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Dark matter candidates and their indirect detection

Thesis for the degree of Philosophiae Doctor

Trondheim, August 2010

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ISBN 978-82-471-2272-3 (printed ver.)
ISBN 978-82-471-2273-0 (electronic ver.)
ISSN 1503-8181

Doctoral theses at NTNU, 2010:152

Printed by NTNU-trykk

Abstract

This thesis deals with the problem of non-baryonic dark matter (DM), which according to the standard model of cosmology is necessary to explain rotation curves of galaxies, the movement of galaxy clusters and the formation of structure in the early universe. The thesis consists of four introductory chapters and then five papers.

The thesis starts with a brief introduction to cosmology and evidence for the existence of non-baryonic dark matter. The following chapter gives some additional explainations and results concerning the topic of the three first papers, namely an application of the neutralino from the minimal supersymmetric model (MSSM) as an implementation of superheavy dark matter (SHDM).

SHDM was introduced in a paper of Chung, Kolb and Riotto, and is a non-thermal relic which could be produced by e.g. gravitational mechanisms at the end of inflation. The theory has the advantage that the correct dark matter abundance $\Omega_\chi \sim 1$ is generated quite independently of the details of concrete particle physics, provided that the dark matter particle is stable and superheavy, of order 10^{13} GeV. On the other hand, a disadvantage is that most SHDM candidates have no tree-level interactions with standard model (SM) particles and would hence be difficult to observe. Moreover, weak interactions generically become strong for masses $m_\chi \gg m_Z$, which infers non-perturbativity. An exception to this is softly broken supersymmetry (SUSY), in particular the MSSM, which remains perturbative for all, also superheavy, masses. The conclusion is that superheavy MSSM allows us to calculate the properties of SHDM. We explore this possibility in papers I-III:

Paper I proposes the neutralino as a well-suited candidate for SHDM, as described above. Moreover, it lays the foundation for the other papers by calculating relevant Feynman amplitudes, energy relaxation times and cross-sections.

In paper II we study the formation and evolution of superdense dark matter clumps. The smallest DM objects in the universe are called clumps and they are produced first. Being produced very early during the radiation

dominated epoch, superdense clumps evolve as isolated objects. They do not belong to hierarchical structures for a long time after production, and therefore they are not destroyed by tidal interactions during the formation of larger structures. Superdense clumps cannot be composed of standard neutralinos, since their annihilations would overproduce the diffuse gamma radiation. On the other hand, superdense clumps (consisting of e.g. SHDM) can be observed by gamma radiation from DM annihilations and by gravitational wave detectors, while the production of primordial black holes and cascade nucleosynthesis constrain this scenario.

When we in Paper III are using the neutralino as candidate for the SHDM, we find that free-streaming allows the formation of DM clumps of all masses down to $\sim 260m_\chi$ in case the neutralino is a Bino. This low-mass cutoff increases the diffuse flux of ultrahigh energy (UHE) particles produced by annihilations. Another effect is the formation of superdense clumps, in which the annihilation rate can be strongly enhanced. In the case of a Higgsino, the annihilation signal is enhanced by the Sommerfeld effect. As a result, annihilations of superheavy neutralinos in dense clumps may lead to observable fluxes of annihilation products in the form of UHE particles, for both cases, Binos and Higgsinos, as the lightest supersymmetric particles.

Chapter 3 provides a background for the topic of paper IV. Here we present an experiment which observed a surplus in the flux of galactic positrons, the so-called PAMELA excess. A corresponding surplus in the flux of antiprotons was not observed. This has led to phenomenological models extending the SM with DM in the TeV range, where the DM at tree-level couples to electrons (and positrons) only. Usually, predictions of annihilation rates of DM into stable particles in these models include tree-level processes and electromagnetic bremsstrahlung, but not electroweak bremsstrahlung. We show that the electroweak bremsstrahlung corrections are substantial, and should be included. Moreover, we derive upper limits on $\langle \sigma v \rangle$ of DM models where the DM annihilates solely into electrons or neutrinos at tree-level. In chapter 3 we also show some figures not included in paper IV, and finally we perform an additional calculation relevant to this paper.

