Compositional Finite-time Stability Analysis

S. Mojtaba Tabatabaeipour[1] and Mogens Blanke[1]

***Abstract*— In this paper, finite-time stability and finite-time boundedness for nonlinear systems with polynomial vector fields are investigated. Finite-time stability requires the states of the system to remain in a given bounded set in a finite-time interval and finite-time boundedness considers the same problem for the system but with bounded disturbance. Sufficient condition for finite-time stability and finite-time boundedness of nonlinear systems as well as a computational method based on sum of squares programming to check the conditions is given. Then, we consider the problem of finite-time stability for a system that consists of an interconnection of subsystems and we show how to decompose the problem into subproblems for each subsystem with coupling constraints. We show how we can solve the problem using sum of squares programming and dual decomposition. The method is demonstrated through some examples.**

I. INTRODUCTION

In many practical applications we are concerned with the behavior of the system over a finite-time interval. For example, consider the problem of transient stability of a synchronous machine in a power system subject to a severe fault such as a short circuit fault. The problem essentially boils down to checking if the trajectories of the faulty system (fault-on trajectories) lie in the region of attraction of the post-fault system at the time of clearing the fault. As another example, consider the problem of certifying that in a chemical process the temperature or pressure stay in given bounds in for a given period of time. These are finite-time stability (FTS) problems since we are checking whether the trajectories of the system remain in a given bounded set in a finite-time. A system is said to be finite-time stable if assuming that the initial states of the system are in a given bounded set, then the trajectories of the systems would remain in a prescribed bounded set for a given finite time interval. It should be noted that the concept is different from Lyapunov stability or asymptotic stability since in the later concepts the behavior of the system over an infinite interval of time is studied. Therefore, a system that is FTS might not be asymptotically stable and a system that is asymptotically stable might not be FTS.

Studies on finite-time stability date back to the 1950s in the Russian literature e.g. [1], [2] and later in [3]. In [4] the problem for linear systems subject to time-varying parametric uncertainties and exogenous constant disturbance is considered. The concept of FTS is extended to finite-time boundedness (FTB) by considering external disturbances into account and sufficient conditions in terms of linear matrix inequalities (LMIs) for FTB are given. Moreover, sufficient conditions for state feedback stabilization of the system using LMIs are provided. Using LMIs these results are extended for different classes of linear systems, see [5] for a survey. In [6] FTS for linear time-varying systems using Lyapunov differential matrix equations is studied and necessary and sufficient conditions for FTS are given. FTS for some classes of switched systems is also studied. In [7] sufficient conditions for FTS of impulsive linear dynamical systems in terms of differential LMIs are given. In [8] FTB of linear switched systems in discrete time is studied and average dwell time of the switching signal to guarantee FTB is obtained. There are very few work that considers the problem for nonlinear systems. In [9] the problem of FTS and finite time stabilziation for quadratic systems is considered. Finally, we point out that in [10], [11] the authors consider a finite-time stability problem which implies both Lyapunov stability and finite-time convergence. The concept investigated in [10], [11] is different from the notion studies in this paper and references [1], [2], [3], [4].

In this work, we consider the problem of FTS and FTB for nonlinear systems. We drive sufficient conditions for FTS and FTB of nonlinear systems and then we show how we can check these conditions for nonlinear systems with polynomial vector fields using sum of squares (SOS) programming. Moreover, we show how we can compute the minimum time that guarantees that the trajectories of the system initiated from an initial set would remain in a bounded set. Then, we consider the problem for a systems given as an interconnection of subsystems. We give compositional conditions for FTB where the overall problem is decomposed to subproblems for each subsystems with some coupling constraints. We shows how to use SOS and dual decomposition to check the conditions.

This paper is organized as follows..

II. PRELIMINARIES

In this section we give the basic definitions and concept that are used throughout the paper.

Definition 1: **monomial** A monomial m_α is a function $m_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ which is defined as: $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_1} \dots x_n^{\alpha_1}$. The degree of a monomial is defined as $deg\ m_\alpha = \Sigma_{i=1}^{n} \alpha_i$.

Definition 2: [12] **Polynomial:** a polynomial $p(x)$ is a linear combination of a finite number of monomials: $p(x) := \sum_{j=1}^{k} c_j m_{\alpha_j}(x)$. The degree of a polynomial is defined as $p := max_j(deg\ m_{\alpha_j})$.

*SOSPO?

[1]S. M. Tabatabaeipour and M. Blanke are with the Department of Electrical Engineering, Technical University of Denmark setaba,mb@elektro.dtu.dk

The set of all polynomials with n variables is denoted by $\mathcal{R}_n$. The set of positive semidefinite polynomials denoted by $\mathcal{P}_n$ are the set of polynomials that are nonnegative on all $\mathbb{R}^n$ which is defined by: $\mathcal{P}_n := \{p \in \mathcal{R}_n : p(x) \geq 0, \forall x \in \mathbb{R}^n\}$.

Definition 3: A polynomial p is said to be sum of squares (SOS) if it can be decomposed to a sum of squared of some polynomials $p_1, \ldots, p_M$ i.e $p = \sum_{i=1}^{M} p_i(x)^2$.

