GLOBAL DISSIPATIVE SOLUTIONS OF THE TWO-COMPONENT CAMASSA–HOLM SYSTEM FOR INITIAL DATA WITH NONVANISHING ASYMPTOTICS

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Abstract. We show existence of a global weak dissipative solution of the Cauchy problem for the two-component Camassa–Holm (2CH) system on the line with nonvanishing and distinct spatial asymptotics. The influence from the second component in the 2CH system on the regularity of the solution, and, in particular, the consequences for wave breaking, is discussed. Furthermore, the interplay between dissipative and conservative solutions is treated.

1. Introduction

We show existence of a weak global dissipative solution of the Cauchy problem for the two-component Camassa–Holm (2CH) system with arbitrary $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$, given by

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x = 0, \tag{1.1a}$$

$$\rho_t + (u\rho)_x = 0, \tag{1.1b}$$

with initial data $u|_{t=0} = u_0$ and $\rho|_{t=0} = \rho_0$. The initial data may have nonvanishing limits at infinity, that is,

$$\lim_{x\to\pm\infty} u_0(x) = u_{\pm\infty} \quad \text{and} \quad \lim_{x\to\pm\infty} \rho_0(x) = \rho_\infty. \tag{1.2}$$

The 2CH system was first analyzed by Constantin and Ivanov [14]. Global existence, well-posedness and blow-up conditions have been further studied in a Sobolev setting in [16, 17] and in Besov spaces in [18]. The scalar CH equation (i.e., with ρ identically equal to zero), was introduced and studied by Camassa and Holm in the fundamental paper [5], see also [6], and its analysis has been pervasive.

The CH equation possesses many intriguing properties. Here we concentrate on global solutions for the Cauchy problem on the line. The challenge is that the CH equation experiences blow-up in finite time, even for smooth initial data, in the sense that the $H^1_{\rm loc}$ norm of the solution remains finite while u_x blows up. Continuation of the solution past blow-up is intricate. It has turned out to be two distinct ways to continue the solution past blow-up, denoted conservative and dissipative solutions, respectively. Conservative solutions are associated with preservation of the H^1 norm, while dissipative solutions are characterized by a sudden drop in H^1

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norm at blow-up. This dichotomy has consequences for the well-posedness of the initial value problem as the two solutions coincide prior to blow-up. Here we focus on the dissipative case.

Three features are novel in this paper: First of all we include distinct and non-vanishing asymptotics of the initial data, and hence of the solution, at infinity. Since prior work has been on solutions in H^1, this has considerable consequences for the analysis. Secondly, we extend previous results for the CH equation to the 2CH system. It is not at all clear a priori that the highly tailored construction for the CH equation extends to the 2CH system. Finally, we greatly simplify the analysis of two of us [32] even in the scalar case of the CH equation with vanishing asymptotics. One advantage of the present approach is that we can use the same change of variables as in the conservative case, in contrast to the approach chosen in [32]. We reformulate the 2CH system in terms of Lagrangian coordinates, and in this respect it relates to [3, 4, 31, 32, 20] for the CH equation. Previous work on the CH equation, covering also the periodic case, includes, e.g., [10, 11, 12, 13, 15, 34, 35, 7, 8, 19, 29, 23]. See also [25].

The intricate problems regarding wave breaking can best be exemplified in the context of multipeakon solutions of the CH equation with $\kappa = 0$. For reasons of brevity, and since this example has been discussed in detail in [32], we omit the discussion here. For additional theory on multipeakons, see [1, 2, 33, 27, 30].

The continuation of the solution past wave breaking has been studied both in the conservative [3, 31] and dissipative [4, 32] case. In both cases the approach has been to reformulate the partial differential equation as a system of Banach space-valued ordinary differential equations, and we follow that approach here. A different approach, based on vanishing viscosity, has been advocated in [34, 35].

If we for a moment assume vanishing asymptotics, the dichotomy can be further understood if one considers the associated energy, that is, the H^1 norm of the solution u for the CH equation. In the case of a symmetric antipeakon-peakon collision, the H^1 norm is constant prior to wave breaking. At collision time it vanishes, and remains zero for dissipative solutions, while returning to the previous value in the conservative case. Thus we need to keep the information about the energy in the conservative case and this is handled by augmenting the solution with the energy. More precisely, we consider as solution the pair (u, μ) where μ is a Radon measure with absolute continuous part $\mu_{\mathrm{ac}} = u_x^2 dx$. This allows for energy concentration in terms of Dirac masses, while keeping the information about the energy. On the other hand, in the dissipative case, energy is not preserved, rather it is strictly decreasing at wave breaking. The extension from scalar CH equation to the two-component 2CH system follows by augmenting the Lagrangian reformulation by an additional variable.

Let us now turn to a more detailed description of the results in this paper. First we observe that we can put $\kappa = 0$ and $\eta = 1$ since if (u, ρ) solves (1.1), then (v, τ) with $v(t, x) = u(t, x - \kappa t/2) + \kappa/2$ and $\tau(t, x) = \sqrt{\eta}\rho(t, x - \kappa t/2)$ will solve (1.1) with $\kappa = 0$ and $\eta = 1$. Note that this only applies since we allow for non decaying initial data at infinity. Furthermore, we assume that $u_{-\infty} = 0$. We reformulate the 2CH system as

$$
\begin{aligned}
u_t + uu_x + P_x &= 0, \\
\rho_t + (u\rho)_x &= 0,
\end{aligned}
$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2.$$

Next we introduce the characteristics $y(t,\xi)$, that is, the solution of $y_t(t,\xi) = u(t,y(t,\xi))$ for a given u and initial data $y(0,\xi)$. The Lagrangian velocity is given by $U(t,\xi) = u(t,y(t,\xi))$. As long as $y(t,\xi)$ remains strictly increasing as a function of ξ, the solution remains smooth, and, in particular, conservative and dissipative solutions coincide. Thus we introduce the time for wave breaking, or energy dissipation, by

$$\tau(\xi) = \begin{cases} 0, & \text{if } y_\xi(0,\xi) = 0, \\ \sup\{t \in \mathbb{R}_+ \mid y_\xi(t',\xi) > 0 \text{ for all } 0 \le t' < t\}, & \text{otherwise.} \end{cases} \tag{1.3}$$

We can rewrite the full 2CH system as a system of ordinary differential equations. First define

$$\begin{aligned} h &= u_x^2 \circ y y_\xi + \bar\rho^2 \circ y y_\xi, \\ \bar r &= \bar\rho \circ y y_\xi, \\ U &= \bar U + c\chi(y), \\ r &= \bar r + k y_\xi, \end{aligned}$$

where $\rho = \bar\rho + k$ with $\bar\rho \in L^2(\mathbb{R})$ and $u = \bar u + c\chi$ with $\bar u \in H^1(\mathbb{R})$. In addition, $\bar U = \bar u \circ y$. The function χ is a smooth increasing function that vanishes for large negative arguments and equals one for large positive arguments.

Next we find that the system obeys the following system of ordinary differential equations

$$\begin{aligned} y_t &= U, \quad U_t = -Q(X), \\ y_{t,\xi} &= \chi_{\{\tau(\xi)>t\}} U_\xi, \\ U_{t,\xi} &= \chi_{\{\tau(\xi)>t\}} \left(\frac{1}{2}h + (U^2 + \frac{1}{2}k^2 - P(X))y_\xi + k\bar r \right), \\ h_t &= \chi_{\{\tau(\xi)>t\}} 2(U^2 + \frac{1}{2}k^2 - P(X))U_\xi, \\ \bar r_t &= \chi_{\{\tau(\xi)>t\}} \left(-kU_\xi\right), \\ c_t &= 0, \\ k_t &= 0, \end{aligned}$$

where $P(X) - U^2 - \frac{1}{2}k^2$ and $Q(X)$ are given by (3.17) and (3.18), respectively (observe the subtle modifications in the dissipative case compared with (3.14) and (3.8), respectively). Introduce q and w for y_ξ and U_ξ, respectively. We find that $X = (\bar U, c, q, w, h, \bar r, k)$ satisfies the system $X_t = \chi_{\{\tau(\xi)>t\}} F(X)$. The function X takes values in a specific Banach space $\bar V$, see (3.19). This system poses two challenges: First of all, due to wave breaking, the right-hand side is discontinuous, and thus existence and uniqueness of solutions cannot follow the standard path. This is the key difficulty compared with the conservative case. Secondly, the system possesses a number of additional constraints in order to be consistent with the original Eulerian formulation. For example, we need to make sure that $q = y_\xi$ and $w = U_\xi$ are satisfied for positive t. We will also need $y_\xi h = U_\xi^2 + \bar r^2$. This is secured by only using initial data from a carefully selected set that makes sure that all additional requirements are preserved for the solution. The set is denoted

$\mathcal{G}$ and is given in Definition 3.2. A key result is the proof of global existence of a solution, $X(t) = S_t(X_0)$ with initial data X_0, in the Lagrangian variables, Theorem 3.13. Next we have to analyze stability of the solution in Lagrangian coordinates. The problem here is to identify a metric that separates conservative solutions (say $X^c(t)$) and dissipative solutions (say $X^d(t)$) near wave breaking. The key is the behavior of the derivative of the characteristics, y_ξ. At wave breaking y_ξ vanishes. For dissipative solutions y_ξ remains constant for all later times, while for the conservative solutions it becomes positive immediately after. For conservative solutions, the metric is induced by the Euclidean norm of the Banach space $\bar{V}$. For dissipative solutions, we need a metric which in addition can separate X^d and X^c after collision. Indeed, if we denote by t_c the collision time, the distance between $X^d(t_c+\epsilon,\xi)$ and $X^c(t_c+\epsilon,\xi)$ with respect with this metric has to be big, for any $\varepsilon > 0$. This is taken care of by the introduction of a function g in Definition 3.1, and the corresponding metric $d_{\mathbb{R}}$ in Definition 4.1. The metric yields the flow Lipschitz continuous in the sense that

$$d_{\mathbb{R}}(X(t), X(\bar{t})) \leq C_T(M)\,|\bar{t}-t|\,, \quad t, \bar{t} \leq T, \tag{1.4}$$

for initial data in the set B_M, given by (3.28), see Lemma 4.9.

Having obtained the solution in Lagrangian coordinates, the next task is to transfer the solution back to Eulerian variables (u,ρ). Here we are confronted with relabeling issue; there are several distinct solutions in Lagrangian variables corresponding to one and the same Eulerian functions (u,ρ). This is the reminiscent of the fact that there are many distinct ways to parametrize the graph of a given function. We identify the functions that give the same Eulerian solution, and show that the semigroup in Lagrangian coordinates respects the relabeling in the sense that $S_t(X \circ f) = S_t(X) \circ f$ where S_t denotes the semigroup of solutions and f denotes the label. We define, for any ξ such that $x = y(\xi)$ (cf. Theorem 5.11),

$$\begin{aligned} u(x) &= U(\xi), \\ \mu &= y_\#(h(\xi)\,d\xi), \\ \bar{\rho}(x)\,dx &= y_\#(\bar{r}(\xi)\,d\xi), \\ \rho(x) &= k + \bar{\rho}(x). \end{aligned}$$

We measure the distance between two Eulerian solutions by their corresponding Lagrangian distance, see (6.1).

The interplay between dissipative and conservative solutions is interesting. As shown in [14, 22] a positive density ρ_0 regularizes the function u. If ρ_0 is positive on the whole line, no wave breaking will take place, and conservative and dissipative solutions coincide. On the other hand, if ρ_0 is identically zero, then u will satisfy the scalar CH equation, and we will have wave breaking generically. A local version of this result is that if ρ_0 is positive on an interval, then the solution will remain regular, i.e., no wave breaking will take place in the interval bounded by the corresponding characteristics. In [22] we have shown that one can obtain conservative solutions of the CH equation by considering solutions of the 2CH system with positive density ρ. If one lets the initial density approach zero appropriately, then the solution u will converge to the conservative solution of the CH equation. However, we do show continuity results for dissipative solutions of the 2CH system, cf. Lemmas 7.3 and 7.4. These results are discussed in Section 7.

2. Eulerian setting

We consider the Cauchy problem for the two component Camassa–Holm system with arbitrary $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$, given by

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x = 0, \tag{2.1a}$$

$$\rho_t + (u\rho)_x = 0, \tag{2.1b}$$

with initial data $u|_{t=0} = u_0$ and $\rho|_{t=0} = \rho_0$. We are interested in global solutions for initial data u_0 with nonvanishing and possibly distinct limits at infinity, that is,

$$\lim_{x\to-\infty} u_0(x) = u_{-\infty} \quad \text{and} \quad \lim_{x\to\infty} u_0(x) = u_\infty. \tag{2.2}$$

Furthermore we assume that the initial density has equal asymptotics which need not to be zero, that is,

$$\lim_{x\to\pm\infty} \rho_0(x) = \rho_\infty. \tag{2.3}$$

More precisely, we introduce the spaces

(2.4)

$$H_\infty(\mathbb{R}) = \{v \in H^1_{\text{loc}}(\mathbb{R}) \mid v(x) = \bar{v}(x) + v_{-\infty}\chi(-x) + v_\infty\chi(x),\ \bar{v} \in H^1(\mathbb{R}), v_{\pm\infty} \in \mathbb{R}\},$$

where χ denotes a smooth partition function with support in $[0, \infty)$ such that $\chi(x) = 1$ for $x \geq 1$ and $\chi'(x) \geq 0$ for $x \in \mathbb{R}$, and

$$L^2_{\text{const}}(\mathbb{R}) = \{g \in L^1_{\text{loc}}(\mathbb{R}) \mid g(x) = g_\infty + \bar{g}(x),\ \bar{g} \in L^2(\mathbb{R}), g_\infty \in \mathbb{R}\}. \tag{2.5}$$

Subsequently, we will assume that

$$u_0 \in H_\infty(\mathbb{R}), \quad \rho_0 \in L^2_{\text{const}}(\mathbb{R}). \tag{2.6}$$

Introducing the mapping I_χ from $H^1(\mathbb{R}) \times \mathbb{R}^2$ into $H^1_{\text{loc}}(\mathbb{R})$ given by

$$I_\chi(\bar{u}, c_-, c_+)(x) = \bar{u}(x) + c_-\chi(-x) + c_+\chi(x)$$

for any $(\bar{u}, c_-, c_+) \in H^1(\mathbb{R}) \times \mathbb{R}^2$, yields that any initial condition $u_0 \in H_\infty(\mathbb{R})$ is defined by an element in $H^1(\mathbb{R}) \times \mathbb{R}^2$ through the mapping I_χ. Hence we see that $H_\infty(\mathbb{R})$ is the image of $H^1(\mathbb{R}) \times \mathbb{R}^2$ by I_χ, that is, $H_\infty(\mathbb{R}) = I_\chi(H^1(\mathbb{R}) \times \mathbb{R}^2)$. The linear mapping I_χ is injective. We equip $H_\infty(\mathbb{R})$ with the norm

$$\|u\|_{H_\infty} = \|\bar{u}\|_{H^1} + |c_-| + |c_+| \tag{2.7}$$

where $u = I_\chi(\bar{u}, c_-, c_+)$. Then $H_\infty(\mathbb{R})$ is a Banach space. Given another partition function $\tilde{\chi}$, we define the mapping $(\tilde{\bar{u}}, \tilde{c}_-, \tilde{c}_+) = \Psi(\bar{u}, c_-, c_+)$ from $H^1(\mathbb{R}) \times \mathbb{R}^2$ to $H^1(\mathbb{R}) \times \mathbb{R}^2$ as $\tilde{c}_- = c_-$, $\tilde{c}_+ = c_+$ and

$$\tilde{\bar{u}}(x) = \bar{u}(x) + c_-(\chi(-x) - \tilde{\chi}(-x)) + c_+(\chi(x) - \tilde{\chi}(x)). \tag{2.8}$$

The linear mapping Ψ is a continuous bijection. Since

$$I_\chi = I_{\tilde{\chi}} \circ \Psi,$$

we can see that the definition of the Banach space $H_\infty(\mathbb{R})$ does not depend on the choice of the partition function χ. The norm defined by (2.7) for different partition functions χ are all equivalent.

Similarly, one can associate to any element $\rho \in L^2_{\text{const}}(\mathbb{R})$ the unique pair $(\bar{\rho}, k) \in L^2(\mathbb{R}) \times \mathbb{R}$ through the mapping J from $L^2(\mathbb{R}) \times \mathbb{R}$ to $L^2_{\text{const}}(\mathbb{R})$ which is defined as

$$J(\bar{\rho}, k) = \bar{\rho} + k. \tag{2.9}$$

In fact J is bijective from $L^2(\mathbb{R})\times\mathbb{R}$ to $L^2_{\rm const}(\mathbb{R})$, which allows us to equip $L^2_{\rm const}(\mathbb{R})$ with the norm

$$\|\rho\|_{L^2_{\rm const}} = \|\bar\rho\|_{L^2} + |k|, \tag{2.10}$$

where we decomposed ρ according to $\rho = J(\bar\rho, k)$. Thus $L^2_{\rm const}(\mathbb{R})$ together with the norm defined in (2.10) is a Banach space.

Note that for smooth solutions, we have the following conservation law

$$(u^2+u_x^2+\eta\rho^2)_t + (u(u^2+u_x^2+\eta\rho^2))_x = (u^3+\kappa u^2 - 2Pu)_x. \tag{2.11}$$

Moreover, if $(u(t,x),\rho(t,x))$ is a solutions of the two-component Camassa–Holm system (2.1), then, for any constant $\alpha\in\mathbb{R}$ we easily find that

$$v(t,x) = u(t,x-\alpha t)+\alpha, \quad \text{and} \quad \tau(t,x) = \sqrt{\eta}\rho(t,x-\alpha t), \tag{2.12}$$

solves the two-component Camassa–Holm system with κ replaced by $\kappa - 2\alpha$ and $\eta = 1$. Therefore, without loss of generality, we assume in what follows, that $\lim_{x\to-\infty} u_0(x) = 0$ and $\eta = 1$. In addition, we only consider the case $\kappa = 0$ as one can make the same conclusions for $\kappa \neq 0$ with slight modifications.

3. Lagrangian setting

In this section we will introduce the set of Lagrangian coordinates we want to work with and the corresponding Banach spaces.

3.1. **Reformulation of the 2CH system in Lagrangian coordinates.**

The 2CH system with $\kappa = 0$ can be rewritten as the following system[1]

$$u_t + uu_x + P_x = 0, \tag{3.1a}$$

$$\rho_t + (u\rho)_x = 0, \tag{3.1b}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2, \tag{3.1c}$$

where $P - u^2 - \frac{1}{2}k^2$ and P_x are given by

(3.2)

$$\begin{aligned} P(t,x) - u^2(t,x) - \frac{1}{2}k^2 &= -2c\chi(x)\bar u(t,x) - \bar u^2(t,x) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(2c\chi\bar u + \bar u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\bar\rho^2 + k\bar\rho)(t,z)dz \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}2c^2(\chi'^2 + \chi\chi'')(z)dz, \end{aligned}$$

and

$$\begin{aligned} P_x(t,x) &= 2c^2\chi\chi'(x) \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x-z)e^{-|x-z|}(2c\chi\bar u + \bar u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\bar\rho^2 + k\bar\rho)(t,z)dz \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(x-z)e^{-|x-z|}2c^2(\chi'^2+\chi\chi'')(z)dz. \end{aligned} \tag{3.3}$$

[1] For κ nonzero (3.1c) is simply replaced by $P - P_{xx} = u^2 + \kappa u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2$.

A close inspection of $P_x(t,x)$, like in [21], reveals that the asymptotic behavior has to be preserved. Thus we write here and later

$$u(t,x)=\bar{u}(t,x)+c\chi(x), \quad \bar{u}\in H^1(\mathbb{R}), \tag{3.4}$$

and

$$\rho(t,x)=\bar{\rho}(t,x)+k, \quad \rho\in L^2(\mathbb{R}). \tag{3.5}$$

Until wave breaking occurs for the first time every dissipative solution coincides with the conservative one, and hence they can be described in the same way. Therefore we summarize the derivation of the Lagrangian coordinates and the corresponding system of ordinary differential equations, which describe the conservative solutions here. For details we refer to [22]. Afterwards, in the next subsection, we will adapt the system describing the time evolution to the dissipative case.

Define the characteristics $y(t,\xi)$ as the solution of

$$y_t(t,\xi)=u(t,y(t,\xi)) \tag{3.6}$$

for a given $y(0,\xi)$. The Lagrangian velocity is given by $U(t,\xi)=u(t,y(t,\xi))$ and we find using (3.1a) that

$$U_t(t,\xi)=-Q(t,\xi), \tag{3.7}$$

where $Q(t,\xi)=P_x(t,\xi)$ is given by

$$\begin{aligned}Q(t,\xi)&=2c^2\chi(y(t,\xi))\chi'(y(t,\xi))\\&\quad-\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(\xi-\eta)e^{-|y(t,\xi)-y(t,\eta)|}(2c\chi\circ y\bar{U}y_\xi+\bar{U}^2y_\xi+\frac{1}{2}h+k\bar{r})(t,\eta)d\eta\\&\quad-\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(\xi-\eta)e^{-|y(t,\xi)-y(t,\eta)|}2c^2(\chi'^2+\chi\chi'')(y(t,\eta))y_\xi(t,\eta)d\eta\end{aligned} \tag{3.8}$$

where we have introduced

$$h(t,\xi)=u_x^2(t,y(t,\xi))y_\xi(t,\xi)+\bar{\rho}^2(t,y(t,\xi))y_\xi(t,\xi), \tag{3.9}$$

$$r(t,\xi)=\rho(t,y(t,\xi))y_\xi(t,\xi), \tag{3.10}$$

$$U(t,\xi)=\bar{U}(t,\xi)+c\chi(y(t,\xi)), \tag{3.11}$$

and

$$r(t,\xi)=\bar{r}(t,\xi)+ky_\xi(t,\xi). \tag{3.12}$$

The time evolution of $h(t,\xi)$ is given by

$$h_t(t,\xi)=2(U^2(t,\xi)+\frac{1}{2}k^2-P(t,\xi))U_\xi(t,\xi), \tag{3.13}$$

where, slightly abusing the notation, $P(t,\xi)-U^2(t,\xi)-\frac{1}{2}k^2=P(t,y(t,\xi))-U^2(t,\xi)-\frac{1}{2}k^2$ is given by

(3.14)
$$\begin{aligned}P(t,\xi)-U^2(t,\xi)-\frac{1}{2}k^2&\\ &=-2c\chi(y(t,\xi))\bar U(t,\xi)-\bar U^2(t,\xi)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}e^{-|y(t,\xi)-y(t,\eta)|}(2c\chi\circ y\bar U y_\xi+\bar U^2y_\xi+\frac{1}{2}h+k\bar r)(t,\eta)d\eta\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}e^{-|y(t,\xi)-y(t,\eta)|}2c^2(\chi'^2+\chi\chi'')(y(t,\eta))y_\xi(t,\eta)d\eta.\end{aligned}$$

Last but not least, according to (3.1b), $r(t,\xi)$ is preserved with respect to time, i.e., $r_t=0$.

3.2. **Necessary adaptations for dissipative solutions.** Wave breaking for the 2CH system means that u_x becomes unbounded, which is equivalent, in the Lagrangian setting, to saying that y_ξ becomes zero. Let therefore $\tau(\xi)$ be the first time when $y_\xi(t,\xi)$ vanishes, i.e.,

$$\tau(\xi)=\begin{cases}0, & \text{if } y_\xi(0,\xi)=0,\\ \sup\{t\in\mathbb{R}_+\mid y_\xi(t',\xi)>0 \text{ for all } 0\leq t'<t\}, & \text{otherwise,}\end{cases} \tag{3.15}$$

where $\mathbb{R}_+=[0,\infty)$. The dissipative solutions will then be described through the solutions of the following system of ordinary differential equations

$$y_t=U,\quad U_t=-Q(X), \tag{3.16a}$$
$$y_{t,\xi}=\chi_{\{\tau(\xi)>t\}}U_\xi, \tag{3.16b}$$
$$U_{t,\xi}=\chi_{\{\tau(\xi)>t\}}\left(\frac{1}{2}h+(U^2+\frac{1}{2}k^2-P(X))y_\xi+k\bar r\right), \tag{3.16c}$$
$$h_t=\chi_{\{\tau(\xi)>t\}}2(U^2+\frac{1}{2}k^2-P(X))U_\xi, \tag{3.16d}$$
$$\bar r_t=\chi_{\{\tau(\xi)>t\}}\left(-kU_\xi\right), \tag{3.16e}$$
$$c_t=0, \tag{3.16f}$$
$$k_t=0, \tag{3.16g}$$

where $P(X)-U^2-\frac{1}{2}k^2$ and $Q(X)$ are given by (observe the subtle modifications in the dissipative case compared with (3.14) and (3.8), respectively)

(3.17)
$$\begin{aligned}P(t,\xi)-U^2(t,\xi)-\frac{1}{2}k^2&\\ &=-2c\chi(y(t,\xi))\bar U(t,\xi)-\bar U^2(t,\xi)\\ &\quad+\frac{1}{2}\int_{\tau(\eta)>t}e^{-|y(t,\xi)-y(t,\eta)|}(2c\chi\circ y\bar U y_\xi+\bar U^2y_\xi+\frac{1}{2}h+k\bar r)(t,\eta)d\eta\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}e^{-|y(t,\xi)-y(t,\eta)|}2c^2(\chi'^2+\chi\chi'')(y(t,\eta))y_\xi(t,\eta)d\eta\end{aligned}$$

and

(3.18)

$Q(t,\xi)=2c^2\chi(y(t,\xi))\chi'(y(t,\xi))$

$$-\frac{1}{2}\int_{\tau(\eta)>t}\operatorname{sgn}(\xi-\eta)e^{-|y(t,\xi)-y(t,\eta)|}(2c\chi\circ y\bar{U}y_\xi+\bar{U}^2y_\xi+\frac{1}{2}h+k\bar{r})(t,\eta)d\eta$$
$$-\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(\xi-\eta)e^{-|y(t,\xi)-y(t,\eta)|}2c^2(\chi'^2+\chi\chi'')(y(t,\eta))y_\xi(t,\eta)d\eta,$$

respectively. (The integrals over the real line in (3.17) and (3.18) could be replaced by $\tau(\eta)>t$ since $y_\xi(t,\eta)=0$ outside this domain.)

We introduce the following notation for the Banach spaces we will often use. Let

$$E=L^2(\mathbb{R})\cap L^\infty(\mathbb{R}),$$

together with the norm

$$\|f\|_E=\|f\|_{L^2}+\|f\|_{L^\infty},$$

and

$$\begin{aligned}W&=L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R}),\\ \bar{W}&=E\times E\times E\times E,\\ V&=L^\infty(\mathbb{R})\times L^2(\mathbb{R})\times L^\infty(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^2(\mathbb{R})\times L^\infty(\mathbb{R}),\\ \bar{V}&=L^\infty(\mathbb{R})\times E\times L^\infty(\mathbb{R})\times E\times E\times E\times E\times L^\infty(\mathbb{R}).\end{aligned}\tag{3.19}$$

For any function $f\in C([0,T],B)$ for $T\geq 0$ and B a normed space, we denote

$$\|f\|_{L^1_TB}=\int_0^T\|f(t,\cdot)\|_B\,dt\quad\text{and}\quad\|f\|_{L^\infty_TB}=\sup_{t\in[0,T]}\|f(t,\cdot)\|_B.$$

Definition 3.1. *For $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{R}^8$, we define the functions $g_1,g_2,g\colon\mathbb{R}^8\to\mathbb{R}$ by*

$$\begin{aligned}g_1(x)&=|x_5|+2|x_7x_8|+2x_4,\\ g_2(x)&=x_4+x_6\end{aligned}$$

and

$$g(x)=\begin{cases}g_1(x), & \text{if } x\in\Omega_1,\\ g_2(x), & \text{otherwise},\end{cases}\tag{3.20}$$

where Ω_1 is the set where $g_1\leq g_2$, x_5 is negative, and $x_7+x_8x_4=0$, thus

$$\Omega_1=\{x\in\mathbb{R}^8\mid |x_5|+2|x_7x_8|+2x_4\leq x_4+x_6,\ x_5\leq 0,\ \text{and } x_7+x_8x_4=0\}.$$

Furthermore, we will split up Ω_1^c as follows. Ω_2 is the complement of Ω_1 restricted to the set where $x_7+x_8x_4=0$, that is,

$$\Omega_2=\Omega_1^c\cap\{x\in\mathbb{R}^8\mid x_7+x_8x_4=0\},$$

and Ω_3 is the set of all points such that $x_7+x_8x_4\neq 0$, that is,

$$\Omega_3=\{x\in\mathbb{R}^8\mid x_7+x_8x_4\neq 0\}.$$

Note the following obvious relations:

$$\begin{aligned}\Omega_1\cap\Omega_2=\emptyset,\qquad \Omega_1\cap\Omega_3&=\emptyset,\qquad \Omega_2\cap\Omega_3=\emptyset,\\ \Omega_1\cup\Omega_2\cup\Omega_3&=\mathbb{R}^8.\end{aligned}\tag{3.21}$$

See Figure 4.1. As long as we are working in Lagrangian coordinates we will identify $x=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ with $X=(y,\bar{U},c,y_\xi,U_\xi,h,\bar{r},k)$.

