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Bosons and Fermions in Curved Spacetime

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To
my father and mother,
sister and brother.

Abstract

The Klein-Gordon and Dirac equations are formulated for a classical curved background. The equations are then expressed in Friedmann-Robertson-Walker Universe. Further the equations are solved in $\mathbb{R} \times S^3$ spacetime. We present general normalized solutions to the Klein-Gordon equation together with non-normalized particular solutions to the Dirac equation in $\mathbb{R} \times S^3$. The discrete energy quantization and degeneracy are found for both cases.

Sammendrag

Klein-Gordon ligningen og Dirac ligningen formuleres for generelt krumme klassiske bakgrunner. Deretter blir disse uttrykt for Friedmann-Robertson-Walker Univers. Videre løses ligningene i $\mathbb{R} \times S^3$ romtid. Vi presenterer generelle normaliserte løsninger av Klein-Gordon ligningen, sammen med ikke-normaliserte partikulære løsninger av Dirac ligningen i $\mathbb{R} \times S^3$. Den diskrete energikvantiseringen sammen med degenerasjonsgraden blir funnet for begge tilfeller.

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Conventions

- We use natural units throughout, setting $\hbar = c = 1$.
- The metric will have the signature $\text{diag}(+ - --)$.
- Einsteins summation convention is used.

Chapter 1

Introduction

The quantum theory of gravity can rightfully be described as the holy grail of modern physics. Since the completion of the theory of quantum mechanics and quantum field theory, physicists have tried to reconcile the theories of gravity and quantum physics, with recent attempts being the famous *string theory*.

The theory of General Relativity shows that gravity is an effect attributed to the curvature of spacetime. In this thesis we hare going to analyze quantum mechanical equations in a classical curved background. In this sense we can call it *classical quantum gravity*. Specifically we are going to analyze the Klein-Gordon equation and the Dirac equation. We will solve these equations in a closed Friedmann-Robertson-Walker Universe to obtain the energy eigenvalues and degeneracies. Such problems have been considered in earlier works, see for example [6] and [15]. This work will be a continuation of the thesis [9].

Chapter 2

Curved Spacetime

As discussed in the introduction, this thesis will provide an analysis of quantum fields in a curved spacetime background. In this chapter we are therefore going to outline the mathematical concepts that are needed to describe curved spacetime. These concepts will describe the nature of space and time itself, the arena if you like, of all physical processes. We take into account the Special Theory of Relativity and describe space and time together as *spacetime*.

Mathematically, the spacetime of our universe make up what is called a 4 dimensional *manifold*. That is, a topological space (which is Hausdorff and has a second countable basis) that is locally euclidean. In fact our spacetime is locally Minkowskian. Such manifolds are called *pseudo-Riemannian* or *Lorentzian* manifolds. We won't venture too far into the mathematics of manifolds. It is only useful for us to have it as a basic mathematical structure on which we are going to attach the properties that will be of importance. For a more thorough treatment on curved spacetime see [7], which is the book we will mostly follow in this chapter.

2.1 Vectors and dual vectors

On a manifold M we define at each point p a *tangent space* T_p, which is the vector space consisting of the tangent vectors to all the curves passing through p on M.

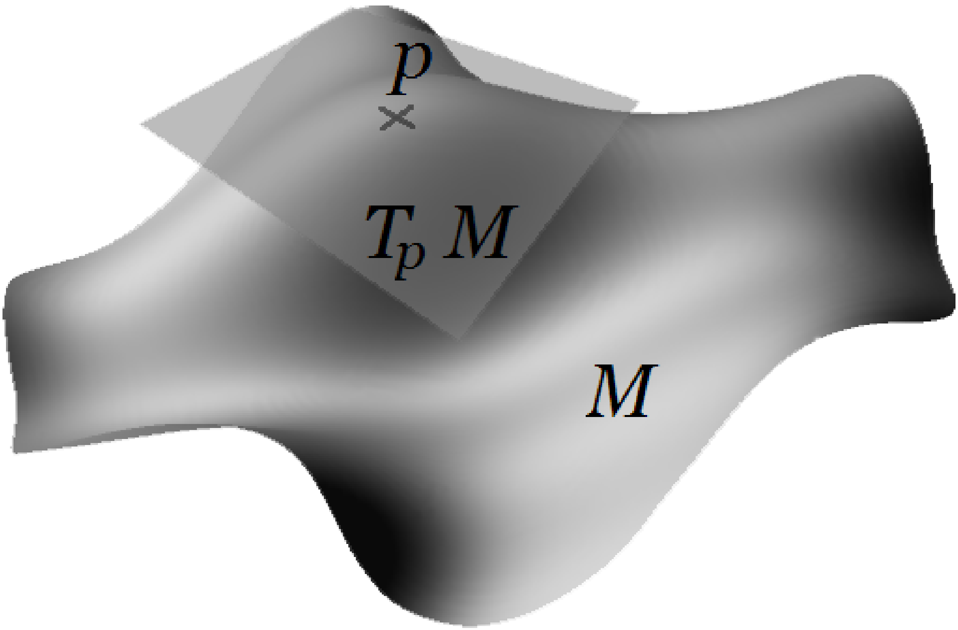

Figure 2.1: The tangent space T_p at the point p on a manifold M [12].

We will refer to a *vector* V as a vector in T_p and denote the components of V by V^μ. These are the components of V with respect to some basis $\hat{e}_{(\mu)}$ such that

$$V = V^\mu \hat{e}_{(\mu)}. \tag{2.1}$$

There is a natural basis that we can define on the tangent space. It is the basis of directional derivatives of the coordinate functions x^μ at p. We will call this basis the *coordinate basis* and it consists of the partial derivatives;

$$\hat{e}_{(\mu)} = \partial_\mu. \tag{2.2}$$

If we change the coordinate system such that $x^\mu \to x^{\mu'}$, the new coordinate basis will be given by the chain rule;

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu. \tag{2.3}$$

Hence, for a coordinate transformation we have

$$V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu. \tag{2.4}$$

A vector is unaffected by a coordinate transformation (only its components will transform), so that the transformation property of the components is given by;

$$V^{\mu'} = V^{\mu} \frac{\partial x^{\mu'}}{\partial x^{\mu}}. \tag{2.5}$$

For each tangent space T_p, we define a *cotangent space* T_p^* consisting of *dual vectors*. These are linear maps from the tangent space to the real numbers. Consider a dual vector ω. Then

$$\omega = \omega_\mu \hat{\theta}^{(\mu)} \tag{2.6}$$

for some basis $\hat{\theta}^{(\mu)}$. This dual basis is constructed such that

$$\hat{\theta}^{(\nu)} \left(\hat{e}_{(\mu)} \right) = \delta^{\nu}_{\mu}, \tag{2.7}$$

so when we act on a vector V by ω we get

$$\begin{aligned} \omega(V) &= \omega_\mu \hat{\theta}^{(\mu)} \left(V^{\nu} \hat{e}_{(\nu)} \right) \\ &= \omega_\mu V^{\nu} \hat{\theta}^{(\mu)} \left(\hat{e}_{(\nu)} \right) \\ &= \omega_\mu V^{\mu}. \end{aligned} \tag{2.8}$$

The gradient of a function f is a dual vector, and is denoted by $\mathrm{d}f$. If we take the gradient of the coordinate functions, $\mathrm{d}x^{\mu}$, these will constitute the basis in the cotangent space corresponding to the coordinate basis in the tangent space. This is because

$$\mathrm{d}x^{\mu} \left(\partial_\nu \right) = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}. \tag{2.9}$$

The dual coordinate basis transforms as

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \mathrm{d}x^{\mu} \tag{2.10}$$

under coordinate transformations, so that a dual vector will transform as

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_\mu. \tag{2.11}$$

2.1.1 Tensors

Of course the concepts of vectors and dual vectors can be generalized to tensors of any rank. A tensor T of rank (k,l) is a multi-linear map from k dual vectors and l vectors to the real numbers. Hence, in the coordinate basis we have

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_l}. \tag{2.12}$$

We will usually suppress the tensor product notation. Under a coordinate transformation the components of a rank-(k,l) tensor transform as

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\,. \tag{2.13}$$

Notation: In this thesis we will almost always refer to a tensor by its components. We will use the convention that when the tensor is written with indices from the middle of the Greek alphabet, $\lambda,\mu,\nu,\rho,\sigma$, the components are with respect to the coordinate basis. When we speak of the coordinate basis we will include both the basis and the dual basis.

At times we will encounter objects that possess indices but is not a tensor of any rank. Such an object might be the Christoffel symbols $\Gamma^{\mu}{}_{\nu\lambda}$, which we will discuss more later. Here the index names and placing serve only to remind us what types of indices the indices of the non-tensorial object usually contract with. The placement of the indices of non-tensorial objects will be of such a nature that we can use the summation convention.

2.2 The metric

When we discuss properties of length or distances, angles or intervals of time, we talk about concepts that are fundamental to physics and the measurements we do in experiments. They are concepts we associate with spacetime, but are not properties that exist on a manifold alone. We need an additional structure to deal with these concepts. That is the role of the *metric*.

The metric will be a rank-$(0,2)$ tensor that is denoted by $g_{\mu\nu}$. It will be symmetric in its two indices and will have the inverse $g^{\mu\nu}$ such that

$$g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}_{\lambda}. \tag{2.14}$$

Repeatedly the metric is written as

$$ds^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \tag{2.15}$$

where we have explicitly written out the basis. The notation ds^2 is inherited from the notion of a *line element* in which the dual basis components $\mathrm{d}x^{\mu}$ is replaced by the infinitesimals dx^{μ}. Because of this we will usually call ds^2 the line element.

With the metric structure we can, for each vector V^μ, define the dual vector corresponding to V^μ in the following way

$$V_\mu \equiv g_{\mu\nu} V^\nu \mathrm{d}x^\mu \mathrm{d}x^\nu \left(\partial_\nu\right) = g_{\mu\nu} V^\nu \mathrm{d}x^\mu = g_{\mu\nu} V^\nu. \tag{2.16}$$

Similarly we get a vector from a dual vector by acting on the dual vector by the inverse metric tensor. The idea is that we define the *inner product* of two vectors V^μ and W^ν as

$$V \cdot W \equiv V_\mu W^\mu = g_{\mu\nu} V^\mu W^\nu. \tag{2.17}$$

Now we see that we have defined a structure that allows us to talk about the length of vectors and other quantities related to the usual notion of an inner product. The procedure of contracting indices by the metric tensor generalizes to tensors of any rank, and we say that we can lower or raise tensorial indices with the metric and its inverse respectively.

In flat spacetime the line element is given by

$$ds^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \equiv \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{2.18}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.19}$$

is the usual *Minkowski metric*.

2.3 Covariant derivatives and the Christoffel symbol

The idea of curvature can exist on a manifold without the structure of a metric. Although we will only deal with spacetime as a manifold equipped with a metric. Curvature however is not assigned to the metric, but to an object known as the *affine connection*. We denote the affine connection by $\Gamma^\mu{}_{\nu\lambda}$, but it is not a tensor. When the manifold is equipped with a metric there is a unique affine connection known as the *Christoffel symbol* and we will use the same notation for this connection as with a general affine connection. This will be no problem since we from this point and out will assume that the metric structure is attached to the spacetime manifold.

The connection is intimately related to the concept of a covariant derivative. In a Minkowskian spacetime the partial derivative of a tensor, for example $\partial_\mu V^\nu$, has the same form in every coordinate system related by a Lorentz transformation. These objects can therefore be used to write physical laws that are Lorentz invariant and therefore obey Special Relativity. In an arbitrary spacetime and for arbitrary coordinate transformations the notion of a partial derivative is generalized to a *covariant derivative*. The covariant derivative is constructed in such a way that when acting on a general tensor, the result will again be a tensor. We denote the covariant derivative by ∇_μ.

By requiring that the operator ∇_μ should be linear and obey the Leibniz rule for operators, it can be written as a partial derivative plus a linear transformation. So for the covariant derivative of a vector we have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda, \tag{2.20}$$

where $\Gamma^\nu{}_{\mu\lambda}$ is a set of linear transformations, or matrices, known as the *connection*. The transformation property of the connection is such that the object $\nabla_\mu V^\nu$ transforms as a tensor. We won't state the explicit form of the transformation property of $\Gamma^\nu{}_{\mu\lambda}$ here, other than that it is not that of a tensor.

Imposing that the covariant derivative of the Kronecker delta should vanish and that

$$\nabla_\mu \phi = \partial_\mu \phi, \tag{2.21}$$

for a scalar ϕ, we get the covariant derivative of a dual vector:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda. \tag{2.22}$$

This is generalized to a rank-(k,l) tensor in the following way:

$$\begin{aligned}
\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} =& \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\
&+ \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}{}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\
&- \Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots .
\end{aligned} \tag{2.23}$$

We will assume that the connection is *torsion-free*, i.e that it is symmetric in the lower indices;

$$\Gamma^\mu{}_{\nu\lambda} = \Gamma^\mu{}_{\lambda\nu} \tag{2.24}$$

In addition we assume that the connection is *metric compatible*, that is, that the covariant derivative of the metric vanishes;

$$\nabla_\mu g_{\nu\lambda} = 0. \tag{2.25}$$

With these assumptions there is a unique expression of the connection that involves the metric tensor;

$$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)} \tag{2.26}$$

When written in this way, with the assumptions we have made, the connection is called the Christoffel symbol.

One identity which will be of use, is the expression for $\Gamma^{\nu}{}_{\mu\nu}$. Contracting the two indices gives the identity

$$\Gamma^{\nu}{}_{\mu\nu} = \partial_\mu \ln \sqrt{-g}, \tag{2.27}$$

where g is the metric determinant.

2.3.1 Parallel transport

When we take the derivative of a vector field along some curve $x^\mu(\lambda)$ in the usual euclidean scenario, we essentially compare the components of the vector at two infinitesimally close points, call them x and $x + dx$. They are separated by the parameter distance $\delta\lambda$. The derivative along the curve in flat spacetime at the point x will then be

$$\left. \frac{d}{d\lambda} V^\mu \right|_x = \lim_{\delta\lambda \to 0} \frac{V^\mu(x + dx) - V^\mu(x)}{\delta\lambda}. \tag{2.28}$$

Now on an arbitrary manifold, the object $V^\mu(x + dx) - V^\mu(x)$ will not in general be a vector. This is because a vector at to different points on a manifold is not part of the same tangent space. The result of this will be that their difference is not necessarily a vector in any tangent space, and does in general not exist. We recognize this as the same reason that the partial derivative of a vector is not itself a vector. This issue is overcome with the concept of *parallel transport*. The idea is to take the vector at one of the two points and transport it along the manifold, while keeping it constant, so that the two vectors coincide. Then the two vectors will be part of the same tangent space and is therefore comparable. In flat spacetime the requirement for a vector to be parallel transported along a curve $x^\mu(\lambda)$, would be that its derivative along the curve should vanish. On a general manifold this requirement becomes

$$\frac{D}{d\lambda} V^\mu = \frac{dx^\nu}{d\lambda} \nabla_\nu V^\mu = 0, \tag{2.29}$$

where we have defined the covariant derivative along the curve as

$$\frac{D}{d\lambda} \equiv \frac{dx^\nu}{d\lambda} \nabla_\nu. \tag{2.30}$$

This generalizes easily to tensors of any rank. It is worth noting that the result of parallel transport is dependent on the path of transport. This can be illustrated as follows:

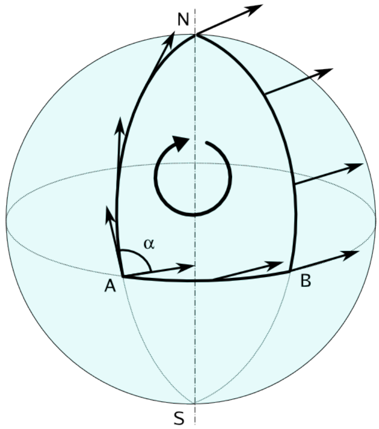

Figure 2.2: Parallel transport of a vector on a 2-sphere [4].

Using the expression for the covariant derivative of a vector, (2.20), we get the *equation of parallel transport* of a vector:

$$\frac{D}{d\lambda}V^\mu = \frac{dx^\nu}{d\lambda}\partial_\nu V^\mu + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}V^\sigma = 0. \tag{2.31}$$

This is a first order ODE that describes how a vector V^μ is parallel transported along a path $x^\nu(\lambda)$ on a manifold. It will be independent of the parametrization λ. On infinitesimal form it is written as

$$\boxed{V^\mu(x \to x + dx) = V^\mu(x) - \Gamma^\mu{}_{\nu\lambda}(x)V^\nu(x)dx^\lambda} \tag{2.32}$$

Here we have written the coordinates explicitly to know where each object is located. $V^\mu(x \to x + dx)$ is the vector $V^\mu(x)$ parallel transported from the point x to the point $x + dx$ along the curve $x^\mu(\lambda)$. We use the notation in which a point in spacetime is written as x with the indices suppressed.

The assumption that the connection is metric compatible, (2.25), yields that the length of a vector is preserved when it is parallel transported. To see this we calculate

$$\begin{aligned}\frac{D}{d\lambda}\left(g_{\mu\nu}V^\mu V^\nu\right) &= \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu V^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)V^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}V^\nu\right) \\ &= 0,\end{aligned} \tag{2.33}$$

where the first term vanishes since $\nabla_\lambda g_{\mu\nu} = 0$. The other terms is zero because of the requirement of parallel transport. We conclude from this that the assumption of metric compatibility is a natural one to make.

Equation (2.32) will be of major importance in this thesis. We will use it to find the connection coefficients of objects that are not necessarily tensors, such is the case with the spinor in the Dirac equation. We see that it is the Christoffel symbols that determine the transport of vectors written in the coordinate basis. For other bases the connection will be different. We will encounter a scenario like that when we discuss local inertial bases. More on that later. First we are going to briefly cover the material of quantifying the curvature on the manifold.

2.3.2 The Riemann curvature tensor

The material covered in this section will be of minor importance to us, so we will only state the general properties of the Riemann tensor.

The exact notion of curvature at each point on a manifold is quantified by the *Riemann curvature tensor*. It is defined as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \tag{2.34}$$

This is a rank-$(1,3)$ tensor that is antisymmetric in the last two lower indices. We see that this tensor is only dependent on the Christoffel symbols. Hence it is the connection that defines a curvature on a manifold, as stated earlier.

From the Riemann curvature tensor we can construct two objects that will be of relevance. First off we have the *Ricci tensor* defined as

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \tag{2.35}$$

This tensor will be symmetric. From this tensor again, we can finally form what is called the *Ricci scalar*;

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}. \tag{2.36}$$

2.4 Local inertial coordinates and the tetrad formalism

Thus far we have been working with the natural choice of the coordinate basis on the spacetime manifold. In the coordinate basis the basis vectors are given by $\hat{e}_{(\mu)} = \partial_\mu$ and the dual basis vectors are given by $\hat{\theta}^{(\mu)} = \mathrm{d}x^\mu$. All the objects and structure considered exist independently of any specific coordinate system, so let's take a slightly different approach of setting up a basis.