The set of all SOS polynomials in n variables is denoted by Σ_n which is defined as: $\Sigma_n := \{s \in \mathcal{R}_n : \exists M, \exists \{p_i\}_{i=1}^M \subset \mathcal{R}_n$ such that $s = \sum_{i=1}^{M} p_i^2\}$.

Proposition 1: A polynomial $p(x) \in \mathcal{R}_n$ of degree $2d$ is SOS if and only if there exist a positive semidefinite matrix $Q \geq 0$ and a vector of monomials $z(x)$ in n variables up to degree d such that $p(x) = z^T(x)Qz(x)$.

Theorem 1: [12] The existence of a SOS decomposition of a polynomial system in n variables of degree $2d$ can be formulated as a linear matrix inequality (LMI) feasibility problem test.

The following lemma is used to check conditions of the form $g_0(x) \geq 0$ whenever $g_1(x), \cdots, g_m(x) \geq 0$ by converting them into sum of square programming.

Lemma 1: (Generalized S-procedure) [13] Given functions $g_0(x), g_1(x), \cdots, g_m(x) \in \mathcal{R}_n$, if there exists $s_1, s_2, \cdots, s_m \in \Sigma[x]$ such that $g_0 - \Sigma_{i=1}^m s_i g_i \in \Sigma[x]$ then, it holds that:

$$\begin{aligned} \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\} \subseteq \\ \{x \in \mathbb{R}^n : g_1(x) \geq 0\} \end{aligned} \tag{1}$$

For a given set D its complement is denoted by D^c, its closure is denoted by $\bar{D}$, and its boundary is denoted by ∂D.

III. Problem statement

Here we consider the following autonomous system:

$$\dot{x} = f(x), \tag{2}$$

$x \in D \subseteq \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is Lipschitz on D.

Definition 4: The nonlinear system $\dot{x} = f(x),\ t \in [0, T]$ is said to be finite-time stable (FTS) with respect to $[D_1, D_2, T]$, where $D_1 \subset D_2 \subseteq D$ iff:

$$x(0) \in D_1 \Rightarrow x(T) \in D_2 \text{ for all } t \in [0, T], \tag{3}$$

where D_1 and D_2 are given sets.

In the following we assume that D, D_1 and D_2 are given as two semi-algebraic sets:

$$D = \{x \in \mathbb{R}^n : g_0(x) \geq 0\} \tag{4}$$

$$D_1 = \{x \in \mathbb{R}^n : g_1(x) \geq 0\} \tag{5}$$

$$D_2 = \{x \in \mathbb{R}^n : g_2(x) < 0\} \tag{6}$$

Remark 1: The definition that we use is more general than the definition used in most of the available literature since the sets D_1 and D_2 are semi-algebraic sets. To the best of our knowledge in all of the works that are based on using LMIs the sets D_1 and D_2 are restricted to ellipsoids e.g. [4], [6] or polytopes [9].

Theorem 2: The system (2) is finite time stable with respect to (D_1, D_2, T) if there exist a continuously differentiable function $B(x)$ and a positive scalar α such that the following conditions are satisfied:

$$B(x) \leq \epsilon \ \forall x \in D_1 \tag{7}$$

$$B(x) \geq 1 \ \forall x \in D_2^c \tag{8}$$

$$\dot{B}(x) - \alpha B(x) \leq 0 \quad \forall x \in \bar{D}_2 \tag{9}$$

$$\alpha < -\frac{1}{T} \ln \epsilon \tag{10}$$

Proof: (By contradiction) Assume that $x(t)$ is a trajectory of the system whose initial point is $x(0)$ such that $x(0) \in D_1$. Suppose there exist a $t_1 \in [0, T]$ such that $x(t_1) \in D_2^c$. Moreover, assume that t_1 is the first time that $x(t)$ enters D_2^c i.e $x(t_0) \in D_2$ for all $t_0 < t_1$. Due to the continuity of the solution, $x(t_1)$ must be on ∂D_2 since D_2 is an open set. Because $\dot{B}(x) - \alpha B(x) \leq 0$ in $\bar{D}_2$, then we have :

$$\frac{\dot{B}(x)}{B(x)} \leq \alpha. \tag{11}$$

Integrating $\dot{B}(x)/B(x)$ from along $x(t)$ we have:

$$\int_0^{t_1} \frac{\dot{B}(x)}{B(x)} dt \leq \int_0^{t_1} \alpha dt,$$

which yields:

$$\ln B(x(t_1)) - \ln B(x_0) \ \leq \alpha t_1.$$

Hence the following bound is obtained:

$$B(x(t_1)) \leq B(x_0) e^{\alpha t_1}.$$

Because $B(x) \leq \epsilon \ \forall x \in D_1$ is satisfied and since $x(0) \in D_1$ then we have:

$$B(x(t_1)) \leq \epsilon e^{\alpha t_1} \leq \epsilon e^{\alpha T} < 1.$$

But since $x(t_1) \in D_2^c$, we have $B(x(t_1)) \geq 1$ which is a contradiction. This means that there does not exist any $t_1 \in [0, T]$ such that $x(t_1) \in D_2^c$. ■

Finding the function $B(x)$ which satisfies the conditions of theorem 2 is in general very hard. In the following we use the generalized S-procedure to find $B(x)$ algorithmically by solving a SOS programming problem. Using the generalized S-procedure (Lemma 1), we get the following proposition.