Definition 3.2. *The set $\mathcal{G}$ consists of all (ζ, U, h, r) such that*

$$X = (\zeta, \bar{U}, c, \zeta_\xi, U_\xi, h, \bar{r}, k) \in \bar{V}, \tag{3.22a}$$

$$g(y, \bar{U}, c, y_\xi, U_\xi, h, \bar{r}, k) - 1 \in E, \tag{3.22b}$$

$$y_\xi \geq 0, h \geq 0 \text{ almost everywhere}, \tag{3.22c}$$

$$\lim_{\xi \to -\infty} \zeta(\xi) = 0, \tag{3.22d}$$

$$\frac{1}{y_\xi + h} \in L^\infty(\mathbb{R}), \tag{3.22e}$$

$$y_\xi h = U_\xi^2 + \bar{r}^2 \text{ almost everywhere}, \tag{3.22f}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$.

The condition (3.22d) will be valid as long as the solutions exist since in that case we must have $\lim_{\xi \to -\infty} U(t, \xi) = 0$ by construction. In addition it should be noted that, due to the definition of $g(X)$, (3.22b) is valid for any X that satisfies (3.22a).

Making the identifications $y_\xi = q$ and $w = U_\xi$, we obtain

$$y_t = U, \quad U_t = -Q(X), \tag{3.23a}$$

$$q_t = \chi_{\{\tau(\xi) > t\}} w, \tag{3.23b}$$

$$w_t = \chi_{\{\tau(\xi) > t\}} \left(\frac{1}{2} h + (U^2 + \frac{1}{2} k^2 - P(X)) q + k\bar{r} \right), \tag{3.23c}$$

$$h_t = \chi_{\{\tau(\xi) > t\}} 2(U^2 + \frac{1}{2} k^2 - P(X)) w, \tag{3.23d}$$

$$\bar{r}_t = \chi_{\{\tau(\xi) > t\}} (-kw), \tag{3.23e}$$

$$c_t = 0, \tag{3.23f}$$

$$k_t = 0, \tag{3.23g}$$

where $P(X) - U^2 - \frac{1}{2} k^2$ and $Q(X)$ are given by

$$\begin{aligned} P(t, \xi) - U^2(t, \xi) - \frac{1}{2} k^2 & \\ = -2c\chi(y(t, \xi)) \bar{U}(t, \xi) - \bar{U}^2(t, \xi) & \\ + \frac{1}{2} \int_{\tau(\eta) > t} e^{-|y(t,\xi) - y(t,\eta)|} (2c\chi \circ y \bar{U} q + \bar{U}^2 q + \frac{1}{2} h + k\bar{r})(t, \eta) d\eta & \\ + \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t,\xi) - y(t,\eta)|} 2c^2 (\chi'^2 + \chi\chi'')(y(t, \eta)) q(t, \eta) d\eta & \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} Q(t, \xi) = 2c^2 \chi(y(t, \xi)) \chi'(y(t, \xi)) & \\ - \frac{1}{2} \int_{\tau(\eta) > t} \operatorname{sgn}(\xi - \eta) e^{-|y(t,\xi) - y(t,\eta)|} & \\ \times (2c\chi \circ y \bar{U} q + \bar{U}^2 q + \frac{1}{2} h + k\bar{r})(t, \eta) d\eta & \\ - \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta) e^{-|y(t,\xi) - y(t,\eta)|} 2c^2 (\chi'^2 + \chi\chi'')(y(t, \eta)) q(t, \eta) d\eta, & \end{aligned} \tag{3.25}$$

respectively.

The definition of τ given by (3.15) (after replacing y_ξ by the corresponding variable q) is not appropriate for $q \in C([0,T], L^\infty(\mathbb{R}))$, and, in addition, it is not clear from this definition whether τ is measurable or not. That is why we replace this definition by the following one. Let $\{t_i\}_{i=1}^\infty$ be a dense countable subset of $[0,T]$. Define

$$A_t = \bigcup_{n\geq 1} \bigcap_{t_i \leq t} \Big\{\xi \in \mathbb{R} \mid q(t_i,\xi) > \frac{1}{n}\Big\}.$$

The sets A_t are measurable for all t, and we have $A_{t'} \subset A_t$ for $t \leq t'$. We consider a dyadic partition of the interval $[0,T]$ (that is, for each n, we consider the set $\{2^{-n}iT\}_{i=0}^{2^n}$) and set

$$\tau^n(\xi) = \sum_{i=0}^{2^n} \frac{iT}{2^n} \chi_{i,n}(\xi)$$

where $\chi_{i,n}$ is the indicator function of the set $A_{2^{-n}iT} \setminus A_{2^{-n}(i+1)T}$. The function τ^n is by construction measurable. One can check that $\tau^n(\xi)$ is increasing with respect to n, it is also bounded by T. Hence, we can define

$$\tau(\xi) = \lim_{n\to\infty} \tau^n(\xi),$$

and τ is a measurable function. The next lemma gives the main property of τ.

Lemma 3.3. *If, for every $\xi \in \mathbb{R}$, $q(t,\xi)$ is positive and continuous with respect to time, then*

$$\tau(\xi) = \begin{cases} 0, & \text{if } y_\xi(0,\xi) = 0, \\ \sup\{t \in \mathbb{R}_+ \mid y_\xi(t',\xi) > 0 \text{ for all } 0 \leq t' < t\}, & \text{otherwise}, \end{cases} \tag{3.26}$$

that is, we retrieve the definition (3.15).

Proof. (From [32].) We denote by $\bar\tau(\xi)$ the right-hand side of (3.26), and we want to prove that $\bar\tau = \tau$. We claim that

$$\text{for all } t < \bar\tau(\xi), \text{ we have } \xi \in A_t, \text{ and for all } t \geq \bar\tau(\xi), \text{ we have } \xi \notin A_t. \tag{3.27}$$

If $t < \bar\tau(\xi)$, then $\inf_{t'\in[0,t]} q(t',\xi) > 0$ because q is continuous in time and positive. Hence, there exists an n such that $\inf_{t'\in[0,t]} q(t',\xi) > \frac{1}{n}$, and we find that $\xi \in \bigcap_{t_i\leq t} \big\{\xi \in \mathbb{R} \mid q(t_i,\xi) > \frac{1}{n}\big\}$ in order that $\xi \in A_t$. If $t \geq \bar\tau(\xi)$, then there exists a sequence $t_{i(k)}$ of elements in the dense family $\{t_i\}$ of $[0,T]$ such that $t_{i(k)} \leq \bar\tau \leq t$ and $\lim_{k\to\infty} t_{i(k)} = \bar\tau$. Since $q(t,\xi)$ is continuous, $\lim_{k\to\infty} q(t_{i(k)},\xi) = q(\bar\tau(\xi),\xi) = 0$ and for any integer $n > 0$, there exists a k such $q(t_{i(k)},\xi) \leq \frac{1}{n}$ and $t_{i(k)} \leq t$. Hence, for any $n > 0$, $\xi \notin \bigcap_{t_i\leq t} \big\{\xi \in \mathbb{R} \mid q(t_i,\xi) > \frac{1}{n}\big\}$ and therefore $\xi \notin A_t$. When $\bar\tau(\xi) > 0$, for any $n > 0$, there exists $0 \leq i \leq 2^n - 1$ such that $2^{-n}iT < \bar\tau \leq 2^{-n}(i+1)T$. From (3.27), we infer that $\xi \in A_{2^{-n}iT} \setminus A_{2^{-n}(i+1)T}$. Hence, $\tau^n(\xi) = 2^{-n}iT$, so that

$$\bar\tau(\xi) - \frac{T}{2^n} \leq \tau^n(\xi) \leq \bar\tau(\xi) + \frac{T}{2^n}.$$

Letting n tend to infinity, we conclude that $\tau(\xi) = \bar\tau(\xi)$. If $\bar\tau(\xi) = 0$, then $\xi \notin A_t$ for all $t \geq 0$ and $\tau^n(\xi) = 0$ for all n. Hence, $\tau(\xi) = \bar\tau(\xi) = 0$. □

So far we have identified q with y_ξ. However, y_ξ does not decay fast enough at infinity to belong to $L^2(\mathbb{R})$, but $y_\xi - 1 = \zeta_\xi$ will be in $L^2(\mathbb{R})$ and we therefore introduce $v = q - 1$. In the case of conservative solutions, we know that $Q(X)$ and $P(X) - U^2 - \frac{1}{2}k^2$ are Lipschitz continuous on bounded sets and that $Q(X)$ and

$P(X)-U^2-\frac{1}{2}k^2$ can be bounded by a constant depending on the bounded set. A slightly different result is true when describing dissipative solutions. Let

(3.28)
$$B_M=\{X\in\bar V \mid \|X\|_{\bar V}+\left\|\frac{1}{q+h}\right\|_{L^\infty}\leq M,\ qh=w^2+\bar r^2,\ q\geq 0, \text{ and } h\geq 0 \text{ a.e.}\}.$$

Remark 3.4. *According to our system of ordinary differential equations* (3.23) *it seems natural to impose for the solution space that if wave breaking occurs, then the functions q, w, h, and $\bar r$ should remain unchanged afterwards, that means $q(t,\xi)=0$, $w(t,\xi)=0$, and $\bar r(t,\xi)=0$ for all $t\geq\tau(\xi)$ and $h(t,\xi)=h(\tau(\xi),\xi)$. Moreover, also the asymptotic behavior is preserved, i.e., $c(t)=c(0)$ and $k(t)=k(0)$. In what follows we will always assume that these properties are fulfilled for any $X\in C([0,T],B_M)$ without stating it explicitly.*

In addition it should be pointed out that for any $X\in C([0,T],B_M)$ the set of all points which enjoy wave breaking within a finite time interval $[0,T]$ is bounded, since

(3.29)
$$\text{meas}(\{\xi\in\mathbb{R}\mid q(T,\xi)=0\})\leq\int_{\mathbb{R}}\frac{h}{q+h}(T,\xi)d\xi\leq\left\|\frac{1}{q(T)+h(T)}\right\|_{L^\infty}\|h\|_{L^1}\leq C(M),$$

where $C(M)$ denotes some constant only depending on M.

Lemma 3.5. *(i) For all $X\in C([0,T],B_M)$, we have*

$$\|Q(X)\|_{L^\infty_T E}+\left\|P(X)-U^2-\frac{1}{2}k^2\right\|_{L^\infty_T E}\leq C(M) \tag{3.30}$$

for a constant $C(M)$ which only depends on M.
(ii) For any X and $\tilde X$ in $C([0,T],B_M)$, we have

$$\begin{aligned}\left\|Q(X)-Q(\tilde X)\right\|_{L^1_T E}&+\left\|\big(P(X)-U^2-\frac{1}{2}k^2\big)-\big(P(\tilde X)-\tilde U^2-\frac{1}{2}\tilde k^2\big)\right\|_{L^1_T E}\\ &\leq C(M)\Big(T\left\|X-\tilde X\right\|_{L^\infty_T\hat V}\\ &\quad+\int_{\mathbb{R}}\big(\int_\tau^{\tilde\tau}\tilde h(t,\xi)\chi_{\{\tau<\tilde\tau\}}(\xi)dt+\int_{\tilde\tau}^{\tau}h(t,\xi)\chi_{\{\tilde\tau<\tau\}}(\xi)dt\big)d\xi\Big),\end{aligned} \tag{3.31}$$

where

(3.32)
$$\begin{aligned}\left\|X-\tilde X\right\|_{\hat V}&=\|y-\tilde y\|_{L^\infty}+\left\|\bar U-\tilde{\bar U}\right\|_E+|c-\tilde c|+\|q-\tilde q\|_{L^2}\\ &\quad+\|w-\tilde w\|_{L^2}+\left\|(h-\tilde h)\chi_{\{\tau(\xi)>t\}}\chi_{\{\tilde\tau(\xi)>t\}}\right\|_{L^2}+\|\bar r-\tilde{\bar r}\|_{L^2}+|k-\tilde k|.\end{aligned}$$

Here $C(M)$ denotes a constant which only depends on M.

Proof. We will only establish the estimates for $P(X)-U^2-\frac{1}{2}k^2$ as the ones for $Q(X)$ can be obtained using the same methods with only slight modifications. The main tool for proving the stated estimates will be Young's inequality which we recall here for the sake of completeness. For any $f\in L^p(\mathbb{R})$ and $g\in L^q(\mathbb{R})$ with $1\leq p,q,r\leq\infty$, we have

$$\|f\star g\|_{L^r}\leq\|f\|_{L^p}\|g\|_{L^q},\quad \text{if}\quad 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}. \tag{3.33}$$

(i): Let $f(\xi)=\chi_{\{\xi>0\}}e^{-\xi}$. Then we have

$$\begin{aligned}
&\left\|-\frac{e^{-\zeta(t,\xi)}}{2}(f\star[\chi_{\{\tau(\xi)>t\}}e^{\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+\frac{1}{2}h+k\bar{r})])(t,\xi)\right\|_{L_T^\infty E}\\
&\quad\leq\frac{1}{2}e^{\|\zeta\|_{L_T^\infty L^\infty}}\left\|(f\star[\chi_{\{\tau(\xi)>t\}}e^{\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+\frac{1}{2}h+k\bar{r})])(t,\xi)\right\|_{L_T^\infty E}\\
&\quad\leq C(M)(\|f\|_{L^1}+\|f\|_{L^2})\left\|e^{\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+\frac{1}{2}h+k\bar{r})\right\|_{L_T^\infty L^2}\\
&\quad\leq C(M).
\end{aligned}$$

Similarly, it follows that

$$\left\|\frac{e^{\zeta(t,\xi)}}{2}\big([\chi_{\{\xi<0\}}e^{\xi}]\star[\chi_{\{\tau(\xi)>t\}}e^{-\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+\frac{1}{2}h+k\bar{r})]\big)(t,\xi)\right\|_{L_T^\infty E}\leq C(M).$$

Analogously one can investigate the other integral term. Indeed, since $y(\xi)=\xi+\zeta(\xi)$, we have $\xi=y(\xi)-\zeta(\xi)$. The support of χ' is contained in $[0,1]$ and this means that the support of $\chi'\circ y$ is contained in the set $\{\xi\in\mathbb{R}\mid 0\leq y(\xi)\leq 1\}$. Inserting this into $\xi=y(\xi)-\zeta(\xi)$, we get that $\operatorname{supp}(\chi'\circ y)\subset\{\xi\in\mathbb{R}\mid -\|\zeta\|_{L^\infty}\leq\xi\leq 1+\|\zeta\|_{L^\infty}\}$. Using that we obtain

$$\begin{aligned}
\|\chi'\circ y(t,\,\cdot\,)\|_{L^2}^2&=\int_{-\|\zeta(t,\,\cdot\,)\|_{L^\infty}}^{1+\|\zeta(t,\,\cdot\,)\|_{L^\infty}}(\chi'\circ y)^2(t,\xi)d\xi\leq\|\chi'\|_{L^\infty}^2\int_{-\|\zeta(t,\,\cdot\,)\|_{L^\infty}}^{1+\|\zeta(t.\,\cdot\,)\|_{L^\infty}}d\xi\\
&\leq C(M),
\end{aligned}\tag{3.34}$$

together with the fact that a similar estimate holds for the $L^2(\mathbb{R})$-norm of $\chi\circ y\chi''\circ y$. It follows immediately that $\bar{U}^2$ and $c\chi\circ y\bar{U}$ both belong to $L^2(\mathbb{R})$ and that they can be bounded by a constant only depending on M. This finishes the proof of the first part.

(ii): The only term which cannot be investigated like in (i) is given by the integral term with domain of integration $\{\xi\mid\tau(\xi)>t\}$. Let $f(\xi)=\chi_{\{\xi>0\}}e^{-\xi}$ as before, and write $z=e^{\zeta}\frac{1}{2}h+e^{\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+k\bar{r})$. Then we can write

$$\begin{aligned}
f\star(\chi_{\{\tau(\xi)>t\}}z-\chi_{\{\tilde{\tau}(\xi)>t\}}\tilde{z})&=f\star\big(\chi_{\{\tau(\xi)>t\}}\chi_{\{\tau(\xi)<\tilde{\tau}(\xi)\}}(z-\tilde{z})\big)\\
&\quad+f\star\big((\chi_{\{\tau(\xi)>t\}}-\chi_{\{\tilde{\tau}(\xi)>t\}})\chi_{\{\tau(\xi)<\tilde{\tau}(\xi)\}}\tilde{z}\big)\\
&\quad+f\star\big(\chi_{\{\tilde{\tau}(\xi)>t\}}\chi_{\{\tau(\xi)\geq\tilde{\tau}(\xi)\}}(z-\tilde{z})\big)\\
&\quad+f\star\big((\chi_{\{\tau(\xi)>t\}}-\chi_{\{\tilde{\tau}(\xi)>t\}})\chi_{\{\tau(\xi)\geq\tilde{\tau}(\xi)\}}z\big).
\end{aligned}\tag{3.35}$$

We estimate each of these terms separately. The first and the third term are similar, thus we only treat the first one. We obtain

$$\begin{aligned}
\big\|f\star\big(\chi_{\{\tau(\xi)>t\}}\chi_{\{\tau(\xi)<\tilde{\tau}(\xi)\}}(z-\tilde{z})\big)\big\|_E&\leq(\|f\|_{L^1}+\|f\|_{L^2})\big\|\chi_{\{\tau(\xi)>t\}}\chi_{\{\tilde{\tau}(\xi)>t\}}(z-\tilde{z})\big\|_{L^2}\\
&\leq C(M)\left\|X-\tilde{X}\right\|_{\hat{V}}.
\end{aligned}$$

The second term can be treated in much the same way as the fourth one. We have $(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\tilde{\tau}(\xi)>t\}})\chi_{\{\tau(\xi)<\tilde{\tau}(\xi)\}}=-\chi_{\{\tau(\xi)\leq t<\tilde{\tau}(\xi)\}}$. Introduce $z=z_1+z_2$ with $z_1=e^{\zeta}\frac{1}{2}h$ and $z_2=e^{\zeta}(2c\chi\circ y\bar{U}q+\bar{U}^2q+k\bar{r})$. Then

$$\begin{aligned}
&\big\|f\star\big((\chi_{\{\tau(\xi)>t\}}-\chi_{\{\tilde{\tau}(\xi)>t\}})\chi_{\{\tau(\xi)<\tilde{\tau}(\xi)\}}\tilde{z}_1\big)\big\|_{L_T^1E}\\
&\quad\leq\big\|f\star\big(-\chi_{\{\tau(\xi)\leq t<\tilde{\tau}(\xi)\}}\tilde{z}_1\big)\big\|_{L_T^1E}
\end{aligned}$$

$$\leq C(M)(\|f\|_{L^\infty}+\|f\|_{L^2})\left\|\chi_{\{\tau(\xi)\leq t<\tilde\tau(\xi)\}}e^{\tilde\zeta}\tilde h\right\|_{L^1_TL^1}$$
$$\leq C(M)\int_{\mathbb{R}}\Big(\int_{\tau(\xi)}^{\tilde\tau(\xi)}\tilde h(t,\xi)\chi_{\{\tau(\xi)\leq t<\tilde\tau(\xi)\}}(\xi)dt\Big)d\xi$$

after applying Fubini's theorem in the last step, which is possible since the set of points which enjoy wave breaking within the time interval $[0,T]$ is bounded. Finally

$$\begin{aligned}
&\big\|f\star\big((\chi_{\{\tau(\xi)>t\}}-\chi_{\{\tilde\tau(\xi)>t\}})\chi_{\{\tau(\xi)<\tilde\tau(\xi)\}}\tilde z_2\big)\big\|_E\\
&\quad\leq\big\|f\star\big(-\chi_{\{\tau(\xi)\leq t<\tilde\tau(\xi)\}}\tilde z_2\big)\big\|_E\\
&\quad\leq(\|f\|_{L^1}+\|f\|_{L^2})\left\|\chi_{\{\tau(\xi)\leq t<\tilde\tau(\xi)\}}\tilde z_2\right\|_{L^2}\\
&\quad\leq C(M)\left\|\chi_{\{\tau(\xi)\leq t<\tilde\tau(\xi)\}}e^{\tilde\zeta}\big(2\tilde c\chi\circ\tilde y\bar{\tilde U}(q-\tilde q)+\bar{\tilde U}^2(q-\tilde q)+\tilde k(\bar r-\bar{\tilde r})\big)\right\|_{L^2}\\
&\quad\leq C(M)\left\|X-\tilde X\right\|_{\hat V}.
\end{aligned}$$

□

To show the short-time existence of solutions we will use an iteration argument for the following system of ordinary differential equations. Denote generically $(\zeta,\bar U,c,q,w,h,\bar r,k)$ by X and $(q,w,h,\bar r)$ by Z, thus $X=(\zeta,\bar U,c,Z,k)$. Then we define the mapping

$$\mathcal{P}\colon C([0,T],\bar V)\to C([0,T],\bar V)$$

as follows: Given X in $C([0,T],B_M)$, we can compute $P(X)-U^2-\frac{1}{2}k^2$ and $Q(X)$ using (3.24) and (3.25). Then $\tilde X=\mathcal{P}(X)$ is given as the modified solution with $\tilde X(0)=X(0)$ of the following system of ordinary differential equations

$$\tilde\zeta_t(t,\xi)=\tilde U(t,\xi),\quad \tilde U_t(t,\xi)=-Q(X)(t,\xi), \tag{3.36a}$$
$$\tilde q_t(t,\xi)=\tilde w(t,\xi), \tag{3.36b}$$
$$\tilde w_t(t,\xi)=\frac{1}{2}\tilde h(t,\xi)+(U^2(t,\xi)+\frac{1}{2}k^2-P(X)(t,\xi))\tilde q(t,\xi)+k_0\bar{\tilde r}(t,\xi), \tag{3.36c}$$
$$\tilde h_t(t,\xi)=2(U^2(t,\xi)+\frac{1}{2}k^2-P(X)(t,\xi))\tilde w(t,\xi), \tag{3.36d}$$
$$\bar{\tilde r}_t(t,\xi)=-k_0\tilde w(t,\xi), \tag{3.36e}$$
$$\tilde c_t=0, \tag{3.36f}$$
$$\tilde k_t=0. \tag{3.36g}$$

Next, we modify $\tilde X$ as follows: Determine the function $\tilde\tau(\xi)$ according to (3.15) (with X replaced by $\tilde X$). Subsequently, we modify the function $\tilde X$ by setting

$$\tilde q(t,\xi)=\tilde q(\tilde\tau(\xi),\xi),\qquad \tilde w(t,\xi)=\tilde w(\tilde\tau(\xi),\xi), \tag{3.37a}$$
$$\tilde h(t,\xi)=\tilde h(\tilde\tau(\xi),\xi),\qquad \bar{\tilde r}(t,\xi)=\bar{\tilde r}(\tilde\tau(\xi),\xi),\quad t\geq\tilde\tau(\xi). \tag{3.37b}$$

Observe that the first two components of $\tilde X$, given by (3.36a), remain unmodified. We write $\tilde Z_t=\chi_{\{\tilde\tau>t\}}F(X)\tilde Z$. Thus we set $\tilde Z(t,\xi)=\tilde Z(\tilde\tau(\xi),\xi)$ for $t>\tilde\tau(\xi)$. We will in the following, to keep the notation reasonably simple, often write X for the vector (ζ,U,v,w,h,r), or even (y,U,v,w,h,r), and correspondingly for $\tilde X$.

We will frequently consider the following spaces. For $X=(\zeta,\bar U,c,q,w,h,\bar r,k)$ and $Z=(q,w,h,\bar r)$, we define

$$\|Z\|_W=\|v\|_{L^2}+\|w\|_{L^2}+\|h\|_{L^2}+\|\bar r\|_{L^2},$$

$$\|Z\|_{\bar{W}} = \|v\|_E + \|w\|_E + \|h\|_E + \|\bar{r}\|_E\,,$$
$$\|X\|_V = \|\zeta\|_{L^\infty} + \left\|\bar{U}\right\|_{L^2} + |c| + \|v\|_{L^2} + \|w\|_{L^2} + \|h\|_{L^2} + \|\bar{r}\|_{L^2} + |k|,$$
$$\|X\|_{\bar{V}} = \|\zeta\|_{L^\infty} + \left\|\bar{U}\right\|_E + |c| + \|v\|_E + \|w\|_E + \|h\|_E + \|\bar{r}\|_E + |k|.$$

The following set

$$\kappa_{1-\gamma} = \{\xi \in \mathbb{R} \mid \frac{h_0}{q_0+h_0}(\xi) \geq 1-\gamma,\ w_0(\xi) \leq 0, \text{ and } \bar{r}_0(\xi)+k_0q_0(\xi)=0\}, \quad \gamma \in [0, \frac{1}{2}], \tag{3.38}$$

will play a key role in the context of wave breaking. (For a motivation on the set $\kappa_{1-\gamma}$, please see the paragraph before Lemma 3.11.) In particular, we have that

$$\text{meas}(\kappa_{1-\gamma}) \leq \frac{1}{1-\gamma}\int_{\mathbb{R}} \frac{h_0}{q_0+h_0}(\xi)d\xi \leq \frac{1}{1-\gamma}\left\|\frac{1}{q_0+h_0}\right\|_{L^\infty}\|h_0\|_{L^1}\,, \tag{3.39}$$

and therefore the set $\kappa_{1-\gamma}$ has finite measure if we choose $\gamma \in [0, \frac{1}{2}]$, and, in particular, $\text{meas}(\kappa_{1-\gamma}) \leq C(M)$.

Lemma 3.6. *Given $X_0 \in \mathcal{G}\cap B_{M_0}$ for some constant M_0, given $X = (\zeta, U, v, w, h, r) \in C([0,T], B_M)$, we denote by $\tilde{X} = (\tilde{\zeta}, \tilde{U}, \tilde{v}, \tilde{w}, \tilde{h}, \tilde{r}) = \mathcal{P}(X)$ with initial data X_0. Let $\bar{M} = \|Q(X)\|_{L^\infty_T L^\infty} + \left\|P(X) - U^2 - \frac{1}{2}k^2\right\|_{L^\infty_T L^\infty} + M_0$. Then the following statements hold:*

(i) For all t and almost all ξ

$$\tilde{q}(t,\xi) \geq 0, \quad \tilde{h}(t,\xi) \geq 0, \tag{3.40}$$

and

$$\tilde{q}\tilde{h} = \tilde{w}^2 + \tilde{\bar{r}}^2. \tag{3.41}$$

Thus, $\tilde{q}(t,\xi) = 0$ implies $\tilde{w}(t,\xi) = 0$ and $\tilde{\bar{r}}(t,\xi) = 0$. We recall the notation $\tilde{q} = \tilde{v} + 1$.

(ii) We have

$$\left\|\frac{1}{\tilde{q}+\tilde{h}}(t,\,\cdot)\right\|_{L^\infty} \leq 2e^{C(\bar{M})T}\left\|\frac{1}{q_0+h_0}\right\|_{L^\infty}, \tag{3.42}$$

and

$$\left\|(\tilde{q}+\tilde{h})(t,\,\cdot)\right\|_{L^\infty} \leq 2e^{C(\bar{M})T}\|q_0+h_0\|_{L^\infty}\,, \tag{3.43}$$

for all $t \in [0,T]$ and a constant $C(\bar{M})$ which depends only on $\bar{M}$. In particular, $\tilde{q}+\tilde{h}$ remains bounded strictly away from zero.