The fundamental principle that underlies the General Theory of Relativity is the *Einstein Equivalence Principle*. It can be stated in many different ways, but let's use the formulation from [7]:

In small enough regions of spacetime, the laws of physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.

We recognize this as one of the criterions for describing spacetime as a Lorentzian manifold; the fact that spacetime is locally Minkowskian. It is a direct consequence of the Principle of Equivalence. Since we know how to formulate the relativistic quantum mechanics for the scalar and spin-$\frac{1}{2}$ particles, i.e, to describe their behaviour in Minkowski space, we know that this is valid in small enough regions of *curved* spacetime. This will be our starting point for our formulation of the Dirac equation in curved spacetime as we will see later.

Let's start with the basics first. Let's set up a basis at each point in spacetime, a spacetime that will be curved, and let the basis be Minkowskian or locally inertial. This means that the metric tensor will be Minkowskian when written in terms of this basis. We will denote these basis vectors as $\hat{e}_{(a)}$ and the dual basis vectors as $\hat{\theta}^{(b)}$, with a latin index. So in the neighbourhood of each point in spacetime we have

$$g_{ab} = g_{ab}\hat{\theta}^{(a)}\hat{\theta}^{(b)} = \eta_{ab}. \tag{2.37}$$

This can be seen as the inner product of the two dual basis vectors, and in this sense they constitute an orthonormal set with respect to the Minkowski metric. Such an orthonormal set is called a *tetrad* or *vierbein*, and this procedure of setting up an orthonormal frame at each point on a manifold is called *the tetrad formalism*.

Notation: We will use the convention that when a tensor is written with indices from the start of the Latin alphabet, a, b, c, d, then the components are with respect to a local inertial basis, such as described in this section. From this point and out we will refer to a vector written in terms of local indices as *local vectors*. Vectors written in terms of the coordinate basis will be referred to as *global vectors*. There is a point to be specified here. A manifold is comprised of so called *charts*, which is coordinate systems that cover "patches" of the manifold. In this sense there might not exist a single coordinate system that can cover the whole manifold, and hence be called global. Nevertheless we will still call the coordinate basis a global basis. It is more global than local anyway.

As before we require that

$$\hat{\theta}^{(a)}\left(\hat{e}_{(b)}\right) = \delta^a_b. \tag{2.38}$$

We can always transform between the old coordinate basis and the local inertial basis;

$$\hat{e}_{(\mu)} = e_{\mu}{}^{a}\hat{e}_{(a)} \tag{2.39}$$

and similar for the dual basis,

$$\hat{\theta}^{(\mu)} = e^{\mu}{}_{b}\hat{\theta}^{(b)}. \tag{2.40}$$

Here ${e_\mu}^a$ and ${e^\mu}_b$ will be transformation matrices. These will be the matrices we will call the *vierbein* and the *inverse vierbein* respectively. That they are inverses follows from the requirements (2.38) and (2.7). We have that

$$ {e_\mu}^a {e^\mu}_b = \delta^a_b \qquad \text{and} \qquad {e_\mu}^a {e^\nu}_a = \delta^\nu_\mu . \tag{2.41} $$

Consider the metric written in terms of the global basis, and transform the global basis to the local one. The result will be the metric written in local coordinates as in equation (2.37). Let's see

$$ g_{ab} = g_{\mu\nu} {e^\mu}_a \hat{\theta}^{(a)} {e^\nu}_b \hat{\theta}^{(b)} = \eta_{ab} . \tag{2.42} $$

Suppressing the dual basis vectors we get that

$$ \eta_{ab} = {e^\mu}_a {e^\nu}_b g_{\mu\nu} \qquad \text{and} \qquad g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab} . \tag{2.43} $$

The metric tensor is not the only tensor we can refer to in terms of the local basis. We can talk of any tensor with respect to the local basis, and the vierbeins and the inverse vierbeins transform between the local and global indices. So for a global vector V^μ, its components in the local basis is

$$ V^a = {e_\mu}^a V^\mu . \tag{2.44} $$

For a general tensor we have

$$ {T^{\mu_1 \cdots a \cdots \mu_k}}_{\nu_1 \cdots b \cdots \nu_l} = {e_\mu}^a {e^\nu}_b {T^{\mu_1 \cdots \mu \cdots \mu_k}}_{\nu_1 \cdots \nu \cdots \nu_l} , \tag{2.45} $$

and we see that we can switch back and fourth between local and global indices. With this notation we have that local indices gets raised and lowered with the Minkowski metric, while global indices gets raised and lowered as usual with the global metric. With the vierbeins at hand we have a way of dealing with physical quantities that are only formulated in a Minkowskian background.

2.5 The spin connection

With the tetrad formalism we have set up a local inertial coordinate system at each point on the curved manifold. We want to know how a vector located at one point with components in the local basis at that point, parallel transports to another point on the manifold with a new local basis. The answer to this question will yield a connection related to this form of parallel transport. It is called *the spin connection* and it will be define a covariant derivative of local tensors. The name *spin connection* derives from the fact that it allows us to formulate the covariant derivative of the spinor, which we will discuss later.

The equation for parallel transport of local vectors should be of the same form as the global vector case, except with a different connection;

$$V^a(x \to x + dx) = V^a(x) - \omega_\mu{}^a{}_b(x) V^b(x) dx^\mu. \tag{2.46}$$

Here $\omega_\mu{}^a{}_b(x)$ is the spin connection. It carries the information to transport the vector itself utilizing the Christoffel symbols, as well as adjusting the local coordinates at the starting point with the local coordinates at the end point via the vierbeins. To find the explicit expression for $\omega_\mu{}^a{}_b(x)$, we first recall that

$$V^\mu(x) = e^\mu{}_a(x) V^a(x). \tag{2.47}$$

Transported from the point x to the point $x + dx$ this reads

$$V^\mu(x \to x + dx) = e^\mu{}_a(x + dx) V^a(x \to x + dx). \tag{2.48}$$

Expanding $e^\mu{}_a(x + dx)$ to first order in dx yields

$$V^\mu(x \to x + dx) = e^\mu{}_a(x) V^a(x \to x + dx) + \partial_\nu e^\mu{}_a(x) V^a(x \to x + dx) dx^\nu. \tag{2.49}$$

Inserting the expression for $V^a(x \to x + dx)$, (2.46), we get (keeping only first order terms in dx)

$$\begin{aligned} V^\mu(x \to x + dx) &= e^\mu{}_a(x) V^a(x) - \left[e^\mu{}_a(x) \omega_\lambda{}^a{}_b(x) - \partial_\lambda e^\mu{}_b(x)\right] V^b(x) dx^\lambda \\ &= V^\mu(x) - \left[e^\mu{}_a(x) \omega_\lambda{}^a{}_b(x) - \partial_\lambda e^\mu{}_b(x)\right] e_\sigma{}^b(x) V^\sigma(x) dx^\lambda. \end{aligned} \tag{2.50}$$

From the parallel transport equation for global vectors, (2.32), we recognize

$$\Gamma^\mu{}_{\sigma\lambda} = \left[e^\mu{}_a \omega_\lambda{}^a{}_b - \partial_\lambda e^\mu{}_b\right] e_\sigma{}^b. \tag{2.51}$$

Solving this for the spin connection, we find it to be

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\sigma{}_b \Gamma^\nu{}_{\sigma\mu} + e_\nu{}^a \partial_\mu e^\nu{}_b. \tag{2.52}$$

Usually we will write it with all indices lowered:

$$\boxed{\omega_{\mu ab} \equiv \eta_{ac} e_\nu{}^c e^\sigma{}_b \Gamma^\nu{}_{\sigma\mu} + \eta_{ac} e_\nu{}^c \partial_\mu e^\nu{}_b} \tag{2.53}$$

Now, since $\Gamma^\nu{}_{\sigma\mu}$ is the Christoffel symbols, we have assumed that the connection is metric compatible. In terms of local coordinates this would mean that the covariant derivative of the Minkowski metric should vanish. With the spin connection at hand we see that

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac}$$

$$= -\omega_{\mu ba} - \omega_{\mu ab} = 0, \tag{2.54}$$

hence the $\omega_{\mu ab}$ is antisymmetric in the last two indices;

$$\omega_{\mu ab} = -\omega_{\mu ba}. \tag{2.55}$$

2.6 The Friedmann–Robertson–Walker metric

In this last section in the chapter on curved spacetime we are going to consider the specific metric in which we will analyze the Klein-Gordon and Dirac fields. It will be the metric that reflects the homogeneity and isotropy of the Universe. The line element is given by

$$ds^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right], \tag{2.56}$$

and it is known as the *Friedmann-Robertson-Walker metric*, the FRW metric for short. Here the expression in the square bracket will be the metric of a maximally symmetric 3-manifold denoted by Σ. In the coordinates we will be using, the size of this manifold will be given by the factor $a(t)$, known as the *scale factor*. The scale factor will then have unit of distance and measure the size of the Universe. Now, the coordinate r will be dimensionless and the factor k in $\frac{1}{1-kr^2}\mathrm{d}r^2$ will take the values

$$k \in \{-1, 0, 1\}. \tag{2.57}$$

It is useful to make the substitution

$$\mathrm{d}\chi = \frac{\mathrm{d}r}{1-kr^2}, \tag{2.58}$$

which yields

$$r = S_k(\chi). \tag{2.59}$$

For each value of k, the function $S_k(\chi)$ will be given by

$$S_k(\chi) = \begin{cases} \sin\chi, & k = +1 \\ \chi, & k = 0 \\ \sinh\chi, & k = -1. \end{cases} \tag{2.60}$$

When $k = -1$ the manifold Σ will exhibit a constant negative curvature and is therefore called *open*. For $k = 0$, Σ will be the manifold of flat space. The case that will be of importance to us will be the case when $k = +1$. In that case the maximally symmetric manifold Σ has a constant positive curvature. It is then called *closed*, and Σ will be the manifold of the 3-sphere.

As mentioned in the introduction we will formulate the Klein-Gordon equation and the Dirac equation in a generally curved spacetime. We will then write out these equations for the FRW metric. At last we are going to solve them on the 3-sphere, and we now have a way of substitution that will give the metric of the 3-sphere from the FRW metric.

Chapter 3

Klein-Gordon Fields in Minkowski Spacetime

One of the two parts of this thesis will be the analysis of the Klein-Gordon field. We begin therefore with a chapter regarding the Klein-Gordon equation and its field in Minkowski spacetime. We will mostly follow L. H. Ryder; *Quantum Field Theory* [13] in the general analysis of the Klein-Gordon equation.

Bosonic particles of spin-0 are described by a scalar field which we will denote as $\phi(x)$. The dynamics of such a scalar field is determined by the so-called *Klein-Gordon equation* of which we will now turn our attention.

3.1 The free Klein-Gordon equation

The neutral and free Klein-Gordon field ϕ will be real and described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi\right)\left(\partial_\mu \phi\right) - \frac{1}{2}m^2\phi^2. \tag{3.1}$$

The corresponding Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 \tag{3.2}$$

Here m is the mass parameter. It is interpreted as the mass of the particles resulting from the quatization of the field. In the case of a charged scalar field, ϕ will be complex and have two independent components; ϕ and its complex conjugate. Inserted into the Euler-Lagrange equation, (3.1) gives the equation of motion known as the Klein-Gordon equation;

$$\left(\partial^{\mu}\partial_{\mu}+m^{2}\right)\phi=0. \tag{3.3}$$

Historically this equation was put forward as the relativistic improvement of the Schrödinger equation. Hence the Klein-Gordon equation was first thought of as a single particle wave equation, with ϕ interpreted as a quantum mechanical wave function obeying Born's probability interpretation. It turns out that such a view cannot be correct. This is because the conserved current density corresponding to the Klein-Gordon equation gives a charge density that is not positive definite, thereby shattering any hope of interpreting it as a probability current in the first place. To see this we observe that the conserved current density in the case of the Klein-Gordon equation will be

$$j^{\mu}=i\left[\phi^{*}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{*}\right]. \tag{3.4}$$

This current density satisfies the continuity equation;

$$\begin{aligned}\partial_{\mu}j^{\mu}=\frac{\partial}{\partial t}\rho+\boldsymbol{\nabla}\cdot\mathbf{j}&=i\partial_{\mu}\left[\phi^{*}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{*}\right]\\&=i\left[\phi^{*}\partial_{\mu}\partial^{\mu}\phi-\phi\partial_{\mu}\partial^{\mu}\phi^{*}\right]=0,\end{aligned}$$

where we have used the Klein-Gordon equation for ϕ and its complex conjugate. With this current density we see that the charge density is given by

$$\rho=i\left[\phi^{*}\frac{\partial}{\partial t}\phi-\phi\frac{\partial}{\partial t}\phi^{*}\right]. \tag{3.5}$$

Now since the Klein-Gordon equation is a second order PDE we are free to choose the initial conditions of ϕ and $\frac{\partial}{\partial t}\phi$. Hence ρ can be negative, positive or zero, thereby establishing the failure of the probability interpretation. We note that for a real field, ρ will always be zero.

With (3.5) in mind, we define the inner product of two functions f and g for the Klein-Gordon case as

$$(f,g)=i\int d^{3}\mathbf{x}\left[f^{*}\frac{\partial}{\partial t}g-g\frac{\partial}{\partial t}f^{*}\right]. \tag{3.6}$$

This inner product will define orthogonality of the eigenfunctions of the Klein-Gordon equation.

Another problem with the single particle interpretation is the appearance of negative energy states for the free particle. This issue, and the current density issue, disappear however when the Klein-Gordon equation is rightfully interpreted as a many-particle equation.

Then the scalar field is taken to be an operator, and the procedure of promoting the field to an operator is called *second quantization*. The result of second quantization will be a proper quantum field theory.

As mentioned, the goal of this thesis will be to calculate the energy spectrum and degeneracy for the Klein-Gordon and Dirac field on the 3-sphere. In this sense we will not go far with the second quantization procedure. The solutions and spectrum we find however, is important in their own right since they will be the staring point of second quantization anyway. With this in mind, let's solve the free Klein-Gordon equation for the scalar field.

3.2 The Klein-Gordon field

To find the solutions to the Klein-Gordon equation we adapt Fourier analysis and write the field as a wave expansion (writing the field as explicitly real):

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \left[\phi(k)\mathrm{e}^{-ikx} + \phi^*(k)\mathrm{e}^{ikx}\right]. \tag{3.7}$$

Here $\phi(k)$ are general Fourier coefficients to be determined. Substituting $\phi(x)$ into (3.3) yields

$$\int \frac{d^4k}{(2\pi)^4} \left(k^2 - m^2\right) \left[\phi(k)\mathrm{e}^{-ikx} + \phi^*(k)\mathrm{e}^{ikx}\right] = 0. \tag{3.8}$$

This holds generally only if $k^2 = m^2$, so we write $\phi(x)$ with the coefficients

$$\phi(k) = 2\pi\delta\left(k^2 - m^2\right) A(k), \tag{3.9}$$

which gives

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3} \delta\left(k^2 - m^2\right) \left[A(k)\mathrm{e}^{-ikx} + A^*(k)\mathrm{e}^{ikx}\right]. \tag{3.10}$$

We are now ready to perform the integration over the k^0-component utilizing the delta function. The integral contributes only when $k^2 = m^2$, or phrased differently, when $\left(k^0\right)^2 = \mathbf{k}^2 + m^2$. Defining $\omega_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m^2}$ and recalling the properties of the delta function, we write

$$\delta\left(k^2 - m^2\right) = \delta\left(\left(k^0\right)^2 - \omega_{\mathbf{k}}^2\right)$$

$$= \frac{1}{2\omega_{\mathbf{k}}}\left[\delta\left(k^0 + \omega_{\mathbf{k}}\right) + \delta\left(k^0 - \omega_{\mathbf{k}}\right)\right]. \tag{3.11}$$

Inserted into (3.10) we get the general solution of the Klein-Gordon equation;

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[a(\mathbf{k})\mathrm{e}^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + a^*(\mathbf{k})\mathrm{e}^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}\right], \tag{3.12}$$

where we have gathered like exponentials after performing the substitution $\mathbf{k} \to -\mathbf{k}$ in two of the terms. We have defined $a(\mathbf{k}) = A(\omega_{\mathbf{k}}, \mathbf{k}) + A^*(-\omega_{\mathbf{k}}, -\mathbf{k})$.

Next we note that the functions

$$f_{\mathbf{k}} = \frac{1}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}}\mathrm{e}^{-i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \tag{3.13}$$

form an orthonormal set with respect to the inner product (3.6);

$$(f_{\mathbf{k}}, f_{\mathbf{k}'}) = i\int d^3\mathbf{x}\left[f_{\mathbf{k}}^*\frac{\partial}{\partial t}f_{\mathbf{k}'} - f_{\mathbf{k}'}\frac{\partial}{\partial t}f_{\mathbf{k}}^*\right] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{3.14}$$

Hence we write

$$\phi(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}}\left[f_{\mathbf{k}}a(\mathbf{k}) + f_{\mathbf{k}}^* a^*(\mathbf{k})\right], \tag{3.15}$$

and the conjugate momentum of the field as

$$\pi(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}(-i)\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[f_{\mathbf{k}}a(\mathbf{k}) - f_{\mathbf{k}}^* a^*(\mathbf{k})\right]. \tag{3.16}$$

3.3 Quantization of the field

Here we will briefly go through the most important points regarding second quantization for this thesis. To quantize the field, the field and its conjugate momentum gets promoted to operators. The result is that the Fourier coefficients $a(\mathbf{k})$ and $a^*(\mathbf{k})$ become the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$.

To quantize the field we impose the equal time commutation relations:

$$\left.\begin{aligned}\left[\phi\left(t,\mathbf{x}\right),\pi\left(t,\mathbf{x}'\right)\right]&=i\delta^3\left(\mathbf{x}-\mathbf{x}'\right),\\ \left[\phi\left(t,\mathbf{x}\right),\phi\left(t,\mathbf{x}'\right)\right]&=0,\\ \left[\pi\left(t,\mathbf{x}\right),\pi\left(t,\mathbf{x}'\right)\right]&=0.\end{aligned}\right\}\tag{3.17}$$

Imposing these commutators yields

$$\phi(x)=\int\frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}}\left[f_{\mathbf{k}}a(\mathbf{k})+f^*_{\mathbf{k}}a^\dagger(\mathbf{k})\right]\tag{3.18}$$

and

$$\pi(x)=\int\frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}(-i)\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[f_{\mathbf{k}}a(\mathbf{k})-f^*_{\mathbf{k}}a^\dagger(\mathbf{k})\right],\tag{3.19}$$

where now $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are operators that can be shown to obey the following commutation relations

$$\left.\begin{aligned}\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right]&=(2\pi)^3 2\omega_{\mathbf{k}}\delta^3\left(\mathbf{k}-\mathbf{k}'\right),\\ \left[a(\mathbf{k}),a(\mathbf{k}')\right]&=0,\\ \left[a^\dagger(\mathbf{k}),a^\dagger(\mathbf{k}')\right]&=0.\end{aligned}\right\}\tag{3.20}$$

Here $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are the usual annihilation and creation operators respectively. Recalling the Hamiltonian density, (3.2), the Hamiltonian will be given by

$$H=\int d^3\mathbf{x}\left[\frac{1}{2}\pi^2+\frac{1}{2}\left(\nabla\phi\right)^2+\frac{1}{2}m^2\phi^2\right]$$

$$=\int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\frac{\omega_{\mathbf{k}}}{2}\left[a^\dagger(\mathbf{k})a(\mathbf{k})+a(\mathbf{k})a^\dagger(\mathbf{k})\right],\tag{3.21}$$

In the last equality have suppressed a rather lengthy calculation with excessive use of the commutators (3.20), and the functional representation of the Diracs delta function. Utilizing (3.20) once more we get

$$H=\int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\frac{\omega_{\mathbf{k}}}{2}\left[2a^\dagger(\mathbf{k})a(\mathbf{k})+(2\pi)^3 2\omega_{\mathbf{k}}\delta^3(0)\right].\tag{3.22}$$

We will use this Hamiltonian to calculate the energy of the vacuum corresponding to the scalar field.