Proposition 2: The system (2) is finite time stable with respect to (D_1, D_2, T) if there exist a polynomial $B(x)$, sum of squares polynomials s_1, s_2 and a positive scalar α such that the following conditions are satisfied:

$$-B(x) + \epsilon - s_1 g_1(x) \in \Sigma_n \tag{12}$$

$$(B(x) - 1) - s_2 g_2(x) \in \Sigma_n \tag{13}$$

$$-\frac{\partial B(x)}{\partial x} f(x) + \alpha B(x) + s_3 g_2(x) \in \Sigma_n \tag{14}$$

$$\alpha < -\frac{1}{T} \ln \epsilon. \tag{15}$$

where $0 \leq \epsilon \leq 1$.

The proof of the proposition is straightforward using the generalized S-procedure. We can think of $B(x)$ as a function

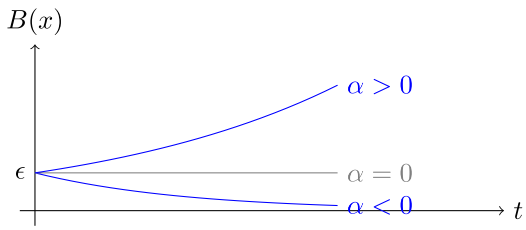

Fig. 1. Upper bound on the value of $B(x)$ for different chooses of α.

that maps trajectories of the system such that that mappings of the trajectories of the system emanating from D_1 are bounded by $\epsilon e^{\alpha t}$. This is shown in Figure 1. Depending on the value of α three cases are possible: if $\alpha > 0$ then the values of $B(x(t))$ increase over time but never surpasses the bound $\epsilon e^{\alpha t}$. This means that the trajectories are guaranteed to remain in D_2 for a finite interval of time. If $\alpha = 0$ the values B over time become never bigger than ϵ which means that the trajectories would remain in D_2 for an infinite time. If $\alpha < 0$ the values of B over time decreases and in this case also the trajectories would remain in D_2 for an infinite time. Note that we have not excluded the possibility of $B(x) \leq$ for $x \in D_2 \setminus D_1$. If the conditions are satisfied with $\epsilon = 0$, then the system's trajectories would always remain in D_2.

The conditions given in the proposition form a nonlinear SOS programming since we have the nonlinear relation $T \leq -\frac{1}{\alpha} \ln \epsilon$. In case T is given and we want to check FTS with respect to (D_1, D_2, T), by fixing the value of ϵ, then the maximum acceptable value for α is given by the relation $\alpha < -\frac{1}{T} \ln \epsilon$. Therefore, by checking the feasibility of the rest of conditions, we can check FTS of the system. The feasibility test can be completed by performing a line search over ϵ.

For some applications, it is interesting to find the maximum value of T such that given D_1, D_2, we can certify that all trajectories initiated in D_1 would remain in D_2. In other words we want to find the maximum T such that the systems is FTS w.r.t (D_1, D_2, D_3). We can find a solution to this problem by searching for the minimum of α that satisfies the following conditions:

$$\min_{B \in \mathcal{R}_n, s_1, s_2, s_3 \in \Sigma_n} \alpha$$
$$s.t. \begin{cases} -B(x) + \epsilon - s_1 g_1(x) \in \Sigma_n \\ (B(x) - 1) - s_2 g_2(x) \in \Sigma_n \\ -\frac{\partial B(x)}{\partial x} f(x) + \alpha B(x) + s_3 g_2(x) \in \Sigma_n \end{cases} \tag{16}$$

Then, the system is stable for all T such that $T \leq -\frac{1}{\alpha} \ln \epsilon$. For a given ϵ, the above problem is bilinear in its variables since we have the term $\alpha B(x)$ in the constraints. But because α is a scalar term, the problem can be solved by bisection on α. Therefore, to find the minimum of the α we perform a line search on ϵ where for each ϵ a bisection on α is performed. The detail of the method is not reported here for the sake of space.

Instead of using a exponential bound on $B(x(t))$, we can use a linear bound. A linear bound yields a more conservative solution but with less computational complexity. The following theorem gives the sufficient conditions for a linear bound.