(iii) There exists a $\gamma \in (0, \frac{1}{2})$ depending only on $\bar{M}$ and T such that if $\xi \in \kappa_{1-\gamma}$, then $\tilde{X}(t,\xi) \in \Omega_1$ for all $t \in [0,T]$, $\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t,\xi)$ is a decreasing function and $\frac{\tilde{w}}{\tilde{q}+\tilde{h}}(t,\xi)$ is an increasing function with respect to time, and therefore we have

$$\frac{w_0}{q_0+h_0}(\xi) \leq \frac{\tilde{w}}{\tilde{q}+\tilde{h}}(t,\xi) \leq 0 \quad \text{and} \quad 0 \leq \frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t,\xi) \leq \frac{q_0}{q_0+h_0}(\xi). \tag{3.44}$$

In addition for γ sufficiently small, depending only on $\bar{M}$ and T, we have

$$\kappa_{1-\gamma} \subset \{\xi \in \mathbb{R} \mid 0 \leq \tilde{\tau}(\xi) < T\}. \tag{3.45}$$

(iv) Moreover, for any given $\gamma \in (0, \frac{1}{2})$, there exists $\hat{T} > 0$ such that

$$\{\xi \in \mathbb{R} \mid 0 < \tilde{\tau}(\xi) < \hat{T}\} \subset \kappa_{1-\gamma}. \tag{3.46}$$

Proof. (*i*) Since $X_0 \in \mathcal{G}$, equations (3.40) and (3.41) hold for almost every $\xi \in \mathbb{R}$ at $t = 0$. We consider such a ξ and will drop it in the notation. From (3.36), we have, on the one hand,

$$(\tilde{q}\tilde{h})_t = \tilde{q}_t\tilde{h} + \tilde{q}\tilde{h}_t = \tilde{w}\tilde{h} + 2(U^2 + \frac{1}{2}k^2 - P(X))\tilde{q}\tilde{w},$$

and, on the other hand,

$$(\tilde{w}^2 + \bar{\tilde{r}}^2)_t = 2\tilde{w}\tilde{w}_t + 2\bar{\tilde{r}}\bar{\tilde{r}}_t = \tilde{w}\tilde{h} + 2(U^2 + \frac{1}{2}k^2 - P(X))\tilde{q}\tilde{w}.$$

Thus, $(\tilde{q}\tilde{h} - \tilde{w}^2 - \bar{\tilde{r}}^2)_t = 0$, and since $\tilde{q}(0)\tilde{h}(0) = \tilde{w}^2(0) + \bar{\tilde{r}}^2(0)$, we have $\tilde{q}(t)\tilde{h}(t) = \tilde{w}^2(t) + \bar{\tilde{r}}^2(t)$ for all $t \in [0, T]$. We have proved (3.41). From the definition of $\tilde{\tau}$, we have that $\tilde{q}(t) > 0$ on $[0, \tilde{\tau}(\xi))$ and by the definition of $\tilde{q}$, we have $\tilde{q}(t) = 0$ for $t \geq \tilde{\tau}(\xi)$. Hence, $\tilde{q}(t) \geq 0$ for $t \geq 0$. From (3.41), it follows that, for $t \in [0, \tilde{\tau}(\xi))$, $\tilde{h}(t) = \frac{\tilde{w}^2+\bar{\tilde{r}}^2}{\tilde{q}}(t)$ and therefore $\tilde{h}(t) \geq 0$. By continuity (with respect to time) of $\tilde{h}$, we have $\tilde{h}(\tilde{\tau}(\xi)) \geq 0$ and, since the variable does not change for $t \geq \tilde{\tau}(\xi)$, we have $\tilde{h}(t) \geq 0$ for all $t \geq 0$.

(*ii*) We consider a fixed ξ that we suppress in the notation. We denote the Euclidean norm of $\tilde{Z} = (\tilde{q}, \tilde{w}, \tilde{h}, \bar{\tilde{r}})$ by $\left|\tilde{Z}\right|_2 = (\tilde{q}^2 + \tilde{w}^2 + \tilde{h}^2 + \bar{\tilde{r}}^2)^{1/2}$. Since $\tilde{Z}_t = F(X)\tilde{Z}$, we have

$$\begin{aligned}\frac{d}{dt}|\tilde{Z}|_2^{-2} &= -2|\tilde{Z}|_2^{-4}\tilde{Z}\cdot\frac{d\tilde{Z}}{dt} = -2|\tilde{Z}|_2^{-4}\tilde{Z}\cdot F(X)\tilde{Z}\\ &\leq C(\bar{M})|\tilde{Z}|_2^{-2},\end{aligned}$$

for a constant $C(\bar{M})$ which depends only $\bar{M}$. Applying Gronwall's lemma, we obtain $\left|\tilde{Z}(t)\right|_2^{-2} \leq e^{C(\bar{M})T}\left|\tilde{Z}(0)\right|_2^{-2}$. Hence,

$$\frac{1}{\tilde{q}^2 + \tilde{w}^2 + \tilde{h}^2 + \bar{\tilde{r}}^2}(t) \leq e^{C(\bar{M})T}\frac{1}{q_0^2 + w_0^2 + h_0^2 + \bar{r}_0^2}. \tag{3.47}$$

Using (3.41), we have

$$\tilde{q}^2 + \tilde{w}^2 + \tilde{h}^2 + \bar{\tilde{r}}^2 = \tilde{q}^2 + \tilde{q}\tilde{h} + \tilde{h}^2.$$

Hence, (3.47) yields

$$\frac{1}{(\tilde{q}+\tilde{h})^2}(t) \leq \frac{1}{\tilde{q}^2 + \tilde{q}\tilde{h} + \tilde{h}^2}(t) \leq e^{C(\bar{M})T}\frac{1}{q_0^2 + q_0h_0 + h_0^2} \leq 2e^{C(\bar{M})T}\frac{1}{(q_0 + h_0)^2}.$$

The second claim can be shown similarly.

(*iii*) Let us consider a given $\xi \in \kappa_{1-\gamma}$. We are going to determine an upper bound on γ depending only on $\bar{M}$ and T such that the conclusions of (*iii*) hold. For γ small enough we have $X_0(\xi) \in \Omega_1$ as otherwise $g(X_0(\xi)) = q_0(\xi) + h_0(\xi)$ and

$$1 = \frac{g(X_0(\xi))}{q_0(\xi) + h_0(\xi)} < \frac{-w_0(\xi) - 2k_0\bar{r}_0(\xi) + 2q_0(\xi)}{q_0(\xi) + h_0(\xi)} \leq (1 + 2|k_0|)\sqrt{\gamma} + 2\gamma$$

would lead to a contradiction since $|k_0| \leq M_0$.

We claim that there exists a constant $\gamma(\bar{M}, T)$ depending only on $\bar{M}$ and T such that for all $\gamma \leq \gamma(\bar{M}, T)$, $\xi \in \mathbb{R}$, and $t \in [0, T]$,

$$\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t,\xi) \leq \gamma \text{ and } \tilde{w}(t,\xi) = 0 \text{ implies } \tilde{q}(t,\xi) = 0 \tag{3.48}$$

and

$$\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t,\xi) \leq \gamma \text{ implies } \left(\frac{\tilde{w}}{\tilde{q}+\tilde{h}}\right)_t (t,\xi) \geq 0. \tag{3.49}$$

We consider a fixed $\xi \in \mathbb{R}$ and suppress it in the notation. If $\tilde{w}(t) = 0$, then (3.41) yields $\tilde{q}(t)\tilde{h}(t) = \bar{\tilde{r}}^2(t) = k_0^2\tilde{q}^2(t)$, where we used that $\tilde{k}(t) = \tilde{k}(0) = k_0$ and $\bar{\tilde{r}}(t) = -\tilde{k}(t)\tilde{q}(t) = -k_0\tilde{q}(t)$. Thus either $\tilde{q}(t) = 0$ or $\tilde{h}(t) = k_0^2\tilde{q}(t)$. Assume that $\tilde{q}(t) \neq 0$, then $\tilde{h}(t) = k_0^2\tilde{q}(t)$. Hence $1-\gamma \leq \frac{\tilde{h}(t)}{\tilde{q}(t)+\tilde{h}(t)} = k_0^2\frac{\tilde{q}(t)}{\tilde{q}(t)+\tilde{h}(t)} \leq C(\bar{M},T)\gamma$, and we are led to a contradiction if we choose γ small enough. Hence, $\tilde{q}(t) = 0$, and we have proved (3.48).

If $\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t) \leq \gamma$, we have

$$\begin{aligned}
\left(\frac{\tilde{w}}{\tilde{q}+\tilde{h}}\right)_t &= \frac{1}{2} + (U^2 + \frac{1}{2}k^2 - P(X) - \frac{1}{2})\frac{\tilde{q}}{\tilde{q}+\tilde{h}} + k_0\frac{\bar{\tilde{r}}}{\tilde{q}+\tilde{h}} \\
&\quad - (2U^2 + k^2 - 2P(X) + 1)\frac{\tilde{w}^2}{(\tilde{q}+\tilde{h})^2} \\
&\geq \frac{1}{2} - C(\bar{M},T)\frac{\tilde{q}}{\tilde{q}+\tilde{h}} - k_0^2\frac{\tilde{q}}{\tilde{q}+\tilde{h}} - C(\bar{M},T)\frac{\tilde{q}\tilde{h}}{(\tilde{q}+\tilde{h})^2} \\
&\geq \frac{1}{2} - C(\bar{M},T)\frac{\tilde{q}}{\tilde{q}+\tilde{h}} \\
&\geq \frac{1}{2} - C(\bar{M},T)\gamma,
\end{aligned} \tag{3.50}$$

where we used that $\tilde{k}(t) = \tilde{k}(0) = k_0$ and $\bar{\tilde{r}}(t) = -\tilde{k}(t)\tilde{q}(t) = -k_0\tilde{q}(t)$. (Recall that we allow for a redefinition of $C(\bar{M},T)$.) By choosing $\gamma(\bar{M},T) \leq (4C(\bar{M},T))^{-1}$, we get $\left(\frac{\tilde{w}}{\tilde{q}+\tilde{h}}\right)_t \geq 0$, and we have proved (3.49).

For any $\gamma \leq \gamma(\bar{M},T)$, we consider a given ξ in $\kappa_{1-\gamma}$ and again suppress it in the notation. We define

$$t_0 = \sup\{t \in [0,\tilde{\tau}] \mid \frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t') < 2\gamma \text{ and } \tilde{w}(t') < 0 \text{ for all } t' \leq t\}.$$

Let us prove that $t_0 = \tilde{\tau}$. Assume the opposite, that is, $t_0 < \tilde{\tau}$. Then, we have either $\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t_0) = 2\gamma$ or $\tilde{w}(t_0) = 0$. We have $\left(\frac{\tilde{q}}{\tilde{q}+\tilde{h}}\right)_t \leq 0$ on $[0,t_0]$ and $\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t)$ is decreasing on this interval. Hence, $\frac{\tilde{q}}{\tilde{q}+\tilde{h}}(t_0) \leq \frac{\tilde{q}}{\tilde{q}+\tilde{h}}(0) \leq \gamma$, and therefore we must have $\tilde{w}(t_0) = 0$. Then, (3.48) implies $\tilde{q}(t_0) = 0$, and therefore $t_0 = \tilde{\tau}$, which contradicts our assumption. From (3.50), we get, for γ sufficiently small,

$$0 = \frac{\tilde{w}}{\tilde{q}+\tilde{h}}(\tilde{\tau}) \geq \frac{\tilde{w}}{\tilde{q}+\tilde{h}}(0) + C(\bar{M},T)\tilde{\tau},$$

and therefore $\tilde{\tau} \leq \frac{\sqrt{\gamma}}{C(\bar{M},T)}$. By taking γ small enough we can impose $\tilde{\tau} < T$, which proves (3.45). It is clear from (3.49) that $\frac{\tilde{w}}{\tilde{q}+\tilde{h}}$ is increasing. Assume that $\tilde{X}(t,\xi)$ leaves Ω_1 for some t. Then, we get

$$1 = \frac{\tilde{q}(t)+\tilde{h}(t)}{\tilde{q}(t)+\tilde{h}(t)} \leq \frac{|\tilde{w}(t)| + 2\,|k_0\bar{\tilde{r}}(t)| + \tilde{2}q(t)}{\tilde{q}(t)+\tilde{h}(t)} \leq (1+2|k_0|)\sqrt{\gamma} + 2\gamma$$

and, by taking γ small enough, we are led to a contradiction.

(*iv*) Without loss of generality we assume $\hat T \leq 1$. From (*iii*) we know that there exists a γ' only depending on $\bar M$ and T such that for $\xi \in \kappa_{1-\gamma'}$, $X(t,\xi) \in \Omega_1$ and in particular we have that the function $\frac{\tilde q}{\tilde q + \tilde h}$ is decreasing and $\frac{\tilde w}{\tilde q+\tilde h}$ is an increasing function both with respect to time on $[0,T]$. Let $\bar\gamma \leq \min(\gamma,\gamma')$. We consider a fixed $\xi \in \mathbb{R}$ such that $\tilde\tau(\xi) < \hat T$ (which means implicitly $\tilde r(t) = 0$ for all t), but $\xi \notin \kappa_{1-\bar\gamma}$. Let us introduce

$$t_0 = \inf\{t \in [0,\tilde\tau) \mid \frac{\tilde h}{\tilde q+\tilde h}(\bar t) \geq 1-\bar\gamma \text{ and } \tilde w(\bar t) \leq 0 \text{ for all } \bar t \in [t,\tilde\tau)\}. \tag{3.51}$$

Since $\tilde w_t(\tilde\tau) = \frac12 \tilde h(\tilde\tau) > 0$ and $\tilde w(\tilde\tau) = \tilde q(\tilde\tau) = \bar{\tilde r}(\tilde\tau) = 0$, the definition of t_0 is well-posed when $\tilde\tau > 0$, and we have $t_0 < \tilde\tau$. By assumption $t_0 > 0$ and $\tilde w(t_0) = 0$ or $\frac{\tilde h}{\tilde q+\tilde h}(t_0) = 1-\bar\gamma$. We cannot have $\tilde w(t_0) = 0$, since it would imply, see (3.48), that $\tilde q(t_0) = 0$ and therefore $t_0 = \tilde\tau$ which is not possible. Thus we must have $\frac{\tilde h}{\tilde q+\tilde h}(t_0) = 1-\bar\gamma$ and in particular $\frac{\tilde q}{\tilde q+\tilde h}(t_0) = \bar\gamma$. According to the choice of $\bar\gamma$ we have that $\frac{\tilde q}{\tilde q+\tilde h}(t) \leq \bar\gamma$ for all $t \geq t_0$ and $\frac{\tilde w}{\tilde q+\tilde h}(t)$ is increasing. Then we have

$$\begin{aligned}
\Big(\frac{\tilde w}{\tilde q+\tilde h}\Big)_t &= \frac12 + (U^2 + \frac12 k^2 - P(X) - \frac12)\frac{\tilde q}{\tilde q+\tilde h} + k_0 \frac{\bar{\tilde r}}{\tilde q+\tilde h} \\
&\quad - (2U^2 + k^2 - 2P(X) + 1)\frac{\tilde w^2}{(\tilde q+\tilde h)^2} \\
&\geq \frac12 - C(\bar M, T)\frac{\tilde q}{\tilde q+\tilde h} \\
&\geq \frac12 - C(\bar M, T)\bar\gamma,
\end{aligned}$$

which yields for $0 \leq t_0 \leq t' \leq 1$

$$\frac{\tilde w}{\tilde q+\tilde h}(t') \geq \frac{\tilde w}{\tilde q+\tilde h}(t_0) + (t'-t_0)(\frac12 - C(\bar M,1)\bar\gamma).$$

Since $\frac{\tilde w}{\tilde q+\tilde h}(t_0) = -\sqrt{\bar\gamma(1-\bar\gamma)}$, we choose $\hat T$ such that $0 > -\sqrt{\bar\gamma(1-\bar\gamma)} + \hat T(\frac12 - C(\bar M,1)\bar\gamma)$. Thus $\frac{\tilde w}{\tilde q+\tilde h}(\hat T) \neq 0$ and therefore all points which enjoy wave breaking before $\hat T$ are contained in $\kappa_{1-\bar\gamma}$, since any point entering $\kappa_{1-\bar\gamma}$ at a later time cannot reach the origin within the time interval $[0,\hat T]$ according to the last estimate. □

Lemma 3.7. *Given $M > 0$, there exists $\bar T$ and $\bar M$ such that for all $T \leq \bar T$ and any initial data $X_0 \in \mathcal{G} \cap B_M$, $\mathcal{P}$ is a mapping from $C([0,T], B_{\bar M})$ to $C([0,T], B_{\bar M})$.*

Proof. To simplify the notation, we will generically denote by $K(M)$ and $C(\bar M)$ increasing functions of M and $\bar M$, respectively. Without loss of generality, we assume $\bar T \leq 1$.

Let $X \in C([0,T], B_{\bar M})$ for a value of $\bar M$ that will be determined at the end as a function of M. We assume without loss of generality $\bar M \geq M$. Let $\tilde X = \mathcal{P}(X)$. From Lemma 3.5, we have

$$\|Q(X)\|_{L^\infty_T E} \leq C(\bar M), \qquad \Big\|P(X) - U^2 - \frac12 k^2\Big\|_{L^\infty_T E} \leq C(\bar M). \tag{3.52}$$

Since $\tilde{U}_t = -Q(X)$ and $U_0 = \bar{U}_0 + c_0\chi \circ y_0$, we get

$$\left\|\tilde{U}\right\|_{L_T^\infty L^\infty} \leq \|U_0\|_{L^\infty} + T\,\|Q(X)\|_{L_T^\infty L^\infty} \leq M + TC(\bar{M}).$$

We use that $\tilde{U} = \bar{\tilde{U}} + c_0\chi \circ \tilde{y}$ to deduce that

$$\left\|\bar{\tilde{U}}\right\|_{L_T^\infty L^\infty} \leq K(M) + TC(\bar{M}). \tag{3.53}$$

Since, $\tilde{\zeta}_t = \tilde{U}$, we get

$$\left\|\tilde{\zeta}\right\|_{L_T^\infty L^\infty} \leq \|\zeta_0\|_{L^\infty} + T\left\|\tilde{U}\right\|_{L_T^\infty L^\infty} \leq M + TC(\bar{M}). \tag{3.54}$$

Moreover, $\bar{\tilde{U}}_t = -Q(X) - c_0\chi' \circ \tilde{y}\tilde{U}$, we have

$$\begin{aligned}\left\|\bar{\tilde{U}}\right\|_{L_T^\infty L^2} &\leq \left\|\bar{U}_0\right\|_{L^2} + T(\|Q(X)\|_{L_T^\infty E} + |c_0|\left\|\tilde{U}\right\|_{L_T^\infty L^\infty}\|\chi' \circ \tilde{y}\|_{L_T^\infty L^2})\\ &\leq K(M) + TC(\bar{M}),\end{aligned} \tag{3.55}$$

by (3.34).

From (3.36), by the Minkowsky inequality for integrals, we get

$$\|\tilde{v}(t,\cdot)\|_E \leq \|v_0\|_E + \int_0^t \|\tilde{w}(t',\cdot)\|_E\,dt', \tag{3.56a}$$

$$\begin{aligned}\|\tilde{w}(t,\cdot)\|_E \leq \|w_0\|_E &+ T\left\|P(X) - U^2 - \frac{1}{2}k^2\right\|_{L_T^\infty E}\\ &+ \int_0^t \Big(\frac{1}{2}\left\|\tilde{h}(t',\cdot)\right\|_E + \left\|U^2 + \frac{1}{2}k^2 - P(X)\right\|_{L_T^\infty E}\|\tilde{v}(t',\cdot)\|_E\\ &\qquad + |k_0|\,\|\tilde{w}(t',\cdot)\|_E\Big)dt',\end{aligned} \tag{3.56b}$$

$$\left\|\tilde{h}(t,\cdot)\right\|_E \leq \|h_0\|_E + 2\int_0^t \left\|U^2 + \frac{1}{2}k^2 - P(X)\right\|_{L_T^\infty E}\|\tilde{w}(t',\cdot)\|_E\,dt', \tag{3.56c}$$

$$\|\bar{\tilde{r}}(t,\cdot)\|_E \leq \|\bar{r}_0\|_E + \int_0^t |k_0|\,\|\tilde{w}(t',\cdot)\|_E\,dt'. \tag{3.56d}$$

These inequalities imply that

$$\left\|\tilde{Z}(t,\cdot)\right\|_{\bar{W}} \leq K(M) + TC(\bar{M}) + C(\bar{M})\int_0^t \left\|\tilde{Z}(t',\cdot)\right\|_E dt', \tag{3.57}$$

and, applying Gronwall's inequality yields

$$\left\|\tilde{Z}\right\|_{L_T^\infty \bar{W}} \leq (K(M) + TC(\bar{M}))e^{C(\bar{M})T}. \tag{3.58}$$

Gathering (3.53), (3.54), (3.55), and (3.58), we get

$$\left\|\tilde{X}\right\|_{L_T^\infty \bar{V}} \leq (K(M) + TC(\bar{M}))e^{C(\bar{M})T}. \tag{3.59}$$

From (3.42) we get

$$\left\|\frac{1}{\tilde{q} + \tilde{h}}\right\|_{L_T^\infty L^\infty} \leq K(M)e^{C(\bar{M})T}.$$

Thus we finally obtain

$$\left\|\tilde{X}\right\|_{L_T^\infty \bar{V}}+\left\|\frac{1}{\tilde{q}+\tilde{h}}\right\|_{L_T^\infty L^\infty}\leq (K(M)+TC(\bar{M}))e^{C(\bar{M})T} \tag{3.60}$$

for some constants $K(M)$ and $C(\bar{M})$ that only depend on M and $\bar{M}$, respectively. We now set $\bar{M}=2K(M)$. Then we can choose T so small that $(K(M)+C(\bar{M})T)e^{C(\bar{M})T}\leq 2K(M)=\bar{M}$ and therefore $\left\|\tilde{X}\right\|_{L_T^\infty \bar{V}}+\left\|\frac{1}{\tilde{q}+\tilde{h}}\right\|_{L_T^\infty L^\infty}\leq \bar{M}$. □

Given $X_0\in\mathcal{G}\cap B_M$, there exists $\bar{M}$ which depends only on M such that $\mathcal{P}$ is a mapping from $C([0,T],B_{\bar{M}})$ to $C([0,T],B_{\bar{M}})$ for T small enough. Therefore we set

$$\mathrm{Im}(\mathcal{P})=\{\mathcal{P}(X)\mid X\in C([0,T],B_{\bar{M}})\}. \tag{3.61}$$

We define the *discontinuity residual* as

$$\Gamma(X,\tilde{X})=\int_{\mathbb{R}}\Big(\int_\tau^{\tilde{\tau}}\tilde{h}(t,\xi)\chi_{\{\tau<\tilde{\tau}\}}(\xi)dt+\int_{\tilde{\tau}}^{\tau}h(t,\xi)\chi_{\{\tilde{\tau}<\tau\}}(\xi)dt\Big)d\xi.$$

Here it should be noted that $\Gamma(X,\tilde{X})$ describes the distance between τ and $\tilde{\tau}$ as the following estimate shows,

$$\int_{\tilde{\tau}}^{\tau}h(t,\xi)dt\leq C(\bar{M})(\tau-\tilde{\tau})\leq C(\bar{M})\left(\int_{\tilde{\tau}}^{\tau}h(t,\xi)dt+\int_{\tilde{\tau}}^{\tau}(q(t,\xi)-\tilde{q}(t,\xi))dt\right).$$

According to Lemma 3.5, we have,

$$\begin{aligned}\left\|Q(X)-Q(\tilde{X})\right\|_{L_T^1E}&+\left\|\big(P(X)-U^2-\frac{1}{2}k^2\big)-\big(P(\tilde{X})-\tilde{U}^2-\frac{1}{2}\tilde{k}^2\big)\right\|_{L_T^1E}\\&\leq C(\bar{M})\Big(T\left\|X-\tilde{X}\right\|_{L_T^\infty\bar{V}}+\Gamma(X,\tilde{X})\Big).\end{aligned} \tag{3.62}$$

In the next lemma we establish some estimates for $\Gamma(X,\tilde{X})$, $\Gamma(\mathcal{P}(X),\mathcal{P}(\tilde{X}))$ and a quasi-contraction property for $\mathcal{P}$.

Lemma 3.8. *Given X, $\tilde{X}\in\mathrm{Im}(\mathcal{P})$ and $\gamma\in(0,\frac{1}{2})$ there exists $T>0$ depending on $\bar{M}$ such that the following inequalities hold*
(i)

$$\Gamma(X,\tilde{X})\leq C(\bar{M})\left\|X-\tilde{X}\right\|_{L_T^\infty\bar{V}}, \tag{3.63}$$

(ii)

$$\Gamma(\mathcal{P}(X),\mathcal{P}(\tilde{X}))\leq C(\bar{M})\left(T(\left\|\mathcal{P}(X)-\mathcal{P}(\tilde{X})\right\|_{L_T^\infty\bar{V}}+\left\|X-\tilde{X}\right\|_{L_T^\infty\bar{V}})+\gamma\Gamma(X,\tilde{X})\right), \tag{3.64}$$

(iii)

$$\left\|\mathcal{P}(X)-\mathcal{P}(\tilde{X})\right\|_{L_T^\infty\bar{V}}\leq C(\bar{M})\left(T\left\|X-\tilde{X}\right\|_{L_T^\infty\bar{V}}+\Gamma(X,\tilde{X})\right), \tag{3.65}$$

where $C(\bar{M})$ denotes some constant which only depends on $\bar{M}$.

Proof. Denote by $X_2 = \mathcal{P}(X)$ and $\tilde{X}_2 = \mathcal{P}(\tilde{X})$. Given $\gamma > 0$ we know from Lemma 3.6 (*iv*) that there exists T small enough such that $\{\xi \in \mathbb{R} \mid \tau_2(\xi) < T \text{ or } \tilde{\tau}_2(\xi) < T\} \subset \kappa_{1-\gamma}$ and we consider such T. Without loss of generality we can assume $T \leq 1$ and $\gamma \leq \gamma(\bar{M}, 1)$.

