60}$$

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We can now integrate eq. (3.60) to find the effective potential

$$V_{\text{eff}}[\phi] = \int_0^\phi d\hat{\phi} \frac{dV_{\text{eff}}[\hat{\phi}]}{d\hat{\phi}} = \int_0^\phi d\hat{\phi} \frac{dV_{b;\text{eff}}[\hat{\phi}, \psi]}{d\psi} \Big|_{\psi=0}. \quad (3.61)$$

In other words if we can compute $\frac{dV_{b;\text{eff}}[\hat{\phi}, \psi]}{d\psi} \Big|_{\psi=0}$, then all we have to do to find the effective potential is to perform one integral over $\hat{\phi}$.

We will now see the reason why this is useful. Looking at eq. (3.58) we see that only one term survives when we set $\psi = 0$,

$$\frac{dV_{b;\text{eff}}[\hat{\phi}, \psi]}{d\psi} \Big|_{\psi=0} = -\Gamma^{(1)}(p_i = 0, \hat{\phi}), \quad (3.62)$$

which is a tadpole diagram.

The tadpole is computed with $\psi = 0$, meaning that we have one external background field $\hat{\phi}$ and any diagram of ψ in the blob just as we did when computing diagrams in section 3.3.1.2.

$$\Gamma^{(1)} = \text{Diagram} \quad (3.63)$$


The explicit calculation for scalar ϕ^4 can be found in Sher [55].

3.3.1.4 Functional Method

So far we have discussed different methods of computing diagrams to find the effective potential. In this section we will use the path integral and explicitly evaluate eq. (3.33) to find the effective potential. This will be our preferred method that we use in chapter 4 and 5.

We recall eq. (3.33) which says that we integrate out ϕ only integrating over the region corresponding to 1PI diagrams

$$e^{i\Gamma[\hat{\phi}]} = \int_{\text{1PI}} \mathcal{D}\phi e^{iS[\phi+\hat{\phi}]}, \quad (3.64)$$

where we will assume that $\hat{\phi}$ is constant. We start by Taylor expanding the action

$$S[\phi + \hat{\phi}] = S[\hat{\phi}] + S'[\hat{\phi}]\phi + \frac{1}{2}S''[\hat{\phi}]\phi^2 + \frac{1}{3!}S'''[\hat{\phi}]\phi^3 + \dots \quad (3.65)$$

If we want the 1-loop effective potential we realize as in eq. (3.54) that only $S[\hat{\phi}]$ and $\frac{1}{2}S''[\hat{\phi}]\phi^2$ are the terms that could contribute to making the 1-loop 1PI diagrams. Hence we are left with the integral

$$\begin{aligned}
 e^{i\Gamma[\hat{\phi}]} &= e^{iS[\hat{\phi}]} \int \mathcal{D}\phi e^{i\frac{1}{2}S''[\hat{\phi}]\phi^2} \\
 &= e^{iS[\hat{\phi}]} \int \mathcal{D}\phi e^{i \int d^4x \frac{1}{2}\phi(x)\mathcal{L}''[\hat{\phi}]\phi(x)} \\
 &= e^{iS[\hat{\phi}]} \frac{1}{\sqrt{\text{Det}\mathcal{L}''[\hat{\phi}]}} \\
 &= \exp\left(i\mathcal{V}_4\mathcal{L}[\hat{\phi}] - \frac{1}{2}\mathcal{V}_4 \int \frac{d^4p}{(2\pi)^4} \ln \det\mathcal{L}''(p)\right),
 \end{aligned} \tag{3.66}$$

where we have used eq. (2.45) to do the Gaussian integral and rewrite the functional determinant. We can rewrite this in terms of the effective potential to 1-loop order as

$$V_{\text{eff}}[\hat{\phi}] = V[\hat{\phi}] - i\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln \det\mathcal{L}''(p), \tag{3.67}$$

where we have used that $\mathcal{L}[\hat{\phi}] = \mathcal{L}_{\text{kin}} - V[\hat{\phi}] = -V[\hat{\phi}]$ for constant $\hat{\phi}$.

Now consider the scalar ϕ^4 theory again with $\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4$. This gives

$$\begin{aligned}
 V[\hat{\phi}] &= \frac{\lambda}{4!}\hat{\phi}^4 \\
 \mathcal{L}''[\hat{\phi}] &= -\Box - \frac{\lambda}{2}\hat{\phi}^2.
 \end{aligned} \tag{3.68}$$

Plugging this into eq. (3.67) we find

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}] &= \frac{\lambda}{4!}\hat{\phi}^4 - i\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln\left(p^2 - \frac{\lambda}{2}\hat{\phi}^2\right) \\
 &= \frac{\lambda}{4!}\hat{\phi}^4 - i\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \ln\left(1 - \frac{\lambda\hat{\phi}^2}{2p^2}\right) + \text{const.},
 \end{aligned} \tag{3.69}$$

where we have subtracted off an infinite constant to rewrite the log. We have again reproduced eq. (3.50) which will give us the final renormalized effective potential in eq. (3.52) as described earlier.

Notice that there was no additional calculation needed beyond taking derivatives of the Lagrangian and Fourier transforming to get the integral in eq. (3.69). In this case we only had one field to deal with, so taking the determinant was trivial. In other more complicated cases this may not be so easy. If the matrix

is diagonal we will get a product of terms similar to what we got above, but if there are off-diagonal terms it will in general be more complicated. This will in fact be the case in both the Abelian Higgs model in chapter 4 and the Standard Model in chapter 5. We will get back to how we resolve this problem in section 4.1.3.

3.3.2 Resummed Effective Potential

The discussion in the last few sections has all been about the fixed order effective potential. Looking at the scalar ϕ^4 potential in eq. (3.52) we see that we have a large logarithm. If $\phi \gg \phi_R$ such that $\frac{3\lambda_R}{32\pi^2} \left(\ln \frac{\phi^2}{\phi_R^2} - \frac{25}{6} \right) \geq 1$ we do not trust our perturbative expansion to hold since higher loop terms would contribute to the same order as the 1-loop result. We will now see how we can use the renormalization group equations from section 2.4 to improve the range of validity for our result for the effective potential.

3.3.2.1 RGE for a Cross Section

Before we discuss the RGE for the effective potential we will remind ourselves of how the RGE improves the cross section with large logarithms for which will be a close analogy for how the RGE improvement we want to study. Assume that we have computed some cross section as a function of some physical scale

$$\sigma(Q) = \sigma_0 \left(1 + \frac{\alpha}{\pi} + a_1 \left(\frac{\alpha}{\pi} \right)^2 + \frac{\alpha}{\pi} \ln \frac{Q}{\mu} + \dots \right), \quad (3.70)$$

for some numbers a_1 and σ_0 that are assumed to be known. We choose $\mu = Q_0$, and at this scale we can measure $\sigma(Q_0)$ and we can determine the coupling constant at this scale $\alpha = \alpha(Q_0)$. The cross section can now be written as

$$\sigma(Q) = \sigma_0 \left(1 + \frac{\alpha(Q_0)}{\pi} + a_1 \left(\frac{\alpha(Q_0)}{\pi} \right)^2 + \frac{\alpha}{\pi} \ln \frac{Q}{Q_0} + \dots \right). \quad (3.71)$$

If we go to some high energy scale $Q \gg Q_0$, the logarithm gets large and we should not trust our perturbative expansion. This is where the RGE comes in. Solving the beta function for α gives us $\alpha(Q)$ for any Q ³. In terms of eq. (3.70) and eq. (3.71), we can choose a different $\mu = Q_1$ and compensate by changing the value of $\alpha(Q_0)$ to $\alpha(Q_1)$. Since we can do this for any $\mu = Q$, we can with the knowledge of $\alpha(Q)$ predict the cross section at any scale

$$\sigma(Q) = \sigma_0 \left(1 + \frac{\alpha(Q)}{\pi} + a_1 \left(\frac{\alpha(Q)}{\pi} \right)^2 + \dots \right), \quad (3.72)$$

³The solution $\alpha(Q)$ is an approximation corresponding to a resummation of the logarithmic terms.

without the fear of any large logarithms. This is the RGE improved cross section. The RGE improves our result since we now only have to worry about coupling α being small and not $\alpha \ln \frac{Q}{\mu}$ to trust our expansion [55]. With this in mind, we will now discuss the RGE improved effective potential using an analogous argument.

3.3.2.2 RGE for Effective Potentials

In section 2.4.2.1 we derived an equation that the Green's functions has to satisfy since a Green's function of bare parameters must be independent of μ . The same is true for the effective potential as described by Sher [55]

$$\frac{d}{d\mu} V_{\text{eff}} = 0. \quad (3.73)$$

Since the potential is only a function of the arbitrary scale μ , some couplings g_i , masses m_i and the field strength ϕ , we can using the chain rule find the RGE equation for the effective potential

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_{g_i} \frac{\partial}{\partial g_i} + \beta_{m_i} \frac{\partial}{\partial m_i} - \gamma \phi \frac{\partial}{\partial \phi} \right] V_{\text{eff}} = 0, \quad (3.74)$$

where we have defined

$$\beta_{g_i} = \mu \frac{\partial g_i}{\partial \mu}, \quad \beta_{m_i} = \mu \frac{\partial m_i}{\partial \mu}, \quad \gamma \phi = -\mu \frac{\partial \phi}{\partial \mu}. \quad (3.75)$$

There are different ways of using eq. (3.74). If we know the effective potential it can be used to extract the beta functions and anomalous dimension, and if we know the beta functions and anomalous dimensions we can find the effective potential. We will see an example of how we can find 1-loop scalar QED effective potential just from the beta functions in section 4.4.1, and an example of how we can extract the beta functions from the effective potential is given in section 3.3.2.3.

We will now use the method of characteristics described in Appendix B to solve eq. (3.74). We parametrize the variables in eq. (3.74) by a new parameter t , i.e. $\mu = \mu(t)$, $g_i = g_i(t)$, $m_i = m_i(t)$ and $\phi = \phi(t)$, and find the following set of differential equations

$$\begin{aligned} \frac{d}{dt} \mu(t) &= \mu, \\ \frac{d}{dt} g_i(t) &= \beta_{g_i}, \\ \frac{d}{dt} m_i(t) &= \beta_{m_i}, \\ \frac{d}{dt} \phi(t) &= -\gamma \phi. \end{aligned} \quad (3.76)$$

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We can easily solve the first and the last equation

$$\begin{aligned}\mu(t) &= \mu e^t, \\ \phi(t) &= \phi e^{-\int_0^t dt' \gamma},\end{aligned}\tag{3.77}$$

and the two middle equations can only be solved once the functions β_{g_i} and β_{m_i} are given. Assuming that there exists a solution, we can always write the solution as $g_i(t)$ and $m_i(t)$. In practice these equations will usually be solved numerically, and we only need to specify the initial values of the couplings and masses for $t = 0$.

3.3.2.3 Massless Scalar ϕ^4 Resummed Effective Potential

In this section we will apply the method described above to find the resummed potential for massless scalar ϕ^4 theory [57]. We have already found the effective potential to 1-loop in eq. (3.50), and in $\overline{\text{MS}}$ it is

$$V_{\text{eff}}[\phi, \lambda_R, \mu] = \frac{\lambda_R}{4!} \phi^4 + \frac{\lambda_R^2 \phi^4}{256\pi^2} \left[\ln \left(\frac{\frac{\lambda_R}{2} \phi^2}{\mu^2} \right) - \frac{3}{2} \right].\tag{3.78}$$

Using that the effective potential is independent of μ , we find

$$\begin{aligned}0 &= \mu \frac{dV_{\text{eff}}}{d\mu} \\ &= \frac{\beta_\lambda}{4!} \phi^4 - \gamma \frac{\lambda_R}{3!} \phi^4 - \frac{\lambda_R^2 \phi^4}{128\pi^2} \\ &\quad + (2\beta_\lambda \lambda_R - 4\gamma \lambda_R^2) \frac{\phi^4}{256\pi^2} \left[\ln \left(\frac{\frac{\lambda_R}{2} \phi^2}{\mu^2} \right) - \frac{3}{2} \right].\end{aligned}\tag{3.79}$$

Solving to first order we find

$$\begin{aligned}\beta_\lambda^{(1)} &= \frac{3\lambda_R^2}{16\pi^2}, \\ \gamma^{(1)} &= 0.\end{aligned}\tag{3.80}$$

Now that we have the beta function, we can solve eq. (3.76) for the beta function

$$\frac{d}{dt} \lambda(t) = \frac{3}{16\pi^2} \lambda^2,\tag{3.81}$$

which has the solution

$$\lambda(t) = \frac{\lambda}{1 - \frac{3}{16\pi^2} \lambda t},\tag{3.82}$$

where $\lambda(0) \equiv \lambda$. Together with eq. (3.77), where $\phi(t) = \phi$ since $\gamma = 0$, we find that the resummed effective potential is

$$V_{\text{eff}}[\phi, \lambda(t), \mu(t)] = \frac{\lambda(t)}{4!} \phi^4 + \frac{\lambda(t)^2 \phi^4}{256\pi^2} \left[\ln \left(\frac{\lambda(t) \phi^2}{\mu(t)^2} \right) - \frac{3}{2} \right]. \quad (3.83)$$

As pointed out by Ford [57], we have the freedom to choose our parametrization t . Following the discussion in section 3.3.2.1 a natural choice, analogous to the choice $\mu = Q$ for the cross section, for massless scalar ϕ^4 theory would be to take

$$\mu(t)^2 = \mu^2 e^{2t} = \phi^2, \quad (3.84)$$

or equivalently, we define t as

$$t = \ln \frac{\phi}{\mu}. \quad (3.85)$$

Since t is defined in terms of ϕ we now rewrite eq. (3.83) as

$$\begin{aligned} V_{\text{eff}}[\phi, \lambda(\phi)] &= \frac{\lambda(\phi)}{4!} \phi^4 + \frac{\lambda(\phi)^2 \phi^4}{256\pi^2} \left[\ln \left(\frac{\lambda(\phi)}{2} \right) - \frac{3}{2} \right] \\ &\equiv \frac{\lambda_{\text{eff}}(\phi)}{4!} \phi^4, \end{aligned} \quad (3.86)$$

where we have defined the effective lambda

$$\lambda_{\text{eff}}(\phi) = \lambda(\phi) + \frac{3\lambda(\phi)^2}{32\pi^2} \left[\ln \left(\frac{\lambda(\phi)}{2} \right) - \frac{3}{2} \right]. \quad (3.87)$$

We see that the $\ln \mu$ terms have vanished in the same way as they did for the cross section, but in this case we also had a logarithm of couplings which are still present.

We will use this same procedure again in section 4.4 for massless scalar QED and in section 5.4 for the Standard Model.

3.3.3 Gauge dependence

In this chapter we have only considered scalar ϕ^4 theory for any of our calculations. It turns out that if we have a quantum field theory with a gauge symmetry, the effective potential will in general depend on the gauge choice. If we gauge fix with the R_ξ gauges, the effective potential will in general depend on ξ .

We will see this gauge dependence throughout chapter 4 and 5 in the calculations of the effective potential. We will now review the Nielsen identity, which we believe may be an important ingredient to understanding the gauge dependence on the Higgs mass bound in chapter 5.

3.3.3.1 Nielsen Identity

In this section we will derive the Nielsen identity which is an important identity regarding the gauge dependence of the effective potential. It was first derived by N.K. Nielsen in 1975 [58] and we will follow the derivation given in [59, 60].

Consider a gauge theory described by the action $S[\phi_i]$ with a set of fields denoted by ϕ_i . Under an infinitesimal gauge transformation

$$\phi_i \rightarrow \phi_i + \Delta_i \alpha \quad (3.88)$$

where Δ_i is a linear operator, the classical action $S[\phi_i]$ is left invariant. In general there can be many gauge parameters α_j , giving $\delta\phi_i = \Delta_i^j \alpha_j$, but we will suppress this index j for simplicity. As described in section 2.3.5 we will gauge fix this theory with a gauge-fixing function $F[\phi_i]$ and introducing Fadeev-Popov ghosts c and \bar{c} . We can now write the W , the generating functional of connected Green's functions, as

$$\exp[iW[J, F]] = \int \mathcal{D}\phi_i \mathcal{D}c \mathcal{D}\bar{c} \exp \left[iI[\phi_i, F] + i \int d^4x J^i(x) \phi_i(x) \right] \quad (3.89)$$

where

$$I[\phi_i, F] = S[\phi_i] - \int d^4x \left[\frac{1}{2\xi} (F[\phi_i])^2 + \bar{c} \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i c \right] \quad (3.90)$$

is invariant under the BRST transformations

$$\delta_B \phi_i = \zeta \Delta_i c, \quad \delta_B \bar{c} = -\zeta \frac{1}{\xi} F[\phi_i], \quad \delta_B c = 0 \quad (3.91)$$

as described in section 2.3.6, where ζ is an arbitrary Grassman number.

Let's start by checking that $I[\phi_i, F]$ is invariant under the given transformation. We know from section 2.3.6 that $S[\phi_i]$ always is invariant under the BRST transformation since it is gauge invariant. The gauge fixing and ghost term transform as

$$\begin{aligned} \delta_B \left[\frac{1}{2\xi} (F[\phi_i])^2 + \bar{c} \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i c \right] &= \frac{F}{\xi} \delta_B F[\phi_i] + (\delta_B \bar{c}) \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i c \\ &+ \bar{c} \left(\delta_B \frac{\delta F[\phi_i]}{\delta \phi_i} \right) \Delta_i c + \bar{c} \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i (\delta_B c). \end{aligned} \quad (3.92)$$

Using the transformations in eq. (3.91) we find

$$\begin{aligned} \frac{F}{\xi} \delta_B F[\phi_i] &= \frac{F}{\xi} \frac{\delta F[\phi_i]}{\delta \phi_i} \delta_B \phi_i = \frac{F}{\xi} \frac{\delta F[\phi_i]}{\delta \phi_i} \zeta \Delta_i c, \\ (\delta_B \bar{c}) \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i c &= -\zeta \frac{F}{\xi} \frac{\delta F[\phi_i]}{\delta \phi_i} \Delta_i c, \end{aligned} \quad (3.93)$$

and we see that the first two terms in eq. (3.92) cancel. The last term is zero since $\delta_B c = 0$ and the third term vanish because

$$\bar{c}\delta_B \left[\frac{\delta F[\phi_i]}{\delta \phi_i} \right] \Delta_i c = \bar{c} \frac{\delta^2 F[\phi_i]}{\delta \phi_i \delta \phi_j} \delta_B \phi_j \Delta_i c = \bar{c} \frac{\delta^2 F[\phi_i]}{\delta \phi_i \delta \phi_j} \zeta(\Delta_j c)(\Delta_i c) = 0, \quad (3.94)$$

since the ghosts anticommute.

To simplify the notation we will denote the expectation value of an operator $\mathcal{O}(\phi_i)$ by

$$\langle \mathcal{O}(\phi_i) \rangle = e^{-iW} \int \mathcal{D}\phi_i \mathcal{D}c \mathcal{D}\bar{c} \mathcal{O}(\phi_i) \exp \left[iI[\phi_i, F] + i \int d^4x J^i(x) \phi_i(x) \right]. \quad (3.95)$$

Now consider the operator $\mathcal{O} = \bar{c}G$ for any functional $G = G[\phi_i]$. Since this is linear in the ghost field the expectation value must vanish

$$\langle \bar{c}(x)G[\phi_i(x)] \rangle = 0. \quad (3.96)$$

Applying the BRST transformation we find

$$\left\langle \delta_B [\bar{c}(x)G[\phi_i(x)]] + i\bar{c}(x)G[\phi_i(x)] \int d^4y J^i(y) \delta_B \phi_i(y) \right\rangle = 0, \quad (3.97)$$

where we have used that the $I[\phi_i, F]$ is invariant and the current does not transform under BRST. Using eq. (3.91) we can rewrite this as

$$\begin{aligned} & \left\langle \frac{1}{\xi} F[\phi_i(x)] G[\phi_i(x)] + \bar{c}(x) \frac{\delta G[\phi_i(x)]}{\delta \phi_i(x)} \Delta_i c(x) \right\rangle \\ & = i \int d^4y J^i(y) \langle \Delta_i c(y) \bar{c}(x) G[\phi_i(x)] \rangle. \end{aligned} \quad (3.98)$$

This is an identity valid for any $G[\phi_i(x)]$. We will use this shortly.

Now we go back to eq. (3.90) and we do an infinitesimal shift $F \rightarrow F + \Delta F$. We find

$$I \rightarrow I - \int d^4x \left[\frac{F}{\xi} \Delta F + \bar{c} \frac{\delta \Delta F}{\delta \phi_i} \Delta_i c \right], \quad (3.99)$$

and we can rewrite eq. (3.89) as

$$\begin{aligned} e^{iW[J, F + \Delta F]} &= e^{iW[J, F] + i\Delta W} = e^{iW[J, F]} (1 + i\Delta W) \\ &= \int \mathcal{D}\phi_i \mathcal{D}c \mathcal{D}\bar{c} e^{iI[\phi_i, F] + i \int d^4x J^i(x) \phi_i(x)} e^{-i \int d^4x \left[\frac{F}{\xi} \Delta F + \bar{c} \frac{\delta \Delta F}{\delta \phi_i} \Delta_i c \right]} \\ &= \int \mathcal{D}\phi_i \mathcal{D}c \mathcal{D}\bar{c} e^{iI[\phi_i, F] + i \int d^4x J^i(x) \phi_i(x)} \left\{ 1 - i \int d^4x \left[\frac{F \Delta F}{\xi} + \bar{c} \frac{\delta \Delta F}{\delta \phi_i} \Delta_i c \right] \right\} \\ &= e^{iW[J, F]} \left(1 - i \int d^4x \left\langle \frac{F \Delta F}{\xi} + \bar{c} \frac{\delta \Delta F}{\delta \phi_i} \Delta_i c \right\rangle \right). \end{aligned} \quad (3.100)$$

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We see that

$$\begin{aligned}\Delta W &= - \int d^4x \left\langle \frac{F\Delta F}{\xi} + \bar{c} \frac{\delta\Delta F}{\delta\phi_i} \Delta_i c \right\rangle \\ &= -i \int d^4x d^4y J^i(y) \langle \Delta_i c(y) \bar{c}(x) \Delta F[\phi_i(x)] \rangle\end{aligned}\tag{3.101}$$

where we have used eq.(3.98) with $G = \Delta F$.

Recall from eq. (3.11) that the effective action $\Gamma[J, F]$ is related to the generating functional $W[J, F]$ by the Legendre transformation

$$\Gamma[\phi_i, F] = W[J, F] - \int d^4x J^i(x) \phi_i(x).\tag{3.102}$$

Varying this with respect to $\phi_i(x)$ as in eq. (3.13) we find

$$\frac{\partial\Gamma[\phi_i, F]}{\partial\phi_i(x)} = -J^i(x).\tag{3.103}$$

Now varying $F \rightarrow F + \Delta F$ we find

$$\begin{aligned}\Delta\Gamma[\phi_i, F] &= \Delta W[J, F] \\ &= i \int d^4x d^4y \frac{\partial\Gamma[\phi_i, F]}{\partial\phi_i(y)} \langle \Delta_i c(y) \bar{c}(x) \Delta F[\phi_i(x)] \rangle_{\text{1PI}},\end{aligned}\tag{3.104}$$

where the subscript 1PI indicates that we only include the one-particle irreducible graphs. Instead of a change in the gauge fixing function, we can equivalently use that an infinitesimal shift in the gauge parameter gives $\Delta F = -\frac{F}{2\xi} d\xi$. We can now write eq. (3.104) as

$$\begin{aligned}\xi \frac{\partial\Gamma[\phi_i, F]}{\partial\xi} &= -\frac{i}{2} \int d^4x d^4y \frac{\partial\Gamma[\phi_i, F]}{\partial\phi_i(y)} \langle \Delta_i c(y) \bar{c}(x) F[\phi_i(x)] \rangle_{\text{1PI}} \\ &= \int d^4y \frac{\partial\Gamma[\phi_i, F]}{\partial\phi_i(y)} H_i[\phi_i(x), y],\end{aligned}\tag{3.105}$$

where we have defined

$$H_i[\phi_i(x), y] \equiv -\frac{i}{2} \int d^4x \langle \Delta_i c(y) \bar{c}(x) F[\phi_i(x)] \rangle_{\text{1PI}}.\tag{3.106}$$

Eq. (3.105) is the Nielsen identity.

We are interested in the Nielsen identity for the effective potential, and for simplicity let us consider the case where we only have one field $\phi(x)$. We start by remembering eq. (3.36) for the derivative expansion of the effective action

$$\Gamma = \int d^4x \left[-V_{\text{eff}}(\phi) + \frac{1}{2} Z(\phi) (\partial_\mu \phi)^2 + \dots \right],\tag{3.107}$$

and we do a similar expansion for $H_i[\phi_i(x), y]$

$$H_i[\phi(x), y] = C(\phi) + D(\phi)(\partial_\mu\phi)^2 + \dots \quad (3.108)$$

where all the terms on the right hand side are evaluated at y . Inserting this into eq. (3.105) we find

$$\begin{aligned} & \xi \frac{\partial}{\partial \xi} \int d^4x \left[-V_{\text{eff}}(\phi) + \frac{1}{2}Z(\phi)(\partial_\mu\phi)^2 + \dots \right] \\ &= \int d^4x \left[C(\phi) + D(\phi)(\partial_\mu\phi)^2 + \dots \right] \left[-\frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} + \frac{1}{2} \frac{\partial Z(\phi)}{\partial \phi} (\partial_\mu\phi)^2 + \dots \right]. \end{aligned} \quad (3.109)$$

Comparing terms with no derivatives we find

$$\xi \frac{\partial V_{\text{eff}}}{\partial \xi} = C(\phi) \frac{\partial V_{\text{eff}}}{\partial \phi} \quad (3.110)$$

which is the Nielsen identity for the effective potential. Comparing the two sides and matching the terms with two derivative we find

$$\xi \frac{\partial Z}{\partial \xi} = C \frac{\partial Z}{\partial \phi} - 2D \frac{\partial V_{\text{eff}}}{\partial \phi} + 2Z \frac{\partial C}{\partial \phi}, \quad (3.111)$$

and in principle we can continue to higher and higher order if we wanted to.

We will not use the Nielsen directly in this thesis, but we will discuss eq. (3.110) in section 5.8 since it might be a useful tool to resolve the gauge dependent Higgs mass bound.

3.4 Stability and Metastability

As described in section 3.3, we can use the effective potential to find the vacuum of the theory. But what happens when the effective potential has multiple minima? If there are multiple local minima with different energies, we have to consider the possibility that we are not in the true vacuum of the theory, but in some false vacuum. If we are in the false vacuum, it is possible to tunnel to the true vacuum analogous to tunneling in quantum mechanics. This was described in the seminal paper on the fate of the false vacuum by Sidney Coleman [1].

The idea of a false vacuum is in fact very relevant for our study of the Standard Model as was pointed out in chapter 1, and we will go much deeper into the details of this in chapter 5. Depending on the top and Higgs mass, it is possible for the Standard Model effective potential to have a lower energy state than the vacuum we currently live in. We know that we have not tunneled into this possible other vacuum since that would destroy the universe as we know it. In the words of Coleman and de Luccia [61]

"The possibility that we are living in a false vacuum has never been a cheering one to contemplate. Vacuum decay is the ultimate ecological catastrophe; in a new vacuum there are new constants of nature; after vacuum decay, not only is life as we know it impossible, so is chemistry as we know it."

Since this catastrophe has not hit us yet, we have two possible explanations. Our universe will be completely stable if our vacuum is the global minimum (the true vacuum), and we will refer to this as absolute stability. The other option is that there is a lower energy state, but the decay rate is so low that the lifetime exceeds the age of our universe. This scenario is called metastability.

In chapter 5 we will see that with the current values of the Standard Model parameters the theory seems to indicate that we are not in the true vacuum of the theory, but in some false vacuum. This is assuming that our universe is described by just the Standard Model up to the Plack scale (10^{20}GeV) which is the energy scale above which we can no longer safely ignore the contributions of gravity.

3.4.1 Absolute Stability

For our vacuum to be absolutely stable, we mean that there is no lower energy state than the current vacuum we are using in our theory. Let E_0 be the vacuum energy density of our current vacuum state $\phi = v$, and the absolute stability condition can be written as

$$V_{\text{eff}}[\phi] > E_0. \quad (3.112)$$

As mentioned earlier, we will only be looking at ϕ up to 10^{20}GeV .

Without setting any of the masses to zero the effective potential will be of the form

$$V_{\text{eff}}[\phi] = -\frac{1}{2}m_{\text{eff}}^2(\phi)\phi^2 + \frac{\lambda_{\text{eff}}(\phi)}{4!}\phi^4, \quad (3.113)$$

where $m_{\text{eff}}^2(\phi)$ is the effective mass and $\lambda_{\text{eff}}(\phi)$ is the effective quartic coupling in the resummed potential. We have here chosen the overall constant such that $V_{\text{eff}}[0] = 0$. To estimate the value of the vacuum energy density at the minimum, we use some rough estimates of the tree level values

$$v = 246\text{GeV}, \quad \lambda \sim 0.1, \quad m \sim 100\text{GeV}, \quad (3.114)$$

and find the value at the minimum to be

$$V_{\text{eff}}[\phi = v] = E_0 \sim -10^8\text{GeV}^4. \quad (3.115)$$

In the Standard Model, we will find in chapter 5 that $\lambda_{\text{eff}}(\phi)$ changes sign somewhere between 10^{10}GeV and 10^{20}GeV . Assume that we find $\lambda_{\text{eff}}(\phi) = -10^{-10}$ at $\phi = 10^{10}\text{GeV}$. In this case the energy density will be

$$V_{\text{eff}}[\phi = 10^{10}\text{GeV}] = -\frac{1}{2}(100)^2 10^{20}\text{GeV}^4 - \frac{1}{4!}10^{-10+40}\text{GeV}^4. \quad (3.116)$$

The second term is 6 orders of magnitude bigger than the first term, so we can safely only consider the last term. In this case the value of the energy density is $V_{\text{eff}}[10^{10}] \sim -10^{30}\text{GeV}^4$, i.e. 22 orders of magnitude bigger than the value at $\phi = v$ give in eq. (3.115).