3.4 Vacuum energy

Remembering that $a(\mathbf{k})\,|0\rangle = 0$ for a vacuum state $|0\rangle$ we calculate the vacuum energy for the scalar field:

$$\langle 0|\, H\, |0\rangle = \frac{1}{2}\delta^3(0) \int d^3\mathbf{k}\omega_{\mathbf{k}}$$

$$= V \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2}\omega_{\mathbf{k}}. \tag{3.23}$$

Here we have used that $\delta^3(0) = V/(2\pi)^3$, where V is the volume of space, clearly infinite in the present case.

In this thesis we will always deal with situations where $\omega_{\mathbf{k}}$ take on a discrete, infinite set of values rather than a continuum as discussed so far. This will be linked to the finiteness of the 3-sphere space. When $\mathbf{k}$ take on discrete values we will have the replacement

$$V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \quad \longrightarrow \quad \sum_{\mathbf{k}}, \tag{3.24}$$

so that the vacuum energy of the scalar field will be given by

$$\boxed{\langle 0|\, H\, |0\rangle = \frac{1}{2}\sum_{\mathbf{k}}\omega_{\mathbf{k}}} \tag{3.25}$$

3.5 The Klein-Gordon equation in spherical coordinates

It will be useful for later reference to consider the free Klein-Gordon equation in spherical coordinates. This will introduce the so-called *radial Klein-Gordon equation* and we will encounter a similar situation for both the Klein-Gordon and the Dirac field when analyzed on the 3-sphere. It will therefore be beneficial to introduce these concepts in the simplest scenario.

Written in terms of spherical coordinates the Klein-Gordon equation reads

$$\frac{\partial^2}{\partial t^2}\phi - \boldsymbol{\nabla}^2\phi + m^2\phi = 0, \tag{3.26}$$

where

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right) \tag{3.27}$$

is the Laplace operator in spherical coordinates. This operator is treated in any good introductory text on quantum mechanics, see for example [10]. Imposing separation of variables, we write the solution to (3.26) as

$$\Phi = R(r) Y_l^{m_l}(\theta, \varphi) \mathrm{e}^{-i\omega t} \tag{3.28}$$

for a radial function $R(r)$, and where $Y_l^{m_l}(\theta, \varphi)$ are the usual *spherical harmonics* with the *orbital quantum number* l and the corresponding *magnetic quantum number* m_l. Being the spherical harmonics they satisfy the orthonormality condition

$$\int d\Omega \left(Y_{l'}^{m_l'} \right)^* Y_l^{m_l} = \delta_{ll'} \delta_{m_l m_l'} \tag{3.29}$$

where the integral goes over the angular part of space. Inserting the ansatz (3.28) into (3.26) yields

$$Y_l^{m_l} \mathrm{e}^{-i\omega t} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) R - \frac{l(l+1)}{r^2} R + \mathbf{k}^2 R \right] = 0, \tag{3.30}$$

where we have utilized the way the angular part of the Laplacian acts on the spherical harmonics. We have also written $\mathbf{k}^2 = \omega^2 - m^2$ as usual. This gives the radial Klein-Gordon equation in spherical coordinates;

$$\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{l(l+1)}{r^2} + \mathbf{k}^2 \right] R = 0, \tag{3.31}$$

which is a second order ODE that determines the radial function $R(r)$. When this equation is solved, the general solution to the Klein-Gordon equation follows by the principle of superposition (this is what we implicitly did when we adapted Fourier analysis for solving the Klein-Gordon equation earlier). We will leave the radial equation here however, because it is only the way it is obtained that will be useful for us in later analysis.

Throughout this thesis we will write up a table of the quantum numbers that we are using. This table will be updated with new quantum numbers as we go along. So far we have:

Quantum number	**Expression**	**Values**
l	l	$0, 1, 2, 3, \ldots$
m_l	$-l, -l+1, \ldots, l-1, l$	$\ldots, -2, -1, 0, 1, 2, \ldots$

Table 3.1: 1st table of quantum numbers.

3.6 Closing remarks

In this chapter we have adapted the notation where we write the angular frequency $\omega_{\mathbf{k}}$ and the wave-vector $\mathbf{k}$. This notation is all due to the wave like form of the Klein-Gordon equation. Of course these quantities will correspond to the energy and momentum of a particle quantization. So for a particle we have the correspondence

$$\mathbf{k} \quad \longrightarrow \quad \mathbf{p}, \qquad \text{and} \qquad \omega_{\mathbf{k}} \quad \longrightarrow \quad E_{\mathbf{p}}. \tag{3.32}$$

Chapter 4

Klein-Gordon Fields in Curved Spacetime

We are now ready to embark on the first major topic of this thesis. The present chapter will be devoted to the formulation of the Klein-Gordon equation in a curved spacetime background. Next we will state its form in the FRW metric, and then solve the equation on the 3-sphere.

With the Klein-Gordon field being a tensorial object (a scalar), the transition from Minkowski space to a general curved spacetime will be quite immediate.

4.1 The covariant Klein-Gordon equation

To write down the coordinate independent (covariant) version of the Klein-Gordon equation we start by recalling its Lagrangian density in Minkowski space;

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\nu\phi\right)\left(\partial_\mu\phi\right) - \frac{1}{2}m^2\phi^2. \tag{4.1}$$

For curved spacetime the Minkowski metric will be replaced by the general metric tensor. Next we consider the appearance of $\partial_\mu\phi$. Now since ϕ is a scalar, its covariant derivative will reduce to the partial derivative, leaving $\partial_\mu\phi$ tensorial. We seem to have covered all points that need consideration when writing $\mathcal{L}$ covariantly, however there is one more circumstance that needs to be investigated. The effect of a curved spacetime could in theory manifest itself through a scalar coupling to ϕ^2. We know of exactly such a scalar quantity, namely the Ricci curvature scalar $\mathcal{R}$. This will add a term proportional to $\mathcal{R}\phi^2$, which in turn leads to the Lagrangian density for a scalar field in curved spacetime on the form

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\left(\partial_\nu\phi\right)\left(\partial_\mu\phi\right) - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\xi\mathcal{R}\phi^2. \tag{4.2}$$

Here ξ will be the gravitational coupling to the scalar field with mass dimension of two. This coupling could in theory be determined by experiments, however the effect of gravity are of such a small magnitude that such an experiment would be next to impossible to conduct.

To find the covariant Klein-Gordon equation from this Lagrangian density we write up the action

$$S = \int d^4x\sqrt{-g}\mathcal{L}, \tag{4.3}$$

where we have recalled the way integration is performed in curved spacetime. Upon requiring that $\delta S = 0$ we get the Euler-Lagrange equations in curved spacetime

$$\partial_\mu \frac{\partial\sqrt{-g}\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} - \frac{\partial\sqrt{-g}\mathcal{L}}{\partial\phi} = 0. \tag{4.4}$$

Inserting the Lagrangian density (4.2) into these Euler-Lagrange equations yields the Klein-Gordon equation in curved spacetime:

$$\boxed{\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + \left(m^2 + \xi\mathcal{R}\right)\phi = 0} \tag{4.5}$$

Here we have used the fact that $\sqrt{-g}$ does not depend on neither $\partial_\mu\phi$ nor ϕ. We are now going to write this equation out for the FRW metric.

4.2 The Klein-Gordon equation in FRW spacetime

For easy reference we start by recalling that the FRW metric is given by (2.56):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix}, \tag{4.6}$$

with the inverse metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{a(t)^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix}. \tag{4.7}$$

We will define $F \equiv \sqrt{1-kr^2}$ from here on, to make notation easier. Also the scale factor is taken to be constant from here on, that is $a(t) = a$. The square root of $-g$ is given by

$$\sqrt{-g} = \frac{a^3 r^2 \sin\theta}{F}. \tag{4.8}$$

We now write the the Klein-Gordon equation, (4.5), as

$$\frac{\partial^2}{\partial t^2}\phi + \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\right)\phi + \left(m^2 + \xi\mathcal{R}\right)\phi = 0. \tag{4.9}$$

Here the Latin indices goes over the spatial components. Written out we have

$$\frac{\partial^2}{\partial t^2}\phi - \frac{1}{a^2}\boldsymbol{\nabla}^2_{\text{FRW}}\phi + \left(m^2 + \xi\mathcal{R}\right)\phi = 0, \tag{4.10}$$

where we have defined the FRW Laplacian

$$\boldsymbol{\nabla}^2_{\text{FRW}} \equiv \frac{F}{r^2}\frac{\partial}{\partial r}\left(Fr^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right). \tag{4.11}$$

We observe that the angular part of this operator is identical to the angular part of the Laplacian for flat spacetime in spherical coordinates, (3.27). The radial part however is different and gives a different radial equation. As done in flat spacetime we impose separation of variables and write

$$\Phi = R(r)Y_l^{m_l}(\theta,\varphi)\mathrm{e}^{-i\omega t}. \tag{4.12}$$

Inserting this into (4.10) gives the radial Klein-Gordon equation in FRW spacetime;

$$\left[\frac{F}{r^2}\frac{d}{dr}\left(Fr^2\frac{d}{dr}\right) - \frac{l(l+1)}{r^2} + a^2\left(\mathbf{k}^2 - \xi\mathcal{R}\right)\right]R = 0. \tag{4.13}$$

4.3 The Klein-Gordon equation on the 3-sphere

We are now going to solve the Klein-Gordon equation for the FRW-metric after performing the substitutions $k = 1$ and $r = \sin\chi$. As discussed in chapter 2, such a substitution results in the metric of the 3-sphere, denoted by $\mathbb{R} \times S^3$. The scale factor a denotes the radius of the sphere. Imposing the substitutions yield

$$F = \cos\chi \quad \text{and} \quad \frac{d}{dr} = \frac{1}{\cos\chi}\frac{d}{d\chi}.$$

Hence the the radial equation becomes

$$\left[\frac{1}{\sin^2\chi}\frac{d}{d\chi}\left(\sin^2\chi\frac{d}{d\chi}\right) - \frac{l(l+1)}{\sin^2\chi} + a^2\left(\mathbf{k}^2 - \xi\mathcal{R}\right)\right]R = 0, \tag{4.14}$$

or with the differential operator written out:

$$\left[\frac{d^2}{d\chi^2} + 2\frac{\cos\chi}{\sin\chi}\frac{d}{d\chi} - \frac{l(l+1)}{\sin^2\chi} + \varsigma^2\right]R = 0. \tag{4.15}$$

To simplify notation further we have introduced $\varsigma^2 \equiv a^2\left(\mathbf{k}^2 - \xi\mathcal{R}\right)$. We can start to recognize this ODE as a hypergeometric differential equation by imposing the following substitution, [8]:

$$u = \sin^2\frac{\chi}{2}. \tag{4.16}$$

Using trigonometric identities yields

$$\cos\chi = \cos^2\frac{\chi}{2} - \sin^2\frac{\chi}{2} = 1 - 2\sin^2\frac{\chi}{2} = \underline{1 - 2u},$$

$$\sin^2\chi = 1 - \cos^2\chi = 1 - (1-2u)^2 = \underline{4u(1-u)},$$

$$\frac{d}{d\chi} = \frac{du}{d\chi}\frac{d}{du} = \sin\frac{\chi}{2}\cos\frac{\chi}{2}\frac{d}{du} = \frac{1}{2}\sin\chi\frac{d}{du} = \underline{\sqrt{u(1-u)}\frac{d}{du}},$$

$$\frac{d^2}{d\chi^2} = \underline{u(1-u)\frac{d^2}{du^2} + \left(\frac{1}{2} - u\right)\frac{d}{du}}.$$

This substituted into the radial equation gives the following ODE for R_l:

$$\left[u(u-1)\frac{d^2}{du^2} - 3\left(\frac{1}{2} - u\right)\frac{d}{du} - \frac{l(l+1) + 4u(u-1)\varsigma^2}{4u(u-1)}\right] R = 0. \tag{4.17}$$

4.3.1 Solving the radial equation

The ODE for R is well behaved except at the points 0, 1 and possibly at infinity. We are now going to analyze these points following appendix A.

The point $u_0 = 0$:

Now $u_0(u_0 - 1) = 0$, so this point is singular. The limits (A.2) become

$$\begin{aligned} \lim_{u\to 0} -\frac{3\left(\frac{1}{2} - u\right)}{u-1} &= \frac{3}{2}, \\ \lim_{u\to 0} -\frac{l(l+1) + 4u(u-1)\varsigma^2}{4(u-1)^2} &= -\frac{l(l+1)}{4}, \end{aligned} \tag{4.18}$$

which are both finite. We conclude that the point $u_0 = 0$ is a regular singular point.

The point $u_0 = 1$:

Now $u_0(u_0 - 1) = 0$, so this point is singular. The limits (A.2) become

$$\begin{aligned} \lim_{u\to 1} -\frac{3\left(\frac{1}{2} - u\right)}{u} &= \frac{3}{2}, \\ \lim_{u\to 1} -\frac{l(l+1) + 4u(u-1)\varsigma^2}{4u^2} &= -\frac{l(l+1)}{4}, \end{aligned} \tag{4.19}$$

which are both finite. We conclude that the point $u_0 = 1$ is a regular singular point.

The point $u_0 = \infty$:

We now make the substitution $u = \frac{1}{v}$. Then

$$\frac{d}{du} = -v^2\frac{d}{dv},$$

and hence

$$\frac{d^2}{du^2} = 2v^3\frac{d}{dv} + v^4\frac{d^2}{dv^2}.$$

Substituted into the original equation (4.17) yields:

$$v^2(1-v)R'' - v\left(1+\frac{1}{2}v\right)R' - \frac{l(l+1)v^2 + 4(1-v)\varsigma^2}{4(1-v)}R = 0. \tag{4.20}$$

Where $'$ now denotes derivative with respect to v. From this we see that the point $u_0 = \infty$ corresponds to the point $v_0 = 0$. Hence $v_0^2(1-v_0) = 0$ so that $v_0 = 0$ is a singular point. The limits (A.2) now become

$$\lim_{v\to 0} -\frac{\left(1+\frac{1}{2}v\right)}{1-v} = -1,$$
$$\lim_{v\to 0} -\frac{l(l+1)v^2 + 4(1-v)\varsigma^2}{4(1-v)^2} = -\varsigma^2, \tag{4.21}$$

which are both finite. We finally conclude that $v_0 = 0$ is a regular singular point. Hence the point $u_0 = \infty$ is a regular singular point.

We have now classified all the singular points of the radial equation, and found that it has the three regular singular points 0, 1 and ∞. The indicial equations for each of these singular points respectively, is given by (A.5):

$$\mu(\mu-1) + \frac{3}{2}\mu - \frac{l(l+1)}{4} = 0, \tag{4.22}$$

$$\nu(\nu-1) + \frac{3}{2}\nu - \frac{l(l+1)}{4} = 0, \tag{4.23}$$

$$\lambda(\lambda-1) - \lambda - \varsigma^2 = 0, \tag{4.24}$$

where we have used the limits calculated above for each singular point. Solving these equations for the indices yields:

$$\mu^2 + \frac{1}{2}\mu = \frac{l(l+1)}{4},$$

$$\Rightarrow \quad \left(\mu + \frac{1}{4}\right)^2 = \frac{1}{16} + \frac{l(l+1)}{4}$$

$$= \frac{1}{16}\left(1 + 4l^2 + 4l\right)$$

$$= \left(\frac{1+2l}{4}\right)^2,$$

which gives

$$\mu^{(\pm)} = \underline{-\frac{1}{4} \pm \frac{1}{4}(1+2l),} \tag{4.25}$$

since l is not negative. Similarly we get

$$\nu^{(\pm)} = \underline{-\frac{1}{4} \pm \frac{1}{4}(1+2l),} \tag{4.26}$$

$$\lambda^{(\pm)} = \underline{1 \pm \sqrt{1+\varsigma^2}.} \tag{4.27}$$

With these indices we have the following tableau (see appendix A) for the ODE for R:

$$\mathcal{T}(R) = \left\{ \begin{matrix} 0 & 1 & \infty \\ \frac{1}{2}l & \frac{1}{2}l & 1+\sqrt{1+\varsigma^2} \\ -\frac{1}{2}-\frac{1}{2}l & -\frac{1}{2}-\frac{1}{2}l & 1-\sqrt{1+\varsigma^2} \end{matrix} \right\}. \tag{4.28}$$

As discussed in appendix A, we can by a suitable factorization of the form

$$f = u^{\alpha_0}(u-1)^{\alpha_1} R, \tag{4.29}$$

shift the indices in this tableau so that we get a new tableau corresponding to the hypergeometric differential equation. The hypergeometric equation has the tableau,

$$\mathcal{T}(\text{hypergeometric}) = \begin{Bmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{Bmatrix}. \tag{4.30}$$

Imposing the factorization, (4.29), the tableau for the ODE for the function f will be given by

$$\mathcal{T}(f) = \begin{Bmatrix} 0 & 1 & \infty \\ \\ \frac{1}{2}l+\alpha_0 & \frac{1}{2}l+\alpha_1 & 1+\sqrt{1+\varsigma^2}-\alpha_0-\alpha_1 \\ \\ -\frac{1}{2}-\frac{1}{2}l+\alpha_0 & -\frac{1}{2}-\frac{1}{2}l+\alpha_1 & 1-\sqrt{1+\varsigma^2}-\alpha_0-\alpha_1 \end{Bmatrix}. \tag{4.31}$$

Equating $\mathcal{T}(f)$ and $\mathcal{T}(\text{hypergeometric})$ results in equations for all the unknowns $\alpha_0, \alpha_1, \alpha, \beta, \gamma$. We get

$$\alpha_0 = -\frac{1}{2}l, \qquad \alpha = l+1+\sqrt{1+\varsigma^2},$$

$$\alpha_1 = -\frac{1}{2}l, \qquad \beta = l+1-\sqrt{1+\varsigma^2},$$

$$\gamma = l+\frac{3}{2}.$$

From appendix (A.3) we see that f satisfies the hypergeometric equation:

$$u(1-u)f'' + [\gamma-(\alpha+\beta+1)u]\,f' - \alpha\beta f = 0, \tag{4.32}$$

with the solution

$$f = A\,{}_2F_1\left(\alpha,\beta;\gamma;u\right) + Bu^{1-\gamma}\,{}_2F_1\left(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; u\right), \tag{4.33}$$

for two complex constants A and B. The radial Klein-Gordon equation then gets the solutions

$$\begin{aligned} R =& Au^{\frac{1}{2}l}(u-1)^{\frac{1}{2}l}\,{}_2F_1\left(\alpha,\beta;\gamma;u\right) \\ &+ Bu^{-\frac{1}{2}-\frac{1}{2}l}(u-1)^{\frac{1}{2}l}\,{}_2F_1\left(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; u\right). \end{aligned} \tag{4.34}$$

Let's analyze the behaviour of this solution near $u = 0$, corresponding to $\chi = 0$. As u approaches zero, the only possible singular factors will be the factors of u. We look at their exponents. Now since

$$\frac{1}{2}l \geq 0, \tag{4.35}$$

the first term is acceptable. For the second term we have the exponent of u as

$$-\frac{1}{2} - \frac{1}{2}l < 0, \tag{4.36}$$

so that we must require $B = 0$ for a physically acceptable solution. Then the solution of the radial equation is

$$R = Au^{\frac{1}{2}l}(u-1)^{\frac{1}{2}l} {}_2F_1\left(l+1+\sqrt{1+\varsigma^2}, l+1-\sqrt{1+\varsigma^2}; l+\frac{3}{2}; u\right). \tag{4.37}$$

Next we analyze the behaviour of this solution near $u = 1$, corresponding to $\chi = \pi$. For this analysis we use relation (A.14) and write

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;u) = &\left[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\right] {}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;1-u) + \\ &(1-u)^{\gamma-\alpha-\beta}\left[\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \\ &{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-u). \end{aligned} \tag{4.38}$$

From this we see that around $u = 1$ there will appear a term proportional to $(u-1)^{\gamma-\alpha-\beta-\alpha_1}$ in the solution R. Since

$$\gamma - \alpha - \beta - \alpha_1 = -\frac{1}{2} - \frac{1}{2}l < 0, \tag{4.39}$$

this term will be too singular near $u = 1$, and we must require that the term vanishes. The only way for this to happen is that

$$\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} = 0. \tag{4.40}$$

The gamma function does not have any zeroes, hence we must have that either $\Gamma(\alpha)$ or $\Gamma(\beta)$ hit a pole and goes to infinity. This happens if the argument of the gamma function is a negative integer counting zero. α will always be positive in our case, so that we must have $\beta = -n_r$ where we have defined *the radial quantum number* $n_r = 0, 1, 2, \ldots$.