(*i*) Let us now consider $\xi \in \kappa_{1-\gamma}$ such that $\tau(\xi) \neq \tilde{\tau}(\xi)$. Without loss of generality we assume $\tau(\xi) < \tilde{\tau}(\xi)$. At time $t = 0$, X and $\tilde{X}$ coincide and therefore we cannot have $\tau(\xi) = 0$ because it would imply $\tilde{\tau}(\xi) = 0$. Hence $0 < \tau(\xi) < \tilde{\tau}(\xi) \leq T$. Since $X(t,\xi)$ and $\tilde{X}(t,\xi)$ both belong to the $\mathrm{Im}(\mathcal{P})$ and $\xi \in \kappa_{1-\gamma}$, we get that $X(t,\xi)$, $\tilde{X}(t,\xi) \in \Omega_1$ and especially $w(t,\xi) \leq 0$ and $\tilde{w}(t,\xi) \leq 0$. Thus we get from (3.36), if $\hat{X} = \mathcal{P}(\hat{X})$, that for $t \in [\tau(\xi), \tilde{\tau}(\xi)]$,

$$
\begin{aligned}
0 \geq \tilde{w}(t,\xi) = \tilde{w}(\tau(\xi),\xi) &+ \frac{1}{2}\int_{\tau}^{t} \tilde{h}(t',\xi)dt' \\
&+ \int_{\tau}^{t} (\hat{U}^2 + \frac{1}{2}\hat{k}^2 - P(\hat{X}))\tilde{q}(t',\xi)dt' + \int_{\tau}^{t} k_0\bar{\tilde{r}}(t',\xi)dt'.
\end{aligned}
\tag{3.66}
$$

Thus, since $\left\|\hat{U}^2 + \frac{1}{2}\hat{k}^2 - P(\hat{X})\right\|_{L^\infty_T E} \leq C(\bar{M})$ and $q(t,\xi) = w(t,\xi) = \bar{r}(t,\xi) = 0$ for $t \geq \tau(\xi)$, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{\tau}^{\tilde{\tau}} \tilde{h}(t,\xi)dt \\
&\leq -\tilde{w}(\tau(\xi),\xi) + C(\bar{M})\int_{\tau}^{\tilde{\tau}} (\tilde{q}(t,\xi) + |\bar{\tilde{r}}(t,\xi)|)dt \\
&\leq w(\tau(\xi),\xi) - \tilde{w}(\tau(\xi),\xi) + TC(\bar{M})(\|q - \tilde{q}\|_{L^\infty_T L^\infty} + \|\bar{r} - \bar{\tilde{r}}\|_{L^\infty_T L^\infty}) \\
&\leq C(\bar{M})\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}}.
\end{aligned}
\tag{3.67}
$$

A similar inequality holds for $0 < \tilde{\tau}(\xi) < \tau(\xi) \leq T$. Since $\mathrm{meas}(\kappa_{1-\gamma}) \leq C(\bar{M})$ and the only points that contribute to the integral are contained in $\kappa_{1-\gamma}$, we get

$$
\begin{aligned}
\Gamma(X,\tilde{X}) &= \int_{\mathbb{R}} \big(\int_{\tau}^{\tilde{\tau}} \tilde{h}(t,\xi)\chi_{\{\tau<\tilde{\tau}\}}(\xi)dt + \int_{\tilde{\tau}}^{\tau} h(t,\xi)\chi_{\{\tilde{\tau}<\tau\}}(\xi)dt\big)d\xi \\
&\leq C(\bar{M})\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}}.
\end{aligned}
$$

(*ii*) We denote $X_2 = \mathcal{P}(X)$ and $\tilde{X}_2 = \mathcal{P}(\tilde{X})$. Let us now consider $\xi \in \kappa_{1-\gamma}$ such that $\tau_2(\xi) \neq \tilde{\tau}_2(\xi)$. Without loss of generality we assume $\tau_2(\xi) < \tilde{\tau}_2(\xi)$. At time $t = 0$, X_2 and $\tilde{X}_2$ coincide and therefore we cannot have $\tau_2(\xi) = 0$ because it would imply $\tilde{\tau}_2(\xi) = 0$. Hence $0 < \tau_2(\xi) < \tilde{\tau}_2(\xi) \leq T$. Since $X_2(t,\xi)$ and $\tilde{X}_2(t,\xi)$ both belong to the $\mathrm{Im}(\mathcal{P})$ and $\xi \in \kappa_{1-\gamma}$, we get that $X_2(t,\xi)$, $\tilde{X}_2(t,\xi) \in \Omega_1$ and especially $w_2(t,\xi) \leq 0$ and $\tilde{w}_2(t,\xi) \leq 0$. Thus we get from (3.36) that for $t \in [\tau_2(\xi), \tilde{\tau}_2(\xi)]$,

$$
\begin{aligned}
0 \geq \tilde{w}_2(t,\xi) = \tilde{w}_2(\tau_2(\xi),\xi) &+ \frac{1}{2}\int_{\tau_2}^{t} \tilde{h}_2(t',\xi)dt' \\
&+ \int_{\tau_2}^{t} (\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X}))\tilde{q}_2(t',\xi)dt' + \int_{\tau_2}^{t} k_0\bar{\tilde{r}}_2(t',\xi)dt',
\end{aligned}
\tag{3.68}
$$

since $\tilde{k}_2(t) = k_0 = \tilde{k}(t)$ for all t. From (3.36), we get following (3.67)

$$\begin{aligned}\int_{\tau_2}^{\tilde{\tau}_2} \tilde{h}_2(t,\xi) &\leq 2(w_2(\tau_2,\xi) - \tilde{w}_2(\tau_2,\xi)) + C(\bar{M})\int_{\tau_2}^{\tilde{\tau}_2}(|q_2 - \tilde{q}_2|(t,\xi) + |\bar{r}_2 - \bar{\tilde{r}}_2|(t,\xi))dt \\ &\leq 2\,|w_2(\tau_2,\xi) - \tilde{w}_2(\tau_2,\xi)| + C(\bar{M})T(\|q_2 - \tilde{q}_2\|_{L^\infty_T L^\infty} + \|\bar{r}_2 - \bar{\tilde{r}}_2\|_{L^\infty_T L^\infty}),\end{aligned} \tag{3.69}$$

where we used that $w_2(\tau_2,\xi) = 0$, $\tilde{w}_2(\tilde{\tau}_2,\xi) \leq 0$, and $q_2(t,\xi) = \bar{r}_2(t,\xi) = 0$ for $t \in [\tau_2(\xi), \tilde{\tau}_2(\xi)]$. A corresponding inequality holds for the case $\tilde{\tau}_2(\xi) < \tau_2(\xi)$. We have, using again (3.36) on the interval $[0, \tau_2(\xi)]$,

$$\begin{aligned}|w_2(\tau_2,\xi) - \tilde{w}_2(\tau_2,\xi)| &\leq \frac{1}{2}\int_0^{\tau_2} |h_2 - \tilde{h}_2|(t,\xi)dt + |k_0|\int_0^{\tau_2} |\bar{r}_2 - \bar{\tilde{r}}_2|(t,\xi)dt \\ &\quad + \int_0^{\tau_2} |\big(U^2 + \frac{1}{2}k_0^2 - P(X)\big) - \big(\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X})\big)||q_2|(t,\xi)dt \\ &\quad + \int_0^{\tau_2} |\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X})||q_2 - \tilde{q}_2|(t,\xi)dt \\ &\leq \frac{1}{2}\int_0^{\tau_2} |h_2 - \tilde{h}_2|(t,\xi)dt \\ &\quad + C(\bar{M})\int_0^{\tau_2}(|q_2 - \tilde{q}_2|(t,\xi) + |\bar{r}_2 - \bar{\tilde{r}}_2|(t,\xi))dt \\ &\quad + C(\bar{M})\gamma\left\|\big(U^2 + \frac{1}{2}k_0^2 - P(X)\big) - \big(\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X})\big)\right\|_{L^1_T E} \\ &\leq C(\bar{M})T(\left\|X_2 - \tilde{X}_2\right\|_{L^\infty_T \bar{V}} + \left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}}) \\ &\quad + C(\bar{M})\gamma\Gamma(X,\tilde{X}),\end{aligned} \tag{3.70}$$

where we used that $\frac{q_2}{q_2+h_2} \leq \gamma$ and $q_2 = (q_2 + h_2)\frac{q_2}{q_2+h_2} \leq C(\bar{M})\gamma$.

(*iii*) First we estimate $\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}(\kappa^c_{1-\gamma})}$. For $\xi \in \kappa^c_{1-\gamma}$, we have $Z_{2,t} = F(X)Z_2$ and $\tilde{Z}_{2,t} = F(\tilde{X})\tilde{Z}_2$ for all $t \in [0,T]$. Hence,

$$\begin{aligned}\left\|(Z_2 - \tilde{Z}_2)(t,\cdot)\right\|_{\bar{W}(\kappa^c_{1-\gamma})} &\leq \int_0^t \left\|\Big(F(X) - F(\tilde{X})\Big)Z_2(t',\cdot)\right\|_{\bar{W}(\kappa^c_{1-\gamma})} dt' \\ &\quad + \int_0^t \left\|F(\tilde{X})(Z_2 - \tilde{Z}_2)(t',\cdot)\right\|_{\bar{W}(\kappa^c_{1-\gamma})} dt'.\end{aligned} \tag{3.71}$$

We have, since $k_2(t) = k(t) = k_0 = \tilde{k}(t) = \tilde{k}_2(t)$ for all t,

$$\begin{aligned}\Big(F(X) - F(\tilde{X})\Big)Z_2 = \Big(0, &\big((U^2 + \frac{1}{2}k_0^2 - P(X)) - (\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X}))\big)q_2, \\ &2\big((U^2 + \frac{1}{2}k_0^2 - P(X)) - (\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X}))\big)w_2, 0\Big),\end{aligned}$$

and therefore

$$\left\|(F(X) - F(\tilde{X}))Z_2\right\|_{L^1_T \bar{W}} \leq C(\bar{M})\left\|\big(U^2 + \frac{1}{2}k_0^2 - P(X)\big) - \big(\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X})\big)\right\|_{L^1_T E}. \tag{3.72}$$

Applying Gronwall's lemma to (3.71), as $\left\|F(\tilde{X})\right\|_{L^\infty_T L^\infty} \leq C(\bar{M})$, we get

$$\left\|Z_2 - \tilde{Z}_2\right\|_{\bar{W}(\kappa^c_{1-\gamma})} \leq C(\bar{M}) \left\|(F(X) - F(\tilde{X}))Z_2\right\|_{L^1_T \bar{W}}. \tag{3.73}$$

Hence, we get by (3.72) that

$$\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}(\kappa^c_{1-\gamma})} \leq C(\bar{M}) \left\|\big(P(X) - U^2 - \frac{1}{2}k_0^2\big) - \big(P(\tilde{X}) - \tilde{U}^2 - \frac{1}{2}k_0^2\big)\right\|_{L^1_T E}. \tag{3.74}$$

Thus, we have by (3.62) that

$$\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}(\kappa^c_{1-\gamma})} \leq C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.75}$$

To estimate $\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}(\kappa_{1-\gamma})}$, we fix $\xi \in \kappa_{1-\gamma}$ and assume without loss of generality that $0 < \tau_2(\xi) < \tilde{\tau}_2(\xi) \leq T$. From Lemma 3.6, we have that $\frac{\tilde{q}_2}{\tilde{q}_2 + \tilde{h}_2}$ is positive decreasing and $\frac{\tilde{w}_2}{\tilde{q}_2 + \tilde{h}_2}$ is negative decreasing so that

$$|\tilde{q}_2(t, \xi)| \leq C(\bar{M}) \, |\tilde{q}_2(\tau_2, \xi)| \ \text{ and } \ |\tilde{w}_2(t, \xi)| \leq C(\bar{M}) \, |\tilde{w}_2(\tau_2, \xi)| \tag{3.76}$$

for $t \in [\tau_2(\xi), T]$, and therefore

$$|\tilde{q}_2(t, \xi) - q_2(t, \xi)| \leq C(\bar{M}) \, |\tilde{q}_2(\tau_2, \xi) - q_2(\tau_2, \xi)| \tag{3.77}$$

and

$$|\tilde{w}_2(t, \xi) - w_2(t, \xi)| \leq C(\bar{M}) \, |\tilde{w}_2(\tau_2, \xi) - w_2(\tau_2, \xi)| \tag{3.78}$$

for $t \in [\tau_2(\xi), T]$ because $q_2(t, \xi) = w_2(t, \xi) = 0$ for $t \in [\tau_2(\xi), T]$. Since $\tilde{\bar{r}}_2(t, \xi) + k_0\tilde{q}_2(t, \xi) = 0$, we know that $\operatorname{sign}(\tilde{\bar{r}}_2(t, \xi)) = -\operatorname{sign}(k_0)$, and therefore $\tilde{\bar{r}}_{2,t}(t, \xi) = -k_0\tilde{w}_2(t, \xi)$ implies that $|\tilde{\bar{r}}_2(t, \xi)|$ decreases on $[\tau_2(\xi), T]$. Thus

$$|\tilde{\bar{r}}_2(t, \xi)| \leq |\tilde{\bar{r}}_2(\tau_2, \xi)| \quad \text{and} \quad |\tilde{\bar{r}}_2(t, \xi) - \bar{r}_2(t, \xi)| \leq |\tilde{\bar{r}}_2(\tau_2, \xi) - \bar{r}_2(\tau_2, \xi)| \tag{3.79}$$

for all $t \in [\tau_2(\xi), T]$ since $\bar{r}_2(t, \xi) = 0$ for all $t \in [\tau_2(\xi), T]$. For $t \in [\tau_2(\xi), T]$, we have $\tilde{h}_{2,t} = 2(\tilde{U}^2 + \frac{1}{2}k_0^2 - P(\tilde{X}))\tilde{w}_2$ and $h_{2,t} = 0$. Hence,

$$\left|(\tilde{h}_2 - h_2)(t, \xi)\right| \leq \left|(\tilde{h}_2 - h_2)(\tau_2, \xi)\right| + C(\bar{M})T \, |(\tilde{w}_2 - w_2)(\tau_2, \xi)| \,, \tag{3.80}$$

from (3.78). For $t \in [0, \tau_2(\xi)]$, we have $Z_{2,t} = F(X)Z_2$ and $\tilde{Z}_{2,t} = F(\tilde{X})\tilde{Z}_2$. We proceed as in the previous step and in the same way as we obtained (3.73), we now obtain

$$|(\tilde{Z}_2 - Z_2)(\tau_2, \xi)| \leq C(\bar{M}) \left\|(F(X) - F(\tilde{X}))Z_2\right\|_{L^1_T L^\infty},$$

and, after using (3.72) together with (3.62), we get

$$|(Z_2 - \tilde{Z}_2)(\tau_2, \xi)| \leq C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.81}$$

Combining (3.77), (3.78), (3.79), (3.80) and (3.81), we get

$$|(Z_2 - \tilde{Z}_2)(t, \xi)| \leq C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big) \tag{3.82}$$

for all $t \in [0, T]$. Since $\operatorname{meas}(\kappa_{1-\gamma}) \leq C(\bar{M})$, (3.82) implies

$$\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}(\kappa_{1-\gamma})} \leq C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.83}$$

Combining (3.75) and (3.83), we get

$$\left\|Z_2 - \tilde{Z}_2\right\|_{L^\infty_T \bar{W}} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.84}$$

From (3.36a), we obtain

$$\left\|U_2 - \tilde{U}_2\right\|_{L^\infty_T L^\infty} \le \left\|Q(X) - Q(\tilde{X})\right\|_{L^1_T E} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big), \tag{3.85}$$

and

$$\left\|\zeta_2 - \tilde{\zeta}_2\right\|_{L^\infty_T L^\infty} \le T\left\|U_2 - \tilde{U}_2\right\|_{L^\infty_T L^\infty} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.86}$$

Combining the last two inequalities yields

$$\left\|\bar{U}_2 - \bar{\tilde{U}}_2\right\|_{L^\infty_T L^\infty} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.87}$$

Finally from (3.36a) we get

$$\left\|\bar{U}_2 - \bar{\tilde{U}}_2\right\|_{L^\infty_T L^2} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.88}$$

Thus adding up (3.84), (3.86), (3.87), and (3.88) we have that

$$\left\|X_2 - \tilde{X}_2\right\|_{L^\infty_T \bar{V}} \le C(\bar{M})\big(T\left\|X - \tilde{X}\right\|_{L^\infty_T \bar{V}} + \Gamma(X, \tilde{X})\big). \tag{3.89}$$

□

Theorem 3.9 (Short time solution)**.** *For any initial data $X_0 = (y_0, U_0, h_0, r_0) \in \mathcal{G}$, there exists a time $T > 0$ such that there exists a unique solution $X = (y, U, h, r) \in C([0, T], \bar{V})$ of (3.16) with $X(0) = X_0$. Moreover $X(t) \in \mathcal{G}$ for all $t \in [0, T]$.*

Proof. In order to prove the existence and uniqueness of the solution we use an iteration argument. Therefore we set $X_{n+1} = \mathcal{P}(X_n)$ and $X_n(0) = X_0$ for all $n \in \mathbb{N}$. This implies that X_n for $n = 1, 2, \dots$ belongs to $\mathrm{Im}(\mathcal{P})$. We have

$$\begin{aligned}
\|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} &\le C(\bar{M})\big(T\|X_n - X_{n-1}\|_{L^\infty_T \bar{V}} + \Gamma(X_n, X_{n-1})\big) \\
&\le C(\bar{M})\Big(T\big(\|X_n - X_{n-1}\|_{L^\infty_T \bar{V}} + \|X_{n-1} - X_{n-2}\|_{L^\infty_T \bar{V}}\big) \\
&\qquad + \gamma\Gamma(X_{n-1}, X_{n-2})\Big) \\
&\le C(\bar{M})(T + \gamma)\big(\|X_n - X_{n-1}\|_{L^\infty_T \bar{V}} + \|X_{n-1} - X_{n-2}\|_{L^\infty_T \bar{V}}\big)
\end{aligned}$$

where we used Lemma 3.8. Hence, for T and γ small enough, we have

$$\|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} \le \frac{1}{4}(\|X_n - X_{n-1}\|_{L^\infty_T \bar{V}} + \|X_{n-1} - X_{n-2}\|_{L^\infty_T \bar{V}}) \quad \text{for } n \ge 2.$$

Summation over all $n \ge 2$ on the left-hand side then yields

$$\sum_{n=2}^{N} \|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} \le \frac{1}{4}(\sum_{n=1}^{N-1} \|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} + \sum_{n=0}^{N-2} \|X_{n+1} - X_n\|_{L^\infty_T \bar{V}})$$

and

$$\frac{1}{2}\sum_{n=0}^{N} \|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} \le \|X_1 - X_0\|_{L^\infty_T \bar{V}} + \|X_2 - X_1\|_{L^\infty_T \bar{V}}$$

independently on N. Since $\|X_{n+1} - X_n\|_{L^\infty_T \bar{V}} \geq 0$ for all $n \in \mathbb{N}$, the series $\sum_{n=0}^\infty \|X_{n+1} - X_n\|_{L^\infty_T \bar{V}}$ is increasing, bounded from above and hence convergent. In particular,

$$\|X_m - X_n\|_{L^\infty_T \bar{V}} \leq \sum_{i=n}^{m-1} \|X_{i+1} - X_i\|_{L^\infty_T \bar{V}} \leq \sum_{i=n}^{\infty} \|X_{i+1} - X_i\|_{L^\infty_T \bar{V}} \,,$$

and therefore $\{X_n\}_{n=1}^\infty$ is a Cauchy sequence and tends to a unique limit $X(t)$. The continuity of $\mathcal{P}$ in $L^\infty_T \bar{V}$ follows from (3.63) and (3.65). Hence, we obtain that $X(t)$ is not only unique but also a fix point of the mapping $\mathcal{P}$ to the initial condition X_0.

It is left to prove that $U_\xi = w$ and $y_\xi = q$. Recall that $Q(X)$ is defined via (3.25) and $Q(X)$ is differentiable if and only if y is differentiable. A formal computation gives us that

(3.90)
$$\begin{aligned} Q_\xi(X) &= \chi_{\{\tau(\xi)>t\}}(-\frac{1}{2}h - (U^2 + \frac{1}{2}k^2 - P(X))q - k\bar{r}) \\ &\quad + \left(2c^2(\chi'^2 + \chi\chi'')(y) + 2c\chi \circ y\bar{U} + \bar{U}^2 - U^2 - \frac{1}{2}k^2 + P(X)\right)(y_\xi - q), \end{aligned}$$

and $Q_\xi \in L^1_{\mathrm{loc}}([0,1] \times \mathbb{R})$ if $\zeta_\xi = y_\xi - 1 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In addition, as $q(t,\xi) = \chi_{\{\tau(\xi)>t\}}(\xi) q(t,\xi)$ and $w(t,\xi) = \chi_{\{\tau(\xi)>t\}}(\xi) w(t,\xi)$, we have

$$(q - y_\xi)_t = (w - U_\xi), \tag{3.91a}$$

$$\begin{aligned} (w - U_\xi)_t &= \big(2c^2(\chi'^2 + \chi\chi'')(y) \\ &\qquad + 2c\chi \circ y\bar{U} + \bar{U}^2 - U^2 - \frac{1}{2}k^2 + P(X)\big)(y_\xi - q). \end{aligned} \tag{3.91b}$$

This means in particular if $q_0 = y_{0,\xi}$ and $w_0 = U_{0,\xi}$, that

$$\begin{aligned} &\|(q - y_\xi)(t,\cdot)\|_E + \|(w - U_\xi)(t,\cdot)\|_E \\ &\qquad \leq C(M) \int_0^t (\|(q - y_\xi)(t',\cdot)\|_E + \|(w - U_\xi)(t',\cdot)\|_E) dt' \end{aligned}$$

and thus using Gronwall's inequality yields that $y_\xi = q$ and $U_\xi = w$.

Let us prove that $X(t) \in \mathcal{G}$ for all t. From (3.40) and (3.41), we get $q(t,\xi) \geq 0$, $h(t,\xi) \geq 0$ and $qh = w^2 + \bar{r}^2$ for all t and almost all ξ and therefore, since $U_\xi = w$ and $y_\xi = q$, the conditions (3.22c) and (3.22f) are fulfilled. Since $\zeta(t,\xi) = \zeta(0,\xi) + \int_0^t U(t,\xi)\, dt$, we obtain by the Lebesgue dominated convergence theorem that $\lim_{\xi \to -\infty} \zeta(t,\xi) = 0$ because $\bar{U}(t,\xi) = U(t,\xi)$ for $\xi \leq -\|\zeta\|_{L^\infty_T L^\infty}$ and $\bar{U} \in H^1(\mathbb{R})$. Hence, since in addition $X(t) \in B_{\bar{M}}$, $X(t)$ fulfills all the conditions listed in (3.22) and $X(t) \in \mathcal{G}$. □

Remark 3.10. *The set $\mathcal{G} \cap B_M$ is closed with respect to the topology of $\bar{V}$. We have*

$$\begin{aligned} y_{\xi,t} &= U_\xi, \\ h_t &= 2(U^2 + \frac{1}{2}k_0^2 - P(X))U_\xi, \end{aligned}$$

and

$$\bar{r}_t = -k_0 U_\xi,$$

for all $\xi \in \mathbb{R}$ and $t \in \mathbb{R}_+$, since $U_\xi(t,\xi) = 0$ for $t \geq \tau(\xi)$. This means in particular, that y_ξ, $\bar{r}$, and h are differentiable almost everywhere with respect to time (in the classical sense).

We have that $(\zeta, U, \zeta_\xi, U_\xi, h, r)$ is a fixed point of $\mathcal{P}$, and the results of Lemma 3.6 hold for $X = \tilde{X} = (\zeta, U, \zeta_\xi, U_\xi, h, r)$. Since this lemma is going to be used extensively we rewrite it for the fixed point solution X. For this purpose, we redefine B_M and $\kappa_{1-\gamma}$, see (3.28) and (3.38), as

$$B_M = \{X \in \bar{V} \mid \|X\|_{\bar{V}} + \left\|\frac{1}{y_\xi + h}\right\|_{L^\infty} \leq M\},$$

with $X = (\zeta, U, \zeta_\xi, U_\xi, h, r)$,

(3.92)

$$\kappa_{1-\gamma} = \{\xi \in \mathbb{R} \mid \frac{h_0}{y_{0,\xi} + h_0}(\xi) \geq 1 - \gamma,\, U_{0,\xi}(\xi) \leq 0, \text{ and } r_0(\xi) = 0\}, \quad \gamma \in [0, \frac{1}{2}].$$

Note that every condition imposed on points $\xi \in \kappa_{1-\gamma}$ is motivated by what is known about wave breaking. If wave breaking occurs at some time t_b energy is concentrated on sets of measure zero in Eulerian coordinates, which correspond to the sets where $\frac{h}{y_\xi + h}(t_b, \xi) = 1$ in Lagrangian coordinates. Furthermore, it is well-known that wave breaking in the context of the 2CH system means that the spatial derivative becomes unbounded from below and hence $U_\xi(t,\xi) \leq 0$ for $t_b - \delta \leq t \leq t_b$ for such points, see [14, 25]. Finally, it has been shown in [22, Theorem 6.1] that wave breaking within finite time can only occur at points ξ where $r_0(\xi) = 0$.

Recall that $g(X)$ denotes $g(y, \bar{U}, c, y_\xi, U_\xi, h, \bar{r}, k)$. Lemma 3.6 rewrites as follows.

Lemma 3.11. *Let M_0 be a constant, and consider initial data $X_0 \in \mathcal{G} \cap B_{M_0}$. Denote the solution of (3.16) with initial data X_0 by $X = (\zeta, U, \zeta_\xi, U_\xi, h, r) \in C([0,T], B_M)$. Introduce $\bar{M} = \|Q(X)\|_{L^\infty_T L^\infty} + \left\|P(X) + \frac{1}{2}k^2 - U^2\right\|_{L^\infty_T L^\infty} + M_0$. Then the following statements hold:*

(i) We have

$$\left\|\frac{1}{y_\xi + h}(t, \cdot)\right\|_{L^\infty} \leq 2e^{C(\bar{M})T} \left\|\frac{1}{y_{0,\xi} + h_0}\right\|_{L^\infty}, \tag{3.93}$$

and

$$\|(y_\xi + h)(t, \cdot)\|_{L^\infty} \leq 2e^{C(\bar{M})T} \|y_{0,\xi} + h_0\|_{L^\infty} \tag{3.94}$$

for all $t \in [0,T]$ and a constant $C(\bar{M})$ which depends on $\bar{M}$.

(iii) There exists a $\gamma \in (0, \frac{1}{2})$ depending only on $\bar{M}$ and T such that if $\xi \in \kappa_{1-\gamma}$, then $X(t,\xi) \in \Omega_1$ for all $t \in [0,T]$, $\frac{y_\xi}{y_\xi + h}(t,\xi)$ is a decreasing function and $\frac{U_\xi}{y_\xi + h}(t,\xi)$ is an increasing function, both with respect to time, and therefore we have

$$\frac{U_{0,\xi}}{y_{0,\xi} + h_0}(\xi) \leq \frac{U_\xi}{y_\xi + h}(t,\xi) \leq 0 \quad \text{and} \quad 0 \leq \frac{y_\xi}{y_\xi + h}(t,\xi) \leq \frac{y_{0,\xi}}{y_{0,\xi} + h_0}(\xi). \tag{3.95}$$

In addition, for γ sufficiently small, depending only on $\bar{M}$ and T, we have

$$\kappa_{1-\gamma} \subset \{\xi \in \mathbb{R} \mid 0 \leq \tau(\xi) < T\}. \tag{3.96}$$

(iv) Moreover, for any given $\gamma \in (0, \frac{1}{2})$, there exists $\hat{T} > 0$ such that

$$\{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < \hat{T}\} \subset \kappa_{1-\gamma}. \tag{3.97}$$

To prove global existence of the solution we will use the estimate contained in the following lemma.

Lemma 3.12. *Given $M_0 > 0$ and $T_0 > 0$, there exists a constant M which only depends on M_0 and T_0 such that, for any $X_0 = (y_0, U_0, h_0, r_0) \in B_{M_0}$, we have $X(t) \in B_M$ for all $t \in [0,T]$, where $X(t)$ denotes the short time solution on $[0,T]$ with $T \le T_0$ given by Theorem 3.9 for initial data X_0.*

Proof. To simplify the notation we will generically denote by C constants which only depend on χ, c_0, and k_0, and by $C(M_0,T_0)$ constants which in addition depend on M_0 and T_0.

Let us introduce

$$\Sigma = \int_{\mathbb{R}} \bar{U}^2 y_\xi \, d\xi + \|h\|_{L^1}.$$

Since $h \ge 0$ and $h = U_\xi^2 + \bar{r}^2 - h\zeta_\xi$, we have $\|h\|_{L^1} = \int_{\mathbb{R}} h \, d\xi < \infty$. We can estimate the $\left\|\bar{U}\right\|_{L^\infty}^2$ as follows:

$$\begin{aligned}
\bar{U}^2(\xi) &= 2\int_{-\infty}^{\xi} \bar{U}\bar{U}_\xi \, d\eta \\
&= 2\int_{-\infty}^{\xi} \bar{U} U_\xi \, d\eta - 2\int_{-\infty}^{\xi} c\bar{U}\chi' \circ y y_\xi \, d\eta \\
&\le \int_{\{\eta | y_\xi(\eta) > 0\}} \Big(\bar{U}^2 y_\xi + \frac{U_\xi^2}{y_\xi}\Big) \, d\eta + 2\int_{\mathbb{R}} |c\bar{U}|\chi' \circ y y_\xi \, d\eta \\
&\le \int_{\{\eta | y_\xi(\eta) > 0\}} (\bar{U}^2 y_\xi + h) \, d\eta + 2C \left\|\bar{U}\right\|_{L^\infty} \\
&\le \Sigma + 2C \left\|\bar{U}\right\|_{L^\infty},
\end{aligned}$$

where we used that $y_\xi(\eta) = 0$ implies $U_\xi(\eta) = 0$ and therefore in the first integral in the second line the integrand is zero whenever $y_\xi(\xi) = 0$. Thus it suffices to integrate over $\{\eta \in \mathbb{R} \mid y_\xi(\eta) > 0\} \cap \{\eta \le \xi\}$ which justifies the subsequent estimate. Finally, inserting that $\left\|\bar{U}\right\|_{L^\infty} \le \frac{1}{4C} \left\|\bar{U}\right\|_{L^\infty}^2 + C$, we get

$$\left\|\bar{U}\right\|_{L^\infty}^2 \le 2\Sigma + C. \tag{3.99}$$

From (3.14), we get

$$\|P(X)\|_{L^\infty} \le C(1 + \Sigma + \|U\|_{L^\infty}^2) \le C(1 + \Sigma). \tag{3.100}$$

Similarly one obtains

$$\|Q(X)\|_{L^\infty} \le C(1 + \Sigma). \tag{3.101}$$

We can now compute the derivative of Σ. From (3.16) we get

$$\begin{aligned}
\frac{d\Sigma}{dt} &= \int_{\mathbb{R}} 2\bar{U}\bar{U}_t y_\xi \, d\xi + \int_{\mathbb{R}} \bar{U}^2 y_{\xi t} \, d\xi + \int_{\mathbb{R}} h_t \, d\xi \\
&= \int_{\mathbb{R}} 2\bar{U}(-Q(X) - cU\chi' \circ y) y_\xi \, d\xi + \int_{\mathbb{R}} \bar{U}^2 U_\xi \, d\xi + \int_{\mathbb{R}} 2(U^2 + \frac{1}{2}k^2 - P(X)) U_\xi \, d\xi \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

Note that we can put the time derivative under the integral, since y_ξ and h are differentiable almost everywhere with respect to time (cf. Remark 3.10). We estimate each of these integrals separately. Thus

$$\begin{aligned}
A_1 &= -2\int_{\mathbb{R}} Q(X)\bar U y_\xi d\xi - 2\int_{\mathbb{R}} \big(c\bar U^2\chi'\circ y y_\xi + c^2\bar U\chi\circ y\chi'\circ y y_\xi\big)\, d\xi \\
&\le -2\int_{\mathbb{R}} P(X)_\xi \bar U d\xi + C\Sigma + C\left\|\bar U\right\|_{L^\infty} \\
&\le 2\int_{\mathbb{R}} P(X)\bar U_\xi d\xi + C\Sigma + C\left\|\bar U\right\|_{L^\infty},
\end{aligned}$$

after integration by parts in the last step, since $P(X)_\xi = Q(X)y_\xi$. Thus

$$\begin{aligned}
A_2 &= \int_{\mathbb{R}} \bar U^2 \bar U_\xi\, d\xi + \int_{\mathbb{R}} c\bar U^2(\chi'\circ y)y_\xi d\xi \\
&= \int_{\mathbb{R}} c\bar U^2(\chi'\circ y)y_\xi d\xi \le C\Sigma.
\end{aligned}$$

Furthermore

$$\begin{aligned}
A_3 &= 2\int_{\mathbb{R}} U^2U_\xi d\xi + \int_{\mathbb{R}} k^2U_\xi d\xi - 2\int_{\mathbb{R}} P(X)\bar U_\xi d\xi - 2c\int_{\mathbb{R}} P(X)\chi'\circ y y_\xi d\xi \\
&= -2\int_{\mathbb{R}} P(X)\bar U_\xi d\xi - 2c\int_{\mathbb{R}} P(X)\chi'\circ y y_\xi d\xi + \frac{2}{3}c^3 + k^2c \\
&\le -2\int_{\mathbb{R}} P(X)\bar U_\xi d\xi + C\left\|P(X)\right\|_{L^\infty} + C.
\end{aligned}$$

Finally, by adding these estimates, we get

$$\begin{aligned}
\frac{d\Sigma}{dt} &\le C\Sigma + C + C\left\|\bar U\right\|_{L^\infty} + C\left\|P(X)\right\|_{L^\infty} \\
&\le C\Sigma + C + C\left\|\bar U\right\|_{L^\infty}^2 + C\left\|P(X)\right\|_{L^\infty} \\
&\le C\Sigma + C,
\end{aligned}$$

by (3.99) and (3.100). Hence, Gronwall's lemma implies that $\max_{t\in[0,T_0]}\Sigma(t)$ can be bounded by some constant only depending on M_0 and T_0. Using now (3.99), (3.100), and (3.101), we immediately obtain that the same is true for $\left\|\bar U(t,\,\cdot\,)\right\|_{L^\infty}$, $\left\|P(t,\,\cdot\,)\right\|_{L^\infty}$, and $\left\|Q(t,\,\cdot\,)\right\|_{L^\infty}$ with $t\in[0,T_0]$. From (3.16), we obtain that

$$|\zeta(t,\xi)| \le |\zeta(0,\xi)| + \int_0^t |U(t',\xi)|\, dt', \tag{3.102}$$

and hence also $\left\|\zeta(t,\,\cdot\,)\right\|_{L^\infty}$ can be bounded on $[0,T_0]$ by a constant only depending on M_0 and T_0.