In the above argument we used $\lambda_{\text{eff}} = -10^{-10}$ as an example. In general it will be a very good approximation to just look at the sign of λ_{eff} to determine if there will exist a state with lower energy than E_0 or not, since for large ϕ the ϕ^4 term will completely dominate. In our analysis in chapter 5 we will use the condition $\lambda_{\text{eff}} > 0$ for absolute stability.

3.4.2 Metastability

The condition for metastability is less strict than the condition for absolute stability as was described by Arnold [8]. There is nothing wrong with the existence of a lower energy state, but since we have not tunneled to it yet, we must require that the lifetime of this system is longer than the age of our universe.

The analysis for metastability is more complicated than just looking at the sign of λ_{eff} . It is based on the work on instantons by Coleman [26] where he starts with a consideration of a classical potential with two minima as in figure 3.1. By following Coleman's analysis it is possible to find the instanton solutions, and use this to compute the decay rate from the false to true vacua.

We have not performed any calculations of metastability in our analysis in chapter 4 or 5. Once we understand the analysis better, we plan to add the metastability calculations in the analysis in chapter 5 in the future.

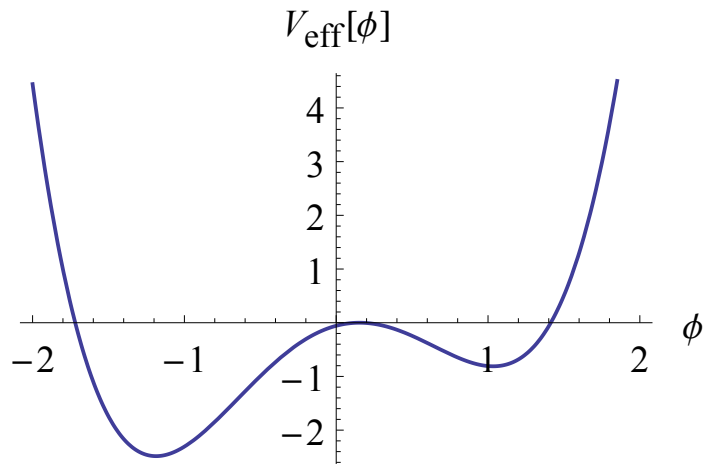


Figure 3.1: A simple potential with two minima. The true vacuum on the left and the false vacuum on the right.

Chapter 4

Abelian Higgs Model

Before we go on to study the Standard Model in the next chapter, we will have a look at a much simpler model with some interesting properties. We will study the Abelian Higgs model which is simple to study and has both symmetry breaking and a gauge symmetry. These two features are important in the Standard Model, and they will play an important role for our study in the next chapter.

The Abelian Higgs model was introduced in section 2.3.9.1 where we studied the symmetry breaking for $m^2 > 0$ in the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi|^2 + m^2|\phi|^2 - \frac{\lambda}{3!}|\phi|^4, \quad (4.1)$$

where $D_\mu = \partial_\mu + ieA_\mu$. In section 2.3.9.1 we chose the unitary gauge, $\pi(x) = 0$, but we will now not make this gauge choice. Instead we will work in the R_ξ gauges without choosing a specific value for ξ .

We will start by setting up the calculation using the background field method and sketch out how Coleman and Weinberg [15] computed the effective potential using Feynman diagrams. We then present Jackiw's functional method [13], and by modifying this method we find a simple way of computing the effective potential to 1-loop that works the same way independently of gauge choice and gauge fixing condition. We will then renormalize the effective potential and reproduce Coleman and Weinberg's result, in a general R_ξ gauge, showing that the theory even in the massless limit will be spontaneously broken by radiative corrections.

In section 4.3 will discuss different ways of gauge fixing and why these choices are made, including a special comment on one way of gauge fixing that is frequently used in the literature. In section 4.4 we will discuss the resummed effective potential for massless scalar QED, and then we study the gauge dependence of this theory numerically in section 4.5.

Much of the analysis done in this chapter is made analogous to the study we will perform for the Standard Model in the next chapter. Since the Abelian Higgs model is a much simpler theory to study, we will use this chapter to introduce the concepts that will be the key ingredients in chapter 5.

4.1 Calculating the 1-loop Effective Potential

Starting with the Lagrangian in eq. (4.1) we choose to fix the gauge with the R_ξ gauges $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A_\mu)^2$ as described in section 2.3.5. The gauge fixed Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 + |D_\mu\phi|^2 + m^2|\phi|^2 - \frac{\lambda}{3!}|\phi|^4, \quad (4.2)$$

where $D_\mu = \partial_\mu + ieA_\mu$. The ghost fields decouple from the theory with this gauge fixing condition, so there is no need to include them here¹. For simplicity we will write $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and work with two real scalar fields instead of one complex, and the Lagrangian can be written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}A_\mu \left[g_{\mu\nu}\square - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu \right] A_\nu + \frac{1}{2}e^2 A_\mu^2 \phi^2 + eA_\mu \varepsilon_{ab} \phi_a \partial_\mu \phi_b \\ & - \frac{1}{2}\phi_a \square \phi_a + \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!}\phi^4, \end{aligned} \quad (4.3)$$

where $\phi^2 = \phi_1^2 + \phi_2^2$, $a \in 1, 2$ and $\varepsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{ab}$. We can now write the gauge fixed classical action as

$$\begin{aligned} S[\phi, A_\mu] = & \int d^4x d^4y \left[-\frac{1}{2}A_\mu(x) i\Delta_{\mu\nu}^{-1}(x-y) A_\nu(y) - \frac{1}{2}\phi_a(x) iD_{ab}^{-1}(x-y)\phi_b(y) \right] \\ & + \int d^4x \left[\frac{1}{2}e^2 A_\mu(x)^2 \phi(x)^2 + eA_\mu(x)\varepsilon_{ab}\phi_a(x)\partial_\mu\phi_b(x) - \frac{\lambda}{4!}(\phi(x))^4 \right], \end{aligned} \quad (4.4)$$

where we have defined

$$\begin{aligned} i\Delta_{\mu\nu}^{-1}(x-y) &= -\delta^4(x-y) \left[g_{\mu\nu}\square - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu \right], \\ iD_{ab}^{-1}(x-y) &= -\delta^4(x-y)\delta_{ab} [m^2 - \square]. \end{aligned} \quad (4.5)$$

¹The decoupled ghosts will only contribute to an overall constant in the effective potential, so they will not play an important role in our study.

4.1.1 Background Fields

Given the action in eq. (4.4), we start by expanding $S[\phi, A_\mu] \rightarrow S[\phi + \hat{\phi}, A_\mu]$ where we take $\hat{\phi}$ to be a constant background field as described in section 3.2. We want to compute the effective potential to 1-loop order, so the relevant terms will be the ones with two propagating fields. Anything else will only contribute to higher order corrections. Expanding eq. (4.4) term by term, we find that the relevant terms are

$$\begin{aligned}
 \frac{1}{2}\phi_a(x)iD_{ab}^{-1}(x-y)\phi_b(y) &\longrightarrow \frac{1}{2}\hat{\phi}_a iD_{ab}^{-1}(x-y)\hat{\phi}_b \\
 &\quad + \frac{1}{2}\phi_a(x)iD_{ab}^{-1}(x-y)\phi_b(y), \\
 \frac{1}{2}e^2 A_\mu(x)^2 \phi(x)^2 &\longrightarrow \frac{1}{2}e^2 A_\mu(x)^2 \hat{\phi}_a \hat{\phi}_a, \\
 eA_\mu(x)\varepsilon_{ab}\phi_a(x)\partial_\mu\phi_b(x) &\longrightarrow eA_\mu(x)\varepsilon_{ab}\hat{\phi}_a\partial_\mu\phi_b(x) \\
 \frac{\lambda}{4!}(\phi_a(x)\phi_a(x))^2 &\longrightarrow \frac{2\lambda}{4!}(\hat{\phi}_a\hat{\phi}_a)(\phi_b(x)\phi_b(x)) \\
 &\quad + \frac{\lambda}{3!}(\hat{\phi}_a\phi_a(x))(\hat{\phi}_b\phi_b(x)) + \frac{\lambda}{4!}\hat{\phi}^4,
 \end{aligned} \tag{4.6}$$

where we have explicitly left out the x dependence in the background fields since they are constant. We can now write the action, including only 1-loop terms, as

$$\begin{aligned}
 S[\phi + \hat{\phi}, A_\mu] &= S[\hat{\phi}]_{\text{tree}} - \int d^4x d^4y \left[\frac{1}{2}A_\mu(x)i\bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; x, y)A_\nu(y) \right. \\
 &\quad \left. + \frac{1}{2}\phi_a i\bar{D}_{ab}^{-1}(\hat{\phi}; x, y)\phi_b(y) + A_\mu(x)M_a^\mu(\hat{\phi}; x, y)\phi_a(y) \right],
 \end{aligned} \tag{4.7}$$

where

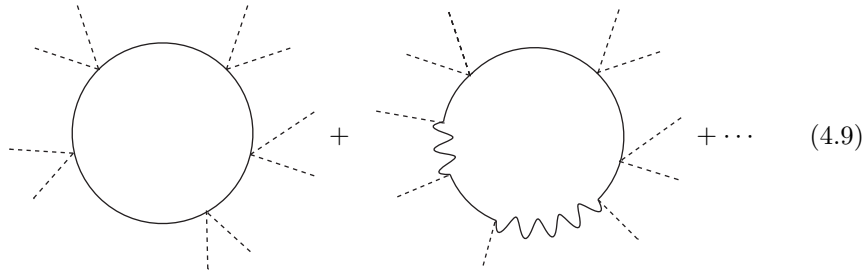
$$\begin{aligned}
 S[\hat{\phi}]_{\text{tree}} &= \frac{1}{2} \int d^4x d^4y \hat{\phi}_a iD_{ab}^{-1}(x-y)\hat{\phi}_b - \mathcal{V}_4 \frac{\lambda}{4!} \hat{\phi}^4, \\
 i\bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; x, y) &= i\Delta_{\mu\nu}^{-1}(x-y) - e^2 \hat{\phi}^2 g_{\mu\nu} \delta^4(x-y), \\
 i\bar{D}_{ab}^{-1}(\hat{\phi}; x, y) &= iD_{ab}^{-1}(x-y) + \lambda \left[\frac{1}{6} \hat{\phi}^2 \delta_{ab} + \frac{1}{3} \hat{\phi}_a \hat{\phi}_b \right] \delta^4(x-y), \\
 M_a^\mu(\hat{\phi}; x, y) &= -\delta^4(x-y) \left[e\varepsilon_{ab} \hat{\phi}_b \partial_\mu \right].
 \end{aligned} \tag{4.8}$$

4.1.2 Feynman Diagrams

In Coleman and Weinberg's original paper [15], they found the effective potential using diagrams following the same procedure as we did in section 3.3.1.1 and

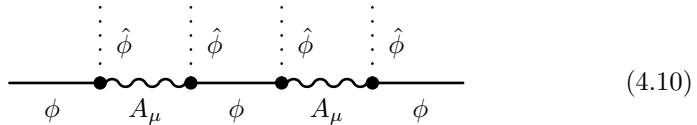
4.1. CALCULATING THE 1-LOOP EFFECTIVE POTENTIAL

3.3.1.2. We will now not only have ϕ propagating in the loop, but we also have to add up all the diagrams with A_μ in the loop. What makes this very complicated is the term $A_\mu(x)M_a^\mu(\hat{\phi}; x, y)\phi_a(y)$ in eq. (4.7). For example, the diagram with 10 external $\hat{\phi}$ we will have diagrams like



$$(4.9)$$

and we have to sum up all of them with the right symmetry factors. It can probably be done in principle, but we will not make an attempt to compute them here. The kind of diagrams where two different fields mix like



$$(4.10)$$

is typically referred to as kinetic mixing.

Coleman and Weinberg did their calculation in the Landau gauge, $\xi = 0$ or equivalently $\partial_\mu A_\mu = 0$, where we see that $A_\mu(x)M_a^\mu(\hat{\phi}; x, y)\phi_a(y) = 0$ by using the derivative in M_a^μ to integrate by parts. Hence, there is no kinetic mixing in the Landau gauge, and summing up all the diagrams becomes significantly easier. We will not reproduce Coleman and Weinberg's calculation here since we will do it another way (without choosing a value for x_i) shortly, but it is in principle just the same calculation as we did in section 3.3.1.2 with another sum with just photons in the loop. The result is given in [15]

$$V_{\text{eff}}[\phi] = \frac{1}{4!}\phi^4 \left[\lambda_R + \frac{1}{8\pi^2} \left(9e_R^4 + \frac{5}{6}\lambda_R^2 \right) \left(\ln \frac{\phi^2}{\phi_R^2} - \frac{25}{6} \right) \right], \quad (4.11)$$

where the renormalization conditions $V'''[0] = 0$ and $V''''[\phi_R] = \lambda_R$ have been used. We will get back to the interpretation of this potential after we have reproduced the result without choosing a gauge, i.e. a value for ξ .

4.1.3 Evaluating the Path Integral

We will now use the path integral to evaluate the effective potential using the background field method as described in section 3.3.1.4. The integral is of the

form

$$e^{i\Gamma[\hat{\phi}]_{1\text{-loop}}} = \int \mathcal{D}\phi \mathcal{D}A e^{iS[\phi + \hat{\phi}, A_\mu]_{1\text{-loop}}}, \quad (4.12)$$

with $S[\phi + \hat{\phi}, A_\mu]_{1\text{-loop}}$ given in eq. (4.7). This calculation was first performed by Jackiw in 1974 [13] and we will quickly review his work, and then we will do the calculation using a simplifying trick.

Jackiw's Approach

Schematically the integral we need to perform is

$$\mathcal{I} = \int \mathcal{D}\phi \mathcal{D}A e^{-\left(\frac{1}{2}A\Delta A + \frac{1}{2}\phi D\phi + AM\phi\right)} \quad (4.13)$$

where we have dropped the integral over x and y for simplicity. Performing the integral over ϕ , treating $AM = J$ as a current, we find using eq. (2.7)

$$\begin{aligned} \mathcal{I} &= \int \mathcal{D}A e^{-\frac{1}{2}A\Delta A} \frac{N}{\sqrt{\text{Det}D}} e^{\frac{1}{2}AMD^{-1}MA} \\ &= \int \mathcal{D}A e^{-\frac{1}{2}A[\Delta - MD^{-1}M]A} \frac{N}{\sqrt{\text{Det}D}} \end{aligned} \quad (4.14)$$

up to some constant N . We can now perform the integral over A

$$\mathcal{I} = N \frac{1}{\sqrt{\text{Det}D} \sqrt{\text{Det}[\Delta - MD^{-1}M]}}. \quad (4.15)$$

It turns out that performing these two functional determinants is a lengthy and somewhat messy process. Instead we will now present the approach we will take.

New Approach

The idea we will present here is very general, so we will give a discussion in general notation, independent of field theory.

Let $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^m$ for $n, m \in \mathbb{N}$. Consider the integral

$$\mathcal{I} = \int d\vec{x} d\vec{y} e^{-\left(\frac{1}{2}\vec{x}A\vec{x} + \frac{1}{2}\vec{y}B\vec{y} + \vec{x}C\vec{y}\right)}. \quad (4.16)$$

Instead of performing one integral at the time, like Jackiw did, we now construct a new vector $z \in \mathbb{R}^{n+m}$ with

$$\vec{z} = \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}, \quad M = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}, \quad (4.17)$$

such that

$$\mathcal{I} = \int d\bar{z} e^{-\frac{1}{2}\bar{z}M\bar{z}} = N \frac{1}{\sqrt{\text{Det}M}}. \quad (4.18)$$

So instead of doing two integrals and a lot of algebra, we can just construct a new vector and a new matrix such that the whole calculation can be done in one step.

We will use this trick in the next section to evaluate the path integral for the Abelian Higgs model. We have not seen this done explicitly anywhere in the literature, but Kang [14] does write his calculation in terms of one combined vector. However he uses that (Assuming A is invertible)

$$\begin{aligned} \det|M| &= \begin{vmatrix} A & C \\ C^T & B \end{vmatrix} = \left| \begin{bmatrix} A & 0 \\ C^T & I \end{bmatrix} \begin{bmatrix} I & A^{-1}C \\ 0 & B - C^T A^{-1}C \end{bmatrix} \right| \\ &= \det|A| \det|B - C^T A^{-1}C|, \end{aligned} \quad (4.19)$$

effectively rewriting this calculation back to Jackiw's result in eq. (4.15). We will however not do the same rewriting as Kang, but we will evaluate the full matrix determinant.

I believe that the reason why Kang and Jackiw is breaking down the calculation into smaller pieces is because computing a 6×6 determinant by hand is much more complicated than a 4×4 determinant. This is only true if we have to do the calculation by hand, and with today's computer technology we find that it's easier to do the whole calculation in one step. We believe that this is well known for those who do these calculations nowadays, but we have not seen it anywhere in the literature.

Note that this method works for any choice of gauge and gauge fixing. We will see in section 4.3 that many people have chosen other gauge fixing conditions to get rid of the kinetic mixing in order to simplify the evaluation of the path integral. With this method we find that evaluating the path integral is the same amount of work independently of the gauge fixing choice.

Evaluating the Path Integral

We will now construct a new field Φ consisting of the ϕ_a and A_μ fields and explicitly construct the new 6×6 matrix. Then we will evaluate the determinant and find the effective potential to 1-loop order.

Working in position space is less convenient in this case, so let's start by

Fourier transforming our Lagrangian. We transform

$$\begin{aligned}
 A_\mu(x) &= \int \frac{d^4 p}{(2\pi)^4} e^{ipx} A_\mu(p), \\
 \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; x-y) &= \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; p), \\
 \phi_a(x) &= \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \phi_a(p), \\
 \bar{D}_{ab}^{-1}(\hat{\phi}; x-y) &= \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \bar{D}_{ab}^{-1}(\hat{\phi}; p),
 \end{aligned} \tag{4.20}$$

and find the 1-loop action in eq. (4.7) can be written as

$$\begin{aligned}
 S[\phi + \hat{\phi}, A_\mu]_{1\text{-loop}} &= - \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \left[A_\mu(-p) i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; p) A_\nu(p) + \phi_a(-p) i \bar{D}_{ab}^{-1}(\hat{\phi}; p) \phi_b(p) \right. \\
 &\quad \left. + A_\mu(p) M_a^\mu(\hat{\phi}; -p) \phi_a(-p) + A_\mu(-p) M_a^\mu(\hat{\phi}; p) \phi_a(p) \right].
 \end{aligned} \tag{4.21}$$

Now we define the new field $\Phi(p)$ and the matrix $\Sigma(\hat{\phi}; p)$ as

$$\Phi(p) = \begin{bmatrix} \phi_a(p) \\ A_\mu(p) \end{bmatrix}, \quad \Sigma(\hat{\phi}; p) = \begin{bmatrix} i \bar{D}_{ab}^{-1}(\hat{\phi}; p) & M_a^\mu(\hat{\phi}; p) \\ (M_a^\mu(\hat{\phi}; -p))^T & i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; p) \end{bmatrix}, \tag{4.22}$$

and we find that we can write

$$S[\phi + \hat{\phi}, A_\mu]_{1\text{-loop}} = - \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \Phi(-p) \Sigma(\hat{\phi}, p) \Phi(p). \tag{4.23}$$

Performing the path integral as in eq. (2.45) tells us that to find the effective potential we simply have to evaluate the matrix determinant $\det \Sigma(\hat{\phi}, p)$ and integrate over all possible momentum p . The components of $\Sigma(\hat{\phi}; p)$ are in momentum space

$$\begin{aligned}
 i \bar{D}_{ab}^{-1}(\hat{\phi}; p) &= -(p^2 + m^2) \delta_{ab} + \frac{\lambda}{6} \hat{\phi}^2 \delta_{ab} + \frac{\lambda}{3} \hat{\phi}_a \hat{\phi}_b, \\
 i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}; p) &= \left[g_{\mu\nu} p^2 - \left(1 - \frac{1}{\xi} \right) p_\mu p_\nu \right] - e^2 \hat{\phi}^2 g_{\mu\nu}, \\
 M_a^\mu(\hat{\phi}; p) &= ip^\mu e \varepsilon_{ab} \hat{\phi}_b.
 \end{aligned} \tag{4.24}$$

Evaluating the matrix determinant we find that

$$\begin{aligned} \det\Sigma(\hat{\phi}; p) &= -\frac{1}{\xi} \left[p^2 - e^2 \hat{\phi}^2 \right]^3 \left[p^2 + m^2 - \frac{1}{2} \lambda \hat{\phi}^2 \right] \\ &\quad \times \left[p^4 + p^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) - e^2 \xi \hat{\phi}^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) \right]. \end{aligned} \quad (4.25)$$

Notice that we have now done the calculation in $d = 4$ dimensions. When we want to perform the integrals using dim-reg, we will be interested in the result in a general dimension d . The result is

$$\begin{aligned} \det\Sigma(\hat{\phi}; p) &= -\frac{1}{\xi} \left[p^2 - e^2 \hat{\phi}^2 \right]^{d-1} \left[p^2 + m^2 - \frac{1}{2} \lambda \hat{\phi}^2 \right] \\ &\quad \times \left[p^4 + p^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) - e^2 \xi \hat{\phi}^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) \right], \end{aligned} \quad (4.26)$$

and the only thing that is different is the power of the first factor.

Using eq. (2.45) and eq. (3.67) we find

$$\begin{aligned} V_{\text{eff}}[\hat{\phi}] &= V[\hat{\phi}]_{\text{tree}} + V[\hat{\phi}]_{\text{1-loop}} \\ &= V[\hat{\phi}]_{\text{tree}} - \frac{i}{2} \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln \det\Sigma(\hat{\phi}; p) \end{aligned} \quad (4.27)$$

where we have multiplied by a factor of μ^{4-d} to make the whole expression have the right mass dimension in any space-time dimension d . The two terms are

$$V[\hat{\phi}]_{\text{tree}} = -\frac{1}{2} m^2 \hat{\phi}^2 + \frac{\lambda}{4!} \hat{\phi}^4 \quad (4.28)$$

$$\begin{aligned} V[\hat{\phi}]_{\text{1-loop}} &= -\frac{i}{2} \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln \det\Sigma(\hat{\phi}; p) \\ &= -\frac{i}{2} \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \left\{ (d-1) \ln \left[p^2 - e^2 \hat{\phi}^2 \right] + \ln \left[p^2 + m^2 - \frac{1}{2} \lambda \hat{\phi}^2 \right] \right. \\ &\quad \left. + \ln \left[p^4 + p^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) - e^2 \xi \hat{\phi}^2 \left(m^2 - \frac{1}{6} \lambda \hat{\phi}^2 \right) \right] + \ln \left[-\frac{1}{\xi} \right] \right\}. \end{aligned} \quad (4.29)$$

4.1.4 Calculation of Log Integrals

In eq. (4.29) we see that the integrals we have to do are of the following form

$$\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln(p^2 - A), \quad \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln(p^4 - Ap^2 + B), \quad (4.30)$$

and we will now compute them in general using dim-reg with $d = 4 - \epsilon$.

We rewrite the first integral as

$$\mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \ln(p^2 - A) = \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \ln\left(1 - \frac{A}{p^2}\right) + C_0 \quad (4.31)$$

where C_0 is some arbitrary infinite constant we will ignore since it will be removed by counter terms. We then find after Wick rotating eq. (4.31)

$$\begin{aligned} i\mu^\epsilon \int \frac{d^d p_E}{(2\pi)^d} \ln\left(1 + \frac{A}{p_E^2}\right) &= \frac{i\mu^\epsilon 2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right) (2\pi)^d} \int d p_E p_E^{d-1} \ln\left(1 + \frac{A}{p_E^2}\right) \\ &= \frac{i\mu^\epsilon 2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right) (2\pi)^d} \frac{A^{\frac{d}{2}} \pi \csc \frac{d\pi}{2}}{d} \\ &= -\frac{iA^2}{16\pi^2} \frac{1}{\epsilon} + \frac{i}{64\pi^2} 2A^2 \left(\ln \frac{A}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) \end{aligned} \quad (4.32)$$

where we have Wick rotated and expanded for small ϵ . In the case where this integral is multiplied by $d - 1$ we find

$$\begin{aligned} (d-1)i\mu^\epsilon \int \frac{d^d p_E}{(2\pi)^d} \ln\left(1 + \frac{A}{p_E^2}\right) \\ = -\frac{3iA^2}{16\pi^2} \frac{1}{\epsilon} + \frac{i}{64\pi^2} 6A^2 \left(\ln \frac{A}{4\pi\mu^2 e^{-\gamma_E}} - \frac{5}{6} \right) \end{aligned} \quad (4.33)$$

To do the second integral we start by Wick rotating and factorizing

$$\mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \ln(p^4 - Ap^2 + B) = i\mu^\epsilon \int \frac{d^d p_E}{(2\pi)^d} \ln[(p_E^2 + C_+)(p_E^2 + C_-)], \quad (4.34)$$

where $C_\pm = \frac{1}{2}(A \pm \sqrt{A^2 - 4B})$. Using the result from eq. (4.32) we find

$$\begin{aligned} \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \ln(p^4 - Ap^2 + B) &= -\frac{i(C_+^2 + C_-^2)}{16\pi^2} \frac{1}{\epsilon} \\ &+ \frac{i}{64\pi^2} 2C_+^2 \left(\ln \frac{C_+}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) \\ &+ \frac{i}{64\pi^2} 2C_-^2 \left(\ln \frac{C_-}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) \end{aligned} \quad (4.35)$$

4.1.5 The Unrenormalized Abelian Higgs Effective Potential

With the general solutions to our integrals we can now find the effective potential for the Abelian Higgs model. We start with the tree level potential given in eq.

(4.28), and we rewrite it in terms of the renormalized couplings and counterterms

$$V[\hat{\phi}]_{\text{tree}} = \Lambda + \delta\Lambda - \frac{1}{2}(m_R^2 + \delta m)\hat{\phi}^2 + \frac{\lambda_R + \delta\lambda}{4!}\hat{\phi}^4, \quad (4.36)$$

where we have also included a cosmological constant and the corresponding counterterm since we only have computed the potential up to a constant. The 1-loop corrections in eq. (4.29) with the integrals in section 4.1.4 are

$$\begin{aligned} V[\hat{\phi}]_{1\text{-loop}} = & -\frac{1}{32\pi^2} \frac{1}{\epsilon} \left[3m_A^4 + m_B^4 + m_{C_+}^4 + m_{C_-}^4 \right] \\ & + \frac{1}{64\pi^2} \left[3m_A^4 \left(\ln \frac{m_A^2}{4\pi\mu^2 e^{-\gamma_E}} - \frac{5}{6} \right) + m_B^4 \left(\ln \frac{m_B^2}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) \right. \\ & \left. + m_{C_+}^4 \left(\ln \frac{m_{C_+}^2}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) + m_{C_-}^4 \left(\ln \frac{m_{C_-}^2}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2} \right) \right] \end{aligned} \quad (4.37)$$

where we have defined

$$\begin{aligned} m_A^2 &= e_R^2 \hat{\phi}^2, \\ m_B^2 &= \frac{1}{2} \lambda_R \hat{\phi}^2 - m_R^2, \\ m_{C_\pm}^2 &= \frac{1}{2} \left[\left(\frac{1}{6} \lambda_R \hat{\phi}^2 - m_R^2 \right) \pm \sqrt{\left(\frac{1}{6} \lambda_R \hat{\phi}^2 - m_R^2 \right)^2 - 4e_R^2 \xi \left(\frac{1}{6} \lambda_R \hat{\phi}^2 - m_R^2 \right) \hat{\phi}^2} \right]. \end{aligned} \quad (4.38)$$

We have here expressed the couplings in terms of the renormalized couplings and dropped the counter terms since they will be higher order than 1-loop. We will in general also renormalize the field strength $\hat{\phi} \rightarrow Z_\phi^{\frac{1}{2}} \hat{\phi}$, but we have not written it out explicitly since it won't be needed to cancel any infinities in the next section. But when we consider the RG improved effective potential in section 4.4 we will include it since the field strength will run with scale μ .

4.2 Renormalization

In this section we will use the counterterms to renormalize the Abelian Higgs model in eq. (4.37). We will first show that we can use $\overline{\text{MS}}$ to remove all the infinities, and we will then look at the massless limit and reproduce Coleman and Weinberg's result for a general value of ξ . We will then see how one goes about predicting the gauge independent scalar to vector mass ratio $\frac{m_S^2}{m_A^2}$ from the gauge dependent effective potential.

4.2.1 The Effective Potential in $\overline{\text{MS}}$

To see that all the infinities can be removed using the given counterterms, we start by simplifying the first term in eq. (4.37) that is proportional to $\frac{1}{\epsilon}$. We find

$$\begin{aligned}
 3m_A^4 + m_B^4 + m_{C_+}^4 + m_{C_-}^4 = & 2m_R^2 + \left[2e_R^2\xi - \frac{4}{3}\lambda \right] m_R^2 \hat{\phi}^2 \\
 & + \left[3e_R^4 + \frac{5}{18}\lambda_R^2 - \frac{1}{3}\lambda_R e_R^2 \xi \right] \hat{\phi}^4.
 \end{aligned} \tag{4.39}$$

We see that we have one constant, one term proportional to $\hat{\phi}^2$ and one term proportional to $\hat{\phi}^4$, so the counterterms δ_Λ , δ_m and δ_λ can be used with $\overline{\text{MS}}$ to cancel all the $\frac{1}{\epsilon}$ divergences and $4\pi e^{-\gamma_E}$ in the logarithms. The $\overline{\text{MS}}$ renormalized effective potential to 1-loop order becomes

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}] = & \Lambda - \frac{1}{2}m_R^2 \hat{\phi}^2 + \frac{\lambda_R}{4!} \hat{\phi}^4 \\
 & + \frac{1}{64\pi^2} \left[3m_A^4 \left(\ln \frac{m_A^2}{\mu^2} - \frac{5}{6} \right) + m_B^4 \left(\ln \frac{m_B^2}{\mu^2} - \frac{3}{2} \right) \right. \\
 & \left. + m_{C_+}^4 \left(\ln \frac{m_{C_+}^2}{\mu^2} - \frac{3}{2} \right) + m_{C_-}^4 \left(\ln \frac{m_{C_-}^2}{\mu^2} - \frac{3}{2} \right) \right],
 \end{aligned} \tag{4.40}$$

where m_A^2 , m_B^2 and $m_{C_\pm}^2$ are defined in eq. (4.38).