Theorem 3: The system (2) is finite-time stable with respect to (D_1, D_2, T) if there exist a continuously differentiable function $B(x)$ and a positive scalar α such that the following conditions are satisfied:

$$B(x) \leq 0 \ \forall x \in D_1 \tag{17}$$
$$B(x) \geq 1 \ \forall x \in D_2^c \tag{18}$$
$$\dot{B}(x) \leq \alpha \quad \forall x \in \bar{D}_1 \tag{19}$$
$$\alpha < \frac{1}{T} \tag{20}$$

Proof: Assume that $x(t)$ is a trajectory whose initial point is $x(0)$ and $x(0) \in D_1$. Assume there exist $0 \leq t_1 \leq T$ such that $x(t_1) \in D_2^c$. Moreover, assume that t_1 is the first time that $x(t)$ enters D_2^c i.e $x(t_0) \in D_2$ for all $t_0 < t_1$. Due to the continuity of the solution, $x(t_1)$ must be on ∂D_2 since D_2 is an open set. If the conditions of the theorem are satisfied, we have:

$$B(x(t_1)) = B(x(0)) + \int_0^{t_1} \dot{B}(x) dt \leq \int_0^{t_1} \alpha dt \leq \alpha T < 1.$$

This contradicts the assumption that $x(t_1) \in D_2^c$ since we have $B(x) \geq 1 \ \forall x \in D_2^c$. Therefore $t_1 \notin [0, T]$. ■

Now, we can check the conditions of theorem by searching for B and the minimum α that satisfies the conditions 17-19. This can be reformulated as the following SOS program:

$$\min_{B \in \mathcal{R}_n, s_1, s_2, s_3 \in \Sigma_n} \alpha$$
$$s.t. \begin{cases} -B(x) - s_1 g_1(x) \in \Sigma_n \\ (B(x) - 1) - s_2 g_2(x) \in \Sigma_n \\ -\frac{\partial B(x)}{\partial x} f(x) + s_3 g_2(x) \in \Sigma_n \end{cases} \tag{21}$$

IV. Finite Time Boundedness

The concept of finite-time boundedness considers the behavior of the system when external disturbances are also taken into account. The dynamic of the system is given as:

$$\dot{x} = f(x, w), \tag{22}$$

where $f : D \to \mathbb{R}^n$ is Lipschitz on D and $w \in W \subseteq \mathbb{R}^m$ is the disturbance and $w \in \mathcal{L}_\infty(\mathbb{R}_{\geq 0}, W)$.

Definition 5: The nonlinear system (22), $t \in [0, T]$ subject to disturbance $w \in W$ is said to be finite-time bounded with respect to (D_1, D_2, W, T) if:

$$x(0) \in D_1 \Rightarrow x(T) \in D_2 \ \forall \ t \in [0, T], \forall w \in W. \tag{23}$$

The following theorem gives the sufficient condition for the systems to be FTB.

Theorem 4: The system (22) is said to be finite-time bounded with respect to (D_1, D_2, T) if there exist a continuously differentiable function $B(x)$ and positive scalars α

and ϵ such that the following conditions are satisfied:

$$B(x) \leq \epsilon \ \forall x \in D_1 \tag{24}$$
$$B(x) \geq 1 \ \forall x \in D_2^c \tag{25}$$
$$\dot{B}(x) \leq \alpha B(x) \ \forall x \in \bar{D}_1 \times \mathcal{W} \tag{26}$$
$$\alpha < -\frac{1}{T} \ln \epsilon \tag{27}$$

The proof is very similar to the proof of previous theorems and is omitted here for the sake of space. It easy to see that using the generalized S-procedure we can solve the following SOS optimization program and check if to find the maximum T such that the systems is FTS w.r.t (D_1, D_2, T). It is assumed that that $w \in W$ where W is a given semi-algebraic set: $W = \{w : g_w(w) \geq 0\}$.

$$\min_{B \in \mathcal{R}_n, s_1, s_2 \in \Sigma_n, s_w \in \Sigma_{(n+m)}} \alpha$$
$$s.t. \begin{cases} -B(x) + \epsilon - s_1 g_1(x) \in \Sigma_n \\ (B(x) - 1) - s_2 g_2(x) \in \Sigma_n \\ -\frac{\partial B(x)}{\partial x} f(x, w) + \alpha B(x) - s_w g_w + s_3 g_2(x) \in \Sigma_{(n+m)} \end{cases} \tag{28}$$

The problem can be solved as before by line search on ϵ and bisection on α.

V. Compositional Finite-time Stability

In this section, we consider the problem for a dynamical system given as an interconnection of subsystems. Our goal is to provide sufficient conditions for FTB of the overall dynamical system. Consider $N \geq 2$ subsystems given as:

$$\dot{x_i} = f_i(x_i, \mathbf{u}_i, w_i), \ i = 1, \cdots, N \tag{29}$$

with $x_i, \mathbb{R}^{n_i}, w_i \in \mathbb{R}^{m_i}$. We assume that the subsystems are coupled as follows. For each i let $u_j = h_j(x_j)$ for $j \neq i$ and $\mathbf{u}_i = \begin{bmatrix} u_1, \cdots, u_N \end{bmatrix}$ where the ith element u_i is removed. This does not mean that each subsystem is coupled with all the other subsystems but it might be coupled to some of them through $u_j = h_j(x_j)$. In case f_i does not depend on u_j the subsystems i and j are not coupled. The composite system is then given as:

$$\dot{x} = f(x, w) \tag{30}$$

where $x = \begin{bmatrix} x_1^T, \cdots, x_N^T \end{bmatrix}^T$ and $w = \begin{bmatrix} w_1^T, \cdots, w_N^T \end{bmatrix}^T$ and

$$f(x, w) = \begin{bmatrix} f_1(x_1, h_2(x_2), \cdots, h_N(x_N)), w_1) \\ f_2(x_2, h_1(x_1), h_3(x_3), \cdots, h_N(x_N), w_2) \\ \vdots \\ f_N(x_N, h_1(x_1), \cdots, h_{N-1}(x_{N-1}), w_N) \end{bmatrix} \tag{31}$$