Applying Young's inequality to (3.17) and (3.18) and following the proof of Lemma 3.5 we get

(3.103)

$$\begin{aligned}
&\left\|(P(X) - U^2 - \frac{1}{2}k^2)(t,\,\cdot\,)\right\|_{L^2} + \left\|Q(X)(t,\,\cdot\,)\right\|_{L^2} \\
&\quad\le C(M_0,T_0) \\
&\qquad + C(M_0,T_0)\big(\left\|\bar U(t,\,\cdot\,)\right\|_{L^2} + \left\|\zeta_\xi(t,\,\cdot\,)\right\|_{L^2} + \left\|h(t,\,\cdot\,)\right\|_{L^2} + \left\|\bar r(t,\,\cdot\,)\right\|_{L^2}\big).
\end{aligned}$$

Let

$$\alpha(t) = \left\|\bar{U}(t,\,\cdot\,)\right\|_{L^2} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2} + \|U_\xi(t,\,\cdot\,)\|_{L^2} + \|h(t,\,\cdot\,)\|_{L^2} + \|\bar{r}(t,\,\cdot\,)\|_{L^2}\,,$$

then

$$\alpha(t) \leq \alpha(0) + C(M_0,T_0) + C(M_0,T_0)\int_0^t \alpha(t')dt'. \tag{3.104}$$

Hence Gronwall's lemma gives us $\alpha(t) \leq C(M_0,T_0)$.

Similarly, one can show that

$$\|\zeta_\xi(t,\,\cdot\,)\|_{L^\infty} + \|U_\xi(t,\,\cdot\,)\|_{L^\infty} + \|h(t,\,\cdot\,)\|_{L^\infty} + \|\bar{r}(t,\,\cdot\,)\|_{L^\infty} \leq C(M_0,T_0).$$

It remains to prove that $\left\|\frac{1}{y_\xi+h}\right\|_{L^\infty_T L^\infty}$ can be bounded by some constant depending on M_0 and T_0, but this follows immediately form (3.93). This completes the proof. □

We can now prove global existence of solutions.

Theorem 3.13 (Global solution)**.** *For any initial data $X_0 = (y_0, U_0, h_0, r_0) \in \mathcal{G}$, there exists a unique global solution $X = (y, U, h, r) \in C(\mathbb{R}_+, \mathcal{G})$ of (3.16) with $X(0) = X_0$.*

Proof. By assumption $X_0 \in \mathcal{G}$, and therefore there exists a constant M_0 such that $X_0 \in B_{M_0}$. By Theorem 3.9 there exists a $T > 0$ such that we can find a unique short time solution $X(t) \in \mathcal{G}$ on $[0,T]$. Moreover, according to Lemma 3.7, the length of the time interval for which the solution exists and is unique, is linked to M_0. Thus we can only find a unique global solution if $\|X(t)\|_{\bar{V}} + \left\|\frac{1}{y_\xi+h}\right\|_{L^\infty}$ does not blow up within a finite time interval, but this follows from Lemma 3.12. □

4. Stability of solutions

Definition 4.1. *The mapping $d_\mathbb{R}\colon \mathcal{G}\times\mathcal{G} \to \mathbb{R}_+$*

$$d_\mathbb{R}(X,\tilde{X}) = \left\|X-\tilde{X}\right\|_V + \left\|g(X)-g(\tilde{X})\right\|_{L^2(\mathbb{R})} + \kappa(X,\tilde{X}), \tag{4.1}$$

for X, $\tilde{X} \in \mathcal{G}$ defines a metric on $\mathcal{G}$. The function $\kappa(X,\tilde{X})$ is defined as follows

$$\kappa(X,\tilde{X}) = \begin{cases} 1, & \textit{if } \operatorname{meas}(\{\xi\in\mathbb{R} \mid (r(\xi)=0 \textit{ and } \tilde{r}(\xi)\neq 0) \\ & \qquad\qquad \textit{or } (r(\xi)\neq 0 \textit{ and } \tilde{r}(\xi)=0)\})>0, \\ 0, & \textit{otherwise.}\end{cases} \tag{4.2}$$

In what follows we will denote

$$d\colon \mathbb{R}^8\times\mathbb{R}^8 \to \mathbb{R}_+, \quad d(X,\tilde{X}) = |Z-\tilde{Z}| + |g(X)-g(\tilde{X})| + \iota(X,\tilde{X}),$$

with

$$\iota(X,\tilde{X}) = \begin{cases} 1, & \text{if } (r=0 \text{ and } \tilde{r}\neq 0) \text{ or } (r\neq 0 \text{ and } \tilde{r}=0), \\ 0, & \text{otherwise.}\end{cases} \tag{4.3}$$

Here it should be noted that for any two solutions X and $\tilde{X}$ of the 2CH system, $\kappa(X,\tilde{X})$ and $\iota(X,\tilde{X})$ are independent of time, since $r_t = 0$ and $\tilde{r}_t = 0$.

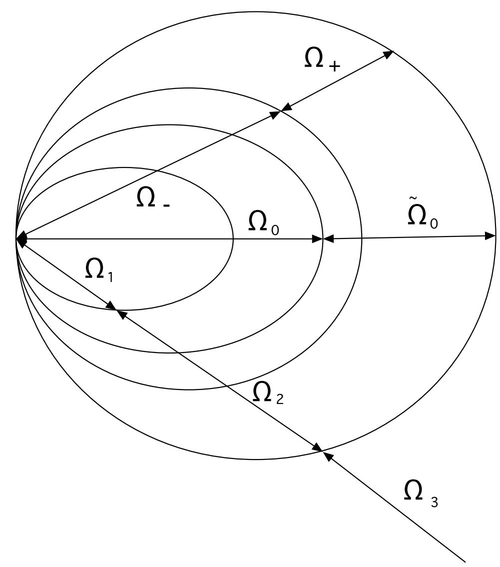

FIGURE 1. The various regions, cf. Definition 3.1 and (4.4).

4.1. **Necessary estimates.** Denote

$$\begin{aligned}\Omega_- &= \{x \in \mathbb{R}^8 \mid x_5 \leq 0 \text{ and } x_7 + x_8x_4 = 0\},\\ \Omega_+ &= \{x \in \mathbb{R}^8 \mid x_5 \geq 0 \text{ and } x_7 + x_8x_4 = 0\}.\end{aligned} \tag{4.4}$$

Note that $\Omega_- \cup \Omega_+ = \Omega_1 \cup \Omega_2 = \{x \in \mathbb{R}^8 \mid x_7 + x_8x_4 = 0\}$ (cf. Definition 3.1). See Figure 4.1.

Lemma 4.2. *The restrictions of g to Ω_2, Ω_3, and Ω_- are Lipschitz on bounded sets. More precisely,*

$$|g(X) - g(\tilde{X})| \leq C(\bar{M}) \left(|Z - \tilde{Z}| + (|\bar{r}| + |\bar{\tilde{r}}|)|k - \tilde{k}|\right), \tag{4.5}$$

for any X, $\tilde{X}$ in $\Omega_2 \cap B_{\bar{M}}$, X, $\tilde{X}$ in $\Omega_3 \cap B_{\bar{M}}$, or X, $\tilde{X}$ in $\Omega_- \cap B_{\bar{M}}$.

Proof. The cases when both X, $\tilde{X} \in \Omega_1$, X, $\tilde{X} \in \Omega_2$, and X, $\tilde{X} \in \Omega_3$ are straightforward. Let us consider the case when $X \in \Omega_1$ and $\tilde{X} \in \Omega_2 \cap \Omega_-$, that is, $-U_\xi - 2k\bar{r} + 2y_\xi \leq y_\xi + h$ and $\tilde{y}_\xi + \tilde{h} \leq -\tilde{U}_\xi - 2\tilde{k}\bar{\tilde{r}} + 2\tilde{y}_\xi$, respectively. We have

(4.6)

$$|g(X) - g(\tilde{X})| = |-U_\xi - 2k\bar{r} + 2y_\xi - \tilde{y}_\xi - \tilde{h}|$$

$$\begin{aligned}
&\leq |y_\xi - \tilde{y}_\xi| + |h - \tilde{h}| + h + U_\xi + 2k\bar{r} - y_\xi \\
&\leq |y_\xi - \tilde{y}_\xi| + |h - \tilde{h}| + h + U_\xi + 2k\bar{r} - y_\xi - \tilde{h} - \tilde{U}_\xi - 2\tilde{k}\bar{\tilde{r}} + \tilde{y}_\xi \\
&\leq C(\bar{M})(|Z - \tilde{Z}| + (|\bar{r}| + |\bar{\tilde{r}}|)|k - \tilde{k}|).
\end{aligned}$$

□

Lemma 4.3. *Given $M > 0$, there exist $\bar{M} > 0$, $T > 0$ and $\delta > 0$ which depend only on M such that for any $\xi \in \mathbb{R}$ satisfying $d(X_0(\xi), \tilde{X}_0(\xi)) < \delta$, we have*

$$|g(X(t,\xi)) - g(\tilde{X}(t,\xi))| \leq C(\bar{M}) \left(|Z(t,\xi) - \tilde{Z}(t,\xi)| + (|\bar{r}(t,\xi)| + |\bar{\tilde{r}}(t,\xi)|)|k - \tilde{k}| \right), \tag{4.7}$$

for all $t \in [0, T]$ where $X(t)$ and $\tilde{X}(t)$ are the solutions to (3.16) *with initial data X_0 and $\tilde{X}_0$ belonging to B_M.*

Proof. Without loss of generality we assume $T \leq 1$. We already know that there exists $\bar{M}$ only depending on M such that $X(t)$, $\tilde{X}(t) \in B_{\bar{M}}$ for all $t \in [0, 1]$. We consider $\xi \in \mathbb{R}$ such that $d(X_0(\xi), \tilde{X}_0(\xi)) < \delta$ for a δ that we are going to determine. For simplicity we drop ξ in the notation from now on. Since $X(t)$ and $\tilde{X}(t)$ are solutions of (3.16), we see that due to Lemma 4.2 the estimate (4.7) will be proved if we can show that either $X(t)$ and $\tilde{X}(t)$ belong to Ω_-, $X(t)$ and $\tilde{X}(t)$ belong to Ω_2, or $X(t)$ and $\tilde{X}(t)$ belong to Ω_3.

If $X_0 \in \Omega_3$ and $\tilde{X}_0 \in \Omega_3^c = \Omega_1 \cup \Omega_2$, by (4.3), we have $d(X_0, \tilde{X}_0) \geq 1$. Thus by choosing $\delta < \frac{1}{2}$, we impose that either $X_0, \tilde{X}_0 \in \Omega_3$ or $X_0, \tilde{X}_0 \in \Omega_3^c$. In addition, since $r(t) = r(0)$ and $\tilde{r}(t) = \tilde{r}(0)$, we can conclude that points which initially lie inside Ω_3 will remain in Ω_3 for all times and points starting outside Ω_3 can never enter Ω_3.

Hence in order to verify the claim it is left to show that either $X(t)$ and $\tilde{X}(t)$ belong to Ω_2 or $X(t)$ and $\tilde{X}(t)$ belong to Ω_-. To this end denote by Ω_0 and $\tilde{\Omega}_0$ the following sets (cf. Figure 4.1)

$$\begin{aligned}
&\Omega_0 = \{x \in \mathbb{R}^8 \mid 2|x_7 x_8| + x_4 \leq x_6,\ x_5 \leq 0 \text{ and } x_7 + x_8 x_4 = 0\}, \\
&\tilde{\Omega}_0 = \Omega_2 \setminus \Omega_0.
\end{aligned} \tag{4.8}$$

We will distinguish three cases:

(i) X_0 and $\tilde{X}_0$ in Ω_0: Since $\Omega_0 \subset \Omega_-$, we infer that X_0 and $\tilde{X}_0$ belong to Ω_-. Hence $X(t)$ and $\tilde{X}(t)$ will satisfy (4.7) as long as both $X(t)$ and $\tilde{X}(t)$ belong to Ω_-. Let us prove that $X(t)$ (and, in the same way, $\tilde{X}(t)$) for T small enough remains in Ω_-. Denote by t_0 the first time when $X(t)$ leaves Ω_-. By continuity we must have $U_\xi(t_0) = 0$. Since $r(t) = r(0) = 0$ for all t in $[0, T]$, it implies $y_\xi(t_0)h(t_0) = \bar{r}^2(t_0) = k_0^2 y_\xi^2(t_0)$, by (3.22f) and the definition (3.12) of $\bar{r}(t)$. Hence, as the origin belongs to Ω_-, either $y_\xi(t_0) = 0$ or $h(t_0) = k_0^2 y_\xi(t_0)$. Points reaching the origin remain there, that is, if $y_\xi(t) = U_\xi(t) = \bar{r}(t) = 0$ for $t = t_0$ then it remains true for $t \geq t_0$. Hence we infer that, $y_\xi(t_0) \neq 0$ and $h(t_0) = k_0^2 y_\xi(t_0)$. Let $z(t) = y_\xi(t) - 2k_0\bar{r}(t) - h(t)$. We have, by assumption, $z(0) \leq 0$, because $X_0 \in \Omega_0$ and $\bar{r}(0) = -k_0 y_\xi(0)$. Using that $\bar{r}(t_0) = -k_0 y_\xi(t_0)$ and $h(t_0) = k_0^2 y_\xi(t_0)$, $z(t_0) = y_\xi(t_0) - 2k_0\bar{r}(t_0) - h(t_0) = y_\xi(t_0) + k_0^2 y_\xi(t_0) = y_\xi(t_0) + h(t_0) \geq \frac{1}{\bar{M}}$. On the other hand we can compute z_t and we obtain $z_t \leq C_1(\bar{M})$ for some constant $C_1(\bar{M})$ only depending on $\bar{M}$ and therefore on M. Thus $z(t) \leq z(0) + C_1(\bar{M})T$. Hence if we choose T small enough, that means $T < (\bar{M}C(\bar{M}))^{-1}$, we obtain $z(t_0) < \frac{1}{\bar{M}}$, which is a contradiction and we have proved that $X(t)$ remains in Ω_-. Similarly one proves that $\tilde{X}(t)$ remains in Ω_-.

(ii) $X_0 \in \Omega_0$ and $\tilde{X}_0 \in \tilde{\Omega}_0$: First of all we want to make sure that (4.7) holds at time $t = 0$. According to Lemma 4.2, this will be the case if we can prove that $d(X_0, \tilde{X}_0) < \delta$ implies either $X_0, \tilde{X}_0 \in \Omega_-$ or $X_0, \tilde{X}_0 \in \Omega_2$. If $X_0 \in \Omega_2$, then (4.7) holds since $\tilde{X}_0 \in \tilde{\Omega}_0 \subseteq \Omega_2$.

If $X_0 \notin \Omega_2$, we have $X_0 \in \Omega_1$ so that $X_0 \in \Omega_-$. Assume that $\tilde{X}_0 \notin \Omega_-$. Then $|U_{0,\xi} - \tilde{U}_{0,\xi}| \leq \delta$ implies $|\tilde{U}_{0,\xi}| \leq \delta$ and $|U_{0,\xi}| \leq \delta$, as $U_{0,\xi}$ and $\tilde{U}_{0,\xi}$ have opposite signs. Since $X_0 \in \Omega_0$, we have $y_{0,\xi} - 2k_0\bar{r}_0 \leq h_0$ which implies using (3.22f) that

$$y_{0,\xi}^2 - 2k_0\bar{r}_0 y_{0,\xi} = y_{0,\xi}^2 + 2\bar{r}_0^2 \leq U_{0,\xi}^2 + \bar{r}_0^2.$$

Thus $y_{0,\xi} \leq \delta$, $|\bar{r}_0| \leq \delta$ and we have

$$\delta \geq g(\tilde{X}_0) - g(X_0) \geq \tilde{y}_{0,\xi} + \tilde{h}_0 - |U_{0,\xi}| - 2|k_0\bar{r}_0| - 2y_{0,\xi} \geq \frac{1}{\bar{M}} - C(\bar{M})\delta. \tag{4.9}$$

Taking δ sufficiently small, we are led to a contradiction. Hence $\tilde{X}_0 \in \Omega_-$.

We have already seen in (i) that we can choose T so small that $X(t)$ remains in Ω_- for $t \in [0, T]$. Let us denote $\tilde{z}(t) = \tilde{y}_\xi(t) - 2\tilde{k}_0\bar{\tilde{r}}(t) - \tilde{h}(t)$. For z as defined in (i), we have $z(0) \leq 0$. Hence $\tilde{z}(0) \leq z(0) + |\tilde{z}(0) - z(0)| \leq |y_\xi(0) - \tilde{y}_\xi(0)| + |h(0) - \tilde{h}(0)| + |\bar{r}(0)||k_0 - \tilde{k}_0| + |\tilde{k}_0||\bar{r}(0) - \bar{\tilde{r}}(0)|$ and therefore $\tilde{z}(0) < C(\bar{M})\delta$. Let us now consider the first time t_0 when $\tilde{X}(t_0)$ leaves Ω_-. Again, as in (i), we obtain that $\tilde{h}(t_0) = k_0^2\tilde{y}_\xi(t_0)$ and $\tilde{z}(t_0) \geq \frac{1}{\bar{M}}$. In addition we know $\tilde{z}(t_0) \leq \tilde{z}_0 + C(\bar{M})T \leq C(\bar{M})(\delta + T)$, which leads to a contradiction if we choose T and δ small enough.

(iii) X_0 and $\tilde{X}_0$ in $\tilde{\Omega}_0$: In this case, since $\tilde{\Omega}_0 \subset \Omega_2$, X_0 and $\tilde{X}_0$ belong to Ω_2. We have $z(0) \geq 0$. Let us prove that $X(t)$ (and, in the same way, $\tilde{X}(t)$) for T small enough remains in Ω_2. Note that because the origin is a repulsive point, solutions cannot reach the origin from within Ω_2. This means in particular that $X(t)$ can only leave Ω_2 if it starts in $\Omega_- \cap \Omega_2$ or after entering $\Omega_- \cap \Omega_2$. Therefore we assume without loss of generality $X_0 \in \Omega_- \cap \Omega_2$. Denote by t_0 the first time when $X(t)$ leaves $\tilde{\Omega}_2$. Then we must have $|U_\xi(t_0)| + 2|k_0\bar{r}|(t_0) + 2y_\xi(t_0) = y_\xi(t_0) + h(t_0)$ which gives $U_\xi(t_0) = y_\xi(t_0) - 2k_0\bar{r}(t_0) - h(t_0) = z(t_0)$. Hence we obtain after some computations using the latter equation together with (3.22f) and $\bar{r}(t_0) + ky_\xi(t_0) = 0$ that $-z(t_0) = \frac{y_\xi(t_0)+h(t_0)}{\sqrt{5+4k_0^2}} \geq \frac{1}{\bar{M}\sqrt{5+4\bar{M}^2}}$. Moreover, $-z(t_0) \leq -z(0) + C_1(\bar{M})T$ which implies $\frac{1}{\bar{M}\sqrt{5+4\bar{M}^2}} \leq \frac{y_\xi(t_0)+h(t_0)}{\sqrt{5+4k_0^2}} \leq C_1(\bar{M})T$, which leads to a contradiction if we choose T small enough. We have thus proved that $X(t)$ remains in $\tilde{\Omega}_2$ and the same result holds for $\tilde{X}(t)$. □

A close look at the proof of the last lemma shows that δ is only turning up in (ii) and can be bounded by a constant only depending on $\bar{M}$. Hence, Lemma 4.3 can be extended to the following lemma.

Lemma 4.4. *Given $M > 0$ and $T > 0$, then the solutions $X(t)$ and $\tilde{X}(t)$ with initial data X_0 and $\tilde{X}_0$ in B_M, respectively, belong to $B_{\bar{M}}$ for $t \in [0, T]$. For all $\bar{t} \in [0, T]$ there exist $\bar{T} \in [0, T]$ and δ positive depending on $\bar{M}$ and independent of $\bar{t}$, such that for any $\xi \in \mathbb{R}$ satisfying $d(X(\bar{t}, \xi), \tilde{X}(\bar{t}, \xi)) < \delta$, we have*

$$|g(X(t,\xi)) - g(\tilde{X}(t,\xi))| \leq C(\bar{M})(|Z(t,\xi) - \tilde{Z}(t,\xi)| + (|\bar{r}| + |\bar{\tilde{r}}|)|k - \tilde{k}|) \tag{4.10}$$

for all $t \in [\bar{t}, \bar{t} + \bar{T}] \cap [0, T]$.

The next lemma shows why the function g is so important and why we choose exactly the metric $d_{\mathbb{R}}$ instead of the norm $\|\cdot\|_V$ we used in the last section.

Lemma 4.5. *Given $M > 0$ and $T > 0$, then the solutions $X(t)$ and $\tilde{X}(t)$ with initial data X_0 and $\tilde{X}_0$ in B_M, respectively, belong to $B_{\bar{M}}$ and we have for any $\xi \in \mathbb{R}$ the following estimates:*
(i) If $\tau(\xi) \leq t_1 < t_2 \leq \tilde{\tau}(\xi)$

$$\begin{aligned}\int_{t_1}^{t_2} \tilde{h}(t,\xi)dt \leq |U_\xi(t_1,\xi) - \tilde{U}_\xi(t_1,\xi)| \\ + C(\bar{M}) \int_{t_1}^{t_2} \big(|Z(t,\xi) - \tilde{Z}(t,\xi)| + |g(X(t,\xi)) - g(\tilde{X}(t,\xi))|\big)dt.\end{aligned} \tag{4.11}$$

(ii) If $\tilde{\tau}(\xi) \leq t_1 < t_2 \leq \tau(\xi)$

$$\begin{aligned}\int_{t_1}^{t_2} h(t,\xi)dt \leq |U_\xi(t_1,\xi) - \tilde{U}_\xi(t_1,\xi)| \\ + C(\bar{M}) \int_{t_1}^{t_2} \big(|Z(t,\xi) - \tilde{Z}(t,\xi)| + |g(X(t,\xi)) - g(\tilde{X}(t,\xi))|\big)dt,\end{aligned} \tag{4.12}$$

for a constant $C(\bar{M})$ depending only on $\bar{M}$.

Proof. We assume without loss of generality $\tau(\xi) < \tilde{\tau}(\xi)$. We distinguish three cases.
(i) If $\tilde{X}(t,\xi) \in \Omega_3$ for $t \in [t_1,t_2]$, we know $g(\tilde{X}) = \tilde{y}_\xi + \tilde{h}$ and $g(X) = 0$ on $[t_1,t_2]$ because $y_\xi(t) = U_\xi(t) = \bar{r}(t) = 0$ so that $X(t,\xi) \in \Omega_1$. Thus we get

$$\int_{t_1}^{t_2} \tilde{h}(t,\xi)dt \leq \int_{t_1}^{t_2} |g(\tilde{X}(t,\xi)) - g(X(t,\xi))|dt. \tag{4.13}$$

As already pointed out several times before, $\tilde{X}(t)$ remains in Ω_3 for all times if it starts in Ω_3.