4.2.2 Massless Limit of the Abelian Higgs Model

In this section we will look at the case where $m_R = 0$, i.e. we are studying massless scalar QED, which is the same case Coleman and Weinberg studied in 1973. In this limit we find

$$m_A^2 = e_R^2 \hat{\phi}^2, \quad m_B^2 = \frac{1}{2}\lambda_R \hat{\phi}^2, \quad m_{C_\pm}^2 = \frac{1}{12} \hat{\phi}^2 \left[\lambda_R \pm \sqrt{\lambda_R^2 - 24e_R^2 \xi \lambda_R} \right]. \tag{4.41}$$

Substituting this into eq. (4.40) we find

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}] = & \Lambda + \delta_\Lambda + \frac{1}{2}\delta_m \hat{\phi}^2 + \frac{\lambda_R + \delta\lambda}{4!} \hat{\phi}^4 + f(e_R, \lambda_R) \hat{\phi}^4 \\
 & + \frac{1}{64\pi^2} \hat{\phi}^4 \left[3e_R^4 + \frac{5}{18}\lambda_R^2 - \frac{1}{3}e_R^2 \lambda_R \xi \right] \ln \frac{\hat{\phi}^2}{\mu^2},
 \end{aligned} \tag{4.42}$$

4.2. RENORMALIZATION

where we have included the counterterms that have absorbed the $\frac{1}{\epsilon}$ infinities, and $f(e_R, \lambda_R)$ is defined as

$$\begin{aligned}
 f(e_R, \lambda_R) = & \frac{1}{64\pi^2} \left[\frac{1}{2} e_R^2 \lambda_R \xi - \frac{5}{2} e_R^4 - \frac{5}{12} \lambda_R^2 + 3e_R^4 \ln e_R^2 + \frac{1}{4} \lambda_R^2 \ln \frac{\lambda_R}{2} \right. \\
 & + \frac{1}{144} \left(\lambda_R + \sqrt{\lambda_R^2 - 24e_R^2 \xi \lambda_R} \right)^2 \ln \left(\lambda_R + \sqrt{\lambda_R^2 - 24e_R^2 \xi \lambda_R} \right) \\
 & \left. + \frac{1}{144} \left(\lambda_R - \sqrt{\lambda_R^2 - 24e_R^2 \xi \lambda_R} \right)^2 \ln \left(\lambda_R - \sqrt{\lambda_R^2 - 24e_R^2 \xi \lambda_R} \right) \right]. \quad (4.43)
 \end{aligned}$$

We will see that we don't really need to know the functional form for $f(e_R, \lambda_R)$ after we apply the renormalization conditions. The conditions Coleman and Weinberg used were

$$V_{\text{eff}}[0] = 0, \quad V_{\text{eff}}''[0] = 0, \quad V_{\text{eff}}''''[\phi_R] = \lambda_R, \quad (4.44)$$

where $\phi_R \neq 0$ is some reference scale used since the fourth derivative is singular at $\hat{\phi} = 0$. Applying these conditions we find

$$V_{\text{eff}}[\hat{\phi}] = \frac{1}{4!} \hat{\phi}^4 \left[\lambda_R + \frac{1}{8\pi^2} \left(9e_R^4 + \frac{5}{6} \lambda_R^2 - e_R^2 \lambda_R \xi \right) \left(\ln \frac{\hat{\phi}^2}{\phi_R^2} - \frac{25}{6} \right) \right], \quad (4.45)$$

which in the Landau gauge $\xi = 0$ reproduces Coleman and Weinberg's result in eq. (4.11).

We see that the effective potential comes out gauge dependent, and you should ask yourself where the ξ dependence comes from. Jackiw [13] pointed out that the source of the gauge dependence is the fact that the calculation of the effective potential only includes 1PI Feynman diagrams. If we were to compute the corresponding S -matrix element for a 4-point scattering, we would include the wave function renormalization diagrams

$$(4.46)$$

which would contribute to making the S -matrix element gauge independent as we saw in section 2.3.7. But since it is not 1PI, it will not be included into the effective potential.

There has been a lot of confusion about how to understand this gauge dependence of the effective potential. One paper by Frere and Nicoletopoulos [62] that we found claimed that all the gauge dependence of the effective potential could be removed by a redefinition of the scalar field strength. Our understanding of the gauge dependence is the opposite: The gauge dependence in the effective potential comes from the running of the scalar field strength. We will see this clearly in a calculation of the 1-loop effective potential from the beta functions in section 4.4.1.

4.2.3 Symmetry Breaking by Radiative Corrections

We have found the effective potential in eq. (4.45) with gauge dependence, and we would like to ask is whether or not we can extract any physical information from the potential. Physical quantities computed from the effective potential should still be gauge invariant, and in this section we will see that this happens when we are careful about the orders of the parameters, only keeping the terms to the order we are working. This section follows the discussion in Schwartz [19].

First note that in the case where $m_R = 0$, there is no spontaneous symmetry breaking in the tree level part of the effective potential. We define the minimum of the 1-loop effective potential $\langle\phi\rangle$ such that $V'_{\text{eff}}[\langle\phi\rangle] = 0$. Taking the derivative of eq. (4.45) we find

$$\ln \frac{\langle\phi\rangle^2}{\phi_R^2} = \frac{11}{3} - \frac{48\pi^2\lambda_R}{5\lambda_R^2 + 54e_R^4 - 6e_R^2\lambda_R\xi}. \quad (4.47)$$

In the case where $e_R^4 \approx \lambda_R \ll 1$ this simplifies to

$$\ln \frac{\langle\phi\rangle^2}{\phi_R^2} = \frac{11}{3} - \frac{8\pi^2\lambda_R}{9e_R^4}. \quad (4.48)$$

Now we choose our reference scale $\phi_R = \langle\phi\rangle \neq 0$, or equivalently we define $\lambda_R \equiv V'''_{\text{eff}}[\langle\phi\rangle]$. Eq. (4.47) then gives us that

$$\lambda_R = \frac{33}{8\pi^2} e_R^4, \quad (4.49)$$

and our assumption that $e_R^4 \approx \lambda_R$ is self consistent. This result was first derived by Coleman and Weinberg [15]. Substituting eq. (4.49) into eq. (4.45) we find

$$V_{\text{eff}}[\hat{\phi}] = \frac{3e_R^4}{64} \hat{\phi}^4 \left[\ln \frac{\hat{\phi}^2}{\langle\phi\rangle^2} - \frac{1}{2} \right] + \mathcal{O}(e_R^6). \quad (4.50)$$

The important point to note here is that the theory which had no spontaneous symmetry breaking at tree level has acquired a non-zero vacuum expectation

value. The symmetry has been broken by the radiative corrections in the 1-loop effective potential.

Since the symmetry is broken we will find, following the same procedure as in section 2.3.9, a scalar with mass

$$m_S^2 = V''[\langle\phi\rangle] = \frac{3e_R^4}{8\pi^2} \langle\phi\rangle^2, \quad (4.51)$$

and the photon will acquire a mass

$$m_A^2 = e_R^2 \langle\phi\rangle^2. \quad (4.52)$$

Hence we predict that the ratio of the scalar mass to photon mass will be

$$\frac{m_S^2}{m_A^2} = \frac{3e_R^2}{8\pi^2}, \quad (4.53)$$

which is independent of $\langle\phi\rangle$ and ξ .

In the above argument we used eq. (4.49) and only kept terms up to $\mathcal{O}(e_R^4)$ in the effective potential, and we have seen that the gauge dependence drops out since $\xi e_R^2 \lambda_R = \mathcal{O}(e_R^6)$. But what would have happened if we did a 2-loop calculation and had to keep all the terms up to $\mathcal{O}(e_R^6)$? This calculation was performed by Kang [14], and the 2-loop calculation contains a ξe_R^6 term that exactly cancels the $\xi e_R^2 \lambda_R$ term from the 1-loop effective potential. Due to this cancellation, Kang found the mass ratio to 2-loops

$$\frac{m_S^2}{m_A^2} = \frac{3e_R^2}{8\pi^2} - \frac{61e_R^4}{768\pi^2}, \quad (4.54)$$

to be gauge independent.

4.2.4 Physical Renormalization of Abelian Higgs Model

We started the renormalization procedure by using $\overline{\text{MS}}$ with the Abelian Higgs model, and when we studied massless scalar QED we used the renormalization conditions given in eq. (4.44). These renormalization conditions are much more physical than $\overline{\text{MS}}$, and in massless scalar QED it gave us a simple expression for the effective potential in eq. (4.45). So why don't we use these renormalization conditions in the Abelian Higgs model? As we now will see, the expression we get for the effective potential is messy and it's easy to lose track of the physics. The only purpose of this section is to justify why we choose to use $\overline{\text{MS}}$ over other more physical renormalization schemes, besides the fact that most of the literature on the subject also uses $\overline{\text{MS}}$.

4.2.4.1 Renormalization of Toy Model

For the purpose of this section, consider the effective potential for some toy model

$$\begin{aligned}
 V_{\text{eff}}[\phi] = & \Lambda + \delta_\Lambda - \frac{1}{2} (m_R^2 + \delta_m) \phi^2 + \frac{\lambda_R + \delta\lambda}{4!} \phi^4 \\
 & + \frac{1}{64\pi^2} \left(\frac{1}{2} \lambda_R \phi^2 - m_R^2 \right)^2 \left[\ln \frac{\frac{1}{2} \lambda_R \phi^2 - m_R^2}{\mu^2} - \frac{3}{2} \right],
 \end{aligned} \tag{4.55}$$

which is the Abelian Higgs model from section 4.1.5 with only m_B^2 included².

Using the renormalization conditions

$$V_{\text{eff}}[\phi_R] = 0, \quad V''_{\text{eff}}[\phi_R] = m_R^2, \quad V''''_{\text{eff}}[\phi_R] = \lambda_R, \tag{4.56}$$

the renormalized effective potential becomes

$$\begin{aligned}
 V_{\text{eff}}[\phi] = & \frac{(\phi^2 - \phi_R^2)}{1536\pi^2 (\lambda_R \phi_R^2 - 2m_R^2)^2} \left[48m_R^6 (\lambda_R + 64\pi^2) \right. \\
 & - 4m_R^4 \lambda_R (9\lambda_R (\phi_R^2 + \phi^2) - 64\pi^2 (\phi^2 - 17\phi_R^2)) \\
 & + 4m_R^2 \lambda_R^2 \phi_R^2 (3\lambda_R (7\phi^2 - 20\phi_R^2) - 64\pi^2 (\phi^2 - 8\phi_R^2)) \\
 & \left. + \lambda_R^3 \phi_R^4 (\lambda_R (83\phi_R^2 - 25\phi^2) + 64\pi^2 (\phi^2 - 5\phi_R^2)) \right] \\
 & + \frac{(\frac{1}{2}\phi^2 \lambda_R - m_R^2)^2}{64\pi^2} \ln \left(\frac{\frac{1}{2}\lambda_R \phi^2 - m_R^2}{\frac{1}{2}\lambda_R \phi_R^2 - m_R^2} \right).
 \end{aligned} \tag{4.57}$$

You can check that this potential satisfies the renormalization conditions, and it's clear that we don't gain much from this renormalization scheme over using $\overline{\text{MS}}$ in terms of understanding the physical content of the effective potential. Remember that we only included one of the four terms from the Abelian Higgs 1-loop effective potential, so the real case is possibly at least four times worse than eq. (4.57).

4.3 Alternative Ways of Gauge Fixing

So far in this chapter we have only considered the Abelian Higgs model with the gauge fixing \mathcal{L}_{gf} and ghost term \mathcal{L}_{gh}

$$\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}} = -\frac{1}{2\xi} (\partial_\mu A_\mu)^2 - \bar{c} \square c. \tag{4.58}$$

²We have omitted m_A^2 and $m_{C_\pm}^2$ to simplify the calculation to make the point clearer. It gets too messy if we included them all.

As mentioned earlier, we have neglected the ghosts so far since they decouple from the theory.

One of the things that complicated the calculation of the effective potential was the kinetic mixing. Coleman and Weinberg chose a specific gauge, and Jackiw's functional method involves some further calculation to find V_{eff} . As it turns out, there is a way of getting the best of both worlds. By choosing a different gauge fixing function, we will see that it is possible to cancel the kinetic mixing and have one free gauge parameter ξ . However, with the new gauge fixing condition, the ghosts will not in general decouple from the theory.

4.3.1 New Gauge Fixing Condition

We start by defining a new gauge fixing condition as proposed by Kastening [63]

$$F[A_\mu, \phi_1, \phi_2] = \partial_\mu A_\mu + \xi e \phi_1 \phi_2. \quad (4.59)$$

Following the approach in section 2.3.5 we want to find the corresponding ghost term. Under an infinitesimal gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha, \\ \phi &\rightarrow \phi + i\alpha \phi, \end{aligned} \quad (4.60)$$

or equivalently

$$\begin{aligned} \phi_1 &\rightarrow \phi_1 \cos \alpha - \phi_2 \sin \alpha, \\ \phi_2 &\rightarrow \phi_2 \cos \alpha + \phi_1 \sin \alpha, \end{aligned} \quad (4.61)$$

and we find

$$\begin{aligned} &\left. \frac{\delta}{\delta \alpha} F \left[A_\mu - \frac{1}{e} \partial_\mu \alpha, \phi_1 \cos \alpha + \phi_2 \sin \alpha, \phi_2 \cos \alpha - \phi_1 \sin \alpha \right] \right|_{\alpha=0} \\ &= \left. \frac{\delta}{\delta \alpha} \left[\partial_\mu A_\mu - \frac{1}{e} \square \alpha + \xi e (\phi_1 \cos \alpha + \phi_2 \sin \alpha) (\phi_2 \cos \alpha - \phi_1 \sin \alpha) \right] \right|_{\alpha=0} \\ &= -\frac{1}{e} \square - \xi e (\phi_1^2 - \phi_2^2). \end{aligned} \quad (4.62)$$

After rescaling the ghost fields we find that the ghost and gauge fixing Lagrangian is

$$\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}} = -\frac{1}{2\xi} (\partial_\mu A_\mu + \xi e \phi_1 \phi_2)^2 + \partial_\mu \bar{c} \partial_\mu c - \xi e^2 (\phi_1^2 - \phi_2^2) \bar{c} c. \quad (4.63)$$

Note that this gauge fixing also breaks the global U(1) symmetry we had earlier for the complex scalar field. In section 4.1 we found that the effective potential in eq. (4.42) only depends on the scalar fields in the combination $\phi_1^2 + \phi_2^2$. This will no longer be the case since we have broken the global U(1) symmetry. Instead of using the background field method for the full complex field $\phi \rightarrow \phi + \hat{\phi}$, we must now choose a specific direction. We will choose to expand the real component $\phi_1 \rightarrow \phi_1 + \hat{\phi}_1$ and leave ϕ_2 unchanged.

As already mentioned, we see that the ghosts do not decouple from the theory with our new gauge fixing condition. They must now be included in the calculation of the effective potential.

4.3.1.1 Cancellation of Kinetic Mixing

If we expand the Abelian Higgs Lagrangian in eq. (4.1) using the background field method for $\phi_1 \rightarrow \phi_1 + \hat{\phi}_1$ we get one kinetic mixing term

$$\mathcal{L}^{\text{kin.mix.}} = e\partial_\mu A_\mu \hat{\phi}_1 \phi_2. \quad (4.64)$$

The same expansion of the cross term in \mathcal{L}_{gf} from eq. (4.63) is

$$\mathcal{L}_{\text{gf}}^{\text{mixing}} = -\frac{1}{2\xi} \partial_\mu A_\mu 2\xi e \hat{\phi}_1 \phi_2, \quad (4.65)$$

and we see that the kinetic mixing is completely removed

$$\mathcal{L}^{\text{kin.mix.}} + \mathcal{L}_{\text{gf}}^{\text{kin.mix.}} = 0. \quad (4.66)$$

4.3.1.2 The Effective Potential with New Gauge Fixing

We will now compute the effective potential with the gauge fixing condition in eq. (4.63). Using the background field method with $\phi_1 \rightarrow \phi_1 + \hat{\phi}_1$, we can in the same way as in eq. (4.7) write the action up to 1-loop terms as

$$S[\phi_1 + \hat{\phi}_1, \phi_2, A_\mu] = S[\hat{\phi}_1]_{\text{tree}} - \int d^4x d^4y \left[\frac{1}{2} A_\mu(x) i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; x, y) A_\nu(y) + \frac{1}{2} \phi_a i \bar{D}_{ab}^{-1}(\hat{\phi}_1; x, y) \phi_b(y) + \bar{c}(x) G(\hat{\phi}_1; x, y) c(y) \right], \quad (4.67)$$

where we have defined

$$\begin{aligned}
 S[\hat{\phi}_1]_{\text{tree}} &= -\mathcal{V}_4 \left[-\frac{1}{2}m^2\hat{\phi}_1^2 + \frac{\lambda}{4!}\hat{\phi}_1^4 \right], \\
 i\bar{D}_{ab}^{-1}(\hat{\phi}_1; p) &= -(p^2 + m^2)\delta_{ab} + \frac{\lambda}{2}\hat{\phi}_1^2\delta_{a1}\delta_{b1} + \left(\frac{\lambda}{6} + \xi e^2 \right) \hat{\phi}_1^2\delta_{a2}\delta_{b2}, \\
 i\bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; p) &= \left[g_{\mu\nu}p^2 - \left(1 - \frac{1}{\xi} \right) p_\mu p_\nu \right] - e^2\hat{\phi}_1^2 g_{\mu\nu}, \\
 G(\hat{\phi}_1; p) &= -\left(p^2 - \xi e\hat{\phi}_1^2 \right),
 \end{aligned} \tag{4.68}$$

and everything is listed in momentum space analogous to eq. (4.24).

To find the 1-loop effective potential we perform the path integral as in section 3.3.1.4, and we can evaluate the different fields independently since there is no kinetic mixing. Using eq. (2.7) twice and eq. (2.9) for the ghost fields, we find

$$e^{i\Gamma[\hat{\phi}_1]} = e^{iS[\hat{\phi}_1]} \frac{\text{Det} \left| G(\hat{\phi}_1; p) \right|}{\sqrt{\text{Det} \left| i\bar{D}_{ab}^{-1}(\hat{\phi}_1; p) \right| \text{Det} \left| i\bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; p) \right|}}, \tag{4.69}$$

where the matrix determinants are (in d dimensions)

$$\begin{aligned}
 \det \left| i\bar{D}_{ab}^{-1}(\hat{\phi}_1; p) \right| &= \left[p^2 - \left(\frac{\lambda}{2}\phi_1^2 - m^2 \right) \right] \left[p^2 - \left(\left(\frac{\lambda}{6} - e^2\xi \right) \phi_1^2 - m^2 \right) \right], \\
 \det \left| i\bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; p) \right| &= -\frac{1}{\xi} [p^2 - e^2\phi_1^2]^{d-1} [p^2 - e^2\xi\phi_1^2], \\
 \det \left| G(\hat{\phi}_1; p) \right| &= [p^2 - \xi e^2\phi_1^2].
 \end{aligned} \tag{4.70}$$

We see that all the integrals we have to perform are of the following form $\int \frac{d^d p}{(2\pi)^d} \ln [p^2 - M^2]$. Using the integrals given in section 4.1.4 we find that the effective potential to 1-loop using $\overline{\text{MS}}$ is

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}_1] &= \Lambda - \frac{1}{2}m_R^2\hat{\phi}_1^2 + \frac{\lambda_R}{4!}\hat{\phi}_1^4 \\
 &+ \frac{1}{64\pi^2} \left[3m_A^4 \left(\ln \frac{m_A^2}{\mu^2} - \frac{5}{6} \right) + m_B^4 \left(\ln \frac{m_B^2}{\mu^2} - \frac{3}{2} \right) \right. \\
 &\quad \left. + m_C^4 \left(\ln \frac{m_C^2}{\mu^2} - \frac{3}{2} \right) - m_D^4 \left(\ln \frac{m_D^2}{\mu^2} - \frac{3}{2} \right) \right],
 \end{aligned} \tag{4.71}$$

where Λ is an overall constant and we have defined

$$\begin{aligned}
 m_A^2 &= e_R^2 \hat{\phi}_1^2, \\
 m_B^2 &= \frac{1}{2} \lambda_R \hat{\phi}_1^2 - m_R^2, \\
 m_C^2 &= \left(\frac{1}{6} \lambda_R + e_R^2 \xi \right) \hat{\phi}_1^2 - m_R^2, \\
 m_D^2 &= \xi e_R^2 \hat{\phi}_1^2.
 \end{aligned} \tag{4.72}$$

Note the minus sign for the m_D^2 term in eq. (4.71). This comes from the fact that the ghost term determinant is in the numerator and not in the denominator. This result matches the one given by Kastening [63].

4.3.2 Comment on Gauge Fixing Used in the Literature

After having gone through many of the papers relevant to this research project, I have seen many different ways of describing how they choose to gauge fix their Lagrangian to cancel the kinetic mixing. I personally find the literature quite confusing at times, and in this section I would like to comment on a few key points I think should be emphasized to get a complete picture of the procedure of choosing a gauge fixing function.

The common feature of many papers [64, 65, 66, 67, 68, 69, 70, 17] (to mention a few), is that they choose the gauge fixing function along the lines of Kastening [63]

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} (\partial_\mu A_\mu + \xi e v \phi_2)^2 \tag{4.73}$$

where v is some constant parameter we can choose as we wish. The idea is that if we expand the Lagrangian around $\phi_1 \rightarrow \phi_1 + v$ the kinetic mixing will vanish. This is sometimes referred to as the 't Hooft R_ξ gauge [71, 72]. The corresponding ghost term will be

$$\mathcal{L}_{\text{gh}} = -\bar{c} \square c - \xi e^2 v \phi_1 \bar{c} c. \tag{4.74}$$

For us, the problem with this approach is that if we wish to use functional methods³ to find the effective potential, we have to use the background field method expanding around $\phi_1 \rightarrow \phi_1 + \hat{\phi}_1$. This will again generate a kinetic mixing term $\mathcal{L}^{\text{kin.mix.}} = e \partial_\mu A_\mu \hat{\phi}_1 \phi_2$, and we are back to where we started.

With the background field method described in section 4.1 we can easily compute the effective potential with the gauge fixing given in eq. (4.73). Shifting

³If you would like to compute diagrams to find the effective potential there is no need to introduce the background field, and the following discussion does not apply.

4.3. ALTERNATIVE WAYS OF GAUGE FIXING

$\phi_1 \rightarrow \phi_1 + \hat{\phi}_1$, the action up to 1-loop terms are

$$\begin{aligned}
 S[\phi_1 + \hat{\phi}_1, \phi_2, A_\mu] = & S[\hat{\phi}_1]_{\text{tree}} - \int d^4x d^4y \left[\frac{1}{2} A_\mu(x) i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; x, y) A_\nu(y) \right. \\
 & + \frac{1}{2} \phi_a i \bar{D}_{ab}^{-1}(\hat{\phi}_1; x, y) \phi_b(y) + A_\mu(x) M_a^\mu(\hat{\phi}_1; x, y) \phi_a(y) \\
 & \left. + \bar{c}(x) G(\hat{\phi}_1; x, y) c(y) \right], \quad (4.75)
 \end{aligned}$$

where we have defined

$$\begin{aligned}
 S[\hat{\phi}_1]_{\text{tree}} = & -\mathcal{V}_4 \left[-\frac{1}{2} m^2 \hat{\phi}_1^2 + \frac{\lambda}{4!} \hat{\phi}_1^4 \right], \\
 i \bar{D}_{ab}^{-1}(\hat{\phi}_1; p) = & -(p^2 + m^2) \delta_{ab} + \frac{\lambda}{2} \hat{\phi}_1^2 \delta_{a1} \delta_{b1} + \left(\frac{\lambda}{6} \hat{\phi}_1^2 + \xi e^2 v^2 \right) \delta_{a2} \delta_{b2}, \\
 i \bar{\Delta}_{\mu\nu}^{-1}(\hat{\phi}_1; p) = & \left[g_{\mu\nu} p^2 - \left(1 - \frac{1}{\xi} \right) p_\mu p_\nu \right] - e^2 \hat{\phi}_1^2 g_{\mu\nu}, \\
 M_a^\mu(\hat{\phi}_1; p) = & i p^\mu e \left(\hat{\phi}_1 - v \right) \delta_{a2}, \\
 G(\hat{\phi}_1; p) = & - \left(p^2 - \xi e v \hat{\phi}_1 \right), \quad (4.76)
 \end{aligned}$$

and everything is listed in momentum space analogous to eq. (4.24). As in eq. (4.22) we define a new field $\Phi(p)$ and the matrix $\Sigma(\hat{\phi}; p)$, and evaluating the determinant of $\Sigma(\hat{\phi}; p)$ and $G(\hat{\phi}; p)$ we find the $\overline{\text{MS}}$ renormalized effective potential to be

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}_1] = & \Lambda - \frac{1}{2} m_R^2 \hat{\phi}_1^2 + \frac{\lambda_R}{4!} \hat{\phi}_1^4 + \frac{1}{64\pi^2} \left[3m_A^4 \left(\ln \frac{m_A^2}{\mu^2} - \frac{5}{6} \right) \right. \\
 & + m_B^4 \left(\ln \frac{m_B^2}{\mu^2} - \frac{3}{2} \right) + m_{C^+}^4 \left(\ln \frac{m_{C^+}^2}{\mu^2} - \frac{3}{2} \right) \\
 & \left. + m_{C^-}^4 \left(\ln \frac{m_{C^-}^2}{\mu^2} - \frac{3}{2} \right) - 2m_D^4 \left(\ln \frac{m_D^2}{\mu^2} - \frac{3}{2} \right) \right], \quad (4.77)
 \end{aligned}$$

where Λ is an overall constant and we have defined

$$\begin{aligned}
 m_A^2 &= e_R^2 \hat{\phi}_1^2, \\
 m_B^2 &= \frac{1}{2} \lambda_R \hat{\phi}_1^2 - m_R^2, \\
 m_{C_\pm}^2 &= \frac{1}{2} \left(\frac{\lambda_R}{6} \hat{\phi}_1^2 - m_R^2 \right) + \xi e_R^2 v \hat{\phi}_1 \\
 &\quad \pm \frac{1}{2} \sqrt{\left(\frac{\lambda_R}{6} \hat{\phi}_1^2 - m_R^2 \right) \left[\left(\frac{\lambda_R}{6} \hat{\phi}_1^2 - m_R^2 \right) + 4\xi e_R^2 \hat{\phi}_1 (v - \hat{\phi}_1) \right]}, \\
 m_D^2 &= \xi e_R^2 v \hat{\phi}_1.
 \end{aligned} \tag{4.78}$$

The potential in eq. (4.77) is a function of the field $\hat{\phi}_1$ for some fixed constant v . Choosing $\hat{\phi}_1 = v$ will reproduce eq. (4.71), but notice that this only means that the two results are the same for one value of $\hat{\phi}_1 = v$, since v is a constant. You might be tempted to say that we can choose $v = \hat{\phi}_1$ when writing down the gauge fixing function, and some papers actually discuss using the background field in the gauge fixing term [73]. However since the only fields available to us when gauge fixing are the fields in the Lagrangian, this is not in my opinion a meaningful thing to do.

There is nothing wrong with the result in eq. (4.77), but it is certainly more complicated to work with than eq. (4.71). This also illustrates that the functional form of the effective potential is different with different gauge fixing functions.

I can only speculate about what procedure the authors of the cited papers have used to find the effective potential, but even though most papers write down eq. (4.73) as their gauge fixing function they always give eq. (4.71) as the general formula for the effective potential. It is also worth noting that if we choose $\xi = 0$, eq. (4.71) and eq. (4.77) are the same. So in this case the result given in e.g. [17, 6] using eq. (4.63) with $\xi = 0$ are still correct. But for a general ξ using the background field method as described above, I believe that one should really be using eq. (4.63) to gauge fix in the proper way.

4.4 Resummed Effective Potential

In this section we will consider the resummed effective potential. Unfortunately there are not that many interesting physical concepts to study for large field values for massless scalar QED, but we will mostly be interested in learning how to resum the potential to be valid at high scales to prepare us for the calculations in chapter 5. In the Standard Model we will see that the quartic coupling goes to zero at high scales, and we are very interested in studying the consequences of this.

We will start by showing how we can reproduce the leading logarithmic terms of the effective potential to 1-loop given the beta functions and anomalous dimension. We will for simplicity consider the massless limit of the Abelian Higgs model given in eq. (4.3) which is the same as massless scalar QED. We will then use the 1-loop beta function, following the procedure described in section 3.3.2, to find the resummed effective potential for massless scalar QED.

4.4.1 1-loop Effective Potential from β functions

In massless scalar QED we know that the beta functions to 1-loop order using $\overline{\text{MS}}$ [20] are

$$\begin{aligned}\mu \frac{de}{d\mu} &= \beta_e = \frac{1}{48\pi^2} e^3, \\ \mu \frac{d\lambda}{d\mu} &= \beta_\lambda = \frac{1}{24\pi^2} (5\lambda^2 - 18\lambda e^2 + 54e^4),\end{aligned}\tag{4.79}$$

where $e = e(\mu)$ and $\lambda = \lambda(\mu)$. The field strength renormalization [74] is

$$Z_\phi = 1 + \frac{1}{8\pi^2} \frac{1}{\epsilon} e^2 (3 - \xi),\tag{4.80}$$

which means that the anomalous dimension will be

$$\frac{\mu}{Z_\phi} \frac{dZ_\phi}{d\mu} = \gamma = \frac{1}{8\pi^2} e^2 (3 - \xi)\tag{4.81}$$

Note that the anomalous dimension is gauge dependent, while the beta functions are gauge independent.