We assume that the sets D_1 and D_2 are given as Cartesian products of $D_{1,i}$ and $D_{2,i}$ as:

$$D_1 = D_{1,1} \times \cdots D_{1,N} \tag{32}$$
$$D_2 = D_{2,1} \times \cdots D_{2,N} \tag{33}$$

where $D_{1,i} \subseteq D_{2,i} \subseteq \mathbb{R}^{n_i}$. It is also assumed that W is given as:

$$W = W_1 \times \cdots W_N. \tag{34}$$

These sets are given as:

$$D_{1,i} = \{x \in \mathbb{R}^n : g_{1,i} \geq 0\}, \tag{35}$$
$$D_{2,i} = \{x \in \mathbb{R}^n : g_{2,i} < 0\}, \tag{36}$$
$$W_i = \{w \in \mathbb{R}^m : g_{w,i} \geq 0\}. \tag{37}$$

Theorem 5: The dynamical system (30) is finite-time bounded with respect to (D_1, D_2, W, T) with D_1 and D_2 defined as in (32),(33), and (34) respectively, if there exist continuously differentiable functions $B_i(x_i)$, functions γ_i, positive scalars ϵ_i's and α's such that:

$$B_i(x_i) \geq 0 \ \forall x \in \mathbb{R}^{n_i} \tag{38}$$
$$B_i(x_i) \leq \epsilon_i \ \forall x \in D_{1,i}, i = 1, \cdots, N \tag{39}$$
$$B_i(x_i) \geq 1 \ \forall x \in D_{2,i}^c, i = 1, \cdots, N \tag{40}$$
$$\frac{\partial B_i(x_i)}{\partial x_i} f_i(x_i, \mathbf{u}_i, w_i) \leq$$
$$\alpha B_i(x_i) + \gamma_i(x_i, \mathbf{u}_i) \ \forall x_i \in \bar{D}_{1,i}, w_i \in W_i,$$
$$\sum_{i=1}^{N} \epsilon_i = \epsilon < 1 \tag{41}$$
$$\sum_{i=1}^{N} \gamma_i(x_i, \mathbf{u}_i) \leq 0, i = 1, \cdots, N \tag{42}$$
$$\alpha < -\frac{1}{T} \ln \epsilon \tag{43}$$

Proof: Let us construct $B(x)$ as $B(x) = \sum_{i=1}^{N} B_i(x_i)$. Then, if the conditions of (41) are satisfied, we have:

$$B(x) \leq \epsilon \ \forall x \in D_1 \tag{44}$$
$$B(x) \geq 1 \ \forall x \in D_2^c \tag{45}$$

Moreover due to (50) and (5), we have:

$$\dot{B}(x) \leq \alpha B(x) \ \forall x \in \bar{D}_1 \times W \tag{46}$$

which proves that the system is FTB with respect to (D_1, D_2, W, T). ■

The above theorem puts an exponential bound on the mappings of trajectories of the systems through $B(x)$. Similarly, we can use a linear bound. This is shown in the following theorem:

Theorem 6: The dynamical system (30) is finite-time bounded with respect to (D_1, D_2, W, T) with D_1, D_2, and W defined as in (32) (33), and (34) respectively, if there exist continuously differentiable functions $B_i(x_i)$, positive scalars ϵ_i's and α_i's such that:

$$B_i(x_i) \leq 0 \ \forall x \in D_{1,i}, i = 1, \cdots, N \tag{47}$$
$$B_i(x_i) \geq 1 \ \forall x \in D_{2,i}^c, i = 1, \cdots, N \tag{48}$$
$$\frac{\partial B_i}{\partial x_i}(x_i)(f_i(x_i, u_1, \cdots, u_N, w_i)) \leq \alpha + \gamma_i(\mathbf{u_i})$$
$$\forall x_i \in \mathbb{R}^{n_i}, w_i \in W_i, u_i \in \mathbb{R}^{n_i} \tag{49}$$
$$\sum_i \gamma_i(x_i, \mathbf{u}_i) \leq 0 \tag{50}$$
$$\alpha < \frac{1}{T} \tag{51}$$

In the following, we show how we can check the conditions of the above theorem by solving a SOS program. The SOS program is only shown for the case of exponential bound on B (theorem 5). The case of linear bound can be solved similarly.

