



NTNU – Trondheim
Norwegian University of
Science and Technology

Symmetry Breaking in ordinary and supersymmetric Models of Quantum Field Theory

Peder Notto Galteland

Physics

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Supervisor: Kåre Olaussen, IFY

Norwegian University of Science and Technology
Department of Physics

Abstract

This thesis will consider two different theories, and apply variational techniques to each, in order to investigate their true ground states and the possibility of symmetry breaking.

A scalar field theory is modified by introducing a variational mass-parameter, which also serves as a regularization. The effective potential is calculated to two loops, and a variational calculation is done to find the extrema of the potential.

Secondly, the model of Wess and Zumino is modified by removing the restrictions on the particle masses. The vacuum energy is then calculated to two loops, and a variational calculation is performed with the renormalized masses as parameters, in order to determine the true ground state of the theory.

It is discovered that the scalar field theory does not develop any symmetry breaking ground states, as was predicted by Coleman and Weinberg. The non-renormalization and vanishing vacuum energy of the Wess-Zumino model is verified to two-loop order. Furthermore, it is discovered that the modified Wess-Zumino model has several ground states which are more energetically favoured than the supersymmetric state, in which supersymmetry is strongly broken. This may begin explain, if they exist, why we do not see any supersymmetric partners in our accelerators.

Sammendrag

Denne oppgaven vil ta for seg to forskjellige teorier, og anvende teknikker for variasjonsregning på begge, for å kunne undersøke teoriens virkelige grunntilstand og muligheten for symmetribrudd.

En skalar feltteori blir modifisert ved å legge til et masseledd i Lagrangetettheten, som brukes som både variasjonsparameter og en regulariseringsteknikk. Det effektive potensialet blir så funnet til første orden som funksjon av variasjonsparameteren, for å undersøke mulige symmetribrytende ekstremalpunkt.

Deretter blir den supersymmetriske Wess-Zumino modellen modifisert ved å fjerne restriksjonene på partikkelmassene. Vakuumenergien blir så regnet ut til andre orden, og en variasjonsregning blir utført med de renormaliserte massene som parametre for å kunne bestemme den virkelige grunntilstanden til teorien.

Det blir konkludert med at den skalare feltteorien ikke utvikler noen symmetribrytende grunntilstander, noe som har vært forutsett av Coleman og Weinberg. Det blir eksplisitt vist at Wess-Zumino modellen kun får et logaritmisk divergent bidrag til mottermene gjennom renormalisering, og at vakuum-energien er identisk lik null, til andre orden i koblingen. Til slutt blir det konkludert med at vakuumenergien til den supersymmetriske teorien har flere tilstander som er mer energetisk gunstig enn den supersymmetriske, hvor supersymmetri er sterkt brutt. Dette kan være en forklaring på hvorfor vi ikke ser supersymmetriske partnere til partiklene i standardmodellen, hvis de eksisterer.

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Conventions

- We will, unless otherwise specified, always use natural units, where:

$$\hbar = c = k_b = 1$$

This means that masses, temperatures, and energies has units of eV, while length and time has units of $(\text{eV})^{-1}$

- The metric has the signature

$$\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

- The Pauli-matrices will be denoted by σ_i , and are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & i \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Tr denotes general traces, while tr denotes traces over gamma matrices only
- We use the Weyl, or chiral, basis for the gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

where we have defined $\sigma^\mu = (I, \sigma_i), \bar{\sigma}^\mu = (I, -\sigma_i)$.

- The fifth gamma matrix is defined by $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which implies that $\{\gamma^5, \gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = I$

Chapter 1

Introduction

Symmetry breaking is used in the standard model to generate masses in a way that preserves gauge symmetries[1, 2, 3]. Here massless gauge fields obtain masses from a scalar field with a spontaneously broken $SU(2)$ symmetry through the Higgs mechanism[4]. This neatly describes all the known particles, but one piece is missing, the Higgs boson. In this thesis we will explore mechanisms for mass generation through symmetry breaking. Julian Schwinger [5, 6] showed how this might occur in a general way, and Sidney Coleman and Erick Weinberg [7] explored several models using the formalism of the effective action. They showed in their paper that a gauged scalar field may generate a mass due to interactions with the gauge field. In addition to the missing Higgs boson, there are other theoretical problems in the standard model. The Higgs has been almost completely ruled out outside of the mass range of 115–130 GeV[8, 9, 10, 11]. To obtain such a light Higgs, remarkable fine-tuning cancellations of the quadratic corrections to the bare mass is required. This is known as the hierarchy problem. Supersymmetry is, if it exists in nature, expected to stabilize this[12]. If nature exhibits supersymmetry, however, it would have to be softly broken, as we have not yet discovered superpartners to any of the standard model particles[13]. We will therefore also explore the possibility of a model with a number of degrees of freedom consistent with supersymmetry, but with broken supersymmetry.

This thesis will be twofold: First we want to explore Coleman and Weinberg's formalism further, by introducing a test-function with a variational parameter into the Lagrangian of a simple scalar field theory. This is done to attempt to probe for a more favorable ground state of the theory, which may break symmetries, and thus introduce massive particles, if the theory is massless to begin with. The scalar field theory was shown by Coleman and Weinberg to obtain a new minima for the effective potential, but it lies in a region in which their approximation breaks down, so it will be interesting to see if the test-function can shed some more light on this.

Secondly, we want to examine the vacuum energy of the simplest supersymmetric model, first introduced by Julius Wess and Bruno Zumino in 1974[14], called the Wess-Zumino model. First we will examine the theory as it is, mainly to verify two aspects of it which makes it rather interesting, namely what is known as the non-renormalization theorem, and the fact that its vacuum energy is equal to zero. This makes them much less divergent than

regular theories, and simplifies some of the calculations.

After this we will modify the theory, but not by adding a test function, as it is already much more complicated than the scalar field theory in its basic form. It will simply be generalized by not putting any restrictions on the particle masses. This can still be viewed as a way to probe a larger class of possible ground states, perhaps less sophisticated. We will then perform a variational calculation, with the renormalized particle masses as our variational parameters. This is done to see if there is possible to locate a ground state more favourable than the supersymmetric point, where all the particles of the theory have the same mass.

The model of Wess and Zumino is chosen because, as mentioned earlier, supersymmetry is an interesting property that nature may exhibit; but if it does exist it has to be broken in some way, as we do not see any supersymmetric partners in our particle accelerators yet. Therefore we find it interesting to explore if a variational calculation may begin to explain why nature does not choose this particularly beautiful symmetry. The hope here is to show that something close to supersymmetry, as we already know that supersymmetry must be softly broken, is favoured by nature. The philosophy behind this approach is that supersymmetry, if it is a natural state, should emerge naturally from a more general class of theories. One could imagine having a large parameter space of bare masses, couplings and other relevant quantities, from which supersymmetry emerges through renormalization. This has both an energetic and a probabilistic component, as one could imagine starting out with a set of bare parameters picked from some distribution, which then evolve to the most energetically favoured set of renormalized parameters. If large areas of the bare parameter space maps onto smaller areas, or maybe just one point, one could argue that the emergent theory is natural in the sense that nature favours it independently of where we start out in the bare parameter space.

The hypothesis here is that the generalized form of the Wess-Zumino model should have stable ground states in which supersymmetry is broken. This may show whether or not supersymmetry is the most natural state, depending on the energetic qualities of the states.

1.1 Path integral formalism

The calculations used in thesis will be based on the path integral formalism of quantum field theory, so the first thing we must do is define the path integral. This derivation follows Zee[15], chapter I.2, and a thorough treatment of the path integral and functional integration in general can be found in Mandl and Shaw[16], chapters 12 and 13.

We start out with quantum mechanics, which can be considered a $0 + 1$ dimensional field theory. We want to calculate the amplitude for a particle to evolve from some initial state to some other final state, *i.e.*

$$\langle q_F | e^{-iHT} | q_I \rangle, \quad (1.1.1)$$

where q_i are the states in the position representation, H is the Hamiltonian describing the system and T is the time the evolution takes place over. To evaluate this we split the time T into N equal parts $\delta_T = \frac{T}{N}$

$$\langle q_F | e^{-iH\delta_T} \dots e^{-iH\delta_T} | q_I \rangle, \quad (1.1.2)$$

and insert $N - 1$ complete sets of states $\int dq |q\rangle \langle q| = 1$ between each exponential

$$\left(\prod_{n=1}^{N-1} \int dq_n \right) \langle q_F | e^{-iH\delta_T} | q_{N-1} \rangle \langle q_{N-1} | \dots | q_1 \rangle \langle q_1 | e^{-iH\delta_T} | q_I \rangle. \quad (1.1.3)$$

Looking at one of these amplitudes, we have

$$\langle q_{j+1} | e^{-iH\delta_T} | q_j \rangle = \langle q_{j+1} | e^{-iH\delta_T} \left(\int \frac{dp}{2\pi} |p\rangle \langle p| \right) | q_j \rangle. \quad (1.1.4)$$

This can be manipulated further by inserting the general one-particle Hamiltonian $H = \frac{\hat{p}^2}{2m} + V(\hat{q})$, using $\langle q|p\rangle = e^{ipq}$ and doing the resulting Gaussian integration over p

$$\begin{aligned} \langle q_{j+1} | e^{-iH\delta_T} | q_j \rangle &= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta_T + ip(q_{j+1} - q_j) - iV(q)\delta_T} \\ &= \left(\frac{-im}{2\pi\delta_T} \right)^{\frac{1}{2}} e^{\frac{im}{2}\delta_T \left(\frac{q_{j+1} - q_j}{\delta_T} \right)^2 - iV(q)\delta_T}. \end{aligned} \quad (1.1.5)$$

Inserting this into Eq. (1.1.3) with $q_F = q_N$ and $q_I = q_0$ we obtain

$$\begin{aligned} \langle q_F | e^{-iHT} | q_I \rangle &= \left[\left(\frac{-im}{2\pi\delta_T} \right)^{\frac{N}{2}} \prod_{n=0}^{N-1} \int dq_n \right] \\ &\quad \times \exp \left\{ i \sum_{n=0}^{N-1} \delta_T \left[\frac{1}{2} m \left(\frac{q_{n+1} - q_n}{\delta_T} \right)^2 - V(q) \right] \right\} \end{aligned} \quad (1.1.6)$$

We now take the limit $N \rightarrow \infty$ and identify $\sum_{n=0}^{N-1} \delta_T$ as $\int_0^T dt$, and $\frac{q_{n+1}-q_n}{\delta_T}$ as \dot{q} . We also define the path-integral measure as

$$\int Dq(t) = \lim_{N \rightarrow \infty} \left(\frac{-im}{2\pi\delta_T} \right)^{\frac{N}{2}} \prod_{n=0}^{N-1} \int dq_n. \quad (1.1.7)$$

It is important to note that this limit is not trivial when working in the real time formalism. Here one multiplies an infinite number of oscillatory factors, and it is not necessarily true that we have convergence. The Euclidean version of the path integral have been defined rigorously though, but it is not that simple for the real time version. In the end it does not matter however, as any factors in the measure will cancel when we calculate normalized Green's functions, whether they are infinite or not. Assuming that the limit can be taken, we are left with

$$\langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t) e^{i \int_0^T dt (\frac{1}{2} m \dot{q}^2 - V(q))} = \int Dq(t) e^{i \int_0^T dt L(q, \dot{q})}. \quad (1.1.8)$$

This expression is straightforward to generalize to multiple particles with position $q_a(t)$. This can be seen as a collection of oscillators spread out on a lattice, where each site is denoted by the index a . This system is described by

$$\left(\prod_a \int Dq_a(t) \right) e^{i \int_0^T dt L(\mathbf{q}, \dot{\mathbf{q}})}. \quad (1.1.9)$$

Where $L(\mathbf{q}, \dot{\mathbf{q}})$ is any Lagrangian with arbitrary interactions between the lattice points q_a collected in the vector \mathbf{q} .

Now we want to make the transition to quantum field theory. To do that we make the spacing between the lattice sites infinitely small, taking the continuum limit. This roughly means that the index a becomes a continuous spatial variable \vec{x} , the time integral over L becomes a D -dimensional integral $\int d^4x$ over the Lagrangian density \mathcal{L} , our particle positions $q_a(t)$ becomes a field $\phi(x)$ and the product of integrals over all paths becomes an integral over all configurations of the field ϕ . That is, the defining quantity of a quantum field theory is

$$Z = \int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}} \quad (1.1.10)$$

which is the path integral.

1.2 Lagrangian field theory

This thesis relies heavily on the Lagrangian formalism, so we will review some of the points needed here. The Lagrangian \mathcal{L} , or Lagrangian density to be more precise, in field theory is a function of the fields and their derivatives. ϕ and $\partial_\mu \phi$ in the language of scalar field theory. The action, defined as

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi), \quad (1.2.1)$$

describes the dynamics of the fields in the theory through the principle of stationary action, or Hamilton's principle. This basically means that if we evolve the system through time, it will follow the path in parameter space where S is stationary. This leads to the Euler-Lagrange equations, which are the equations of motion for the fields. To find the extremum of S we demand that its variation is zero when we vary $\phi \rightarrow \phi + \delta\phi$.

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\} = 0. \quad (1.2.2)$$

We assume that the variation of a derivative is the same as the derivative of a variation, then we can rewrite the second part.

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\} = 0. \quad (1.2.3)$$

The last term is a total derivative, so it can be removed through a partial integration. This will potentially leave a boundary term, but that will not affect the variational properties. This leaves us with

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\} \delta\phi = 0. \quad (1.2.4)$$

As this should hold for arbitrary variations of the fields, we must conclude that

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right), \quad (1.2.5)$$

which is the Euler-Lagrange equation. It is worth noting that this equation, and the principle used to derive it, can be viewed as a stationary phase approximation to the path integral.

To write down the Lagrangian that describes the theory one is interested in correctly, one has to, in principle, write down every term of the right dimension which contains the fields and derivatives of fields in question. In four space-time dimensions this means that every term has to have mass dimension 4. The free part for each particle has to give the correct equations of motion through the Euler-Lagrange equation, for instance, a scalar field Lagrangian has to give the Klein-Gordon equation. Then one can simply add interaction terms of the correct dimension, that respects the wanted symmetries. One also has to take into account that conventional renormalizability demands that the parameters for each term must have a non-negative mass dimension.

As a simple example we will look at a free scalar field, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2. \quad (1.2.6)$$

Applying Eq. (1.2.5) to this yields

$$-m^2 \phi = \partial_\mu (\partial^\mu \phi),$$

or

$$(\square + m^2)\phi = 0, \quad (1.2.7)$$

which is the Klein-Gordon equation.

1.3 Symmetries and Noethers theorem

A symmetry of the action is some transformation

$$\phi \rightarrow \phi + \alpha \Delta \phi, \quad (1.3.1)$$

which leaves the Lagrangian invariant up to a total divergence, which can always be removed by a partial integration. More precisely, the symmetry must be a one-parameter group of continuous symmetries, connected to the identity transformation. This means that we can consider an infinitesimal change from the identity transformation. That is, under the transformation, the Lagrangian changes as such

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu. \quad (1.3.2)$$

Noether's theorem [17, 18] simply and beautifully states that if the Lagrangian is invariant under such a transformation, there exists a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu, \quad (1.3.3)$$

i.e. $\partial_\mu j^\mu = 0$. To prove this we look at the change of \mathcal{L} under the symmetry $\mathcal{L} \rightarrow \mathcal{L} + \alpha \Delta \mathcal{L}$

$$\begin{aligned} \alpha \Delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi} \alpha \Delta \phi + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi) \\ &= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \alpha \Delta \phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta \phi \\ &= \alpha \partial_\mu \mathcal{J}^\mu. \end{aligned} \quad (1.3.4)$$

The term in the bracket is equal to zero from the Euler-Lagrange equation, Eq. (1.2.5), which means that we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu \right) \equiv \partial_\mu j^\mu = 0, \quad (1.3.5)$$

q.e.d. Another way of looking at this is to consider the charge, which actually is the generator of the symmetry in the first place when one uses the Hamiltonian formalism,

$$Q = \int d^3 x j^0, \quad (1.3.6)$$

which is constant in time.

As an example, consider a Lagrangian invariant under translations in space-time

$$x^\mu \rightarrow x^\mu + a^\mu. \quad (1.3.7)$$

Derivatives generate space-time translations, so the action on the fields are

$$\phi \rightarrow \phi + a^\nu \partial_\nu \phi. \quad (1.3.8)$$

The Lagrangian is also a scalar, so it must transform in the same way

$$\mathcal{L} \rightarrow \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}). \quad (1.3.9)$$

As we have four translations, we will get four conserved currents, $T^\mu{}_\nu$. Inserting $\delta\phi \hat{=} \partial_\nu \phi$ and $\mathcal{J}^\mu \hat{=} \delta^\mu{}_\nu \mathcal{L}$ into Eq. (1.3.3) we obtain

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}, \quad (1.3.10)$$

which is the energy-momentum tensor. This means that a theory invariant under space-time translations respects local energy-momentum conservation, and that the momentum operator generates translations in time and space.

1.3.1 Chiral symmetries

A quantum field theory with chiral symmetry is basically a theory which treats left-handed and right-handed particles independently. Chiral is the Greek word for hand, and chirality then means "handedness". The fifth gamma matrix can be used to project out the chiral components of a Dirac field ψ

$$\begin{aligned} \psi_L &= \frac{1}{2}(1 - \gamma^5)\psi \\ \psi_R &= \frac{1}{2}(1 + \gamma^5)\psi. \end{aligned} \quad (1.3.11)$$

Consider a theory of free massless fermions, described by the Lagrangian

$$\mathcal{L} = \bar{\psi} (i\cancel{\partial} - m) \psi. \quad (1.3.12)$$

If we write out this in terms of left- and right-handed fields, $\psi = \psi_L + \psi_R$, using the projections from Eq. (1.3.11), we obtain

$$\mathcal{L} = \bar{\psi}_L i\cancel{\partial} \psi_L + \bar{\psi}_R i\cancel{\partial} \psi_R - m \bar{\psi}_L \psi_R - m \bar{\psi}_R \psi_L. \quad (1.3.13)$$

As we see, the mass term mixes left- and right-handed fields.

If we set $m = 0$, the action is invariant under the transformation $\psi \rightarrow e^{i\alpha\gamma^5}\psi$, leading to the conserved Noether current $j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. This is what is known as a chiral symmetry.

1.3.2 Scale invariance

A theory with no relevant mass scale is also known as a scale invariant theory. This is not as simple as stating that the theory is describing massless particles with dimensionless couplings however. For a theory to be scale invariant, the couplings have to be independent of any energy scale. This means that the beta-functions, which describe the flow of the couplings over energy scales, of the renormalization group equations describing the theory vanish. This

is what is known as a fixed point in the renormalization group flow, and is important in *e.g.* the theory of phase transitions.

Free massless quantum electrodynamics, while rather trivial, is an example of a theory with scale invariance. No masses nor couplings means that the theory is automatically scale invariant. Massless ϕ^4 theory is scale invariant on the classical level, but quantizing it introduces a scale in the aforementioned manner.

1.4 Symmetry breaking

There are essentially three kinds symmetry breaking in physics, and the differentiation between them is rather unclear at times. One talks about spontaneous, dynamical and anomalous symmetry breakdown. Spontaneous symmetry breaking was the first one to appear historically, and is the most familiar one. The usual description is that the classical minima of the potential is not a symmetric state, which may lead to massless Goldstone bosons. This is what is used in the standard model, where the $SU(2)$ symmetry of the Higgs field is broken. The mechanism behind this is thoroughly explained in the following section.

Dynamical symmetry breakdown on the other hand, is often defined as a symmetry breaking which arises due to the dynamics of the fields themselves, and the dynamics with other fields on a quantum level. This may still lead to Goldstone bosons however. Massless quantum chromodynamics (QCD) is an example of a theory where this happens. Here a quark anti-quark condensate is formed, with a non-zero expectation value, which breaks the chiral symmetry. This in turn gives rise to pions as Goldstone bosons.

As you may already see, there is not really any difference between the two situations on the quantum level. The symmetry breakdown in the standard model can also be considered as a dynamical effect. The nomenclature can be rather confusing, when there really is not much of a difference between the two phenomena.

Finally, one also talks about anomalous symmetry breaking. This is when one has a symmetry on the classical level, but it is broken when quantizing the theory, where the source of the breaking is the integration measure. This can be considered more of an explicit breaking, as we did not really have a symmetry to begin with. Goldstone bosons will therefore not appear in a theory with a anomalously broken continuous symmetry, as it would with a dynamically broken symmetry. An example of this is the anomalous breakdown of chiral symmetry, known as the chiral, or Adler-Bell-Jackiw, anomaly[19, 20]. Another example of an anomalous symmetry breaking is the scale, or conformal anomaly. Again in massless QCD, one has a scale invariance which is broken by an anomaly. This leads to a scale being introduced, namely the scale at which colour confinement occurs, which again determines the masses of the quarks and the hadrons. This is what is known as dimensional transmutation. The theory is originally described by a dimensionless parameter, the coupling, but the anomaly introduces a typical energy scale at which the coupling is defined. Hence, one can "trade" the dimensionless coupling for the dimensionful scale.

As we see, these three are all similar effects, and it can be difficult to keep track of the nomenclature. Nevertheless, in this thesis we will explore different models with different

kinds of symmetries and the possible breaking of said symmetries. We will therefore first review the formalism of (spontaneous) symmetry breaking, Goldstone's theorem and the effective action, which will be needed later on, in the language of scalar field theory.

1.4.1 Example of SSB

It is convenient to first look at a simple example to illustrate the concept of SSB. Consider a set of N fields described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2, \quad (1.4.1)$$

where ϕ^i is a set of N scalar fields. The opposite sign of the mass term, which is what leads to the symmetry breaking, is put in by hand. As the Lagrangian only involves the length of ϕ^i , it is invariant under the transformation

$$\phi^i \rightarrow R^{ij} \phi^j, \quad (1.4.2)$$

where R^{ij} is any $N \times N$ orthogonal matrix. This group of transformations is simply the rotation group, or $O(N)$, which preserves the length of vectors.

The lowest energy configuration is a constant field ϕ_0^i , whose value minimizes the potential

$$V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}[(\phi^i)^2]^2. \quad (1.4.3)$$

This potential is minimized for any set of fields ϕ_0^i that satisfies

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}. \quad (1.4.4)$$

This equation only determines the length of the vector, so it is convenient to choose a vacuum that points along one of the directions of the field vector. It is conventional to choose coordinates so that ϕ_0^i is pointing in the N th direction

$$\phi_0^i = \left(0, \dots, \sqrt{\frac{\mu^2}{\lambda}}\right) \quad (1.4.5)$$

Now we define a new set of fields, with the N th field expanded around the vacuum

$$\phi^i(x) = (\phi^k(x), v + \xi(x)), \quad (1.4.6)$$

where k now runs from 1 to $N - 1$. Then we insert this, and the value $v = \sqrt{\frac{\mu^2}{\lambda}}$ for the vacuum expectation value into the Lagrangian, and obtain

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \phi^k)^2 + \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}(2\mu^2)\xi^2 + \frac{1}{4}\frac{\mu}{\lambda} \\ & - \sqrt{\lambda}\mu\xi^2 - \sqrt{\lambda}\mu\xi(\phi^k)^2 - \frac{\lambda}{2}\xi^2(\phi^k)^2 - \frac{\lambda}{4}[(\phi^k)^2]^2 - \frac{\lambda}{4}\xi^4 \end{aligned} \quad (1.4.7)$$

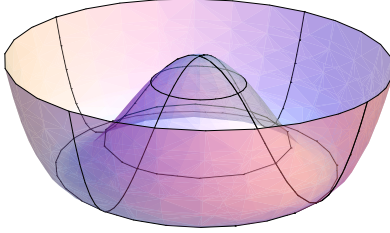


Figure 1.1: A plot of the classical potential for $N = 2$, illustrating the massless excitation "rolling along the gutter" and the massive excitation in the orthogonal direction.

This Lagrangian describes $N - 1$ massless fields and a single massive field, with cubic and quartic interactions. The vacuum energy term is again introduced, and it is worth noting that this term is classical, introduced due to the symmetry breaking, and not from quantum effects.

The $O(N)$ symmetry is no longer apparent, leaving the subgroup $O(N - 1)$, which rotates the ϕ^k fields among themselves. This rotation describes movements along directions where the potential has a vanishing second derivative, while the massive field corresponds to oscillations in the radial direction of V . This can be visualized for $N = 2$, where we get what is often called the "Mexican hat" potential, shown in Fig. 1.1

1.4.2 Goldstone's theorem

The phenomenon where massless particles appear in theories with spontaneously broken *continuous* symmetries is a completely general result, known as *Goldstone's theorem*[21].