Using our functional method in section 4.1 we saw that the 1-loop effective potential will be of the form

$$V_{\text{eff}} = \frac{\lambda(\mu)}{4!} Z_\phi^2(\mu) \phi^4 + F(e(\mu), \lambda(\mu), \xi) Z_\phi^2(\mu) \phi^4 \ln \frac{Z_\phi(\mu) \phi^2}{\mu^2},\tag{4.82}$$

where $\lambda(\mu)$ and $e(\mu)$ are the solutions to eq. (4.79) and solving for $Z_\phi(\mu)$ gives the renormalization of the scalar field strength. $F = F(e(\mu), \lambda(\mu), \xi)$ is some function of the couplings and ξ that we want to find. There will also be some finite terms in eq. (4.82) that we have not included, but they will be of higher order when solving for F to 1-loop order.

We know that the effective potential is independent of the scale μ , i.e. $\mu \frac{dV_{\text{eff}}}{d\mu} = 0$, and we can do a infinitesimal change from a scale μ to μ_0

$$\begin{aligned}\lambda(\mu) &= \lambda(\mu_0) + \beta_\lambda \ln \frac{\mu}{\mu_0}, \\ Z_\phi(\mu) &= Z_\phi(\mu_0) \left(1 + \gamma \ln \frac{\mu}{\mu_0} \right).\end{aligned}\tag{4.83}$$

We will use this property to specify the function $F(e(\mu), \lambda(\mu), \xi)$ since the μ dependence must cancel when we expand as in eq. (4.83). We find

$$V_{\text{eff}} = \frac{1}{4!} \left(\lambda(\mu_0) + \beta_\lambda \ln \frac{\mu}{\mu_0} \right) Z_\phi^2(\mu_0) \left(1 + 2\gamma \ln \frac{\mu}{\mu_0} \right) \phi^4 \\ + F(e(\mu_0), \lambda(\mu_0), \xi) Z_\phi^2(\mu_0) \phi^4 \ln \frac{Z_\phi(\mu_0) \phi^2}{\mu^2}, \quad (4.84)$$

where we have only kept the leading terms since we are only working to 1-loop order. Also remember that the couplings in the beta functions are evaluated at μ_0 . To find F , we collect the terms that are proportional to $\ln \mu^2$ and set them equal to zero,

$$0 = \frac{1}{4!} \beta_\lambda + \frac{1}{4!} \lambda(\mu_0) 2\gamma - 2F \\ \Downarrow \\ F = \frac{\beta_\lambda + 2\lambda(\mu_0)\gamma}{4!2} \quad (4.85)$$

Using eq. (4.79) and eq. (4.81) with $\lambda = \lambda(\mu_0)$ and $e = e(\mu_0)$

$$\frac{1}{4!2} (\beta_\lambda + 2\gamma\lambda) = \frac{1}{48\pi^2} \left(\frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24} + \frac{2\lambda(3 - \xi)e^2}{8} \right) \\ = \frac{1}{64\pi^2} \left(\frac{5}{18}\lambda^2 + 3e^4 - \xi \frac{1}{3}\lambda e^2 \right). \quad (4.86)$$

Substituting this back into eq. (4.84) we find

$$V_{\text{eff}} = \frac{1}{4!} \lambda(\mu_0) Z_\phi^2(\mu_0) \phi^4 \\ + \frac{1}{64\pi^2} \left(\frac{5}{18}\lambda(\mu_0)^2 + 3e(\mu_0)^4 - \xi \frac{1}{3}\lambda(\mu_0)e(\mu_0)^2 \right) Z_\phi^2(\mu_0) \phi^4 \ln \frac{Z_\phi(\mu_0) \phi^2}{\mu_0^2} \quad (4.87)$$

which reproduces the 1-loop result as in eq. (4.42). Notice that we do not reproduce the finite terms we called $f(e, \lambda)$ in eq. (4.42). This is okay since we only expect the RGE in general to reproduce the leading logarithmic terms and not the finite terms. Notice that all the gauge dependence of the leading logarithmic terms in the effective potential comes from the gauge dependence of the anomalous dimension. This matches our understanding of the gauge dependence discussed in section 4.2.2.

4.4.2 Scalar QED Resummed Effective Potential

Consider the fixed order effective potential in eq. (4.40) with $\Lambda = 0$ and $m_R = 0$. Each of the constants m_A^2 , m_B^2 and $m_{C_\pm}^2$ will for $m_R = 0$ be proportional to ϕ^2 ,

so we define the a corresponding variable with the subscript in lower case without the ϕ^2 factor, e.g. $m_a^2 = \frac{m_a^2}{\phi^2}$. The potential will be of the form

$$V_{\text{eff}}[\phi] = \frac{\lambda}{4!}\phi^4 + \sum_{i=a,b,c_{\pm}} \frac{n_i}{64\pi^2} m_i^4 \phi^4 \left[\ln \frac{m_i^2 \phi^2}{\mu^2} - c_i \right], \quad (4.88)$$

where $n_a = 3$, $n_b = n_{c_{\pm}} = 1$, $c_a = \frac{5}{6}$ and $n_b = n_{c_{\pm}} = \frac{3}{2}$.

Given the beta functions and anomalous dimension in eq. (4.79) and eq. (4.81) we can find how the couplings and scalar field strength changes with scale. We will not attempt an analytic solution, but these equations can easily be solved numerically. Solving the renormalization group equations for the effective potential as in section 3.3.2 and choosing $t = \ln \frac{\phi}{\mu}$ we find that the resummed effective potential be written in terms of an effective λ ,

$$V_{\text{eff}}[\phi] = \frac{\lambda_{\text{eff}}(\phi)}{4!}\phi^4, \quad (4.89)$$

$$\lambda_{\text{eff}}(\phi) \equiv \lambda(\phi) + \sum_{i=a,b,c_{\pm}} \frac{n_i}{64\pi^2} m_i^4 \phi^4 \left[\ln (Z_{\phi}(\phi) m_i^2) - c_i \right],$$

where the field strength renormalization is

$$Z_{\phi}(\phi) = \exp \left[\int \gamma(\mu) d \ln \mu \right]. \quad (4.90)$$

In the next section we will make plots of λ_{eff} to get a clear picture of how the different choices of ξ and gauge fixing conditions change the form of the effective potential.

4.5 Gauge Dependence

In this section we will compute and plot the effective potential for massless scalar QED numerically using `NDSolve` in Mathematica [75]. We are also in this section motivated by the analysis we will perform in chapter 5 for the Standard Model, and we will plot the effective potential up to the Planck scale 10^{20}GeV where one expects new physics to be important.

In this chapter we have covered many different ways of finding the effective potential, and in essence we want to show that the shape of the potential will be different depending on gauge choice (the value of ξ) and gauge fixing condition.

We will be using the resummed potential described in section 4.4 and we will for simplicity⁴ factor out the ϕ^4 and just plot λ_{eff} . For convenience, we will

⁴The reason for this is that it is to hard to distinguish the different curves when we multiply by the factor of ϕ^4 . If we included this factor the plot range would be roughly from zero to 10^{80} , and $\mathcal{O}(1)$ factors in front of 10^{80} are practically invisible.

summarize our results in the next section for all the different λ_{eff} we will study. All the couplings and Z_ϕ are the solutions to the RGE equations, but we will simply write λ , e and Z_ϕ instead of $\lambda(\phi)$, $e(\phi)$ and $Z_\phi(\phi)$ to make the equations more compact and to increase readability.

4.5.1 Different Definitions of λ_{eff}

We will start by looking at the effective potential associated with the gauge fixing condition $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A_\mu)^2$. The result was given in eq. (4.40) and we denote the effective λ with a superscript "mix" since this was the gauge where we had kinetic mixing. The function we will plot is as a function of ξ

$$\begin{aligned} \lambda_{\text{eff}}^{\text{mix}}(\xi) = & \lambda Z_\phi^2 + \frac{3}{8\pi^2} Z_\phi^2 \left[3m_a^4 \left(\ln Z_\phi m_a^2 - \frac{5}{6} \right) + m_b^4 \left(\ln Z_\phi m_b^2 - \frac{3}{2} \right) \right. \\ & \left. + m_{c_+}^4 \left(\ln Z_\phi m_{c_+}^2 - \frac{3}{2} \right) + m_{c_-}^4 \left(\ln Z_\phi m_{c_-}^2 - \frac{3}{2} \right) \right] \end{aligned} \quad (4.91)$$

where we have defined

$$m_a^2 = e^2, \quad m_b^2 = \frac{1}{2}\lambda, \quad m_{c_\pm}^2 = \frac{1}{12} \left[\lambda \pm \sqrt{\lambda^2 - 24e^2\xi\lambda} \right]. \quad (4.92)$$

In section 4.4.1 we saw how we could, with just the knowledge of the beta functions, find the leading logarithmic term of the effective potential, but not the finite terms. This result was given in eq. (4.42). We will denote the effective lambda in this case with a superscript " β " since this result comes from the beta functions

$$\lambda_{\text{eff}}^\beta(\xi) = \lambda Z_\phi^2 + \frac{3}{8\pi^2} Z_\phi^2 \left[3e^4 + \frac{5}{18}\lambda^2 - \frac{1}{3}e^2\lambda\xi \right] \ln Z_\phi \quad (4.93)$$

The third case we will consider is the alternative gauge fixing discussed in section 4.3. The effective lambda will be denoted by a superscript "alt" since this is an alternative way of gauge fixing. The function is

$$\begin{aligned} \lambda_{\text{eff}}^{\text{alt}}(\xi) = & \lambda Z_\phi^2 + \frac{3}{8\pi^2} Z_\phi^2 \left[3m_a^4 \left(\ln Z_\phi m_a^2 - \frac{5}{6} \right) + m_b^4 \left(\ln Z_\phi m_b^2 - \frac{3}{2} \right) \right. \\ & \left. + m_c^4 \left(\ln Z_\phi m_c^2 - \frac{3}{2} \right) - m_d^4 \left(\ln Z_\phi m_d^2 - \frac{3}{2} \right) \right], \end{aligned} \quad (4.94)$$

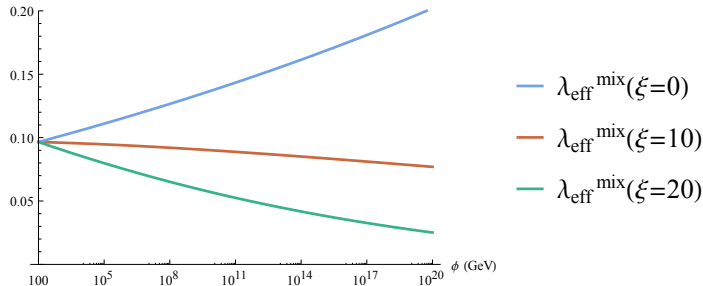


Figure 4.1: Plot of $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ for $\xi = 0, 10, 20$. The curves are shifted by a constant to start at the same point at $\phi = \mu_0 = 100\text{GeV}$.

where we have defined

$$m_a^2 = e^2, \quad m_b^2 = \frac{1}{2}\lambda, \quad m_c^2 = \frac{1}{6}\lambda + e^2\xi, \quad m_d^2 = \xi e^2. \quad (4.95)$$

4.5.2 Results and Discussion

In all cases we have to choose some initial condition. For all calculations in this section we will start at an initial scale $\mu_0 = 100\text{ GeV}$ with conditions $e(\mu_0) = 0.3$, $\lambda(\mu_0) = 0.1$ and $Z(\mu_0) = 1$. These numbers are chosen more or less arbitrarily and other numbers will give similar results. As mentioned earlier, we will integrate up to the Planck scale 10^{20} GeV motivated by the analysis in chapter 5.

With these initial conditions we find that all the $\lambda_{\text{eff}}(\xi)$ will have different values at μ_0 for different ξ . In reality the potential is only given up to some constant that we fix by applying some initial condition at some given scale. To make it easier to interpret the plots, we have used this freedom to shift some of the curves vertically so that they start at the same point at $\phi = \mu_0$.

4.5.2.1 Comparing $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ for $\xi = 0, 10, 20$

Figure 4.1 shows a plot of $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ for three different values $\xi = 0, 10, 20$. The plot tells us that the shape of $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ depends heavily on the value of ξ . $\lambda_{\text{eff}}^{\text{mix}}(\xi = 0)$ is monotonically increasing from the initial value and up to $\phi = 10^{20}\text{ GeV}$, while $\lambda_{\text{eff}}^{\text{mix}}(\xi = 10)$ and $\lambda_{\text{eff}}^{\text{mix}}(\xi = 20)$ are monotonically decreasing with the $\xi = 20$ curve below the $\xi = 10$ curve.

One might be surprised that $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ as defined in eq. (4.91) depends this heavily on the value of ξ as we see in figure 4.1. In reality what is happening is that $Z_\phi = \exp\left[\int \gamma(\mu) d \ln \mu\right]$ gets a large negative exponent for high values of ϕ . To see how big effect Z_ϕ plays, we show the plot of $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{mix}}(\xi)$ in figure 4.2. This

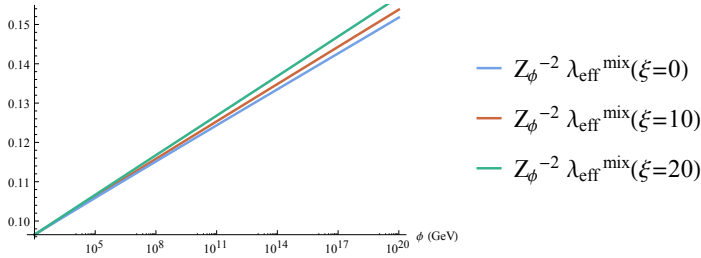


Figure 4.2: Plot of $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{mix}}(\xi)$ for $\xi = 0, 10, 20$. The curves are shifted by a constant to start at the same point at $\phi = \mu_0 = 100\text{GeV}$.

corresponds to not including the field strength renormalization factor for the ϕ^4 term we have factored off in when we defined $\lambda_{\text{eff}}^{\text{mix}}(\xi)$.

In figure 4.2 we see that the curve with $\xi = 20$ has a greater slope than $\xi = 10$ and $\xi = 0$. Note especially that the order is completely reversed compared to the curves in figure 4.1.

We conclude that the effective potential will be dependent on the value of ξ , but in analyzing the dependence one must be careful about what curves one chose to look at. The gauge dependence changes the curves in two different ways. It will change the slope of $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{mix}}(\xi)$ and it will suppress the curve with the overall factor Z_ϕ^2 .

4.5.2.2 Comparing $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ and $\lambda_{\text{eff}}^\beta(\xi)$

In this section we will compare $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ and $\lambda_{\text{eff}}^\beta(\xi)$ for $\xi = 0$ and for $\xi = 10$. We are interested in this result to see how big the difference is between the full effective potential we found from the path integral and the just the leading logarithmic terms of the potential that we found just using the beta functions and the anomalous dimension.

We first consider the case where $\xi = 0$. $\lambda_{\text{eff}}^{\text{mix}}(0)$ and $\lambda_{\text{eff}}^\beta(0)$ and the difference between these two are shown in figure 4.3.

We see from the top plot that the two curves are separated, and this comes from the finite terms that are missing for $\lambda_{\text{eff}}^\beta(0)$. Looking at the difference between the two, we see that the separation is not constant, but varies as we increase ϕ . However the difference is almost two orders of magnitude smaller than the values of $\lambda_{\text{eff}}^\beta(0)$ and $\lambda_{\text{eff}}^{\text{mix}}(0)$, so we are roughly off by $\mathcal{O}(1)\%$. In figure 4.4 we find the same general behavior where we have done the comparison for $\xi = 10$.

Comparing figure 4.3 and 4.4 we again see that the value of the effective potential is gauge dependent, as we saw in section 4.5.2.1. In figure 4.4 we also

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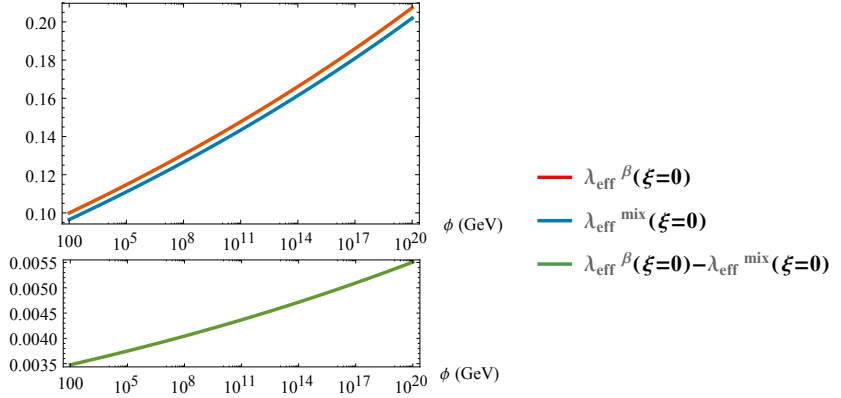


Figure 4.3: The top plot is of $\lambda_{\text{eff}}^{\text{mix}}(0)$ and $\lambda_{\text{eff}}^{\beta}(0)$ and the bottom plot shows the difference $\lambda_{\text{eff}}^{\beta}(0) - \lambda_{\text{eff}}^{\text{mix}}(0)$.

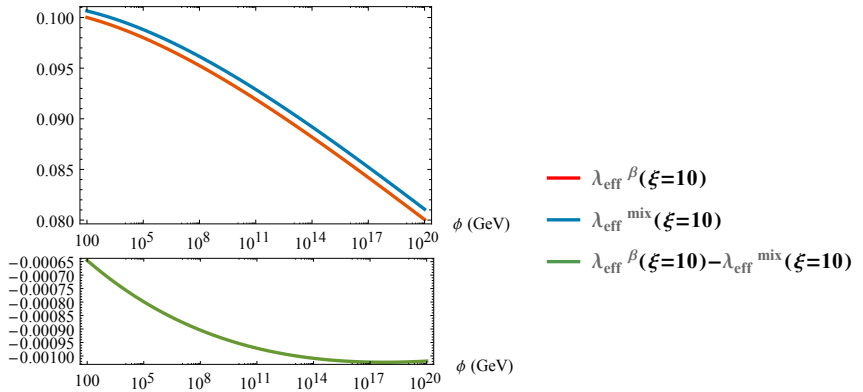


Figure 4.4: The top plot is of $\lambda_{\text{eff}}^{\text{mix}}(10)$ and $\lambda_{\text{eff}}^{\beta}(10)$ and the bottom plot shows the difference $\lambda_{\text{eff}}^{\beta}(10) - \lambda_{\text{eff}}^{\text{mix}}(10)$.

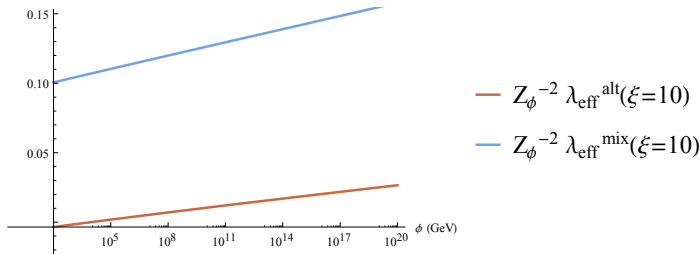


Figure 4.5: The plot shows the difference between $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{mix}}(10)$ and $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{alt}}(10)$ for the same choice of our gauge fixing parameter $\xi = 10$.

notice that the the difference $\lambda_{\text{eff}}^\beta(10) - \lambda_{\text{eff}}^{\text{mix}}(10)$ is negative, and that this quantity was positive in figure 4.3. Hence it is not possible to say for certain if we are over or underestimating the numerical value of the effective potential by considering just the leading logarithmic terms.

The conclusion we draw from this is that if we care about the numerical value of $\lambda_{\text{eff}}(\xi)$, we should really consider the full effective potential we find from doing the path integral. If we only care about the general shape of the potential, it may be enough to find the leading logarithmic terms using the beta functions and anomalous dimension.

4.5.2.3 Comparing $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ and $\lambda_{\text{eff}}^{\text{alt}}(\xi)$

The last thing we are interested in checking is if the two different ways of gauge fixing gives different results for the same choice of ξ . The result for $\xi = 0$ will always be the same. This can easily be seen by comparing eq. (4.91) and eq. (4.94) for $\xi = 0$.

For $\xi = 10$ the result is very different as can be seen in figure 4.5. We have not included the factor of Z_ϕ^2 for the reasons mentioned in section 4.5.2.1. We see that $\lambda_{\text{eff}}^{\text{mix}}(\xi)$ and $\lambda_{\text{eff}}^{\text{alt}}(\xi)$ have different initial values, and in figure 4.5.2.3 we have shifted the curves so that they start at the same point. Figure 4.6(a) shows the result for $\xi = 10$, and in figure 4.6(b) we have done the same for $\xi = 20$. Comparing the curves, we see that the shape of the potential will in general be different for different ways of gauge fixing. Only in the Landau gauge, $\xi = 0$, will give the same result for the two cases we have studied in this chapter.

4.5.2.4 Concluding Remarks

In this section we have plotted λ_{eff} for different choices of gauge fixing and different choices of the gauge fixing parameter ξ . From looking at the functions as they were defined in section 4.5.1 it is not surprising that the plots came out

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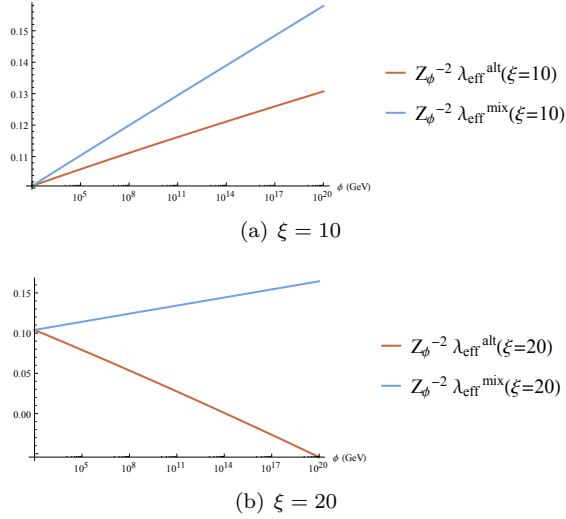


Figure 4.6: The two plots shows the difference between $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{mix}}(\xi)$ and $Z_\phi^{-2} \lambda_{\text{eff}}^{\text{alt}}(\xi)$ for the same choice of our gauge fixing parameter ξ , and we have shifted the two curves for easier comparison between the slopes of the curve.

differently for the different cases, but this section has given us some intuition about where and how the plots differ and how big the effects are.

There is not much interesting physics to draw from these curves, but in the next chapter we will study the equivalent λ_{eff} for the Standard Model. Seeing how the gauge dependence changes λ_{eff} for massless scalar QED tells us that we should be expecting the Standard Model effective potential to also depend on gauge fixing and the value of ξ . In the Standard Model, λ_{eff} is used to find a bound on the Higgs mass from requiring stability conditions [6]. We will use the tools we have developed and discussed in this chapter to analyze how this bound depends on ξ .

Chapter 5

The Standard Model

In this chapter we will use the tools developed in the previous chapters to study the Higgs effective potential in the Standard Model. We will compute the effective potential to 1-loop in a general R_ξ gauge and use the renormalization group equations to find the resummed effective potential as we did in chapter 4. This will be used to study the stability of the Standard Model as we discussed in section 3.4.

The history of the stability considerations of a quantum field theory started with Sidney Coleman [1] describing fate of the false vacuum in 1977, and in 1979 Cabbibio et al. [2] and Hung [3] described how requiring stability of our theory would give bounds on the masses in the model. We will see explicitly how we get these bounds in this chapter. These papers described the absolute stability condition, and in 1989 Arnold [8] discussed the concept of meta-stability by noting that it is possible to have a sensible theory describing our universe in a false vacuum as long as the lifetime of the false vacuum is longer than the age of the universe.

Improvements on this analysis have been done continuously over the last two decades, and it is still very much relevant today. Some of the relevant papers in this timeperiod are Lindner et al. [4] which analyzed the stability of the Standard Model in the context of the top search at Fermilab in 1989. Altarelli and Isidori [5] updated the Higgs mass bound in 1994, and they concluded that the Higgs would be too heavy to be found at LEP due to the large top mass. The most recent works are the analysis by Degraasi et al. [6] from 2012 and the latest update from July 2013 by Buttazzo et al. [7].

The bound of most interest has been the lower bound on the Higgs mass since its value was unknown until the Higgs discovery [10, 11] in 2012, with the current measured value being $m_H = 125.66 \pm 0.34$ GeV [12]. In fact, we will see in this chapter the stability question has become even more interesting since a 125GeV Higgs mass indicates that we live in a false vacuum if the Standard

Model is the theory describing our universe up to the Planck scale. We are only interested studying the theory up to the Planck scale because we expect gravity contributions to be significant beyond this scale.

In the most recent studies by Buttazzo et al. [7] the analysis of the Standard Model effective potential has reached impressive precision. The Higgs potential and the top Yukawa coupling are computed with 2-loop NNLO (next-to-next-to-leading order) precision, and the Standard Model parameters are computed with full 3-loop NNLO RGE precision up to the Planck scale.

In this chapter we will compute the Standard Model effective potential to 1-loop in the R_ξ gauges, use the RGE to find the resummed effective potential and define λ_{eff} as we did in chapter 4. Using the results from Degraasi et al. [6] and Buttazzo et al. [7] we will also define an effective quartic coupling in the Landau gauge.

After a discussion of the initial conditions and threshold correction in section 5.5.3 we will reproduce the results from Buttazzo et al. [7]. Then finally, we will add the main ingredient to our analysis. We will use the gauge dependent Standard Model effective potential to 1-loop together with what we have learned about the gauge dependence in chapter 4 to redo the calculation of the Higgs mass bound from absolute stability requirements and see how this bound depends on the gauge parameter ξ .

We will end this chapter with a discussion on our findings and speculate about the interpretation and possible solution to unsolved problems.

5.1 Simplified Calculation

Before we compute the full 1-loop effective potential for the Standard Model, we will do a short calculation explaining the essential points of this chapter. We extract only the most essential terms from the effective potential which give the main contributions, and we will see how to use this to find a rough bound on the Higgs mass from absolute stability requirements [19].

We will consider the Lagrangian

$$\mathcal{L} = |\partial_\mu H|^2 + m^2 |H|^2 - \lambda |H|^4 + \bar{\psi}_t (i\not{\partial} + m_t) \psi_t \quad (5.1)$$

where $H = \begin{pmatrix} 0 \\ \frac{\phi + \hat{\phi}}{\sqrt{2}} \end{pmatrix}$ and $m_t = \frac{y_t \hat{\phi}}{\sqrt{2}}$, which includes only the Higgs and top in the Lagrangian given in eq. (5.6). Expanding this and only keeping the terms contributing to 1-loop we find

$$\mathcal{L} = \frac{1}{2} m^2 \hat{\phi}^2 - \frac{\lambda}{4} \hat{\phi}^4 - \frac{1}{2} \phi \left[\square - m^2 + 3\lambda \hat{\phi}^2 \right] \phi + \bar{\psi}_t (i\not{\partial} + m_t) \psi_t. \quad (5.2)$$

We will not perform the details of this calculation since it will be done in detail in the next section, but instead we can extract the relevant terms from the

result given in eq. (5.35). The effective potential to 1-loop for the Lagrangian given in eq. (5.2) is

$$\begin{aligned}
 V_{\text{eff}}[\hat{\phi}] = & -\frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^4 + \frac{(-m^2 + 3\lambda\hat{\phi}^2)^2}{64\pi^2} \left[\ln \frac{(-m^2 + 3\lambda\hat{\phi}^2)}{\mu^2} - \frac{3}{2} \right] \\
 & - \frac{12}{64\pi^2} \left(\frac{1}{2}y_t^2\hat{\phi}^2 \right)^2 \left[\ln \frac{(\frac{1}{2}y_t^2\hat{\phi}^2)}{\mu^2} - \frac{3}{2} \right]
 \end{aligned} \tag{5.3}$$

Following the discussion in section 3.4 we take the limit $\lambda\hat{\phi}^2 \gg m^2$, and we see that the leading contribution to the effective potential in eq. (5.3) simplifies to

$$V_{\text{eff}}[\hat{\phi}] = \frac{\lambda}{4}\hat{\phi}^4 + \frac{1}{64\pi^2}\hat{\phi}^4 (9\lambda^2 - 3y_t^4) \ln \frac{\hat{\phi}^2}{\mu^2}. \tag{5.4}$$

The essential point is to note that if $9\lambda^2 - 3y_t^4 > 0$, the effective potential will always be positive for large $\hat{\phi}$, and hence the theory must be stable. Relating the values of the coupling to the Higgs and top mass we find $y_t = \sqrt{2}\frac{m_t}{v}$, $m = \sqrt{2}m_H$ and $m = \sqrt{\lambda}v$ the Higgs mass bound is

$$m_H > \frac{1}{\sqrt[4]{3}}m_t. \tag{5.5}$$

Using the $\overline{\text{MS}}$ top mass 163GeV [19], we find the lower bound on the Higgs mass to be $m_H > 123.9\text{GeV}$. This is just barely below the measured value $m_H = 125.66 \pm 0.34\text{GeV}$ [12], and we see that a precision calculation is necessary to determine if the Standard Model vacuum is really stable or not. The rest of the chapter will be spent making this argument precise.

We also see here that if we take the Higgs mass to be known, we could have used eq. (5.5) to find an upper bound on the top mass for the Standard Model to be absolutely stable.