Proposition 3: The dynamical system (30) is finite-time bounded with respect to (D_1, D_2, W, T) with D_1, D_2, and W defined as in (32), (33), and (34) respectively, if there exist continuously differentiable functions $B_i(x_i)$, positive scalars ϵ_i's and α's such that:

$$-B_i(x_i) - s_{1,i}g_{1,i}(x_i) \in \Sigma, i = 1, \cdots, N \tag{52}$$

$$(B_i(x_i) - 1) - s_{2,i}g_{2,i}(x_i) \in \Sigma, i = 1, \cdots, N \tag{53}$$

$$-\frac{\partial B_i(x_i)}{\partial x_i}(f_i(x_i, u_1, \cdots, u_N, w_i)) + \alpha B_i(x_i) + \gamma_i(x_i, \mathbf{u_i})$$
$$-s_{w,i}g_{w,i}(w_i) + s_{3,i}g_{2,i}(x_i) \in \Sigma \tag{54}$$

$$\sum_i \gamma_i(x_i, \mathbf{u}_i) \le 0, i = 1, \cdots, N \tag{55}$$

where is α chosen as $\alpha < \frac{1}{T}\ln\epsilon$.
Therefore, we solve the following feasibility problem for a fixed value of ϵ and choosing $\alpha < \frac{1}{T}\ln\epsilon$:

$$-B_i(x_i) - s_{1,i}g_{1,i} \in \Sigma, i = 1, \cdots, N \tag{56}$$

$$(B_i(x_i) - 1) - s_{2,i}g_{1,i} \in \Sigma, i = 1, \cdots, N \tag{57}$$

$$-\frac{\partial B_i}{\partial x_i}(x_i)(f_i(x_i, u_1, \cdots, u_N, w_i)) + \alpha B_i + \gamma_i(x_i, \mathbf{u_i})$$
$$-s_{w,i}g_{w,i} + s_{3,i}g_{2,i} \in \Sigma, i = 1, \cdots, N \tag{58}$$

$$\sum_{i=1}^{N} \gamma_i(x_i, \mathbf{u}_i) \le 0. \tag{59}$$

The problem is not still decomposed due to the coupling of the constraints through function γ_i. To decompose the problem we use dual decomposition [14], [15]. In order to decompose the problem into subproblems, we constrain the γ_i to have the following structure:

$$\gamma_i = Z(\begin{bmatrix} x_i & \mathbf{u}_i \end{bmatrix})^T P_i Z(\begin{bmatrix} x_i & \mathbf{u}_i \end{bmatrix}), \tag{60}$$

where $Z(\begin{bmatrix} x_i & \mathbf{u}_i \end{bmatrix})$ is a vector of monomials in $\begin{bmatrix} x_i & \mathbf{u}_i \end{bmatrix}$ and $P_i = \operatorname{diag}(\Gamma_i)$ where Γ_i is a vector of constants with an appropriate dimension. Therefore, the coupling constraint $\sum_i \gamma_i(x_i, \mathbf{u}_i) \le 0$ is reduced to $\sum_i \Gamma_i \le 0$. The Largrangian is formed as:

$$\sum_i \lambda^T (\sum_i \Gamma_i) = \sum_i \lambda^T \Gamma_i) \tag{61}$$

and the Lagrangian dual function is :

$$g(\lambda) = \sum_i g_i(\lambda) \tag{62}$$

where

$$g_i(\lambda) := \min\ \lambda^T \Gamma_i$$

$$-B_i(x_i) - s_{1,i}g_{1,i} \in \Sigma, i = 1, \cdots, N \tag{63}$$

$$(B_i(x_i) - 1) - s_{2,i}g_{1,i} \in \Sigma, i = 1, \cdots, N \tag{64}$$

$$-\frac{\partial B_i}{\partial x_i}(x_i)(f_i(x_i, u_1, \cdots, u_N, w_i)) + \alpha B_i(x_i) - \gamma_i(x_i, \mathbf{u_i})$$
$$-s_{w,i}g_{w,i} \in \Sigma \tag{65}$$

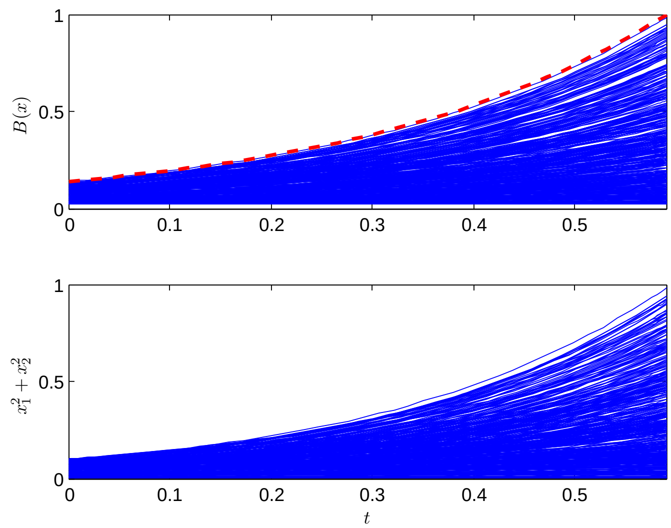

Fig. 2. $B(x)$,$\epsilon e^{\alpha x}$

and the dual problem is given as:

$$\max_{\lambda \ge 0} g(\lambda). \tag{66}$$

For a given λ we can find $g_i(\lambda)$'s separately. If the coupling constraint $\sum_i \Gamma_i \le 0$ is satisfied then the problem is solved, otherwise, we need to update the value of the Lagrangian multiplier λ. This is done by using the subgradient method as shown in Algorithm V (see [14] or[15] for details.).