The theorem is proven easily on the classical level: Consider a Lagrangian with a symmetry G and a vacuum state invariant under a subgroup of G , H . If we expand the potential around the vacuum state ϕ_0 which minimizes it

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^i(\phi - \phi_0)^j \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} + \dots \quad (1.4.8)$$

The second derivative term is a symmetric matrix whose eigenvalues are the masses squared of the fields.

If we now assume that the potential is invariant under the general transformation

$$\phi^i \rightarrow \phi^i + \alpha \Delta^i(\phi), \quad (1.4.9)$$

where $\Delta^a(\phi)$ is an arbitrary function involving all the fields, we have the following relation

$$V(\phi^i + \alpha \Delta^i \phi) = V(\phi^i), \quad (1.4.10)$$

or if we expand

$$\Delta^i(\phi) \frac{\partial V}{\partial \phi^i} = 0. \quad (1.4.11)$$

If we differentiate this equation with respect to ϕ^j and evaluate it at the minima ϕ_0 we obtain

$$\frac{\partial \Delta^i(\phi)}{\partial \phi^j} \frac{\partial V}{\partial \phi^i} + \Delta^i(\phi) \frac{\partial^2 V}{\partial \phi^j \partial \phi^i} \Big|_{\phi=\phi_0} = 0 \quad (1.4.12)$$

and we see that, as the first term vanishes since ϕ_0 is the minimum of the potential, the second term must also vanish. If the ground state is a symmetry, $\Delta^i(\phi)$ is zero and the relation is trivial. If, on the other hand, the ground state is not left unchanged by the transformation, the second derivative must be zero. This derivative is just the mass matrix, this means that if a symmetry is not a symmetry of the ground state, we must have a zero eigenvalue of the mass matrix in the direction of the symmetry transformation; and hence, particles with zero mass.

In this class of models, that is relativistic models with broken *internal* symmetries, we have the following: If the vacuum is invariant under the subgroup H of G , the number of massless particles, or Goldstone bosons, is equal to the dimension of the left coset G/H , or the number of symmetries broken.

1.4.3 The effective action

When discussing symmetry breakdown in quantum field theories, it is useful to introduce the effective action, Γ . It will allow us to use geometrical arguments, as we did on the classical level, when looking at symmetry breakdown on the quantum level. This will, as was previously implied, blur the line between spontaneous and dynamical symmetry breaking. This is because the effective action will take both quantum and classical effects into account.

We start with our familiar generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\{i \int d^4x (\mathcal{L} + J(x)\phi(x))\}, \quad (1.4.13)$$

then we define an energy functional $E[J]$ by $Z[J] = \exp\{-iE[J]\}$. If we differentiate this with respect to the source J we obtain

$$\begin{aligned} -\frac{\delta E[J(y)]}{\delta J(x)} &= \frac{1}{iZ} \frac{\delta Z[J(y)]}{\delta J(x)} \\ &= \frac{1}{Z} \int \mathcal{D}\phi \phi(x) \exp i \int d^4y (\mathcal{L} + J\phi) \\ &= \frac{\langle 0 | \phi(x) | 0 \rangle_J}{\langle 0 | 0 \rangle_J} = \langle \phi(x) \rangle_J = \phi_c(x). \end{aligned} \quad (1.4.14)$$

The classical field is defined as the vacuum expectation value of the field ϕ , in the presence of the source J

Now we define the effective action as the Legendre transform of $E[J]$.

$$\Gamma[\phi_c] = -E[J] - \int d^4y J(y)\phi_c(y). \quad (1.4.15)$$

Computing the functional derivative of this new quantity

$$\frac{\delta\Gamma[\phi_c(x)]}{\delta\phi_c(y)} = - \int d^4x \frac{\delta J(x)}{\delta\phi_c(y)} \frac{\delta E}{\delta J(x)} - \int d^4x \frac{\delta J(x)}{\delta\phi_c(y)} \phi_c(x) - J(x). \quad (1.4.16)$$

$\frac{\delta E}{\delta J(x)} = -\phi_c(x)$ by definition, so the first and second term cancels, and we end up with

$$\frac{\delta\Gamma[\phi_c(x)]}{\delta\phi_c(y)} = -J(x). \quad (1.4.17)$$

This procedure is completely analogous to *e.g.* the construction of the Hamilton function from the Lagrange function in classical mechanics.

This can be simplified slightly if we only consider translation- and Lorentz-invariant vacuum states $\phi_c(x) = \phi_c$. We know that Γ is an extensive quantity, so it must be proportional to VT , so we write

$$\Gamma[\phi_c] = -(VT)V_{\text{eff}}(\phi_c). \quad (1.4.18)$$

Where V_{eff} is defined as the effective potential. The Legendre transform Eq. (1.4.15) now looks like

$$V_{\text{eff}}(\phi_c) = \frac{E[J]}{VT} + J\phi_c. \quad (1.4.19)$$

Which can be interpreted as, if we remember that $Z = \exp\{-iE[J]\}$ is basically $\langle\Omega|e^{-iHT}|\Omega\rangle$

$$V_{\text{eff}}(\phi_c) = \varrho_{vac} + J\phi_c, \quad (1.4.20)$$

where ϱ_{vac} is the vacuum energy density.

The effective potential is basically the quantum version of the classical potential. Or rather, the classical potential is the classical limit of the quantum object that is the effective potential. This implies that we can use the same geometrical arguments as in Subsection 1.4.1 on the effective potential, with the added benefit that quantum fluctuations are taken into consideration. Finding the true stable vacuum state then reduces to solving the equation

$$\frac{\partial V_{\text{eff}}}{\partial\phi_c} = 0. \quad (1.4.21)$$

To gain some more information of the meaning of Γ , we can expand it in ϕ_c ; just as we do for Z , which gives us the n-point Green's functions.

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \dots, x_n) \phi_c(x_1) \cdots \phi_c(x_n). \quad (1.4.22)$$

The functions $\Gamma^{(n)}$ are the *one-particle-irreducible*(1PI) Green's functions. Hence, the effective action is the generating functional for 1PI Green's functions.

So far we have done this for a simple coupling $J\phi$, but it is straightforward to generalize this to couplings to arbitrary composite operators of all the fields involved. The only difference here would be that Γ no longer generates 1PI Green's functions.

1.4.4 Further motivation for the use of Legendre transforms

Let us assume we want to calculate some integral given by

$$\int \mathcal{D}\phi e^{-S(\phi) + \vec{x} \cdot \vec{f}(\phi)} \equiv e^{-F(\vec{x})}. \quad (1.4.23)$$

Here $\vec{f}(\phi)$ is some arbitrary function of the variables ϕ , which we want to calculate the expectation value of, while \vec{x} is a set of variables used to probe said expectation values. Now we insert unity on the form

$$1 = \int \mathcal{D}\vec{f} \delta(\vec{f} - \vec{f}(\phi)), \quad (1.4.24)$$

and rewrite the integral

$$e^{-F(\vec{x})} = \int \mathcal{D}\vec{f} e^{\vec{x} \cdot \vec{f}} \int \mathcal{D}\phi e^{-S(\phi)} \delta(\vec{f} - \vec{f}(\phi)). \quad (1.4.25)$$

Now we define the quantity $G(\vec{f})$

$$e^{-G(\vec{f})} \equiv \int \mathcal{D}\phi e^{-S(\phi)} \delta(\vec{f} - \vec{f}(\phi)). \quad (1.4.26)$$

This may be used to probe the relevant contributions to the integral (1.4.23). Evaluating Eq. (1.4.26) directly will most likely be difficult, but it is often the case that we may evaluate Eq. (1.4.23) (at least perturbatively), and assume that Eq. (1.4.25) is determined by its maximum value. That is

$$\begin{aligned} e^{-F(\vec{x})} &\approx \max_{\vec{f}} \left(e^{\vec{x} \cdot \vec{f} - G(\vec{f})} \right) \\ F(\vec{x}) &\approx \min_{\vec{f}} \left(G(\vec{f}) - \vec{x} \cdot \vec{f} \right) \end{aligned} \quad (1.4.27)$$

This means that $\vec{x} = \nabla_{\vec{f}} G$, and

$$F(\vec{x}) = G(\vec{f}) - \vec{f} \cdot \nabla_{\vec{f}} G(\vec{f}) \quad (1.4.28)$$

By examining how the minimum changes under a change of \vec{x} and the resulting change in G one can deduce the opposite transformation

$$G(\vec{f}) = F(\vec{x}) + \vec{x} \cdot \nabla_{\vec{x}} F, \quad (1.4.29)$$

where $\vec{f} = -\nabla_{\vec{x}} F$.

As we see, this leads to equations identically to the Legendre transformations introduced in the previous subsection. This means that the Legendre transform is an excellent way to approximate the effective potential. A proper and intuitive overview of the Legendre transform can be found in [22].

Chapter 2

Scalar field theory

2.1 Free scalar field theory

We will first discuss the simplest quantum field theory, free scalar field theory. The path integral, or partition function, in Euclidean space is

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \left\{ - \int d^4x \mathcal{L}_0 \right\} \equiv \exp[-ET], \quad (2.1.1)$$

where the free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2 \phi^2. \quad (2.1.2)$$

This integral can be solved directly using Eq. (A.1.8)

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \int d^4x \phi [-(\square - m^2)] \phi \right\} \\ &= C \det [-(\square - m^2)]^{-\frac{1}{2}}, \end{aligned} \quad (2.1.3)$$

which gives us the energy density of the theory

$$ET = -\log Z = \frac{1}{2} \log \det [\square + m^2] + C = \frac{1}{2} \text{Tr} \log [\square + m^2] + C. \quad (2.1.4)$$

Such traces over general operators can be calculated by integrating it over all space and inserting two completeness relations of properly normalized momentum eigenstates.

$$\begin{aligned} \text{Tr} \mathcal{O} &= \int d^4x \langle x | \mathcal{O} | x \rangle \\ &= \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \langle x | k \rangle \langle k | \mathcal{O} | q \rangle \langle q | x \rangle \\ &= VT \int \frac{d^4k}{(2\pi)^4} \lambda, \end{aligned} \quad (2.1.5)$$

where λ are the eigenvalues of \mathcal{O} . Applying Eq. (2.1.5) to Eq. (2.1.4) we obtain

$$\varrho = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \log(k^2 + m^2) + C. \quad (2.1.6)$$

The factor C depends on the normalization of the measure in the path-integral, which is dependent on the regularization used. It should be independent of any physical parameters, *e.g.* masses or couplings, and we can therefore overlook it. This result should be equivalent to the result obtained through the canonical formalism, *i.e.*

$$\varrho = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2} = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \hbar \omega_k. \quad (2.1.7)$$

To show this we have to do the $k^4 \equiv \omega$ integral. First we partially integrate to get rid of the logarithm

$$\varrho = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \log(\omega^2 + \vec{k}^2 + m^2) = -\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{2\omega^2}{\omega^2 + \omega_k^2} \quad (2.1.8)$$

The ω integral can now be done using contour integration, after we rotate it away from the imaginary axis $\omega \rightarrow -i\omega$

$$I_\omega \equiv - \int \frac{d\omega}{2\pi} \frac{2\omega^2}{\omega^2 + \omega_k^2} = \int \frac{d\omega}{2\pi} \frac{2\omega}{\omega^2 - \omega_k^2 + i\epsilon}. \quad (2.1.9)$$

The contour can now be closed in the upper half plane, picking up the pole at $\omega = -\omega_k + i\epsilon$, and we obtain

$$I_\omega = i \frac{2\pi i}{2\pi} \text{Res} \left\{ \frac{2\omega^2}{\omega^2 - \omega_k^2}; \omega = -\omega_k \right\}. \quad (2.1.10)$$

The residue is $-\omega_k$ and the result is

$$\varrho = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \omega_k, \quad (2.1.11)$$

identical to the canonical result.

2.2 Interacting scalar field theory

This is about as much as we can do with the free theory, so to make it more interesting we add a self-interaction term, the Lagrangian is then

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (2.2.1)$$

The an-harmonic term $\lambda \phi^4$ means that we can not solve the integral directly, as before. To solve this problem, we add a source term $J(x)\phi(x)$ to the action, and get what we call the

generating functional $\mathcal{Z}[J]$. The an-harmonic term can then be dealt with by expanding it and replacing it with functional derivatives with respect to J

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\phi e^{-\int d^4x [\mathcal{L} - J(x)\phi(x)]} \\
&= \int \mathcal{D}\phi e^{-\int d^4x [\mathcal{L}_0 - J(x)\phi(x)]} \left(1 - \int d^4x \frac{\lambda}{4!} \phi^4 + \dots \right) \\
&= \int \mathcal{D}\phi \left(1 - \frac{\lambda}{4!} \int d^4w \left(\frac{\delta}{\delta J(w)} \right)^4 + \dots \right) e^{-\int d^4x [\mathcal{L}_0 - J(x)\phi(x)]} \\
&= e^{-\frac{\lambda}{4!} \int d^4w \left(\frac{\delta}{\delta J(w)} \right)^4} \int \mathcal{D}\phi e^{-\int d^4x [\mathcal{L}_0 - J(x)\phi(x)]}.
\end{aligned} \tag{2.2.2}$$

Since the integral remaining is Gaussian, it can now be solved using Eq. (A.1.9), yielding

$$\mathcal{Z}[J] = \mathcal{Z}_0 e^{-\frac{\lambda}{4!} \int d^4w \left(\frac{\delta}{\delta J(w)} \right)^4} e^{\frac{1}{2} \int d^4x \int d^4y J(x) D(x,y) J(y)}, \tag{2.2.3}$$

where $D(x, y)$ is the inverse of the operator $-\square + m^2$, or the position space propagator, and \mathcal{Z}_0 is the generating functional with both the sources and the couplings set to zero. $D(x, y)$ is the Green's function of the operator, so we can use the following relation

$$(-\square + m^2)D(x, y) = \delta^{(4)}(x - y), \tag{2.2.4}$$

to represent it as a momentum integral, inserting the definition of the delta-function and Fourier transforming, we obtain

$$D(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 + m^2}, \tag{2.2.5}$$

where

$$D(k) = \frac{1}{k^2 + m^2} \tag{2.2.6}$$

is the momentum space propagator.

2.3 Feynman rules

2.3.1 Feynman rules for correlation functions

Now we have all we need to find n -point correlation functions, or Green's functions. These are defined as

$$G^{(n)}(x_1, \dots, x_n) = \langle \phi(x_1) \dots \phi(x_n) \rangle \equiv \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi e^{-S[\phi]} \phi(x_1) \dots \phi(x_n). \tag{2.3.1}$$

As before we can functionally differentiate $\mathcal{Z}[J]$ with respect to $J(x_i)$ to bring down a factor of $\phi(x_i)$, giving us

$$G^{(n)}(x_1, \dots, x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \frac{\mathcal{Z}[J]}{\mathcal{Z}_0}. \quad (2.3.2)$$

This means that we can, by using Eq. (2.3.2) and expanding Eq. (2.2.3) in λ , systematically calculate any Green's function to any order in λ . We simply perform the functional differentiations to the desired order, then set the remaining J 's to zero. As an example we will calculate $G^{(2)}$ explicitly to first order, and derive the Feynman rules by example. First we must define some shorthand notation: We write sources at x , $J(x)$, as J_x ; propagators between x and y , $D(x-y)$, as D_{xy} ; functional derivatives with respect to the source at x , $\frac{\delta}{\delta J(x)}$, as δ_x and integrals over x , $\int d^4x$, as \int_x . We then have to calculate

$$G^{(2)}(x_1, x_2) = \left(1 - \frac{\lambda}{4!} \int_w \delta_w^4\right) \delta_1 \delta_2 e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y}. \quad (2.3.3)$$

The x_1 and x_2 derivatives are easily calculated

$$G^{(2)}(x_1, x_2) = \left(1 - \frac{\lambda}{4!} \int_w \delta_w^4\right) \left[D_{12} + \int_x J_x D_{x1} \int_x J_x D_{x2} \right] e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y}.$$

The $\mathcal{O}(\lambda^0)$ can now be read off, giving just a free propagator $D(x_1 - x_2)$. Continuing, we have the first order contribution

$$\begin{aligned} G^{(2)}(x_1, x_2)_1 &= -\frac{\lambda}{4!} \int_w \delta_w^4 \left[D_{12} + \int_x J_x D_{x1} \int_x J_x D_{x2} \right] e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y} \\ &= -\frac{\lambda}{4!} \int_w \delta_w^3 \left[D_{12} \int_x J_x D_{xw} + D_{1w} \int_x J_x D_{x2} + D_{2w} \int_x J_x D_{x1} \right. \\ &\quad \left. + \int_x J_x D_{x1} \int_x J_x D_{x2} \int_x J_x D_{xw} \right] e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y} \\ &= -\frac{\lambda}{4!} \int_w \delta_w^2 \left[D_{12} D_{ww} + D_{12} \left(\int_x J_x D_{xw} \right)^2 + 2D_{1w} D_{2w} \right. \\ &\quad \left. + 2 \left(D_{1w} \int_x J_x D_{x2} + D_{2w} \int_x J_x D_{x1} \right) \int_x J_x D_{xw} \right. \\ &\quad \left. + D_{ww} \int_x J_x D_{x1} \int_x J_x D_{x2} + \dots \right] e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y} \\ &= -\frac{\lambda}{4!} \int_w \delta_w \left[3D_{12} D_{ww} \int_x J_x D_{xw} + 6D_{1w} D_{1w} \int_x J_x D_{xw} \right. \\ &\quad \left. + 3 \left(D_{1w} \int_x J_x D_{x2} + D_{2w} \int_x J_x D_{x1} \right) D_{ww} + \dots \right] \\ &\quad \times e^{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y} \end{aligned} \quad (2.3.4)$$

here the ellipsis denotes terms of higher order in J than there is derivatives, as they will vanish in the end. The last derivative is carried out and gives, after setting J to zero,

$$G^{(2)}(x_1, x_2)_1 = -\frac{\lambda}{8} \int d^4w D_{12} D_{ww} - \frac{\lambda}{2} \int d^4w D_{1w} D_{ww} D_{w2}. \quad (2.3.5)$$


The first term is disconnected, and therefore not interesting; but the second is a correction to the propagator, which must be handled by renormalization. The connected Green's function is then, to first order

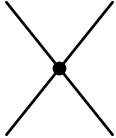
$$G_C^{(2)}(x_1, x_2) = D(x_1 - x_2) - \frac{\lambda}{2} \int d^4w D(x_1 - w) D(0) D(w - x_2) + \dots \quad (2.3.6)$$

These terms can be represented by Feynman diagrams:

$$G_C^{(2)}(x_1, x_2) = \text{---} + \text{---} \bigcirc \text{---} . \quad (2.3.7)$$

The factor 2 the second diagram is divided by is called the symmetry factor. It arises due to the factor of $\frac{1}{2}$ not being cancelled when the derivatives from the λ term hits $\exp\{\frac{1}{2} \int_x \int_y J_x D_{xy} J_y\}$. The different possibilities has already been accounted for with the factor of $4!$. With this definition, the symmetry factor can be read of the diagram, it is simply the number of times one can alter the diagram without changing its topology. For instance, in our example, the factor is 2 because we can twist the loop around, and we have two topologically equivalent diagrams. We can now deduce the Feynman rules in position space for correlation functions:

- Draw all topologically possible diagrams.
- Assign a label to each vertex, *e.g.* $x_1, x_2, w, z \dots$
- For each link  associate a factor $D(x - y)$.

- For each vertex  associate a factor $-\lambda \int d^4w$.

- Divide each diagram by its symmetry factor.

It is also important to note that the *connected* Green's functions can also be obtained by differentiating $\log \mathcal{Z}[J]$ with respect to $J(x_i)$ the appropriate amount of times. It is therefore customary to call $\log \mathcal{Z}[J]$ the generating functional of connected correlation functions.

2.3.2 Momentum space Feynman rules

We would now like to deduce the Feynman rules for momentum space Green's functions, as calculations are more easily done here. From the definition of the Fourier transform we have

$$G^{(2)}(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} G^{(2)}(k), \quad (2.3.8)$$

so we should be able to read off the momentum space version of $G^{(2)}(x_1, x_2)$ from Eq. (2.3.6). After inserting copies of Eq. (2.2.5) into Eq. (2.3.6) we have

$$\begin{aligned} G^{(2)}(x_1, x_2) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x_1-x_2)}}{k^2 + m^2} \\ &\quad - \frac{\lambda}{2} \int d^4w \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x_1-w)}}{k^2 + m^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(w-x_2)}}{q^2 + m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x_1-x_2)}}{k^2 + m^2} \\ &\quad - \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x_1-x_2)}}{k^2 + m^2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \frac{1}{k^2 + m^2}. \end{aligned} \quad (2.3.9)$$

Now it is straightforward to read off $G^{(2)}(k)$

$$G^{(2)}(k) = \frac{1}{k^2 + m^2} - \frac{\lambda}{2} \frac{1}{k^2 + m^2} \left(\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} \right) \frac{1}{k^2 + m^2}. \quad (2.3.10)$$

It is customary to remove the external propagators $\frac{1}{k^2+m^2}$, as we have in the second term. These show up in all diagrams, and are implied. It is also convenient to only consider what is called one particle irreducible, or 1PI, diagram. That is, diagrams that cannot be separated into two pieces by cutting one internal line.

This implies the following Feynman rules for 1PI amputated Green's functions in momentum space:

- Draw all relevant diagrams.
- Assign a momentum flowing through each link, *e.g.* $p, k, q \dots$
- Associate a factor $\frac{1}{k^2+m^2}$ for each link.
- Associate a factor $-\lambda$ for each vertex.
- Integrate over the momenta flowing in each loop.
- Remove all external propagators.
- Divide each diagram by its symmetry factor.

2.3.3 Feynman rules for vacuum energy

To calculate the vacuum energy one has to, in principle, solve the integral $\mathcal{Z}[J]$, as $ET = -\log \mathcal{Z}$. This is, as we saw earlier, impossible for any other than the trivial free theory, so let us try to see if $\log \mathcal{Z}$ can be expressed through Feynman diagrams. Starting from Eq. (2.2.3) we have to calculate

$$\begin{aligned} \log \mathcal{Z} &= \log \mathcal{Z}_0 \\ &+ \log \left[\left(1 - \frac{\lambda}{4!} \int d^4 w \left(\frac{\delta}{\delta J(w)} \right)^4 + \dots \right) e^{\frac{1}{2} \int d^4 x \int d^4 y J(x) D(x,y) J(y)} \right] \\ &= \log \mathcal{Z}_0 + \log \mathcal{Z}_I = -ET, \end{aligned} \quad (2.3.11)$$

with J set to zero in the end. $\log \mathcal{Z}_0$ is just the energy associated with the free part of the theory, which we have already calculated in Eq. (2.1.4). The zeroth order term in the coupling will just give 1 after we set J to zero, so the first order term we have to calculate is

$$-\frac{\lambda}{4!} \int_w \delta_w^4 e^{\frac{1}{2} \int_x \int_y j_x D_{xy} j_y}, \quad (2.3.12)$$

again with obvious shorthand. Performing the derivatives as before we end up with, after setting J to zero

$$-\frac{\lambda}{8} \int d^4 w [D(w-w)]^2. \quad (2.3.13)$$

The next term in the expansion will be

$$\frac{1}{2} \left(-\frac{\lambda}{4!} \right)^2 \int_z \delta_z^4 \int_w \delta_w^4 e^{\frac{1}{2} \int_x \int_y j_x D_{xy} j_y}, \quad (2.3.14)$$

which after a long calculation yields

$$\begin{aligned} &\frac{1}{2} \left(\frac{-\lambda}{8} \int d^4 w [D(0)]^2 \right)^2 + \frac{(-\lambda)^2}{16} \int d^4 w \int d^4 z D(0) D(w-z) D(0) \\ &+ \frac{(-\lambda)^2}{48} \int d^4 w \int d^4 z [D(w-z)]^4. \end{aligned} \quad (2.3.15)$$

Each term has an overall integration over space-time, which gives a factor of VT , and the first term in the second order part has in fact two. The rest of the result can be expressed as Feynman diagrams with no external legs, *i.e.* vacuum diagrams. For example

$$-\frac{\lambda}{8} \int d^4 w [D(w-w)]^2 = VT \cdot \left(\begin{array}{c} \text{8} \end{array} \right). \quad (2.3.16)$$

This follows the conventions already established in the rules for correlation functions. Applying this to the full second order result we have

$$\log \mathcal{Z}_I = \log \left\{ 1 + VT \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} \right) + \frac{1}{2}(VT)^2 \left(\text{diagram 1} \right)^2 + \dots \right\}. \quad (2.3.17)$$

If we carry out the calculations to higher and higher order, we will see clearly what we are beginning to see here. Each diagram will show up in higher powers, with the appropriate factor in front, in such a way that we can exponentiate.

$$\log \mathcal{Z}_I = \log \exp \left\{ VT \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right) \right\}. \quad (2.3.18)$$

or

$$\frac{E_I}{V} = -\frac{\log \mathcal{Z}_I}{VT} = - \left\{ \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right\}. \quad (2.3.19)$$

This also shows that $\log \mathcal{Z}$ only consists of connected diagrams. This implies the following rules for calculating vacuum energy diagrams:

- Draw all possible diagrams to the appropriate order with no external legs.
- Assign a label to each vertex, *e.g.* $x_1, x_2, w, z \dots$
- Associate a factor $D(x_1 - x_2)$ for each link.
- Associate a factor $-\lambda \int d^4 w$ for each vertex.
- There will be an overall integration over space, resulting in a factor of VT , which must be removed.
- Divide each diagram by its symmetry factor.
- Multiply each diagram by -1 .