5.2 The Standard Model Lagrangian

We will now derive the 1-loop effective potential for the Standard Model including the SU(2) and U(1) gauge bosons, the complex scalar doublet H and the top quark. The SU(3) gauge bosons are not included since they will not contribute to 1-loop because the Higgs is not colored. We only include the top quark since it will be the most important contribution due to its high mass. The relevant Lagrangian was given in eq. (2.110)

$$\mathcal{L} = -\frac{1}{4}(W_{\mu\nu}^a)^2 - \frac{1}{4}(B_{\mu\nu})^2 + |D_\mu H|^2 + m^2|H|^2 - \lambda|H|^4 + \mathcal{L}_{\text{fermion}} \tag{5.6}$$

where $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2 f^{abc} W_\mu^b W_\nu^c$ where f^{abc} are the SU(2) structure constants, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $\mathcal{L}_{\text{fermion}}$ is the part of the Lagrangian including the top, and the covariant derivative is

$$D_\mu H = \partial_\mu H - \frac{i}{2} [g_2 W_\mu^a \sigma^a + g_1 B_\mu] H, \quad (5.7)$$

where σ^a are the Pauli matrices. g_2 is the SU(2) coupling constant and g_1 is the U(1) coupling constant. We will look at the details of the fermion part of the Lagrangian $\mathcal{L}_{\text{fermion}}$ a bit later. We gauge fix using the R_ξ gauges, so the gauge fixing Lagrangian is

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi_W} (\partial_\mu W_\mu^a)^2 - \frac{1}{2\xi_B} (\partial_\mu B_\mu)^2. \quad (5.8)$$

The ghost Lagrangian is

$$\mathcal{L}_{\text{gh}} = -\bar{c}_B \square c_B - \bar{c}_W^a \partial_\mu (\partial_\mu \delta^{ac} + g_2 f^{abc} W_\mu^b) c_W^c, \quad (5.9)$$

where \bar{c}_W^a and c_W^a , and \bar{c}_B and c_B are the ghosts corresponding to the SU(2) and U(1) gauge fixing, respectively.

5.3 Computing the Effective Potential to 1-loop

5.3.1 Notation

For simplicity we will introduce some new notation that will be used in the derivation of the 1-loop effective potential. We will write the complex doublet H in terms of 4 real variables

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\psi_1 \\ \phi_2 + i\psi_2 \end{pmatrix}. \quad (5.10)$$

We will later write this as a real four component vector for the scalars

$$\Psi_a = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix}, \quad (5.11)$$

and we will construct a vector of the gauge bosons

$$V_\mu = \begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix}. \quad (5.12)$$

Eventually we will also construct a vector of all the fields that we will denote by

$$\Phi = \begin{pmatrix} \Psi_a \\ V_\mu \end{pmatrix}. \quad (5.13)$$

5.3.2 The 1-loop Lagrangian

To find the 1-loop effective potential, we will use the background field method as described in section 3.2. We will expand $H \rightarrow H + \hat{H}$ for some constant background field \hat{H} . We are interested in the effective potential for the Higgs field, and it's conventional to choose $\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix}$. After we have expanded around the background field, we will integrate out all the other fields using the path integral as we did in section 3.3.1.4.

Since we are only working to 1-loop, we will only include the terms that contribute to this order. The relevant terms will be the ones with two propagating fields, just as we did in section 4.1.1 for the Abelian Higgs model. To 1-loop the ghost term decouples from the theory, so we need not include it.

We start by looking at the kinetic terms for the gauge bosons. We find that the relevant terms are

$$\begin{aligned} -\frac{1}{4} (W_{\mu\nu}^a)^2 - \frac{1}{2\xi_W} (\partial_\mu W_\mu^a)^2 &\rightarrow -\frac{1}{2} W_\mu^a \Delta_{\mu\nu}^{ab} W_\nu^b \\ -\frac{1}{4} (B_{\mu\nu})^2 - \frac{1}{2\xi_B} (\partial_\mu B_\mu)^2 &\rightarrow -\frac{1}{2} B_\mu \Delta_{\mu\nu} B_\nu \end{aligned} \quad (5.14)$$

where we have defined

$$\begin{aligned} \Delta_{\mu\nu}^{ab} &= - \left[g_{\mu\nu} \square - \left(1 - \frac{1}{\xi_W} \right) \partial_\mu \partial_\nu \right] \delta^{ab}, \\ \Delta_{\mu\nu} &= - \left[g_{\mu\nu} \square - \left(1 - \frac{1}{\xi_B} \right) \partial_\mu \partial_\nu \right]. \end{aligned} \quad (5.15)$$

We can write this compactly as

$$\mathcal{L}_{\text{gauge kin.}} = -\frac{1}{2} V_\mu \begin{bmatrix} \Delta_{\mu\nu}^{11} & 0 & 0 & 0 \\ 0 & \Delta_{\mu\nu}^{22} & 0 & 0 \\ 0 & 0 & \Delta_{\mu\nu}^{33} & 0 \\ 0 & 0 & 0 & \Delta_{\mu\nu} \end{bmatrix} V_\nu. \quad (5.16)$$

Next we look at the third term in eq. (5.6), $|D_\mu H|^2$. We expand $H \rightarrow H + \hat{H}$ and find that the relevant terms are

$$\begin{aligned} |D_\mu H|^2 &\rightarrow |\partial_\mu H|^2 + 2\text{Re} \left[\frac{i}{2} \hat{H}^\dagger (g_2 W_\mu^a \sigma^a + g_1 B_\mu)^\dagger \partial_\mu H \right] \\ &\quad + \frac{1}{4} \left| [g_2 W_\mu^a \sigma^a + g_1 B_\mu] \hat{H} \right|^2. \end{aligned} \quad (5.17)$$

The first pure derivative term can simply be rewritten as

$$|\partial_\mu H|^2 = -\frac{1}{2} \Psi_a \square \delta^{ab} \Psi_b. \quad (5.18)$$

To simplify the other two terms we first note that we can write

$$\begin{aligned}
 [g_2 W_\mu^a \sigma^a + g_1 B_\mu] \hat{H} &= \begin{bmatrix} g_1 B_\mu + g_2 W_\mu^3 & g_2 W_\mu^1 - i g_2 W_\mu^2 \\ g_2 W_\mu^1 + i g_2 W_\mu^2 & g_1 B_\mu - g_2 W_\mu^3 \end{bmatrix} \hat{H} \\
 &= \frac{\hat{\phi}_2}{\sqrt{2}} \begin{bmatrix} g_2 W_\mu^1 - i g_2 W_\mu^2 \\ g_1 B_\mu - g_2 W_\mu^3 \end{bmatrix}.
 \end{aligned} \tag{5.19}$$

Using this we find

$$\begin{aligned}
 &2\text{Re} \left[\frac{i}{2} \hat{H}^\dagger (g_2 W_\mu^a \sigma^a + g_1 B_\mu)^\dagger \partial_\mu H \right] \\
 &= -\frac{1}{2} \hat{\phi}_2 [g_2 W_\mu^1 \partial_\mu \psi_1 + g_2 W_\mu^2 \partial_\mu \phi_1 + g_1 B_\mu \partial_\mu \psi_2 - g_2 W_\mu^3 \partial_\mu \psi_2] \\
 &= -\frac{1}{2} V_\mu \begin{bmatrix} 0 & 0 & \frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu & 0 \\ \frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu \\ 0 & 0 & 0 & \frac{1}{2} g_1 \hat{\phi}_2 \partial_\mu \end{bmatrix} \Psi_a \\
 &- \frac{1}{2} \Psi_a \begin{bmatrix} 0 & -\frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} g_2 \hat{\phi}_2 \partial_\mu & -\frac{1}{2} g_1 \hat{\phi}_2 \partial_\mu \end{bmatrix} V_\mu \\
 &\equiv -\frac{1}{2} V_\mu M_\mu^{a\dagger} \Psi_a - \frac{1}{2} \Psi_a M_\mu^a V_\mu,
 \end{aligned} \tag{5.20}$$

where in the last line we have defined the matrix M_μ^a . The third term becomes

$$\begin{aligned}
 &\frac{1}{4} | [g_2 W_\mu^a \sigma^a + g_1 B_\mu] \hat{H} |^2 \\
 &= \frac{\hat{\phi}_2^2}{8} (g_2^2 (W_\mu^1)^2 + g_2^2 (W_\mu^2)^2 + g_1^2 (B_\mu)^2 + g_2^2 (W_\mu^3)^2 - 2g_1 g_2 B_\mu W_\mu^3) \\
 &= -\frac{1}{2} V_\mu \begin{bmatrix} -\frac{1}{4} g_2^2 \hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} g_2^2 \hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} g_2^2 \hat{\phi}_2^2 g_{\mu\nu} & \frac{g_1 g_2}{4} \hat{\phi}_2^2 g_{\mu\nu} \\ 0 & 0 & \frac{g_1 g_2}{4} \hat{\phi}_2^2 g_{\mu\nu} & -\frac{1}{4} g_1^2 \hat{\phi}_2^2 g_{\mu\nu} \end{bmatrix} V_\nu.
 \end{aligned} \tag{5.21}$$

After expanding $H \rightarrow H + \hat{H}$, the mass term in eq. (5.6) becomes simply

$$m^2 H^\dagger H \rightarrow m^2 \hat{H}^\dagger \hat{H} + m^2 H^\dagger H = \frac{1}{2} m^2 \hat{\phi}_2^2 + \frac{1}{2} m^2 \Psi_a \Psi_a, \tag{5.22}$$

and the quartic coupling can be written as

$$\begin{aligned}
 -\lambda(H^\dagger H)^2 &\rightarrow -\lambda \left[(\hat{H}^\dagger \hat{H})^2 + (H^\dagger \hat{H})^2 + (\hat{H}^\dagger H)^2 \right. \\
 &\quad \left. + 2(H^\dagger H)(\hat{H}^\dagger \hat{H}) + 2(\hat{H}^\dagger H)(H^\dagger \hat{H}) \right] \\
 &= -\frac{\lambda}{4}\hat{\phi}_2^4 - \frac{1}{2}\Psi_a \begin{bmatrix} \lambda\hat{\phi}_2^2 & 0 & 0 & 0 \\ 0 & 3\lambda\hat{\phi}_2^2 & 0 & 0 \\ 0 & 0 & \lambda\hat{\phi}_2^2 & 0 \\ 0 & 0 & 0 & \lambda\hat{\phi}_2^2 \end{bmatrix} \Psi_b.
 \end{aligned} \tag{5.23}$$

With all the 1-loop terms in the Lagrangian written in matrix form in terms of Ψ_a and V_μ , we can write

$$\begin{aligned}
 \mathcal{L}_{1\text{-loop}} &= \frac{1}{2}m^2\hat{\phi}_2^2 - \frac{\lambda}{4}\hat{\phi}_2^4 - \frac{1}{2}\Phi \begin{bmatrix} D^{ab} & M_\mu^a \\ M_\mu^{a\dagger} & \bar{\Delta}_{\mu\nu} \end{bmatrix} \Phi + \mathcal{L}_{\text{fermion}} \\
 &= \frac{1}{2}m^2\hat{\phi}_2^2 - \frac{\lambda}{4}\hat{\phi}_2^4 - \frac{1}{2}\Phi\Sigma\Phi + \mathcal{L}_{\text{fermion}},
 \end{aligned} \tag{5.24}$$

where the components of the matrix Σ are

$$M_\mu^a = \begin{bmatrix} 0 & -\frac{1}{2}g_2\hat{\phi}_2^2\partial_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}g_2\hat{\phi}_2^2\partial_\mu & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}g_2\hat{\phi}_2^2\partial_\mu & -\frac{1}{2}g_1\hat{\phi}_2^2\partial_\mu \end{bmatrix} \tag{5.25}$$

$$D^{ab} = \begin{bmatrix} \square - m^2 + \lambda\hat{\phi}_2^2 & 0 & 0 & 0 \\ 0 & \square - m^2 + 3\lambda\hat{\phi}_2^2 & 0 & 0 \\ 0 & 0 & \square - m^2 + \lambda\hat{\phi}_2^2 & 0 \\ 0 & 0 & 0 & \square - m^2 + \lambda\hat{\phi}_2^2 \end{bmatrix} \tag{5.26}$$

$$\bar{\Delta}_{\mu\nu} = \begin{bmatrix} \Delta_{\mu\nu}^{11} - \frac{1}{4}g_2^2\hat{\phi}_2^2g_{\mu\nu} & 0 & 0 & 0 \\ 0 & \Delta_{\mu\nu}^{22} - \frac{1}{4}g_2^2\hat{\phi}_2^2g_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Delta_{\mu\nu}^{33} - \frac{1}{4}g_2^2\hat{\phi}_2^2g_{\mu\nu} & \frac{g_1g_2}{4}\hat{\phi}_2^2g_{\mu\nu} \\ 0 & 0 & \frac{g_1g_2}{4}\hat{\phi}_2^2g_{\mu\nu} & \Delta_{\mu\nu} - \frac{1}{4}g_1^2\hat{\phi}_2^2g_{\mu\nu} \end{bmatrix}. \tag{5.27}$$

5.3.3 Fermion Contribution

In our study of the Standard Model we will only include the top quark since this will be the leading contribution because the top Yukawa coupling is $y_t \approx 1$. We

found in section 2.3.11.1 that the relevant top quark Lagrangian is

$$\mathcal{L}_{\text{fermion}} = \bar{\psi}_t(i\hat{\not{\phi}} - m_t)\psi_t, \quad (5.28)$$

where $m_t = \frac{1}{\sqrt{2}}y_t\hat{\phi}_2$ and ψ_t is a colored Dirac spinor.

5.3.4 Determinants

To find the effective potential we will evaluate the determinant of the matrices following the same procedure as in section 4.1.3. We know how to do the Gaussian integral over Φ from chapter 4 but now we also need to take into account the Dirac fermion. The two integrals can be done separately, and we find

$$\begin{aligned} e^{i\Gamma[\hat{\phi}_2]} &= \int \mathcal{D}\Phi e^{i \int d^4x -\frac{1}{2}\Phi\Sigma\Phi} \int \mathcal{D}\bar{\psi}_t \mathcal{D}\psi_t e^{i \int d^4x \bar{\psi}_t(i\hat{\not{\phi}} - m_t)\psi_t} \\ &= \frac{\text{Det} |i\hat{\not{\phi}} - m_t|}{\sqrt{\text{Det} |\Sigma|}}, \end{aligned} \quad (5.29)$$

up to some overall constant. Using eq. (3.67) we can write the effective potential as

$$\begin{aligned} V_{\text{eff}}[\hat{\phi}_2] &= V_{\text{tree}}[\hat{\phi}_2] \\ &\quad - i\frac{1}{2}\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln \det \Sigma \\ &\quad + i\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln \det |\not{p} + m|. \end{aligned} \quad (5.30)$$

5.3.4.1 Determinant of Dirac Matrices

In d dimensions we have to be a bit careful about taking the determinant of the Dirac matrices. We start by rewriting this as

$$\begin{aligned} \ln \det(m + \not{p}) &= \text{tr} \ln(m + \not{p}) = \text{tr} \left[\ln m + \ln \left(1 + \frac{\not{p}}{m} \right) \right] \\ &= \text{tr} \left[\ln m - \sum_{n=1,2,\dots} \frac{(-1)^n}{n} \left(\frac{\not{p}}{m} \right)^n \right], \end{aligned} \quad (5.31)$$

where we have done a series expansion of the log. Since the trace of an odd number of Dirac matrices is zero and $\not{p}^2 = p^2$, we can cancel all the odd terms and write this as a logarithm again. The trace gives just a factor of $2^{\frac{d}{2}}$, which is

the dimension of the Dirac matrices in d dimensions [76].

$$\begin{aligned}\ln \det(m + \not{p}) &= 2^{\frac{d}{2}-1} \left[\ln m^2 - \sum_{n=1,2,\dots} \frac{1}{n} \left(\frac{p^2}{m^2} \right)^n \right] \\ &= 2^{\frac{d}{2}-1} \left[\ln m^2 + \frac{1}{2} \ln \left(1 - \frac{p^2}{m^2} \right) \right] = 2^{\frac{d}{2}-1} \ln(m^2 - p^2).\end{aligned}\quad (5.32)$$

5.3.4.2 Integrals

We can now evaluate the integral, and we find

$$i\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \ln \det|\not{p} + m_t| = -3 \frac{4m_t^4}{64\pi^2} \left[\ln \frac{m_t^2}{\mu^2} - \frac{3}{2} \right], \quad (5.33)$$

where the factor of 3 comes from the fact that the top has three different colors. Here we have removed the infinities that will be removed using $\overline{\text{MS}}$. The matrix determinant over Σ becomes

$$\begin{aligned}\ln \det|\Sigma| &= d \ln k^2 + 2(d-1) \ln \left[k^2 - \frac{g_2^2 \hat{\phi}_2^2}{4} \right] \\ &+ (d-1) \ln \left[k^2 - \frac{(g_1^2 + g_2^2) \hat{\phi}_2^2}{4} \right] + \ln \left[k^2 - (3\lambda \hat{\phi}_2^2 - m^2) \right] \\ &+ 2 \ln \left[k^4 - k^2 (\lambda \hat{\phi}_2^2 - m^2) + \frac{1}{4} \xi_W g_2^2 \hat{\phi}_2^2 (\lambda \hat{\phi}_2^2 - m^2) \right] \\ &+ \ln \left[k^4 - k^2 (\lambda \hat{\phi}_2^2 - m^2) + \frac{1}{4} (g_1^2 \xi_B + g_2^2 \xi_W) \hat{\phi}_2^2 (\lambda \hat{\phi}_2^2 - m^2) \right].\end{aligned}\quad (5.34)$$

Using the integrals performed in section 4.1.4 we find that the effective potential for the Standard Model to 1-loop can be written as

$$\begin{aligned}V_{\text{eff}}[\hat{\phi}_2] &= -\frac{1}{2} m^2 \hat{\phi}_2^2 + \frac{\lambda}{4} \hat{\phi}_2^4 + \frac{1}{64\pi^2} \left[6m_A^4 \left(\ln \frac{m_A^2}{\mu^2} - \frac{5}{6} \right) + 3m_B^4 \left(\ln \frac{m_B^2}{\mu^2} - \frac{5}{6} \right) \right. \\ &+ m_C^4 \left(\ln \frac{m_C^2}{\mu^2} - \frac{3}{2} \right) + 2m_{D^+}^4 \left(\ln \frac{m_{D^+}^2}{\mu^2} - \frac{3}{2} \right) + 2m_{D^-}^4 \left(\ln \frac{m_{D^-}^2}{\mu^2} - \frac{3}{2} \right) \\ &\left. + m_{E^+}^4 \left(\ln \frac{m_{E^+}^2}{\mu^2} - \frac{3}{2} \right) + m_{E^-}^4 \left(\ln \frac{m_{E^-}^2}{\mu^2} - \frac{3}{2} \right) - 12m_t^4 \left(\ln \frac{m_t^2}{\mu^2} - \frac{3}{2} \right) \right],\end{aligned}\quad (5.35)$$

where we have defined

$$\begin{aligned}
 m_A^2 &= \frac{1}{4} g_2^2 \hat{\phi}_2^2 \\
 m_B^2 &= \frac{1}{4} (g_1^2 + g_2^2) \hat{\phi}_2^2 \\
 m_C^2 &= 3\lambda \hat{\phi}_2^2 - m^2 \\
 m_{D\pm}^2 &= \frac{1}{2} \left(\lambda \hat{\phi}_2^2 - m^2 \pm \sqrt{(\lambda \hat{\phi}_2^2 - m^2) (\lambda \hat{\phi}_2^2 - m^2 - g_2^2 \xi_W \hat{\phi}_2^2)} \right) \\
 m_{E\pm}^2 &= \frac{1}{2} \left(\lambda \hat{\phi}_2^2 - m^2 \pm \sqrt{(\lambda \hat{\phi}_2^2 - m^2) (\lambda \hat{\phi}_2^2 - m^2 - (g_1^2 \xi_B + g_2^2 \xi_W) \hat{\phi}_2^2)} \right) \\
 m_t^2 &= \frac{1}{2} y_t^2 \hat{\phi}_2^2
 \end{aligned} \tag{5.36}$$

5.4 Resummed Effective Potential

In the previous section we found the Standard Model effective potential to 1-loop. We would now like to use the beta functions and anomalous dimension to resum this result following the same procedure as in the Abelian Higgs model in section 4.4. The 1-loop beta functions are [77, 78]

$$\begin{aligned}
 \beta_\lambda^{(1)} &= \frac{1}{16\pi^2} \left(\lambda (12y_t^2 - 3g_1^2 - 9g_2^2) + \frac{3}{8} (g_1^2 + g_2^2)^2 + \frac{3g_2^4}{4} + 24\lambda^2 - 6y_t^4 \right) \\
 \beta_{g_1}^{(1)} &= \frac{1}{16\pi^2} \left(\frac{41}{6} g_1^3 \right) \\
 \beta_{g_2}^{(1)} &= \frac{1}{16\pi^2} \left(-\frac{19}{6} g_2^3 \right) \\
 \beta_{g_3}^{(1)} &= \frac{1}{16\pi^2} (-7g_3^3) \\
 \beta_{y_t}^{(1)} &= \frac{1}{16\pi^2} \left(y_t \left(-\frac{17}{12} g_1^2 - \frac{9g_2^2}{4} - 8g_3^2 \right) + \frac{9y_t^3}{2} \right) \\
 \gamma_h^{(1)} &= \frac{1}{16\pi^2} \left(-\frac{1}{2} g_1^2 \xi_B - \frac{3}{2} g_2^2 \xi_W + \frac{3g_1^2}{2} + \frac{9g_2^2}{2} - 6y_t^2 \right)
 \end{aligned} \tag{5.37}$$

and the full 3-loop beta functions are given in appendix C.

5.4.1 Running of Standard Model Couplings

Before we resum the effective potential, we will have a quick look at the running of the Standard Model coupling constants. Figure 5.1 shows the running of the

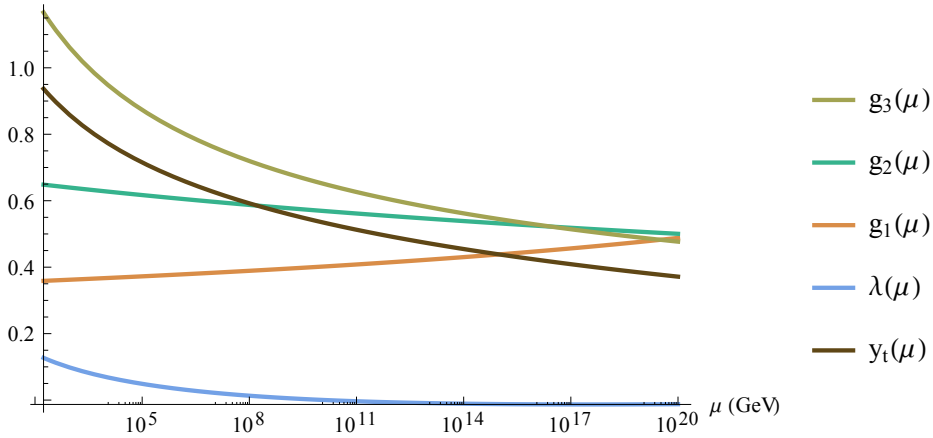


Figure 5.1: Running of the Standard Model coupling constants with 3-loop beta functions.

couplings solved numerically using `NDSolve` in Mathematica [75] with the initial conditions given by Buttazzo et al. [7]

$$\begin{aligned}
 \lambda(m_t) &= 0.12710, & y_t(m_t) &= 0.93697, & m_t &= 173.36\text{GeV}, \\
 g_1(m_Z) &= 0.35745, & g_2(m_Z) &= 0.65171, & g_3(m_Z) &= 1.1666, \\
 m_Z &= 91.1876\text{GeV}, & m_h &= 125.66\text{GeV}.
 \end{aligned}
 \tag{5.38}$$

To us, the most relevant feature of the running coupling constants in figure 5.1 is the running of the quartic coupling $\lambda(\mu)$. It is very interesting that it goes so to close to zero and almost flattens out at the Planck scale. Since we are interested in the stability of the Standard Model, we will be interested in seeing if $\lambda(\mu)$ goes negative or is always positive up to the Planck scale. This feature of the coupling depends strictly on the different initial conditions on the Higgs and top mass, and this will essentially be how we find the bound on the Higgs mass by requiring absolute stability of the Standard Model vacuum.

In addition to the running of the Standard Model couplings, we will also have to include the wave function renormalization

$$Z_\phi(\mu) = \exp \left[\int \gamma_h(\mu) d \ln \mu \right].
 \tag{5.39}$$

Note that γ_h is gauge dependent as can be seen in eq. (5.37) for $\gamma_h^{(1)}$.

5.4.2 Defining λ_{eff}

To study the stability of the Standard Model, we will follow the discussion given in section 3.4 and define an effective quartic coupling

$$V_{\text{eff}}[\hat{\phi}] = \frac{\lambda_{\text{eff}}(\hat{\phi})}{4} \hat{\phi}^4, \quad (5.40)$$

where we write $\hat{\phi}$ instead of $\hat{\phi}_2$ as the argument/scale dependence for simplicity and to get more compact notation. We saw in section 3.4 that this involves an approximation where we ignore the Higgs bare mass and running of this mass. This has been done by Ford [57] and Degrassi et al. [6], and we will follow this procedure. We have not checked the validity of this approximation, and we plan to do this in future work.

We will now define two different versions of the effective quartic coupling that are resummed using the RGE as described in section 3.3.2. We will first use the result from our calculation in section 5.3 and define $\lambda_{\text{eff}}^{(1)}$ which will be the 1-loop effective quartic coupling with gauge dependence. We will also include the result for the 2-loop calculation in the Landau gauge given by Degrassi et al. [6], and we will denote this by $\lambda_{\text{eff}}^{(2)}$.

In defining the effective quartic coupling in terms of the resummed coupling constants, we leave out the scale dependence to simplify the notation. I.e. we write just λ and not $\lambda(\phi)$ etc.

5.4.2.1 $\lambda_{\text{eff}}^{(1)}$: 1-loop with Gauge Dependence

Using eq. (5.35) we find that the resummed effective quartic coupling is

$$\begin{aligned} \lambda_{\text{eff}}^{(1)} = & \lambda Z_\phi^2 + \frac{1}{16\pi^2} Z_\phi^2 \left[6m_a^4 \left(\ln Z_\phi m_a^2 - \frac{5}{6} \right) + 3m_b^4 \left(\ln Z_\phi m_b^2 - \frac{5}{6} \right) \right. \\ & + m_c^4 \left(\ln Z_\phi m_c^2 - \frac{3}{2} \right) + 2m_{d_+}^4 \left(\ln Z_\phi m_{d_+}^2 - \frac{3}{2} \right) + 2m_{d_-}^4 \left(\ln Z_\phi m_{d_-}^2 - \frac{3}{2} \right) \\ & \left. + m_{e_+}^4 \left(\ln Z_\phi m_{e_+}^2 - \frac{3}{2} \right) + m_{e_-}^4 \left(\ln Z_\phi m_{e_-}^2 - \frac{3}{2} \right) - 12m_t^4 \left(\ln Z_\phi m_t^2 - \frac{3}{2} \right) \right] \end{aligned} \quad (5.41)$$

where we have defined

$$\begin{aligned} m_a^2 = \frac{1}{4}g_2^2, \quad m_b^2 = \frac{1}{4}(g_1^2 + g_2^2), \quad m_c^2 = 3\lambda, \quad m_t^2 = \frac{1}{2}y_t^2 \\ m_{d_\pm}^2 = \frac{1}{2} \left(\lambda \pm \sqrt{\lambda^2 - \lambda g_2^2 \xi_W} \right), \quad m_{e_\pm}^2 = \frac{1}{2} \left(\lambda \pm \sqrt{\lambda^2 - \lambda (g_1^2 \xi_B + g_2^2 \xi_W)} \right). \end{aligned} \quad (5.42)$$

We can simplify the notation, and write this compactly as

$$\lambda_{\text{eff}}^{(1)} = \lambda Z_\phi^2 + \frac{1}{16\pi^2} Z_\phi^2 \sum_i N_i m_i^2 (\ln [Z_\phi m_i^2] - C_i) \quad (5.43)$$

where $i \in \{a, b, c, d_\pm, e_\pm, t\}$ and the constants can be organized in a simple table:

i	a	b	c	d_\pm	e_\pm	t
N_i	6	3	1	2	1	-12
C_i	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

5.4.2.2 $\lambda_{\text{eff}}^{(2)}$: 2-loop in the Landau Gauge

Degrassi et al. [6] give the resummed effective potential to 2-loops in the Landau gauge. Due to the complexity of the 2-loop result, we only report 2-loop contribution for the strong and top Yukawa couplings, and we will denote this by $\lambda_{\text{eff}}^{(2)}$,

$$\begin{aligned} \lambda_{\text{eff}}^{(2)} = & \lambda Z_\phi^2 + \frac{1}{16\pi^2} Z_\phi^2 \sum_i N_i \kappa_i^2 (\ln [Z_\phi \kappa_i] - C_i) \\ & + \frac{1}{(16\pi^2)^2} y_t^4 \left[8g_s^2 (3 \ln^2 [Z_\phi \kappa_t] - 8 \ln [Z_\phi \kappa_t] + 9) \right. \\ & \left. - \frac{3}{2} y_t^2 \left(3 \ln^2 [Z_\phi \kappa_t] - 16 \ln [Z_\phi \kappa_t] + 23 + \frac{\pi^2}{3} \right) \right] \end{aligned} \quad (5.44)$$

where all the constants are given compactly in a table:

i	t	W	Z	h	χ
N_i	-12	6	3	1	3
C_i	$\frac{3}{2}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{3}{2}$	$\frac{3}{2}$
κ_i	$\frac{y_t}{2}$	$\frac{g_2^2}{4}$	$\frac{g_1^2 + g_2^2}{4}$	3λ	λ

Note that the strong coupling g_s first comes in at 2-loop order.

The gauge fixing procedure used by Degrassi et al. [6] is different from the one we have used in eq. (5.8). They have instead used the gauge fixing described in section 4.3.1 that cancels the kinetic mixing. This is however not a problem since we showed in section 4.3.2 that the effective potential with this gauge fixing matches the one we have used in the Landau gauge.

We have here intentionally chosen different indices for the index i for the sum in $\lambda_{\text{eff}}^{(2)}$ compared to the indices in $\lambda_{\text{eff}}^{(1)}$ to avoid any confusion as to what we are referring to. It is also convenient to keep the notation used by Degrassi et al. [6] in comparing our equations to the ones in their paper.

It is easy to match the indices: $\{a \leftrightarrow W\}$, $\{b \leftrightarrow Z\}$, $\{c \leftrightarrow h\}$ and $\{t \leftrightarrow t\}$. d_{\pm} and e_{\pm} depends on ξ_W and ξ_B and can only be compared when choosing $\xi_W = \xi_B = 0$. In this case the $m_{d_-} = m_{e_-} = 0$ and d_+ and e_+ combine to give the same result as χ . Seeing that the results match to 1-loop in the Landau gauge serves as a check that we have done our calculations in the previous section correctly.