It is left to show what happens if $\tilde{X}(t,\xi) \in \Omega_3^c$ for all times. The result will follow from the following two estimates.
(ii) If $\tilde{X}(t,\xi) \in \Omega_+$ for $t \in [t_1,t_2]$, we know $g(\tilde{X}) = \tilde{y}_\xi + \tilde{h}$ and $g(X) = 0$ on $[t_1,t_2]$. Hence we have

$$\int_{t_1}^{t_2} \tilde{h}(t,\xi)dt \leq \int_{t_1}^{t_2} |g(\tilde{X}(t,\xi)) - g(X(t,\xi))|dt. \tag{4.14}$$

(iii) If $\tilde{X}(t_2,\xi) \in \Omega_-$, we know that $\tilde{U}_\xi(t_2,\xi) \leq 0$. Thus (3.16) implies that

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2} \tilde{h}(t,\xi)dt = \tilde{U}_\xi(t_2,\xi) - \tilde{U}_\xi(t_1,\xi) \\ - \int_{t_1}^{t_2} (\tilde{U}^2 + \frac{1}{2}\tilde{k}^2 - P(\tilde{X}))\tilde{y}_\xi(t,\xi)dt - \int_{t_1}^{t_2} \tilde{k}\tilde{\bar{r}}(t,\xi)dt.\end{aligned}$$

Since $\tilde{U}_\xi(t_2,\xi) \leq 0$ and $U_\xi(t,\xi) = y_\xi(t,\xi) = \bar{r}(t,\xi) = 0$ for all $t \in [t_1,t_2]$, we get

$$\frac{1}{2}\int_{t_1}^{t_2} \tilde{h}(t,\xi)dt \leq |U_\xi(t_1,\xi) - \tilde{U}_\xi(t_1,\xi)| \\ + C(\bar{M})\int_{t_1}^{t_2} |y_\xi(t,\xi) - \tilde{y}_\xi(t,\xi)|dt + C(\bar{M})\int_{t_1}^{t_2} |\bar{r}(t,\xi) - \bar{\tilde{r}}(t,\xi)|dt.$$

In general $\tilde{X}(t_2,\xi) \notin \Omega_-$, does not imply that $\tilde{X}(t,\xi) \in \Omega_+$ for all $t \in [t_1,t_2]$. Therefore we define t_3 as

$$t_3 = \inf\{t \in [t_1,t_2] \mid X(t',\xi) \in \Omega_+ \text{ for all } t' \geq t\}.$$

Then, $\tilde{X}(t,\xi) \in \Omega_+$ for all $t \in [t_3,t_2]$ and $U_\xi(t_3,\xi) \leq 0$ so that the general case is proved by combining the cases (ii) and (iii). □

4.2. **Stability results.**

Theorem 4.6. *Given $M > 0$ there exist constants $\bar{T} \leq 1$ and K depending only on M such that for any initial data X_0 and $\tilde{X}_0$ in B_M*

$$\sup_{t\in[0,\bar{T}]} d_{\mathbb{R}}(X(t),\tilde{X}(t)) \leq K d_{\mathbb{R}}(X_0,\tilde{X}_0). \tag{4.15}$$

Proof. We assume without loss of generality $T \leq 1$. We already know that there exists $\bar{M}$ only depending on M (since $T \leq 1$) such that $X(t)$ and $\tilde{X}(t)$ belong to $B_{\bar{M}}$ for all $t \in [0,T]$. Let us introduce for the moment the following metric

$$\begin{aligned}\tilde{d}_{\mathbb{R}}(X,\tilde{X}) &= \|y-\tilde{y}\|_{L^\infty_T L^\infty} + \left\|\bar{U} - \bar{\tilde{U}}\right\|_{L^\infty_T L^2} + |c-\tilde{c}| \\ &\quad + \left\|Z - \tilde{Z}\right\|_{L^\infty_T W} + |k-\tilde{k}| + \left\|g(X) - g(\tilde{X})\right\|_{L^\infty_T L^2} + \kappa(X,\tilde{X}).\end{aligned} \tag{4.16}$$

Then

$$d_{\mathbb{R}}(X(t),\tilde{X}(t)) \leq \tilde{d}_{\mathbb{R}}(X,\tilde{X}), \quad t \in [0,T]. \tag{4.17}$$

This means in particular that the theorem is proved once we show that

$$\tilde{d}_{\mathbb{R}}(X,\tilde{X}) \leq K d_{\mathbb{R}}(X_0,\tilde{X}_0). \tag{4.18}$$

We estimate each of the terms in (4.16). First, we want to show that

$$\begin{aligned}&\left\|Q(X) - Q(\tilde{X})\right\|_{L^1_T E} + \left\|\big(P(X) - U^2 - \frac{1}{2}k^2\big) - \big(P(\tilde{X}) - \tilde{U}^2 - \frac{1}{2}\tilde{k}^2\big)\right\|_{L^1_T E} \\ &\qquad \leq C(\bar{M})(d_{\mathbb{R}}(X_0,\tilde{X}_0) + T\tilde{d}_{\mathbb{R}}(X,\tilde{X})).\end{aligned} \tag{4.19}$$

To this end we first observe that by following closely the proof of Lemma 3.5 one can show that

$$\begin{aligned}&\left\|Q(X) - Q(\tilde{X})\right\|_{L^1_T E} + \left\|\big(P(X) - U^2 - \frac{1}{2}k^2\big) - \big(P(\tilde{X}) - \tilde{U}^2 - \frac{1}{2}\tilde{k}^2\big)\right\|_{L^1_T E} \\ &\quad \leq C(\bar{M})\Big(T\tilde{d}_{\mathbb{R}}(X,\tilde{X}) \\ &\qquad + \int_{\mathbb{R}}\big(\int_\tau^{\tilde{\tau}} \tilde{h}(t,\xi)\chi_{\{\tilde{\tau}(\xi)>\tau(\xi)\}}dt + \int_{\tilde{\tau}}^{\tau} h(t,\xi)\chi_{\{\tau(\xi)>\tilde{\tau}(\xi)\}}dt\big)d\xi\Big).\end{aligned} \tag{4.20}$$

The first step towards proving the claim is to estimate properly the integral term on the right-hand side. Applying Lemma 4.5 yields

$$
\begin{aligned}
(4.21)\quad & \int_{\mathbb{R}} \left(\int_{\tau}^{\tilde{\tau}} \tilde{h}(t,\xi)\chi_{\{\tilde{\tau}(\xi)>\tau(\xi)\}}dt + \int_{\tilde{\tau}}^{\tau} h(t,\xi)\chi_{\{\tau(\xi)>\tilde{\tau}(\xi)\}}dt \right) d\xi \\
&\leq \int_{\mathbb{R}} (|U_\xi(\tau,\xi) - \tilde{U}_\xi(\tau,\xi)|\chi_{\{\tau<\tilde{\tau}\}} + |U_\xi(\tilde{\tau},\xi) - \tilde{U}_\xi(\tilde{\tau},\xi)|\chi_{\{\tilde{\tau}<\tau\}})d\xi \\
&\quad + C(\bar{M}) \int_{\mathbb{R}} \int_0^T \big(|Z(t,\xi) - \tilde{Z}(t,\xi)| \\
&\qquad\qquad + |g(X(t,\xi)) - g(\tilde{X}(t,\xi))|\big)(\chi_{\{\tau<\tilde{\tau}\}} + \chi_{\{\tilde{\tau}<\tau\}}\big)dtd\xi \\
&\leq \int_{\mathbb{R}} (|U_\xi(\tau,\xi) - \tilde{U}_\xi(\tau,\xi)|\chi_{\{\tau<\tilde{\tau}\}} + |U_\xi(\tilde{\tau},\xi) - \tilde{U}_\xi(\tilde{\tau},\xi)|\chi_{\{\tilde{\tau}<\tau\}})d\xi \\
&\quad + C(\bar{M})T\big((\mathrm{meas}(\kappa_{1-\gamma}))^{1/2} + (\mathrm{meas}(\tilde{\kappa}_{1-\gamma}))^{1/2}\big)\tilde{d}_{\mathbb{R}}(X,\tilde{X}),
\end{aligned}
$$

where we in the last step used that T is chosen so small that all points such that $\tau(\xi) < T$ or $\tilde{\tau}(\xi) < T$ belong to $\kappa_{1-\gamma} \cup \tilde{\kappa}_{1-\gamma}$. This is possible according to Lemma 3.11 (iv).

Fix $\xi \in \mathbb{R}$ such that $0 < \tau(\xi) < \tilde{\tau}(\xi) \leq T$. Then $U_{\xi,t} = \frac{1}{2}h + (U^2 + \frac{1}{2}k^2 - P(X))y_\xi + k\bar{r}$ and $\tilde{U}_{\xi,t} = \frac{1}{2}\tilde{h} + (\tilde{U}^2 + \frac{1}{2}\tilde{k}^2 - P(\tilde{X}))\tilde{y}_\xi + \tilde{k}\bar{\tilde{r}}$ for $t \in [0,\tau(\xi)]$. Hence

(4.22)

$$
\begin{aligned}
&|U_\xi(\tau(\xi),\xi) - \tilde{U}_\xi(\tau(\xi),\xi)| \\
&\quad \leq |U_\xi(0,\xi) - \tilde{U}_\xi(0,\xi)| + \frac{1}{2}\int_0^{\tau(\xi)} |h - \tilde{h}|(t,\xi)dt \\
&\qquad + \int_0^{\tau(\xi)} \big|\big(U^2 + \frac{1}{2}k^2 - P(X)\big) - \big(\tilde{U}^2 + \frac{1}{2}\tilde{k}^2 - P(\tilde{X})\big)\big||y_\xi|(t,\xi)dt \\
&\qquad + \int_0^{\tau(\xi)} |\tilde{U}^2 + \frac{1}{2}\tilde{k}^2 - P(\tilde{X})||y_\xi - \tilde{y}_\xi|(t,\xi)dt \\
&\qquad + \int_0^{\tau(\xi)} |k||\bar{r} - \bar{\tilde{r}}|(t,\xi)dt + \int_0^{\tau(\xi)} |k - \tilde{k}||\bar{\tilde{r}}|(t,\xi)dt \\
&\quad \leq |U_\xi(0,\xi) - \tilde{U}_\xi(0,\xi)| \\
&\qquad + C(\bar{M})\int_0^{\tau(\xi)} |Z - \tilde{Z}|(t,\xi)dt + C(\bar{M})T|k - \tilde{k}| \\
&\qquad + \gamma C(\bar{M})\left\| \big(U^2 + \frac{1}{2}k^2 - P(X)\big) - \big(\tilde{U}^2 + \frac{1}{2}\tilde{k}^2 - P(\tilde{X})\big) \right\|_{L^1_T E} \\
&\quad \leq |U_\xi(0,\xi) - \tilde{U}_\xi(0,\xi)| \\
&\qquad + C(\bar{M})\int_0^{\tau(\xi)} |Z - \tilde{Z}|(t,\xi)dt + C(\bar{M})T|k - \tilde{k}| \\
&\qquad + \gamma C(\bar{M})\Big(T\tilde{d}_{\mathbb{R}}(X,\tilde{X}) \\
&\qquad\qquad + \int_{\mathbb{R}} \big(\int_{\tau(\xi)}^{\tilde{\tau}(\xi)} \tilde{h}(t,\xi)\chi_{\{\tilde{\tau}(\xi)>\tau(\xi)\}}dt
\end{aligned}
$$

$$+\int_{\tilde\tau(\xi)}^{\tau(\xi)} h(t,\xi)\chi_{\{\tau(\xi)>\tilde\tau(\xi)\}}dt\Big)d\xi\Big).$$

Here we used that for $\xi\in\kappa_{1-\gamma}$, we have $y_\xi(t,\xi)=(y_\xi(t,\xi)+h(t,\xi))\frac{y_\xi(t,\xi)}{y_\xi(t,\xi)+h(t,\xi)}\le \bar M\gamma$. In the last estimate we applied (4.20). For the case $0<\tilde\tau(\xi)<\tau(\xi)\le T$, a similar treatment as in (4.22) yields an estimate of the same form with every τ replaced by $\tilde\tau$ and vice versa. Inserting this estimate and the estimate (4.22) into (4.21) implies, since $\mathrm{meas}(\tilde\kappa_{1-\gamma}),\mathrm{meas}(\kappa_{1-\gamma})\le C(\bar M)$, that

$$(1-\gamma C(\bar M))\int_{\mathbb R}\Big(\int_\tau^{\tilde\tau}\tilde h(t,\xi)\chi_{\{\tilde\tau(\xi)>\tau(\xi)\}}dt+\int_{\tilde\tau}^{\tau}h(t,\xi)\chi_{\{\tau(\xi)>\tilde\tau(\xi)\}}dt\Big)d\xi \\ \le C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)). \tag{4.23}$$

Choosing γ small enough we find that

$$\int_{\mathbb R}\Big(\int_\tau^{\tilde\tau}\tilde h(t,\xi)\chi_{\{\tilde\tau(\xi)>\tau(\xi)\}}dt+\int_{\tilde\tau}^{\tau}h(t,\xi)\chi_{\{\tau(\xi)>\tilde\tau(\xi)\}}dt\Big)d\xi \\ \le C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)), \tag{4.24}$$

and especially

$$\Big\|Q(X)-Q(\tilde X)\Big\|_{L^1_TE}+\Big\|\big(P(X)-U^2-\frac12k^2\big)-\big(P(\tilde X)-\tilde U^2-\frac12\tilde k^2\big)\Big\|_{L^1_TE} \\ \le C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)), \tag{4.25}$$

which is (4.19).

We now return to the proof of (4.18), where we carefully investigate all terms in $\tilde d_{\mathbb R}(X,\tilde X)$ separately using (4.25). The equations $U_t-\tilde U_t=Q(\tilde X)-Q(X)$ and $U_0=\bar U_0+c\chi\circ y_0$ imply

$$\begin{aligned}\Big\|U(t,\,\cdot\,)-\tilde U(t,\,\cdot\,)\Big\|_{L^\infty}&\le\Big\|U_0-\tilde U_0\Big\|_{L^\infty}+\Big\|Q(X)-Q(\tilde X)\Big\|_{L^1_TE}\\&\le\Big\|U_0-\tilde U_0\Big\|_{L^\infty}+C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X))\\&\le C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)),\end{aligned} \tag{4.26}$$

and $y_t-\tilde y_t=U-\tilde U$ yields

$$\|y(t,\,\cdot\,)-\tilde y(t,\,\cdot\,)\|_{L^\infty}\le\|y_0-\tilde y_0\|_{L^\infty}+T\Big\|U-\tilde U\Big\|_{L^\infty_TL^\infty} \\ \le C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+Td_{\mathbb R}(X,\tilde X)). \tag{4.27}$$

Since $\bar U_t=-Q(X)-c\chi'\circ yU$ and $\bar{\tilde U}_t=-Q(\tilde X)-\tilde c\chi'\circ\tilde y\tilde U$, after combining (4.25), (4.26) and (4.27), we obtain

$$\Big\|\bar U(t,\,\cdot\,)-\bar{\tilde U}(t,\,\cdot\,)\Big\|_{L^2}\le C(\bar M)\big(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)\big). \tag{4.28}$$

To estimate $\Big\|Z-\tilde Z\Big\|_{L^\infty_TW}$ we split $\mathbb R$ into two sets. Let us introduce $\mathcal N=\kappa_{1-\gamma}\cup\tilde\kappa_{1-\gamma}$. For $\xi\in\mathcal N^c$, we have $Z_t=F(X)Z$ and $\tilde Z_t=F(\tilde X)\tilde Z$ for all $t\in[0,T]$. Thus $Z_t-\tilde Z_t=(F(X)-F(\tilde X))Z+F(\tilde X)(Z-\tilde Z)$, and

$$\Big\|(Z-\tilde Z)(t,\,\cdot\,)\Big\|_{W(\mathcal N^c)}\le\Big\|Z_0-\tilde Z_0\Big\|_W$$

$$+\int_0^t \Big(\Big\|(F(X)-F(\tilde X))Z(t',\cdot)\Big\|_{W(\mathcal{N}^c)} + \Big\|F(\tilde X)(Z-\tilde Z)(t',\cdot)\Big\|_{W(\mathcal{N}^c)}\Big)dt'.$$

We get after applying Gronwall's lemma

$$\Big\|(Z-\tilde Z)(t,\cdot)\Big\|_{W(\mathcal{N}^c)} \leq C(\bar M)\left(\Big\|Z_0-\tilde Z_0\Big\|_W + \int_0^t \Big\|(F(X)-F(\tilde X))Z(t',\cdot)\Big\|_{W(\mathcal{N}^c)}\,dt'\right) \tag{4.29}$$

for $t\in[0,T]$. By definition,

$$\begin{aligned}&(F(X)-F(\tilde X))Z\\ &\quad= \Big(0, (U^2+\frac12 k^2 - P(X) - \tilde U^2 - \frac12\tilde k^2 + P(\tilde X))y_\xi + (k-\tilde k)\bar r,\\ &\qquad 2(U^2+\frac12 k^2 - P(X) - \tilde U^2 - \frac12 \tilde k^2 + P(\tilde X))U_\xi, -(k-\tilde k)U_\xi\Big).\end{aligned}$$

Moreover, y_ξ, U_ξ, and $\bar r$ are bounded by some constants only depending on $\bar M$ and $\|\bar r\|_{L^\infty_T L^2} + \|U_\xi\|_{L^\infty_T L^2} \leq C(\bar M)$, so that,

$$\begin{aligned}\Big\|(F(X)-F(\tilde X))Z\Big\|_{L^1_T E} &\leq C(\bar M)\Big\|\big(U^2+\frac12 k^2 - P(X)\big) - \big(\tilde U^2 + \frac12\tilde k^2 - P(\tilde X)\big)\Big\|_{L^1_T E}\\ &\quad + C(\bar M)T|k-\tilde k|,\end{aligned}$$

and hence

$$\begin{aligned}\Big\|(Z-\tilde Z)(t,\cdot)\Big\|_{W(\mathcal{N}^c)} &\leq C(\bar M)(\Big\|Z_0-\tilde Z_0\Big\|_W + T|k-\tilde k|)\\ &\quad + C(\bar M)\Big\|\big(U^2+\frac12 k^2-P(X)\big) - \big(\tilde U^2+\frac12\tilde k^2 - P(\tilde X)\big)\Big\|_{L^1_T E}\\ &\leq C(\bar M)(d_{\mathbb{R}}(X_0,\tilde X_0) + T\tilde d_{\mathbb{R}}(X,\tilde X)).\end{aligned} \tag{4.30}$$

For $\xi\in\mathcal{N}$, we assume without loss of generality $0\leq\tau(\xi)\leq\tilde\tau(\xi)\leq T$. Recall that $\mathrm{meas}(\kappa_{1-\gamma})\leq C(\bar M)$ and $\mathrm{meas}(\tilde\kappa_{1-\gamma})\leq C(\bar M)$ and hence $\mathrm{meas}(\mathcal{N})\leq C(\bar M)$. For $t\in[\tau,\tilde\tau]$, we have $Z(t,\xi)=Z(\tau,\xi)$ and $\tilde Z_t = F(\tilde X)\tilde Z$. Thus

$$\frac{d}{dt}(\tilde Z - Z) = F(\tilde X)\tilde Z = F(\tilde X)(\tilde Z - Z) + F(\tilde X)Z, \tag{4.31}$$

and after applying Gronwall's lemma we obtain

$$\begin{aligned}|\tilde Z(t,\xi)-Z(t,\xi)| &\leq C(\bar M)\big(|Z(\tau,\xi)-\tilde Z(\tau,\xi)| + \int_\tau^{\tilde\tau}|F(\tilde X)Z(t',\xi)|dt'\big)\\ &\leq C(\bar M)\big(|Z(\tau,\xi)-\tilde Z(\tau,\xi)| + \frac12\int_\tau^{\tilde\tau} h(\tau,\xi)dt'\big)\end{aligned} \tag{4.32}$$

for $t\in[\tau,\tilde\tau]$. In particular,

$$\int_{\tau(\xi)}^{\tilde\tau(\xi)} h(\tau(\xi),\xi)dt = \int_{\tau(\xi)}^{\tilde\tau(\xi)} h(t,\xi)dt = \int_{\tau(\xi)}^{\tilde\tau(\xi)}(h(t,\xi)-\tilde h(t,\xi))dt + \int_{\tau(\xi)}^{\tilde\tau(\xi)}\tilde h(t,\xi)dt \tag{4.33}$$

$$\leq C(\bar{M})\Big(|U_\xi(\tau(\xi),\xi)-\tilde{U}_\xi(\tau(\xi),\xi)| + \int_{\tau(\xi)}^{\tilde{\tau}(\xi)}(|Z(t,\xi)-\tilde{Z}(t,\xi)|+|g(X(t,\xi))-g(\tilde{X}(t,\xi))|)dt\Big),$$

where we used Lemma 4.5 in the last step, and we get

$$|\tilde{Z}(t,\xi)-Z(t,\xi)| \leq C(\bar{M})\Big(|Z(\tau(\xi),\xi)-\tilde{Z}(\tau(\xi),\xi)| + \int_{\tau(\xi)}^{\tilde{\tau}(\xi)}(|Z(t',\xi)-\tilde{Z}(t',\xi)|+|g(X(t',\xi))-g(\tilde{X}(t',\xi))|)dt'\Big) \tag{4.34}$$

when $t\in[\tau,\tilde{\tau}]$. For $t\leq\tau(\xi)$, we have $Z_t=F(X)Z$ and $\tilde{Z}_t=F(\tilde{X})\tilde{Z}$, so that we obtain after applying Gronwall's lemma once more

$$\begin{aligned}|Z(\tau(\xi),\xi)-\tilde{Z}(\tau(\xi),\xi)| &\leq C(\bar{M})(|Z(0,\xi)-\tilde{Z}(0,\xi)|+\left\|(F(X)-F(\tilde{X}))Z\right\|_{L^1_TE})\\ &\leq C(\bar{M})\big(|Z(0,\xi)-\tilde{Z}(0,\xi)|+d_{\mathbb{R}}(X_0,\tilde{X}_0)+T\tilde{d}_{\mathbb{R}}(X,\tilde{X})\big).\end{aligned} \tag{4.35}$$

Finally, combining (4.34) and (4.35) we end up with

$$\begin{aligned}|Z(t,\xi)-\tilde{Z}(t,\xi)| \leq C(\bar{M})\Big(&|Z(0,\xi)-\tilde{Z}(0,\xi)|\\ &+\int_{\tau(\xi)}^{\tilde{\tau}(\xi)}(|Z(t',\xi)-\tilde{Z}(t',\xi)|+|g(X(t',\xi))-g(\tilde{X}(t',\xi))|)dt'\\ &+d_{\mathbb{R}}(X_0,\tilde{X}_0)+T\tilde{d}_{\mathbb{R}}(X,\tilde{X})\Big),\quad t\in[0,\tilde{\tau}].\end{aligned} \tag{4.36}$$

Integrating (4.36) over the bounded domain $\mathcal{N}$ and applying Minkowski's inequality for integrals, then yields

$$\left\|Z(t,\,\cdot\,)-\tilde{Z}(t,\,\cdot\,)\right\|_{W(\mathcal{N})} \leq C(\bar{M})(d_{\mathbb{R}}(X_0,\tilde{X}_0)+T\tilde{d}_{\mathbb{R}}(X,\tilde{X})). \tag{4.37}$$

Adding up (4.30) and (4.37), we have

$$\left\|Z-\tilde{Z}\right\|_{L^\infty_TW} \leq C(\bar{M})(d_{\mathbb{R}}(X_0,\tilde{X}_0)+T\tilde{d}_{\mathbb{R}}(X,\tilde{X})). \tag{4.38}$$

Finally, it is left to estimate $\left\|g(X(t,\,\cdot\,))-g(\tilde{X}(t,\,\cdot\,))\right\|_{L^2}$. We have to distinguish several cases and therefore we introduce the set

$$I=\{\xi\in\mathbb{R}\mid \iota(X_0(\xi),\tilde{X}_0(\xi))=0\}.$$

(i) If $\xi\in I$, we can choose $\delta\leq\frac{1}{2}$ depending on T and M as in Lemma 4.4. If ξ is such that $d(X_0(\xi),\tilde{X}_0(\xi))<\delta$, then $|g(X(t,\xi))-g(\tilde{X}(t,\xi))|\leq C(\bar{M})(|Z(t,\xi)-\tilde{Z}(t,\xi)|+(|\bar{r}(t,\xi)|+|\bar{\tilde{r}}(t,\xi)|)|k-\tilde{k}|)$. On the other hand, if ξ is such that $d(X_0(\xi),\tilde{X}_0(\xi))\geq\delta$, we have $|g(X(t,\xi))-g(\tilde{X}(t,\xi))|\leq C(\bar{M})\frac{d(X_0(\xi),\tilde{X}_0(\xi))}{\delta}$ since $|g(X(t,\xi))|$ and $|g(\tilde{X}(t,\xi))|$ can be bounded by a constant only depending on $\bar{M}$. Thus we get that, since δ only depends on $\bar{M}$,

$$\begin{aligned}&|g(X(t,\xi))-g(\tilde{X}(t,\xi))|\\ &\quad\leq C(\bar{M})\big(|Z(t,\xi)-\tilde{Z}(t,\xi)|+(|\bar{r}|+|\bar{\tilde{r}}|)|k-\tilde{k}|+d(X_0(\xi),\tilde{X}_0(\xi))\big).\end{aligned} \tag{4.39}$$

Note that since $\iota(X_0(\xi),\tilde X_0(\xi))=0$ by assumption, $d(X_0(\xi),\tilde X_0(\xi))$ is square integrable on I. Hence

$$\begin{aligned}\Big\|g(X(t,\cdot\,))-g(\tilde X(t,\cdot\,))\Big\|_{L^2(I)}&\leq C(\bar M)\big(\Big\|Z(t,\cdot\,)-\tilde Z(t,\cdot\,)\Big\|_W+d_{\mathbb R}(X_0,\tilde X_0)\big)\\&\leq C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)).\end{aligned}\tag{4.40}$$

(*ii*) If $\iota(X_0(\xi),\tilde X_0(\xi))=1$, either $X(t,\xi)\in\Omega_2$ or $X(t,\xi)\in\Omega_1$ and $\tilde X(t,\xi)\in\Omega_3$ (or the symmetric case where either $\tilde X(t,\xi)\in\Omega_2$ or $\tilde X(t,\xi)\in\Omega_1$ and $X(t,\xi)\in\Omega_3$). If $X(t,\xi)\in\Omega_2$, then $g(X(t,\xi))=y_\xi(t,\xi)+h(t,\xi)$ and $g(\tilde X(t,\xi))=\tilde y_\xi(t,\xi)+\tilde h(t,\xi)$, and we have

$$|g(X(t,\xi))-g(\tilde X(t,\xi))|\leq|y_\xi(t,\xi)-\tilde y_\xi(t,\xi)|+|h(t,\xi)-\tilde h(t,\xi)|.$$

If $X(t,\xi)\in\Omega_1$, we know that $d(X_0(\xi),\tilde X_0(\xi))\geq1$ and hence

$$|g(X(t,\xi))-g(\tilde X(t,\xi))|\leq C(\bar M)d(X_0(\xi),\tilde X_0(\xi)).$$

Since the set Ω_1 has finite measure, we get

$$\begin{aligned}\Big\|g(X(t,\cdot\,))-g(\tilde X(t,\cdot\,))\Big\|_{L^2(I^c)}&\leq C(\bar M)\big(\Big\|Z(t,\cdot\,)-\tilde Z(t,\cdot\,)\Big\|_W+d_{\mathbb R}(X_0,\tilde X_0)\big)\\&\leq C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)).\end{aligned}\tag{4.41}$$

Combining (4.27), (4.28), (4.38), (4.40), and (4.41) yields

$$\tilde d_{\mathbb R}(X,\tilde X)\leq C(\bar M)(d_{\mathbb R}(X_0,\tilde X_0)+T\tilde d_{\mathbb R}(X,\tilde X)),\tag{4.42}$$

which implies, if we choose T small enough, that

$$\tilde d_{\mathbb R}(X,\tilde X)\leq C(\bar M)d_{\mathbb R}(X_0,\tilde X_0).\tag{4.43}$$

This finishes the proof. □

Theorem 4.7. *Given $M>0$ and $T>0$, then the solutions $X(t)$ and $\tilde X(t)$ with initial data X_0 and $\tilde X_0$, respectively, in B_M, belong to $B_{\bar M}$ for $t\in[0,T]$. For any given $\tilde t\in[0,T]$, there exist K and $\tilde T$ depending on $\bar M$ and independent of $\tilde t$, such that*

$$\sup_{t\in[\tilde t,\tilde t+\tilde T]\cap[0,T]}d_{\mathbb R}(X(t),\tilde X(t))\leq Kd_{\mathbb R}(X(\tilde t),\tilde X(\tilde t)).\tag{4.44}$$

Proof. A close inspection of the proof of Theorem 4.6 shows that all our estimates use upper bounds, and hence we can replace the constants therein depending on $\bar M$ by the $\bar M$ in the statement of the present theorem. Since $X(t)$ belongs to $B_{\bar M}$ for all $t\in[0,T]$, we can apply Lemma 3.11 (*iv*) which tells us that for any initial time $\tilde t$ we can find for any γ a time interval such that all points enjoying wave breaking are contained in $\kappa_{1-\gamma}$ and this time interval is independent of the initial time $\tilde t$. Hence, after this observation we can follow the proof of Theorem 4.6. □

Theorem 4.8. *For any time $T>0$ there exists a constant K only depending on M and T such that*

$$\sup_{t\in[0,T]}d_{\mathbb R}(X(t),\tilde X(t))\leq Kd_{\mathbb R}(X_0,\tilde X_0)\tag{4.45}$$

for any solutions $X(t)$ and $\tilde{X}(t)$ in $B_{\bar{M}}$, $t \in [0,T]$, with initial data X_0 and $\tilde{X}_0$, respectively, in B_M.

Proof. There exists $\bar{M}$ only depending on M and T such that $X(t)$ and $\tilde{X}(t)$ belong to $B_{\bar{M}}$ for all $t \in [0,T]$, cf. Theorem 3.13. From the short time stability result Theorem 4.7 we know that there exist constants K and $\bar{T}$ depending only on $\bar{M}$ such that

$$d_{\mathbb{R}}(X(t), \tilde{X}(t)) \leq K d_{\mathbb{R}}(X(\tilde{t}), \tilde{X}(\tilde{t})) \tag{4.46}$$

for any $t \in [0,T] \cap [\tilde{t}, \tilde{t} + \bar{T}]$. To obtain global stability we therefore split up the interval $[0,T]$ into smaller time intervals where the last inequality is valid. For any $T > 0$ there exists $N \in \mathbb{N}$ such that $T \leq (N+1)\bar{T}$ and accordingly we define $t_0 = 0$, $t_1 = \bar{T}, \dots, t_N = N\bar{T}$, and $t_{N+1} = T$. Hence $d_{\mathbb{R}}(X(t), \tilde{X}(t)) \leq K d_{\mathbb{R}}(X(t_i), \tilde{X}(t_i))$ for all $t \in [t_i, t_{i+1}]$, due to the last lemma. Hence we finally obtain

$$d_{\mathbb{R}}(X(t), \tilde{X}(t)) \leq K^{N+1} d_{\mathbb{R}}(X_0, \tilde{X}_0), \tag{4.47}$$

which proves the claim. □

In addition to the last stability result, one can also show the Lipschitz continuity of every solution with respect to time.

Lemma 4.9. *Given $M > 0$ and $T > 0$, then we have for any solution $X(t)$ with initial data $X_0 \in B_M$,*

$$d_{\mathbb{R}}(X(t), X(\bar{t})) \leq C_T(M) \left| \bar{t} - t \right|, \quad t, \bar{t} \leq T, \tag{4.48}$$

for a constant $C_T(M)$ which only depends on M and T.