If we want to work in momentum space we can apply exactly the same logic as when we considered momentum space correlation functions. Hence, we have the following Feynman rules for vacuum energy diagrams in momentum space:

- Draw all possible diagrams to the appropriate order with no external legs.

- Assign a momentum to each link, *e.g.* $p, k, q \dots$
- Associate a factor $\frac{1}{k^2+m^2}$ for each link.
- Associate a factor $-\lambda$ for each vertex.
- Divide each diagram by its symmetry factor.
- Multiply each diagram by -1 .

2.3.4 A note on Euclidean space and Minkowski space

So far in this thesis we have worked in Euclidean space, rather than Minkowski space. This is basically just a question of when to Wick-rotate, which we have to do anyway when we want to perform our integrals, to avoid poles. Later on, when we introduce fermions, we will work exclusively in Minkowski space when considering diagrams. This is because defining the Euclidean path integral for fermions is troublesome, due to difficulties defining the gamma-matrices in Euclidean space.

If we wanted to start out with a Minkowski space path integral for our scalar particles, the only few differences would be in the propagators, vertex factors and the factor we have to multiply each diagram with to obtain vacuum energies. Specifically, the propagator would be

$$D(k) = \frac{i}{k^2 - m^2 + i\epsilon}, \quad (2.3.20)$$

and the vertex factor would now be $-i\lambda$. The presence of $i\epsilon$ in the denominator is now necessary to avoid the pole at $p^2 = m^2$ on the real axis. This factor is implied in every Minkowski space propagator, and will not always be written explicitly. To figure out the overall sign of the vacuum energy diagrams, we look at the definition of the energy functional. In Minkowski space we would have

$$Z[J] = \exp \{-iET\} \quad (2.3.21)$$

$$ET = i \log Z[J] \quad (2.3.22)$$

and hence have to multiply each diagram by i instead of -1 .

2.4 Renormalization

To make sure that our theory will give meaningful predictions, it is important to show that it is renormalizable. Proving this in general is quite involved, but we can show it term by term to the order we are interested in. We use renormalized perturbation theory with a mass counterterm only, as coupling constant renormalization is not necessary at this level. We therefore split the bare mass m^2 into a renormalized mass m_R^2 and a counterterm $\delta m^2 = m^2 - m_R^2$. This does not change anything, we simply rearrange the terms and consider

the renormalized mass as a part of the free theory and the counterterm as a perturbation. The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_R)^2 + \frac{1}{2}m_R^2\phi_R^2 + \frac{\lambda}{4!}\phi_R^4 + \frac{1}{2}\delta m^2\phi_R^2, \quad (2.4.1)$$

where we have suppressed the subscript R on the coupling, or rather; the bare and the renormalized coupling are the same at this level of discussion. The counterterm gives the following addition to the Feynman rules

$$\text{---}\times\text{---} = -\delta m^2. \quad (2.4.2)$$

The renormalization condition at this level is simply that the full propagator is equal to the inverse zeroth order propagator, plus finite terms, diagrammatically

$$\text{---}\textcircled{\text{---}}\text{---} = \left(\text{---} \right)^{-1} + \text{terms regular at } p^2 = m^2. \quad (2.4.3)$$

To first order in the coupling, we have three diagrams

$$\begin{aligned} & \left(\text{---} \right)^{-1} + \text{---}\times\text{---} + \text{---}\textcircled{\text{---}}\text{---} \\ & = p^2 + m_R^2 - \frac{1}{p^2 + m_R^2} \delta m^2 \frac{1}{p^2 + m_R^2} \\ & \quad - \frac{1}{p^2 + m_R^2} \left(\frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_R^2} \right) \frac{1}{p^2 + m_R^2}. \end{aligned} \quad (2.4.4)$$

The renormalization condition leads us to demand that the one-loop diagram is cancelled completely by the counterterm (it may have finite parts also, but it is very convenient to define it this way here), which fixes it at this order to

$$\delta m^2 = -\frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_R^2} = -\frac{\lambda}{2} I_4. \quad (2.4.5)$$

This basically amounts to renormalizing the 2-point 1PI green's function, which is defined as the inverse zeroth order propagator, *minus* the sum of all 1PI diagrams with two external legs. As we simply demand that the counterterm cancels the 1PI diagram(s), we will in this thesis not bother with the extra overall negative sign.

Next we turn to the vacuum energy, where we to zeroth order have the following diagram

$$\text{---}\textcircled{\text{---}}\text{---} = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m_R^2} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log(k^2 + m_R^2). \quad (2.4.6)$$

To first order in the coupling there are two contributions, as the counterterm is proportional to λ

$$\begin{aligned}
\text{Diagram 1} + \text{Diagram 2} &= -\frac{1}{2}\delta m^2 I_l - \frac{1}{8}\lambda I_l \\
&= -\frac{1}{4}\lambda I_l^2 - \frac{1}{8}\lambda I_l^2.
\end{aligned}
\tag{2.4.7}$$

Now, both of these expressions are divergent, but that is not an issue here, as we will only be interested in the energy difference of these expressions compared to some reference energy, or in our case, this energy will be the reference energy as we will see in the next section.

Before we move on it is interesting to note that the value for δm^2 found here is also the one corresponding to the value of the physical mass which minimizes the energy. This is not a general result, it just happens to come out this way here. The vacuum energy density associated with the three diagrams calculated is

$$\varrho(m_R^2) = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \log(k^2 + m_R^2) + \frac{1}{2}\delta m^2 I_l + \frac{\lambda}{8} I_l^2.
\tag{2.4.8}$$

The minima is given by

$$\frac{d\varrho}{dm_R^2} = \frac{1}{2} I_l - \frac{1}{2} I_l + \frac{1}{2}\delta m^2 \frac{dI_l}{dm_R^2} + \frac{\lambda}{4} I_l \frac{dI_l}{dm_R^2} = 0,
\tag{2.4.9}$$

which again gives us

$$\delta m^2 = -\frac{\lambda}{2} I_l,
\tag{2.4.10}$$

or if we set the bare mass to zero

$$m_R^2 = \frac{\lambda}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m_R^2}.
\tag{2.4.11}$$

This is similar to the gap equation which appears in BCS-theory[23].

2.5 The modified theory

In order to attempt to probe the parameter space for a state closer to the real ground state of the scalar field theory, we add a quadratic coupling to an external source, μ^2 . We may not find the exact ground state, but the state we find will hopefully be similar to, and most importantly respond to symmetries in the same way as the real one. To avoid possible singularities from evaluating two fields in the same point we also include a function connecting fields at separate points. That is, we add a term $\mu^2 \mathcal{O}[\phi]$ to the Lagrangian, where $\mathcal{O}[\phi]$ is some operator in position space. Specifically, the action is now

$$S = \int d^4 x \mathcal{L}_0 + \mu^2 \int d^4 x d^4 y \Delta(x-y) \phi(x) \phi(y)
\tag{2.5.1}$$

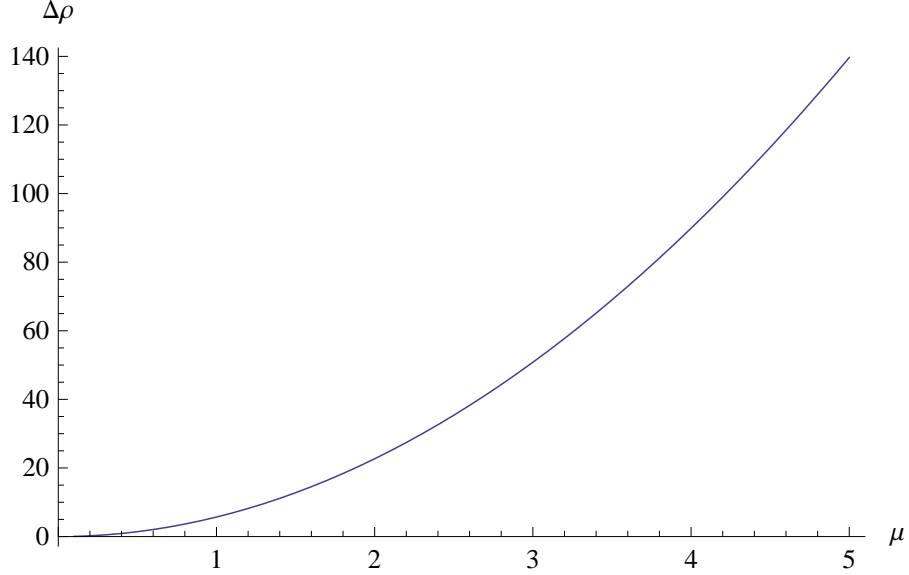


Figure 2.1: Vacuum energy density as a function of the variational parameter μ for $m_R = 0$.

We now introduce the formalism of the effective action from Sec. 1.4.3, and it is straightforward to see that we now have

$$\frac{\delta E}{\delta \mu^2} = -\langle \mathcal{O} \rangle \equiv -\xi^2, \quad (2.5.2)$$

Compared to Eq. (1.4.15), μ^2 is equivalent to the source J and ξ^2 is equivalent to the classical field ϕ_c .

This modification lead to a new set of momentum space Feynman rules

$$\text{====} = \frac{1}{k^2 + m^2 + \mu^2 \tilde{\Delta} m^2}, \quad (2.5.3)$$

$$\text{X} = -\lambda, \quad (2.5.4)$$

where $\tilde{\Delta} m$ is the momentum space version of Δm , and will be chosen such that the graphs we are interested in yield finite results.

2.5.1 Renormalization

In renormalizing the modified theory, we still use the old renormalization condition, *i.e.* the counterterm is still fixed as given in Eq. (2.4.5). The two point function to first order in the

coupling is now

$\text{====} + \text{====} \times \text{====} + \text{====} \circ \text{====} .$

The 0. order part is nice and finite, so we only have to consider the two last contributions. Using Eq. (2.4.5) we see that these two are equal to

$$\begin{aligned}
& -\delta m^2 - \frac{1}{2}\lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_R^2 + \mu^2 \tilde{\Delta} m^2} \equiv -\delta m^2 - \frac{1}{2}\lambda I'_l \\
& = -\frac{1}{2}\lambda \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{k^2 + m_R^2 + \mu^2 \tilde{\Delta} m^2} - \frac{1}{k^2 + m_R^2} \right] \\
& = \frac{1}{2}\lambda \int \frac{d^4k}{(2\pi)^4} \frac{\mu^2 \tilde{\Delta} m^2}{(k^2 + m_R^2 + \mu^2 \tilde{\Delta} m^2)(k^2 + m_R^2)}, \tag{2.5.5}
\end{aligned}$$

which is finite for an appropriate choice of $\tilde{\Delta} m^2$.

We now turn to the difference in vacuum energy compared to the unmodified theory. To zeroth order we have the following expression

$$\begin{aligned}
\Delta \varrho_0 & = \varrho_0[\tilde{\Delta} m^2] - \varrho_0[\tilde{\Delta} m^2 = 0] \\
& = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\sqrt{\vec{k}^2 + m_R^2 + \mu^2 \tilde{\Delta} m^2} - \sqrt{\vec{k}^2 + m_R^2} \right] \\
& = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m_R^2} \left[\sqrt{1 + \frac{\mu^2 \tilde{\Delta} m^2}{\vec{k}^2 + m_R^2}} - 1 \right] \\
& = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log \left(\frac{k^2 + m_R^2 + \mu^2 \tilde{\Delta} m^2}{k^2 + m_R^2} \right), \tag{2.5.6}
\end{aligned}$$

Which again is finite for the appropriate choice of $\tilde{\Delta} m^2$. To first order we have four diagrams contributing

$$\begin{aligned}
& \text{Diagram 1} + \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} \\
& = -\frac{1}{2}\delta m^2 I'_l - \frac{1}{8}\lambda I'_l + \frac{1}{2}\delta m^2 I_l + \frac{1}{8}\lambda I_l. \tag{2.5.7}
\end{aligned}$$

We know from Eq. (2.5.5) that the combination $I'_l - I_l$ is finite, so this expression will hopefully only depend on that particular combination. Inserting again Eq. (2.4.5) into the

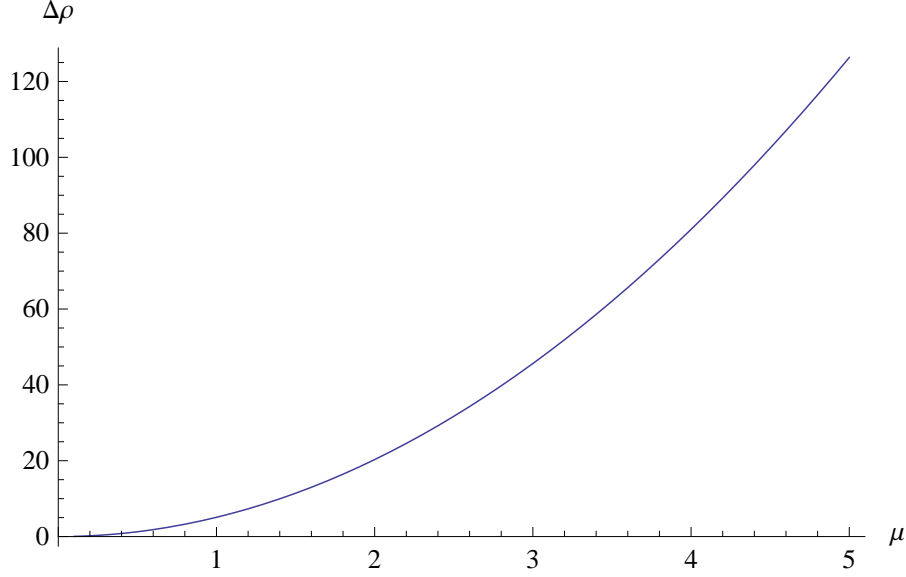


Figure 2.2: Vacuum energy density as a function of the variational parameter μ for $m_R \neq 0$.

associated vacuum energy, we have the following

$$\begin{aligned}
\Delta\varrho_1 &= \varrho_1(\tilde{\Delta}m^2) - \varrho_1(\tilde{\Delta}m^2 = 0) \\
&= -\frac{\lambda}{4}I'_l I_l + \frac{\lambda}{8}I_l'^2 - \frac{\lambda}{4}I_l^2 - \frac{\lambda}{8}I_l^2 \\
&= -\frac{\lambda}{4}I_l(I'_l - I_l) + \frac{\lambda}{8}(I_l'^2 - I_l^2) \\
&= -\frac{\lambda}{8}I_l(I'_l - I_l) + \frac{\lambda}{8}I_l'(I'_l - I_l) = \frac{\lambda}{8}(I'_l - I_l)^2
\end{aligned} \tag{2.5.8}$$

$$\tag{2.5.9}$$

As we see, the renormalization is successful, and we have a finite result.

2.6 Calculations

In order to calculate the vacuum energy density ϱ we must first choose $\tilde{\Delta}m^2$ in such a way that the integrals converge. To make sure that what we calculate are actually energy densities, we choose $\Delta m^2(x - y)$ to be a function of $\vec{x} - \vec{y}$ only, or conversely $\tilde{\Delta}m^2(k)$ a function of \vec{k} only. Then one can still use contour integration to prove that the four dimensional version of Eq. (2.5.6) is equal to the three dimensional one. The choice

$$\tilde{\Delta}m^2(\vec{k}) = \frac{M^4}{(\vec{k}^2 + M^2)^2}. \tag{2.6.1}$$

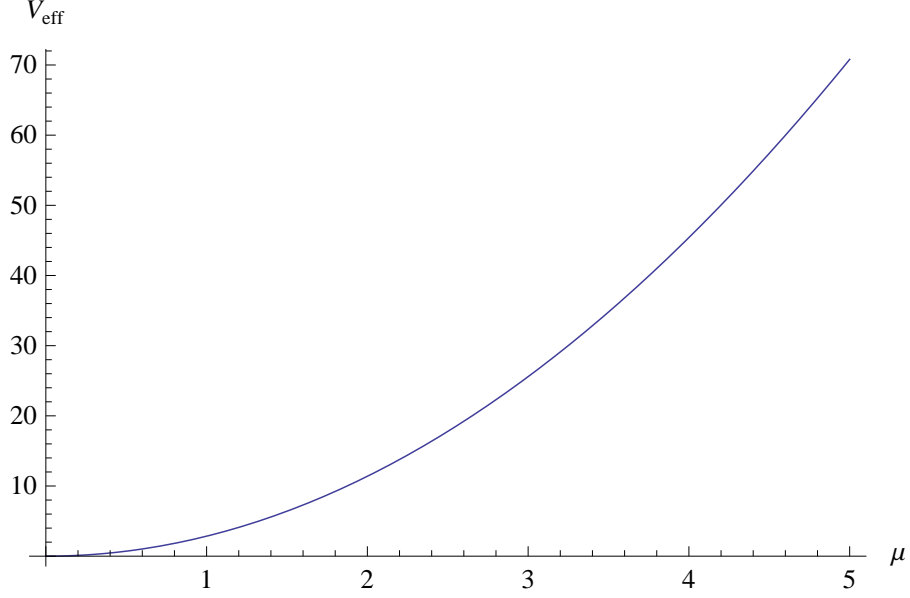


Figure 2.3: The effective potential as a function of the variational parameter μ for $m_R = 0$.

is sufficient to make all of our diagrams finite. Using contour integration we can convert Eq. (2.5.9) into a 3 dimensional integral also, giving us the following two expressions to calculate

$$\Delta\varrho_0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\sqrt{\vec{k}^2 + m_R^2 + \frac{\mu^2 M^4}{(\vec{k}^2 + M^2)^2}} - \sqrt{\vec{k}^2 + m_R^2} \right) \equiv \frac{1}{2} I_0, \quad (2.6.2)$$

$$\Delta\varrho_1 = \frac{\lambda}{8} \left[\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\sqrt{\vec{k}^2 + m_R^2 + \frac{\mu^2 M^4}{(\vec{k}^2 + M^2)^2}} - \frac{1}{\sqrt{\vec{k}^2 + m_R^2}} \right) \right]^2 \equiv \frac{\lambda}{8} I_1^2. \quad (2.6.3)$$

From Eq. (1.4.20) we know that

$$V_{\text{eff}} = \varrho + \mu^2 \xi^2, \quad (2.6.4)$$

where

$$\xi^2 = -\frac{\partial\varrho}{\partial\mu^2}. \quad (2.6.5)$$

Performing the derivatives, we get

$$\begin{aligned} \xi_0 &= -\frac{1}{2} \frac{\partial I_0}{\partial\mu^2} \\ &= -\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \frac{M^4}{(\vec{k}^2 + M^2)^2} \left(\vec{k}^2 + m_R^2 + \frac{\mu^2 M^4}{(\vec{k}^2 + M^2)^2} \right)^{-1/2}, \end{aligned} \quad (2.6.6)$$

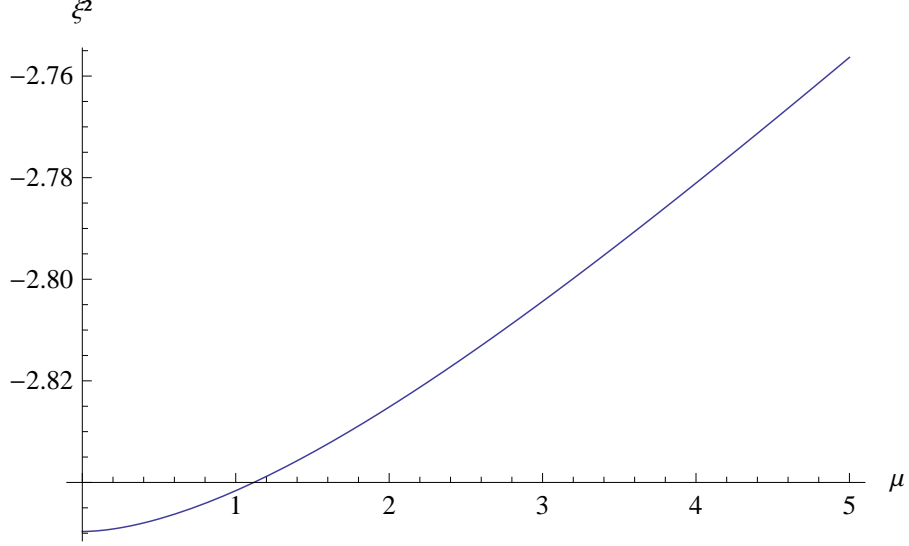


Figure 2.4: The conjugated variable ξ^2 as a function of the variational parameter μ for $m_R = 0$.

and

$$\begin{aligned} \xi_1 &= -\frac{\lambda}{4} I_1 \frac{\partial I_1}{\partial \mu^2} \\ &= \frac{\lambda}{8} I_1 \left[\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{M^4}{(\vec{k}^2 + M^2)^2} \left(\vec{k}^2 + m_R^2 + \frac{M^4}{(\vec{k}^2 + M^2)^2} \right)^{-3/2} \right]. \end{aligned} \quad (2.6.7)$$

As these integrals are rather tricky, they are performed numerically with the aid of Mathematica. The Mathematica notebook is included in Appendix B.

2.6.1 Results and Discussion

We choose to plot the energy for $m_R = 0$ and $m_R \neq 0$ and the effective potential for $m_R = 0$ (using arbitrary units of energy, as only the qualitative appearance is important here). The plots for the energy density is shown in Figs. 2.1 and 2.2. With the plot of the effective action we run in to some difficulties, as we see from Figs. 2.3 and 2.4 for $m_R = 0$, the conjugated variable ξ^2 changes so very slowly in comparison to the effective potential. This means that if we want to plot V_{eff} as a function of ξ^2 parametrically, we get a very steep line for all ranges of μ .

As we see in both plots of the vacuum energy, Figs. 2.1 and 2.2, there is no ground state more favourable than for the unmodified theory. $\Delta\varrho$ is always larger than zero, which means that the energy of the modified theory is always larger than the unmodified one. Hence, the addition of the test function fails to probe a better ground state. Even though we are not

able to create a proper plot of the effective potential as a function of ξ^2 , we can draw the same conclusion from the two separate plots of the potential and ξ^2 as a function of μ . In Figs. 2.3 and 2.4 we see that both of these functions are monotonically increasing functions of μ , and therefore no new minima outside of $\mu = 0$ can be obtained.

It seems, from these results, that Coleman and Weinberg's conclusion; that the minima they observed[7] can not be trusted, was correct. The technique used here is apparently good at probing out the true ground state of the theory, even if it leads to slightly more complicated integrals to solve. The upside to this is that the theory is automatically regularized by a proper choice of test function. It is also renormalizable, given that the unmodified theory is renormalizable in the first place. It seems that a scalar ϕ^4 theory is too simple to exhibit any interesting symmetry breaking behaviour. This leads us to the question of whether more complicated theories will. One could gauge the theory, but that would bring in other difficulties with regularization and renormalization due to gauge invariance. We therefore wish to apply the same ideas to a theory without gauge-fields, but with fermions and Yukawa couplings, namely the supersymmetric Wess-Zumino model.