Using the expression for β yields

$$\sqrt{1+\varsigma^2} = l + n_r + 1,$$

$$\Rightarrow \quad \varsigma^2 = n_{\mathrm{B}}^2 - 1, \tag{4.41}$$

where we have defined *the principal quantum number* for the Bosonic case: $n_{\mathrm{B}} \equiv l+n_r+1 = 1, 2, 3, \ldots$. Inserting the expression for ς we get the energy quantization for a scalar field on $S^3 \times \mathbb{R}$:

$$\omega = \pm\sqrt{\frac{(n_{\mathrm{B}}^2 - 1) + 6\xi}{a^2} + m_{\mathrm{B}}^2} \tag{4.42}$$

Here we have used the expression for the Ricci scalar in FRW spacetime. This is calculated in appendix B.1. For $\frac{da}{dt} = 0$ and $k = 1$ it is given by

$$\mathcal{R} = \frac{6}{a^2}. \tag{4.43}$$

Let's go back to the solution of the radial equation, (4.37). With the quantum numbers defined we can write it as

$$R = A u^{\frac{1}{2}l} (u-1)^{\frac{1}{2}l} {}_2F_1\left(n_r + 2(l+1), -n_r; (l+1) + \frac{1}{2}; u\right). \tag{4.44}$$

From appendix A.3 we see that if we define $b = l + 1$, then

$${}_2F_1\left(n_r + 2b, -n_r; b + \frac{1}{2}; u\right) = \frac{n_r!}{(2b)_{n_r}} C_{n_r}^{(b)}(1-2u), \tag{4.45}$$

where $C_{n_r}^{(b)}$ is the Gegenbauer polynomial of degree n_r. Recalling that $u = \sin^2\frac{\chi}{2}$ we finally arrive at the solution to the radial Klein-Gordon equation on S^3:

$$\boxed{R(\chi) = \sin^l \chi \, C_{n_r}^{(b)}(\cos\chi)} \tag{4.46}$$

Here we have suppressed all constants.

4.4 Normalization of the solutions to the Klein-Gordon equation on the 3-sphere

We proceed by normalizing the the positive energy solutions

$$\Phi = R(\chi) Y^{l}_{m_l}(\theta, \varphi) \mathrm{e}^{-i\omega_{\mathrm{B}} t}, \tag{4.47}$$

where the function R was calculated in the previous section. We have defined ω_{B} as the positive root of (4.42). Recalling the normalization condition for the scalar field, which follows from (3.6)), we should have

$$2\omega_{\mathrm{B}} \int \left|\Phi\right|^2 d^3\mathbf{x} = 1. \tag{4.48}$$

Now since the spherical harmonics are orthonormal, it suffices to normalize the radial functions. We should have, for a normalization constant A,

$$2\omega_{\mathrm{B}} \left|A\right|^2 \int_0^\pi \left|R(\chi)\right|^2 \sin^2 \chi d\chi = 1,$$

$$\Rightarrow \quad 2\omega_{\mathrm{B}} \left|A\right|^2 \int_0^\pi \sin^{2l+2} \chi \left[C^{(b)}_{n_r}(\cos\chi)\right]^2 d\chi = 1. \tag{4.49}$$

From appendix A.3 we have the orthogonality relation for the Gegenbauer polynomials, (A.16). With the variable substitution $x = \cos\chi$ we get

$$\int_0^\pi \sin^{2l+2} \chi \left[C^{(b)}_{n_r}(\cos\chi)\right]^2 d\chi = \frac{\pi^{-2l-1}\Gamma(n_r + 2(l+1))}{n_r!(n_r + l + 1)\left[\Gamma(l+1)\right]^2}, \tag{4.50}$$

where we have used that $b = l + 1$. Inserted into (4.49) yields

$$A = \pi^{l+\frac{1}{2}} \Gamma(l+1) \sqrt{\frac{n_{\mathrm{B}}}{2\omega_{\mathrm{B}}} \frac{(n_{\mathrm{B}} - l - 1)!}{\Gamma(n_{\mathrm{B}} + l + 1)}} \tag{4.51}$$

4.5 General solution of the Klein-Gordon equation on the 3-sphere

With the normalization of the special solutions to the Klein-Gordon on $\mathbb{R} \times S^3$ we are now ready to write down the general solution. By the principle of superposition the general solution to the Klein-Gordon equation on the 3-sphere is

$$\boxed{\phi(t,\chi,\theta,\varphi) = \sum_{n_{\mathrm{B}},l,m_l} \sqrt{\frac{1}{2\omega_{\mathrm{B}}}} \left[c_{n_{\mathrm{B}}lm_l} \Phi_{n_{\mathrm{B}}lm_l}(t,\chi,\theta,\varphi) + c^{\dagger}_{n_{\mathrm{B}}lm_l} \Phi^{*}_{n_{\mathrm{B}}lm_l}(t,\chi,\theta,\varphi) \right]} \tag{4.52}$$

where $c_{n_{\mathrm{B}}lm_l}$ and $c^{\dagger}_{n_{\mathrm{B}}lm_l}$ are Fourier coefficients. We have also written

$$\Phi_{n_{\mathrm{B}}lm_l}(t,\chi,\theta,\varphi) = \pi^{l+\frac{1}{2}} \Gamma(l+1) \sqrt{\frac{n_{\mathrm{B}}(n_{\mathrm{B}}-l-1)!}{\Gamma(n_{\mathrm{B}}+l+1)}} R(\chi) Y^{l}_{m_l}(\theta,\varphi) \mathrm{e}^{-i\omega_{\mathrm{B}}t}, \tag{4.53}$$

with

$$\boxed{\omega_{\mathrm{B}} = \sqrt{\frac{(n_{\mathrm{B}}^2-1)+6\xi}{a^2} + m_{\mathrm{B}}^2}} \tag{4.54}$$

Let's update the table of quantum numbers thus far:

Quantum number	**Expression**	**Values**
l	l	$0,1,2,3,\ldots$
m_l	$-l,-l+1,\ldots,l-1,l$	$\ldots,-2,-1,0,1,2,\ldots$
n_r	n_r	$0,1,2,3,\ldots$
n_{B}	$l+n_r+1$	$1,2,3,\ldots$

Table 4.1: 2nd table of quantum numbers.

The quantum number n_{B} completely determines the value of the energy level ω_{B}. For each value of n_{B}, the possible values of l is given by

$$l = 0,1,2,\ldots,n_{\mathrm{B}}-1. \tag{4.55}$$

For each of these values for l, there is again $(2l+1)$ values of the magnetic quantum number m_l. Hence the degeneracy, $d(n_\mathrm{B})$, of each energy level is given by

$$\boxed{d(n_\mathrm{B}) = \sum_{l=0}^{n_\mathrm{B}-1} (2l+1) = n_\mathrm{B}^2} \tag{4.56}$$

4.6 Quantization on the 3-sphere

In this last section we will briefly outline how one can quantize the field solutions (4.52). The conjugate momentum of the field, given by $\pi = \frac{\partial \mathcal{L}}{\partial_t \phi}$, will be equal to $\dot{\phi}$ also in $\mathbb{R} \times S^3$ spacetime. This arises of course, from the fact that $g_{00} = 1$. As we did in Minkowski space we impose the commutation relations

$$\left.\begin{aligned} &[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'), \\ &[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = 0, \\ &[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = 0. \end{aligned}\right\} \tag{4.57}$$

With these relations, it can be shown that the coefficients $c_{n_\mathrm{B} l m_l}$ and $c^\dagger_{n_\mathrm{B} l m_l}$ will obey

$$\left.\begin{aligned} &\left[c_{n_\mathrm{B} l m_l}, c^\dagger_{n'_\mathrm{B} l' m'_l}\right] = \delta_{n_\mathrm{B} n'_\mathrm{B}} \delta_{l l'} \delta_{m'_l m_l}, \\ &\left[c_{n_\mathrm{B} l m_l}, c_{n'_\mathrm{B} l' m'_l}\right] = 0, \\ &\left[c^\dagger_{n_\mathrm{B} l m_l}, c^\dagger_{n'_\mathrm{B} l' m'_l}\right] = 0, \end{aligned}\right\} \tag{4.58}$$

suitable for annihilation and creation operators.

Chapter 5

Dirac Fields in Minkowski Spacetime

We have now finished our study of the scalar field on the 3-sphere, and it is therefore time to begin the second part of this thesis. It is time to turn the our attention to the spin-$\frac{1}{2}$ field.

Fermionic particles of spin-$\frac{1}{2}$ are described by the so-called *Dirac field*, denoted by $\psi(x)$, and its dynamical behaviour is determined by the *Dirac equation*. Being a field of non-zero spin, the Dirac field must be of a different form than a scalar field, in that it must incorporate the additional degrees of freedom that comes with non-zero spin. We will see how this turns out in a bit.

Since we are going to analyze the Dirac equation in curved spacetime, specifically on the 3-sphere, we start with an outline of the Dirac theory in Minkowski space. We will for the most part use the book of A. Zee; *Quantum field theory in a nutshell* [16] for reference on the background theory presented in this chapter.

5.1 The free Dirac equation

The physics of the Dirac field $\psi(x)$ is described by the Dirac equation;

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0. \tag{5.1}$$

Here the objects γ^{μ} are *gamma matrices*, that is, 4×4 matrices satisfying the so-called *Clifford algebra*:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \times I_4, \tag{5.2}$$

where I_4 is the four dimensional identity matrix. We will almost always use the more compact notation of writing I_4 as 1 when working with the Dirac equation. From the

Clifford algebra we see that

$$\left(\gamma^0\right)^2 = 1, \qquad \left(\gamma^i\right)^2 = -1. \tag{5.3}$$

We also impose the Hermiticity conditions

$$\left(\gamma^0\right)^\dagger = \gamma^0, \qquad \left(\gamma^i\right)^\dagger = -\gamma^i, \tag{5.4}$$

which conveniently can be combined as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{5.5}$$

However, the Hermiticity condition together with the Clifford algebra, does not define the gamma matrices uniquely. Thus we have to choose a representation for the gamma matrices. One example will be the *standard (or Dirac) representation* defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{5.6}$$

although we will choose another representation when we are going to solve the Dirac equation on the 3-sphere. The two by two matrices σ^i is the *Pauli matrices*:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.7}$$

It is sometimes useful to define what is called *the Dirac matrices*;

$$\beta \equiv \gamma^0, \qquad \boldsymbol{\alpha} \equiv \gamma^0\vec{\gamma}. \tag{5.8}$$

Different representations of gamma matrices are related by a similarity transformation S, that is, a non-singular unitary transformation. Thus a new representation can be obtained by

$$\gamma^{\mu'} = S\gamma^\mu S^{-1}, \tag{5.9}$$

with the new solution $\psi' = S\psi$ satisfying the Dirac equation expressed in terms of the new gamma matrices $\gamma^{\mu'}$.

With γ^μ being a four by four matrix, the Dirac field ψ is a four component object called a *spinor*;

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{5.10}$$

In Klein-Gordon theory we frequently came across the quantity ϕ^* or $\phi^\dagger$. In the Dirac theory however $\psi^\dagger$ will not be as important as the so-called *Dirac adjiont* . The Dirac adjoint is defined as follows:

$$\bar{\psi} = \psi^\dagger \gamma^0. \tag{5.11}$$

As with the Klein-Gordon field we can transform ψ to momentum space by setting

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4} \psi(k) \mathrm{e}^{-ikx} \tag{5.12}$$

so that the components $\psi(k)$ satisfies the momentsum space Dirac equation

$$(\gamma^\mu k_\mu - m)\, \psi(k) = 0. \tag{5.13}$$

Historically the Dirac equation was put forward by Dirac as a first order relativistic wave equation, as opposed to the second order Klein-Gordon equation. The original motivation was to obtain a single particle wave equation with a positive definite probability density. To see that this will be the case we simply note that the conserved current density of the Dirac equation will be the quantity

$$j^\mu = \bar{\psi} \gamma^\mu \psi. \tag{5.14}$$

This current density is shown to obey the continuity equation with the use of the Dirac equation. From this we define the probability density as $j^0 = \psi^\dagger \psi$, giving

$$j^0 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2, \tag{5.15}$$

which is clearly positive definite.

While solving the issue of the probability density, the Dirac equation still suffers the same fate as the Klein-Gordon equation in giving both positive and negative energies for the free particle. This was first resolved by Dirac by proposing that all the negative energy states are filled up by means of the Pauli exclusion principle, but the modern interpretation will be that the negative energy states will correspond to a positive energy state of the antiparticle.

5.2 Dirac bilinears

We can construct a basis for all 4×4 matrices out of the gamma matrices. The basis will be given by the following set $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$, where the fifth gamma matrix is defined as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{5.16}$$

and $\sigma^{\mu\nu}$ as

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{5.17}$$

The Dirac bilinears will be quantities of the form $\bar{\psi}\Gamma\psi$, where Γ is a four by four matrix. From basis set presented above we see that we can form 16 different Dirac bilinears. Each of these bilinears will transform differently under Lorentz transformations as presented in this table:

Dirac bilinear	**Transformation property**
$\bar{\psi}\psi$	Scalar
$\bar{\psi}\gamma^5\psi$	Pseudoscalar
$\bar{\psi}\gamma^\mu\psi$	Vector
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Axial vector
$\bar{\psi}\sigma^{\mu\nu}\psi$	Antisymmetric tensor

Table 5.1: The Dirac bilinears with transformation properties under Lorentz transformations.

The vector and scalar Dirac bilinear will be used later for the development of the covariant derivative of the spinor. The transformation of the spinor itself under Lorentz transformations is also worth noting. It will certainly not be that of a 4-vector. Under rotations the spinor transforms as a spin-$\frac{1}{2}$ field, whereas a vector field will rotate as a spin-1 field. We will not go further into the transformation properties of the spinor, it only suffices for us to note that it is not a vector field (it is a spinor field).

5.3 Quantization of the field

We will also briefly discuss the quantization of the Dirac field. Similar to the Klein-Gordon equation, the Dirac equation has to be rightfully interpreted as a many-particle equation.

For the free wave solution let $u(k,s)\mathrm{e}^{-ikx}$ denote the positive energy solution and $v(k,s)\mathrm{e}^{ikx}$ denote the negative. Here $k^0 = \omega_{\mathbf{k}} = \pm\sqrt{\mathbf{k}^2 + m^2}$ as before. We have also included the spin degrees of freedom for the spinor, that is, $s = \pm\frac{1}{2}$. Inserted into the Dirac equation we get

$$(\gamma^\mu k_\mu - m)\, u(k,s) = 0 \quad \text{and} \quad (\gamma^\mu k_\mu + m)\, v(k,s) = 0. \tag{5.18}$$

We will impose the normalization conditions

$$\sum_s u^\dagger(k,s)u(k,s) = \sum_s v^\dagger(k,s)v(k,s) = \omega_{\mathbf{k}} \tag{5.19}$$

together with

$$\left.\begin{aligned} \bar{u}(k,s)u(k,s') &= -\bar{v}(k,s)v(k,s') = m\delta_{ss'}, \\ \bar{u}(k,s)v(k,s') &= -\bar{v}(k,s)u(k,s') = 0. \end{aligned}\right\} \tag{5.20}$$

We then obtain

$$\psi(x) = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sqrt{\frac{1}{\omega_{\mathbf{k}}}} \left[b(k,s)u(k,s)\mathrm{e}^{-ikx} + d^\dagger(k,s)v(k,s)\mathrm{e}^{ikx} \right] \tag{5.21}$$

and

$$\bar{\psi}(x) = \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sqrt{\frac{1}{\omega_{\mathbf{k}}}} \left[d(k,s)\bar{v}(k,s)\mathrm{e}^{-ikx} + b^\dagger(k,s)\bar{u}(k,s)\mathrm{e}^{ikx} \right], \tag{5.22}$$

where b and $b^\dagger$ are annihilation and creation operators for the particle, and d and $d^\dagger$ are corresponding operators for the antiparticle. Since the Dirac field obeys the Fermi-Dirac statistics (being a fermion field) we must impose *anticommutators* in order to quantize the field;

$$\left.\begin{aligned} \{\psi(t,\mathbf{x}), \psi^\dagger(t,\mathbf{x}')\} &= i\delta^3(\mathbf{x}-\mathbf{x}'), \\ \{\psi(t,\mathbf{x}), \psi(t,\mathbf{x}')\} &= 0, \\ \{\psi^\dagger(t,\mathbf{x}), \psi^\dagger(t,\mathbf{x}')\} &= 0. \end{aligned}\right\} \tag{5.23}$$

It can be shown that these anticommutators gives the following anticommutators for the creation and annihilation operators:

$$\{b(k,r), b^\dagger(k',s)\} = \{d(k,r), d^\dagger(k',s)\} = \delta_{kk'}\delta ss', \tag{5.24}$$

with all other combinations equal to zero. This will be suitable for creation and annihilation operators. We specifically note that for a vacuum state $|0\rangle$ we have

$$b\,|0\rangle = d\,|0\rangle = 0. \tag{5.25}$$

5.4 Vacuum energy

The Lagrangian and Hamiltonian density for the Dirac field will be

$$\mathcal{L} = \bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi \tag{5.26}$$

and

$$\mathcal{H} = \bar{\psi}\left(i\vec{\gamma}\cdot\vec{\partial} + m\right)\psi \tag{5.27}$$

respectively. From this it is possible to obtain the Hamiltonian

$$H = \int d^3\mathbf{k}\sum_{s}\omega_{\mathbf{k}}\left[b^{\dagger}(k,s)b(k,s) + d^{\dagger}(k,s)d(ks) - \delta^3(0)\right], \tag{5.28}$$

from which we will have the vacuum energy for a Dirac field for a discrete spectrum of $\mathbf{k}$:

$$\boxed{\langle 0|\,H\,|0\rangle = -2\sum_{\mathbf{k}}\omega_{\mathbf{k}}} \tag{5.29}$$

We observe that it comes with a minus sign.