Proof. We already know that for any $T > 0$, we have $\|X(t,\,\cdot\,)\|_{\bar{V}} + \|g(X(t,\,\cdot\,)) - 1\|_{L^2}$ for $t \in [0,T]$, can be bounded by some constant $C_T(M)$ depending on M and T. Then we have

$$\|\zeta(t,\,\cdot\,) - \zeta(\bar{t},\,\cdot\,)\|_{L^\infty} \leq \int_{\bar{t}}^{t} \|U(t',\,\cdot\,)\|_{L^\infty}\, dt' \leq C_T(M)|\bar{t} - t|, \tag{4.49a}$$

$$\begin{aligned} \left\|\bar{U}(t,\,\cdot\,) - \bar{U}(\bar{t},\,\cdot\,)\right\|_{L^2} &\leq \int_{\bar{t}}^{t} \left\|-Q(X)(t',\,\cdot\,) - c\chi' \circ y(t',\,\cdot\,)\bar{U}(t',\,\cdot\,)\right\|_{L^2} dt' \\ &\leq C_T(M)|\bar{t} - t|, \end{aligned} \tag{4.49b}$$

$$\|Z(t,\,\cdot\,) - Z(\bar{t},\,\cdot\,)\|_{W} \leq \int_{\bar{t}}^{t} \|F(X)(t',\,\cdot\,)Z(t',\,\cdot\,)\|_{L^2}\, dt' \leq C_T(M)|\bar{t} - t|. \tag{4.49c}$$

Hence

$$\|X(t,\,\cdot\,) - X(\bar{t},\,\cdot\,)\|_{V} \leq C_T(M)|\bar{t} - t|. \tag{4.50}$$

It remains to estimate $\|g(X(\bar{t},\xi)) - g(X(t,\xi))\|_{L^2}$. If for a given $\xi \in \mathbb{R}$, $X(t,\xi) \in \Omega_1$ and $X(\bar{t},\xi) \in \Omega_1$, we have

$$|g(X(t,\xi)) - g(X(\bar{t},\xi))| = \int_{\bar{t}}^{t} |g_{1,t}(X(t',\xi))| dt', \tag{4.51}$$

where, slightly abusing the notation, $g_{1,t}$ denotes the time derivative of $-U_\xi - 2k\bar{r} + 2y_\xi$. Similarly if $X(t,\xi) \in \Omega_1^c$ and $X(\bar{t},\xi) \in \Omega_1^c$, we get

$$|g(X(t,\xi)) - g(X(\bar{t},\xi))| = \int_{\bar{t}}^{t} |g_{2,t}(X(t',\xi))| dt', \tag{4.52}$$

where $g_{2,t}$ denotes the time derivative of $y_\xi + h$. If $X(t,\xi) \in \Omega_1$ and $X(\bar{t},\xi) \in \Omega_1^c$ (the case $X(t,\xi) \in \Omega_1^c$, $X(\bar{t},\xi) \in \Omega_1$ can be treated in much the same way), we can find a $\tilde{t} \in (\bar{t}, t)$ such that $g_1(X(\tilde{t},\xi)) = g_2(X(\tilde{t},\xi))$ and therefore

(4.53)

$$\begin{aligned} |g(X(t,\xi)) - g(X(\bar{t},\xi))| &\leq |g_1(X(t,\xi)) - g_1(X(\tilde{t},\xi))| + |g_2(X(\tilde{t},\xi)) - g_2(X(\bar{t},\xi))| \\ &\leq \int_{\bar{t}}^{\tilde{t}} |g_{2,t}(X(t',\xi))| dt' + \int_{\tilde{t}}^{t} |g_{1,t}(X(t',\xi))| dt' \\ &\leq \int_{\bar{t}}^{t} |g_{1,t}(X(t',\xi))| + |g_{2,t}(X(t',\xi))| dt'. \end{aligned}$$

Since $\|g_{1,t}(X(t',\cdot))\|_{L^2}$ and $\|g_{2,t}(X(t',\cdot))\|_{L^2}$, can be uniformly bounded by a constant $C_T(M)$ for all $t' \in [\bar{t}, t]$, we get after using (4.51)-(4.53) together with applying Minkowski's inequality for integrals, that

$$\begin{aligned} \|g(X(t,\xi)) - g(X(\bar{t},\xi))\|_{L^2} &\leq \int_{\bar{t}}^{t} \|g_{1,t}(X(t',\cdot))\|_{L^2} + \|g_{2,t}(X(t',\cdot))\|_{L^2}\, dt' \\ &\leq C_T(M)|\bar{t} - t|. \end{aligned} \tag{4.54}$$

□

5. From Eulerian to Lagrangian variables and vice versa

So far the derivation of our system of ordinary differential equations (3.16) in Lagrangian coordinates is only valid for initial data u_0 in Eulerian coordinates, where no concentration of mass takes place. However, it is well known that in the case of conservative solutions, concentration of mass is linked to wave breaking. Since our description of dissipative solutions in Lagrangian variables until wave breaking occurs, coincides with the one used in [22] for conservative solutions, one might hope that the sets of Eulerian and Lagrangian coordinates can be described in much the same way, and that the mappings from Eulerian to Lagrangian coordinates and vice versa can be defined using the same ideas. It turns out that we can do so. For the sake of completeness we summarize these results here. We start by introducing the set of Eulerian and Lagrangian coordinates together with the set of relabeling functions which allows us to identify equivalence classes in the Lagrangian variables.

Definition 5.1 (Eulerian coordinates)**.** *The set $\mathcal{D}$ is composed of all triplets (u, ρ, μ) such that $u \in H_{0,\infty}(\mathbb{R})$, $\rho \in L^2_{\mathrm{const}}(\mathbb{R})$ and μ is a positive finite Radon measure whose absolutely continuous part, μ_{ac}, satisfies*

$$\mu_{\mathrm{ac}} = (u_x^2 + \bar{\rho}^2)\, dx. \tag{5.1}$$

Definition 5.2 (Relabeling functions)**.** *We denote by G the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$f - \mathrm{Id} \text{ and } f^{-1} - \mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}), \tag{5.2a}$$

$$f_\xi - 1 \text{ belongs to } L^2(\mathbb{R}), \tag{5.2b}$$

where Id *denotes the identity function. Given $\kappa > 0$, we denote by G_κ the subset of G defined by*

$$G_\kappa = \{ f \in G \mid \|f - \mathrm{Id}\|_{W^{1,\infty}} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}} \leq \kappa \}. \tag{5.3}$$

Definition 5.3 (Lagrangian coordinates)**.** *The subsets $\mathcal{F}$ and $\mathcal{F}_\kappa$ of $\mathcal{G}$ are defined as*

$$\mathcal{F}_\kappa = \{X = (y, U, h, r) \in \mathcal{G} \mid y + H \in G_\kappa\},$$

and

$$\mathcal{F} = \{X = (y, U, h, r) \in \mathcal{G} \mid y + H \in G\},$$

where $H(t,\xi)$ is defined by

$$H(t,\xi) = \int_{-\infty}^{\xi} h(t,\tilde{\xi})d\tilde{\xi}.$$

Note that $H(t,\xi)$ is finite, since from (3.22f), we have $h = U_\xi^2 + \bar{r}^2 - \zeta_\xi h$ and therefore $h \in L^1(\mathbb{R})$. In addition it should be pointed out that the condition on $y + H$ is closely linked to $\left\|\frac{1}{y_\xi + h}\right\|_{L^\infty}$ as the following lemma shows.

Lemma 5.4 ([31, Lemma 3.2])**.** *Let $\kappa \geq 0$. If f belongs to G_κ, then $1/(1+\kappa) \leq f_\xi \leq 1 + \kappa$ almost everywhere. Conversely, if f is absolutely continuous, $f - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$, f satisfies* (5.2b) *and there exists $d \geq 1$ such that $1/d \leq f_\xi \leq d$ almost everywhere, then $f \in G_\kappa$ for some κ depending only on d and $\|f - \mathrm{Id}\|_{W^{1,\infty}}$.*

An immediate consequence of (3.22b) is therefore the following result.

Lemma 5.5. *The space $\mathcal{G}$ is preserved by the governing equations* (3.16)*.*

For the sake of simplicity, for any $X = (y, U, h, r) \in \mathcal{F}$ and any function $f \in G$, we denote $(y \circ f, U \circ f, h \circ f f_\xi, r \circ f f_\xi)$ by $X \circ f$.

Proposition 5.6. *The map from $G \times \mathcal{F}$ to $\mathcal{F}$ given by $(f, X) \mapsto X \circ f$ defines an action of the group G on $\mathcal{F}$.*

Since G is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/G$ of $\mathcal{F}$ with respect to the action of the group G. The equivalence relation on $\mathcal{F}$ is defined as follows: For any $X, X' \in \mathcal{F}$, we say that X and X' are equivalent if there exists a relabeling function $f \in G$ such that $X' = X \circ f$. We denote by $\Pi(X) = [X]$ the projection of $\mathcal{F}$ into the quotient space $\mathcal{F}/G$, and introduce the mapping $\Gamma\colon \mathcal{F} \to \mathcal{F}_0$ given by

$$\Gamma(X) = X \circ (y + H)^{-1}$$

for any $X = (y, U, h, r) \in \mathcal{F}$. We have $\Gamma(X) = X$ when $X \in \mathcal{F}_0$. It is not hard to prove that Γ is invariant under the action of G, that is, $\Gamma(X \circ f) = \Gamma(X)$ for any $X \in \mathcal{F}$ and $f \in G$. Hence, there corresponds to Γ a mapping $\tilde{\Gamma}$ from the quotient space $\mathcal{F}/G$ to $\mathcal{F}_0$ given by $\tilde{\Gamma}([X]) = \Gamma(X)$ where $[X] \in \mathcal{F}/G$ denotes the equivalence class of $X \in \mathcal{F}$. For any $X \in \mathcal{F}_0$, we have $\tilde{\Gamma} \circ \Pi(X) = \Gamma(X) = X$. Hence, $\tilde{\Gamma} \circ \Pi|_{\mathcal{F}_0} = \mathrm{Id}\,|_{\mathcal{F}_0}$. Any topology defined on $\mathcal{F}_0$ is naturally transported into $\mathcal{F}/G$ by this isomorphism. We equip $\mathcal{F}_0$ with the metric induced by the E-norm, i.e., $d_{\mathcal{F}_0}(X, X') = d_{\mathbb{R}}(X, X')$ for all $X, X' \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is closed in E, this metric is complete. We define the metric on $\mathcal{F}/G$ as

$$d_{\mathcal{F}/G}([X], [X']) = d_{\mathbb{R}}(\Gamma(X), \Gamma(X')),$$

for any $[X], [X'] \in \mathcal{F}/G$. Then, $\mathcal{F}/G$ is isometrically isomorphic with $\mathcal{F}_0$ and the metric $d_{\mathcal{F}/G}$ is complete. As in [31], we can prove the following lemma.

Lemma 5.7. *Given $\alpha \geq 0$. The restriction of Γ to $\mathcal{F}_\alpha$ is a continuous mapping from $\mathcal{F}_\alpha$ to $\mathcal{F}_0$.*

Remark 5.8. *The mapping Γ is not continuous from $\mathcal{F}$ to $\mathcal{F}_0$. The spaces $\mathcal{F}_\alpha$ were precisely introduced in order to make the mapping Γ continuous.*

We denote by $S\colon \mathcal{F}\times[0,\infty)\to\mathcal{F}$ the continuous semigroup which to any initial data $X_0\in\mathcal{F}$ associates the solution $X(t)$ of the system of differential equations (3.16) at time t. As indicated earlier, the two-component Camassa–Holm system is invariant with respect to relabeling. More precisely, using our terminology, we have the following result.

Theorem 5.9. *For any $t>0$, the mapping $S_t\colon\mathcal{F}\to\mathcal{F}$ is G-equivariant, that is,*

$$S_t(X\circ f)=S_t(X)\circ f \tag{5.4}$$

for any $X\in\mathcal{F}$ and $f\in G$. Hence, the mapping $\tilde{S}_t$ from $\mathcal{F}/G$ to $\mathcal{F}/G$ given by

$$\tilde{S}_t([X])=[S_tX]$$

is well-defined. It generates a continuous semigroup.

We have the following diagram:

$$\begin{array}{ccc}
\mathcal{F}_0 & \xrightarrow{\ \Pi\ } & \mathcal{F}/G \\
\uparrow{\scriptstyle \Gamma} & & \\
\mathcal{F}_\alpha & & \uparrow{\scriptstyle \tilde{S}_t} \\
\uparrow{\scriptstyle S_t} & & \\
\mathcal{F}_0 & \xrightarrow{\ \Pi\ } & \mathcal{F}/G
\end{array} \tag{5.5}$$

Next we describe the correspondence between Eulerian coordinates (functions in $\mathcal{D}$) and Lagrangian coordinates (functions in $\mathcal{F}/G$). In order to do so, we have to take into account the fact that the set $\mathcal{D}$ allows the energy density to have a singular part and a positive amount of energy can concentrate on a set of Lebesgue measure zero.

We first define the mapping L from $\mathcal{D}$ to $\mathcal{F}_0$ which to any initial data in $\mathcal{D}$ associates an initial data for the equivalent system in $\mathcal{F}_0$.

Theorem 5.10. *For any (u,ρ,μ) in $\mathcal{D}$, let*

$$y(\xi)=\sup\left\{y \mid \mu((-\infty,y))+y<\xi\right\}, \tag{5.6a}$$

$$h(\xi)=1-y_\xi(\xi), \tag{5.6b}$$

$$U(\xi)=u\circ y(\xi), \tag{5.6c}$$

$$r(\xi)=\rho\circ y(\xi)y_\xi(\xi). \tag{5.6d}$$

Then $(y,U,h,r)\in\mathcal{F}_0$. We denote by $L\colon\mathcal{D}\to\mathcal{F}_0$ the mapping which to any element $(u,\rho,\mu)\in\mathcal{D}$ associates $X=(y,U,h,r)\in\mathcal{F}_0$ given by (5.6)*.*

On the other hand, to any element in $\mathcal{F}$ there corresponds a unique element in $\mathcal{D}$ which is given by the mapping M defined below.

Theorem 5.11. *Given any element $X=(y,U,h,r)\in\mathcal{F}$. Then, the measure $y_\#(\bar{r}(\xi)\,d\xi)$ is absolutely continuous, and we define (u,ρ,μ) as follows, for any ξ such that $x=y(\xi)$,*

$$u(x)=U(\xi), \tag{5.7a}$$

$$\mu = y_\#(h(\xi)\,d\xi), \tag{5.7b}$$

$$\bar\rho(x)\,dx = y_\#(\bar r(\xi)\,d\xi), \tag{5.7c}$$

$$\rho(x) = k + \bar\rho(x). \tag{5.7d}$$

We have that (u,ρ,μ) belongs to $\mathcal{D}$. We denote by $M\colon \mathcal{F} \to \mathcal{D}$ the mapping which to any X in $\mathcal{F}$ associates the element $(u,\rho,\mu) \in \mathcal{D}$ as given by (5.7). In particular, the mapping M is invariant under relabeling.

Finally, one has to declare the connection between the equivalence classes in Lagrangian coordinates and the set of Eulerian coordinates.

Theorem 5.12. *The mappings M and L are invertible. We have*

$$L \circ M = \mathrm{Id}_{\mathcal{F}/G} \text{ and } M \circ L = \mathrm{Id}_{\mathcal{D}}.$$

6. Continuous semigroup of solutions

In the last section we defined the connection between Eulerian and Lagrangian coordinates, which is the main tool when defining weak solutions of the 2CH system. Also stability results will heavily depend on this relation since we want to measure distances between solutions of the 2CH system by measuring the distance in Lagrangian coordinates rather than in Eulerian coordinates.

Accordingly, we define T_t as

$$T_t = M \circ S_t \circ L.$$

The metric $d_{\mathcal{D}}$ is defined as

$$d_D((u_1,\rho_1,\mu_1),(u_2,\rho_2,\mu_2)) = d_{\mathcal{F}_0}(L(u_1,\rho_1,\mu_1), L(u_2,\rho_2,\mu_2)). \tag{6.1}$$

Definition 6.1. *Assume that $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $\rho\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfy*
(i) $u \in L^\infty_{\mathrm{loc}}([0,\infty), H_\infty(\mathbb{R}))$ and $\rho \in L^\infty_{\mathrm{loc}}([0,\infty), L^2_{\mathrm{const}}(\mathbb{R}))$,
(ii) the equations

$$\iint_{[0,\infty)\times\mathbb{R}} \Big[-u(t,x)\phi_t(t,x) + \big(u(t,x)u_x(t,x) + P_x(t,x)\big)\phi(t,x)\Big] dxdt = \int_{\mathbb{R}} u(0,x)\phi(0,x)dx, \tag{6.2}$$

$$\iint_{[0,\infty)\times\mathbb{R}} \Big[(P(t,x) - u^2(t,x) - \frac12 u_x^2(t,x) - \frac12\rho^2(t,x))\phi(t,x) + P_x(t,x)\phi_x(t,x)\Big] dxdt = 0, \tag{6.3}$$

and

$$\iint_{[0,\infty)\times\mathbb{R}} \Big[-\rho(t,x)\phi_t(t,x) - u(t,x)\rho(t,x)\phi_x(t,x)\Big] dxdt = \int_{\mathbb{R}} \rho(0,x)\phi(0,x)dx, \tag{6.4}$$

hold for all $\phi \in C_0^\infty([0,\infty)\times\mathbb{R})$. Then we say that (u,ρ) is a weak global solution of the two-component Camassa–Holm system.

Theorem 6.2. *The mapping T_t is a continuous semigroup of solutions with respect to the metric $d_{\mathcal{D}}$. Given any initial data $(u_0,\rho_0,\mu_0) \in \mathcal{D}$, let $(u(t,\cdot\,),\rho(t,\cdot\,),\mu(t,\cdot\,)) = T_t(u_0,\rho_0,\mu_0)$. Then (u,ρ) is a weak solution to (3.16) and (u,ρ,μ) is a weak solution to*

$$(u^2+\mu+\rho^2-\bar\rho^2)_t + (u(u^2+\mu+\rho^2-\bar\rho^2))_x = (u^3 - 2Pu)_x. \tag{6.5}$$

The function

$$F(t) = \int_{\mathbb{R}} d\mu(t,x) - \int_{\mathbb{R}} \big(u_x^2(t,x) + \bar\rho^2(t,x)\big)dx \tag{6.6}$$

which is an increasing function, equals the amount of energy that has concentrated at sets of measure zero up to time t. Moreover, for every $t \in [0,\infty)$, we clearly have

$$\int_{\mathbb{R}} \big(u_x^2(t,x) + \bar\rho^2(t,x)\big)dx = \int_{\mathbb{R}} d\mu_{ac}(t,x) \le \int_{\mathbb{R}} d\mu(t,x). \tag{6.7}$$

Remark 6.3. *(i) Equation* (6.5) *is also valid in the conservative case, cf.* [22, Thm. 5.2]*. However, note that there is a difference in the definition of the quantity P in the two cases.*
(ii) An example that illustrates this theorem is given by the symmetric peakon–antipeakon collision in case of the CH equation. Consider the case of $n = 2$ in and let $p_1(0) = -p_2(0)$, and $q_1(0) = -q_2(0) < 0$. Then the solution u will vanish pointwise at a collision time t^* when $q_1(t^*) = q_2(t^*)$, that is, $u(t^*,x) = 0$ for all $x \in \mathbb{R}$. At time $t = t^*$, the total energy, $\mu(\mathbb{R})$, has concentrated at the origin. Using our mapping from Eulerian to Lagrangian coordinates, we obtain after collision, that is, for $t > t^*$, that $y_\xi(t,\xi) = 1$ for $\xi \in \mathbb{R}\setminus[0,\mu(\mathbb{R})]$ and $y_\xi(t,\xi) = 0$ for $\xi \in [0,\mu(\mathbb{R})]$. Going back from Lagrangian to Eulerian coordinates we have $u(t,x) = 0$, while the whole energy still is concentrated at the origin.

Proof. Recall that $P(t,x) - u^2(t,x) - \frac{1}{2}k^2$ is defined by (3.2) and since meas($\{x \in \mathbb{R} \mid \mu(t,\{x\}) \neq 0\}) = 0$, we get

(6.8)

$$\begin{aligned}
P(t,x) - u^2(t,x) - \frac{1}{2}k^2&\\
&= -2c\chi(x)\bar u(t,x) - \bar u^2(t,x)\\
&\quad + \frac{1}{2}\int_{\{x\in\mathbb{R}\mid\mu(t,\{x\})=0\}} e^{-|x-z|}(2c\chi\bar u + \bar u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\bar\rho^2 + k\bar\rho)(t,z)dz\\
&\quad + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}2c^2(\chi'^2 + \chi\chi'')(z)dz.
\end{aligned}$$

Thus setting $x = y(t,\xi)$ and performing the change of variables $z = y(t,\eta)$ yields that $P(t,\xi) - U^2(t,\xi) - \frac{1}{2}k^2 = P(t,y(t,\xi)) - u^2(t,y(t,\xi)) - \frac{1}{2}k^2$ coincides with (3.17). Similar considerations yield that $Q(t,\xi) = P_x(t,y(t,\xi))$ coincides with (3.18).

We will only prove (6.2) and (6.5) here since (6.3) and (6.4) can be shown in much the same way. Using the change of variables $x = y(t,\xi)$ and $U_\xi = u_x \circ y y_\xi$ we get

$$\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} (-u\phi_t + uu_x\phi)(t,x)dxdt\\
&\quad = \iint_{\mathbb{R}_+\times\mathbb{R}} (-Uy_\xi(t,\xi)\phi_t(t,y(t,\xi)) + UU_\xi(t,\xi)\phi(t,y(t,\xi)))d\xi dt\\
&\quad = \iint_{\mathbb{R}_+\times\mathbb{R}} (-Uy_\xi(t,\xi)(\phi(t,y(t,\xi)))_t\\
&\qquad\qquad + U^2y_\xi(t,\xi)\phi_x(t,y(t,\xi)) + UU_\xi(t,\xi)\phi(t,y(t,\xi)))d\xi dt\\
&\quad = \int_{\mathbb{R}} Uy_\xi(0,\xi)\phi(0,y(0,\xi))d\xi - \iint_{\mathbb{R}_+\times\mathbb{R}} Qy_\xi(t,\xi)\phi(t,y(t,\xi))d\xi dt
\end{aligned} \tag{6.9}$$

$$= \int_{\mathbb{R}} u(0,x)\phi(0,x)dx - \iint_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)dx.$$

Note that we do not have to restrict the domain in Lagrangian coordinates to $\mathbb{R}_+ \times \{\xi \in \mathbb{R} \mid y_\xi(t,\xi) \neq 0\}$, since the integrand vanishes whenever $y_\xi(t,\xi) = 0$, and therefore integrating in addition over the set $\mathbb{R}_+ \times \{\xi \in \mathbb{R} \mid y_\xi(t,\xi) = 0\}$ has no influence on the value of the integral.

We now turn to the proof of (6.5), which is equivalent to showing

$$(u^2 + \mu + k^2 + 2k\bar{\rho})_t + (u(u^2 + \mu + k^2 + 2k\bar{\rho}))_x = (u^3 - 2Pu)_x, \tag{6.10}$$

in the sense of distributions. Using the change of variables $x = y(t,\xi)$ and $U_\xi = u_x \circ y y_\xi$ we get

(6.11)
$$\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} u^2\phi_t(t,x)dxdt \\
&\quad = \iint_{\mathbb{R}_+\times\mathbb{R}} [(\phi(t,y(t,\xi)))_t - \phi_x(t,y(t,\xi))U(t,\xi)]U^2(t,\xi)y_\xi(t,\xi)d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} (U^2(t,\xi)y_{t,\xi}(t,\xi) + 2U(t,\xi)U_t(t,\xi)y_\xi(t,\xi))\phi(t,y(t,\xi))d\xi dt \\
&\qquad - \iint_{\mathbb{R}_+\times\mathbb{R}} U^3(t,\xi)\phi_\xi(t,y(t,\xi))d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} (U^2(t,\xi)U_\xi(t,\xi) - 2U(t,\xi)Q(t,\xi)y_\xi(t,\xi))\phi(t,y(t,\xi))d\xi dt \\
&\qquad - \iint_{\mathbb{R}_+\times\mathbb{R}} U^3(t,\xi)\phi_\xi(t,y(t,\xi))d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} u^2(t,x)u_x(t,x)\phi(t,x)dxdt - \iint_{\mathbb{R}_+\times\mathbb{R}} u^3(t,x)\phi_x(t,x)dxdt \\
&\qquad + \iint_{\mathbb{R}_+\times\mathbb{R}} 2U(t,\xi)P_\xi(t,\xi)\phi(t,y(t,\xi))d\xi dt.
\end{aligned}$$

Next

(6.12)
$$\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} \phi_t(t,x)d\mu(t,x)dt = \iint_{\mathbb{R}_+\times\mathbb{R}} \phi_t(t,y(t,\xi))h(t,\xi)d\xi dt \\
&\quad = \iint_{\mathbb{R}_+\times\mathbb{R}} [(\phi(t,y(t,\xi)))_t - \phi_x(t,y(t,\xi))y_t(t,\xi)]h(t,\xi)d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} h_t(t,\xi)\phi(t,y(t,\xi))d\xi dt - \iint_{\mathbb{R}_+\times\mathbb{R}} U(t,\xi)h(t,\xi)\phi_x(t,y(t,\xi))d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} 2(U^2(t,\xi) + \frac{1}{2}k^2 - P(t,\xi))U_\xi(t,\xi)\phi(t,y(t,\xi))d\xi dt \\
&\qquad - \iint_{\mathbb{R}_+\times\mathbb{R}} U(t,\xi)h(t,\xi)\phi_x(t,y(t,\xi))d\xi dt \\
&\quad = -\iint_{\mathbb{R}_+\times\mathbb{R}} 2(u^2 + \frac{1}{2}k^2 - P)(t,x)u_x(t,x)\phi(t,x)dxdt
\end{aligned}$$

$$- \iint_{\mathbb{R}_+ \times \mathbb{R}} u(t,x)\phi_x(t,x) d\mu(t,x).$$

Since $k(t) = k(0)$ we get

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} k^2 \phi_t(t,x) dx dt = 0. \tag{6.13}$$

Finally

(6.14)

$$\begin{aligned}
&\iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{\rho}(t,x)\phi_t(t,x) dx dt \\
&\quad = \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{r}(t,\xi)\phi_t(t,y(t,\xi)) d\xi dt \\
&\quad = \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{r}(t,\xi)[(\phi(t,y(t,\xi)))_t - \phi_x(t,y(t,\xi))U(t,\xi)] d\xi dt \\
&\quad = - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{r}_t(t,\xi)\phi(t,y(t,\xi)) d\xi dt - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{r}(t,\xi)U(t,\xi)\phi_x(t,y(t,\xi)) d\xi dt \\
&\quad = \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k^2 U_\xi(t,\xi)\phi(t,y(t,\xi)) d\xi dt - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{\rho}(t,x)u(t,x)\phi_x(t,x) dx dt \\
&\quad = - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k^2 U(t,\xi)\phi_\xi(t,y(t,\xi)) d\xi dt - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{\rho}(t,x)u(t,x)\phi_x(t,x) dx dt \\
&\quad = - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k^2 u(t,x)\phi_x(t,x) dx dt - \iint_{\mathbb{R}_+ \times \mathbb{R}} 2k\bar{\rho}(t,x)u(t,x)\phi_x(t,x) dx dt.
\end{aligned}$$

Adding up the equalities (6.11)–(6.14) proves (6.10).

As far as (6.7) is concerned, we observe that

$$\begin{aligned}
\int_{\mathbb{R}} (u_x^2 + \bar{\rho}^2)(t,x) dx &= \int_{\{x \in \mathbb{R} \mid \mu(t,\{x\})=0\}} (u_x^2 + \bar{\rho}^2)(t,x) dx = \int_{\{\xi \in \mathbb{R} \mid y_\xi(t,\xi) \neq 0\}} h(t,\xi) d\xi \\
&\leq \int_{\mathbb{R}} h(t,\xi) d\xi = \int_{\mathbb{R}} d\mu(t,x) dx.
\end{aligned}$$

□

7. Stability with respect to the initial data

Already in [22] we showed that ρ has a regularizing effect and that wave breaking is closely linked to the initial value of ρ. In particular, we showed that discontinuities travel at finite speed although the 2CH system has an infinite speed of propagation, see [26]. Moreover, we obtained that wave breaking can only occur at points x which satisfy $\rho_0(x) = 0$. For completeness and to motivate assumptions that were made throughout this paper, we recall this result here.