Chapter 3

Fermions

3.1 Free fermions

Before looking at the full supersymmetric theory, we will have to consider free fermions alone, as there are quite a few differences here from bosons. The free Dirac Lagrangian is as follows

$$\mathcal{L}_D = \bar{\psi}(i\cancel{\partial} - m)\psi. \quad (3.1.1)$$

In order to get the Fermi-Dirac statistics of fermions properly in the path integral, one considers the fields ψ and $\bar{\psi}$ as anti-commuting Grassmann fields. This is covered in Appendix A.4. The path integral is then (we now move away from the imaginary time formalism, and work in Minkowski space),

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x \mathcal{L}_D} = e^{-iET}. \quad (3.1.2)$$

This can be solved directly using the Gaussian Grassmann integral, Eq. (A.4.10), yielding

$$Z = C \det(i\cancel{\partial} - m) = C e^{\text{Tr} \log(i\cancel{\partial} - m)}. \quad (3.1.3)$$

The trace can be massaged a bit to get a more familiar expression

$$\begin{aligned} \text{Tr} \log(i\cancel{\partial} - m) &= \text{Tr} \log \gamma^5 (i\cancel{\partial} - m) \gamma^5 = \text{Tr} \log(-i\cancel{\partial} - m) \\ &= \frac{1}{2} [\text{Tr} \log(i\cancel{\partial} - m) + \text{Tr} \log(-i\cancel{\partial} - m)] \\ &= \frac{1}{2} \text{Tr} \log(\partial^2 + m^2). \end{aligned} \quad (3.1.4)$$

Performing the trace, remembering that we now have to do the operator trace and the Dirac trace, we get the vacuum energy

$$\frac{E}{V} = -\frac{4}{2} \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m^2 + i\epsilon). \quad (3.1.5)$$

As we see, the fermion vacuum energy has the opposite sign of bosons. The factor of 4 from the Dirac trace reflects that the fermions have 4 degrees of freedom, 2 for particles and 2 for antiparticles, both of spin $\frac{1}{2}$. The propagator can also be deduced directly from the path integral, it is the inverse of the operator $i\not{\partial} - m$. Doing the usual routine for Green's functions, we have

$$(i\not{\partial} - m)S_F(x - y) = i\delta^{(4)}(x - y)$$

$$S_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{\not{p} - m + i\epsilon}. \quad (3.1.6)$$

Thus, the momentum space propagator is

$$S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (3.1.7)$$

3.2 Feynman rules for fermions

To determine the Feynman rules for fermions, let us consider Yukawa theory. Take the free Lagrangians of a scalar field and a spinor field, and to that add the three-point coupling (and the four-point interaction of the scalars, but that part is not relevant for the discussion at hand)

$$\mathcal{L}_Y = -g\phi\bar{\psi}\psi. \quad (3.2.1)$$

To calculate correlation functions involving fermions, we have to add two new Grassmannian source terms, $\eta(x)$ and $\bar{\eta}(x)$. These have to be spinors, in order to get a scalar when multiplied by the fermion field.

$$Z[J, \eta, \bar{\eta}] = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{i \int d^4x [\mathcal{L}_\phi + \mathcal{L}_D + \mathcal{L}_Y + J\phi + \bar{\eta}\psi + \bar{\psi}\eta]}. \quad (3.2.2)$$

This can be solved exactly as the scalar theory was, by replacing the coupling terms by functional derivatives with respect to the sources, and then solving the remaining Gaussian. This gives us

$$Z[J, \eta, \bar{\eta}] = Z_0 \exp \left\{ -ig \int d^4w \left(\frac{1}{i^3} \frac{\delta^3}{\delta J(w) \delta \eta(w) \delta \bar{\eta}(w)} \right) \right\}$$

$$\times e^{-\frac{i}{2} \int d^4x d^4y J(x) D(x-y) J(y)} e^{i \int d^4x d^4y \bar{\eta}(x) S(x-y) \eta(y)}, \quad (3.2.3)$$

where Z_0 is the solution to the combined free scalar and spinor path integral.

Now one can expand the coupling term in g and perform functional derivatives in almost exactly the same way as for the scalar field to calculate correlation functions or vacuum energies. One caveat is that one has to be mindful of the overall sign, as the functional derivatives are now anti-commuting. For instance, the three point function

$$G^{(1,2)}(x_1, x_2, x_3) = \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta \eta(x_2)} \frac{\delta}{i\delta \bar{\eta}(x_3)} \frac{Z[J, \eta, \bar{\eta}]}{Z_0}. \quad (3.2.4)$$

This can be calculated in the same way as for the scalar theory, just by going through the algebra of performing the functional derivatives. In doing this to order g , one ends up with

$$G^{(1,2)}(x_1, x_2, x_3) = (-ig \int d^4w) [D(x_1 - w)S(x_2 - w)S(x_3 - w) - D(x_1 - w)S(0)S(x_2 - x_3)]. \quad (3.2.5)$$

As we see, there are a few differences from a purely scalar theory. First of all, the last term, which involve a fermion loop, from $S(0)$, carry an extra minus sign. This arises due to the anti-commutation of the spinor sources, and will always arise whenever there is a closed fermion loop. Secondly, something we do not see here explicitly, is that we have to perform a trace over the fermion loop. If we were to write out the spinor indexes of the sources and the propagator, which it will have as it involves the gamma matrices; we will see that whenever there is a fermion loop, the indices of the involved propagators (and possible matrices from the vertex, if we are doing QED for example) will be set equal to each other. Hence there will be a trace over these matrices. Going to momentum space is completely analogous to the scalar case.

This leads us to define the following additions to the momentum space Feynman rules whenever there are fermions involved.

- For each spinor link, assign a momentum and associate a factor $S_F(p)$.
- For each fermion loop, multiply the diagram by -1 .
- Spinor indexes in a loop must be summed over to account for all possible spin states, which leads to a trace over Dirac matrices in the loop.

Chapter 4

Supersymmetry

4.1 Motivation

So far we have only discussed the more mundane symmetries of nature, but there is another one which is very interesting in its own right, but also leads to many simplifications if it is realized, namely supersymmetry. Supersymmetry was originally considered by Hironari Miyazawa in 1966[24] as a symmetry relating mesons and baryons, but was largely ignored. It was later rediscovered simultaneously in the 1970's by J. L. Gervais and B. Sakita[25], Yu. A. Gol'fand and E.P. Likhtman[26], D. V. Volkov and V.P Akulov[27], and by Julius Wess and Bruno Zumino[28] in 1974, which are perhaps most famous for the discovery. This remarkable new symmetry relates the fundamentally different particles fermions and bosons to each other, and also space-time with an internal set of grassmannian coordinates, which together forms what is known as superspace. The symmetry is loosely stated as follows: For every particle there should exist one or several superpartners, fermions should be paired with bosons, and vice versa.

Supersymmetry implies two very important consequences: First of all, the renormalization needed is only a logarithmically divergent integral, this is known as the non-renormalization theorem[14, 29, 30]. Secondly, the vacuum energy is identically equal to zero [14] These two effects means that a supersymmetric theory is much less divergent than any other realistic QFT model. This is also true to some extent if we choose to break the symmetry, which we want to do. The reason for this being that, if we regularize the theory with a cutoff, any contribution to the vacuum energy dependent only on Λ will automatically vanish. And these terms are the most divergent ones.

These implications, combined with the the fact that many researchers believes that supersymmetry might solve several theoretical problems in the standard model, motivates us to attempt an application of the ideas used on the scalar field theory on a supersymmetric theory. We will not attempt to use the full formalism of the effective action however, as the supersymmetric model we will consider is much more complicated than a simple scalar field theory. We will still perform a variational calculation, but only on the vacuum energy, with the renormalized masses of the particles as parameters. This is done in order to probe for

stable ground states away from the supersymmetric point.

4.2 The supersymmetry algebra

Before discussing the specific supersymmetric model we are interested in, the Wess-Zumino model, we will discuss some of the general properties of the supersymmetry algebra. A thorough treatment can be found in [31], Chapters I-V. The supersymmetry algebra is defined as follows

$$\begin{aligned} \{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} &= 2\sigma_{\alpha\dot{\beta}}{}^\mu P_\mu \delta_B^A, \\ \{Q_\alpha^A, Q_\beta^B\} &= \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0, \\ [P_\mu, Q_\alpha^A] &= [P_\mu, \bar{Q}_{\dot{\alpha}A}] = 0, \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{4.2.1}$$

Some notation needs to be explained here: Q , \bar{Q} and P are generators for translations in grassmannian space and real space, respectively. Together they generate translations in superspace, which is defined by the coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, where x is an ordinary coordinate and $\theta, \bar{\theta}$ are the coordinates of the grassmannian space. α and $\dot{\beta}$ are spinor indices running from one to two. The indices A and B refer to the number of supersymmetry generators we have, and run from 1 to N . We will only discuss $N = 1$ supersymmetry in this text, so these indices will be suppressed. The dotted and undotted notation is explained further in for instance Zee[15], Appendix E.

The notation and algebra can be simplified by introducing anti-commuting spinor parameters ξ^α and $\bar{\xi}_{\dot{\alpha}}$

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \dots = [P_\mu, \xi^\alpha] = 0. \tag{4.2.2}$$

The algebra can now be written entirely in terms of commutators

$$\begin{aligned} [\xi Q, \bar{\xi}\bar{Q}] &= 2\xi\sigma^\mu\bar{\xi}P_\mu, \\ [\xi Q, \xi Q] &= [\bar{\xi}\bar{Q}, \bar{\xi}\bar{Q}] = 0, \\ [P^\mu, \xi Q] &= [P^\mu, \bar{\xi}\bar{Q}] = 0. \end{aligned} \tag{4.2.3}$$

The supersymmetry algebra is a Lie-algebra with commuting and anti-commuting parameters, $(x, \theta, \bar{\theta})$. An element of the group generates motion in this parameter space. This motion may be generated by the differential operators Q and \bar{Q} , note that this is not the same as the generators introduced in Eq. (4.2.1), but they do indeed represent the algebra, together with ∂_μ ;

$$\xi Q + \bar{\xi}\bar{Q} = \xi^\alpha \left(\frac{\partial}{\partial\theta^\alpha} - i\sigma_{\alpha\dot{\alpha}}{}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) + \bar{\xi}_{\dot{\alpha}} \left(\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma_{\alpha\dot{\beta}}{}^\mu \epsilon^{\dot{\beta}\alpha} \partial_\mu \right). \tag{4.2.4}$$

We can now define a chiral superfield, $\Phi(y)$, the most general function of superspace, which also satisfies the equation

$$\left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}{}^\mu \partial_\mu \right) \Phi(y) = 0. \tag{4.2.5}$$

From this the chiral superfield is as follows

$$\Phi(y) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \quad (4.2.6)$$

which can be Taylor expanded around x

$$\begin{aligned} &= A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\nu\partial_\mu A(x) \\ &\quad + \sqrt{2}\theta\psi(x) + i\sqrt{2}\theta\theta\sigma^\mu\bar{\theta}\partial_\mu\psi(x) + \theta\theta F(x). \end{aligned} \quad (4.2.7)$$

Here A is a complex scalar, ψ is a spinor and F is a complex field of dimension two. As F is already of dimension two, we cannot construct a kinetic term for it, so it is clear that it is a non-propagating field, called an auxiliary field. This field is introduced in order to obtain a linear representation of supersymmetry, and eliminating it will give us a non-linear *realization*. This is analogous to how introducing p^0 in $p^\mu = (p^0, \vec{p})$ enable us to represent Lorentz transformations linearly, while eliminating it in favor of $\sqrt{\vec{p}^2 + m^2}$ leads to a non-linear realization. Applying Eq. (4.2.4) on Eq. (4.2.6) gives us the transformation rules of the component fields by comparing terms:

$$\begin{aligned} (\xi Q + \bar{\xi}\bar{Q}) A &= \sqrt{2}\xi\psi, \\ (\xi Q + \bar{\xi}\bar{Q}) \psi &= i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu A + \sqrt{2}\xi F, \\ (\xi Q + \bar{\xi}\bar{Q}) F &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi. \end{aligned} \quad (4.2.8)$$

As Q, \bar{Q} are linear operators, products of Φ 's, are also chiral superfields, the same holds for its conjugate Φ^\dagger . $\Phi^\dagger\Phi$ on the other hand, is not a chiral superfield, it is a vector superfield, defined by $V^\dagger = V$. To find supersymmetric combinations of the component fields (A, ψ, F) we have to look at how Φ, Φ^\dagger and products of these transform under the supersymmetry transformation. Only terms invariant, or that transforms into a total derivative can be used to construct a Lagrangian invariant under supersymmetry.

From Eq. (4.2.8) we see that F transforms into a total derivative, this means that every term proportional to $\theta\theta$ in a chiral superfield transforms into a total derivative, and is therefore invariant under supersymmetry. This is what is called the F -term. Writing out $\Phi_i\Phi_j$ and $\Phi_i\Phi_j\Phi_k$ we can see explicitly what these terms looks like in terms of the component fields

$$\begin{aligned} \Phi_i\Phi_j &= A_iA_j + \sqrt{2}\theta[A_i\psi_j + \psi_iA_j] \\ &\quad + \theta\theta[A_iF_j + F_iA_j - \psi_i\psi_j], \end{aligned} \quad (4.2.9)$$

$$\begin{aligned} \Phi_i\Phi_j\Phi_k &= A_iA_jA_k + \sqrt{2}\theta[A_iA_j\psi_k + A_i\psi_jA_k + \psi_iA_jA_k] \\ &\quad + \theta\theta[A_iA_jF_k + A_iF_jA_k + F_iA_jA_k \\ &\quad - A_i\psi_j\psi_k - A_j\psi_i\psi_k - A_k\psi_i\psi_j]. \end{aligned} \quad (4.2.10)$$

If we specialize to one superfield,

$$\frac{1}{2}m [\Phi\Phi]_F + \frac{1}{3}g [\Phi\Phi\Phi]_F = m \left(AF - \frac{1}{2}\psi\psi \right) + g (A^2F - A\psi\psi) \quad (4.2.11)$$

and its hermitian conjugate are invariant under supersymmetry. A term linear in Φ could also be added, but it can be eliminated by a shift of the fields.

As we see we now have something that looks like mass terms and three-point couplings, but no kinetic term. The vector superfield $\Phi^\dagger\Phi$ will provide that. The $\theta\theta\bar{\theta}\bar{\theta}$ transforms into a total derivative by dimensional analysis, and hence it is invariant under supersymmetry. This is called the D -term. Explicitly,

$$[\Phi^\dagger\Phi]_D = \partial^\mu A^\dagger \partial_\mu A + i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F. \quad (4.2.12)$$

To summarize: We have found the most general supersymmetric Lagrangian one can construct given a chiral superfield Φ

$$\mathcal{L} = [\Phi^\dagger\Phi]_D + \left(\left[\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 + \dots \right]_F + \text{h.c.} \right). \quad (4.2.13)$$

4.2.1 The superpotential

Before we move on it is interesting to discuss the potential of A , or the superpotential, on a classical level. If we write out the terms in the Lagrangian containing only A and F

$$\mathcal{L} = \partial^\mu A^\dagger \partial_\mu A + F^\dagger F + [mA F + gA^2 F + \text{h.c.}], \quad (4.2.14)$$

and eliminate F through its Euler-Lagrange equations

$$F = -m^\dagger A^\dagger - g^\dagger A^{\dagger 2}, \quad (4.2.15)$$

$$F^\dagger = -mA - gA^2, \quad (4.2.16)$$

we obtain the Lagrangian

$$\begin{aligned} \mathcal{L} &= \partial^\mu A^\dagger \partial_\mu A - (mA + gA^2) (m^\dagger A^\dagger + g^\dagger A^{\dagger 2}) \\ &= \partial^\mu A^\dagger \partial_\mu A - V(A). \end{aligned} \quad (4.2.17)$$

Where $V(A)$ is the superpotential. If we now shift A to $A-v$, where $v = \frac{m}{2g}$ the superpotential is on the form

$$V(A) = (|g|^2)^2 (A^2 - v^2) (A^{\dagger 2} - v^{\dagger 2}) \quad (4.2.18)$$

The action is now invariant under the reflections $A \rightarrow -A$, $A^\dagger \rightarrow -A^\dagger$. This must also be combined with a similar reflection of the fermion fields, which together make a Z_2

transformation[30]. This symmetry is isomorphic to a symmetry called R -parity, or R -symmetry[32] when the Z_2 group is extended to a global continuous $U(1)$ symmetry. If we consider a theory with several supercharges, that is $N \geq 1$, the R -symmetry will be a non-Abelian symmetry group. This transformation rotates different supercharges into each other, and is required if we want a theory which exhibits broken supersymmetry.

This potential will look exactly like the Mexican hat potential discussed in Section 1.4.3, only shifted upwards by v^4 , but the qualitative discussion is the same. If we choose coordinates so that A points in the purely real direction (and choose v to be real), we can write the potential as

$$V(A) = (|g|^2)^2 (A^2 - v^2)^2. \quad (4.2.19)$$

Which has extrema at $A = 0$ and $A = \pm v$. The first extremum is a saddle point with $V = g^4 v^4$, and the second is a minimum with $V = 0$, or rather an infinite number of minima. As the second minima has lower energy, our system will fall into one of these states. This will break the Z_2 symmetry described above, but as this is not a continuous symmetry, Goldstone's theorem does not apply. It does however indicate an interesting property of supersymmetric models, which we will show explicitly later, which is that the vacuum energy prefers to be at zero.

4.3 The Wess-Zumino model

If we write the complex scalar A as $\frac{1}{\sqrt{2}}(A' - iB')$, where A' is a real scalar and B' is a real pseudoscalar; F as $\frac{1}{\sqrt{2}}(F' + iG')$, where F and G both are real scalars (and forget about the primes, as we from now on only talk about the real fields); combine the two Weyl spinors into a Majorana spinor and re-scale the coupling g by a factor of $\sqrt{2}$, we can rewrite the Lagrangian (4.2.13) as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2 + \frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \\ & + mFA + mGB - \frac{1}{2}m\bar{\psi}\psi \\ & + gF(A^2 - B^2) + 2gGAB - g\bar{\psi}(A + i\gamma^5 B)\psi. \end{aligned} \quad (4.3.1)$$

This model, called the Wess-Zumino model, was first introduced in [14].

The fields F, G can be eliminated through their Euler-Lagrange equations

$$F = -mA - g(A^2 - B^2), \quad (4.3.2)$$

$$G = -mB - 2gAB \quad (4.3.3)$$

yielding the physical Lagrangian

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}i\bar{\psi}\gamma^\mu\partial_\mu\psi \\
 & - \frac{1}{2}m^2A^2 - \frac{1}{2}m^2B^2 - \frac{1}{2}m\bar{\psi}\psi \\
 & - gmA(A^2 + B^2) - \frac{1}{2}g^2(A^2 + B^2)^2 - g\bar{\psi}(A + i\gamma^5 B)\psi.
 \end{aligned} \tag{4.3.4}$$

Multiplying the interaction part of the Lagrangian by one in various ways allows us to use our familiar rules for symmetry factors

$$\begin{aligned}
 -\mathcal{L}_{int} = & \frac{6}{3!}gmA^3 + \frac{2}{2!}gmAB^2 \\
 & + \frac{12}{4!}g^2(A^4 + B^4) + \frac{4}{2!2!}g^2A^2B^2 \\
 & + \frac{2}{2!}g\bar{\psi}(A + i\gamma^5 B)\psi.
 \end{aligned} \tag{4.3.5}$$

Due to our previous work with scalar and spinor fields it is now straightforward to read off propagators and vertex factors from the Lagrangian:

Scalar propagator: $\text{—————} = \frac{i}{p^2 - m^2},$

Pseudoscalar propagator: $\text{-----} = \frac{i}{p^2 - m^2},$

Fermion propagator: $\text{====} = \frac{i}{\not{p} - m}.$

	= -6igm,		= -2igm,
	= -2ig,		= 2g\gamma^5,
	= -12ig^2,		= -12ig^2,
	= -4ig^2.		

There is one subtle difference though, as we are working with Majorana spinors, which is its own antiparticle, there is no direction on the fermion lines. This means that the fermions can be treated as real scalars when considering symmetry factors.

4.3.1 Renormalization

In several papers it is shown that remarkable cancellations occur during the renormalization procedure of the Wess-Zumino model [14][29]; only a simple wave function renormalization is required. Here we would like to show this explicitly to lowest order using the Lagrangian where the auxiliary fields have been eliminated, Eq. (4.3.4), which with bare fields and parameters reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu A_0)^2 + \frac{1}{2}(\partial_\mu B_0)^2 + \frac{1}{2}i\bar{\psi}_0\gamma^\mu\partial_\mu\psi_0 \\
&\quad - \frac{1}{2}m_0^2A_0^2 - \frac{1}{2}m_0^2B_0^2 - \frac{1}{2}m_0\bar{\psi}_0\psi_0 \\
&\quad - g_0m_0A_0(A_0^2 + B_0^2) - \frac{1}{2}g_0^2(A_0^2 + B_0^2)^2 - g_0\bar{\psi}_0(A_0 + i\gamma^5 B_0)\psi_0 \\
&= \mathcal{L}_{\text{kin}}^{(0)} + \mathcal{L}_{\text{int}}^{(0)}.
\end{aligned} \tag{4.3.6}$$

The only thing we can do to this Lagrangian, which still preserves supersymmetry, is to multiply each field, the mass parameter and the coupling by a three respective factors. That is, each field must be multiplied by the same factor. This can be seen from the basic form of the supersymmetric Lagrangian Eq. (4.3.1). Explicitly we make the substitutions $A_0 = Z^{1/2}A$, $B_0 = Z^{1/2}A$, $\psi_0 = Z^{1/2}\psi$, $m_0 = Z_m m$ and $g_0 = Z_g g$. This of course assumes that we regularize the theory in way that preserves supersymmetry. Doing this and splitting the Lagrangian into a free part with physical parameters and a part with counterterms we get, focusing on the kinetic part first

$$\begin{aligned}
\mathcal{L}_{Z,\text{kin}} &= \mathcal{L}_{\text{kin}} + (Z - 1) \left[\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}i\bar{\psi}\gamma^\mu\partial_\mu\psi \right] \\
&\quad - \frac{1}{2}(ZZ_m^2 - 1)m^2(A^2 + B^2) - \frac{1}{2}(ZZ_m - 1)m\bar{\psi}\psi,
\end{aligned} \tag{4.3.7}$$

where \mathcal{L}_Z denotes a part of the Lagrangian with factors of Z inserted.

From Eq. (4.3.7) we can read of the counterterms for the three propagators

$$\begin{aligned}
\text{---}\times\text{---} &= \text{----}\times\text{----} = i(Z - 1)p^2 - i(ZZ_m^2 - 1)m^2, \\
\text{====}\times\text{====} &= i(Z - 1)\not{p} - i(ZZ_m - 1)m.
\end{aligned} \tag{4.3.8}$$

This looks ugly at the moment, but it will soon be cleared up to a nice relation. We must also define our renormalization conditions for this to make sense. The standard definition when using counterterms is that the full propagator equals the zeroth order propagator plus finite terms, *i.e.* it has poles at the same location, with the same residue as the zeroth order propagator. Diagrammatically we have

$$\text{---} \textcircled{\text{---}} \text{---} = \left(\text{---} \right)^{-1} + \text{terms regular at } p^2 = m^2, \quad (4.3.9)$$

and similarly for the B and ψ propagator. In other words, Z and Z_m has to be determined consistently in such a way that all divergent contributions to the full propagators are removed.

The simplest way to calculate Z and Z_m is to first consider the spinor self-energy. There are 5 contributions of order g^2 :

$$\begin{aligned} \Sigma(\not{p}) = & \text{---} \textcircled{\text{---}} \text{---} + \text{---} \textcircled{\text{---}} \text{---} \\ & + \text{---} \textcircled{\text{---}} \text{---} + \text{---} \textcircled{\text{---}} \text{---} + \text{---} \textcircled{\text{---}} \text{---} . \end{aligned} \quad (4.3.10)$$

Looking at the tadpole diagrams first, these are clearly quadratically divergent by power counting, and will contribute to a shift in the mass. So we better hope these divergences cancels. These diagrams are proportional to the quadratically divergent integral

$$I_l \equiv \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2}, \quad (4.3.11)$$

times the common vertex factor $\frac{-2ig}{2}$ and with relative factors from the loop (to be taken inside the integral)

$$(-6igm) + (-2igm) + (-2ig)(-\text{tr}[\not{k} + m]) = 0. \quad (4.3.12)$$

The remaining diagrams are equal to (we use the convention for the momentum flow shown in Fig. 4.1 in all diagrams of this form)

$$\begin{aligned} \Sigma(\not{p}) &= 4g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\not{p} + \not{k} + m - \gamma^5 (\not{p} + \not{k} + m) \gamma^5}{[k^2 - m^2][(p+k)^2 - m^2]} \\ &= 8g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\not{p} + \not{k}}{[k^2 - m^2][(p+k)^2 - m^2]}. \end{aligned} \quad (4.3.13)$$

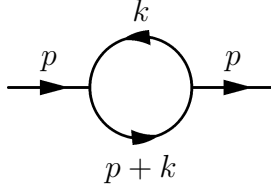


Figure 4.1: Momentum flow convention used in momentum transfer dependent self-energy diagrams. Arrows on the lines shows the direction of the momentum.