Finally, chapter 4 in the introduction serves as a comment to paper V. First we discuss how a N-Higgs-doublet model (NHDM) can provide a dark matter candidate by letting one of the doublets be "inert": The inert doublet doesn't couple directly to fermions, and has no vacuum expectation value. Hence, the lightest particle of the inert doublet becomes stable, and could be a good DM candidate. In this chapter we also show that there is no discrete symmetry we can impose on the NHDM Lagrangian discussed in paper V, which removes certain troublesome $SO(4)$-violating terms (and only them). In paper V we analyze the symmetries of the NHDM, and also state a mass degeneration result, which says that the mass spectra of the $\mathcal{C}$-odd and charged sectors will be (exactly) degenerate in the limit $g' \to 0$, if we have vacuum alignment, the Higgs potential is $\mathcal{C}$-invariant, and the aforementioned troublesome terms are set to zero.

In summary, we have in this thesis studied the features of some hypothetical elementary particles which possibly could constitute (parts of) the dark matter. Moreover, we have studied how they could be detected by observations of the by-products of their annihilations in the galaxy. Future surveys and experiments will have to decide if any of these DM candidates have a right to live.

Acknowledgements

The author would like to thank supervisor prof. Michael Kachelrieß for good guidance, fast feedback and for being able to answer physics questions in a nearly oracle-like manner.

..and God separated between the light and the darkness.

Genesis 1:4

List of papers

Paper I: V. Berezinsky, M. Kachelriess and M. A. Solberg,
"Supersymmetric superheavy dark matter,"
Phys. Rev. D **78**, 123535 (2008)
arXiv:0810.3012 [hep-ph]

Paper II: V. Berezinsky, V. Dokuchaev, Yu. Eroshenko, M. Kachelriess and M. Aa. Solberg,
"Superdense cosmological dark matter clumps,"
Phys. Rev. D **81**, 103529 (2010)
arXiv:1002.3444 [astro-ph.CO]

Paper III: V. Berezinsky, V. Dokuchaev, Yu. Eroshenko, M. Kachelriess and M. Aa. Solberg,
"Annihilations of superheavy dark matter in superdense clumps,"
Phys. Rev. D **81**, 103530 (2010)
arXiv:1002.3445 [astro-ph.GA]

Paper IV: M. Kachelriess, P. D. Serpico and M. Aa. Solberg,
"On the role of electroweak bremsstrahlung for indirect dark matter signatures,"
Phys. Rev. D **80**, 123533 (2009)
arXiv:0911.0001 [hep-ph]

Paper V: K. Olaussen, P. Osland and M. Aa. Solberg
"Symmetry and Mass Degeneration in Multi-Higgs-Doublet Models"
arXiv:1007.1424 [hep-ph]

Contents

Part I

Introduction

Chapter 1

Dark matter

The Lambda-Cold Dark Matter (ΛCDM) model of cosmology is often referred to as the standard model of big-bang cosmology. The model attempts to explain the existence and temperature fluctuations of the cosmic microwave background (CMB), the large scale structure and motion of galaxies and galaxy clusters, the synthesis of light elements as hydrogen, helium and lithium, and also the accelerating expansion of the universe observed in the light from distant galaxies and supernovae. It is the simplest model that is in general agreement with observed phenomena.

The universe is assumed to be spatially homogeneous and isotropic at large scales—scales comparable with the observable universe. This is the so-called cosmological principle. The metric for a space which is spatially homogeneous and isotropic is the maximally-symmetric Robertson-Walker (RW) metric,

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{1.1}$$

where (t, r, θ, ϕ) are the comoving coordinates, $a(t)$ is the cosmic scale factor and with an appropriate rescaling of the coordinates κ can be chosen to be $-1, 0$ or $+1$ for spaces of negative, constant or positive spatial curvature, respectively. Since the universe is spatially homogeneous and isotropic, the scale factor $a(t)$ can only be a function of time. The time coordinate of (1.1) is the proper time measured by an observer at rest in the comoving coordinates (i.e. $(r, \theta, \phi) =$ constant), and observers at rest in the comoving frame stay at rest, that is, (r, θ, ϕ) remains constant (hence the term "comoving"). It is often convenient to express the metric (1.1) in terms of conformal time η, which is defined by $d\eta = dt/a(t)$, and

$$ds^2 = a^2(\eta)\left(d\eta^2 - \frac{dr^2}{1-\kappa r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2\right). \tag{1.2}$$

The redshift z is defined

$$z = \frac{\nu_1 - \nu_2}{\nu_2}, \tag{1.3}$$

where ν_1 and ν_2 are the frequencies of the emitted and observed photon, respectively. For the metric (1.1) we get

$$1 + z = \frac{\nu_1}{\nu_2} = \frac{a_2}{a_1}, \tag{1.4}$$

where a_2 and a_1 is the scale factor of the universe at the time of observation and emission, respectively.