5.5 The free Dirac equation in spherical coordinates

It will be useful for later discussions to analyze the Dirac equation in spherical coordinates. In spherical coordinates the Dirac equation takes the form

$$i\frac{\partial}{\partial t}\Psi = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\Psi + \beta m\Psi, \tag{5.30}$$

where we have introduced Diracs $\boldsymbol{\alpha}$ and β matrices. As done in the case of the Dirac equation in a central potential $V(r)$, we introduce the angular momentum operator $\mathbf{L} =$

$-i\mathbf{r} \times \boldsymbol{\nabla}$, following chapter 4 of [14]. By looking at

$$\hat{e}_{(r)} \times \mathbf{L} = -i\hat{e}_{(r)} \times (\mathbf{r} \times \boldsymbol{\nabla})$$

$$= -i \left[\hat{e}_{(r)} r \frac{\partial}{\partial r} - r\boldsymbol{\nabla} \right], \tag{5.31}$$

we can get an expression for $\boldsymbol{\nabla}$ involving $\mathbf{L}$;

$$\boldsymbol{\nabla} = \hat{e}_{(r)} \frac{\partial}{\partial r} - i\frac{1}{r}\hat{e}_{(r)} \times \mathbf{L}. \tag{5.32}$$

Hence

$$-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} = -i \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) \frac{\partial}{\partial r} - \frac{1}{r}\boldsymbol{\alpha} \cdot \left(\hat{e}_{(r)} \times \mathbf{L} \right). \tag{5.33}$$

For two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ we have the identity

$$(\boldsymbol{\alpha} \cdot \mathbf{A})(\boldsymbol{\alpha} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\Sigma} \cdot (\mathbf{A} \times \mathbf{B}), \tag{5.34}$$

where $\boldsymbol{\Sigma} = \gamma^5 \boldsymbol{\alpha} = 2\mathbf{S}$ is twice the spin operator. Substituting $\mathbf{A} = \hat{e}_{(r)}$ and $\mathbf{B} = \mathbf{L}$ and using that $\hat{e}_{(r)} \cdot \mathbf{L} = 0$, yields

$$\boldsymbol{\alpha} \cdot \left(\hat{e}_{(r)} \times \mathbf{L} \right) = -i\gamma^5 \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) (\boldsymbol{\alpha} \cdot \mathbf{L})$$

$$= -i \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) (2\mathbf{S} \cdot \mathbf{L}), \tag{5.35}$$

since γ^5 commutes with $\boldsymbol{\alpha}$. Inserting this into (5.33) gives

$$-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} = -i \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) \left[\frac{\partial}{\partial r} - \frac{1}{r} (2\mathbf{S} \cdot \mathbf{L}) \right]$$

$$= -i \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) \left[\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r}\beta^2 (2\mathbf{S} \cdot \mathbf{L} + 1) \right]. \tag{5.36}$$

Remembering that $\mathbf{S}^2 = \frac{1}{2}(\frac{1}{2} + 1)$ because we are dealing with a spin-$\frac{1}{2}$ field, we can write

$$2\mathbf{S} \cdot \mathbf{L} + 1 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2 + 1 = \mathbf{J}^2 - \mathbf{L}^2 + \frac{1}{4}. \tag{5.37}$$

Where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum. We now define what we can call a spin-orbit operator [14]:

$$K = \beta \left(\mathbf{J}^2 - \mathbf{L}^2 + \frac{1}{4} \right). \tag{5.38}$$

With this operator at hand the Dirac equation in spherical coordinates can finally be written as

$$i \frac{\partial}{\partial t} \Psi = -i \left(\boldsymbol{\alpha} \cdot \hat{e}_{(r)} \right) \left[\frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r} \beta K \right] \Psi + \beta m \Psi = H_0 \Psi. \tag{5.39}$$

To proceed we will look for eigenfunctions of the Dirac Hamiltonian H_0. As standard in quantum mechanics we can find these eigenfunctions by acquiring a complete set of commuting observables. This set is given by the Hamiltonian, together with total angular momentum and its thrid component, spinor parity and the spin-orbit operator. With this set $\{H_0, \mathbf{J}^2, J_3, \mathcal{P}, K\}$, there is a set of simultaneous eigenfunctions that is constructed out of separation of radial and spherical coordinates, and time.

5.5.1 Spinor spherical harmonics

The angular part of the spinor Ψ is given by spinor spherical harmonics. These are eigenfunctions of $\mathbf{J}^2$ and J_3, and parity. For a Dirac field the total angular momentum is the orbital angular momentum combined with spin angular momentum equal to $\frac{1}{2}$. The eigenfunctions of $\mathbf{L}^2$ and L_3 are the usual spherical harmonics $Y_l^{m_l}$, and the eigenfunctions of $\mathbf{S}^2$ and S_3 are given by two-component spinors χ_{m_s}, where $m_s = -\frac{1}{2}, \frac{1}{2}$ corresponds to spin down and up respectively. These two-component spinors take the form

$$\chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.40}$$

With this in mind, the construction of the spinor spherical harmonics $\Omega^j_{lm_j}$ is given by

$$\Omega^j_{lm_j} = \sum_{m_s = -\frac{1}{2}, \frac{1}{2}} C \left(l, \frac{1}{2}, j; m_j - m_s, m_s, m_j \right) Y_l^{m_l} \chi_{m_s}. \tag{5.41}$$

Here $C(j_1, j_2, j_3; m_1, m_2, m_3)$ are the Clebsch-Gordan coefficients for combining angular momenta j_1 and j_2 into a state with total angular momentum j_3. m_1, m_2 and m_3 are the corresponding magnetic quantum numbers. The functions $\Omega^j_{lm_j}$ form an orthonormal set;

$$\int d\Omega \left(\Omega^{j'}_{l'm'_j} \right)^\dagger \Omega^j_{lm_j} = \delta_{j'j} \delta_{l'l} \delta_{m'_j m_j}. \tag{5.42}$$

From addition of angular momenta we get that $j = l \pm \frac{1}{2}$ (let's assume for the moment that $l > 0$, otherwise j will just be equal to $\frac{1}{2}$ since there is no angular momentum present). For these two values of j the spinor spherical harmonics is given by [1]:

$$\Omega_{lm_j}^{l+1/2} = \begin{pmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}} Y_l^{m_j-1/2} \\ \sqrt{\frac{l-m_j+1/2}{2l+1}} Y_l^{m_j+1/2} \end{pmatrix} \tag{5.43}$$

and

$$\Omega_{lm_j}^{l-1/2} = \begin{pmatrix} -\sqrt{\frac{l-m_j+1/2}{2l+1}} Y_l^{m_j-1/2} \\ \sqrt{\frac{l+m_j+1/2}{2l+1}} Y_l^{m_j+1/2} \end{pmatrix}. \tag{5.44}$$

It is useful to determine the eigenvalues of the operator $K' = \left(\mathbf{J}^2 - \mathbf{L}^2 + \frac{1}{4}\right)$ when acting on these spinor spherical harmonics. We get

$$\begin{aligned} K'\Omega_{lm_j}^{l+1/2} &= \left(\mathbf{J}^2 - \mathbf{L}^2 + \frac{1}{4}\right) \Omega_{lm_j}^{l+1/2} \\ &= \left[\left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right) - l(l+1) + \frac{1}{4}\right] \Omega_{lm_j}^{l+1/2} \\ &= (l+1)\Omega_{lm_j}^{l+1/2}. \end{aligned} \tag{5.45}$$

And similarly

$$K'\Omega_{lm_j}^{l-1/2} = -l\Omega_{lm_j}^{l-1/2}. \tag{5.46}$$

We will call these eigenvalues $-\kappa$, so that $\kappa = \mp\left(j + \frac{1}{2}\right)$ for $j = l \pm \frac{1}{2}$. We will encounter the number κ later when we deal with the Dirac field on the 3-sphere. When κ is set, so is j and l, so we will write $\Omega_{lm_j}^{j} \equiv \Omega_{m_j}^{\kappa}$.

Updating the table of quantum numbers gives

Quantum number	**Expression**	**Values**
l	l	$0, 1, 2, 3, \ldots$
m_l	$-l, -l+1, \ldots, l-1, l$	$\ldots, -2, -1, 0, 1, 2, \ldots$
n_r	n_r	$0, 1, 2, 3, \ldots$
n_{B}	$l + n_r + 1$	$1, 2, 3, \ldots$
m_s	m_s	$-\frac{1}{2}, \frac{1}{2}$
j	$l \pm \frac{1}{2}$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$
m_j	$-j, -j+1, \ldots, j-1, j$	$\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$
κ	$\mp\left(j + \frac{1}{2}\right)$	$\ldots, -2, -1, 1, 2, \ldots$

Table 5.2: 3rd table of quantum numbers.

Chapter 6

Dirac Fields in Curved Spacetime

When we wrote up the covariant version of the Klein-Gordon equation earlier, we began with the Lagrangian density. We altered this Lagrangian by adjusting each term appropriately for a scalar field. While this approach was fairly straightforward because of the tensorial nature of the scalar field, we will need to take another viewpoint when formulating the covariant Dirac equation.

Since the Dirac field is not a tensorial field we cannot simply interchange the partial derivative by the covariant derivative presented in chapter 2. We need to find a completely new covariant derivative appropriate for spinors. To do this we take advantage of the fact that we can formulate the Dirac equation in flat spacetime. With the tetrad formalism at hand we now how to connect separate neighbourhoods in a general curved spacetime via the spin connection. This will be our starting point for writing down the covariant version of the Dirac equation.

6.1 The spinor covariant derivative

First we need to find the covariant derivative of the spinor field,

$$\nabla_\mu \Psi(x) = \left[\partial_\mu + \Omega_\mu(x)\right]\Psi(x). \tag{6.1}$$

Here $\Omega_\mu(x)$ is the connection coefficient for the spinor field. With this connection the spinor should obey the following rule for parallel transport:

$$\Psi(x \to x + dx) = \Psi(x) - \Omega_\mu(x)\Psi(x)dx^\mu. \tag{6.2}$$

To find the spinor connection we look at the parallel transport properties of certain Dirac bilinears, to implicitly derive the rule for parallel transport of the spinor. We will follow chapter 7 of [11] closely. From table 5.1 we have the scalar quantity $S(x) = \overline{\Psi}(x)\Psi(x)$, which should remain unchanged under parallel transport;

$$S(x \to x + dx) = \overline{\Psi}(x \to x + dx)\Psi(x \to x + dx)$$
$$= \left[\Psi^\dagger(x)\gamma^0 - \Psi^\dagger(x)\Omega_\mu^\dagger(x)\gamma^0 dx^\mu\right]\left[\Psi(x) - \Omega_\nu(x)\Psi(x)dx^\nu\right]$$
$$= S(x) - \overline{\Psi}(x)\left[\Omega_\mu(x) + \gamma^0\Omega_\mu^\dagger(x)\gamma^0\right]\Psi(x)dx^\mu. \tag{6.3}$$

Here we have used (6.2), the definition of the Dirac adjoint and that $\left(\gamma^0\right)^2 = 1$. Also, the term proportional to $dx^\mu dx^\nu$ have been neglected since dx^μ is infinitesimal. For (6.3) to hold, we must have

$$\gamma^0\Omega_\mu^\dagger\gamma^0 = -\Omega_\mu. \tag{6.4}$$

Next we look at the local vector $j^a(x) = \overline{\Psi}(x)\gamma^a\Psi(x)$ which should transport in the same way as any other local vector (2.46);

$$j^a(x \to x + dx) = \left[\Psi^\dagger(x) - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\right]\gamma^0\gamma^a\left[\Psi(x) - \Omega_\mu(x)\Psi(x)dx^\mu\right]$$
$$= \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\left[\gamma^a\Omega_\mu(x) - \Omega_\mu(x)\gamma^a\right]\Psi(x)dx^\mu, \tag{6.5}$$

where we have taken into account the first condition on Ω_μ (6.4). By the requirement that this must obey (2.46), we get our second condition on the spinor connection:

$$\left[\gamma^a, \Omega_\mu\right] = \omega_\mu{}^a{}_b\gamma^b. \tag{6.6}$$

From this commutator we conclude that Ω_μ should be composed of some combination of the spin connection, together with a gamma matrix product satisfying the commutator. Recalling that $\left[\gamma^a, \sigma^{bc}\right] = 2i\left(\gamma^c\eta^{ba} - \gamma^b\eta^{ca}\right)$, and making the indices come out right in (6.6), we make the following ansatz for Ω_μ:

$$\Omega_\mu = C\omega_{\mu bc}\sigma^{bc}, \tag{6.7}$$

where C is a complex constant. Inserted into (6.6):

$$\left[\gamma^a, \Omega_\mu\right] = 2iC\omega_{\mu bc}\left(\gamma^c\eta^{ba} - \gamma^b\eta^{ca}\right) = 4iC\omega_\mu{}^a{}_b\gamma^b. \tag{6.8}$$

Here we have used the antisymmetry property of the spin connection in the second equality. From this the constant C is determined, $C = \frac{1}{4i}$. Remembering that $\left(\sigma^{bc}\right)^{\dagger} = \gamma^0\sigma^{bc}\gamma^0$, we see that with the constant C the spin connection also satisfies (6.4). We have then derived the covariant derivative of the spinor. It has the following connection:

$$\Omega_{\mu} = -\frac{1}{4}i\omega_{\mu bc}\sigma^{bc} = \frac{1}{8}\omega_{\mu bc}\left[\gamma^b, \gamma^c\right]. \tag{6.9}$$

6.2 Dirac equation in curved spacetime

To make the Dirac equation valid in curved spacetime we must also consider the gamma matrices in the Minkowskian form of the equation (5.1). These matrices are written in terms of local coordinates, so to write them in terms of global coordinates we must contract with the inverse vierbeins;

$$\gamma^{\mu} = e^{\mu}{}_{a}\gamma^{a}. \tag{6.10}$$

The global gamma matrices satisfy the generalized Clifford algebra:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{6.11}$$

We are now ready to write down the generalized Dirac equation for a curved spacetime background. It is

$$\boxed{\left[ie^{\mu}{}_{a}\gamma^{a}\left(\partial_{\mu} + \Omega_{\mu}\right) - m\right]\Psi = 0} \tag{6.12}$$

The connection term in this equation can be further analyzed. This term when written out,

$$ie^{\mu}{}_{c}\gamma^{c}\Omega_{\mu} = i\gamma^{c}\Omega_{c} = \frac{1}{8}i\omega_{cab}\left(\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a\right), \tag{6.13}$$

involves products of three gamma matrices. Here we have defined $\omega_{cab} \equiv e^{\mu}{}_{c}\omega_{\mu ab}$. Utilizing the identity

$$\gamma^c\gamma^a\gamma^b = \eta^{ca}\gamma^b + \eta^{ab}\gamma^c - \eta^{cb}\gamma^a - i\epsilon^{dcab}\gamma_d\gamma^5 \tag{6.14}$$

yields

$$\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a = 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a - 2i\epsilon^{cabd}\gamma_d\gamma^5. \tag{6.15}$$

The Dirac equation, (6.12), is now written more explicitly as

$$ie^{\mu}{}_{a}\gamma^{a}\partial_{\mu}\Psi + \frac{1}{4}i\omega_{cab}\left(\eta^{ca}\gamma^{b} - \eta^{cb}\gamma^{a}\right)\Psi - \frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_{d}\gamma^{5}\Psi - m\Psi = 0, \qquad (6.16)$$

where the properties of the Levi-Civita symbol have been used. We will later show that the term involving this symbol actually vanishes for the FRW metric.

6.3 The reduced Dirac equation

To further simplify (6.16) we will follow [3], and impose a factorizability ansatz on the spinor. Such an ansatz will work when analyzing the Dirac equation in spacetimes that are sufficiently symmetric. In the case of the FRW metric, spacetime exhibits azimuthal symmetry and the metric only depends on the coordinates r and θ, and time in the case of an expanding universe. Also the metric is diagonal in the FRW case and then consequently so is the vierbein. The vierbein then inherit the same symmetry as the metric. With this in mind we make the following ansatz. Let

$$\Psi = f(x^0, x^1, x^2)\psi, \qquad (6.17)$$

such that Φ satisfies the reduced Dirac equation

$$ie^{\mu}{}_{a}\gamma^{a}\partial_{\mu}\psi - \frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_{d}\gamma^{5}\psi - m\psi = 0. \qquad (6.18)$$

As mentioned, in the FRW case we will show that the term involving the Levi-Civita symbol is equal to zero. Consequently, if there exists an f such that (6.18) holds, we have got rid of the connection term altogether.

The factorization (6.17) inserted into the Dirac equation (6.16), yields

$$ie^{\mu}{}_{a}\gamma^{a}\partial_{\mu}f + \frac{1}{4}i\omega_{cab}\left(\eta^{ca}\gamma^{b} - \eta^{cb}\gamma^{a}\right)f = 0, \qquad (6.19)$$

where we have used (6.18). This set of PDEs for f can be further simplified by multiplying by γ^e to left. We then take the trace of both sides utilizing the trace identity $\mathrm{tr}\left(\gamma^a\gamma^e\right) = 4\eta^{ae}$:

$$4\eta^{ae}e^{\mu}{}_{a}\partial_{\mu}f + \omega_{cab}\left(\eta^{ca}\eta^{eb} - \eta^{cb}\eta^{ea}\right)f = 0, \qquad (6.20)$$

which after some manipulations becomes

$$\partial_{\mu}\log(f) + \frac{1}{2}e_{\mu}{}^{c}\eta_{bc}\omega_{a}{}^{ab} = 0. \qquad (6.21)$$

From the expression for the spin connection (2.52) we have

$$e_\mu{}^c \eta_{bc} \omega_a{}^{ab} = \Gamma^\nu{}_{\mu\nu} + e_\mu{}^a \partial_\nu e^\nu{}_a. \tag{6.22}$$

Using the identity for $\Gamma^\nu{}_{\mu\nu}$, (2.27), we get the following set of PDEs for f:

$$\partial_\mu \log(f) = -\partial_\mu \log\left(\sqrt{e}\right) - \frac{1}{2} e_\mu{}^a \partial_\nu e^\nu{}_a. \tag{6.23}$$

Here e is the determinant of $e_\mu{}^a$. Defining $f \equiv h e^{-1/2}$ gives finally

$$\boxed{\partial_\mu \log(h) = -\frac{1}{2} e_\mu{}^a \partial_\nu e^\nu{}_a.} \tag{6.24}$$

which will determine the existence of the factorization function f, with the condition being that this set of PDEs have an analytic solution.