In order to figure out the divergent structure of this integral, we have to combine the propagators using the Feynman parameter integral Eq. (A.2.2)

$$\Sigma(\not{p}) = 8g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\not{p} + \not{k}}{[(k+px)^2 - m^2 - x(x-1)p^2]^2}. \quad (4.3.14)$$

We now shift the integration variable $k \rightarrow k - px$ and remove terms proportional to \not{k} in the nominator, as they give zero under symmetric integration.

$$\Sigma(\not{p}) = 8g^2 \not{p} \int_0^1 dx (1-x) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2 - x(x-1)p^2]^2}. \quad (4.3.15)$$

The divergent part of the logarithmically divergent integral is equal to its value at zero momentum transfer, this can be easily seen by performing a partial integration. Doing this, and the parameter integral, which now simply gives a factor of one half, we obtain

$$\Sigma(\not{p}) = 4g^2 \not{p} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2} + \text{finite terms}. \quad (4.3.16)$$

We can now make two conclusions: Firstly, there is no need for the counterterm proportional to m for the fermions, *i.e.* $Z_m = Z^{-1}$; Secondly, the wave function renormalization Z is only logarithmically divergent

$$Z = 1 + 4ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \equiv 1 - 4g^2 I_{\log}, \quad (4.3.17)$$

which is exactly the result obtained by Wess and Zumino in [14].

As we see, the claim of Wess and Zumino holds up for the fermion self-energy. But, to make sure that the theory is consistent with supersymmetry, we better get the same value for Z when considering the boson self-energies. First of all, following the result for the fermions, the boson counterterm is now equal to

$$\begin{aligned} \text{---}\times\text{---} &= i(Z-1)p^2 - i(Z^{-1}-1)m^2 \\ &= i(Z-1)(p^2 + m^2). \end{aligned} \quad (4.3.18)$$

The second equality follows from Taylor expanding Z : $Z = 1 - g^2 Z_2 + \mathcal{O}(g^4) \rightarrow Z^{-1} = 1 + g^2 Z_2 + \mathcal{O}(g^4)$.

First we look at the scalar, A , which has the following diagrams to order g^2 :

$$\begin{aligned}
\Pi_A(p^2) = & \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \\
& + \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \\
& + \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} .
\end{aligned} \tag{4.3.19}$$

Again, the tadpoles clearly cancel each other, as they have the same relative factors as the fermion diagrams.

Now we still have some quadratic divergences, from the diagrams

$$\begin{aligned}
\text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} &= \left(\frac{-12ig^2}{2} + \frac{-4ig^2}{2} \right) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
&= -i8g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} = -i8g^2 I_t,
\end{aligned} \tag{4.3.20}$$

which will be cancelled by parts of the remaining fermionic loop.

The remaining boson loops are equal to

$$\begin{aligned}
& \text{---} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \text{---} \\
&= \left(\frac{(-6igm)^2}{2} + \frac{(-2igm)^2}{2} \right) \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{[k^2 - m^2][(p+k)^2 - m^2]} \\
&= 20g^2 m^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2][(p+k)^2 - m^2]} \equiv 20g^2 m^2 I_{\text{sun}}.
\end{aligned} \tag{4.3.21}$$

Finally, the fermionic loop contributes

$$\begin{aligned}
\text{---} \text{---} \text{---} \text{---} &= \frac{(-2ig)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i^2 \text{tr}[(\not{k} + m)(\not{p} + \not{k} + m)]}{[k^2 - m^2][(p+k)^2 - m^2]} \\
&= -8g^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + p \cdot k + m^2}{[k^2 - m^2][(p+k)^2 - m^2]}.
\end{aligned} \tag{4.3.22}$$

The nominator of the integral can be rewritten as follows:

$$k^2 + p \cdot k = \frac{1}{2}(k^2 - m^2) + \frac{1}{2}((k+p)^2 - m^2) - \frac{1}{2}p^2 + m^2, \quad (4.3.23)$$

and we do this because the first two terms cancel a propagator in the denominator, which gives two identical quadratically divergent integrals after a shift in the integration variable. This gives

$$\text{---} \circ \text{---} = i8g^2 I_q + 4g^2 p^2 I_s - 16g^2 m^2 I_{\text{sun}}.$$

As we see, the remaining quadratic divergences cancels, and the self-energy is after adding all the diagrams

$$\begin{aligned} \Pi_A(p^2) &= 4g^2(p^2 + m^2) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2][(p+k)^2 - m^2]} \\ &= 4g^2(p^2 + m^2) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2} + \text{finite terms}, \end{aligned} \quad (4.3.24)$$

hence the counterterm is the same as for the fermion self-energy.

The pseudoscalar self-energy is quite similar to the scalar, we have the diagrams

$$\begin{aligned} \Pi_B(p^2) &= \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \\ &+ \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \\ &+ \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} . \end{aligned} \quad (4.3.25)$$

The tadpoles cancel by the same argument as before, and the remaining simple quadratic diagrams has the same value as for the scalar,

$$\text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} = -i8g^2 I_l \quad (4.3.26)$$

In this case there is only one remaining boson loop, which is equal to

$$\text{---} \text{---} \text{---} = 4g^2 m^2 I_{\text{sun}} \quad (4.3.27)$$

The fermionic loop is a bit different from the scalar case, due to the γ^5 matrix

$$\begin{aligned}
 \text{---}\textcircled{\textcircled{\bullet}}\text{---} &= \frac{(2g)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i^2 \text{tr}[\gamma^5(\not{k} + m)\gamma^5(\not{p} + \not{k} + m)]}{[k^2 - m^2][(k+p)^2 - m^2]} \\
 &= 8g^2 \int \frac{d^4k}{(2\pi)^4} \frac{m^2 - (k^2 + p \cdot k)}{[k^2 - m^2][(p+k)^2 - m^2]}, \tag{4.3.28}
 \end{aligned}$$

which is equal to, after again rewriting the nominator

$$\text{---}\textcircled{\textcircled{\bullet}}\text{---} = i8g^2 I_q + 4g^2 p^2 I_{\text{sun}} \tag{4.3.29}$$

In total

$$\begin{aligned}
 \Pi_B(p^2) &= 4g^2(p^2 + m^2)I_s = \Pi_A(p^2) \\
 &= 4g^2(p^2 + m^2) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2} + \text{finite terms}, \tag{4.3.30}
 \end{aligned}$$

which clearly means that the counterterm for the pseudoscalar is equal to the scalar counterterm.

In conclusion, renormalization leads to the same logarithmically divergent wave function renormalization Z for each field A, B, ψ

$$Z = 1 - 4g^2 I_{\log}, \tag{4.3.31}$$

and the mass is scaled by the same factor Z , $m = Zm_0$. It can also be shown that the coupling also gets a simple scaling factor, $g = Z^{3/2}g_0$, but this will only be necessary at higher orders.

4.3.2 Vacuum energy

An interesting property of a supersymmetric theory is that the vacuum energy is identically equal to zero to all orders. In this section we would like to verify this explicitly to one-loop in the Wess-Zumino model. First we consider the zeroth order contribution, which is easily inferred from the results of Eqs. (2.1.6) and (3.1.5). The bosonic contribution is

$$\frac{-i}{2} \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m^2 + i\epsilon), \tag{4.3.32}$$

for each bosonic field. The fermionic contribution is

$$\frac{2i}{2} \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m^2 + i\epsilon), \tag{4.3.33}$$

which is a factor of 2 smaller than Eq. (3.1.5) due to these fermions being Majorana fermions, and thus having only two degrees of freedom. Since we have two boson fields and one fermion field these contributions add up to zero.

There are four classes of diagrams contributing to the first order vacuum expectation value. Tadpole diagrams, sunset diagrams, figure-eight diagrams and diagrams containing the wave-function-renormalization counterterms. As we will see, the tadpole diagrams cancel each other completely, similarly for the counterterms, while the sunset diagrams and figure-eight diagrams cancels among each other after manipulating the integrands slightly.

Denote the sum of tadpole diagrams by A , and we have

$$\begin{aligned}
A = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
& + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} . \quad (4.3.34)
\end{aligned}$$

These all have two simple loops with a scalar zero momentum propagator connecting them, so they are all proportional to the integral

$$\frac{i}{m^2} \left(\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \right)^2 \equiv \frac{i}{m^2} I_l^2. \quad (4.3.35)$$

The pure boson diagrams give the following relative factors

$$\left(\frac{(6gm)^2}{2^3} + \frac{(2gm)^2}{2^3} + \frac{(6gm)(2gm)}{2^2} \right) = 8g^2m^2. \quad (4.3.36)$$

Each fermion loop gives a factor of $-\text{tr}[\not{k} + m] = -4m$, while the fermion loop from the fifth diagram gives zero, due to the trace over γ^5 . Therefore the fermion contributions add up to

$$\left(\frac{(2g)^2}{2^3} (-4m)^2 + \frac{(2g)(6gm)}{2^2} (-4m) + \frac{(2g)(2gm)}{2^2} (-4m) \right) = -8g^2m^2. \quad (4.3.37)$$

In total we have $A = 0$.

The diagrams involving counterterms will also cancel independently, denote the sum of these by B .

$$B = \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} . \quad (4.3.38)$$

This is equal to

$$\begin{aligned}
B &= i^2(Z-1) \int \frac{d^4k}{(2\pi)^4} \left[\frac{2(k^2 + m^2) - \text{tr}[k + m]}{k^2 - m^2} \right] \\
&= i^2(Z-1) \int \frac{d^4k}{(2\pi)^4} \left[\frac{4m^2 - 4m^2}{k^2 - m^2} + 2 \right] \\
&= 2i^2(Z-1) \int \frac{d^4k}{(2\pi)^4} = 0
\end{aligned} \tag{4.3.39}$$

There are several arguments to why this last equality is valid. First of all, as there are no scale left in the integral, one simply reasons that it can have no other value. And in dimensional regularization it is precisely zero, but if one utilizes a cutoff, this integral would be proportional to Λ^4 , which is bad. The second argument is that this term never should have appeared in the first place. It arises due to the rescaling of the fields, which also means we have to re-scale the path integral measure by the same factor. Normally this is no problem, because when one considers Green's functions, all normalization factors in the measure automatically cancels. But when calculating vacuum energies, there is no normalization as in Green's function calculations, and the *re*-normalization introduces a factor of Z^∞ . This corresponds to the integral over 1 we see in B , and we have no other choice than to discard it.

Next we turn to the figure-eight diagrams, denote them sum of all these diagrams by C

$$C = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} . \tag{4.3.40}$$

These are all proportional to $-iI_l^2$, and their relative factors are

$$\frac{12g^2}{2^3} + \frac{12g^2}{2^3} + \frac{4g^2}{2^2} = 4g^2. \tag{4.3.41}$$

that is, $C = -4ig^2I_l^2$.

Now we look at the boson sunset diagrams, of which there are two. Denote the sum by D

$$D = \text{diagram 4} + \text{diagram 5} . \tag{4.3.42}$$

Using the momentum-flow conventions shown in Fig. 4.2, these are both proportional to the integral

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{-i^3}{[k^2 - m^2][(k - q)^2 - m^2][q^2 - m^2]} \equiv -iI_s, \tag{4.3.43}$$

and their relative factors are

$$\frac{(6gm)^2}{3! \cdot 2} + \frac{(2gm)^2}{2! \cdot 2} = 4g^2m^2. \tag{4.3.44}$$

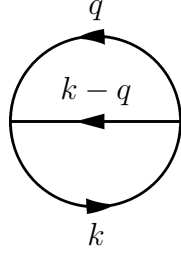


Figure 4.2: Momentum-flow conventions used in vacuum diagrams of the sunset variety.

In total we have $D = -4ig^2m^2I_s$

Now all that remains are the fermion sunset diagrams, which as we see have to cancel $C + D$. At first glance they seem to differ in structure from the figure-eight for this to be possible, but we will see that after some algebra it cancels perfectly. Denote the sum of these diagrams by E

$$E = \text{Diagram 1} + \text{Diagram 2} . \quad (4.3.45)$$

These are also proportional to $-iI_s$, with relative factors

$$\frac{(2g)^2}{2! \cdot 2} (-\text{tr}[(\not{k} + m)(\not{q} + m)]) - \frac{(2g)^2}{2! \cdot 2} (-\text{tr}[\gamma^5(\not{k} + m)\gamma^5(\not{q} + m)]), \quad (4.3.46)$$

where it is implied that the traces have to be taken inside the integrals. Passing the γ^5 through the \not{k} we see that the $m(\not{q} + m)$ parts cancel.

$$E = iI_s g^2 (2\text{tr}[\not{k}(\not{q} + m)]) = i8g^2 I_s kq.$$

We can rewrite $8kq$ as a sum of the three propagators in I_s , plus a remainder.

$$8kq = 4(k^2 - m^2) + 4(q^2 - m^2) - 4((k - q)^2 - m^2) + 4m^2.$$

Now we see that the first three terms conspire to turn I_s into an integral on the form of I_l^2 , after we shift the integration variables. We then have

$$E = 4ig^2m^2I_s + 4ig^2I_l^2, \quad (4.3.47)$$

and indeed, $C + D + E = 0$.

We now see, as was claimed, that the vacuum energy to one loop order is identically equal to zero.

4.4 Asymmetric Wess-Zumino model

Next we want to investigate the result of having a model with the same number of degrees of freedom as the Wess-Zumino model, *i.e.* it has a scalar, a pseudoscalar and a Majorana

fermion, but with no restrictions on the three masses. The idea is to examine the dynamics through a variational calculation on the vacuum energy of the model, and determine whether supersymmetry is favoured by nature or not. The model is very similar to the previous, we only change the $\Phi\Phi$ term to account for differing masses. The Lagrangian is

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2 + \frac{1}{2}i\bar{\psi}\gamma^\mu\partial_\mu\psi \\ & + m_A F A + m_B G B - \frac{1}{2}m_\psi\bar{\psi}\psi \\ & + gF(A^2 - B^2) + 2gGAB - g\bar{\psi}(A + i\gamma^5 B)\psi,\end{aligned}\tag{4.4.1}$$

and after eliminating the auxiliary fields

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}i\bar{\psi}\gamma^\mu\partial_\mu\psi \\ & - \frac{1}{2}m_A^2 A^2 - \frac{1}{2}m_B^2 B^2 - \frac{1}{2}m_\psi\bar{\psi}\psi \\ & - gm_A A^3 - gAB^2(2m_B - m_A) - \frac{1}{2}g^2(A^2 + B^2)^2 - g\bar{\psi}(A + i\gamma^5 B)\psi.\end{aligned}\tag{4.4.2}$$

This gives us a new set of Feynman rules

$$\text{Scalar propagator: } \text{—————} = \frac{i}{p^2 - m_A^2}$$

$$\text{Pseudoscalar propagator: } \text{-----} = \frac{i}{p^2 - m_B^2}$$

$$\text{Fermion propagator: } \text{====} = \frac{i}{\not{p} - m_\psi}$$

	= $-6igm_A$,		= $-2ig(2m_B - m_A)$,
	= $-2ig$,		= $2g\gamma^5$,
	= $-12ig^2$,		= $-12ig^2$,
	= $-4ig^2$.		

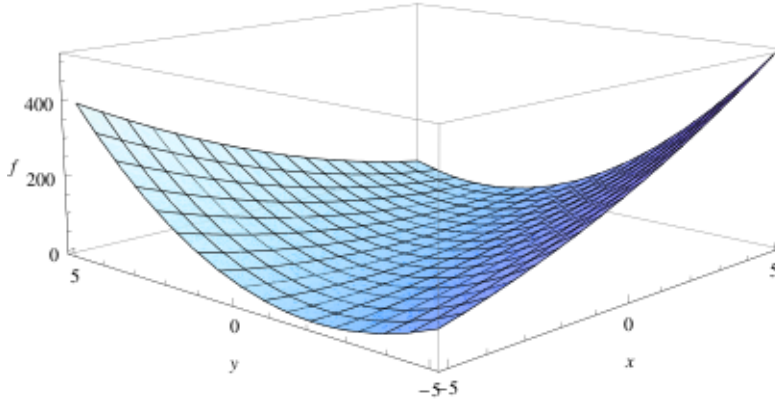


Figure 4.3: A plot of the dominating contribution to the vacuum energy of the modified Wess-Zumino model.

This modification obviously spoils supersymmetry, which means that the vacuum energy is no longer zero. What we want to do is to calculate the leading contributions to this energy, which will be a function of the renormalized masses, and examine possible extrema as a function of these masses. To do this we have to deal with the infinities, and the simplest solution is to utilize a cutoff on k^2 , Λ^2 . Λ is then some large energy, but not infinity, which as we will see will set the scale of the whole theory.

4.4.1 Calculation of the vacuum energy

Now we are ready to begin calculating the vacuum energy of the asymmetric theory. As we are using a cutoff to regularize the integrals, we will calculate the vacuum energy as a double series in g and Λ . That is, we have

$$\begin{aligned} \varrho &= \varrho_0 + \varrho_1 + \mathcal{O}(g^4) \\ &= \varrho_{0,\Lambda^4} + \varrho_{0,\Lambda^2} + \varrho_{1,\Lambda^4} + \varrho_{1,\Lambda^2} + \mathcal{O}(\log \Lambda) + \mathcal{O}(g^4) \end{aligned} \quad (4.4.3)$$

Most of the work was already done in Section 4.3.2, we just have to adjust for the different vertex factors and propagators and perform the resulting integrals. First we handle the zeroth order contribution, the three contributions are all on the form

$$-\frac{i}{2}K_X \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m_X^2), \quad (4.4.4)$$

where $X = \{A, B, \psi\}$ and $K_A = K_B = 1$, $K_\psi = -2$. If we Wick rotate it we get a more familiar form

$$\frac{1}{2}K_X \int \frac{d^4k_E}{(2\pi)^4} (\log(k^2 + m^2) + \log(-1)). \quad (4.4.5)$$

The last imaginary term is discarded as it must be unphysical, it must be swept under the definition of the measure in the path integral. We are then left with an integral identical to Eq. (2.1.6) We have already solved this integral, Eq. (A.3.7) The terms which are only dependent on the cutoff will vanish when we combine the three contributions, as they should since the vacuum energy must vanish in the limit $m_A = m_B = m_\psi$. Adding up the three terms gives us the dominating contribution (*i.e.* terms dependent on the cutoff)

$$\begin{aligned} \varrho_0 = \frac{1}{32\pi^2} & \left[\Lambda^2(m_A^2 + m_B^2 - 2m_\psi^2) - \frac{1}{2}m_A^4 \log\left(\frac{\Lambda^2 + m_A^2}{m_A^2}\right) \right. \\ & \left. - \frac{1}{2}m_B^4 \log\left(\frac{\Lambda^2 + m_B^2}{m_B^2}\right) + m_\psi^4 \log\left(\frac{\Lambda^2 + m_\psi^2}{m_\psi^2}\right) \right] + \mathcal{O}(\Lambda^0). \end{aligned} \quad (4.4.6)$$

Or, to order Λ^2

$$\varrho_{0,\Lambda^2} = \frac{\Lambda^2}{32\pi^2} (m_A^2 + m_B^2 - 2m_\psi^2) \quad (4.4.7)$$

It is easy to see that these expressions are zero for $m_A = m_B = m_\psi$.

Turning to the diagrams of order g^2 , there are basically four classes of diagrams: Quartic two loop diagrams, Eqs. (4.3.34) and (4.3.40); quadratic sunset diagrams, Eq. (4.3.42) and the counterterm diagrams, Eq. (4.3.38). The fermionic sunset diagrams, Eq. (4.3.45) can be rewritten as before into quartic and quadratic contributions. First some notation: The quartic diagrams all have two loop integrals on the form

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_X^2} \equiv I_l(m_X^2), \quad (4.4.8)$$

while the sunset diagrams all have a double integral on the form

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{i^3}{[k^2 - m_X^2][(k-q)^2 - m_Y^2][q^2 - m_X^2]} \equiv iI_s(m_X^2, m_Y^2). \quad (4.4.9)$$

The loop integral is easily solved by Wick rotating and substituting $x = k^2$:

$$\begin{aligned} I_l(m_X^2) &= \frac{1}{16\pi^2} \int_0^\Lambda \frac{2dkk^3}{k^2 + m_x^2} = \frac{1}{16\pi^2} \int_0^\Lambda dx \frac{x + m_X^2 - m_X^2}{x + m_X^2} \\ &= \frac{1}{16\pi^2} \left[\Lambda^2 - m_X^2 \log\left(\frac{\Lambda^2 + m_X^2}{m_X^2}\right) \right] \end{aligned} \quad (4.4.10)$$

Solving the second integral is a bit more involved, and we will be content with just finding the leading behaviour, rather than solving it exactly. First we Wick rotate it, and rewrite it as two integrals, as such

$$\begin{aligned} I_s(m_X^2, m_Y^2) &= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{-1}{[k^2 + m_X^2][(k-q)^2 + m_Y^2][q^2 + m_X^2]} \\ &= - \int \frac{d^4k}{(2\pi)^4} \frac{I(k, m_X^2, m_Y^2)}{k^2 + m_X^2}. \end{aligned} \quad (4.4.11)$$

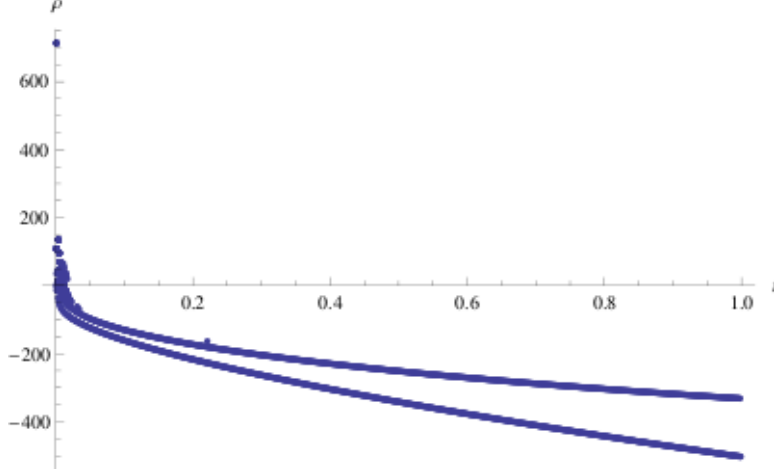


Figure 4.4: An overview of the value of the energy minima as a function of t , with $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$

The inner integral is now a simpler logarithmically divergent integral, hence we can deduce the divergent structure by taking the limit $k = 0$. So, to leading order in Λ , we have

$$I_s(m_X^2, m_Y^2) = - \int \frac{d^4k}{(2\pi)^4} \frac{I(0, m_X^2, m_Y^2)}{k^2 + m_X^2}. \quad (4.4.12)$$

This means that we can solve the integrals separately, handling the logarithmic one first.

$$\begin{aligned} I(0, m_X^2, m_Y^2) &= \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 + m_X^2][q^2 + m_Y^2]} \\ &= \int_0^1 dz \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 + (1-z)m_X^2 + zm_Y^2]^2} \\ &= \int_0^1 dz \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 + M^2]^2}. \end{aligned} \quad (4.4.13)$$

Here we have used Eq. (A.2.2) to combine the propagators, and defined $M = m_X^2(1-z) + m_Y^2z$. This is solved with the same substitution $x = q^2$

$$\begin{aligned} I(0, m_X^2, m_Y^2) &= \frac{1}{16\pi^2} \int_0^1 dz \int_0^{\Lambda^2} dx \frac{x + M^2 - M^2}{x + M^2} \\ &= \frac{1}{16\pi^2} \int_0^1 dz \left[\log \left(\frac{\Lambda^2 + M^2}{M^2} \right) + \frac{M^2}{\Lambda^2 + M^2} - 1 \right] \\ &= \frac{1}{16\pi^2} \int_0^1 dz \left[\log \frac{\Lambda^2}{M^2} + \frac{M^2}{\Lambda^2} + \frac{M^2}{\Lambda^2 + M^2} - 1 + \mathcal{O}\left(\frac{1}{\Lambda^4}\right) \right]. \end{aligned} \quad (4.4.14)$$

The second integral over k is identical to the loop integral in Eq. (4.4.8), which we have already solved. The leading contribution to the sunset integral is therefore

$$I_s(m_X^2, m_Y^2) = - \left(\frac{1}{16\pi^2} \right)^2 \int_0^1 dz \Lambda^2 \left[\log \left(\frac{\Lambda^2}{m_X^2(1-z) + m_Y^2 z} \right) - 1 \right] + \mathcal{O}(\log \Lambda^2). \quad (4.4.15)$$

We can now solve the parameter integral, which gives us the final result for the sunset integral

$$I_s(m_X^2, m_Y^2) = - \left(\frac{1}{16\pi^2} \right)^2 \Lambda^2 \left[\frac{m_X^2 \log \frac{\Lambda^2}{m_X^2} - m_Y^2 \log \frac{\Lambda^2}{m_Y^2}}{m_X^2 - m_Y^2} \right] + \mathcal{O}(\log \Lambda^2), \quad (4.4.16)$$

and with $m_X^2 = m_Y^2$ we get, which is most easily seen from the form of Eq. (4.4.15),

$$I_s(m_X^2, m_X^2) = - \left(\frac{1}{16\pi^2} \right)^2 \left[\Lambda^2 \log \left(\frac{\Lambda^2}{m_X^2} \right) - \Lambda^2 \right] + \mathcal{O}(\log \Lambda^2). \quad (4.4.17)$$

Now we are ready to look at the diagrams, first we handle the quartic diagrams, from Eqs. (4.3.34) and (4.3.40):

$$\begin{aligned} A + C = & -ig^2 \left[\frac{3}{2} I_l(m_A^2)^2 + \frac{3}{2} I_l(m_B^2)^2 + I_l(m_A^2) I_l(m_B^2) \right. \\ & + 12 \frac{m_\psi}{m_A} I_l(m_\psi^2) I_l(m_A^2) + 4 \frac{m_\psi}{m_A} \left(2 \frac{m_B}{m_A} - 1 \right) I_l(m_\psi^2) I_l(m_B^2) \\ & - 8 \frac{m_\psi^2}{m_A^2} I_l(m_\psi^2)^2 - 3 \left(2 \frac{m_B}{m_A} - 1 \right) I_l(m_A^2) I_l(m_B^2) \\ & \left. - \frac{1}{2} \left(2 \frac{m_B}{m_A} - 1 \right)^2 I_l(m_B^2)^2 - \frac{9}{2} I_l(m_A^2)^2 \right]. \end{aligned} \quad (4.4.18)$$

The bosonic sunset diagrams, Eq. (4.3.42), look like the following

$$D = -ig^2 \left[3m_A^2 I_s(m_A^2, m_A^2) + (2m_B - m_A)^2 I_s(m_B^2, m_A^2) \right]. \quad (4.4.19)$$

And the fermionic sunset diagrams, Eq. (4.3.45), after some rewriting

$$\begin{aligned} E = & -ig^2 \left[4I_l(m_\psi^2)^2 - 4I_l(m_\psi^2) I_l(m_A^2) - 4I_l(m_\psi^2) I_l(m_B^2) \right. \\ & \left. + 2m_B^2 I_s(m_\psi^2, m_B^2) - (8m_\psi^2 - 2m_A^2) I_s(m_\psi^2, m_A^2) \right]. \end{aligned} \quad (4.4.20)$$

This is not the whole story though, we also have to take into account contributions to the vacuum energy from diagrams containing counterterms. This means that we have to consider renormalization of the modified theory.