5.5 Higgs Mass Bound

With the calculations completed in the previous sections we have all the ingredients we need to perform the analysis to find the Higgs mass bound. We will now start by discussing the stability conditions used to get the lower bound and details of the initial conditions used.

5.5.1 Absolute Stability Conditions

In section 3.4 we discussed the stability of the Standard Model and we saw that looking at the sign of λ_{eff} is a very good approximation for determining if the Standard Model is absolutely stable or not. We will now revisit this discussion looking at how the sign of λ_{eff} is determined by the initial condition on $\lambda(m_t)$.

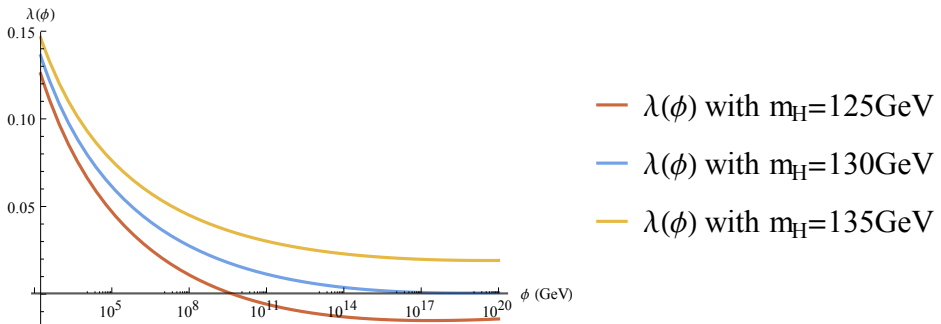


Figure 5.2: The plot shows the running of $\lambda(\phi)$ found from solving the 3-loop beta function for three different Higgs masses and fixed top mass $m_t = 173.35\text{GeV}$. Notice that for low Higgs masses, $\lambda(\phi)$ becomes negative. For high Higgs masses, $\lambda(\phi)$ is greater than zero for all ϕ up to the Planck scale.

In figure 5.2 we have plotted the resummed $\lambda(\phi)$ that we found from solving the 3-loop beta functions using `NDSolve` in Mathematica [75]. We have used the initial conditions described in section 5.5.3, and since the initial condition on $\lambda(m_t)$ depends on m_H we have plotted $\lambda(\phi)$ for three different values of the Higgs mass. We see that lowering the Higgs mass lowers the curve, and if the

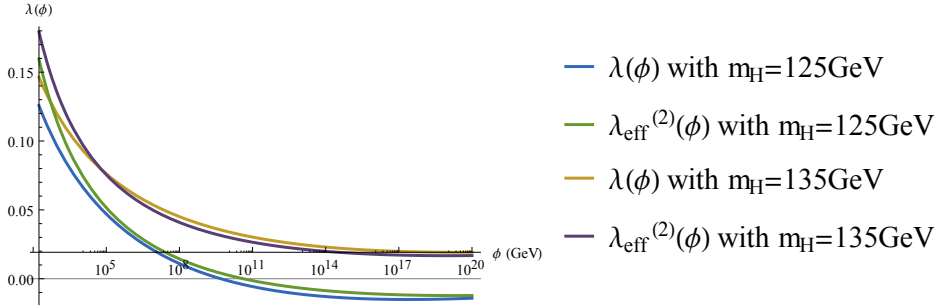


Figure 5.3: Comparison of $\lambda_{\text{eff}}^{(2)}(\phi)$ and $\lambda(\phi)$ for $m_H = 125\text{GeV}$ and $m_H = 135\text{GeV}$.

Higgs mass gets too low λ will go negative at some scale. For the 125GeV Higgs mass this happens around 10^{10}GeV , and for $m_H = 130\text{GeV}$ the curve approaches zero in the range up to the Planck scale.

In reality we are interested in looking at the sign of $\lambda_{\text{eff}}(\phi)$ up to the Planck scale. Figure 5.3 shows a comparison of $\lambda_{\text{eff}}^{(2)}(\phi)$ and $\lambda(\phi)$ for two different Higgs masses, $m_H = 125\text{GeV}$ and $m_H = 135\text{GeV}$. We see that for $m_H = 125\text{GeV}$ $\lambda_{\text{eff}}(\phi) > \lambda(\phi)$ for all ϕ up to the Planck scale, and for $m_H = 135\text{GeV}$ the two curves cross at around 10^5GeV . $\lambda_{\text{eff}}^{(2)}(\phi)$ and $\lambda(\phi)$ are however very similar, and Degrassi et al. [6] report that requiring $\lambda_{\text{eff}}^{(2)}(\phi) > 0$ or $\lambda(\phi) > 0$ only give 0.1GeV a difference in the Higgs mass bound.

The condition for absolute stability used by Degrassi et al. [6] and Buttazzo et al. [7] is $\lambda_{\text{eff}}(\phi) > 0$, and this is also the condition we will use in our analysis. Finding the Higgs mass bound is now reduced to changing the initial condition (Higgs mass) for a fixed top mass and finding the limiting value where $\lambda_{\text{eff}}(\phi) > 0$ for all ϕ up to the Planck scale.

5.5.2 Complex Effective Potential

Using the condition that $\lambda_{\text{eff}}^{(2)}(\phi) > 0$ for absolute stability, we see from figure 5.3 that we can end up in a situation where $\lambda_{\text{eff}}(\phi) > 0$ but $\lambda(\phi) < 0$. Looking at the expression for $\lambda_{\text{eff}}^{(1)}(\phi)$ and $\lambda_{\text{eff}}^{(2)}(\phi)$ in eq. (5.43) and eq. (5.44), respectively, we find a $\ln \lambda$ term. In other words, we are taking the logarithm of a negative number which will make the effective potential complex. For a complex $\lambda_{\text{eff}}(\phi)$ we do not know what the right condition for absolute stability is.

Erick Weinberg wrote a paper in 1987 [79] on how to understand the complex perturbative effective potentials, and he shows how the imaginary part has a natural interpretation as a decay rate per unit volume. This picture of imaginary parts relating to an instability matches on to our intuition from unstable particles

getting imaginary parts when the particles in a loop can go on-shell.

We wonder if the instability from the imaginary part may not be a problem for the stability of our vacuum since the imaginary part happens at a much higher scale than the vacuum we live in. Maybe this instability only arises if we tunnel to a lower energy state where $\lambda < 0$. We are still unsure about the physical interpretation of this to our analysis, and a further understanding will be a part of future work. We will simply ignore this imaginary part and effectively use the stability condition $\text{Re}[\lambda_{\text{eff}}(\phi)] > 0$.

From the definition of $\lambda_{\text{eff}}^{(1)}$ in eq. (5.43), it looks like we might also get a complex effective potential for certain values of ξ_W and ξ_B from the terms $m_{d_{\pm}}^2 = \frac{1}{2} \left(\lambda \pm \sqrt{\lambda^2 - \lambda g_2^2 \xi_W} \right)$ and $m_{e_{\pm}}^2 = \frac{1}{2} \left(\lambda \pm \sqrt{\lambda^2 - \lambda (g_1^2 \xi_B + g_2^2 \xi_W)} \right)$. We will now see that this will not happen due to the structure of the effective potential. Since we have one term with the + sign and one term with the - sign, the imaginary parts from the square roots will always cancel.

In general we can denote one of the $m_{i_{\pm}}^2$ in eq. (5.42) as some complex number since the square root can in general contribute to an imaginary part. Let $x, y, r, \alpha \in \mathbb{R}$ and write $m_{i_+}^2 = x + iy \equiv r e^{i\alpha}$ and $m_{i_-}^2 = x - iy \equiv r e^{-i\alpha}$. We can then write

$$\begin{aligned} & \text{Im} \left[(x + iy)^2 (\ln(x + iy) - c) + (x - iy)^2 (\ln(x - iy) - c) \right] \\ &= \text{Im} \left[(x^2 - y^2 + 2ixy) (\ln r e^{i\alpha} - c) + (x^2 - y^2 - 2ixy) (\ln r e^{-i\alpha} - c) \right] \quad (5.45) \\ &= \text{Im} \left[i\alpha(x^2 - y^2) + 2ixy(\ln r - c) - i\alpha(x^2 - y^2) - 2ixy(\ln r - c) \right] = 0. \end{aligned}$$

Due to the structure of the effective potential, the terms with square roots always come in pairs, resulting in the potential always being real. We only have to worry about the term $\ln \lambda$ that becomes complex when lambda goes negative.

5.5.3 Threshold Corrections and Initial Conditions

Much of the recent work by Degrandi et al. [6] and Buttazzo et al. [7] on the Higgs mass bound has been to improve the analysis of the 2-loop effective potential, 3-loop (NNLO) beta functions and 2-loop (NNLO) threshold corrections at the weak scale. The details of the current level of precision is neatly presented in Buttazzo et al. [7].

We have adopted the 2-loop results for the effective potential from Degrandi et al. [6] and the 3-loop beta function used is listed for completeness in appendix C. The details of the threshold corrections are not really important, and they all enter our calculation in terms of the initial conditions applied in solving the RGE equations for the Standard Model couplings. We have included the most

recent results reported in Buttazzo et al. [7]. The relevant pole masses are

$$\begin{aligned}
 m_t &= 173.36\text{GeV}, \\
 m_Z &= 91.1876\text{GeV}, \\
 m_H &= 125.66\text{GeV},
 \end{aligned}
 \tag{5.46}$$

and the initial conditions on the couplings are given at m_Z or m_t to be

$$\begin{aligned}
 \alpha_3(m_Z) &= 0.1184, \\
 g_1(m_Z) &= \sqrt{\frac{4\pi}{98.35}} \approx 0.35745, \\
 g_2(m_Z) &= \sqrt{\frac{4\pi}{29.587}} \approx 0.65171, \\
 g_3(m_t) &= 1.1666 + 0.00314 \left(\frac{\alpha_3(m_Z) - 0.1184}{0.0007} \right) \\
 &\quad - 0.00046 \left(\frac{m_t - 173.36\text{GeV}}{\text{GeV}} \right), \\
 \lambda(m_t) &= 0.12710 + 0.00206 \left(\frac{m_H - 125.66\text{GeV}}{\text{GeV}} \right) \\
 &\quad - 0.00004 \left(\frac{m_t - 173.36\text{GeV}}{\text{GeV}} \right), \\
 y_t(m_t) &= 0.93697 + 0.00550 \left(\frac{m_t - 173.36\text{GeV}}{\text{GeV}} \right) \\
 &\quad - 0.00042 \left(\frac{\alpha_3(m_Z) - 0.1184}{0.0007} \right).
 \end{aligned}
 \tag{5.47}$$

Notice how the initial condition on λ and y_t are functions of the pole masses m_H and m_t . Changing the Higgs or top mass essentially enters the calculation by changing this initial condition.

5.6 Higgs Mass Bound in the Landau Gauge

We start by analyzing the Higgs mass bound in the Landau gauge using the initial conditions in section 5.5.3 to compute $\lambda_{\text{eff}}^{(2)}(\phi)$ and setting $\text{Re} \left[\lambda_{\text{eff}}^{(2)}(\phi) \right] > 0$ for absolute stability. By varying the top mass and Higgs mass we find the phase diagram given in figure 5.4. The same diagram was computed by Buttazzo et al. [7] in figure 5.5, but note that figure 5.4 is plotted over a smaller range of the masses and does not include the boundary between instability and meta-stability.

We note that the fuzzy boarder between the absolute stability and non-perturbativity region in figure 5.4 are just numerical artifacts. These artifacts

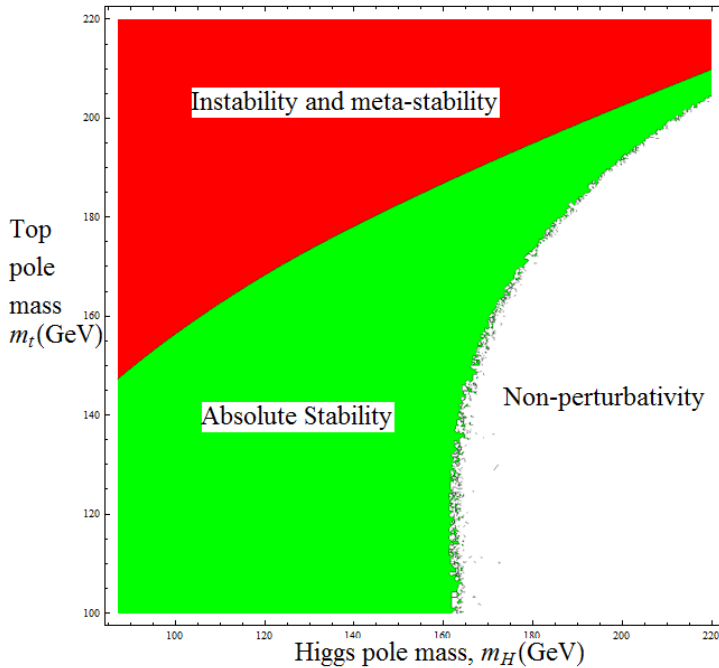


Figure 5.4: The Standard Model phase diagram in terms of the Higgs and top masses. The three sections are instability and meta-stability, absolute stability and non-perturbativity of the Higgs quartic coupling.

are due precision limitations in our calculation where `NDSolve` needs to decide if the Higgs quartic coupling converges or not. Since this boarder is not the main focus of our study, we have not gone out of our way to improve the numerical results in this region.

Fixing the top pole mass to $m_t = 173.36\text{GeV}$ we can read off figure 5.4 the allowed region for the Higgs pole mass m_H for the Standard Model to be absolutely stable. We find the bound to be

$$m_H > 129.6\text{GeV}, \quad (5.48)$$

which is exactly the result found by Buttazzo et al. [7]. For future reference, when we will compare this result to the bound found by $\lambda_{\text{eff}}^{(1)}$, we note that if we only include the 1-loop effective potential in $\lambda_{\text{eff}}^{(2)}$, the bound changes to 129.7GeV .

With this bound on the Higgs mass Buttazzo et al. [7] conclude that absolute vacuum stability of the Standard Model up to the Planck scale is excluded at 2.5σ

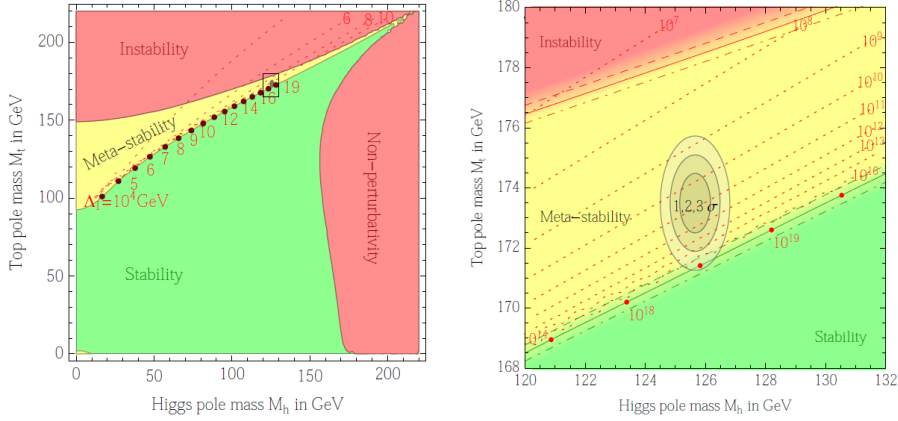


Figure 5.5: The left plot is the Standard Model phase diagrams in terms of the Higgs and top pole masses. The plane is divided into four different regions: Instability, Meta-stability, absolute stability and non-perturbativity of the Higgs quartic coupling.

The right plot shows the region of interest with the preferred experimental ranges indicated by rings corresponding to 1,2 or 3σ . The dotted lines indicate at which scale the instability occurs. The figure is taken from Buttazzo et al. [7].

(99.3% confidence level one-sided) with a Higgs mass $m_H = 125.66 \pm 0.34 \text{ GeV}$ [12]. This is indeed a very interesting result, indicating that we are living in a very interesting place in the phase diagram of the Standard Model. Looking at the right plot in figure 5.5 we live at the boundary between absolute stability and meta-stability. If we live in a meta-stable universe with a false vacuum, it means that there is a configuration with lower energy that we can decay to, but the time it takes to tunnel to the true vacuum is greater than the age of our universe.

It is also very interesting to note that the experimental values lie well within the range of parameters in which we can extrapolate the Standard Model up to the Planck scale without having to add any new physics to make the theory consistent. This provides us with a way to check the consistency of the Standard Model at energy scales way beyond what is possible in today's collider experiments.

From figure 5.4 we only see that the current measured values of the Higgs and top masses lie in the unstable and meta-stable region. We have not yet computed the boundary between the instability and meta-stability region, and this calculation will be included in future work.

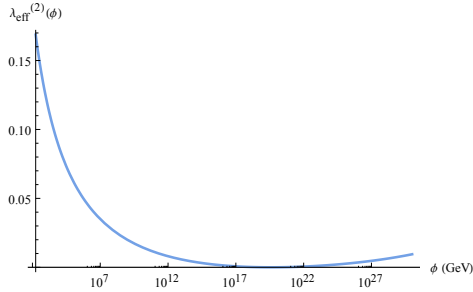


Figure 5.6: Plotting $\lambda_{\text{eff}}^{(2)}(\phi)$ beyond the Planck scale for the limiting value $m_H = 129.6\text{GeV}$, we find that $\lambda_{\text{eff}}^{(2)}(\phi)$ has a minimum around the Planck scale and then increases again.

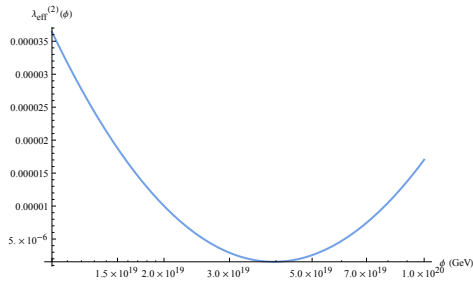


Figure 5.7: For the limiting value $m_H = 129.6\text{GeV}$ the minimum of $\lambda_{\text{eff}}^{(2)}(\phi)$ is located around $4 \times 10^{19}\text{GeV}$.

5.6.1 Sensitivity of Planck Scale Cutoff

We have argued that we are only interested in the energy scales up to the Planck scale since we expect new physics to be relevant in the calculation of the effective potential. Regardless of this argument, we are interested in whether the Higgs mass bound is sensitive to changing the Planck scale cutoff.

To investigate the sensitivity to the cutoff, we plotted $\lambda_{\text{eff}}^{(2)}(\phi)$ beyond 10^{20}GeV to look closer at the behavior at large scales. The result for the limiting value $m_H = 129.6\text{GeV}$ is shown in figure 5.6, and we see that $\lambda_{\text{eff}}^{(2)}(\phi)$ has a minimum around the Planck scale before it increases again. Figure 5.7 shows a close-up on the region of interest, and we see that the minimum is located at $\phi = 4 \times 10^{19}\text{GeV}$. The location of the minimum is remarkably close to the Planck scale, but since it is below 10^{20}GeV , we conclude that the Higgs mass bound is insensitive to increase of the cutoff. However, lowering the cutoff would affect the Higgs mass stability bound.

5.7 Gauge Dependence of the Higgs Mass Bound

As noted several times earlier, the effective potential for a theory with a gauge symmetry will in general be gauge dependent. We see this explicitly in the definition of $\lambda_{\text{eff}}^{(1)}$ in eq. (5.43) where we have explicit dependence on ξ_W and ξ_B in $m_{d_{\pm}}^2$ and $m_{e_{\pm}}^2$, and implicit dependence on ξ_W and ξ_B in Z_{ϕ} since γ_h is gauge dependent. We are interested in seeing if the ξ_B and ξ_W dependence affects the value of the Higgs mass bound found by requiring absolute stability.

In this section we will for simplicity only consider the case $\xi = \xi_W = \xi_B$. We will compute the bound on the Higgs mass using $\lambda_{\text{eff}}^{(1)}$ defined in eq. (5.43), and we will use the 3-loop beta function for all the Standard Model coupling constants. The procedure for finding the Higgs mass bound is the same as described in the previous section for the Landau gauge.

In figure 5.8 we have computed the Higgs mass bound as a function of ξ . The important thing to note is that we see a gauge dependence in the Higgs mass bound. Quantitatively we see a variation of about 0.1GeV over the chosen range of ξ parameters, starting at 129.7GeV for $\xi = 0$ and increasing to 129.8GeV for $\xi = 50$.

To see the importance of the different loop order contributions of γ_h to the mass bound, we have made three different plots in figure 5.8. These three plots include the anomalous dimension up to 1-loop, 2-loop or 3-loops. We observe that the results are very similar, and in the rest of the analysis we will always use the 3-loop anomalous dimension.

Before we go into the discussion about the gauge dependence in the Higgs mass bound, we want to give some more results from our analysis on the behavior of the Higgs mass bound when changing the parameter ξ .

In figure 5.8 we see that the mass bound starts to plateau for large values of ξ . Figure 5.9 shows a plot of the bound for ξ up to 150, and in this range this tendency is even clearer.

We have investigated the technical reason behind the increase of the Higgs mass bound in terms of plotting $Z_{\phi}^{-2}\lambda_{\text{eff}}^{(1)}(\phi)$ ¹. Understanding when $Z_{\phi}^{-2}\lambda_{\text{eff}}^{(1)}(\phi)$ goes negative will give us a better understanding of the gauge dependence of the Higgs mass bound.

First, we consider $m_H = 129.80\text{GeV}$ in figure 5.10 which is a typical case where $\lambda_{\text{eff}}^{(1)}$ is greater than zero for $\xi = 0$ and becomes negative for large ξ values. To keep $\lambda_{\text{eff}}^{(1)} > 0$, we must increase m_H for the larger ξ values, which is exactly what we see in figure 5.8 with the bound increasing for higher values of ξ .

Now we consider the Higgs mass, $m_H = 129.83\text{GeV}$ in figure 5.11. For this

¹We saw when studying the ξ dependence on λ_{eff} in chapter 4 that for large ξ the Z_{ϕ}^2 factor in λ_{eff} complicated the analysis by suppressing λ_{eff} for large scales. Since Z_{ϕ} always will be positive and we only care about the sign of $\lambda_{\text{eff}}^{(1)}$, we will simply plot $Z_{\phi}^{-2}\lambda_{\text{eff}}^{(1)}$.

5.7. GAUGE DEPENDENCE OF THE HIGGS MASS BOUND

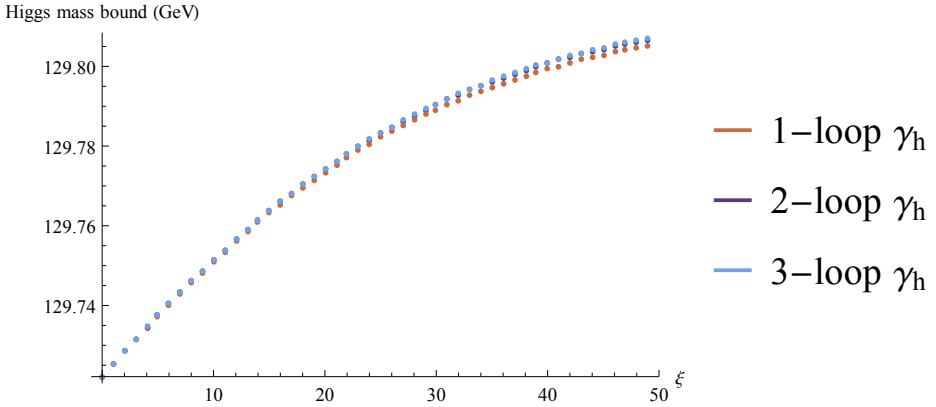


Figure 5.8: Plot of the ξ dependence of the Higgs mass bound including the anomalous dimension up to 1-loop, 2-loop and 3-loop of γ_h . Note that the 2-loop and 3-loop results are almost overlapping, making it hard to distinguish the dots from the different curves.

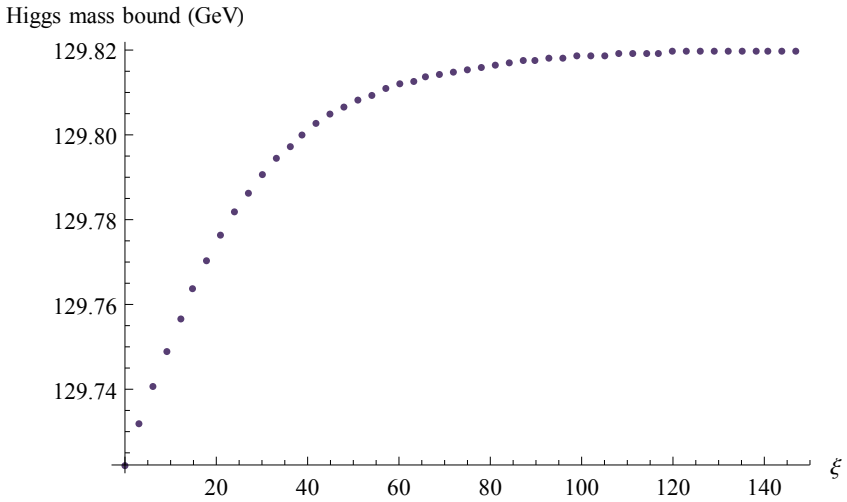


Figure 5.9: We find that the Higgs mass bound as a function of the gauge parameter ξ plateaus for large ξ . The values are computed using 3-loop anomalous dimension γ_h .

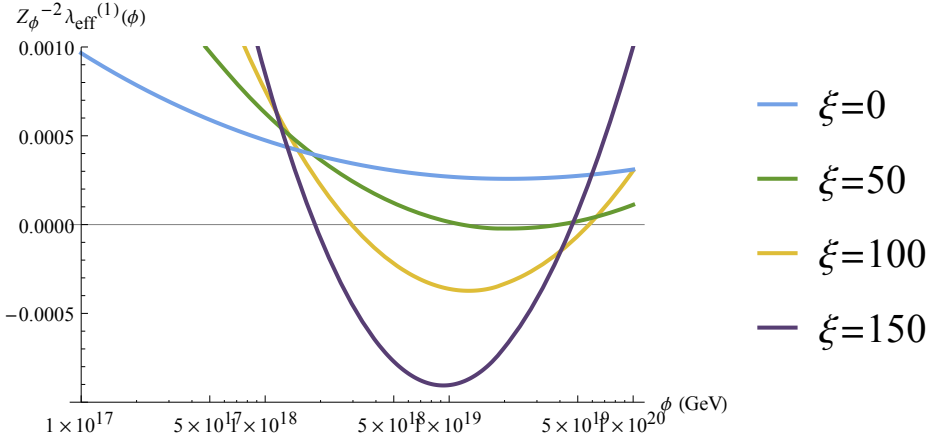


Figure 5.10: For fixed Higgs mass, $m_H = 129.80\text{GeV}$, we see the ξ dependence on $\lambda_{\text{eff}}^{(1)}$ in the region close to the Planck scale. We find that the curve is lowered by increasing ξ .

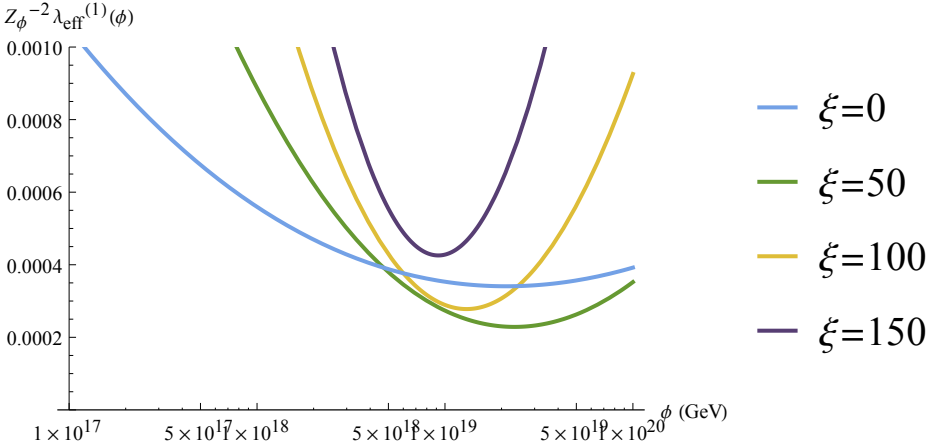


Figure 5.11: For fixed Higgs mass, $m_H = 129.83\text{GeV}$, we see the ξ dependence on $\lambda_{\text{eff}}^{(1)}$ in the region close to the Planck scale. For this value of the Higgs mass the curve is lowered when going from $\xi = 0$ to $\xi = 50$, but we find that it increases again for $\xi = 100$ and $\xi = 150$.

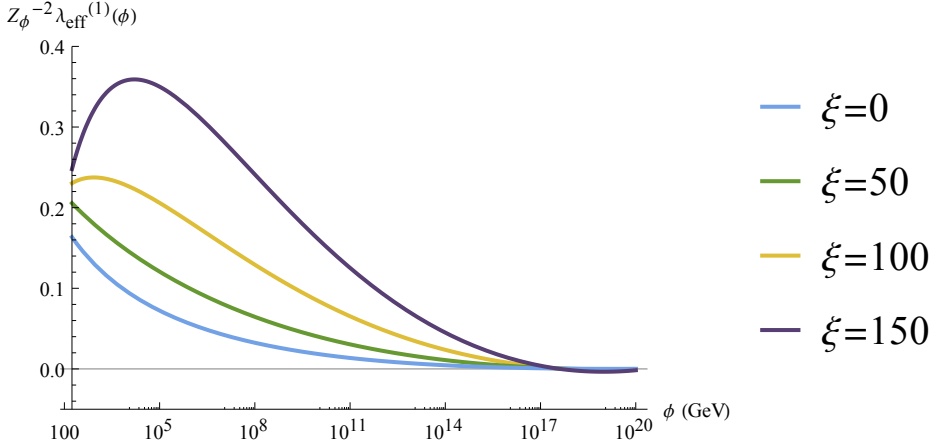


Figure 5.12: For a fixed Higgs mass, $m_H = 129.7\text{GeV}$, we find very different qualitative features of $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ for different values of ξ . But notice that the location of the minimum changes remarkably little.

fixed value of the Higgs mass, qualitative change in the curve when increasing ξ is very different from what we found in figure 5.10. For the $m_H = 129.83\text{GeV}$ case, the curve with $\xi = 0$ is at first lowered by increasing ξ . We see this by comparing the $\xi = 0$ and $\xi = 50$ curves. However, when increasing ξ to 100 or 150 the curve is now raised instead of lowered, and for no value of ξ will the curve intersect the ϕ axis. In this case $\lambda_{\text{eff}}^{(1)}$ will be greater than zero for all values of ξ .