In the case of a diagonal vierbein, the set (6.24) is easy to write out:

$$\left.\begin{aligned}
\partial_t \log(h) &= -\frac{1}{2} e_t{}^t \partial_t e^t{}_t, \\
\partial_r \log(h) &= -\frac{1}{2} e_r{}^r \partial_r e^r{}_r, \\
\partial_\theta \log(h) &= -\frac{1}{2} e_\theta{}^\theta \partial_\theta e^\theta{}_\theta, \\
\partial_\varphi \log(h) &= 0.
\end{aligned}\right\} \tag{6.25}$$

Here we have kept in mind that f does not depend on φ.

This is as far as we can get without specifying exactly what the metric is, although we have stated some general properties like symmetry and diagonality. The existence of f depends highly on the exact expression for the metric.

6.4 The Dirac equation in FRW spacetime

For easy reference we recall that the metric for a FRW universe is given by the line element (2.56), and on matrix form we have

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix}, \tag{6.26}$$

hence the vierbeins and their inverse are respectively given by

$$e_\mu{}^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a(t)}{\sqrt{1-kr^2}} & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)r\sin\theta \end{pmatrix} \tag{6.27}$$

and

$$e^\nu{}_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1-kr^2}}{a(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a(t)r} & 0 \\ 0 & 0 & 0 & \frac{1}{a(t)r\sin\theta} \end{pmatrix}. \tag{6.28}$$

Imposing the factorization (6.17), this set of vierbeins give the following set of PDEs for h:

$$\left.\begin{aligned} &\partial_t \log(h) = \partial_\theta \log(h) = \partial_\varphi \log(h) = 0, \\ &\partial_r \log(h) = \frac{1}{2}\frac{kr}{1-kr^2}, \end{aligned}\right\} \tag{6.29}$$

so h depends only on r. Hence the last equation becomes an ODE in r. Integrating yields

$$h = \left(1 - kr^2\right)^{-1/4}, \tag{6.30}$$

and we get

$$f = he^{-1/2} = \frac{1}{a^{3/2} r \sin^{1/2}\theta}. \tag{6.31}$$

The factorization of the spinor is then given by

$$\Psi = \frac{1}{a^{3/2} r \sin^{1/2}\theta}\psi, \tag{6.32}$$

where ψ satisfies the reduced Dirac equation (6.18).

In the appendix (B.1) we have calculated the spin connection ω_{cab} for the FRW metric. Its non-zero components up to antisymmetry are:

Indices $\{c, a, b\}$	**Expression for** ω_{cab}
$\{1, 0, 1\}$	$\frac{1}{a}\frac{da}{dt}$
$\{2, 0, 2\}$	$\frac{1}{a}\frac{da}{dt}$
$\{2, 1, 2\}$	$\frac{1}{ar}\sqrt{1-kr^2}$
$\{3, 0, 3\}$	$\frac{1}{a}\frac{da}{dt}$
$\{3, 1, 3\}$	$\frac{1}{ar}\sqrt{1-kr^2}$
$\{3, 2, 3\}$	$\frac{1}{ar}\frac{\cos\theta}{\sin\theta}$

Table 6.1: Non-zero components of the spin connection in a FRW Universe up to antisymmetry.

First of all it is worth taking a moment just to mention the high degree of symmetry this connection exhibits. The patterns in this table are highly regular. The expressions for equal middle index are equal. We have gotten only index combinations on the form $\{i, j, i\}$, where $i = 1, 2, 3$ and $j < i$. Secondly we notice that we have no components with all different indices. Hence

$$\frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_d\gamma^5 = 0 \tag{6.33}$$

because of the properties of the Levi-Civita symbol.

Collecting these results we get that ψ obeys a highly reduced Dirac equation on the form

$$ie^{\mu}{}_{a}\gamma^a\partial_\mu\psi - m\psi = 0, \tag{6.34}$$

completely without connection terms. This equation was studied by Brill and Wheeler in 1957 for a Schwarzschild type metric [6], and later by Villalba and Percoco in 1990 for the FRW spacetime [15]. These articles have a more direct approach on obtaining a connectionless equation, whereas we in this thesis have obtained (as in [3]) a way of calculating a factorization function from a given metric to get rid of the connection terms. We will however follow the ideas presented in the mentioned articles upon solving the reduced Dirac equation on the 3-sphere by the method of separation of variables.

Let's begin by writing out equation (6.34):

$$ia\frac{\partial}{\partial t}\psi = -i\gamma^t\left[\gamma^r\frac{\partial}{\partial r} + \gamma^\theta\frac{\partial}{\partial\theta} + \gamma^\varphi\frac{\partial}{\partial\varphi}\right]\psi + am\gamma^t\psi, \tag{6.35}$$

where we have written $\gamma^\mu = e^\mu{}_a\gamma^a$.

As discussed by Brill and Wheeler we have some freedom in choosing the matrices γ^μ. This is attributed to the fact that we can choose different coordinate systems for the vierbeins. Now the FRW metric is written in terms of spherical coordinates, so that the obvious choice for the vierbeins will be the diagonal matrices that we have chosen to work with, where the vierbein axes point in the directions t, r, θ, φ. We will call this choice *the diagonal tetrad gauge* (borrowing the term from [15]). In this gauge we have

$$\gamma^t_d = \gamma^0, \qquad \gamma^r_d = F\gamma^1,$$

$$\gamma^\theta_d = \frac{1}{r}\gamma^2, \qquad \gamma^\varphi_d = \frac{1}{r\sin\theta}\gamma^3,$$

where $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are the usual flat spacetime gamma matrices in Cartesian coordinates. We have put the subscript d for *diagonal* and also defined $F = \sqrt{1-kr^2}$ as before.

Another choice will be the scenario where the vierbein axes point along the directions t, x, y, z. In this *Cartesian tetrad gauge* the gamma matrices will be given by

$$\gamma^t_c = \gamma^0, \qquad \gamma^r_c = F\left[\left(\gamma^1\cos\varphi + \gamma^2\sin\varphi\right)\sin\theta + \gamma^3\cos\theta\right],$$

$$\gamma^\theta_c = \frac{1}{r}\left[\left(\gamma^1\cos\varphi + \gamma^2\sin\varphi\right)\cos\theta - \gamma^3\sin\theta\right],$$

$$\gamma^\varphi_c = \frac{1}{r\sin\theta}\left(-\gamma^1\sin\varphi + \gamma^2\cos\varphi\right).$$

We have put the subscript c for *Cartesian*.

Now these two sets of gamma matrices satisfy the Clifford algebra and are therefore linked to each other by a similarity transformation S. S will be the usual transformation for transforming the gamma matrices from Cartesian coordinates to spherical coordinates and it is given by

$$S = \exp\left(-\frac{\varphi}{2}\gamma^1\gamma^2\right)\exp\left(-\frac{\theta}{2}\gamma^3\gamma^1\right)\mathcal{S}, \tag{6.36}$$

where the transformation

$$\mathcal{S} \equiv \frac{1}{2}\left(\gamma^1\gamma^2 - \gamma^1\gamma^3 + \gamma^2\gamma^3 + 1\right) \tag{6.37}$$

acts on the spatial gamma matrices in the following way

$$\mathcal{S}\gamma^1\mathcal{S}^{-1} = \gamma^3, \quad \mathcal{S}\gamma^2\mathcal{S}^{-1} = \gamma^1, \quad \mathcal{S}\gamma^3\mathcal{S}^{-1} = \gamma^2, \tag{6.38}$$

γ^0 is preserved under this transformation. The two different choices of gamma matrices are related by $\gamma_c^\mu = S\gamma_d^\mu S^{-1}$, and the solutions by $\psi_c = S\psi_d$. The real and measurable quantities should of course be the same for both solutions, so that they are in this sense equivalent, but clearly the diagonal gauge seems easier to use for explicitly solving the equation. However the solutions resulting from this choice will not be as physically transparent as the solution in the Cartesian gauge [6]. Firstly the function Ψ_d will not be a single valued function of position, and secondly, the angular part of the solution will be harder to interpret in terms of physically measurable quantities such as angular momentum. The radial equations will be the same for both choices.

Following this discussion we choose to solve the reduced Dirac equation in the diagonal tetrad gauge and write (6.34) as

$$ia\frac{\partial}{\partial t}\psi = \mathbf{T}\psi \tag{6.39}$$

where we have defined the operators

$$\mathbf{T} = -i\gamma^0\gamma^1 F\frac{\partial}{\partial r} + \frac{\gamma^1}{r}\mathbf{K} - ia\gamma^0 m \tag{6.40}$$

and

$$\mathbf{K} = i\gamma^1\gamma^0\gamma^2\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\gamma^1\gamma^0\gamma^3\frac{\partial}{\partial\varphi}. \tag{6.41}$$

The angular operator $\mathbf{K}$ is Hermitian and commutes with $\mathbf{T}$. It is convenient to work in the representation

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{6.42}$$

where σ^i are the usual Pauli matrices. In the end we will be interested in the solution

$$\Psi = \frac{1}{a^{3/2} r \sin^{1/2}\theta} S\psi, \tag{6.43}$$

so that it will be useful to have the explicit expression for the similarity transformation S. Using the representation (6.42), S will be given in block form as

$$S = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \tag{6.44}$$

for the matrix

$$Z = \begin{pmatrix} \mathrm{e}^{-i\varphi/2} & 0 \\ 0 & \mathrm{e}^{i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \mathcal{S} \tag{6.45}$$

calculated using the series expansion for the exponential operator. Actually it will turn out to be easier to solve the Dirac equation of we apply the transformation $\mathcal{S}$ right away [15]. We will then solve for the solutions $\widetilde{\psi} = \mathcal{S}\psi$. Using the transformation properties of the gamma matrices (6.38), we get

$$\widetilde{\mathbf{T}} = -i\gamma^0\gamma^3 F\frac{\partial}{\partial r} + \frac{\gamma^3}{r}\widetilde{\mathbf{K}} - ia\gamma^0 m \tag{6.46}$$

and

$$\widetilde{\mathbf{K}} = i\gamma^3\gamma^0\gamma^1\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\gamma^3\gamma^0\gamma^2\frac{\partial}{\partial\varphi}. \tag{6.47}$$

Now since the operators $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{K}}$ commutes, we can proceed to find simultaneous eigenfunctions of these two operators and impose separation of variables with the following ansatz

$$\widetilde{\psi} = R(r)\Theta(\theta)\mathrm{e}^{im_j\varphi - i\omega t}. \tag{6.48}$$

The notation m_j will become apparent later. Firstly we look at the azimuthal dependence. For the solution Ψ to be single valued and continuous we should require that

$$\Psi(\varphi) = \Psi(\varphi + 2\pi). \tag{6.49}$$

Since $\Psi \sim S\psi$, and we see from the azimuthal part of the transformation S that

$$S(\varphi + 2\pi) = -S(\varphi), \tag{6.50}$$

we must require that

$$\widetilde{\psi}(\varphi + 2\pi) = -\widetilde{\psi}(\varphi). \tag{6.51}$$

This is satisfied if $m_j = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$.

Having determined the values of the quantum number m_j we go on to analyze the eigenvalue problem for the angular operator. Using the representation (6.42) we get

$$\widetilde{\mathbf{K}}\widetilde{\psi} = i\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}\frac{\partial}{\partial\theta}\psi - \frac{m_j}{\sin\theta}\begin{pmatrix}-\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}\widetilde{\psi} = \kappa\widetilde{\psi}, \tag{6.52}$$

denoting the eigenvalue by κ. Now this suggests that we should write the angular part of the spinor as

$$\widetilde{\psi} \sim \begin{pmatrix}\Theta(\theta)\\ \sigma^3\Theta(\theta)\end{pmatrix} = \begin{pmatrix}\Theta_1\\ \Theta_2\\ \Theta_1\\ -\Theta_2\end{pmatrix}, \tag{6.53}$$

so that we have two free components to be determined. Inserted into the angular eigenvalue equation (6.52) yields a set of two coupled first order ODEs for the θ-dependent angular functions;

$$\boxed{\frac{d}{d\theta}\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix} = \begin{pmatrix}\frac{m_j}{\sin\theta} & -\kappa\\ \kappa & -\frac{m_j}{\sin\theta}\end{pmatrix}\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix}} \tag{6.54}$$

This set will be the angular Dirac equation in FRW spacetime.

We now turn our attention to the radial part of the Dirac equation. Utilizing what we have found for the angular equation we have

$$ia\frac{\partial}{\partial t}\widetilde{\psi} = \widetilde{\mathbf{T}}\widetilde{\psi} = -i\gamma^0\gamma^3 F\frac{\partial}{\partial r}\widetilde{\psi} + \frac{\gamma^3}{r}\kappa\widetilde{\psi} - ia\gamma^0 m\widetilde{\psi}, \tag{6.55}$$

which yields

$$a\omega\widetilde{\psi} = -i\begin{pmatrix}0 & -i\sigma^3\\ i\sigma^3 & 0\end{pmatrix}F\frac{\partial}{\partial r}\widetilde{\psi} + \frac{\kappa}{r}\begin{pmatrix}0 & \sigma^3\\ \sigma^3 & 0\end{pmatrix}\widetilde{\psi} - iam\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\widetilde{\psi}. \tag{6.56}$$

From this we see that we should write the solution as

$$\widetilde{\psi} = \begin{pmatrix}P(r)\Theta(\theta)\\ \sigma^3 Q(r)\Theta(\theta)\end{pmatrix}\mathrm{e}^{im_j\varphi - i\omega t}, \tag{6.57}$$

for two (one-component) radial functions P and Q. Inserting this solution into (6.56) gives the following set of coupled first order ODEs for the radial functions:

$$F\frac{d}{dr}\begin{pmatrix}P\\Q\end{pmatrix}=\begin{pmatrix}-\frac{\kappa}{r} & a\,(\omega+m)\\ -a\,(\omega-m) & \frac{\kappa}{r}\end{pmatrix}\begin{pmatrix}P\\Q\end{pmatrix}\tag{6.58}$$

which will be the radial Dirac equation in FRW spacetime.

We will now turn our attention to solving the angular and radial Dirac equations on the 3-sphere.

6.5 The Dirac equation on the 3-sphere

Similar to when we studied the Klein-Gordon equation in curved spacetime we are now going to solve the Dirac equation for the FRW-metric after performing the substitutions $k=1$ and $r=\sin\chi$, yielding the metric for $\mathbb{R}\times S^3$. As before we have

$$F=\cos\chi,$$

$$\frac{d}{dr}=\frac{1}{\cos\chi}\frac{d}{d\chi}.$$

Let's begin with the radial equations. We have

$$\frac{d}{d\chi}\begin{pmatrix}P\\Q\end{pmatrix}=\begin{pmatrix}-\frac{\kappa}{\sin\chi} & a\,(\omega+m)\\ -a\,(\omega-m) & \frac{\kappa}{\sin\chi}\end{pmatrix}\begin{pmatrix}P\\Q\end{pmatrix}.\tag{6.59}$$

This set can be decoupled to give ODEs for the functions P and Q alone. Isolating P and Q yields

$$P=-\frac{1}{a(\omega-m)}\left(\frac{d}{d\chi}-\frac{\kappa}{\sin\chi}\right)Q,\tag{6.60}$$

$$Q=\frac{1}{a(\omega+m)}\left(\frac{d}{d\chi}+\frac{\kappa}{\sin\chi}\right)P.\tag{6.61}$$

Inserting Q into the upper equation gives the second order ODE for P;

$$\left[\frac{d^2}{d\chi^2}-\frac{\kappa(\kappa+\cos\chi)}{\sin^2\chi}+\varepsilon^2\right]P=0,\tag{6.62}$$

where we have defined $\varepsilon^2 \equiv a^2(\omega^2 - m^2)$. The equation for the function Q will be the same when we let $\kappa \to -\kappa$.

When it comes to the angular equations they remain unchanged by the transition to $\mathbb{R} \times S^3$ spacetime, and we observe that they become identical in form to the radial equations. Hence it suffices to solve the ODE for the radial function P.

6.5.1 The radial solution

The radial Dirac equations are very similar to the radial Klein-Gordon equation on the 3-sphere. We can therefore, by the same substitution of variable, recognize these equations as hypergeometric differential equations. As in the Klein-Gordon case we impose the substitution

$$u = \sin^2 \frac{\chi}{2}. \tag{6.63}$$

For easy reference let's list the trigonometric identities

$$\cos\chi = \cos^2\frac{\chi}{2} - \sin^2\frac{\chi}{2} = 1 - 2\sin^2\frac{\chi}{2} = \underline{1 - 2u},$$

$$\sin^2\chi = 1 - \cos^2\chi = 1 - (1-2u)^2 = \underline{4u(1-u)},$$

$$\frac{d}{d\chi} = \frac{du}{d\chi}\frac{d}{du} = \sin\frac{\chi}{2}\cos\frac{\chi}{2}\frac{d}{du} = \frac{1}{2}\sin\chi\frac{d}{du} = \underline{\sqrt{u(1-u)}\frac{d}{du}},$$

$$\frac{d^2}{d\chi^2} = \underline{u(1-u)\frac{d^2}{du^2} + \left(\frac{1}{2} - u\right)\frac{d}{du}}.$$

This inserted into the ODE for P results in

$$u(u-1)P'' - \left(\frac{1}{2} - u\right)P' - \frac{\kappa(\kappa + 1 - 2u) + 4u(u-1)\varepsilon^2}{4u(u-1)}P = 0, \tag{6.64}$$

where $'$ denotes $\frac{d}{du}$.

As mentioned, we need only solve the equation for P. The solution for Q is obtained by letting $\kappa \to -\kappa$ in the solution for P. Following what we did for the radial Klein-Gordon equation, we begin by classifying the possibly singular points.

The point $u_0 = 0$:

Now $u_0(u_0 - 1) = 0$, so this point is singular. The limits (A.2) become

$$\lim_{u \to 0} -\frac{\left(\frac{1}{2} - u\right)}{u - 1} = \frac{1}{2},$$
$$\lim_{u \to 0} -\frac{\kappa(\kappa + 1 - 2u) + 4u(u-1)\varepsilon^2}{4(u-1)^2} = -\frac{\kappa(\kappa+1)}{4}, \tag{6.65}$$

which are both finite. We conclude that the point $u_0 = 0$ is a regular singular point.

The point $u_0 = 1$:

Now $u_0(u_0 - 1) = 0$, so this point is singular. The limits (A.2) become

$$\lim_{u \to 1} -\frac{\left(\frac{1}{2} - u\right)}{u} = \frac{1}{2},$$
$$\lim_{u \to 1} -\frac{\kappa(\kappa + 1 - 2u) + 4u(u-1)\varepsilon^2}{4u^2} = -\frac{\kappa(\kappa-1)}{4}, \tag{6.66}$$

which are both finite. We conclude that the point $u_0 = 1$ is a regular singular point.