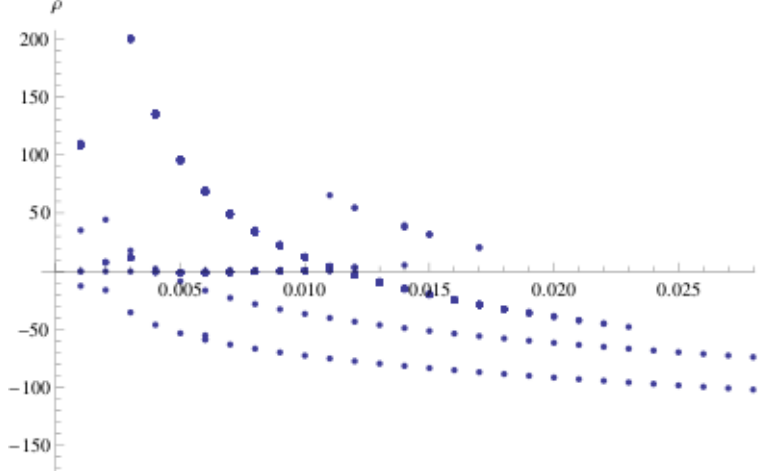


Figure 4.5: An overview of the value of the energy minima as a function of t , with $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$, zoomed in on the lower end of the scale

4.4.2 Renormalizing the asymmetric model

To renormalize the modified theory we will use exactly the same arguments as in Section 4.3.1. The difference now is of course that, as the particle masses are allowed to be different, we will have to allow the renormalization factors Z and Z_M to take on different values for each field. This will of course complicate things immensely, as the non-renormalization theorem no longer applies.

To simplify things, we will not bother with calculating each of the renormalization factors explicitly, we will simply demand that each counterterm cancels each self-energy expression completely. This is because we are only interested in the contribution from the counterterms to the vacuum energy. This leads us to demand that

$$\text{---}\times\text{---} = i(Z_A - 1)p^2 - i(Z_A Z_{m_A}^2 - 1)m_A^2 = -\Pi_A(p^2),$$

$$\text{----}\times\text{----} = i(Z_B - 1)p^2 - i(Z_B Z_{m_B}^2 - 1)m_B^2 = -\Pi_B(p^2),$$

$$\text{====}\times\text{====} = i(Z_\psi - 1)\not{p} - i(Z_\psi Z_{m_\psi} - 1)m = -\Sigma(\not{p}), \quad (4.4.21)$$

where Π_A , Π_B and Σ are given by Eqs. (4.3.19), (4.3.25) and (4.3.10), respectively.

We could calculate each self-energy diagram explicitly, and then close the external propagators into a loop to get the vacuum energy diagram, but it turns out that there is a much

more elegant solution. If we consider the diagrammatic representation of the self-energies, and simply join the external propagators and account for the difference in symmetry factors, we will get a rather nice relation for the vacuum energy contribution from the counterterms. To be precise, we have to calculate the three diagrams

$$B = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} , \quad (4.4.22)$$

where the crosses are determined by Eq. (4.4.21). For instance, the contribution from closing the A -propagator from the following self-energy diagram,

$$\text{[Diagram 4]} \quad (4.4.23)$$

will give -2 times the following vacuum diagram.

$$\text{[Diagram 5]} \quad (4.4.24)$$

The factor of two arises because the self energy diagram has a symmetry factor of 2, as do the vacuum energy diagram involving the counterterm, for a total of 4. The original vacuum diagram, on the other hand, has a symmetry factor of 8, resulting in an overall factor of 2. The negative sign arises from the definition of the counterterms. Similarly, closing the propagator of

$$\text{[Diagram 6]} \quad (4.4.25)$$

will give -3 times the following diagram.

$$\text{[Diagram 7]} \quad (4.4.26)$$

Now the factor is 3, as the counterterm diagram has an overall symmetry factor of 4, while the original vacuum diagram has a factor of $3! \cdot 2$.

Going through similar arguments for all the diagrams, we end up with a nice diagram-

matic relation

$$\begin{aligned}
B &= \text{[Sunset diagram with solid lines]} + \text{[Sunset diagram with dashed lines]} + \text{[Sunset diagram with double lines]} \\
&= -2 \cdot \left(\text{[Two circles connected by a solid line]} + \text{[Two dashed circles connected by a dashed line]} + \text{[Solid circle and dashed circle connected by a solid line]} \right. \\
&\quad \left. + \text{[Two double circles connected by a solid line]} + \text{[Double circle and solid circle connected by a solid line]} + \text{[Double circle and dashed circle connected by a solid line]} \right) \\
&\quad - 2 \cdot \left(\text{[Figure-eight with solid lines]} + \text{[Figure-eight with dashed lines]} + \text{[Figure-eight with double lines]} \right) \\
&\quad - 3 \cdot \left(\text{[Circle with a solid horizontal line]} + \text{[Circle with a dashed horizontal line]} + \text{[Circle with a double horizontal line]} + \text{[Circle with a dashed horizontal line]} \right) \\
&= -2(A + C) - 3(D + E), \tag{4.4.27}
\end{aligned}$$

where A , C , D and E now are given in Eqs. (4.4.18), (4.4.19) and (4.4.20).

There is one problem with this though, $B \neq 0$ in the limit $m_A = m_B = m_\psi$. As the sunset diagrams and the figure-eight diagrams have different relative factors, the quartic part of the fermionic sunset diagrams will no longer cancel the figure-eight diagrams. This has to be the same problem as the one we ran into when considering the vacuum energy of the unmodified theory. In Eq. (4.3.39) we did not get the cancellation we expected immediately, there was a quartic part left over. There we argued that the part left over was an artifact of the renormalization, and should not have appeared in the first place. The same argument has to apply here, as we have considered the counterterms on a purely diagrammatic level, no assumption of the particle masses have been made. This leads us to simply discard the part left over from the fermionic sunset diagrams.

4.4.3 Results and discussion

Now we are ready to combine all the terms to get the full contribution to the vacuum energy, to order Λ^2 . The zeroth order contribution is given by Eq. (4.4.7), repeated here for convenience

$$\varrho_{0,\Lambda^2} = \frac{\Lambda^2}{32\pi^2} (m_A^2 + m_B^2 - 2m_\psi^2) \tag{4.4.28}$$

The full two-loop result is, remembering to multiply by i to get a vacuum energy

$$\varrho_1 = -i(A + C + 2D + 2E) \tag{4.4.29}$$

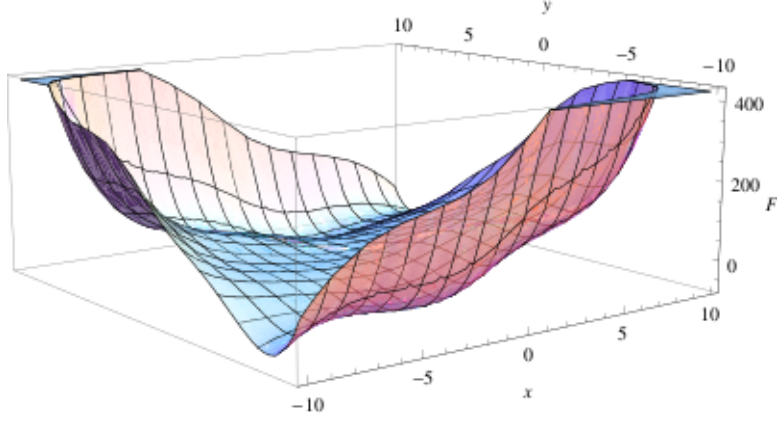


Figure 4.6: Plot of the vacuum energy as a function of x and y for $t = 0.01$, $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$.

Inserting the results from Eqs. (4.4.18), (4.4.19) and (4.4.20) for $A + C$, D and E gives us the following expression

$$\begin{aligned}
\varrho_1 = g^2 & \left[\frac{9}{2} I_l(m_A^2)^2 + \frac{1}{2} \left(2 \frac{m_B}{m_A} - 1 \right)^2 I_l(m_B^2)^2 + 8 \frac{m_\psi^2}{m_A^2} I_l(m_\psi^2)^2 \right. \\
& + 3 \left(2 \frac{m_b}{m_A} - 1 \right) I_l(m_A^2) I_l(m_B^2) - 12 \frac{m_\psi}{m_A} I_l(m_A^2) I_l(m_\psi^2) \\
& - 4 \frac{m_\psi}{m_A} \left(2 \frac{m_B}{m_A} - 1 \right) I_l(m_A^2) I_l(m_\psi^2) \\
& - \frac{3}{2} I_l(m_A^2)^2 - \frac{3}{2} I_l(m_B^2)^2 - I_l(m_A^2) I_l(m_B^2) \\
& - 4 I_l(m_\psi^2)^2 + 4 I_l(m_A^2) I_l(m_\psi^2) + 4 I_l(m_B^2) I_l(m_\psi^2) \\
& - 6 m_A^2 I_s(m_A^2, m_A^2) - 2 m_A^2 \left(2 \frac{m_B}{m_A} - 1 \right)^2 I_s(m_B^2, m_A^2) \\
& \left. + (16 m_\psi^2 - 4 m_A^2) I_s(m_\psi^2, m_A^2) - 4 m_B^2 I_s(m_\psi^2, m_B^2) \right] \quad (4.4.30)
\end{aligned}$$

If we now insert the solutions to the integrals I_l and I_s from Eqs. (4.4.10) and (4.4.16 and sort ϱ_1 according to order of Λ we get, with $\varrho_1 \equiv \varrho_{1,\Lambda^4} + \varrho_{1,\Lambda^2} + \mathcal{O}(\log \Lambda^2)$:

$$\varrho_{1,\Lambda^4} = \left(\frac{g\Lambda^2}{16\pi^2} \right)^2 \left[2\frac{m_B^2}{m_A^2} + 8\frac{m_\psi^2}{m_A^2} - 8\frac{m_B m_\psi}{m_A^2} + 4\frac{m_B}{m_A} - 8\frac{m_\psi}{m_A} + 2 \right], \quad (4.4.31)$$

$$\begin{aligned} \varrho_{1,\Lambda^2} = & \left(\frac{g\Lambda}{16\pi^2} \right)^2 \left[m_A^2 \log \frac{\Lambda^2}{m_A^2} \left(12\frac{m_\psi}{m_A} - 6\frac{m_B}{m_A} - 6 \right) \right. \\ & + m_B \log \frac{\Lambda^2}{m_B^2} \left(8\frac{m_B m_\psi}{m_A^2} - 4\frac{m_B^2}{m_A^2} - 2\frac{m_B}{m_A} - 4\frac{m_\psi}{m_A} + 2 \right) \\ & + m_\psi \log \frac{\Lambda^2}{m_\psi^2} \left(8\frac{m_B m_\psi}{m_A^2} - 16\frac{m_\psi^2}{m_A^2} + 8\frac{m_\psi}{m_A} \right) \\ & + 6m_A^2 \left(\log \frac{\Lambda^2}{m_A^2} - 1 \right) \\ & + 2m_A^2 \left(2\frac{m_B}{m_A} - 1 \right)^2 \left(\frac{m_A^2 \log \frac{\Lambda^2}{m_A^2} - m_B^2 \log \frac{\Lambda^2}{m_B^2}}{m_A^2 - m_B^2} \right) \\ & - (16m_\psi^2 - 4m_A^2) \left(\frac{m_A^2 \log \frac{\Lambda^2}{m_A^2} - m_\psi^2 \log \frac{\Lambda^2}{m_\psi^2}}{m_A^2 - m_\psi^2} \right) \\ & \left. + 4m_B^2 \left(\frac{m_B^2 \log \frac{\Lambda^2}{m_B^2} - m_\psi^2 \log \frac{\Lambda^2}{m_\psi^2}}{m_B^2 - m_\psi^2} \right) \right] \end{aligned} \quad (4.4.32)$$

At first glance it seems that the dominating contribution, *i.e* the piece proportional to Λ^4 is a quadratic form, so let us examine that one first. If we take a closer look on Eq. (4.4.31) it can be rewritten by introducing $m_B = xm_A$ and $m_\psi = ym_A$.

$$\varrho_{1,\Lambda^4} = \left(\frac{g\Lambda^2}{16\pi^2} \right)^2 f(x, y), \quad (4.4.33)$$

where

$$\begin{aligned} f(x, y) &= 2x^2 + 8y^2 - 8xy + 4x - 8y + 2 \\ &= 2(1 + x - 2y)^2 \end{aligned} \quad (4.4.34)$$

This equation has an infinite number of extrema, Taking partial derivatives gives us

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4x - 8y + 4 = 0, \\ \frac{\partial f}{\partial y} &= 16y - 8x - 8 = 0 \\ &= 4x - 8y + 4 = 0. \end{aligned} \quad (4.4.35)$$

This means that we have an infinite number of solutions, which as we see from Figure 4.3, are minima. This is more easily seen if we substitute $x = \frac{1}{\sqrt{5}}(2r + s)$ and $y = \frac{1}{\sqrt{5}}(r - 2s)$:

$$f(r, s) = 10\left(s + \frac{1}{\sqrt{5}}\right)^2. \quad (4.4.36)$$

Hence, we have a line in the (r, s) plane through $s = -\frac{1}{\sqrt{5}}$ of degenerate global minima. The interesting thing here is that the line of degenerate minima is rotated away from the supersymmetric points with $x = y$, so already at this level we see indications that the theory have stable ground states shifted away from the supersymmetric point. At this point, however, it is not possible to predict which of these points that are favoured.

To determine this we add the full Λ^2 contribution. By introducing additional parameters $m_A^2 = t\Lambda^2$, $\alpha = \frac{g^2}{8\pi^2}$ and $Q = \frac{\Lambda^4}{32\pi^2}$ we can rewrite ϱ as

$$\begin{aligned} \varrho &= Q(1 + x^2 - 2y^2) + \alpha Q(1 + x - 2y)^2 \\ &+ \alpha Q t \left[-\log t(12y - 6x - 6) - x^2 \log tx^2(8xy - 4x^2 - 2x - 4y + 2) \right. \\ &\quad - y^2 \log ty^2(12y + 8xy - 4y - 16y^2) - 6(\log t + 1) \\ &\quad - 2(2x - 1)^2 \left(\frac{\log t - x^2 \log tx^2}{1 - x^2} \right) \\ &\quad + 2(8y^2 - 2) \left(\frac{\log t - y^2 \log ty^2}{1 - y^2} \right) \\ &\quad \left. - 4x^2 \left(\frac{x^2 \log tx^2 - y^2 \log ty^2}{x^2 - y^2} \right) \right] + \mathcal{O}(\alpha^2) + \mathcal{O}(t^2) \\ &\equiv F(x, y, t, \alpha, Q) + \mathcal{O}(\alpha^2) + \mathcal{O}(t^2), \end{aligned} \quad (4.4.37)$$

where t is now small, as the terms of $\mathcal{O}(\log \Lambda)$ now will be proportional to t^2 . As this is a function of 5 parameters, it is very difficult to visualize. In order to probe this function for minima, we choose to use numerical methods from Mathematica, the notebook is added in Appendix B. The procedure goes as follows. Use numerical methods to scan over the coordinates (x, y) , for several values for t and fixed Q and α , for minima. We start out at the arbitrarily chosen values of $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$. First, we gather all these minima in a list, and plot the values of the vacuum energy as a function of t , shown in Figs. 4.4 and 4.5. This allows us to get a general idea about the energy landscape, and where we might find minima. From these plots we see that for each value of t , there are a pattern of several minima. For each t , there seem to be a minima of rather large energy, and two other minima with a lower, generally negative, energy. There are also minima with an energy close to zero, but the minima with negative energy will of course be favoured. The minima of large energy, and the minima with zero energy seem to disappear as t grows, which may indicate that they are simply artifacts of the numerics. The energy of these minima will generally decrease with increasing t , so there does not seem to be any favoured value of t . The problem here

Parameters	Original values	Parameters	Original values
$t = 0.001$	$m_A^2 = 0.1778$	$t = 0.005$	$m_A^2 = 0.8886$
$Q = 100$	$\Lambda^2 = 177.715$	$Q = 100$	$\Lambda^2 = 177.715$
$\alpha = \frac{1}{16\pi^2}$	$g^2 = \frac{1}{2}$	$\alpha = \frac{1}{16\pi^2}$	$g^2 = \frac{1}{2}$
$x = 1.1208$	$m_B^2 = 0.2232$	$x = -4.1346$	$m_B^2 = 15.1901$
$y = 1.1198$	$m_\psi^2 = 0.2285$	$y = -8.5405$	$m_\psi^2 = 64.8125$
	$\varrho = -0.0153$		$\varrho = -8.5614$
$x = 17.7601$	$m_B^2 = 56.0549$	$x = 4.0423$	$m_B^2 = 14.5193$
$y = 6.4614$	$m_\psi^2 = 7.4194$	$y = 9.0442$	$m_\psi^2 = 72.684$
	$\varrho = -0.1369$		$\varrho = -53.421$
$x = -19.0167$	$m_B^2 = 64.2683$	$x = -5.8923$	$m_B^2 = 30.8507$
$y = -5.2846$	$m_\psi^2 = 4.9631$	$y = 7.9034$	$m_\psi^2 = 55.5031$
	$\varrho = -12.6599$	$\det H \approx 0$	$\varrho = -53.421$
$x = -12.3364$	$m_B^2 = 27.0457$		
$y = -15.3461$	$m_\psi^2 = 41.8524$		
$\det H \approx 0$	$\varrho = 108.603$		

Table 4.1: The minima of the vacuum energy for $t = 0.001$ and $t = .005$, with $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$.

is of course that as t grows, so does all three particle masses. At some point they will be of the order of Λ^2 , and here we must expect new physics to enter.

Next we pick out a few specific values for t , to determine the physical masses we might expect, and plot the vacuum energy as a function of x and y . We choose the values $t = 0.001$, $t = 0.005$, $t = 0.01$ and $t = 0.05$. The plot for $t = 0.01$ is shown in Fig. 4.6 to illustrate the qualitative energy landscape, while the other three are omitted as not much information can be drawn from them. Using the same numerical method to scan for the minima, we find several minima for each t -value, shown in Tables 4.1 and 4.2. From the tables we see more clearly what Fig. 4.5 indicated. Each point in t -space has two points with energy lower than the supersymmetric point, which correspond to roughly the same set of masses. The masses of the particles at these points are generally such that at least one particle is significantly heavier than the others. We also see the higher energy minima, but as indicated in the tables, the determinant of the Hessian matrix at these points is very close to zero. This means that we cannot say for certain whether these points are actual minima. They are probably relatively flat areas, or possibly saddle points, which look like minima due to the numerics. We also observe from Fig. 4.5 that these points gradually vanish as t increases. Lastly, we also find an approximately supersymmetric minima in the plot for $t = 0.001$. It is interesting to note that there are always two clear minima for each plot, and both of them correspond to roughly the same masses for each particle. It is possible that these two points are in fact the same point, they just show up separately because we probe for minima as a function of (x, y) , as opposed to (x^2, y^2) .

Parameters	Original values	Parameters	Original values
$t = 0.01$	$m_A^2 = 0.1778$	$t = 0.05$	$m_A^2 = 0.8886$
$Q = 100$	$\Lambda^2 = 177.715$	$Q = 100$	$\Lambda^2 = 177.715$
$\alpha = \frac{1}{16\pi^2}$	$g^2 = \frac{1}{2}$	$\alpha = \frac{1}{16\pi^2}$	$g^2 = \frac{1}{2}$
$x = 0.3622$	$m_B^2 = 0.233$	$x = -0.8134$	$m_B^2 = 5.87522$
$y = 0.6589$	$m_\psi^2 = 0.7716$	$y = -3.9851$	$m_\psi^2 = 141.116$
$\det H \approx 0$	$\varrho = 0.3446$		$\varrho = -96.9848$
$x = -2.5428$	$m_B^2 = 11.4903$	$x = 1.0458$	$m_B^2 = 9.71824$
$y = -6.6732$	$m_\psi^2 = 79.1397$	$y = 4.2576$	$m_\psi^2 = 161.078$
	$\varrho = -36.7481$		$\varrho = -123.971$
$x = 2.5913$	$m_B^2 = 11.933$		
$y = 7.03812$	$m_\psi^2 = 88.0314$		
	$\varrho = -72.6839$		

Table 4.2: The minima of the vacuum energy for $t = 0.01$ and $t = 0.05$, with $Q = 100$ and $\alpha = \frac{1}{16\pi^2}$.

Next we want to examine how the behaviour of the minima changes with the other parameters, Q and α . Changing Q would only scale the whole problem, as every term is multiplied overall by Q . This means that Q , or Λ sets the energy scale of the whole theory. Which of course also means that the model has little predictive power by itself, there will have to be some input from experiments to determine the overall scale. That is if the model can be extended to be realistic of course. Changing α , on the other hand, may lead to interesting effects, as this parameter determines how strongly the one loop contribution dominates over the one from two loops. Decreasing α will only make matters worse, the particles will be less bound, and the energy will decrease even more at large masses. We will therefore examine the vacuum energy and its minima as we increase the coupling: Going through the same procedure to locate the minima as before, we obtain the minima shown in Fig. 4.7 as a function of t for α -values of $\frac{2}{\pi^2}$, $\frac{1}{\pi^2}$, $\frac{1}{2\pi^2}$, $\frac{1}{4\pi^2}$ and $\frac{1}{8\pi^2}$. Here we see clearly that, as α grows larger, that both sets of minima will start to stabilize and obtain a minimum as a function of t . It is clear however, that they do not correspond to similar states. They actually evolve into two radically different states as α grows large. We also see that the set of minima which vanished for higher t , now evolve into a set of unstable minima at all considered values of t . It seems that the theory actually has local, stable minima for α -values $\frac{1}{4\pi^2}$ and larger. This corresponds to a coupling $g \geq \sqrt{2}$, so whether our perturbation series can be trusted in this area is questionable, but it is not completely far-fetched, as α is still relatively small. We also see that a minima with approximately zero energy appear at low t -values, which then must correspond to a supersymmetric state of light particles. This will not be the most favoured state though, as there are other minima with lower energy.