Figure 5.10 and 5.11 explains the qualitative behavior of the Higgs mass bound as a function of ξ in figure 5.9. These are however just the numerical results we have found, and our understanding these results is in no way complete.

Lastly, we would like to point out one more feature with the plots of $\lambda_{\text{eff}}^{(1)}$ that we find interesting. In figure 5.12 we plot $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ for the limiting value $m_H = 129.7\text{GeV}$ that just barely touches zero for $\xi = 0$. We note that the qualitative features of $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ for different values of ξ are very different. For $\xi = 0$, $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ is monotonically decreasing, while for $\xi = 150$, $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ increases and reaches a maximum before decreasing all the way to the Planck scale. However we note that compared to the differences in $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)}$ from full range of ϕ , the variation of the location of the minimum is surprisingly small.

This feature becomes even more interesting when we remake the calculation in figure 5.12 for $m_H = 125\text{GeV}$ and $m_H = 115\text{GeV}$ in figure 5.13 and 5.14, respectively. Notice that the points where $Z_\phi^{-2}\lambda_{\text{eff}}^{(1)} = 0$ for the different values of ξ are remarkably close to each other.

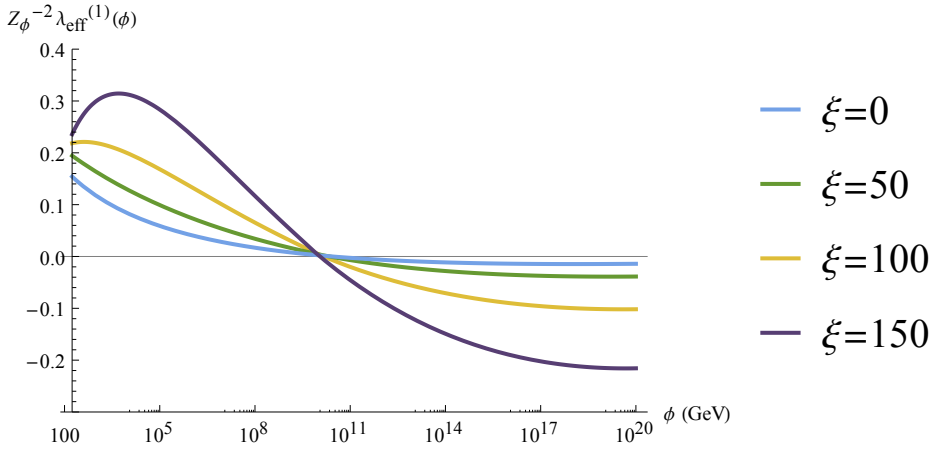


Figure 5.13: With a Higgs mass, $m_H = 125$ GeV, we notice how surprisingly close location of the point where $Z_\phi^{-2} \lambda_{\text{eff}}^{(1)} = 0$ is for the four different values of ξ .

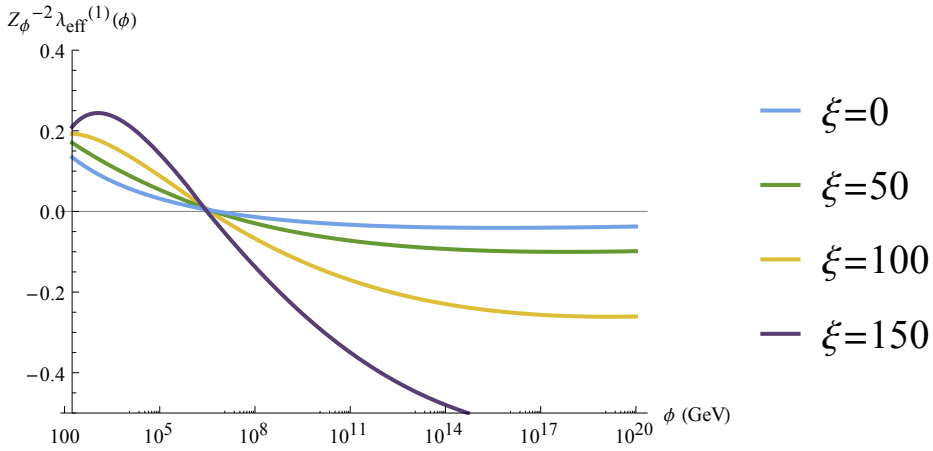


Figure 5.14: With a Higgs mass, $m_H = 115$ GeV, we notice how surprisingly close location of the point where $Z_\phi^{-2} \lambda_{\text{eff}}^{(1)} = 0$ is for the four different values of ξ .

5.8 Discussion

In the previous section we found that the bound on the Higgs mass from an absolute stability requirement is gauge dependent, as shown in figure 5.8. This means that for a given Higgs mass, determining if the potential has a minimum for large field values will not be a gauge invariant statement, as we found in figure 5.10. This observation is merely an identification of a problem with the current method of analyzing the Standard Model by extrapolating the effective potential up to the Planck scale, as used most recently by Buttazzo et al. [7]. We think this problem deserves some attention and further analysis should attempt to find a gauge independent procedure for determining the stability of the Standard Model.

To solve this problem of gauge dependence, the most natural place to start looking for a solution is analyzing the results found in the previous section.

While it is true that the Higgs mass bound we found was gauge dependent, the fact that the curve in figure 5.9 is flat for large ξ is very interesting. Compared to a proper calculation of the Higgs mass bound, one can imagine that our current procedure is inconsistent in some way for small ξ , and this is what is giving us the gauge dependence we are seeing in figure 5.8. Fixing such an inconsistency might give us a gauge independent bound on the Higgs mass. It is also possible for the flat region we are seeing to be an inconsistent artifact of our current calculation, and that the Higgs mass bound really should be growing without an upper bound. The latter scenario would leave us with the conclusion that the current procedure really is meaningless in terms of analyzing the stability of the Standard Model. This is all speculation and the resolution might be something completely different we have yet not thought about. We will attempt to answer these questions in future work.

One remarkable feature of figure 5.12, 5.13 and 5.14 is the observation that for a fixed value of the Higgs mass, the points ϕ where $\lambda_{\text{eff}}^{(1)}(\phi) = 0$ for different values of ξ are almost equal. We say *almost* equal because we know from figure 5.10 that they do not actually cross the ϕ axis at exactly the same point, which is why we found the Higgs mass bound to be gauge dependent in the first place. However, looking at figure 5.12, 5.13 and 5.14 it is hard to imagine that this is happening by pure accident.

Based on these findings, we speculate if there might something gauge invariant about the point where $\lambda_{\text{eff}}^{(1)} = 0$, or at least a gauge invariant property related to this observation. If one can formally find such a gauge invariant property, we will have a formal proof showing that the absolute stability bound is gauge independent since we computed the Higgs mass bound by finding the limiting value when $\lambda_{\text{eff}}^{(1)} > 0$. It is very plausible that some of the gauge artifacts we are seeing in the Higgs mass bound are effects due to the fact that we are working to a fixed loop order in the effective potential, beta functions and threshold

corrections. Seeing how the gauge dependence on the Higgs mass bound changes by adding in higher order effects might give an indication on what the resolution to this problem is.

We know from section 3.3.3.1 of one formal condition related to the gauge dependence of the effective potentials. In eq. (3.110) we found the Nielsen identity

$$\xi \frac{\partial V_{\text{eff}}}{\partial \xi} = C(\phi) \frac{\partial V_{\text{eff}}}{\partial \phi}, \quad (5.49)$$

where $C(\phi)$ is defined in eq. (3.108). However, we do not see at this point how this applies to our calculation. The application of this identity is typically that at the minimum of the potential

$$\left. \frac{\partial V_{\text{eff}}}{\partial \phi} \right|_{\phi_0} = 0 \quad \implies \quad \left. \xi \frac{\partial V_{\text{eff}}}{\partial \xi} \right|_{\phi_0} = 0, \quad (5.50)$$

and this interpreted as saying that the effective potential at the minimum is gauge independent. However, by comparing to the plots in the previous section it is not clear how this enters our calculation since we observe both a change in the value of the effective potential at the minimum, and a change in the location of the minimum. We will again note that these effects might be artifacts from not including higher order contributions, and we are aware of the fact that the Nielsen identity must be satisfied order by order in perturbation theory as described by Nielsen [58].

We would again like to review how we in massless scalar QED with symmetry breaking by radiative corrections was able to predict a gauge independent scalar to vector mass ratio from the gauge dependent effective potential in section 4.2.3. By renormalizing the effective potential at the minimum, Coleman and Weinberg [15] found $\lambda = \mathcal{O}(e^4)$, and by only keeping the terms to leading order in e the scalar to vector mass ratio came out gauge invariant. It is possible that such a procedure can be performed to find the bound on the Higgs mass, but due to the numerical solutions it is difficult to such a calculation at this stage. Instead of trying to solve the problem in the Standard Model, it might be possible to construct a simpler theory with the same features that can be solved analytically to address these questions.

Chapter 6

Summary, Conclusion and Outlook

6.1 Summary

In this thesis, we have studied the gauge dependence of the Standard Model effective potential and how it relates to the Higgs mass bound derived from requiring stability of the Standard Model vacuum.

We have presented a detailed derivation of the 1PI effective action including different approaches for computing the effective potential. We found the background field method together with the path integral particularly useful and applied it throughout chapter 4 and 5.

Before studying the Standard Model, we used the Abelian Higgs model to learn how to work with the gauge dependent 1-loop effective potential. We modified Jackiw's [13] functional method of computing the gauge-dependent effective potential, and with our method we computed the 1-loop effective potential in basically one step, with the same amount of work with or without kinetic mixing in the Lagrangian.

In the Abelian Higgs model, we also performed a detailed comparison of different ways of gauge fixing. We considered $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A_\mu)^2$, $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A_\mu + \xi e v \phi_2)^2$ and $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A_\mu + \xi e v \phi_1)^2$. The second and third gauge fixing Lagrangians are chosen to cancel the kinetic mixing, but we showed that the third one does not work as intended when using the background field method since v is different from the background field.

Motivated by the analysis in chapter 5 on the Higgs mass bound, we used the beta functions and anomalous dimension of the Higgs boson to find the resummed effective potential valid up to the Planck scale for massless scalar QED. We did

a numerical comparison of the different effective quartic couplings and saw that the ξ -dependence was significant for scale dependence of the effective potential, but most of the effect came from the field strength renormalization.

We have also taken the massless limit of the Abelian Higgs model to study some historically important calculations relating to spontaneous symmetry breaking generated by radiative corrections. We reproduced the 1-loop gauge-invariant scalar-to-vector mass ratio, and we believe this knowledge might be useful in attempting to solve the problem of the gauge dependence found in the Higgs mass bound.

With the complete analysis for the Abelian Higgs model, we extended the calculation of the 1-loop effective potential with gauge dependence to the Standard Model. Using the 3-loop beta functions and the 2-loop threshold corrections, we found the resummed Standard Model effective potential.

We reproduced the most recent Higgs mass bound by Buttazzo et al. [7], $m_H > 129.6\text{GeV}$ for absolute stability using the 2-loop effective quartic coupling, and we used the same analysis on our gauge-dependent 1-loop effective potential.

6.2 Conclusions and Outlook

We have computed the Higgs mass bound for different values of the gauge parameter ξ using the same procedure that Buttazzo et al. [7] used in the Landau gauge. The main result from this thesis is that we are seeing a gauge dependence on this bound. We found a variation in the Higgs mass bound of 0.1GeV varying when ξ from 0 to 50. We also found that the bound plateaus for roughly $\xi > 100$. We find the plateau to be very interesting, and we will investigate these results further in future work.

There are basically two kinds of future work to be done on this analysis of the gauge dependence of the Higgs mass bound. First, there are multiple improvements that can be made on our current analysis. We want to go back and check that all the approximations are valid, and see how big the effects are. These approximations include dropping the mass terms in the effective potential and ignoring the imaginary part of the effective potential when $\lambda < 0$. We also want to improve the numerical solutions and get better control over the numerical errors. Quantifying the errors beyond the standard Mathematica [75] accuracy will be very important. Since the ξ -dependence is small, even small numerical errors can affect our result, and we want to make sure that the effects that we see are not numerical artifacts of any sort.

Future work also includes adding new calculations to our analysis. We want to add in the metastability bound and compute the gauge dependence of this bound as well. Understanding the metastability bound also involves understanding new theoretical concepts, and this might take some time to accomplish. It would also be interesting to compute the effective potential with a different gauge fixing

function, for example, one that cancels the kinetic mixing, and see how sensitive the Higgs mass bound is to ξ , and then compare this to our current gauge fixing functions.

A problem with the Standard Model calculations is that we are in many of cases forced to solve the differential equations numerically. So in the future we will be interested in writing down simple toy models with the right properties so we can analyze the gauge dependence analytically.

Eventually, the goal is to formulate gauge-invariant stability and metastability bounds for the resummed potential.

Appendices

Appendix A

Gaussian Integrals

In this appendix we will calculate Gaussian integrals in a range of different contexts. We start with the basic one dimensional case and go on to do multiple dimensions with both real, complex and Grassmann variables [80, 31].

A.1 Gaussian Integral in 1 dimension

Consider the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + Jx}, \quad (\text{A.1})$$

where a and J are some real constants. The first step is to complete the square and then we shift $x \rightarrow x + \frac{J}{a}$

$$\begin{aligned} \mathcal{I} &= \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}a(x-\frac{J}{a})^2 + \frac{J^2}{2a}} \\ &\Downarrow \\ \mathcal{I} &= \frac{1}{\sqrt{a}} e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} dx' e^{-\frac{1}{2}x'^2}, \end{aligned} \quad (\text{A.2})$$

where we have changed variables to $x' = \sqrt{a}x$ in the last step giving the $\frac{1}{\sqrt{a}}$ in front. To compute the integral we use the following trick

$$\begin{aligned} \left[\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} \right]^2 &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}(x^2+y^2)} \\ &= 2\pi \int_0^{\infty} dr r e^{-\frac{1}{2}r^2} = 2\pi \end{aligned} \quad (\text{A.3})$$

where we have changed from Cartesian to radial coordinates. In conclusion we find

$$\mathcal{I} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}. \quad (\text{A.4})$$

A.2 Gaussian Integral in n dimensions

Let x_a and J_a be n dimensional vectors and M_{ab} be a $n \times n$ dimensional real and symmetric matrix. We want to perform the n dimensional Gaussian integral

$$\mathcal{I} = \int d^n x e^{-\frac{1}{2}x_a M_{ab} x_b + J_a x_a}. \quad (\text{A.5})$$

Since M_{ab} is real and symmetric we can always diagonalize it, i.e. find an orthonormal matrix P such that $P_{ab}^{-1} M_{bc} P_{cd} = d_a \delta_{ad}$ where d_a are the eigenvalues of M_{ab} . Now choose a new set of coordinates $y_a = P_{ab}^{-1} x_b$. Changing coordinates doesn't change the measure since $d^y = |\text{Det} P^{-1}| d^n x = d^n x$, where we have used that the absolute value of the determinant of an orthonormal matrix is one. We find

$$\mathcal{I} = \int d^n y e^{-\frac{1}{2}y_a d_a \delta_{ab} y_b + J_a P_{ab} y_b}. \quad (\text{A.6})$$

We see that we can now separate this integral into a product of one dimensional Gaussian integrals.

$$\mathcal{I} = \prod_i \int dy_i e^{-\frac{1}{2}d_i y_i^2 + (J_a P_{ai}) y_i}. \quad (\text{A.7})$$

Using equation (A.4) we find

$$\mathcal{I} = \prod_i \sqrt{\frac{2\pi}{d_i}} e^{\frac{(J_a P_{ai})^2}{2d_i}}. \quad (\text{A.8})$$

To simplify this, we want to write the result in terms of the original matrices and vectors. First, note that the product of the eigenvalues d_i is equal to the determinant of the original matrix M_{ab} , so we can write $\prod_i d_i = \det M$. Second, remember that $P_{ab}^{-1} M_{bc} P_{cd} = d_a \delta_{ad}$ has an equivalent expression for the inverse of the matrix $P_{ab}^{-1} M_{bc}^{-1} P_{cd} = d_a^{-1} \delta_{ad}$. Rewriting this in terms of the inverse matrix we find $M_{ab}^{-1} = P_{ai} d_i^{-1} P_{ib}^{-1}$. Now we can simplify the exponential in eq. (A.8)

$$\prod_i e^{\frac{(J_a P_{ai})^2}{2d_i}} = e^{\sum_i \frac{1}{2} J_a P_{ai} d_i^{-1} P_{ib}^{-1} J_b} = e^{\frac{1}{2} J_a M_{ab}^{-1} J_b}, \quad (\text{A.9})$$

where we have used that P is orthonormal such that $P_{ab}^{-1} = P_{ab}^T = P_{ba}$.

In conclusion we find

$$\mathcal{I} = \int d^n x e^{-\frac{1}{2}x_a M_{ab} x_b + J_a x_a} = \sqrt{\frac{(2\pi)^n}{\det M}} e^{\frac{1}{2}J_a M_{ab}^{-1} J_b}. \quad (\text{A.10})$$

A.3 Gaussian Integral over complex coordinates

Let $a \in \mathbb{R}$, $J \in \mathbb{C}$ and consider the integral

$$\mathcal{I} = \int dz dz^* e^{-a z z^* + J z^* + J^* z}. \quad (\text{A.11})$$

We change coordinates to x and y in the normal way $z = x + iy$, and we define $J = J_x + iJ_y$. Changing the coordinates we get a Jacobian factor that is equal to 2. The integral becomes

$$\begin{aligned} \mathcal{I} &= 2 \int dx dy e^{-a(x^2+y^2)+2J_x x+2J_y y} \\ &= 2 \int dx e^{-ax^2+2J_x x} \int dy e^{-ay^2+2J_y y} \\ &= 2 \sqrt{\frac{2\pi}{2a}} e^{\frac{4J_x^2}{4a}} \times \sqrt{\frac{2\pi}{2a}} e^{\frac{4J_y^2}{4a}} \\ &= 2 \frac{\pi}{a} e^{\frac{JJ^*}{a}} \end{aligned} \quad (\text{A.12})$$

where we have used the result from eq. (A.4). In conclusion

$$\mathcal{I} = \int dz dz^* e^{-a z z^* + J z^* + J^* z} = 2 \frac{\pi}{a} e^{\frac{JJ^*}{a}} \quad (\text{A.13})$$

A.4 Gaussian Integral over multiple complex coordinates

Let H_{ij} be a $n \times n$ dimensional Hermitian matrix, $z_i \in \mathbb{C}$ for $i = 1, 2, \dots, n$, and consider the Gaussian integral

$$\mathcal{I} = \int d^n z d^n z^* e^{-z_i^* H_{ij} z_j + J_i z_i^* + J_i^* z_i}. \quad (\text{A.14})$$

Since H is a Hermitian matrix, we know that we can diagonalize it as $H_{ij} = P_{ik} d_k \delta_{kl} P_{lj}^{-1}$ where P is unitary ($P^\dagger = P^{-1}$) and $d_k \in \mathbb{R}$ are the eigenvalues of

H . We change coordinates to $w \in \mathbb{C}$ such that $z_i = P_{ij}w_j$. The integral becomes

$$\begin{aligned}
\mathcal{I} &= \int d^n z d^n z^* e^{-w_i^* d_i \delta_{ij} w_j + J_i P_{ij}^* w_j^* + J_i^* P_{ij} w_j} \\
&= \prod_i \int dz_i dz_i^* e^{-d_i w_i^* w_i + (J_j P_{ji}^*) w_i^* + (J_j^* P_{ji}) w_i} \\
&= \prod_i 2 \frac{\pi}{d_i} e^{J_k^* P_{ki} \frac{1}{d_i} P_{ij}^* J_j} \\
&= \frac{(2\pi)^n}{\det H} e^{J_i^* H_{ij}^{-1} J_j}
\end{aligned} \tag{A.15}$$

where we have used that $\prod_i d_i = \det H$ and $H_{kj}^{-1} = P_{ki} \frac{1}{d_i} P_{ij}^*$. In conclusion

$$\mathcal{I} = \int d^n z d^n z^* e^{-z_i^* H_{ij} z_j + J_i z_i^* + J_i^* z_i} = \frac{(2\pi)^n}{\det H} e^{J_k^* H_{kj}^{-1} J_j} \tag{A.16}$$

A.5 Gaussian Integral over Grassmann Variables

In this section we will compute the Gaussian integral over Grassmann variables. Since Grassmann variables are very different from numbers we usually deal with, we will start by reviewing some of the basic properties.

A.5.1 Properties of Grassmann Variables

We define a set of n Grassmann variables θ_i , $i = 1, 2, \dots, n$ that satisfy

$$\{\theta_i, \theta_j\} = 0, \quad [x, \theta_i] = 0, \tag{A.17}$$

where x is a normal c -number. We want to define a function of Grassmann variables, and then we want to define differentiation and integration of this function. Any function of Grassmann variables can be defined in terms of its Taylor series. For simplicity let's consider the case where we only have two Grassmann numbers η and θ . Since they satisfy

$$\eta^2 = 0, \quad \theta^2 = 0, \quad \eta\theta = -\theta\eta, \tag{A.18}$$

the most general function can be written as

$$f(\eta, \theta) = a + b\eta + c\theta + d\theta\eta \tag{A.19}$$

where $a, b, c, d \in \mathbb{C}$. For n Grassmann variables we need $\frac{1}{2}(n^2 + n + 2)$ complex numbers to define the most general function. We define differentiation of a c -number and Grassmann number to satisfy

$$\frac{da}{d\theta_i} = 0, \quad \frac{d\theta_i}{d\theta_j} = \delta_{ij}. \tag{A.20}$$

If we are differentiating a product of Grassmann variables, we must remember to include a minus sign when $\frac{d}{d\theta_k}$ passes a Grassmann variable to make the definition take into account the anticommuting nature of the Grassmann variables. For the case of two variables, we find

$$\frac{d(\theta_i\theta_j)}{d\theta_k} = \frac{d\theta_i}{d\theta_k}\theta_j - \theta_i\frac{d\theta_j}{d\theta_k} = \theta_j\delta_{ik} - \theta_i\delta_{jk}. \quad (\text{A.21})$$

Integration is defined to be exactly the same as differentiation,

$$\int d\theta_i a = 0, \quad \int d\theta_i\theta_j = \delta_{ij}, \quad \int d\theta_i(\theta_i\theta_j) = \theta_j. \quad (\text{A.22})$$

To calculate the Gaussian integral, we will need to know how to integrate over multiple variables. Consider the integral

$$\mathcal{I} = \int d\theta_n d\theta_{n-1} \cdots d\theta_2 d\theta_1 [\theta_{i_1}\theta_{i_2} \cdots \theta_{i_n}]. \quad (\text{A.23})$$

This integral can only be nonzero if for every $d\theta_j$ there is one θ_j among the θ_{i_n} (and only one since $\theta_j^2 = 0$). If we relabel two neighboring indices $\theta_{i_j}\theta_{i_{j+1}}$ we must get an overall minus sign since they are Grassmann variables. We conclude that the result must be the totally antisymmetric and the result is

$$\mathcal{I} = \int d\theta_n d\theta_{n-1} \cdots d\theta_2 d\theta_1 [\theta_{i_1}\theta_{i_2} \cdots \theta_{i_n}] = \epsilon_{i_1 i_2 \cdots i_n}. \quad (\text{A.24})$$

Note that the ordering is consistent with the ordering convention $\int d\eta d\theta\theta\eta = 1$ and the fact that $\epsilon_{123 \cdots n} = 1$.

A.5.1.1 Ordering of Grassmann variables

We will now prove the general ordering result

$$\eta_1\theta_1\eta_2\theta_2 \cdots \eta_n\theta_n = (-1)^{\frac{1}{2}n(n-1)}\eta_1\eta_2 \cdots \eta_n\theta_1\theta_2 \cdots \theta_n \quad (\text{A.25})$$

using induction. Let's check it for $n = 3$.

$$\eta_1\theta_1\eta_2\theta_2\eta_3\theta_3 = \eta_1(-\eta_2\theta_1)(-\eta_3\theta_2)\theta_3 = -\eta_1\eta_2\eta_3\theta_1\theta_2\theta_3, \quad (\text{A.26})$$

so it is correct for $n = 3$ since $(-1)^{\frac{1}{2}3(3-1)} = -1$. Now we assume that the result is true for $n = j - 1$, and we want to check the result for $n = j$. We start with the expression with j θ s and η s and note that the only thing missing is moving η_j past $j - 1$ θ s giving a factor of $(-1)^{j-1}$,

$$\begin{aligned} \eta_1\theta_1 \cdots \eta_j\theta_j &= \left[(-1)^{\frac{1}{2}(j-1)(j-2)}\eta_1 \cdots \eta_{j-1}\theta_1 \cdots \theta_{j-1} \right] \eta_j\theta_j \\ &= (-1)^{\frac{1}{2}(j-1)(j-2)+(j-1)} [\eta_1 \cdots \eta_{j-1}\eta_j] [\theta_1 \cdots \theta_{j-1}\theta_j] \\ &= (-1)^{\frac{1}{2}j(j-1)} [\eta_1 \cdots \eta_{j-1}\eta_j] [\theta_1 \cdots \theta_{j-1}\theta_j]. \end{aligned} \quad (\text{A.27})$$

We could also have ordered the final result with decreasing n . It's easy to see that we get the same result. Reordering $\eta_1 \cdots \eta_n \rightarrow \eta_n \cdots \eta_1$ can at most give an overall minus sign. But since we also will reorder the θ s, the overall sign will always cancel. In conclusion we have

$$\begin{aligned} \eta_1 \theta_1 \eta_2 \theta_2 \cdots \eta_n \theta_n &= (-1)^{\frac{1}{2}n(n-1)} \eta_1 \eta_2 \cdots \eta_n \theta_1 \theta_2 \cdots \theta_n \\ &= (-1)^{\frac{1}{2}n(n-1)} \eta_n \eta_{n-1} \cdots \eta_1 \theta_n \theta_{n-1} \cdots \theta_1 \end{aligned} \quad (\text{A.28})$$

A.5.1.2 Complex Grassmann Variables

Sometimes we want to consider complex Grassmann variables. We do this in the same way as with c -numbers, we define

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \quad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}, \quad (\text{A.29})$$

where θ_1 and θ_2 are real Grassmann variables. With this definition we can treat θ and θ^* as independent complex variables.

A.5.2 Gaussian Integral

We will now compute the Gaussian integral over Grassmann variables θ_i and θ_i^* for $i = 1, 2, \dots, n$

$$\mathcal{I} = \int d\theta_n^* d\theta_1 \cdots d\theta_n^* d\theta_n e^{-\theta_i^* A_{ij} \theta_j} \quad (\text{A.30})$$

where $A_{ij} = -A_{ji}$ insures that the exponent is a real number. We Taylor expand the exponential and realize that the only term that will contribute is the $\frac{1}{n!} (-\theta_i^* A_{ij} \theta_j)^n$. This is the only term that will have one and only one copy of every θ_i and θ_i^* . We find

$$\begin{aligned} \mathcal{I} &= \frac{1}{n!} \int d\theta_1^* d\theta_1 \cdots d\theta_n^* d\theta_n (-\theta_{i_1}^* A_{i_1 j_1} \theta_{j_1}) \cdots (-\theta_{i_n}^* A_{i_n j_n} \theta_{j_n}) \\ &= \frac{(-1)^n}{n!} \int [d\theta_1^* d\theta_1 \cdots d\theta_n^* d\theta_n] [\theta_{i_1}^* \theta_{j_1} \cdots \theta_{i_n}^* \theta_{j_n}] A_{i_1 j_1} \cdots A_{i_n j_n} \\ &= \frac{(-1)^{n^2}}{n!} \int [d\theta_n^* \cdots d\theta_1^* d\theta_n \cdots d\theta_1] [\theta_{i_1}^* \cdots \theta_{i_n}^* \theta_{j_1} \cdots \theta_{j_n}] A_{i_1 j_1} \cdots A_{i_n j_n} \\ &= \frac{(-1)^{2n^2}}{n!} \left[\int d\theta_n^* \cdots d\theta_1^* \theta_{i_1}^* \cdots \theta_{i_n}^* \right] \left[\int d\theta_n \cdots d\theta_1 \theta_{j_1} \cdots \theta_{j_n} \right] A_{i_1 j_1} \cdots A_{i_n j_n} \\ &= \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 j_2 \cdots j_n} A_{i_1 j_1} \cdots A_{i_n j_n} \\ &= \det A \end{aligned} \quad (\text{A.31})$$

Here's an explanation of what we have done. Between line 1 and 2 we have collected the n minus signs out to the front. From line 2 to 3 we have used eq. (A.28) twice to reorder the $d\theta_s, d\theta^*_s, \theta_s$ and θ^*_s into the desired order. Simplifying the overall minus signs we find $(-1)^n (-1)^{\frac{1}{2}n(n-1)} (-1)^{\frac{1}{2}n(n-1)} = (-1)^{n^2}$. To get from line 3 to 4 we have moved the $d\theta_n \cdots d\theta_1$ past $\theta^*_{i_1} \cdots \theta^*_{i_n}$. This gives another factor of $(-1)^{n^2}$. Between line 4 and 5 we have used $(-1)^{2n^2} = 1$ and used eq. (A.24) twice to perform the integrals. Finally we have recognized the well known formula for the determinant of a matrix in terms of the Levi-Civita symbols.