Given $(u, \rho, \mu) \in \mathcal{D}$, $p \in \mathbb{N}$ and an open set I, we say that (u, ρ, μ) is p-regular on an open set I if

$$u \in W^{p,\infty}(I), \;\; \rho \in W^{p-1,\infty}(I) \;\; \text{and } \mu_{\text{ac}} = \mu \text{ on } I.$$

By notation, we set $W^{0,\infty}(I) = L^\infty(I)$.

Theorem 7.1. [22, Theorem 6.1] *We consider the initial data* (u_0, ρ_0, μ_0)*. Assume that* (u_0, ρ_0, μ_0) *is p-regular on a given interval* (x_0, x_1) *and*

$$\rho_0(x)^2 \geq c > 0 \tag{7.1}$$

for $x \in (x_0, x_1)$*. Then, for any* $t \in \mathbb{R}_+$*,* $(u, \rho, \mu)(t, \cdot)$ *is p-regular on the interval* $(y(t, \xi_0), y(t, \xi_1))$*, where* ξ_0 *and* ξ_1 *satisfy* $y(0, \xi_0) = x_0$ *and* $y(0, \xi_1) = x_1$ *and are defined as*

$$\xi_0 = \sup\{\xi \in \mathbb{R} \mid y(0, \xi) \leq x_0\} \text{ and } \xi_1 = \inf\{\xi \in \mathbb{R} \mid y(0, \xi) \geq x_1\}.$$

In the case of conservative solutions we have been able to prove in [22, Theorem 6.3] that any conservative solution (u, μ) of the CH equation with initial data (u_0, μ_0) such that $\mu_0 = \mu_{0,\text{ac}}$ can be approximated by smooth conservative solutions (u_n, ρ_n, μ_n) of the 2CH system with $\rho_n(0, x) \neq 0$ for all $x \in \mathbb{R}$. Observe that the approximate solutions of the 2CH system do not experience wave breaking.

In the context of dissipative solutions we cannot hope that we can approximate dissipative solutions of the CH equation by solutions of the 2CH system which do not enjoy wave breaking according to the definition of our metric in Lagrangian coordinates. This is illustrated in Figure 2. Note that, in [22], we show the regularizing effect of strictly positive ρ. This effect is also present in the example shown in the figure because, for strictly positive ρ_0, dissipative and conservative solutions coincide and the second solution (dashed line) is $C^\infty(\mathbb{R})$. However, ρ_0 is small and we also have numerical errors, so that the solution appears as if it contains peaks. In that way, it looks very much like the conservative solution of the scalar CH equation with initial data u_0. The numerical scheme used to compute these solutions is an adaptation of the scheme studied in [9] and the code is available at [24].

Lemma 7.2. *Given* $(u, \rho) \in H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$*, let us denote by* $X_e(x)$ *the following vector*

$$X_e(x) = (x, \bar{u}(x), c, 1, u_x(x), u_x^2(x) + \bar{\rho}^2(x), \bar{\rho}(x), k), \tag{7.2}$$

then for g *as in Definition* 3.20*, we have* $g(X_e(\,\cdot\,)) - 1 \in L^1(\mathbb{R})$ *and the following holds:*

(i) Given two elements (u_1, ρ_1) *and* (u_2, ρ_2) *in* $H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$ *such that*

$$\text{meas}(\{x \in \mathbb{R} \mid (\rho_1(x) = 0 \text{ and } \rho_2(x) \neq 0) \text{ or } (\rho_1(x) \neq 0 \text{ and } \rho_2(x) = 0)\}) = 0, \tag{7.3}$$

then

$$\|g(u_1, \rho_1) - g(u_2, \rho_2)\|_{L^1} \leq C(\|u_{1,x} - u_{2,x}\|_{L^2} + \|\bar{\rho}_1 - \bar{\rho}_2\|_{L^2}), \tag{7.4}$$

where C *denotes a constant dependent on the* $L^2(\mathbb{R})$*-norm of* $u_{1,x}$*,* $u_{2,x}$*,* $\bar{\rho}_1$*, and* $\bar{\rho}_2$*.*

(ii) Given any sequence (u_n, ρ_n) *in* $H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$ *which converges to* (u, ρ) *in* $H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$ *such that for all* $n \in \mathbb{N}$*,*

$$\text{meas}(\{x \in \mathbb{R} \mid (\rho_0(x) = 0 \text{ and } \rho_n(x) \neq 0) \text{ or } (\rho_0(x) \neq 0 \text{ and } \rho_n(x) = 0)\}) = 0,$$

then $g(u_n, \rho_n) - 1$ *converges to* $g(u, \rho) - 1$ *in* $L^1(\mathbb{R})$*.*

Proof. We will only prove (i) since (ii) is an immediate consequence of (i). We will have to consider different cases and therefore we introduce the following sets:

$$\Omega_1 = \{x \in \mathbb{R} \mid |u_x(x)| + 2|k\bar{\rho}(x)| + 2 \leq 1 + u_x^2(x) + \bar{\rho}^2(x), u_x(x) \leq 0, \text{ and } \rho(x) = 0\},$$

$$\Omega_{2-} = \{x \in \mathbb{R} \mid 1 + u_x^2(x) + \bar{\rho}^2(x) \leq |u_x(x)| + 2|k\bar{\rho}(x)| + 2, u_x(x) \leq 0, \text{ and } \rho(x) = 0\},$$

$$\Omega_{2+} = \{x \in \mathbb{R} \mid 1 + u_x^2(x) + \bar\rho^2(x) \le |u_x(x)| + 2|k\bar\rho(x)| + 2, 0 < u_x(x), \text{ and } \rho(x) = 0\},$$
$$\Omega_{2*} = \{x \in \mathbb{R} \mid |u_x(x)| + 2|k\bar\rho(x)| + 2 \le 1 + u_x^2(x) + \bar\rho^2(x), 0 < u_x(x), \text{ and } \rho(x) = 0\},$$
$$\Omega_3 = \{x \in \mathbb{R} \mid \rho(x) \neq 0\}.$$

Note that $\Omega_{2-} \cup \Omega_{2+} \cup \Omega_{2*} = \Omega_2$. In addition we will denote the sets which correspond to (u_1, ρ_1) and (u_2, ρ_2) by superscripts 1 and 2, respectively.

Observe that we have by the definition of Ω_1 for $x \in \Omega_1^i$, $i = 1, 2$, since $0 = \bar\rho_i + k_i$,

$$|u_{i,x}(x)| + \bar\rho_i^2(x) + 1 \le u_{i,x}^2(x), \quad i = 1, 2,$$

which implies that $|u_{i,x}(x)| \ge 1$ for $i = 1, 2$ and therefore

$$\text{meas}(\{x \in \Omega_1^1\}) + \text{meas}(\{x \in \Omega_1^2\}) \le \|u_{1,x}\|_{L^2}^2 + \|u_{2,x}\|_{L^2}^2 < \infty. \tag{7.5}$$

(i) If $x \in \Omega_1^{1,c} \cap \Omega_1^{2,c} = I_1$, then

$$\begin{aligned}&|g(u_1, \rho_1)(x) - g(u_2, \rho_2)(x)|\\ &\qquad = |(u_{1,x} + u_{2,x})(u_{1,x} - u_{2,x})(x) + (\bar\rho_1 + \bar\rho_2)(\bar\rho_1 - \bar\rho_2)(x)|.\end{aligned}$$

Thus, using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\int_{I_1} |g(u_1, \rho_1)(x) - g(u_2, \rho_2)(x)| dx\\ &\quad \le (\|u_{1,x}\|_{L^2} + \|u_{2,x}\|_{L^2}) \|u_{1,x} - u_{2,x}\|_{L^2} + (\|\bar\rho_1\|_{L^2} + \|\bar\rho_2\|_{L^2}) \|\bar\rho_1 - \bar\rho_2\|_{L^2}.\end{aligned} \tag{7.6}$$

(ii) If $x \in \Omega_1^1 \cap \Omega_1^2 = I_2$, then we have

$$|g(u_1, \rho_1)(x) - g(u_2, \rho_2)(x)| = |(u_{2,x} - u_{1,x})(x) + 2(\bar\rho_1 + \bar\rho_2)(\bar\rho_1 - \bar\rho_2)(x)|,$$

and accordingly

$$\begin{aligned}&\int_{I_2} |g(u_1, \rho_1)(x) - g(u_2, \rho_2)(x)| dx\\ &\quad \le \int_{I_2} |(u_{1,x} - u_{2,x})(x)|\, dx + 2(\|\bar\rho_1\|_{L^2} + \|\bar\rho_2\|_{L^2}) \|\bar\rho_1 - \bar\rho_2\|_{L^2}\\ &\quad \le \text{meas}(I_2)^{1/2} \|u_{1,x} - u_{2,x}\|_{L^2} + 2(\|\bar\rho_1\|_{L^2} + \|\bar\rho_2\|_{L^2}) \|\bar\rho_1 - \bar\rho_2\|_{L^2}\\ &\quad \le (\|u_{1,x}\|_{L^2} + \|u_{2,x}\|_{L^2}) \|u_{1,x} - u_{2,x}\|_{L^2} + 2(\|\bar\rho_1\|_{L^2} + \|\bar\rho_2\|_{L^2}) \|\bar\rho_1 - \bar\rho_2\|_{L^2},\end{aligned} \tag{7.7}$$

using (7.5).
(iii) $x \in (\Omega_1^1 \cap \Omega_{2-}^2) \cup (\Omega_{2-}^1 \cap \Omega_1^2) = I_3$: Without loss of generality, we assume $x \in \Omega_1^1 \cap \Omega_{2-}^2$ (the other case follows similarly). Then,

$$\begin{aligned}&|g(u_1, \rho_1)(x) - g(u_2, \rho_2)(x)|\\ &\qquad = |-u_{1,x}(x) + 2\bar\rho_1^2(x) + 2 - 1 - u_{2,x}^2(x) - \bar\rho_2^2(x)|\\ &\qquad \le |(u_{1,x}^2 - u_{2,x}^2)(x) + (\bar\rho_1^2 - \bar\rho_2^2)(x)| + \left|-u_{1,x}^2(x) - u_{1,x}(x) + \bar\rho_1^2(x) + 1\right|\\ &\qquad = |(u_{1,x}^2 - u_{2,x}^2)(x) + (\bar\rho_1^2 - \bar\rho_2^2)(x)| + u_{1,x}^2(x) + u_{1,x}(x) - \bar\rho_1^2(x) - 1\\ &\qquad \le |(u_{1,x}^2 - u_{2,x}^2)(x) + (\bar\rho_1^2 - \bar\rho_2^2)(x)|\\ &\qquad\quad + u_{1,x}^2(x) + u_{1,x}(x) - \bar\rho_1^2(x) - 1 + 1 + \bar\rho_2^2(x) - u_{2,x}(x) - u_{2,x}^2(x)\\ &\qquad \le 2|(u_{1,x}^2 - u_{2,x}^2)(x)| + 2|(\bar\rho_1^2 - \bar\rho_2^2)(x)| + |u_{1,x}(x) - u_{2,x}(x)|,\end{aligned}$$

and hence, using (7.5), we obtain

(7.8)
$$\int_{I_3} |g(u_1,\rho_1)(x)-g(u_2,\rho_2)(x)|dx \\ \leq 3(\|u_{1,x}\|_{L^2}+\|u_{2,x}\|_{L^2})\|u_{1,x}-u_{2,x}\|_{L^2}+2(\|\bar\rho_1\|_{L^2}+\|\bar\rho_2\|_{L^2})\|\bar\rho_1-\bar\rho_2\|_{L^2}.$$

(iv) $x\in(\Omega_1^1\cap\Omega_{2+}^2)\cup(\Omega_{2+}^1\cap\Omega_1^2)=I_4$: Without loss of generality, we assume $x\in\Omega_1^1\cap\Omega_{2+}^2$ (the other case follows similarly). By assumption we have $-u_{1,x}(x)+2\bar\rho_1^2(x)+2\leq 1+u_{1,x}^2(x)+\bar\rho_1^2(x)$ and $1+u_{2,x}^2(x)+\bar\rho_2^2(x)\leq u_{2,x}(x)+2\bar\rho_2^2(x)+2$, which implies that either

$$\begin{aligned}0&\leq g(u_1,\rho_1)(x)-g(u_2,\rho_2)(x)\\&=-u_{1,x}(x)+2\bar\rho_1^2(x)+2-1-u_{2,x}^2(x)-\bar\rho_2^2(x)\\&\leq(u_{1,x}^2-u_{2,x}^2)(x)+(\bar\rho_1^2-\bar\rho_2^2)(x)\\&\leq(u_{1,x}+u_{2,x})(u_{1,x}-u_{2,x})(x)+(\bar\rho_1+\bar\rho_2)(\bar\rho_1-\bar\rho_2)(x),\end{aligned}$$

or

$$\begin{aligned}0&\leq g(u_2,\rho_2)(x)-g(u_1,\rho_1)(x)\\&=1+u_{2,x}^2(x)+\bar\rho_2^2(x)+u_{1,x}(x)-2\bar\rho_1^2(x)-2\\&\leq u_{2,x}(x)+2\bar\rho_2^2(x)+u_{1,x}(x)-2\bar\rho_1^2(x)\\&\leq(u_{2,x}-u_{1,x})(x)+2(\bar\rho_1+\bar\rho_2)(\bar\rho_2-\bar\rho_1)(x),\end{aligned}$$

where we used in the last step that $0\leq -u_{1,x}(x)$ since $x\in\Omega_1^1$. Thus applying (7.5) yields

(7.9)
$$\int_{I_4} |g(u_1,\rho_1)(x)-g(u_2,\rho_2)(x)|dx \\ \leq (\|u_{1,x}\|_{L^2}+\|u_{2,x}\|_{L^2})\|u_{1,x}-u_{2,x}\|_{L^2}+2(\|\bar\rho_1\|_{L^2}+\|\bar\rho_2\|_{L^2})\|\bar\rho_1-\bar\rho_2\|_{L^2}.$$

(v) $x\in(\Omega_1^1\cap\Omega_{2*}^2)\cup(\Omega_{2*}^1\cap\Omega_1^2)=I_5$: Without loss of generality, we assume $x\in\Omega_1^1\cap\Omega_{2*}^2$ (the other case follows similarly). By assumption we have $-u_{1,x}(x)+2\bar\rho_1^2(x)+2\leq 1+u_{1,x}^2(x)+\bar\rho_1^2(x)$ and $u_{2,x}(x)+2\bar\rho_2^2(x)+2\leq 1+u_{2,x}^2(x)+\bar\rho_2^2(x)$, which implies that either

$$\begin{aligned}0&\leq g(u_1,\rho_1)(x)-g(u_2,\rho_2)(x)\\&\leq-u_{1,x}(x)+2\bar\rho_1^2(x)+2-1-u_{2,x}^2(x)-\bar\rho_2^2(x)\\&\leq(u_{1,x}^2-u_{2,x}^2)(x)+(\bar\rho_1^2-\bar\rho_2^2)(x)\\&=(u_{1,x}+u_{2,x})(u_{1,x}-u_{2,x})(x)+(\bar\rho_1+\bar\rho_2)(\bar\rho_1-\bar\rho_2)(x),\end{aligned}$$

or

$$\begin{aligned}0&\leq g(u_2,\rho_2)(x)-g(u_1,\rho_1)(x)\\&=1+u_{2,x}^2(x)+\bar\rho_2^2(x)+u_{1,x}(x)-2\bar\rho_1^2(x)-2\\&\leq u_{2,x}^2(x)+\bar\rho_2^2(x)-\bar\rho_1^2(x)\\&\leq u_{2,x}(u_{2,x}-u_{1,x})(x)+(\bar\rho_1+\bar\rho_2)(\bar\rho_2-\bar\rho_1)(x),\end{aligned}$$

where we used in the last step that $0 \leq -u_{1,x}u_{2,x}(x)$. Thus we get

(7.10)
$$\int_{I_5} |(g(u_1, \rho_1) - g(u_2, \rho_2))(x)|dx \\ \leq (\|u_{1,x}\|_{L^2} + \|u_{2,x}\|_{L^2}) \|u_{1,x} - u_{2,x}\|_{L^2} + (\|\bar\rho_1\|_{L^2} + \|\bar\rho_2\|_{L^2}) \|\bar\rho_1 - \bar\rho_2\|_{L^2}.$$

Finally adding (7.6)–(7.10), we end up with (7.4). □

Lemma 7.3. *Given a sequence (u_n, ρ_n) in $H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$ which converges to (u, ρ) in $H_{0,\infty}(\mathbb{R}) \times L^2_{\text{const}}(\mathbb{R})$ such that for all $n \in \mathbb{N}$,*

$$\text{meas}(\{x \in \mathbb{R} \mid (\rho_0(x) = 0 \text{ and } \rho_n(x) \neq 0) \text{ or } (\rho_0(x) \neq 0 \text{ and } \rho_n(x) = 0)\}) = 0,$$

then $(u_n, \rho_n, (u_{n,x}^2 + \bar\rho_n^2)dx)$ converges to $(u, \rho, (u_x^2 + \bar\rho^2)dx)$ in $\mathcal{D}$.

Proof. First of all note that according to Lemma 7.2 the function $g(X_{e,n}) - 1$ converges to $g(X_e) - 1$ in $L^1(\mathbb{R})$. Since the set of Eulerian and Lagrangian coordinates and the mappings between them coincide with the ones used in [22], one can prove everything as in [22, Lemma 6.4], except $g(X_n) \to g(X) \in L^2(\mathbb{R})$. Thus we will only show that $g(X_n) \to g(X)$ in $L^2(\mathbb{R})$.

Let $X_n = (y_n, U_n, h_n, r_n)$ and $X = (y, U, h, r)$ be the representatives in $\mathcal{F}_0$ given by (5.6) of $L(u_n, \rho_n, (u_{n,x}^2 + \bar\rho_n^2)dx)$ and $L(u, \rho, (u_x^2 + \bar\rho^2)dx)$, respectively and assume that $X_n \to X$ in V. Abbreviate by $b_n = u_{n,x}^2 + \bar\rho_n^2$ and $b = u_x^2 + \bar\rho^2$ and note that $b_n \to b$ in $L^1(\mathbb{R})$. Since the measures $(u_{n,x}^2 + \bar\rho_n^2)dx$ and $(u_x^2 + \bar\rho^2)dx$ are purely absolutely continuous, we obtain that $y_\xi(\xi) > 0$ almost everywhere and in particular

$$y_\xi = \frac{1}{b \circ y + 1} \quad \text{and} \quad y_{n,\xi} = \frac{1}{b_n \circ y_n + 1}. \tag{7.11}$$

This implies in particular that $g_n \circ y_n y_{n,\xi} := g(X_{e,n}) \circ y_n y_{n,\xi} = g(X_n)$ almost everywhere and $g \circ y y_\xi := g(X_e) \circ y y_\xi = g(X)$ almost everywhere so that

(7.12)
$$\begin{aligned} g(X_n) - g(X) &= g_n \circ y_n y_{n,\xi} - g \circ y y_\xi \\ &= (g_n \circ y_n (b \circ y + 1) - g \circ y(b_n \circ y_n + 1)) y_\xi y_{n,\xi} \\ &= (g_n \circ y_n - g \circ y) y_\xi y_{n,\xi} \\ &\quad + \big(g_n \circ y_n (b \circ y - b_n \circ y_n) + (g_n \circ y_n - g \circ y) b_n \circ y_n\big) y_\xi y_{n,\xi}. \end{aligned}$$

We will study the first term on the right-hand side in detail and explain afterwards how the other terms can be treated similarly. We have

$$(g_n \circ y_n - g \circ y) y_\xi y_{n,\xi} = (g_n - g) \circ y_n y_\xi y_{n,\xi} + (g \circ y_n - g \circ y) y_\xi y_{n,\xi}. \tag{7.13}$$

Using now the change of variables $x = y_n(\xi)$, since $y_\xi(\xi) \leq 1$, we get

$$\|(g_n - g) \circ y_n y_\xi y_{n,\xi}\|_{L^1} \leq \|(g_n - g) \circ y_n y_{n,\xi}\|_{L^1} \leq \|g_n - g\|_{L^1}. \tag{7.14}$$

Since $g \in L^1(\mathbb{R})$, we can find to any $\varepsilon > 0$ a continuous function l with compact support such that $\|g - l\|_{L^1} \leq \varepsilon/3$. Hence we can decompose the second term on the right-hand side of (7.13) into

$$\begin{aligned} (g \circ y_n - g \circ y) y_\xi y_{n,\xi} &= (g \circ y_n - l \circ y_n) y_\xi y_{n,\xi} \\ &\quad + (l \circ y_n - l \circ y) y_\xi y_{n,\xi} + (l \circ y - g \circ y) y_\xi y_{n,\xi}. \end{aligned} \tag{7.15}$$

By arguing as in (7.14), one can show that

$$\|(g\circ y_n - l\circ y_n)y_\xi y_{n,\xi}\|_{L^1} + \|(g\circ y - l\circ y)y_\xi y_{n,\xi}\|_{L^1} \le \frac{2}{3}\varepsilon. \tag{7.16}$$

Moreover, since $y_n \to y$ in $L^\infty(\mathbb{R})$ and l is continuous with compact support, we obtain by applying the Lebesgue dominated convergence theorem, that $l\circ y_n \to l\circ y$ in $L^1(\mathbb{R})$ and thus we can choose n big enough so that

$$\|(l\circ y_n - l\circ y)y_\xi y_{n,\xi}\| \le \|l\circ y_n - l\circ y\|_{L^1} \le \frac{\varepsilon}{3}. \tag{7.17}$$

Hence, we get that $\|(g\circ y_n - g\circ y)y_\xi y_{n,\xi}\| \le \varepsilon$ so that

$$\lim_{n\to\infty} \|(g\circ y_n - g\circ y)y_\xi y_{n,\xi}\| = 0. \tag{7.18}$$

As far as the second term on the right-hand side of (7.12) is concerned, we observe that $g_n\circ y_n y_{n,\xi} \le 2(y_\xi + h) = 2$ and $b_n\circ y_n \le h \le 1$, so that we can follow the same procedures as for the first term. Thus we finally obtain $g(X_n)\to g(X)$ in $L^1(\mathbb{R})$ and since $g(X_n)\le 2$ and $g(X)\le 2$ this implies $g(X_n)\to g(X)$ in $L^2(\mathbb{R})$. □

Lemma 7.4. *Let (u_n,ρ_n,μ_n) be a sequence in $\mathcal{D}$ that converges to (u,ρ,μ) in $\mathcal{D}$. Then*

$$u_n \to u \text{ in } L^\infty(\mathbb{R}), \qquad \bar\rho_n \overset{*}{\rightharpoonup} \bar\rho, \qquad k_n \to k \in \mathbb{R},$$
$$\mu_n \overset{*}{\rightharpoonup} \mu, \qquad g(X_{e,n}) \overset{*}{\rightharpoonup} g(X_e),$$

where $X_{e,n}$ and X_e are defined by (7.2)*, for (u_n,ρ_n,μ_n) and (u,ρ,μ), respectively.*

Proof. Since the set of Eulerian and Lagrangian coordinates and the mappings between them coincide with the ones used in [22], one can prove everything as in [22, Lemma 6.4] except for $g(X_{e,n}) \overset{*}{\rightharpoonup} g(X_e)$, which we will prove now.

We denote by $X_n = (y_n, U_n, h_n, r_n)$ and $X = (y, U, h, r)$ the representative of $L(u_n,\rho_n,\mu_n)$ and $L(u,\rho,\mu)$ given by (5.6). By weak-star convergence we mean that

$$\lim_{n\to\infty}\int_{\mathbb{R}} g(X_{e,n})\phi dx = \int_{\mathbb{R}} g(X_e)\phi dx \tag{7.19}$$

for all continuous functions with compact support. It follows from (3.20), that $y_\xi(\xi) = 0$ implies that $g(X(\xi)) = 0$ and therefore

$$\begin{aligned}\int_{\mathbb{R}} g(X_e)\phi dx &= \int_{\mathbb{R}} g(X_e)\phi dx \\ &= \int_{\{\xi\in\mathbb{R} \mid y_\xi\neq 0\}} g(X)\phi\circ y d\xi = \int_{\mathbb{R}} g(X)\phi\circ y d\xi.\end{aligned} \tag{7.20}$$

Similarly one obtains that $\int_{\mathbb{R}} g(X_{e,n})\phi dx = \int_{\mathbb{R}} g(X_n)\phi\circ y_n d\xi$. Since $y_n\to y$ in $L^\infty(\mathbb{R})$ and $y - \mathrm{Id} \in L^\infty(\mathbb{R})$, the support of $\phi\circ y_n$ is contained in some compact set which can be chosen independently of n and, from Lebesgue's dominated convergence theorem, we have $\phi\circ y_n \to \phi\circ y$ in $L^2(\mathbb{R})$. Hence, since $g(X_n)\to g(X)$ in $L^2(\mathbb{R})$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} g(X_n)\phi\circ y_n d\xi = \int_{\mathbb{R}} g(X)\phi\circ y d\xi. \tag{7.21}$$

Combining all the equalities we obtained so far yields (7.19). □

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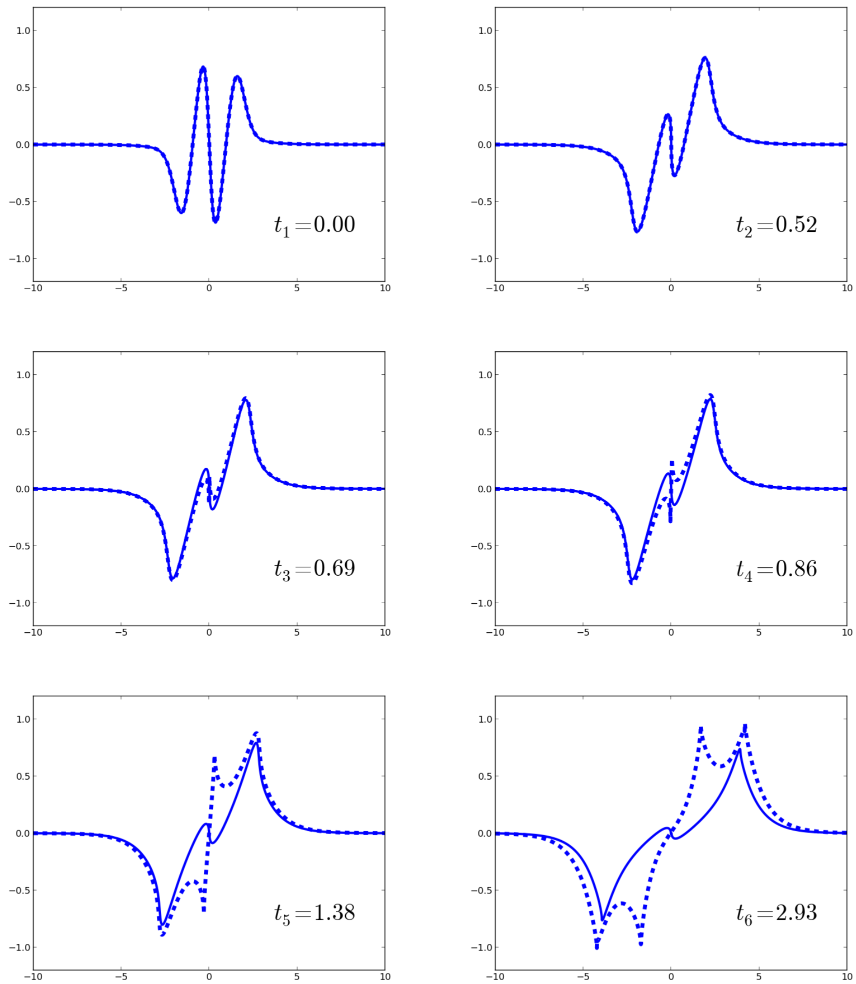

FIGURE 2. Computed dissipative solutions for initial data $u_0(x) = \alpha e^{-x^2} x(x-1)(x+1)$ and $\rho_0(x) = \varepsilon e^{-x^2/10}$ for $\varepsilon = 0$ (solid line) and $\varepsilon = 0.01$ (dashed line). The figures display the function u only. When $\varepsilon = 0$, the function u solves the dissipative CH equation as ρ will remain identically zero. In the case with $\varepsilon > 0$, the fact that ρ_0 is strictly positive implies that no dissipation of energy takes place. In contrast, in the case $\varepsilon = 0$, dissipation starts occurring between the times t_1 and t_2. Prior to that, ρ_0 is so small that we cannot see the difference between the two solutions. However, the solutions look very different for $t \geq t_4$, after the first solution has experienced dissipation. This example show that the semigroup of dissipative solution is not continuous with respect to ρ in a standard sense. This fact is reflected in the construction of our metric, which completely separates initial data for which ρ_0 vanishes and initial data for which ρ_0 is strictly positive in the same region.