Let us examine these apparent minima closer, to see what range of particle masses we might expect here. We pick out slices at the favoured points in t -space, and use the same

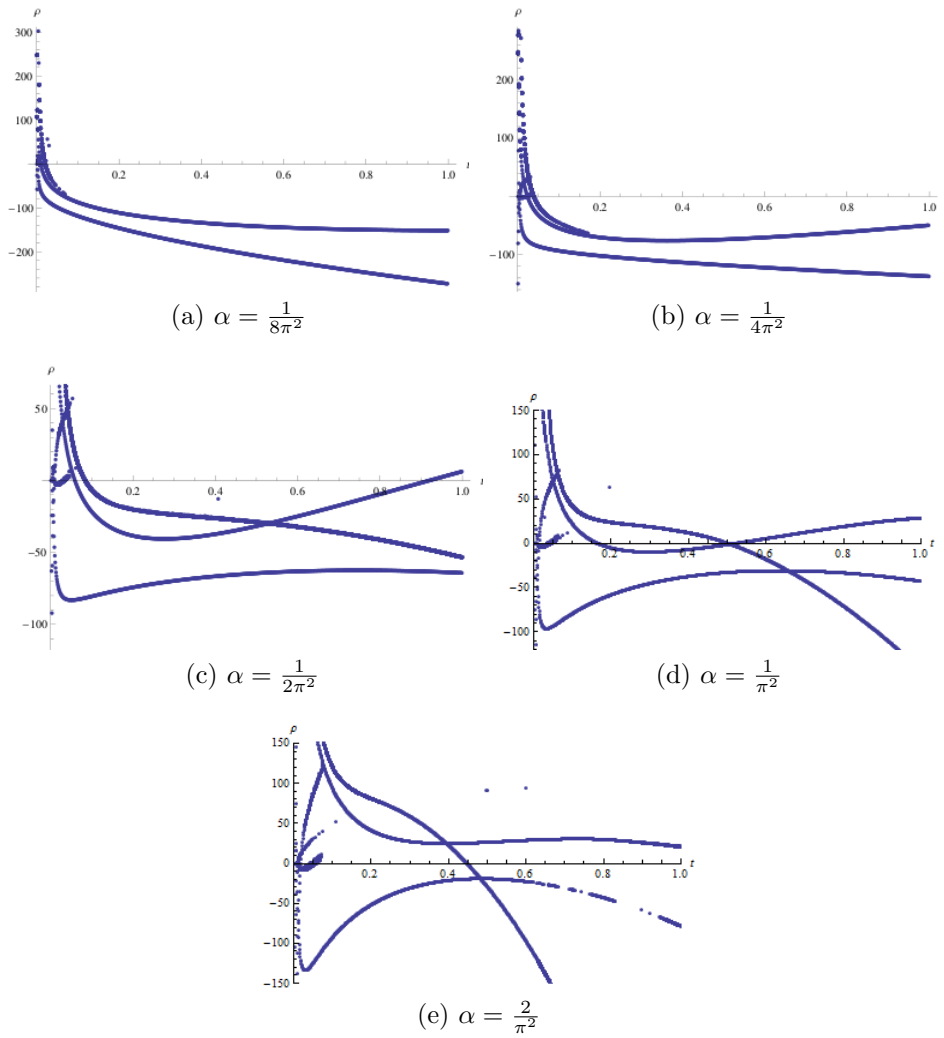


Figure 4.7: Plots of the energies of the minima, as a function of t , with $Q = 100$ and α -values $\frac{2}{\pi^2}$, $\frac{1}{\pi^2}$, $\frac{1}{2\pi^2}$, $\frac{1}{4\pi^2}$ and $\frac{1}{8\pi^2}$.

Parameters	Original values
$t = 0.363$ $Q = 100$ $\alpha = \frac{1}{4\pi^2}$	$m_A^2 = 64.5107$ $\Lambda^2 = 177.715$ $g^2 = 2$
$x = -0.1645$ $y = -1.5588$	$m_B^2 = 1.7459$ $m_\psi^2 = 156.745$ $\varrho = -75.7405$
$x = 0.4442$ $y = 1.7920$	$m_B^2 = 12.784$ $m_\psi^2 = 207.159$ $\varrho = -111.35$

Table 4.3: The minima of the vacuum energy for $t = 0.363$, with $Q = 100$ and $\alpha = \frac{1}{4\pi^2}$.

numerical routine to scan for minima in (x, y) . The details of each point is shown in Tables 4.3 and 4.4. From this we see that the minima at $\alpha = \frac{1}{4\pi^2}$ and $t = 0.363$, corresponding to the stable state, we have one very light particle and two with a mass squared of at least one order of magnitude more. We have also showed the unstable point here for comparison. Turning to the $\alpha = \frac{1}{2\pi^2}$ plot, we find as expected two minima. One at $t = 0.003$ and another at $t = 0.276$. From the values of the masses at these points, we see that the minimum at $t = 0.276$ is probably the same minimum as the one we found for $\alpha = \frac{1}{4\pi^2}$, as we have the same qualitative behaviour. That is, one light pseudo-scalar, and a heavy scalar and fermion. The minimum at $t = 0.003$ on the other hand, is radically different. Here we have a heavy pseudo-scalar, and a light scalar and fermion.

Sadly, none of the discovered minima match up exactly with what we see in nature, that is a light fermion with the rest of the particles heavy, but we did not necessarily expect it to. This is of course because we have completely neglected the gauge sector, which will couple to the particle multiplet we have been discussing so far. An important question now arises, why should the theory pick out the stable minima, when there are more favourable, but unstable states elsewhere. One argument is that it must do so for the theory to make any sense, but this is not a very natural reason. It is feasible, however, that the model will stay in one of the minima if it manages to settle itself in there. This is because tunneling between vacua in QFT is generally very unlikely to happen. The hope here is of course that inclusion of the gauge sector will stabilize the theory further, leading to a stable absolute global minimum, and that this minimum will have qualities similar to what we observe in nature.

These results leads us to a very interesting conclusion: If we allow the Wess-Zumino model to drift away from the supersymmetric point, it may settle itself into minima where supersymmetry is broken to a large degree. It is also apparent that the nature of this symmetry breaking can be radically different at the different minima. If we imagine living in a world described by this model, we would observe one particle, while its superpartners would be out of reach of our accelerators. This is of course not conclusive proof that there

Parameters	Original values	Parameters	Original values
$t = 0.003$	$m_A^2 = 0.53315$	$t = 0.276$	$m_A^2 = 49.0494$
$Q = 100$	$\Lambda^2 = 177.715$	$Q = 100$	$\Lambda^2 = 177.715$
$\alpha = \frac{1}{2\pi^2}$	$g^2 = 4$	$\alpha = \frac{1}{2\pi^2}$	$g^2 = 4$
$x = -11.768$	$m_B^2 = 73.8375$	$x = -0.1804$	$m_B^2 = 1.5962$
$y = -2.9762$	$m_\psi^2 = 4.7225$	$y = -1.4322$	$m_\psi^2 = 100.608$
	$\varrho = -92.1792$		$\varrho = -40.5425$

Table 4.4: The minima of the vacuum energy for $t = 0.003$ and $t = 0.276$, with $Q = 100$ and $\alpha = \frac{1}{2\pi^2}$.

exist superpartners of the fermions of the standard model, as this model is in no way realistic. It is, however, an indication that the existence of heavy superpartners is possible, and that a world with the correct number of degrees of freedom may dynamically choose a configuration similar to this.

Chapter 5

Conclusion and outlook

The goal of this thesis was to first examine whether a variational calculation on the ground state and the effective potential of a scalar field theory could find possible symmetry breaking ground states. We have shown that the method works, and no minima more suitable are discovered in a simple ϕ^4 theory. This was what we expected, as the minimum discovered in Coleman and Weinberg's paper [7] was suggested to be untrustworthy.

Secondly, we wanted to use similar methods to examine a supersymmetric theory, the Wess-Zumino model. First we discussed the model as it was presented by Wess and Zumino, and verified explicitly the non-renormalization theorem and the vanishing vacuum energy to two loops. Then, by allowing the model to move away from the supersymmetric point where all the masses are equal, we have shown that the model may obtain stable minima which is shifted away from the supersymmetric point. The nature of this shift is such that at least one of the particles is significantly heavier than the others. This means that supersymmetry is strongly broken in these states, but not in the same way in each state; we have shown that for sufficiently large coupling we have several minima with radically different characteristics. We have also shown that there are additional, unstable, states. If the theory were to settle itself into these, we would have a runaway growth of all three particle masses. The question now arises, why should the theory pick the stable state, and what decides which stable state we end up with? It seems that pure chance would be responsible for deciding what kind of universe we have, and perhaps more worrisome, whether the universe makes any sense at all.

This warrants further research, to resolve this. The most obvious first step is to extend the formalism of the effective action developed for the scalar field to the supersymmetric model. One could introduce three variational parameters coupled quadratically to each field, with a non-local regularization function. This could then be used to probe further for stable global minima of the model. The problem here is of course that first of all, the parameter space is three-dimensional, and secondly the theory itself is much more complicated than the scalar theory. Either of these complications alone would make the calculations very difficult indeed.

A second, quite different approach is to integrate out one of the fields of the theory, most likely the fermion field. This would allow one to compare the theory to the superpotential, and may shed some light on the problem from a different angle.

The next, and most important step, is to generalize this method to a supersymmetric

theory involving gauge fields. This would allow one to examine a theory which is very much like what is thought to be realized in nature, namely the Minimal Supersymmetric Standard Model[12]. This would of course bring a host of complications, beyond increasing the dimension of the parameter space. As this would be a gauged model, one would no longer be able to use a simple cutoff to regularize the theory. Dimensional regularization, which is widely used to handle ordinary gauge-theories, has its own problems when it comes to supersymmetry[33, 34]. Overlooking these technical difficulties, it is possible that adding the gauge sector to the theory may solve at least some of the problems we have encountered so far. It may provide a mechanism that prevents the theory from falling into the unstable states, or simply remove these states altogether. The most interesting solution would of course be a stable and unambiguous global minimum. This state would of course have to make sense when compared with the standard model. In the standard model we have a very particular pattern, where the fermions always are the lightest partners of the Wess-Zumino sector, and likewise for the bosons of the gauge sector. If in the case of a gauged theory, we get several minima with different characteristics as we did with the Wess-Zumino model alone, it would be difficult to explain why nature always chooses the same puzzling pattern for each generation. Nevertheless, if such a state was found, one could use input from experiments to perhaps determine some of the free parameters of the theory, and examine if this model is in any way realistic. If this is possible, one might be able to actually predict the masses of the superpartners to the known standard model particles.

It is also worth noting that we have only considered the energetic part of our argument of naturalness from the introduction. It could also be interesting to consider the probabilistic side, that is examine the size of the areas in the bare parameter space which corresponds to the minima found, to determine which minima is more likely. Both for the Wess-Zumino sector, which we already have considered, and for a potential calculation of a full, gauged theory. This could clear up the question of whether pure chance determines the state of our universe, if one discovered that one minima is probabilistically favoured.

The ultimate goal of this thesis was to examine the ground state of a supersymmetric model through a variational calculation with the renormalized masses as parameters. As we hypothesized in the introduction, there are indications that a model with a number of degrees of freedom consistent with supersymmetry may have minima more favourable than the supersymmetric point. If we return to our argument of naturalness, we must conclude that supersymmetry is not necessarily the most likely state to emerge from our theory. This is ironic, as the argument for introducing supersymmetry in the standard model in the first place was just that of naturalness. The model seem to show a kind of dynamical symmetry breaking, as we in no way lay any restrictions on the particle masses, we simply allow them to flow as the dynamics of the theory directs them to. This result fits very well with observations from our particle accelerators[13], in which we do not yet see any superpartners of our known standard model particles. Whether this is actually what happens in nature is difficult to say without considering a full supersymmetric theory. It is however, a strong indication and a proof of concept for the idea that supersymmetry, while beautiful, is not the most natural state of affairs.

Appendix A

Mathematical techniques

The identities in this appendix can be found in any proper QFT textbook, *e.g.* [15],[35] or [16].

A.1 Gaussian integrals

As the free path integral, and approximating the path-integral with interactions basically is one big Gaussian integral, we will review some of the mathematics behind these here. The simplest Gaussian integral over one variable can be solved by squaring and transform it to polar coordinates.

$$\begin{aligned} \left(\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} \right)^2 &= 2\pi \int_0^{\infty} dr r e^{-\frac{1}{2}r^2} \\ &= 2\pi \int_0^{\infty} dt e^{-t} = 2\pi, \\ \implies \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} &= \sqrt{2\pi}. \end{aligned} \tag{A.1.1}$$

We can scale this integral by a constant a :

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}. \tag{A.1.2}$$

We will also need the shifted integral

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+bx}, \tag{A.1.3}$$

which can be solved by completing the square in the exponent

$$-\frac{1}{2}ax^2 + bx = -\frac{1}{2}(x - ba^{-1}) a (x - ba^{-1}) + \frac{1}{2}ba^{-1}b, \tag{A.1.4}$$

and shifting the integration variable to $x + ba^{-1}$. This yields

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+bx} &= \int_{-\infty}^{\infty} dx' e^{-\frac{1}{2}ax'^2} e^{\frac{1}{2}ba^{-1}b} \\ &= \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}ba^{-1}b}. \end{aligned} \quad (\text{A.1.5})$$

This can be generalized to N variables: Let x and b both be N -component vectors and A an $N \times N$ symmetric matrix. We then have

$$\int d^N x e^{-\frac{1}{2}x^T A x} = \sqrt{\frac{(2\pi)^N}{\det A}}, \quad (\text{A.1.6})$$

and

$$\int d^N x e^{-\frac{1}{2}x^T A x + b^T x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2}b^T A^{-1}b}. \quad (\text{A.1.7})$$

Now we can generalize this further by letting $N \rightarrow \infty$, which gives us the *functional* integrals we will need in this thesis:

$$\int \mathcal{D}\phi \exp \left[-\frac{1}{2} \int d^4 x \int d^4 y \phi(x) K(x, y) \phi(y) \right] = C(\det K)^{-\frac{1}{2}}, \quad (\text{A.1.8})$$

and

$$\begin{aligned} \int \mathcal{D}\phi \exp \left[-\frac{1}{2} \int d^4 x \int d^4 y \phi(x) K(x, y) \phi(y) + \int d^4 x J(x) \phi(x) \right] \\ = C' \exp \left[\frac{1}{2} \int d^4 x \int d^4 y J(x) K^{-1}(x, y) J(y) \right]. \end{aligned} \quad (\text{A.1.9})$$

The integrals now run over all possible forms of the function $\phi(x)$, integration over x and y is the functional analogue of matrix multiplication, and the operator $K(x, y)$ now has to be symmetric under exchange of x and y . C and C' are just constants which are generally uninteresting to the discussion. For our purposes, K is the propagator of the theory in question, which is always symmetric.

It is obvious that we have glossed over a lot of details when generalizing to functional integrals, but we only need to use the results in this thesis. A more thorough account of functional integration can be found in [16], Chapter 13.

A.2 Feynman parameter integrals

We often encounter loop integrals on the form

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\epsilon][(k+p)^2 - m^2 + i\epsilon]}. \quad (\text{A.2.1})$$

These can be simplified by using the identity ($i\epsilon$ is from now on not written explicitly)

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2}, \quad (\text{A.2.2})$$

yielding

$$\begin{aligned} & \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[(k^2 + p^2 + 2p \cdot k - m^2)x + (k^2 - m^2)(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[(k + px)^2 - m^2 - x(x-1)p^2]^2} \end{aligned} \quad (\text{A.2.3})$$

We can now shift the integration variable by defining $\bar{k} = k + px$, and we end up with the integral

$$\int_0^1 dx \int \frac{d^4 \bar{k}}{(2\pi)^4} \frac{1}{[\bar{k}^2 - M^2]^2}, \quad (\text{A.2.4})$$

where $M^2 = m^2 + x(x-1)p^2$ for notational convenience. This integral can now be readily solved with use of the proper regularization technique.

A.3 Regularization

When performing loop calculations one usually encounters divergent integrals on the form

$$\int \frac{d^4 k}{(2\pi)^4} \frac{(k^2)^\alpha}{(k^2 - m^2 + i\epsilon)^\beta} \quad (\text{A.3.1})$$

More complex denominators can always be simplified through the use of Feynman parameters, see Appendix A.2. These will diverge as long as $\alpha + 2 \geq \beta$. To handle these it is necessary to regularize them in some way, so that their analytic form can be preserved and dealt with by renormalization. First of all, the denominator

$$k^2 - m^2 + i\epsilon = (k^0)^2 - (\vec{k}^2 + m^2) + i\epsilon \equiv (k^0)^2 - \omega_k^2 + i\epsilon \quad (\text{A.3.2})$$

has poles at $k^0 = \pm(\omega_k - i\epsilon)$. Their location in the complex k^0 plane, shown in Figure (A.1) are such that they can be avoided by rotating the integration path 90 degrees counterclockwise to the imaginary axis. That is, we substitute $k^0 = ik_E^0$ and $\vec{k} = \vec{k}_E$. This is what is called a Wick rotation into Euclidean space. In doing this we have that

$$\begin{aligned} k^2 &= (k^0)^2 - \vec{k}^2 = -(k_E^0)^2 - \vec{k}_E^2 = -k_E^2 \\ d^4 k &= dk^0 d^3 k = idk_E^0 d^3 k_E = id^4 k_E \\ \implies & \int \frac{d^4 k}{(2\pi)^4} \frac{(k^2)^\alpha}{(k^2 - m^2 + i\epsilon)^\beta} \\ &= i(-1)^{\alpha+\beta} \int \frac{d^4 k_E}{(2\pi)^4} \frac{(k_E^2)^\alpha}{(k_E^2 + m^2)^\beta} \end{aligned} \quad (\text{A.3.3})$$

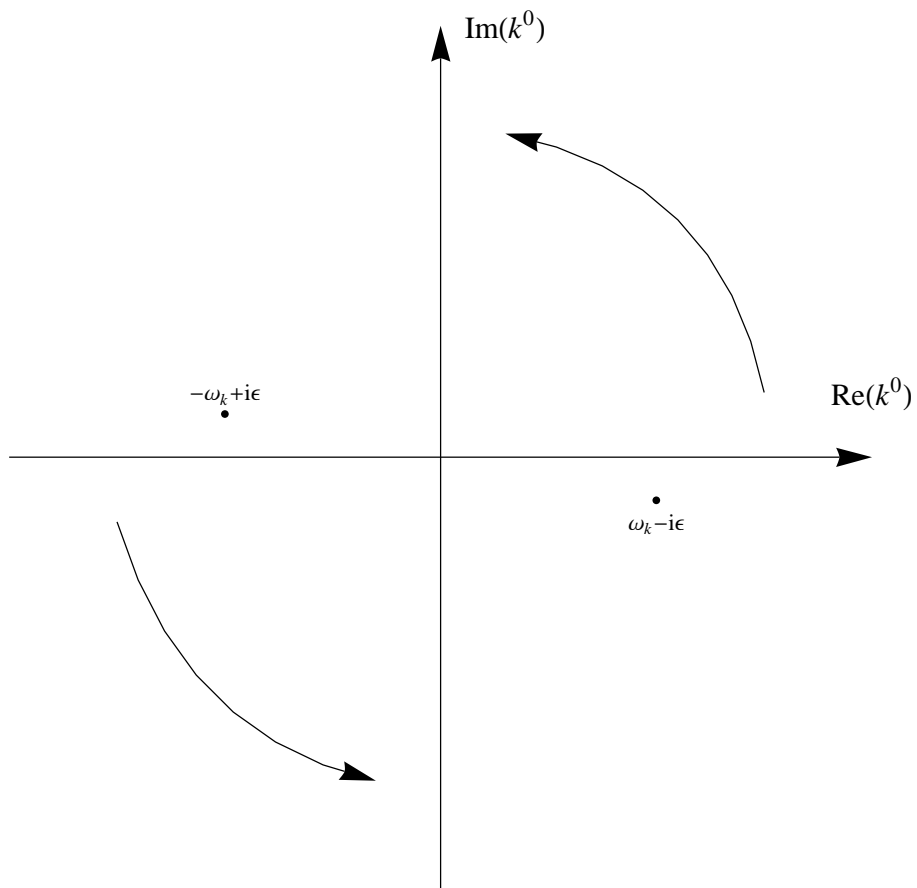


Figure A.1: By rotating the integration path 90 degrees counterclockwise we avoid the poles on the real axis.

Now there is no pole, so ϵ can be set to 0.

The integral can now be solved, and there are several regularization techniques available to do this. The simplest solution is to just cut off the momentum k_E at some high value Λ . This does break gauge symmetry however, but since we will not consider gauge theories in this thesis, a simple cut-off will suit our purposes just fine. The integral will then take the form, suppressing the subscript E from now on, as the sign of m^2 implies that we are in Euclidean space:

$$\int \frac{d^4k}{(2\pi)^4} \frac{(k^2)^\alpha}{(k^2 + m^2)^\beta} = \frac{1}{8\pi^2} \int_0^\Lambda dk k^3 \frac{(k^2)^\alpha}{(k^2 + m^2)^\beta} \quad (\text{A.3.4})$$

Which can be easily solved, depending on the values of α and β .

This is all well and good for most of our integrals, but we have to be careful if we want to compare a four-dimensional integral to a three-dimensional counterpart. There is a particular example in this thesis: In section 2.1 we showed that

$$\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log k^2 + m^2 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2}. \quad (\text{A.3.5})$$

But we better make sure that this still makes sense when we cut off the integrals. So, we want to solve the integral, substituting $x = k^2$,

$$I_4 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log(k^2 + m^2) = \frac{1}{2} \frac{1}{16\pi^2} \int_0^{\Lambda^2} dx x \log(x + m^2) \quad (\text{A.3.6})$$

A partial integration yields

$$\begin{aligned} 32\pi^2 I_4 &= \frac{1}{2} \Lambda^4 \log(\Lambda^2 + m^2) - \frac{1}{2} \int_0^{\Lambda^2} dx x \frac{x + m^2 - m^2}{x + m^2} \\ &= \frac{1}{2} \Lambda^4 \log(\Lambda^2 + m^2) - \frac{1}{4} \Lambda^4 + \frac{1}{2} m^2 \int_0^{\Lambda^2} dx \frac{x + m^2 - m^2}{x + m^2 - m^2} \\ &= \frac{1}{2} \Lambda^4 \log(\Lambda^2 + m^2) - \frac{1}{4} \Lambda^4 + \frac{1}{2} m^2 \Lambda^2 - \frac{1}{2} m^4 \log\left(\frac{\Lambda^2 + m^2}{m^2}\right). \end{aligned} \quad (\text{A.3.7})$$

Next we want to compare this to solving the integral on the three-dimensional form. Now we have to be careful about the cutoff though, cutting off a four dimensional integral at $k^2 = \Lambda^2$ is not necessarily the same as cutting off a three dimensional integral at $|\vec{k}|^2 = \Lambda^2$. So let us write the three-dimensional cutoff as $\tilde{\Lambda}$, just to be safe. The integral we have to solve now is (we now write the length of \vec{k} as k)

$$I_3 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2} = \frac{1}{4\pi^2} \int_0^{\tilde{\Lambda}} dk k^2 \sqrt{k^2 + m^2}. \quad (\text{A.3.8})$$

This integral is a bit trickier than I_4 , so we will let Mathematica do the job.

$$\begin{aligned} 32\pi^2 I_3 &= 8 \int_0^{\tilde{\Lambda}} dk k^2 \sqrt{k^2 + m^2} \\ &= \tilde{\Lambda} (m^2 + 2\tilde{\Lambda}^2) \sqrt{\Lambda^2 + m^2} - m^4 \operatorname{arsinh} \frac{\tilde{\Lambda}}{m}. \end{aligned} \quad (\text{A.3.9})$$

If we now expand both results in powers of m^2/Λ^2 , we can compare

$$\begin{aligned} 32\pi^2 I_4 &= -\frac{1}{4}\Lambda^4 + \frac{1}{2}\Lambda^4 \log \Lambda^2 + \Lambda^2 m^2 - \frac{1}{2}m^4 \log \frac{\Lambda^2}{m^2} - \frac{1}{4}m^4 + \mathcal{O}(\Lambda^{-2}), \\ 32\pi^2 I_3 &= 2\tilde{\Lambda}^4 + 2\tilde{\Lambda}^2 m^2 - \frac{1}{2}m^4 \log \frac{2\tilde{\Lambda}^2}{m^2} + \frac{1}{4}m^4 - \frac{1}{2}m^4 \log 2 + \mathcal{O}(\tilde{\Lambda}^{-2}). \end{aligned} \quad (\text{A.3.10})$$

Sadly, this is not equal, but all is not lost. If we replace $2\tilde{\Lambda}^2 = \Lambda^2$ we get the same result for the terms which are dependent on both Λ and m . As we will see in Chapter 4, these are the only terms we are interested in, or rather the terms which are physical. The only problem is the terms which only depend on m , they are equal with the approximation that $\log 2 = 1$, which is a pretty bad approximation. We are not interested in these terms in this thesis, but they will obviously have a physical implication. It is possible that this difference can be accounted for by shifting $\tilde{\Lambda}$, but that shift would have to be dependent on the particle masses, which of course leads to complications. This indicates that it is not trivial to transform divergent integrals from four to three dimensions by contour integration when employing a cutoff.