We conclude that

$$\mathcal{I} = \int d\theta^*_1 d\theta_1 \cdots d\theta^*_n d\theta_n e^{-\theta^*_i A_{ij} \theta_j} = \det A. \quad (\text{A.32})$$

Adding in linear terms in the exponential, we find [80]

$$\mathcal{I} = \int d\theta^*_1 d\theta_1 \cdots d\theta^*_n d\theta_n e^{-\theta^*_i A_{ij} \theta_j + \xi_i \theta^*_i + \eta_i \theta_i} = e^{\eta_i A_{ij}^{-1} \xi_j} \det A. \quad (\text{A.33})$$

A.6 Gaussian Integrals over Fields

In this section we want to evaluate Gaussian integrals over fields. Consider a real field ϕ_a , a real current $J_a(x)$ and a real symmetric matrix $M_{ab}(x-y)$ in the integral

$$\mathcal{I} = \int \mathcal{D}\phi e^{iS[\phi]} = \int \mathcal{D}\phi e^{-i \int d^4x d^4y \frac{1}{2} \phi_a(x) M_{ab}(x-y) \phi_b(y) + i \int d^4x J_a(x) \phi_a(x)}. \quad (\text{A.34})$$

We will put our system in a finite size box of volume $L^3 T = \mathcal{V}_4$, and we will later take the limit of infinite volume later. We expand the field, matrix and current in a Fourier series

$$\begin{aligned} \phi_a(x) &= \sum_k \frac{1}{\mathcal{V}_4} \phi_a(k) e^{ikx}, \\ M_{ab}(x-y) &= \sum_k \frac{1}{\mathcal{V}_4} M_{ab}(k) e^{ik(x-y)}, \\ J_a(x) &= \sum_k \frac{1}{\mathcal{V}_4} J_a(k) e^{ikx}, \end{aligned} \quad (\text{A.35})$$

where $k = 2\pi \left(\frac{n_t}{T}, \frac{\vec{n}}{L} \right)$, and $n = (n_t, \vec{n})$ is an integer vector. The condition that $\phi(x)$ and $J(x)$ are real give $\phi(k) = \phi^*(-k)$ and $J(k) = J^*(-k)$, and $M_{ab}(x-y)$ being real and symmetric ($M_{ab}(x-y) = M_{ba}(y-x)$) gives that $M_{ab}(k)$ is Hermitian $M_{ab}(k) = M_{ab}^\dagger(k)$.¹

¹To be more specific, being real gives $M_{ab}(k) = M_{ab}^*(-k)$ and being symmetric gives $M_{ab}(k) = M_{ba}(-k)$, which together gives that $M_{ab}(k)$ is Hermitian.

Using the Fourier series, we can rewrite the first term in the action as

$$\begin{aligned}
S[\phi_a]_1 &= - \int d^4x d^4y \frac{1}{\mathcal{V}_4^3} \sum_{k,p,q} \frac{1}{2} \phi_a(k) M_{ab}(p) \phi_b(q) e^{ikx} e^{ip(x-y)} e^{iqy} \\
&= - \frac{1}{\mathcal{V}_4} \sum_{k,p,q} \frac{1}{2} \phi_a(k) M_{ab}(p) \phi_b(q) \frac{1}{\mathcal{V}_4} \int d^4x e^{ix(k+p)} \frac{1}{\mathcal{V}_4} \int d^4y e^{iy(q-p)} \\
&= - \frac{1}{\mathcal{V}_4} \sum_k \frac{1}{2} \phi_a(-k) M_{ab}(k) \phi_b(k),
\end{aligned} \tag{A.36}$$

where we have used that $\frac{1}{\mathcal{V}_4} \int d^4x e^{ix(k+p)} = \delta_{k,-p}$. The second term becomes

$$\begin{aligned}
S[\phi_a]_2 &= \frac{1}{\mathcal{V}_4^2} \int d^4x \sum_{k,p} J_a(k) \phi_a(p) e^{ix(p+k)} \\
&= \frac{1}{\mathcal{V}_4} \sum_k \frac{1}{2} (J_a(k) \phi_a(-k) + J_a(-k) \phi_a(k)).
\end{aligned} \tag{A.37}$$

Due to the symmetry between $k > 0$ and $k < 0$ we can restrict the sum to only positive k . We find

$$\begin{aligned}
&\sum_k \frac{1}{2} \phi_a(-k) M_{ab}(k) \phi_b(k) \\
&= \sum_{k>0} \frac{1}{2} \phi_a(-|k|) M_{ab}(|k|) \phi_b(|k|) + \sum_{k<0} \frac{1}{2} \phi_a(|k|) M_{ab}(-|k|) \phi_b(-|k|) \\
&\quad + \frac{1}{2} \phi_a(0) M_{ab}(0) \phi_b(0) \\
&= \sum_{k>0} \frac{1}{2} \phi_a(-|k|) M_{ab}(|k|) \phi_b(|k|) + \sum_{k<0} \frac{1}{2} \phi_b(-|k|) M_{ba}(|k|) \phi_a(|k|) \\
&\quad + \frac{1}{2} \phi_a(0) M_{ab}(0) \phi_b(0) \\
&= \sum_{k>0} \phi_a(-k) M_{ab}(k) \phi_b(k) + \frac{1}{2} \phi_a(0) M_{ab}(0) \phi_b(0),
\end{aligned} \tag{A.38}$$

where we have used that $M_{ab}(k) = M_{ba}(-k)$ between line 1 and 2, and we relabeled $a \leftrightarrow b$ in the second term in going from line 2 to 3. We have also removed the absolute sign in the last line since we are only summing over positive

k . Doing the same procedure to eq. (A.37) we can write the action as

$$S[\phi] = \frac{1}{\mathcal{V}_4} \sum_{k>0} [-\phi_a^*(k) M_{ab}(k) \phi_b(k) + J_a^*(k) \phi_a(k) + J_a(k) \phi_a^*(k)] - \frac{1}{2\mathcal{V}_4} \phi_a(0) M_{ab}(0) \phi_b(0) + \frac{1}{\mathcal{V}_4} J_a(0) \phi_a(0), \quad (\text{A.39})$$

where we have used $\phi_a(k) = \phi_a^*(-k)$ and $J_a(k) = J_a^*(-k)$. The measure $\mathcal{D}\phi_a$ is now defined as

$$\mathcal{D}\phi \equiv \prod_{k>0} \frac{d^n \phi(k)}{(2\pi\mathcal{V}_4)^{\frac{n}{2}}} \frac{d^n \phi^*(k)}{(2\pi\mathcal{V}_4)^{\frac{n}{2}}} \frac{d^n \phi(0)}{(2\pi\mathcal{V}_4)^{\frac{n}{2}}}, \quad (\text{A.40})$$

and we find

$$\begin{aligned} \mathcal{I} &= \int \mathcal{D}\phi e^{iS[\phi]} \\ &= \prod_{k>0} \int \frac{d^n \phi(k) d^n \phi^*(k)}{(2\pi\mathcal{V}_4)^n} e^{\left[-\phi_a^*(k) \frac{M_{ab}(k)}{\mathcal{V}_4} \phi_b(k) + \frac{J_a^*(k)}{\mathcal{V}_4} \phi_a(k) + \frac{J_a(k)}{\mathcal{V}_4} \phi_a^*(k) \right]} \\ &\quad \times \frac{1}{(2\pi\mathcal{V}_4)^{\frac{n}{2}}} \int d^n \phi(0) e^{-\frac{1}{2\mathcal{V}_4} \phi_a(0) M_{ab}(0) \phi_b(0) + \frac{1}{\mathcal{V}_4} J_a(0) \phi_a(0)} \\ &= \prod_{k>0} \frac{1}{(2\pi\mathcal{V}_4)^n} \frac{(2\pi\mathcal{V}_4)^n}{\det M(k)} e^{\frac{1}{\mathcal{V}_4} J_a^*(k) M_{ab}^{-1}(k) J_b(k)} \\ &\quad \times \frac{1}{(2\pi\mathcal{V}_4)^{\frac{n}{2}}} \frac{(2\pi\mathcal{V}_4)^{\frac{n}{2}}}{\sqrt{\det M(0)}} e^{\frac{1}{\mathcal{V}_4} J_a(0) M_{ab}^{-1}(0) J_b(0)} \\ &= \prod_{k>0} \frac{1}{\det M(k)} e^{\frac{1}{\mathcal{V}_4} J_a^*(k) M_{ab}^{-1}(k) J_b(k)} \frac{1}{\sqrt{\det M(0)}} e^{\frac{1}{\mathcal{V}_4} J_a(0) M_{ab}^{-1}(0) J_b(0)} \end{aligned} \quad (\text{A.41})$$

where we have used eq. (A.16) and eq. (A.6) to perform the integrals.

Since $M_{ab}(k) = M_{ba}(-k)$ we must have $\det M(k) = \det M(-k)$, and we see that we can write

$$\begin{aligned} \prod_{k>0} \frac{1}{\det M(k)} \frac{1}{\sqrt{\det M(0)}} &= \prod_{k<0} \frac{1}{\sqrt{\det M(-k)}} \frac{1}{\sqrt{\det M(0)}} \prod_{k>0} \frac{1}{\sqrt{\det M(k)}} \\ &= \prod_k \frac{1}{\sqrt{\det M(k)}} = e^{-\frac{1}{2} \sum_k \ln \det M(k)}. \end{aligned} \quad (\text{A.42})$$

We can similarly write

$$\prod_{k>0} e^{\frac{1}{\mathcal{V}_4} J_a^*(k) M_{ab}^{-1}(k) J_b(k)} e^{\frac{1}{\mathcal{V}_4} J_a(0) M_{ab}^{-1}(0) J_b(0)} = e^{\frac{1}{\mathcal{V}_4} \sum_k J_a^*(k) M_{ab}^{-1}(k) J_b(k)}. \quad (\text{A.43})$$

Putting everything together, we find

$$\int \mathcal{D}\phi e^{iS[\phi]} = e^{-\frac{1}{2} \sum_k \ln \det M(k)} e^{\frac{1}{\mathcal{V}_4} \sum_k J_a^*(k) M_{ab}^{-1}(k) J_b(k)}. \quad (\text{A.44})$$

Taking the limit $L \rightarrow \infty$, $T \rightarrow \infty$, the sum over k becomes an integral $\sum_k \rightarrow \mathcal{V}_4 \int \frac{d^4 k}{(2\pi)^4}$. The final result is

$$\int \mathcal{D}\phi e^{iS[\phi]} = e^{-\frac{1}{2} \mathcal{V}_4 \int \frac{d^4 p}{(2\pi)^4} \ln \det M(k)} e^{\int \frac{d^4 p}{(2\pi)^4} J_a^*(k) M_{ab}^{-1}(k) J_b(k)}. \quad (\text{A.45})$$

Appendix B

Method of Characteristics

In this appendix we will review the method of characteristics that is used in section 3.3.2.2 to find the resummed effective potential using RGE.

The method of characteristics [81] is a technique used to solve partial differential equations (PDE). The idea is to rewrite the PDE in terms of ordinary differential equations (ODE) that in general are easier to solve.

Consider the PDE

$$a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} = c(x, y, u) \quad (\text{B.1})$$

where $u = u(x, y)$. We want to parametrize x and y in terms of a new parameter t , i.e. $x = x(t)$, $y = y(t)$ and $u = u(x(t), y(t))$. Differentiating u with respect to t we find using the chain rule

$$\frac{du(x(t), y(t))}{dt} = \dot{x} \frac{\partial u}{\partial x} + \dot{y} \frac{\partial u}{\partial y} \quad (\text{B.2})$$

where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$. This is remarkably similar to eq. (B.1), and we get exactly this equation if we choose to define

$$\begin{aligned} \dot{x} &= a(x(t), y(t), u(x(t), y(t))), \\ \dot{y} &= b(x(t), y(t), u(x(t), y(t))), \\ \dot{u} &= c(x(t), y(t), u(x(t), y(t))). \end{aligned} \quad (\text{B.3})$$

To verify this we see by explicit calculation that eq. (B.2) becomes

$$\frac{du(x(t), y(t))}{dt} = a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = c(x(t), y(t), u(x(t), y(t))). \quad (\text{B.4})$$

Now consider a surface $\mathcal{S} = \{(x, y, u(x, y))\}$ in \mathbb{R}^3 . The normal vector at each point (x, y) is $N(x, y) = [u_x(x, y), u_y(x, y), -1]$. Let $V(x, y) = [a(x, y), b(x, y), c(x, y)]$,

and note that eq. (B.1) can be written as $V(x, y) \cdot N(x, y) = 0$. We conclude that $V(x, y)$ is always in the tangent plane to the surface \mathcal{S} . Since $a(x, y)$, $b(x, y)$ and $c(x, y)$ are given we know $V(x, y) \forall (x, y) \in \mathbb{R}^2$. We now want to construct \mathcal{S} from V to find our solution $u(x, y)$.

Consider the curve $\mathcal{C} = [x(t), y(t), u(x(t), y(t))] \in \mathbb{R}^3$ that is parametrized by $t \in \mathbb{R}$. The tangent vector to a curve is described by the derivative with respect to t , and in our case we see by eq. (B.3) that this vector is $V_{\mathcal{C}}(x(t), y(t)) = [a(x(t), y(t)), b(x(t), y(t)), c(x(t), y(t))]$. In other words, we can construct the curve by solving the systems of ODEs in eq. (B.3), which are sometimes referred to as the characteristic equations for eq. (B.1). The curve \mathcal{C} is called the characteristic curve for our PDE in eq. (B.1).

To form the surface \mathcal{S} , we simply take the union of the characteristic curves. One way this is done, is to specify an initial curve that is non-parallel to the characteristic curve. If we know the value along this curve we have enough information to describe the whole surface \mathcal{S} , and we can find $u(x, y)$. To illustrate the method of characteristics, let's do a couple of examples.

B.1 Examples

B.1.1 Example 1

Let $u = u(x, y)$ and consider the PDE

$$u_x + ku_y = 0 \tag{B.5}$$

with the initial condition

$$u(0, y) = \sin(y), \tag{B.6}$$

where the subscript means partial derivative, e.g. $u_x = \frac{\partial u}{\partial x}$. Following the method described above we parametrize $x = x(t)$, $y = y(t)$. The characteristic equations are

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= k, \\ \dot{u} &= 0. \end{aligned} \tag{B.7}$$

The solution is

$$\begin{aligned} x(t) &= t + c_1, \\ y(t) &= kt + c_2, \\ u(x(t), y(t)) &= c_3. \end{aligned} \tag{B.8}$$

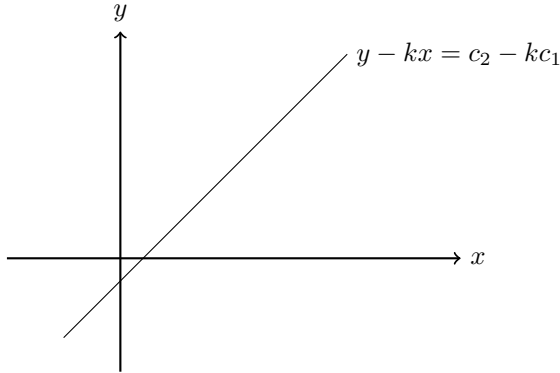


Figure B.1: Plot of one characteristic line projected into the $x - y$ plane.

Note that $u(x(t), y(t))$ is constant along the characteristic line. We can eliminate t from these equations and we find

$$y - kx = c_2 - kc_1. \tag{B.9}$$

A sketch of one characteristic line can be seen in figure B.1. Since we know that at along the y -axis the value of $u(0, y) = \sin y$, we can find the value of c_3 for a specific curve where the line crosses the y -axis. For this characteristic line we find using the initial condition that

$$u(0, y) = \sin(y) = \sin(c_2 - kc_1) = c_3, \tag{B.10}$$

and $\sin(c_2 - kc_1) = \sin(y - kx)$ is the value of $u(x, y)$ along this whole characteristic line. In fact, we can do this for any characteristic and we find

$$u(x, y) = \sin(y - kx) \tag{B.11}$$

B.1.2 Example 2

In this example we will take a slightly different approach to solving the PDE. We will look at the method of characteristics as a method of changing coordinates from (x, y) to (t, s) . Since t and s are independent, the derivation of the characteristic equations follow in exactly the same way as before, just making the total derivatives with respect to t into partial derivatives. Consider the PDE

$$u_x + ku_y = byu \tag{B.12}$$

with the initial condition

$$u(0, y) = f(y), \tag{B.13}$$

where $f(y)$ is a general function. Let $x = x(t, s)$, $y = y(t, s)$ and the characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= 1, \\ \frac{\partial y}{\partial t} &= k, \\ \frac{\partial u}{\partial t} &= bxu. \end{aligned} \tag{B.14}$$

The solution is

$$\begin{aligned} x(t, s) &= t + c_1(s), \\ y(t, s) &= kt + c_2(s), \\ u(t, s) &= c_3(s) \exp \left[\frac{1}{2}bt^2 + bc_1(s)t \right] \end{aligned} \tag{B.15}$$

where we had to first solve for $x(s, t)$ and put that back in to the differential equation for $u(t, s)$ to solve. We now have the freedom to choose s in whatever way we like as long as the coordinate transformation is invertible. We will make a choice such that the initial condition in $(t, s, u(t, s))$ coordinates is $(0, s, f(s))$. This gives

$$\begin{aligned} x(0, s) &= 0 + c_1(s) = 0, \\ y(0, s) &= 0 + c_2(s) = s, \\ u(0, s) &= c_3(s) = f(s). \end{aligned} \tag{B.16}$$

Having fixed the functions $c_i(s)$ we have $x = t$ and $y = kt + s$. Expressing u in (x, y) coordinates we find

$$u(x, y) = \exp \left[\frac{1}{2}bx^2 \right] f(y - kx) \tag{B.17}$$

which is the solution to our PDE with the initial condition $u(0, y) = f(y)$.

Appendix C

Standard Model Beta Functions

In this appendix we list the beta functions used in chapter 5. The beta function for a coupling $g_i(\mu)$ is defined as

$$\mu \frac{d}{d\mu} g_i(\mu) \equiv \beta_{g_i}(g_j(\mu)), \quad (\text{C.1})$$

where $\beta_{g_i}(g_j(\mu))$ in general depends on other couplings $g_j(\mu)$. We will list the beta functions as

$$\beta_g \equiv \beta_g^{(1)} + \beta_g^{(3)} + \beta_g^{(3)} + \dots, \quad (\text{C.2})$$

where the subscript stands for the loop order where the contribution comes from. For simplicity we will drop the μ dependence and just write g_i for $g_i(\mu)$. All values are given using the $\overline{\text{MS}}$ subtraction scheme.

The beta functions $\beta_\lambda^{(1)}$, $\beta_\lambda^{(2)}$, $\beta_{g_1}^{(1)}$, $\beta_{g_1}^{(2)}$, $\beta_{g_2}^{(1)}$, $\beta_{g_2}^{(2)}$, $\beta_{g_3}^{(1)}$, $\beta_{g_3}^{(2)}$, $\beta_{y_t}^{(1)}$ and $\beta_{y_t}^{(2)}$ are taken from [77]. The three loop results $\beta_{g_1}^{(3)}$, $\beta_{g_2}^{(3)}$ and $\beta_{g_3}^{(3)}$ are taken from [82]. The three loop results for $\beta_\lambda^{(3)}$ and $\beta_{y_t}^{(3)}$ are from [83] and note that this is only known in the limit $g_1, g_2 \rightarrow 0$. All the results for the anomalous dimension for Higgs γ_H are from [78].

$$\beta_\lambda^{(1)} = \frac{1}{16\pi^2} \left(\lambda (12y_t^2 - 3g_1^2 - 9g_2^2) + \frac{3}{8} (g_1^2 + g_2^2)^2 + \frac{3g_2^4}{4} + 24\lambda^2 - 6y_t^4 \right) \quad (\text{C.3})$$

$$\begin{aligned} \beta_\lambda^{(2)} = \frac{1}{(16\pi^2)^2} & \left(\lambda y_t^2 \left(\frac{85g_1^2}{6} + \frac{45g_2^2}{2} + 80g_3^2 \right) + \frac{39}{4} g_1^2 g_2^2 \lambda - \frac{559}{48} g_1^4 g_2^2 \right. \\ & + 36\lambda^2 (g_1^2 + 3g_2^2) - \frac{289}{48} g_1^2 g_2^4 + \frac{21}{2} g_1^2 g_2^2 y_t^2 + \frac{629}{24} g_1^4 \lambda \\ & - \frac{1}{48} 379g_1^6 - \frac{19}{4} g_1^4 y_t^2 - \frac{8}{3} g_1^2 y_t^4 - \frac{73}{8} g_2^4 \lambda + \frac{305g_2^6}{16} \\ & \left. - \frac{9}{4} g_2^4 y_t^2 - 32g_3^2 y_t^4 - 312\lambda^3 - 3\lambda y_t^4 - 144\lambda^2 y_t^2 + 30y_t^6 \right) \quad (\text{C.4}) \end{aligned}$$

$$\begin{aligned} \beta_\lambda^{(3)} = \frac{2}{(16\pi^2)^3} & \left((895 - 1296\zeta(3)) g_3^2 \lambda y_t^4 + (1152\zeta(3) - 1224) g_3^2 \lambda^2 y_t^2 \right. \\ & + \left(\frac{1244}{3} - 48\zeta(3) \right) g_3^4 \lambda y_t^2 + (240\zeta(3) - 38) g_3^2 y_t^6 \\ & + \left(32\zeta(3) - \frac{266}{3} \right) g_3^4 y_t^4 + \left(\frac{117}{8} - 198\zeta(3) \right) \lambda y_t^6 \\ & + \left(756\zeta(3) + \frac{1719}{2} \right) \lambda^2 y_t^4 + \left(-36\zeta(3) - \frac{1599}{8} \right) y_t^8 \\ & \left. + 873\lambda^3 y_t^2 + (2016\zeta(3) + 3588)\lambda^4 \right) \quad (\text{C.5}) \end{aligned}$$

$$\beta_{g_1}^{(1)} = \frac{1}{16\pi^2} \left(\frac{41}{6} g_1^3 \right) \quad (\text{C.6})$$

$$\beta_{g_1}^{(2)} = \frac{1}{(16\pi^2)^2} g_1^3 \left(\frac{199g_1^2}{18} + \frac{9g_2^2}{2} + \frac{44g_3^2}{3} - \frac{17y_t^2}{6} \right) \quad (\text{C.7})$$

$$\begin{aligned} \beta_{g_1}^{(3)} = \frac{1}{(16\pi^2)^3} g_1^3 & \left(\frac{205}{96} g_1^2 g_2^2 - \frac{137}{27} g_1^2 g_3^2 + \frac{3}{2} g_1^2 \lambda - \frac{388613g_1^4}{5184} - \frac{2827}{288} g_1^2 y_t^2 \right. \\ & - g_2^2 g_3^2 + \frac{3}{2} g_2^2 \lambda + \frac{1315g_2^4}{64} - \frac{785}{32} g_2^2 y_t^2 + 99g_3^4 \\ & \left. - \frac{29}{3} g_3^2 y_t^2 - 3\lambda^2 + \frac{315y_t^4}{16} \right) \quad (\text{C.8}) \end{aligned}$$

$$\beta_{g_2}^{(1)} = \frac{1}{16\pi^2} \left(-\frac{19}{6} g_2^3 \right) \quad (\text{C.9})$$

$$\beta_{g_2}^{(2)} = \frac{1}{(16\pi^2)^2} g_2^3 \left(\frac{3g_1^2}{2} + \frac{35g_2^2}{6} + 12g_3^2 - \frac{3y_t^2}{2} \right) \quad (\text{C.10})$$

$$\begin{aligned} \beta_{g_2}^{(3)} = \frac{1}{(16\pi^2)^3} g_2^3 \left(\frac{291}{32} g_1^2 g_2^2 - \frac{1}{3} g_1^2 g_3^2 + \frac{1}{2} g_1^2 \lambda + \frac{1}{576} (-5597) g_1^4 \right. \\ \left. - \frac{593}{96} g_1^2 y_t^2 + 39g_2^2 g_3^2 + \frac{3}{2} g_2^2 \lambda + \frac{324953g_2^4}{1728} - \frac{729}{32} g_2^2 y_t^2 \right. \\ \left. + 81g_3^4 - 7g_3^2 y_t^2 - 3\lambda^2 + \frac{147y_t^4}{16} \right) \quad (\text{C.11}) \end{aligned}$$

$$\beta_{g_3}^{(1)} = \frac{1}{16\pi^2} (-7g_3^3) \quad (\text{C.12})$$

$$\beta_{g_3}^{(2)} = \frac{1}{(16\pi^2)^2} g_3^3 \left(\frac{11g_1^2}{6} + \frac{9g_2^2}{2} - 26g_3^2 - 2y_t^2 \right) \quad (\text{C.13})$$

$$\begin{aligned} \beta_{g_3}^{(3)} = \frac{1}{(16\pi^2)^3} g_3^3 \left(-\frac{1}{8} g_1^2 g_2^2 + \frac{77}{9} g_1^2 g_3^2 + \frac{1}{216} (-2615) g_1^4 - \frac{101}{24} g_1^2 y_t^2 \right. \\ \left. + 21g_2^2 g_3^2 + \frac{109g_2^4}{8} - \frac{93}{8} g_2^2 y_t^2 + \frac{65g_3^4}{2} - 40g_3^2 y_t^2 + 15y_t^4 \right) \quad (\text{C.14}) \end{aligned}$$

$$\beta_{y_t}^{(1)} = \frac{1}{16\pi^2} \left(y_t \left(-\frac{17}{12} g_1^2 - \frac{9g_2^2}{4} - 8g_3^2 \right) + \frac{9y_t^3}{2} \right) \quad (\text{C.15})$$

$$\begin{aligned} \beta_{y_t}^{(2)} = \frac{1}{(16\pi^2)^2} y_t \left(y_t^2 \left(\frac{131g_1^2}{16} + \frac{225g_2^2}{16} + 36g_3^2 - 12\lambda \right) - \frac{3}{4} g_1^2 g_2^2 \right. \\ \left. + \frac{19}{9} g_1^2 g_3^2 + \frac{1187g_1^4}{216} + 9g_2^2 g_3^2 - \frac{23g_2^4}{4} - 108g_3^4 + 6\lambda^2 - 12y_t^4 \right) \quad (\text{C.16}) \end{aligned}$$

$$\begin{aligned} \beta_{y_t}^{(3)} = \frac{1}{(16\pi^2)^3} y_t \left(\frac{1}{3} (-4166) g_3^6 + 16g_3^2 \lambda y_t^2 + \frac{3827}{6} g_3^4 y_t^2 - 157g_3^2 y_t^4 \right. \\ \left. - 228\zeta(3) g_3^4 y_t^2 + 640\zeta(3) g_3^6 - 36\lambda^3 + \frac{15}{4} \lambda^2 y_t^2 \right. \\ \left. + 198\lambda y_t^4 + \frac{339y_t^6}{8} + \frac{27}{2} \zeta(3) y_t^6 \right) \quad (\text{C.17}) \end{aligned}$$

$$\gamma_h^{(1)} = \frac{1}{16\pi^2} \left(-\frac{1}{2}g_1^2\xi_B - \frac{3}{2}g_2^2\xi_W + \frac{3g_1^2}{2} + \frac{9g_2^2}{2} - 6y_t^2 \right) \quad (\text{C.18})$$

$$\gamma_h^{(2)} = \frac{1}{(16\pi^2)^2} \left(-\frac{85}{12}g_1^2y_t^2 - \frac{45}{4}g_2^2y_t^2 - 40g_3^2y_t^2 - \frac{3}{4}g_2^4\xi_W^2 - 6g_2^4\xi_W \right. \\ \left. - \frac{1}{48}431g_1^4 - \frac{9}{8}g_2^2g_1^2 + \frac{271g_2^4}{16} - 12\lambda^2 + \frac{27y_t^4}{2} \right) \quad (\text{C.19})$$

$$\gamma_h^{(3)} = \frac{1}{(16\pi^2)^3} \left(-30g_1^2\lambda^2 - 90g_2^2\lambda^2 - \frac{39g_1^4\lambda}{8} - \frac{39}{4}g_2^2g_1^2\lambda - \frac{117g_2^4\lambda}{8} \right. \\ - \frac{1}{6}g_1^4\zeta(3)y_t^2 + 6g_1^2\zeta(3)y_t^4 - 27g_2^2g_1^2\zeta(3)y_t^2 \\ - 136g_3^2g_1^2\zeta(3)y_t^2 - 54g_2^2\zeta(3)y_t^4 + 144g_3^2\zeta(3)y_t^4 \\ + \frac{189}{2}g_2^4\zeta(3)y_t^2 + 48g_3^4\zeta(3)y_t^2 - 216g_2^2g_3^2\zeta(3)y_t^2 \\ + \frac{144271g_1^4y_t^2}{1728} + \frac{319}{8}g_1^2y_t^4 - \frac{371}{32}g_2^2g_1^2y_t^2 + \frac{2419}{18}g_3^2g_1^2y_t^2 \\ + \frac{1161}{8}g_2^2y_t^4 - 15g_3^2y_t^4 - \frac{4275}{64}g_2^4y_t^2 - \frac{1244}{3}g_3^4y_t^2 \\ + \frac{489}{2}g_2^2g_3^2y_t^2 - \frac{15}{8}g_2^6\xi_W^3 - \frac{117}{16}g_2^6\xi_W^2 - \frac{297}{32}g_2^6\xi_W \\ - \frac{9}{4}g_2^6\zeta(3)\xi_W^2 + 9g_1^4\lambda\zeta(3) + 18g_2^2g_1^2\lambda\zeta(3) \\ + 27g_2^4\lambda\zeta(3) + \frac{1529g_1^6\zeta(3)}{24} + \frac{135}{8}g_2^2g_1^4\zeta(3) \\ + 88g_3^2g_1^4\zeta(3) + \frac{153}{8}g_2^4g_1^2\zeta(3) - \frac{1953g_2^6\zeta(3)}{8} \\ + 216g_2^4g_3^2\zeta(3) - \frac{27053g_1^6}{216} - \frac{153}{16}g_2^2g_1^4 - \frac{165}{2}g_3^2g_1^4 \\ - \frac{543}{32}g_2^4g_1^2 - \frac{405}{2}g_2^4g_3^2 - \frac{6785g_2^6}{288} + 72\lambda^3 \\ \left. + 135\lambda^2y_t^2 - 90\lambda y_t^4 - 18\zeta(3)y_t^6 - \frac{789y_t^6}{8} \right) \quad (\text{C.20})$$

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