The point $u_0 = \infty$:

We now make the substitution $u = \frac{1}{v}$. Then

$$\frac{d}{du} = -v^2 \frac{d}{dv},$$

and hence

$$\frac{d^2}{du^2} = 2v^3 \frac{d}{dv} + v^4 \frac{d^2}{dv^2}.$$

Substituted into the original equation (6.64) yields:

$$v^2(1-v)P''_\kappa + v\left(1 - \frac{3}{2}v\right)P'_\kappa - \frac{\kappa(\kappa + 1 - \frac{2}{v})v^2 + 4(1-v)\varepsilon^2}{4(1-v)}P = 0. \tag{6.67}$$

Where $'$ now denotes derivative with respect to v. From this we see that the point $u_0 = \infty$ corresponds to the point $v_0 = 0$. Hence $v_0^2(1 - v_0) = 0$ so that $v_0 = 0$ is a singular point.

The limits (A.2) now become

$$\lim_{v\to 0} \frac{\left(1-\frac{3}{2}v\right)}{1-v} = 1,$$
$$\lim_{v\to 0} -\frac{\kappa(\kappa+1-\frac{2}{v})v^2+4(1-v)\varepsilon^2}{4(1-v)^2} = -\varepsilon^2, \tag{6.68}$$

which are both finite. We finally conclude that $v_0 = 0$ is a regular singular point. Hence the point $u_0 = \infty$ is a regular singular point.

We have now classified all the singular points of the ODE for P, and found that it has the three regular singular points 0, 1 and ∞. The indicial equations for each of these singular points respectively, is given by (A.5):

$$\mu(\mu-1)+\frac{1}{2}\mu-\frac{\kappa(\kappa+1)}{4} = 0, \tag{6.69}$$

$$\nu(\nu-1)+\frac{1}{2}\nu-\frac{\kappa(\kappa-1)}{4} = 0, \tag{6.70}$$

$$\lambda(\lambda-1)+\lambda-\varepsilon^2 = 0, \tag{6.71}$$

where we have used the limits calculated above for each singular point. Solving these equations for the indices yields:

$$\mu^2-\frac{1}{2}\mu = \frac{\kappa(\kappa+1)}{4}$$

$$\Rightarrow \quad \left(\mu-\frac{1}{4}\right)^2 = \frac{1}{16}+\frac{\kappa(\kappa+1)}{4}$$

$$= \frac{1}{16}\left(1+4\kappa^2+4\kappa\right)$$

$$= \left(\frac{1+2\kappa}{4}\right)^2.$$

Which gives

$$\mu^{(\pm)} = \underline{\frac{1}{4} \pm \frac{1}{4}\big|1+2\kappa\big|.} \tag{6.72}$$

And similarly we get

$$\nu^{(\pm)} = \underline{\frac{1}{4} \pm \frac{1}{4}\big|1-2\kappa\big|,} \tag{6.73}$$

$$\lambda^{(\pm)} = \underline{\pm|\varepsilon|.} \tag{6.74}$$

As a control we note that these indices add up to one, as they should. With these indices we have the following tableau for our differential equation:

$$\mathcal{T}(P) = \left\{ \begin{matrix} 0 & 1 & \infty \\ \\ \frac{1}{4}+\frac{1}{4}\big|1+2\kappa\big| & \frac{1}{4}+\frac{1}{4}\big|1-2\kappa\big| & +|\varepsilon| \\ \\ \frac{1}{4}-\frac{1}{4}\big|1+2\kappa\big| & \frac{1}{4}-\frac{1}{4}\big|1-2\kappa\big| & -|\varepsilon| \end{matrix} \right\}. \tag{6.75}$$

As in the case of the radial Klein-Gordon equation, we can by a suitable factorization of the form

$$f = u^{\alpha_0}(u-1)^{\alpha_1}P, \tag{6.76}$$

shift the indices in this tableau so that we get a new tableau corresponding to the hypergeometric differential equation. For simple reference the hypergeometric equation has the tableau,

$$\mathcal{T}(\text{hypergeometric}) = \left\{ \begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix} \right\}. \tag{6.77}$$

Imposing the factorization (6.76) yields

$$\mathcal{T}(f) = \left\{ \begin{matrix} 0 & 1 & \infty \\ \\ \frac{1}{4}+\kappa^{+}+\alpha_0 & \frac{1}{4}+\kappa^{-}+\alpha_1 & +|\varepsilon|-\alpha_0-\alpha_1 \\ \\ \frac{1}{4}-\kappa^{+}+\alpha_0 & \frac{1}{4}-\kappa^{-}+\alpha_1 & -|\varepsilon|-\alpha_0-\alpha_1 \end{matrix} \right\}. \tag{6.78}$$

Here we have defined the two quantities $\kappa^{\pm} \equiv \frac{1}{4}\left|1 \pm 2\kappa\right|$. The notation will become evident in a bit.

Setting $\mathcal{T}(f) = \mathcal{T}(\text{hypergeometric})$ gives

$$\alpha_0 = -\frac{1}{4} - \kappa^+, \qquad \alpha = \kappa^+ + \kappa^- + \frac{1}{2} + |\varepsilon|,$$

$$\alpha_1 = -\frac{1}{4} - \kappa^-, \qquad \beta = \kappa^+ + \kappa^- + \frac{1}{2} - |\varepsilon|,$$

$$\gamma = 2\kappa^+ + 1,$$

The function f then satisfies the hypergeometric differential equation:

$$u(1-u)f'' + \left[\gamma - (\alpha+\beta+1)u\right] f' - \alpha\beta f = 0, \tag{6.79}$$

with the solution

$$f = A\, {}_2F_1\left(\alpha, \beta; \gamma; u\right) + B\, u^{1-\gamma}\, {}_2F_1\left(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; u\right). \tag{6.80}$$

Here A and B are complex constants. The radial equation for P then gets the solutions

$$\begin{aligned} P =& Au^{-\alpha_0}(u-1)^{-\alpha_1}\, {}_2F_1\left(\alpha, \beta; \gamma; u\right) \\ &+ Bu^{1-\gamma-\alpha_0}(u-1)^{-\alpha_1}\, {}_2F_1\left(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; u\right). \end{aligned} \tag{6.81}$$

We have to leave the radial solution here to intermediately solve the angular equations. This is needed to get more information on the values of κ.

6.5.2 The angular solution

The angular equations are given by

$$\frac{d}{d\theta}\begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix} = \begin{pmatrix}\frac{m_j}{\sin\theta} & \kappa \\ -\kappa & -\frac{m_j}{\sin\theta}\end{pmatrix}\begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix}, \tag{6.82}$$

so that we see that these will yield the same solutions as P and Q with the changes

$$\chi \to \theta, \quad \kappa \to -m_j, \quad \varepsilon^2 \to \kappa^2. \tag{6.83}$$

Now writing $w = \sin^2 \frac{\theta}{2}$ we have

$$\begin{aligned}\Theta_1 =& Cw^{-\alpha_0}(w-1)^{-\alpha_1}{}_2F_1\left(\alpha,\beta;\gamma;w\right)\\ &+ Dw^{1-\gamma-\alpha_0}(w-1)^{-\alpha_1}{}_2F_1\left(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;w\right)\end{aligned} \tag{6.84}$$

with

$$\alpha_0 = -\frac{1}{4} - m_j^-, \qquad \alpha = |m_j| + \frac{1}{2} + |\kappa|,$$

$$\alpha_1 = -\frac{1}{4} - m_j^+, \qquad \beta = |m_j| + \frac{1}{2} - |\kappa|,$$

$$\gamma = 2m_j^- + 1,$$

where

$$m_j^+ + m_j^- = |m_j| = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \tag{6.85}$$

and $m_j^\pm \equiv \frac{1}{4}\left|1 \pm 2m_j\right|$ as before. The solution Θ_2 will be obtained by letting $m_j \to -m_j$.

Let's analyze the behaviour of Θ_1 near $w = 0$, corresponding to $\theta = 0$. We seek non-singular eigenfunctions on the interval $0 \le \theta \le \pi$. This requires that the exponents of the factors of u in the solution should be greater than or equal to $\frac{1}{4}$ (taking into account the factorization of the solution Ψ). Now since

$$-\alpha_0 = \frac{1}{4} + m_j^- \ge \frac{1}{4}, \tag{6.86}$$

the first term is acceptable. For the second term we have

$$1 - \gamma - \alpha_0 = \frac{1}{4} - m_j^-. \tag{6.87}$$

This may seem acceptable for the particular value $m_j = \frac{1}{2}$, but here we have to remember that while this holds for Θ_1, it will not be acceptable for Θ_2 since then $m_j \to -m_j$. Hence we must require that $D = 0$. Following this we get the solution

$$\begin{aligned}\Theta_1 =& Cw^{\frac{1}{4}+m_j^-}(w-1)^{\frac{1}{4}+m_j^+} \\ & \times {}_2F_1\left(|m_j|+\frac{1}{2}+|\kappa|, |m_j|+\frac{1}{2}-|\kappa|;\, 2m_j^-+1; w\right).\end{aligned} \tag{6.88}$$

Next we analyze the behaviour of this solution near $w = 1$, corresponding to $\theta = \pi$. As with the Klein-Gordon case we use relation (A.14) and write

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;u) = & \left[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\right] {}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;1-u)+ \\ & (1-u)^{\gamma-\alpha-\beta}\left[\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right]\times \\ & {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-u).\end{aligned} \tag{6.89}$$

From this we see that around $w = 1$ there will appear a term proportional to $(w-1)^{\gamma-\alpha-\beta-\alpha_1}$ in the solution Θ_1. Since

$$\gamma-\alpha-\beta-\alpha_1 = \frac{1}{4}-m_j^+, \tag{6.90}$$

this term will be too singular near $w = 1$, and we must require that the term vanishes. We recall that for this to be the case we must have $\beta = -n_r$, which gives the quantization of κ:

$$\kappa = \pm 1, \pm 2, \pm 3, \ldots. \tag{6.91}$$

With the appearance of $\beta = -n_r$ we see from appendix A.3 that if we define $a = 2m_j^-$, we have

$${}_2F_1\left(|m_j|+\frac{1}{2}+|\kappa|, |m_j|+\frac{1}{2}-|\kappa|;\, 2m_j^-+1; w\right) = \frac{n_r!}{(a+1)_{n_r}}P_{n_r}^{(a,b)}(1-2w), \tag{6.92}$$

where $b = 2m_j^+$ and $P_{n_r}^{(a,b)}(1-2w)$ will be the n_r'th order Jacobi polynomial. This gives us the solution

$$\Theta_1 = Cw^{\frac{1}{4}+\frac{1}{2}a}(w-1)^{\frac{1}{4}+\frac{1}{2}b}P_{n_r}^{(a,b)}(1-2w) \tag{6.93}$$

for a new constant C dependent on the quantum numbers. For Θ_2 we will have $m_j \to -m_j$ so that it has the same solution as Θ_1 but with a and b interchanged;

$$\Theta_2 = Dw^{\frac{1}{4}+\frac{1}{2}b}(w-1)^{\frac{1}{4}+\frac{1}{2}a}P_{n_r}^{(b,a)}(1-2w). \tag{6.94}$$

Having established the angular solutions we can go on to determine the full angular part of the particular solutions Ψ. Recalling that $w = \sin^2\frac{\theta}{2}$ and using the properties of the Jacobi polynomials, it can be shown [15] that

$$\Psi = \frac{1}{a^{3/2}\sin\chi\sin^{\frac{1}{2}}\theta}\begin{pmatrix}\cos(\theta/2)\mathrm{e}^{-i\varphi/2} & -\sin(\theta/2)\mathrm{e}^{-i\varphi/2}\\ \sin(\theta/2)\mathrm{e}^{i\varphi/2} & \cos(\theta/2)\mathrm{e}^{i\varphi/2}\end{pmatrix}\begin{pmatrix}P\Theta\\ \sigma^3 Q\Theta\end{pmatrix}\mathrm{e}^{im_j\varphi-i\omega t} \tag{6.95}$$

will be given by

$$\Psi = \frac{1}{a^{3/2}\sin\chi}\begin{pmatrix}P(\chi)\Omega^{\kappa}_{lm_j}(\theta,\varphi)\\ \\ Q(\chi)\Omega^{-\kappa}_{lm_j}(\theta,\varphi)\end{pmatrix}\mathrm{e}^{-i\omega t}. \tag{6.96}$$

Here we have performed the similarity transformation discussed earlier (6.43). The angular functions $\Omega^{\kappa}_{lm_j}(\theta,\varphi)$ are the normalized spinor spherical harmonics that were presented in section 5.5.1 with the quantum number κ taking the values $\mp(j+\frac{1}{2})$, where j is the total angular momentum.

6.5.3 The radial solution revisited

With the exact nature of the values of κ we can go back to the radial solution

$$\begin{aligned}P =& Au^{-\alpha_0}(u-1)^{-\alpha_1}{}_2F_1(\alpha,\beta;\gamma;u)\\ &+ Bu^{1-\gamma-\alpha_0}(u-1)^{-\alpha_1}{}_2F_1(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;u).\end{aligned} \tag{6.97}$$

As with the angular case we have to require that the second term should vanish for a physically acceptable solution, i.e that $B = 0$. In addition the behaviour around $u = 1$ also require that $\beta = -n_r$ in this case. Then

$$\begin{aligned}P =& Au^{\frac{1}{4}+\kappa^+}(u-1)^{\frac{1}{4}+\kappa^-}\\ &\times {}_2F_1\left(|\kappa|+\frac{1}{2}+|\varepsilon|, |\kappa|+\frac{1}{2}-|\varepsilon|;\, 2\kappa^+ + 1; u\right),\end{aligned} \tag{6.98}$$

and similarly for Q but with the change $\kappa^+ \to \kappa^-$. With $\beta = -n_r$ we get

$$|\varepsilon| = |\kappa| + n_r + \frac{1}{2}, \tag{6.99}$$

giving us (remembering the definition of ε) the energy quantization for the Dirac field in $\mathbb{R} \times S^3$ spacetime:

$$\omega = \pm\sqrt{\frac{\left(n_\mathrm{F} + \frac{1}{2}\right)^2}{a^2} + m_\mathrm{F}^2}. \tag{6.100}$$

Here we have defined the principal quantum number for the Fermionic case;

$$n_\mathrm{F} \equiv |\kappa| + n_r = 1, 2, 3, \ldots. \tag{6.101}$$

The solution P is also in this case expressed in terms of Jacobi polynomials. Now defining $a = 2\kappa^+$ and $b = 2\kappa^-$ yields

$$P(\chi) = A \sin^{\frac{1}{2}}(\chi) \sin^a\left(\frac{\chi}{2}\right) \cos^b\left(\frac{\chi}{2}\right) P_{n_r}^{(a,b)}(\cos\chi) \tag{6.102}$$

and

$$Q(\chi) = B \sin^{\frac{1}{2}}(\chi) \sin^b\left(\frac{\chi}{2}\right) \cos^a\left(\frac{\chi}{2}\right) P_{n_r}^{(b,a)}(\cos\chi), \tag{6.103}$$

for two new constants A and B.

6.6 The solutions to the Dirac equation on the 3-sphere

We now have the particular solutions to the Dirac equation on the 3-sphere. They are given by

$$\boxed{\Psi = \frac{1}{a^{3/2}\sin\chi}\begin{pmatrix} P(\chi)\Omega^{\kappa}_{lm_j}(\theta,\varphi) \\ Q(\chi)\Omega^{-\kappa}_{lm_j}(\theta,\varphi)\end{pmatrix} \mathrm{e}^{\pm i\omega_\mathrm{F} t}} \tag{6.104}$$

where $\Omega^{\kappa}_{lm_j}$ are the spinor spherical harmonics and the radial functions are given in the previous section. This set of particular solutions can as in the case of the Klein-Gordon field, be expanded as a sum over the quantum numbers yielding the full general solution to the problem. We have not normalized the radial solutions but it can be done utilizing the orthogonality properties of the Jacobi polynomials. The Dirac field energies is quantized and given by

$$\boxed{\omega_{\mathrm{F}} = \sqrt{\frac{\left(n_{\mathrm{F}} + \frac{1}{2}\right)^2}{a^2} + m_{\mathrm{F}}^2}} \tag{6.105}$$

The last and full table of the quantum numbers that occurs in this thesis is then:

Quantum number	**Expression**	**Values**
l	l	$0, 1, 2, 3, \ldots$
m_l	$-l, -l+1, \ldots, l-1, l$	$\ldots, -2, -1, 0, 1, 2, \ldots$
n_r	n_r	$0, 1, 2, 3, \ldots$
n_{B}	$l + n_r + 1$	$1, 2, 3, \ldots$
m_s	m_s	$-\frac{1}{2}, \frac{1}{2}$
j	$l \pm \frac{1}{2}$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$
m_j	$-j, -j+1, \ldots, j-1, j$	$\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$
κ	$\mp\left(j + \frac{1}{2}\right)$	$\ldots, -2, -1, 1, 2, \ldots$
n_{F}	$\lvert\kappa\rvert + n_r$	$1, 2, 3, \ldots$

Table 6.2: 4th table of quantum numbers.

The energy levels ω_{F} is completely determined by the value of n_{F}. For each value of n_{F}, $|\kappa|$ ranges over the values

$$|\kappa| = 1, 2, \ldots, n_{\mathrm{F}}. \tag{6.106}$$

For each of these values there are two values of κ corresponding to the two spin states. For a given $|\kappa|$ there corresponds a value of j for which there are $(2j+1)$ or $2|\kappa|$ values of m_j. Combining all these possibilities yield the degeneracy, $d(n_{\mathrm{F}})$ for each energy level:

$$\boxed{d\left(n_{\mathrm{F}}\right) = \sum_{|\kappa|=1}^{n_{\mathrm{F}}} 4|\kappa| = 2n_{\mathrm{F}}\left(n_{\mathrm{F}} + 1\right)} \tag{6.107}$$

Chapter 7

Results

In this final chapter we will summarize and discuss the main results obtained in this thesis. We will refer to the Klein-Gordon field and the Dirac field simply as bosons and fermions, respectively.

7.1 Field equations on curved spacetime

The Klein-Gordon equation was obtained by adjusting the corresponding Lagrangian density, yielding the equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + \left(m_{\mathrm{B}}^2 + \xi\mathcal{R}\right)\phi = 0, \tag{7.1}$$

where the coupling to the scalar curvature of spacetime was included with the coupling constant ξ. We will discuss this coupling in more detail below. This equation was then written out for the specific FRW metric and then solved for $\mathbb{R}\times S^3$ spacetime. It turned out that the solutions came with a discrete spectrum of energies.

For the Dirac equation we had to find the covariant derivative corresponding to the spinor. This where found using the tetrad formalism and the equation could be written out as

$$ie^\mu{}_a\gamma^a\partial_\mu\Psi + \frac{1}{4}i\omega_{cab}\left(\eta^{ca}\gamma^b - \eta^{cb}\gamma^a\right)\Psi - \frac{1}{4}\epsilon^{abcd}\omega_{cab}\gamma_d\gamma^5\Psi - m_{\mathrm{F}}\Psi = 0. \tag{7.2}$$

This equation was greatly simplified by the means of a factorization ansatz. The ansatz considered to work in sufficiently symmetric spacetimes. As with the bosonic case we solved this equation in $\mathbb{R}\times S^3$ spacetime and found that the energies for the field were quantized.