Algorithm 1 Dual Decomposition

$k \leftarrow 0$, Choose a λ_0
repeat
Solve each subproblem (possibly in parallel) to find α_i^* and Γ_i^*
$\lambda_{k+1} \leftarrow (\lambda_k - \Delta_k(\sum_i \Gamma_i^*))_+$
until $|\lambda_k - \lambda_{k-1}| \le \epsilon$

VI. Example

A. Example 1

Consider the nonlinear system:

$$\begin{cases} \dot{x}_1 = x2, \\ \dot{x}_2 = (2 - 0.5x_1^2)x_2 - x_1 \end{cases} \tag{67}$$

We want to to check finite-time stability of the system with $D_1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 0.1\}$ and $D_2 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ To solve the optimization problem we use YALMIP toolbox. B is chosen to be a polynomial of degree 6 and s_1, s_2, s_3 are of degree 4. The problem is solved with $\epsilon = 0.14$. Th minimum α is obtained as 3.3301 which means that the system is finite-time stable for all $T < 0.5904$. Using extensive simulation, we find that $T = 0.5935$ which shows the tightness of the bound obtained by our approach.

To test the sensivity of solution to the choice of ϵ we solve the problem by performing a line search over ϵ. The values of T for different choices of ϵ are shown in Figure 5.

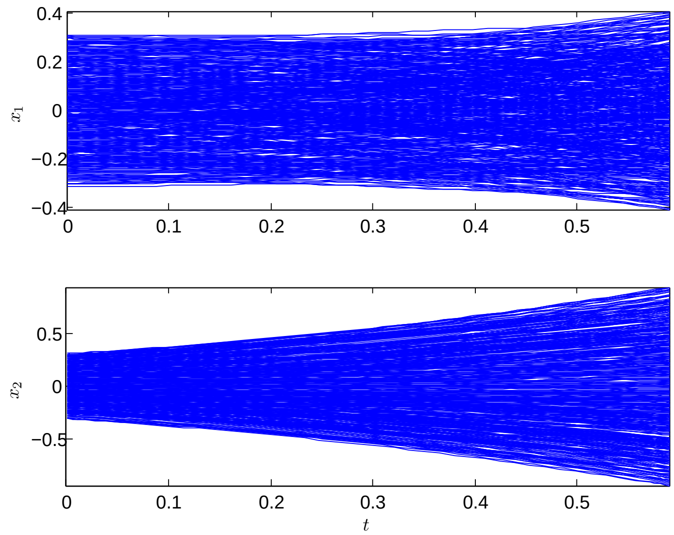

Fig. 3. Trajectories of the system

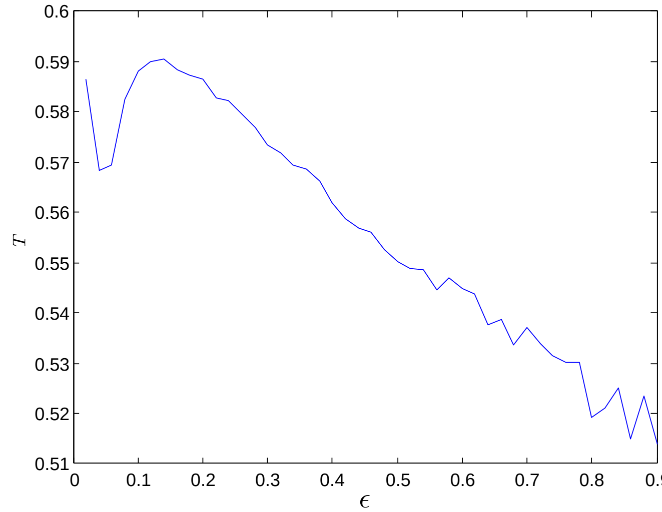

Fig. 5.

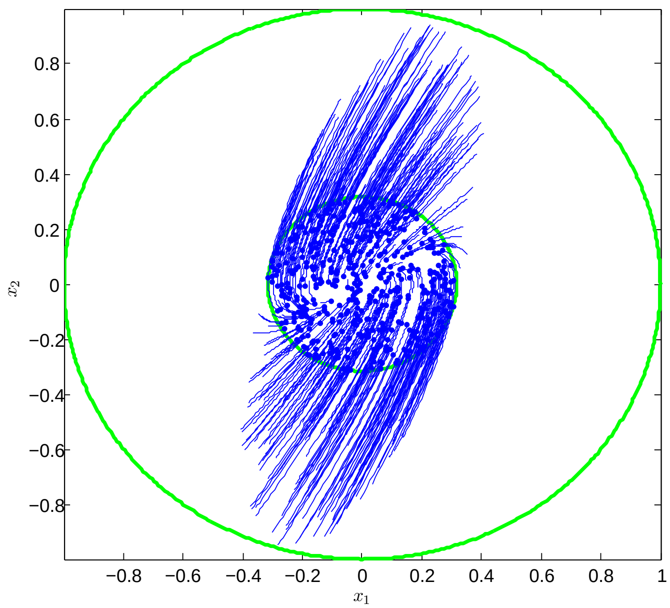

Fig. 4.