A.4 Grassmann algebra

To allow us to construct a path integral for fermions, it is necessary to introduce a mathematical construct called Grassmann numbers, named after Hermann Grassmann. These numbers are also called anti-commuting numbers, due to their algebra. If we have a set of Grassmann numbers θ_i and real number a , they obey the following algebra

$$\{\theta_i, \theta_j\} = 0, \quad [\theta_i, a] = 0. \quad (\text{A.4.1})$$

That is, Grassmann numbers anti-commute with themselves and commute with ordinary numbers. In particular, if $i = j$ we have that

$$\theta_i^2 = -\theta_i^2 \quad (\text{A.4.2})$$

$$\implies \theta_i^2 = 0, \quad (\text{A.4.3})$$

which makes calculations particularly simple, as we can have a maximum of only two powers of any Grassmann number.

Next we want to define integrals over Grassmann numbers. To do this we demand that an integral over an arbitrary function of Grassmann numbers $f(\theta)$ can be shifted and still yield the same result, as for integrals over ordinary integrals. That is

$$\int d\theta f(\theta) = \int d\theta f(\theta + \eta). \quad (\text{A.4.4})$$

If we expand both functions to the most general form possible, which is severely restricted by the algebra, we obtain

$$\int d\theta (a + b\theta) = \int d\theta (a + b(\theta + \eta)). \quad (\text{A.4.5})$$

This leads us to demand that $\int d\theta b\eta = 0$, or as η is arbitrary,

$$\int d\theta = 0, \quad (\text{A.4.6})$$

As a product of two Grassmann numbers commutes with any other Grassmann number we can say that a product of two Grassmann numbers is an ordinary number, which also means that $\int d\theta\theta$ should be an ordinary number. We then simply define

$$\int d\theta\theta = 1, \quad (\text{A.4.7})$$

which fixes the normalization of $d\theta$

This is all we need to do the simplest Gaussian

$$\int d\theta d\bar{\theta} e^{\bar{\theta}a\theta}. \quad (\text{A.4.8})$$

This can now be rewritten as a Taylor series, which will be cut off after only two terms, due to Eq. (A.4.1)

$$\int d\theta d\bar{\theta} e^{\bar{\theta}a\theta} = \int d\theta d\bar{\theta} (1 + \bar{\theta}a\theta) = a. \quad (\text{A.4.9})$$

This can easily be generalized to $2N$ Grassmann variables collected in a column vector θ and a row vector $\bar{\theta}$ each of dimension N . With A some hermitian $N \times N$ matrix we have

$$\int d\theta d\bar{\theta} e^{\bar{\theta}A\theta} = \det A. \quad (\text{A.4.10})$$

We also need the integral

$$\int d\theta d\bar{\theta} e^{\bar{\theta}A\theta + \bar{\eta}\theta + \bar{\theta}\eta}. \quad (\text{A.4.11})$$

This can again can be solved by completing the square and shifting the integration variables

$$\int d\theta d\bar{\theta} e^{\bar{\theta}A\theta + \bar{\eta}\theta - \bar{\theta}\eta} = \int d\theta d\bar{\theta} e^{\bar{\theta}A\theta} e^{\bar{\eta}A^{-1}\eta} \quad (\text{A.4.12})$$

$$= \det A e^{\bar{\eta}A^{-1}\eta} A. \quad (\text{A.4.13})$$

Appendix B

Mathematica calculations

The following pages contain the Mathematica notebooks *phi_calc.nb* and *wz_calc.nb* used in the calculations in this thesis.

(*Definitions for numerical precision*)

Acc = 8; Rec = 40;

(*Define vacuum energy density, E, and the conjugated variable ξ^2 , (ξ here)*)

E0[μ _, M_, mr_] :=
1 / (8 π^2) * NIntegrate[Sqrt[x] (Sqrt[x + mr^2 + $\mu^2 * M^4 / (x + M^2)^2$] - Sqrt[x + mr^2]),
{x, 0, Infinity}, AccuracyGoal \rightarrow Acc, MaxRecursion \rightarrow Rec, WorkingPrecision \rightarrow 50]

E1[μ _, M_, mr_, λ] := $\lambda / 8 * (1 / (8 \pi^2))^2 * (NIntegrate[Sqrt[x] (1 / Sqrt[x + mr^2 + $\mu^2 * M^4 / (x + M^2)^2$] - 1 / Sqrt[x + mr^2]),
{x, 0, Infinity}, AccuracyGoal \rightarrow Acc, MaxRecursion \rightarrow Rec])^2$

ξ 0[μ _, M_, mr_] := -1 / 4 * 1 / (8 π^2) *
NIntegrate[Sqrt[x] * $M^4 / (x + M^2)^2 / Sqrt[x + mr^2 + $\mu^2 * M^4 / (x + M^2)^2$],
{x, 0, Infinity}, AccuracyGoal \rightarrow Acc, MaxRecursion \rightarrow Rec]$

ξ 1[μ _, M_, mr_, λ] := $\lambda / 8 * (1 / (8 \pi^2))^2 * (NIntegrate[Sqrt[x] (1 / Sqrt[x + mr^2 + $\mu^2 * M^4 / (x + M^2)^2$] - 1 / Sqrt[x + mr^2]),
{x, 0, Infinity}, AccuracyGoal \rightarrow Acc, MaxRecursion \rightarrow Rec] *
NIntegrate[Sqrt[x] * $M^4 / (x + M^2)^2 / (x + mr^2 + $\mu^2 * M^4 / (x + M^2)^2$)^(3/2),
{x, 0, Infinity}, AccuracyGoal \rightarrow Acc, MaxRecursion \rightarrow Rec])$$

En[μ _, M_, mr_, λ] := E0[μ , M, mr] + E1[μ , M, mr, λ]

ξ [μ _, M_, mr_, λ] := ξ 0[μ , M, mr] + ξ 1[μ , M, mr, λ]

(*Define the effective potential*)

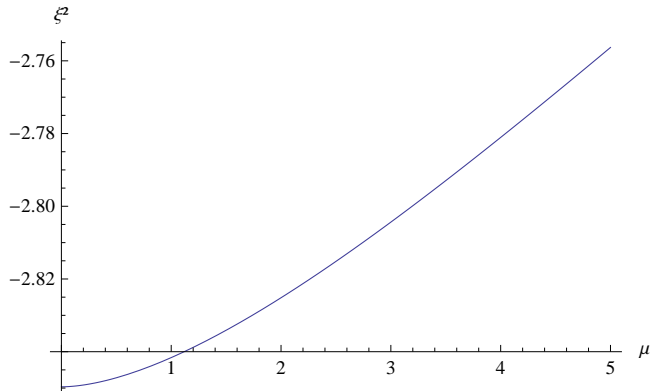
V[μ _, M_, mr_, λ] := En[μ , M, mr, λ] + $\mu^2 * \xi$ [μ , M, mr, λ]

(*Choose some arbitrary values for M and λ *)

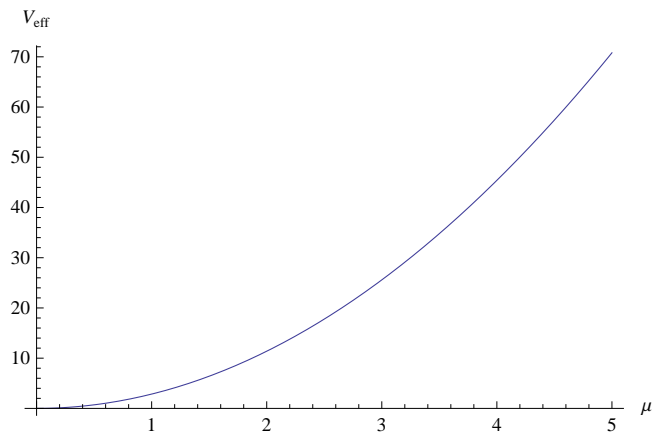
a = 30; b = 1.5;

(*Plotting the effective potential as a function of ξ^2 , for zero and non-zero mass*)

Plot[ξ [μ , a, 0, b], { μ , 0, 5}, AxesLabel \rightarrow { μ , ξ^2 }]



```
Plot[V[μ, a, 0, b], {μ, 0, 5}, AxesLabel → {μ, Veff}]
```



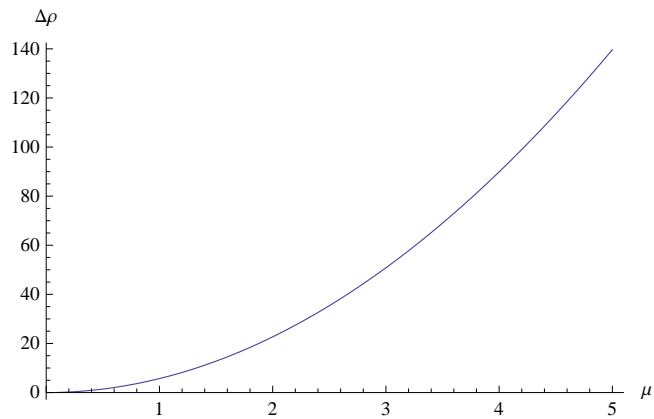
(*Here we run into trouble, as ξ changes very slowly with μ *)

```
ParametricPlot[{ξ[μ, a, 0, b], V[μ, a, 0, b]}, {μ, .1, 1.5}]
```

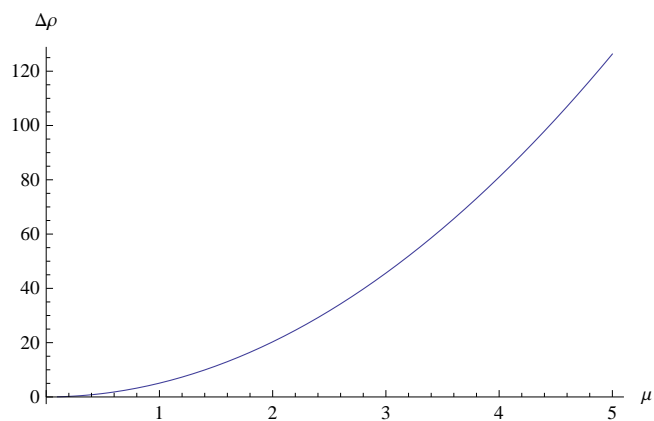
(*Plot not shown, as it uses a lot of space*):

(*Plotting the vacuum energy for zero and non-zero mass*)

```
Plot[En[μ, a, 0, b], {μ, .1, 5}, AxesLabel → {μ, Δρ}]
```



```
Plot[En[μ, a, 10, b], {μ, .1, 5}, AxesLabel → {μ, Δρ}]
```



(*This notebook contains the code to generate the plots and data used in this thesis. The actual plots are removed from the notebook for performance issues. Each plot and data set can be generated by using this code with the parameters listed in the thesis.*)

(*Generating plot over dominating contribution to vacuum energy*)

(*Definitions: $m_A^2 = t \Lambda^2, m_B^2 = x^2 m_A^2, m_\psi^2 = y^2 m_A^2, \alpha = g/8\pi^2$ *)

```
In[30]:= f[x_, y_] := 8 y^2 + 2 x^2 - 8 y - 8 x * y + 4 x + 2
```

```
Plot3D[f[x, y], {x, -5, 5}, {y, -5, 5}, AxesLabel -> {x, y, f}]
```

(*Rotate the axes for better visualization*)

```
In[31]:= h[s_, t_] := Simplify[f[x, y] /. {x -> 1/Sqrt[5](2s+t), y -> 1/Sqrt[5](s-2t)}]
```

```
In[32]:= h[s, t]
```

```
Out[32]= 2 + 4 Sqrt[5] t + 10 t^2
```

(*Full expressions for the vacuum energy*)

(* $l \sim \Lambda, a \sim m_A, b \sim m_B, p \sim m_\psi$ *)

```
In[1]:= E0[l_, a_, b_, p_] := 1 / (32 \pi^2) (l^2 (a^2 + b^2 - 2 p^2))
```

```
In[2]:= E14[l_, a_, b_, p_, g_] :=
  g^2 l^4 / (16 \pi^2)^2 (8 p^2 / a^2 + 2 b^2 / a^2 + 4 b / a - 8 p / a - 8 b * p / a^2 + 2)
```

```
In[3]:= E12[l_, a_, b_, p_, g_] :=
  g^2 l^2 / (16 \pi^2)^2 (a^2 Log[l^2 / a^2] * (12 p / a - 6 b / a - 6) +
  b^2 Log[l^2 / b^2] * (8 p * b / a^2 - 4 b^2 / a^2 - 2 b / a - 4 p / a + 2) +
  p^2 Log[l^2 / p^2] * (12 p / a + 8 p * b / a^2 - 4 p / a - 16 p^2 / a^2) +
  6 a^2 (Log[l^2 / a^2] - 1) +
  2 (2 b - a)^2 ((b^2 Log[l^2 / b^2] - a^2 Log[l^2 / a^2]) / (b^2 - a^2)) +
  4 b^2 ((b^2 Log[l^2 / b^2] - p^2 Log[l^2 / p^2]) / (b^2 - p^2)) -
  2 (8 p^2 - 2 a^2) ((a^2 Log[l^2 / a^2] - p^2 Log[l^2 / p^2]) / (a^2 - p^2))
```

```
In[4]:= En[l_, a_, b_, p_, g_] := E0[l, a, b, p] + E14[l, a, b, p, g] + E12[l, a, b, p, g]
```

(*Define new variables*)

```
In[6]:= F[x_, y_, t_, Q_, \alpha_] := En[l, a, b, p, g] /. {b -> x * a, p -> y * a} /. a -> Sqrt[t] l /.
  {l -> (32 \pi^2 Q)^(1/4), g -> Sqrt[8 \pi^2 \alpha]}
```

```
In[7]:= Simplify[F[x, y, t, Q, α]]
```

$$\text{Out[7]= } Q \left(t (1 + x^2 - 2 y^2) + 2 (1 + x - 2 y)^2 \alpha + \right. \\ \left. 2 t \alpha \left(3 \left(-1 + \text{Log}\left[\frac{1}{t}\right] \right) - 3 (1 + x - 2 y) \text{Log}\left[\frac{1}{t}\right] - x^2 (-1 + 2 x) (1 + x - 2 y) \text{Log}\left[\frac{1}{t x^2}\right] + \right. \right. \\ \left. \frac{(1 - 2 x)^2 \left(-\text{Log}\left[\frac{1}{t}\right] + x^2 \text{Log}\left[\frac{1}{t x^2}\right] \right)}{-1 + x^2} + 4 (1 + x - 2 y) y^3 \text{Log}\left[\frac{1}{t y^2}\right] - \right. \\ \left. \left. \frac{2 (-1 + 4 y^2) \left(-\text{Log}\left[\frac{1}{t}\right] + y^2 \text{Log}\left[\frac{1}{t y^2}\right] \right)}{-1 + y^2} + \frac{2 x^4 \text{Log}\left[\frac{1}{t x^2}\right] - 2 x^2 y^2 \text{Log}\left[\frac{1}{t y^2}\right]}{x^2 - y^2} \right) \right) \right)$$

```
(*Locating minima of the vacuum energy*)
```

```
(*Define the determinant of the Hessian matrix as H*)
```

```
In[8]:= H[x_, y_, t_, Q_, α_] :=
```

```
D[F[x1, y1, t, Q, α], {x1, 2}] * D[F[x1, y1, t, Q, α], {y1, 2}] - \\ D[F[x1, y1, t, Q, α], x1, y1]^2 /. {x1 → x, y1 → y}
```

```
(*Define function to search for roots around (x0,y0) with given t, \\ Q and α, check the hessian and print out the squared particle masses, \\ the value of the cutoff and the value of the vacuum energy*)
```

```
In[9]:= Rootcheck[x0_, y0_, t_, Q_, α_] :=
```

```
{roots = FindRoot[{D[F[x, y, t, Q, α], x], D[F[x, y, t, Q, α], y]}, \\ {x, x0}, {y, y0}], {H[x, y, t, Q, α] /. roots, \\ D[F[x1, y1, t, Q, α], {x1, 2}] /. {x1 → x, y1 → y} /. roots}, \\ {Sqrt[32 π^2 Q] * t, x^2 * Sqrt[32 π^2 Q] * t, y^2 * Sqrt[32 π^2 Q] * t, \\ N[Sqrt[32 π^2 * Q]]} /. roots, F[x, y, t, Q, α] /. roots}
```

```
(*Loop over values of (x,y,t) to scan for \\ minima and save (mA2, mB2, mψ2, Λ2, ρ) to the list "sols"*)
```

```
sols := {}; For[t0 = 0, t0 < 1, t0 = t0 + 0.001, \\ Print[t0]; For[x0 = -10.11, x0 < 10, x0 = x0 + 1, \\ For[y0 = -10.1, y0 < 10, y0 = y0 + 1, H1 = Rootcheck[x0, y0, t0, 100, 1 / (16 π^2)]; \\ If[H1[[2, 1]] > 0 && H1[[2, 2]] > 0, sols = \\ Append[sols, {H1[[3, 1]], H1[[3, 2]], H1[[3, 3]], H1[[3, 4]], H1[[4]]}]]]]]
```

```
(*Delete duplicate points and export to .dat file for backup*)
```

```
sols = DeleteDuplicates[Round[sols, .0001]];
```

```
Export["Points2.dat", sols, "List"];
```

```
(*Generate a listplot of the energy values for increasing t, \\ to get an overall feel for the data*)
```

```
Values = {}; For[i = 1, i < Length[sols], i++,
```

```
Values = Append[Values, {sols[[i, 1]] / (Sqrt[32 π^2 * 100]), sols[[i, 5]]}]]]
```

```
ListPlot[{Values}, AxesLabel → {t, ρ}, PlotRange → All]
```

(*Generate a few plots for different values of t to examine the minima closer*)

```
Plot3D[F[x, y, .01, 100, 1 / (16  $\pi^2$ )],
  {x, -10, 10}, {y, -10, 10}, AxesLabel -> {x, y, F}]
```

(*Locate specific minima for the plot*)

```
In[34]:= solst := {}; For[x0 = -20.11, x0 < 20, x0 = x0 + 1,
  For[y0 = -20.1, y0 < 20, y0 = y0 + 1, H1 = Rootcheck[x0, y0, .01, 100, 1 / (16  $\pi^2$ )];
  If[H1[[2, 1]] > 0 && H1[[2, 2]] > 0, solst = Append[solst, H1]]]]
```

```
In[35]:= solst
```

A very large output was generated. Here is a sample of it:

```
{{{x -> -2.54275, y -> -6.67321}, {159.532, 7.00863},
  {1.77715, 11.4903, 79.1397, 177.715}, -36.7481},
 <<641>>, {{x -> 2.5913, y -> 7.03812}, {181.951, 7.02705},
  {1.77715, 11.9333, 88.0314, 177.715}, -72.6839}}
```

```
Out[35]=
```

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Bibliography

- [1] Sheldon L. Glashow. Partial-symmetries of weak interactions. *Nuclear Physics*, 22(4):579 – 588, 1961.
- [2] Steven Weinberg. A model of leptons. *Phys. Rev. Lett.*, 19:1264–1266, Nov 1967.
- [3] Abdus Salam. Weak and Electromagnetic Interactions. Originally printed in *Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden*, Stockholm 1968, 367-377.
- [4] Peter W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, Oct 1964.
- [5] Julian Schwinger. Gauge invariance and mass. *Phys. Rev.*, 125:397–398, Jan 1962.
- [6] Julian Schwinger. Gauge invariance and mass. ii. *Phys. Rev.*, 128:2425–2429, Dec 1962.
- [7] Sidney Coleman and Erick Weinberg. Radiative corrections as the origin of spontaneous symmetry breaking. *Phys. Rev. D*, 7:1888–1910, Mar 1973.
- [8] ATLAS Collaboration. Search for the standard model higgs boson in the diphoton decay channel with 4.9 fb^{-1} of pp collision data at $\sqrt{s} = 7 \text{ TeV}$ with atlas. *Phys. Rev. Lett.*, 108:111803, Mar 2012.
- [9] CMS Collaboration. Combined results of searches for the standard model higgs boson in pp collisions at $\sqrt{s} = 7 \text{ TeV}$. *Physics Letters B*, 710(1):26 – 48, 2012.
- [10] ATLAS Collaboration. Search for the standard model higgs boson in the decay channel with 4.8 fb^{-1} of pp collision data at with atlas. *Physics Letters B*, 710(3):383 – 402, 2012.
- [11] ATLAS Collaboration. Combined search for the standard model higgs boson using up to 4.9 fb^{-1} of pp collision data at with the atlas detector at the lhc. *Physics Letters B*, 710(1):49 – 66, 2012.
- [12] Savas Dimopoulos and Howard Georgi. Softly broken supersymmetry and $su(5)$. *Nuclear Physics B*, 193(1):150 – 162, 1981.
- [13] K. Nakamura *et.al.* (Particle Data Group). *J. Phys.*, G **37**, 075021 (2010) and 2011 partial update for the 2012 edition.

- [14] J. Wess and B. Zumino. A lagrangian model invariant under supergauge transformations. *Physics Letters B*, 49(1):52 – 54, 1974.
- [15] A. Zee. *Quantum Field Theory in a Nutshell*. Princeton, second edition, 2010.
- [16] F. Mandl and G Shaw. *Quantum Field Theory*. Wiley, second edition, 2010.
- [17] E. Noether. Invariante variationsprobleme. *Nachr. v.d. Ges. d. Wiss. zu Göttingen*, pages 235–257, 1918.
- [18] E. Noether and M.A. Tavel. Invariant variation problems. *Transport Theory and Statistical Mechanics*, 1(3):183–207, 1971.
- [19] Stephen L. Adler. Axial-vector vertex in spinor electrodynamics. *Phys. Rev.*, 177:2426–2438, Jan 1969.
- [20] J. S. Bell and R. Jackiw. A PCAC puzzle: $\pi^0 \rightarrow \gamma\gamma$ in the σ -model. *Nuovo Cimento A Serie*, 60:47–61, March 1969.
- [21] J. Goldstone. Field theories with superconductor solutions. *Il Nuovo Cimento (1955-1965)*, 19:154–164, 1961. 10.1007/BF02812722.
- [22] R. K. P. Zia, Edward F. Redish, and Susan R. McKay. Making sense of the legendre transform. *American Journal of Physics*, 77(7):614–622, 2009.
- [23] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. Theory of superconductivity. *Phys. Rev.*, 108:1175–1204, Dec 1957.
- [24] Hironari Miyazawa. Baryon number changing currents. *Progress of Theoretical Physics*, 36(6):1266–1276, 1966.
- [25] J.-L. Gervais and B. Sakita. Field theory interpretation of supergauges in dual models. *Nuclear Physics B*, 34(2):632 – 639, 1971.
- [26] Yu A Golfand and E P Likhtman. Extension of the algebra of poincare group generators and violation of p invariance. *Pisma Zh Eksp Teor Fiz*, 13:452–455, 1971.
- [27] D. Volkov and V. Akulov. Possible universal neutrino interaction. 509:383–385, 1998. 10.1007/BFb0105270.
- [28] J. Wess and B. Zumino. Supergauge transformations in four dimensions. *Nuclear Physics B*, 70(1):39 – 50, 1974.
- [29] J. Iliopoulos and B. Zumino. Broken supergauge symmetry and renormalization. *Nuclear Physics B*, 76(2):310 – 332, 1974.
- [30] Hidenori Sonoda and Kayhan Ülker. An elementary proof of the non-renormalization theorem for the wess-zumino model. *Progress of Theoretical Physics*, 123(6):989–1002, 2010.

- [31] J. Wess and J. Bagger. *Supersymmetry and Supergravity*. Princeton, 1992.
- [32] G. Lazarides and Q. Shafi. r symmetry in the minimal supersymmetric standard model and beyond with several consequences. *Phys. Rev. D*, 58:071702, Sep 1998.
- [33] Warren Siegel. Supersymmetric dimensional regularization via dimensional reduction. *Physics Letters B*, 84(2):193 – 196, 1979.
- [34] W. Siegel. Inconsistency of supersymmetric dimensional regularization. *Physics Letters B*, 94(1):37 – 40, 1980.
- [35] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 1995.