7.2 Energies and degeneracies on the 3-sphere

In $\mathbb{R} \times S^3$ spacetime the bosons acquired the energy states

$$\omega_{\mathrm{B}} = \sqrt{\frac{(n_{\mathrm{B}}^2 - 1) + 6\xi}{a^2} + m_{\mathrm{B}}^2} \tag{7.3}$$

with the state degeneracy

$$d\left(n_{\mathrm{B}}\right) = \sum_{l=0}^{n_{\mathrm{B}}-1} (2l+1) = n_{\mathrm{B}}^2. \tag{7.4}$$

Here the bosonic quantum number takes the values $n_{\mathrm{B}} = 1, 2, 3, \ldots$. If we had considered a charged scalar field, the degeneracies had been doubled.

Fermions on the other hand was subject to the energies

$$\omega_{\mathrm{F}} = \sqrt{\frac{\left(n_{\mathrm{F}} + \frac{1}{2}\right)^2}{a^2} + m_{\mathrm{F}}^2} \tag{7.5}$$

and degeneracies

$$d\left(n_{\mathrm{F}}\right) = \sum_{|\kappa|=1}^{n_{\mathrm{F}}} 4|\kappa| = 2n_{\mathrm{F}}\left(n_{\mathrm{F}} + 1\right). \tag{7.6}$$

We note that the fermionic quantum number n_{F} take on the same values as the bosonic.

There is no peculiarities about the discrete quantization of the energies for both bosons and fermions in the closed FRW Universe. In this case, space has a finite volume and the wave solutions of the field equations are subject to periodic boundary conditions.

We will now explore the correlation between the bosonic degrees of freedom and the fermionic degrees of freedom (including charged scalar fields). For fermions and bosons we plot the degeneracy versus the energy value (units are irrelevant here).

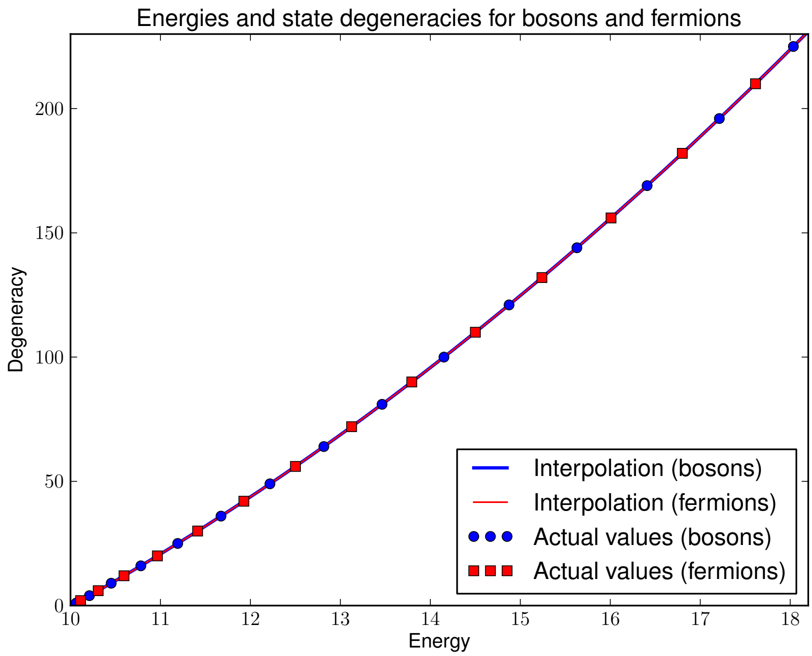

Figure 7.1: Energies and state degeneracies for bosons and fermions for $m_{\mathrm{B}} = m_{\mathrm{F}} = 10$, $a = 1$ and $\xi = 5/24$ (the factors of 2 in the degeneracies have been omitted).

We observe from this plot an intimate relation between the bosonic and fermionic degrees of freedom. Although the energies does not coincide, they follow the same curve. Indeed the interpolation of bosons and fermions coincide for the specific value of the scalar curvature coupling constant, namely $\xi = 5/24$, as opposed to the canonical value $\xi = 1/6 = 4/24$.

7.3 Conclusions and outlook

We conclude that there are some correlation between the bosonic and the fermionic degrees of freedom on the 3-sphere. The energy degeneracies follow the same trajectory when plotted against energy values for a specific value of the scalar curvature coupling constant of the scalar field. This can therefore be seen loosely as a theoretical determination of this constant.

There are a lot of extensions to this work. First and foremost will be the transition to quantum field theory, rather than quantum mechanics. Second will be the inclusion of a time dependent scale factor, with dynamic spacetime rather than static.

Appendix A

Singular Points and the Hypergeometric Differential Equation

In this appendix we are going to briefly discuss some concepts regarding ODEs. These concepts being singular points, the Frobenius method and indices. We will also discuss some properties of the hypergeometric differential equation. The topics discussed here will be relevant for the thesis at hand.

A.1 Singular points

A second order ODE of the form

$$p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \tag{A.1}$$

is said to have a *singular point* at x_0 if $p_2(x_0) = 0$. That singular point is called *regular* if the following limits exists and are finite:

$$\left.\begin{aligned} \lim_{x\to x_0} (x - x_0)\frac{p_1(x)}{p_2(x)} &= c_1, \\ \lim_{x\to x_0} (x - x_0)^2\frac{p_0(x)}{p_2(x)} &= c_2, \end{aligned}\right\} \tag{A.2}$$

where c_1 and c_2 are finite. If this is not the case, then the singular point x_0 is called *irregular*. To classify $x_0 = \infty$ we make the substitution $x = \frac{1}{z}$ and study the point

$z_0 = 0$.

An ODE whose all singular points are regular is called a *Fuchsian differential equation*. Any second order Fuchsian differential equation can, after suitable substitutions, be written as a hypergeometric differential equation.

A.2 The Frobenius method and indices

If an ODE of the form (A.1) has one or more regular singular points, there is a method of finding a series solution to the equation around that singular point. This method is known as the Frobenius method [5]. The idea is to write the series solution of the form

$$y = (x - x_0)^{\mu} \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{A.3}$$

around the regular singular point x_0, and require that $a_0 \neq 0$.

We are now going to consider a second order ODE. Calculating the first and second derivative of y yields

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\mu},$$

$$y' = \sum_{n=0}^{\infty} a_n (n + \mu) (x - x_0)^{n+\mu-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n + \mu)(n + \mu - 1) (x - x_0)^{n+\mu-2}.$$

Inserting this into (A.1) and dividing through $p_2(x)$ gives

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \bigg\{ (n + \mu)(n + \mu - 1) + (x - x_0) \frac{p_1(x)}{p_2(x)} (n + \mu) \\ + (x - x_0)^2 \frac{p_0(x)}{p_2(x)} \bigg\} (x - x_0)^{n+\mu-2} = 0. \end{aligned} \tag{A.4}$$

We now see the reason for defining regular singular points the way we have done. In the limit $x \to x_0$ the expression in the square bracket will be finite for finite n. An equation for μ is given if we write out the $n = 0$ term and require that the coefficient is zero. This will be the most singular term. In the limit of $x \to x_0$ this gives us what is called the *indicial equation* for μ around $x = x_0$. From the limits (A.2) we get that, for a second order ODE, the indicial equation for the regular singular point x_0 is given by

$$\mu(\mu - 1) + c_1\mu + c_2 = 0. \tag{A.5}$$

Here we have remembered that by definition, $a_0 \neq 0$. This equation will have two roots in general, if we count multiplicity. Lets call them $\mu^{(+)}$ and $\mu^{(-)}$. These are the indices of x_0.

The case of regular singular points at 0, 1 and ∞

There is a useful way of organizing the regular singular points together with there indices. In the case where $x_0 = 0, 1, \infty$ consider the following tableau:

$$\mathcal{T}(y) = \begin{Bmatrix} 0 & 1 & \infty \\ \mu^{(+)} & \nu^{(+)} & \lambda^{(+)} \\ \mu^{(-)} & \nu^{(-)} & \lambda^{(-)} \end{Bmatrix}. \tag{A.6}$$

Here the regular singular points are listed in the first row, with their corresponding indices in each column. Of course this tableau generalizes to arbitrary regular singular points and arbitrary order of the ODE, but the case of $x_0 = 0, 1, \infty$ will be of special importance to us.

Factorizations of the form

$$z(x) = \prod_i (x - x_i)^{\alpha_i}\, y(x), \tag{A.7}$$

where i goes through the points $x_i = 0, 1$, are of special importance. Expressing $y(x)$ in terms of $z(x)$ we get a new ODE for $z(x)$ whose indices have been shifted compared to the ODE for $y(x)$. The new tableau for $z(x)$ will be:

$$\mathcal{T}(z) = \begin{Bmatrix} 0 & 1 & \infty \\ \mu^{(+)} + \alpha_0 & \nu^{(+)} + \alpha_1 & \lambda^{(+)} - \sum_i \alpha_i \\ \mu^{(-)} + \alpha_0 & \nu^{(-)} + \alpha_1 & \lambda^{(-)} - \sum_i \alpha_i \end{Bmatrix}. \tag{A.8}$$

Since the indicial tableau completely characterizes the differential equation, we now have a method of substitution that allows us to transform one ODE into another with a known tableau. This is important if we want to know if a particular ODE can be written as say, the hypergeometric differential equation, which will be the case in this thesis. So lets take a look now at some properties of the hypergeometric differential equation.

A.3 The hypergeometric differential equation

The following second order ODE is called the *hypergeometric differential equation*:

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]\, y' - \alpha\beta y = 0. \tag{A.9}$$

Here α, β and γ are complex constants, and the nature of these constants determine what kind of solutions this equation will have. This ODE has regular singular points at $x_0 = 0, 1, \infty$, with the following tableau of indices:

$$\mathcal{T}(\text{hypergeometric}) = \begin{Bmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{Bmatrix}. \tag{A.10}$$

The solution of (A.9) is constructed out of the *hypergeometric series* (or function)[2]. Around the point $x = 0$, we will have the solution

$$y = C\,{}_2F_1\left(\alpha, \beta; \gamma; x\right) + Dx^{1-\gamma}\,{}_2F_1\left(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; x\right). \tag{A.11}$$

Here ${}_2F_1\left(\alpha, \beta; \gamma; x\right)$ is the hypergeometric series, defined as

$${}_2F_1\left(\alpha, \beta; \gamma; x\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)} \frac{x^m}{m!}, \tag{A.12}$$

where $|x| < 1$ and $\gamma \neq 0, -1, -2, \ldots$. To make notation easier we can introduce the *Pochhammer symbol*:

$$(\alpha)_m \equiv \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}. \tag{A.13}$$

The hypergeometric function around the point $x = 0$ is related to the hypergeometric function around $x = 1$ in the following way

$${}_2F_1\left(\alpha, \beta; \gamma; x\right) = \left[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\right] {}_2F_1\left(\alpha, \beta; \alpha+\beta-\gamma+1; 1-x\right) +$$

$$(1-x)^{\gamma-\alpha-\beta} \left[\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times$$

$${}_2F_1\left(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-x\right). \tag{A.14}$$

In the case where α or β is a negative integer $-n = 0, -1, -2, \ldots$, the hypergeometric series reduces to a polynomial of degree n.

Let's assume that $\beta = -n$, then there is two cases that we will encounter in this thesis:

1. If we for some a can write $\alpha = n + 2a$ and $\gamma = a + \frac{1}{2}$, then the hypergeometric function is reduced to

$$ {}_2F_1\left(n + 2a, -n; a + \frac{1}{2}; x\right) = \frac{n!}{(2a)_n} C_n^{(a)}(1 - 2x). \tag{A.15} $$

   The polynomials $C_n^{(a)}$ is the n'th order *Gegenbauer polynomials*. These obey the following orthogonality relation:

$$ \int_{-1}^{1} \left(1 - x^2\right)^{a-\frac{1}{2}} C_n^{(a)}(x) C_{n'}^{(a)}(x) dx = \frac{\pi^{1-2a}\Gamma(n + 2a)}{n!(n + a)\left[\Gamma(a)\right]^2} \delta_{nn'}, \tag{A.16} $$

   where $a \neq 0$ and $a > -\frac{1}{2}$.

2. If we for some a and b can write $\alpha = a + 1 + b + n$ and $\gamma = a + 1$, then the hypergeometric function is reduced to

$$ {}_2F_1\left(a + 1 + b + n, -n; a + 1; x\right) = \frac{n!}{(a + 1)_n} P_n^{(a,b)}(1 - 2x). \tag{A.17} $$

   The polynomials $P_n^{(a,b)}$ is the n'th order *Jacobi polynomials*. These obey the following orthogonality relation:

$$ \begin{aligned} \int_{-1}^{1} (1 - x)^a (1 + x)^b P_n^{(a,b)}(x) P_{n'}^{(a,b)}(x) dx =& \frac{2^{a+b+1}}{2n + a + b + 1} \times \\ & \frac{\Gamma(n + a + 1)\Gamma(n + b + 1)}{n!\Gamma(n + a + b + 1)} \delta_{nn'}, \end{aligned} \tag{A.18} $$

   where $a, b > -1$.

Appendix B

Maple Calculations

Here we will list Maple code for various calculations.

B.1 The spin connection and the Ricci scalar for the FRW metric

Loading the tensor package and defining for the FRW metric, $g_{\mu\nu}$, $g^{\mu\nu}$, $e_{\mu}{}^{a}$, $e^{\mu}{}_{a}$, together with the Minkowski metric:

```
>  with(tensor):
>  coord:=[t,r,theta,phi]:
>  g_compts:=array(symmetric,sparse,1..4,1..4):
>  g_compts[1,1] := 1:
>  g_compts[2,2] := -(a(t)^2)/(1-k*r^2):
>  g_compts[3,3] := -a(t)^2*r^2:
>  g_compts[4,4] := -a(t)^2*r^2*sin(theta)^2:
>  g := create( [-1,-1], eval(g_compts)):
>  ginv:= invert (g, $'$detg'):
>  e_compts:=array(symmetric,sparse,1..4,1..4):
>  e_compts[1,1] := 1:
>  e_compts[2,2] := a(t)/sqrt(1-k*r^2):
>  e_compts[3,3] := a(t)*r:
>  e_compts[4,4] := a(t)*r*sin(theta):
>  e := create( [-1,1], eval(e_compts)):
>  einv := invert( e, 'dete'):
>  eta_compts:=array(symmetric,sparse,1..4,1..4):
>  eta_compts[1,1] := 1:
>  eta_compts[2,2] := -1:
```

```
> eta_compts[3,3] := -1:
> eta_compts[4,4] := -1:
> eta := create( [-1,-1], eval(eta_compts)):
```

Calculation of the Christoffel symbols of first and second kind:

```
> D1g:= d1metric (g, coord):
> Cf1:= Christoffel1 (D1g):
> `tensor/Christoffel2/simp`:= proc(x)
> simplify(x, trig) end proc:
> Cf2:= Christoffel2 (ginv, Cf1):
```

Calculating the Ricci scalar and printing it in LaTeX-code:

```
> D2g := d2metric ( D1g, coord ):
> RMN := Riemann( ginv, D2g, Cf1 ):
> RICCI := Ricci( ginv, RMN ):
> RS := Ricciscalar( ginv, RICCI ):
> latex( 'RS' );
```

$\mathtt{table}([index\_char = [], compts = 6\,\frac{a(t)\,\frac{d^2}{dt^2}a(t) + \left(\frac{d}{dt}a(t)\right)^2 + k}{(a(t))^2}]).$

Calculating the term $e_\nu{}^a e^\sigma{}_b \Gamma^\nu{}_{\sigma\mu}$ in $\omega_\mu{}^a{}_b$:

```
> Cf2local := prod(Cf2,e,[1,1]):
> first := prod(Cf2local,einv,[1,1]):
```

Calculating the term $e_\nu{}^a \partial_\mu e^\nu{}_b$ in $\omega_\mu{}^a{}_b$:

```
> part_einv := partial_diff(einv, coord):
> second := prod(e,part_einv,[1,1]):
> permutesecond := permute_indices(second,[3,1,2]):
```

Forming $\omega_\mu{}^a{}_b$ and then calculating ω_{cab}. Finally we print the components of ω_{cab} in LaTeX-code:

```
> omega := lin_com(first, permutesecond):
> omegalower := lower(eta, omega,2):
> simplomegalower := simplify( omegalower, 'symbolic' ):
> omegalocal := prod(einv, simplomegalower, [1,1]):
> latex( 'omegalocal' ):
```

$\texttt{table}([index\_char = [-1,-1,-1], compts = \texttt{array}([4,4,1 = -\frac{\frac{d}{dt}a(t)}{a(t)}, 4,4,4 = 0, 3,2,2 = 0, 3,4,3 = 0, 1,4,3 = 0, 3,1,3 = \frac{\frac{d}{dt}a(t)}{a(t)}, 2,2,4 = 0, 3,3,3 = 0, 2,1,3 = 0, 3,2,1 = 0, 1,2,3 = 0, 2,1,2 = \frac{\frac{d}{dt}a(t)}{a(t)}, 1,1,1 = 0, 1,4,4 = 0, 2,3,1 = 0, 3,1,2 = 0, 4,2,3 = 0, 1,1,3 = 0, 2,4,3 = 0, 3,4,4 = 0, 1,3,1 = 0, 4,3,1 = 0, 2,3,4 = 0, 1,2,4 = 0, 2,1,1 = 0, 2,4,1 = 0, 2,1,4 = 0, 1,3,3 = 0, 1,4,1 = 0, 1,1,4 = 0, 1,3,4 = 0, 4,1,3 = 0, 4,2,1 = 0, 2,4,4 = 0, 4,4,2 = -\frac{\sqrt{1-kr^2}}{a(t)r}, 2,2,2 = 0, 3,3,4 = 0, 4,1,2 = 0, 1,3,2 = 0, 4,2,4 = \frac{\sqrt{1-kr^2}}{a(t)r}, 3,1,4 = 0, 2,4,2 = 0, 3,2,3 = \frac{\sqrt{1-kr^2}}{a(t)r}, 1,4,2 = 0, 2,2,3 = 0, 3,2,4 = 0, 4,3,3 = 0, 2,2,1 = -\frac{\frac{d}{dt}a(t)}{a(t)}, 2,3,2 = 0, 3,4,2 = 0, 1,2,1 = 0, 3,3,1 = -\frac{\frac{d}{dt}a(t)}{a(t)}, 4,3,4 = \frac{\cos(\theta)}{a(t)r\sin(\theta)}, 3,1,1 = 0, 1,1,2 = 0, 1,2,2 = 0, 4,1,4 = \frac{\frac{d}{dt}a(t)}{a(t)}, 4,2,2 = 0, 3,4,1 = 0, 4,4,3 = -\frac{\cos(\theta)}{a(t)r\sin(\theta)}, 2,3,3 = 0, 4,1,1 = 0, 3,3,2 = -\frac{\sqrt{1-kr^2}}{a(t)r}, 4,3,2 = 0]))$.

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