B. Example 2

In this example we consider the following systems which consists of two coupled systems as follows:

$$S_1 : \begin{cases} \dot{x}_{11} = x_{12} + x_{21}, \\ \dot{x}_{12} = (2 - 0.5x_{11}^2)x_{12} - x_{11}, \end{cases} \tag{68}$$

and

$$S_2 : \begin{cases} \dot{x}_{21} = x_{22} + x_{11}, \\ \dot{x}_{22} = (2 - 0.5x_{21}^2)x_{22} - x_{21}, \end{cases} \tag{69}$$

The states of S_1 are initiated in $D_{11} = \{x_{11}, x_{12} : x_{11}^2 + x_{12}^2 \leq 0.1\}$ and similarly states of S_2 are initiated in $D_{12} = \{x_{21}, x_{22} : x_{21}^2 + x_{22}^2 \leq 0.1\}$. Also, $D_{21} = \{x_{11}, x_{12} : x_{11}^2 + x_{12}^2 \leq 1\}$ and $D_{22} = \{x_{21}, x_{22} : x_{21}^2 + x_{22}^2 \leq 1\}$ We want to find the maximum T such that the systems is FTS w.r.t (D_1, D_2, T). We choose the $\gamma_i = \begin{bmatrix} x_{11} & x_{21} & x_{11}^2 & x_{21}^2 \end{bmatrix} \operatorname{diag}(\Gamma_i) \begin{bmatrix} x_{11} & x_{21} & x_{11}^2 & x_{21}^2 \end{bmatrix}^T$ and $\epsilon = 0.1$. We want to check FTS of the system with respect to $D_1, D_2, 0.4$. We choose $\epsilon_1 = \epsilon_2 = 0.1$ and therefore we must have $\alpha \leq -\frac{1}{0.4} \ln 0.2 = 0.4023$. We choose $\alpha = 4$. Moreover, Δ_k in the Algorithm V is set as $\frac{0.01}{10+k}$. The dual variables Γ_0 are initialized with $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$. The algorithm is stopped as soon as the constraint $\Gamma_i \leq 0$ is satisfied. In this example the constraint is satisfied in 3 iterations where $\Gamma_3 = \begin{bmatrix} -1.9626 & -1.9237 & -0.1841 & -0.1839 \end{bmatrix}^T$.

VII. Conclusion

References

[1] G. Kamenkov, "On stability of motion over a finite interval of time (in russian)," *Jounral of Applied Mathematics and Mechanics*, vol. 17, pp. 529–540, 1953.

[2] A. Lebedev, "The problem of stability in a finite interval of time (in russian)," *Journal of Applied Mathematics and Mechanics*, vol. 18, pp. 75–94, 1954.

[3] P. Dorato, "Short time stability in linear time varying systems," in *in proceedings of the IRE International Convention Record Part 4*, New York, 1961, pp. 83–87.

[4] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, no. 9, pp. 1459 – 1463, 2001.

[5] F. Amato, M. Ariola, M. Carbone, and C. Cosentino, "Finite-time control of linear systems: a survey," in *Current trends in nonlinear systems and control*. Springer, 2006, pp. 195–213.

[6] G. Garcia, S. Tarbouriech, and J. Bernussou, "Finite-time stabilization of linear time-varying continuous systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 364–369, 2009.

[7] R. Ambrosino, F. Calabrese, C. Cosentino, and G. De Tommasi, "Sufficient conditions for finite-time stability of impulsive dynamical systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 861–865, 2009.

[8] X. Lin, H. Du, S. Li, and Y. Zou, “Finite-time boundedness and finite-time l2 gain analysis of discrete-time switched linear systems with average dwell time,” *Journal of the Franklin Institute*, vol. 350, no. 4, pp. 911 – 928, 2013.

[9] F. Amato, C. Cosentino, and A. Merola, “Sufficient conditions for finite-time stability and stabilization of nonlinear quadratic systems,” *Automatic Control, IEEE Transactions on*, vol. 55, no. 2, pp. 430–434, 2010.

[10] S. P. Bhat and D. S. Bernstein, “Finite-time stability of continuous autonomous systems,” *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[11] S. G. Nersesov, C. Nataraj, and J. M. Avis, “Design of finite-time stabilizing controllers for nonlinear dynamical systems,” *International Journal of Robust and Nonlinear Control*, vol. 19, no. 8, pp. 900–918, 2009.

[12] P. A. Parrilo, “Semidefinite programming relaxations for semialgebraic problems,” *Mathematical programming*, vol. 96, no. 2, pp. 293–320, 2003.

[13] Z. W. Jarvis-Wloszek, “Lyapunov based analysis and controller synthesis for polynomial systems using sum-of-squares optimization,” Ph.D. dissertation, University of California, 2003.

[14] S. Boyd, L. Xiao, A. Mutapcic, and J. Mattingley, “Notes on decomposition methods,” Notes for EE364B, Stanford University.

[15] G. B. Dantzig and P. Wolfe, “The decomposition algorithm for linear programs,” *Econometrica: Journal of the Econometric Society*, pp. 767–778, 1961.