The dynamics of the expanding universe only appears implicitly in the time dependence of the scale factor $a(t)$ of the RW metric. To make the time dependence of the scale factor explicit, one must solve the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{1.5}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor and Λ is a cosmological constant. In the case of the *RW* metric $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$, where ρ is the energy density and p is the isotropic pressure. The $\mu = 0$ component of the conservation of stress energy ($T^{\mu\nu}_{;\nu} = 0$) for the stress energy tensor of a perfect fluid yields the 1st law of thermodynamics (for adiabatic evolution)

$$d(\rho a^3) = -p d(a^3). \tag{1.6}$$

For the simple equation of state $p = w\rho$, where w is assumed to be constant with time, eq. (1.6) gives us

$$\rho \propto a^{-3(1+w)} \tag{1.7}$$

for a single component universe. On the other hand, the $\mu = \nu = 0$ part of Einstein equations for the RW metric with a perfect fluid source, yields the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho. \tag{1.8}$$

The energy density ρ is the energy density contributed by all the components of the universe, including a possible cosmological constant. The ratio $\dot{a}/a = H(t)$ is called the Hubble parameter and the Hubble constant H_0 is the present value of this expansion rate. For early times when a is small, one can neglect the cosmological and curvature terms [the latter as long as $w > -1/3$, which makes ρ more singular, cf. (1.7)] of the Friedmann equation (1.8). Combining this with with eq. (1.7) we get

$$a(t) \propto t^{\frac{2}{3(1+w)}} \tag{1.9}$$

for a single-component universe. Then we for a radiation-dominated (early) universe have $w = 1/3$ and hence $\rho \propto a^{-4}$ and

$$\text{RD:} \quad a(t) \propto t^{\frac{1}{2}} \quad H(t) = \frac{1}{2t}, \tag{1.10}$$

while for a matter-dominated universe $w = 0$ (pressureless gas) gives $\rho \propto a^{-3}$ and if $\kappa = 0$

$$\text{MD:} \quad a(t) \propto t^{\frac{2}{3}} \quad H(t) = \frac{2}{3t}. \tag{1.11}$$

If the universe is dominated by a cosmological constant ($w = -1$) we get $\rho = \rho_\Lambda = \frac{\Lambda}{8\pi G}$ = constant, and the Friedmann equation leads to an exponential expansion of the universe

$$\Lambda\text{D}: \quad a(t) \propto e^{\sqrt{\frac{\Lambda}{3}}\,t} \quad H = \sqrt{\frac{\Lambda}{3}}. \tag{1.12}$$

The Friedmann equation (1.8) can be rewritten as

$$\frac{\kappa}{H^2 a^2} = \Omega - 1 \tag{1.13}$$

where Ω and the critical energy density ρ_c equals

$$\Omega = \frac{\rho}{\rho_c} \qquad \rho_c = \frac{3H^2}{8\pi G}. \tag{1.14}$$

Equation (1.13) is valid for all times, although Ω and ρ_c changes as the universe expands. Moreover, we see from eq. (1.13) that if $\Omega > 1$ at one time, Ω is always greater than one. The same is true for the cases $\Omega = 1$ and $\Omega < 1$. The spatial geometry of the universe in RW models as the ΛCDM model is hence dictated by the energy content of the universe: If the energy density ρ is (at some time) greater than the critical density ρ_c, κ has to equal

+1 and the universe has positive spatial curvature and has hence a spherical spatial geometry. Such a universe is finite and is called "closed". If $\rho < \rho_c$ then $\kappa = -1$ and the spatial curvature of the universe is negative and the corresponding spatial geometry hyperbolic. Such an universe is infinite, and is called "open". Finally, if $\rho = \rho_c$ then $\kappa = 0$ and the spatial geometry of the universe is flat. Such an universe is also infinite. In the (benchmark) ΛCDM model the energy content of the universe is at the critical density. In this scenario the universe is infinite. Both dark (non-luminous) and luminous matter, radiation and the so-called dark energy (often identified with a cosmological constant Λ, hence the Λ in ΛCDM) contributes to the total energy density. At the present time the energy content of the universe is believed to be summed up of about 73% dark energy, 23% dark matter, 4% luminous ("baryonic") matter and almost one part of a ten thousand radiation.

We will in this thesis focus on the second most common component, the dark matter.

1.1 The existence of dark matter

In this section we will present some arguments for the existence of dark matter.

1.1.1 Dark matter in galaxies

Assume a star is in a circular orbit around the center of its galaxy. Let the distance to the center be r and let the orbital speed be v. Then the acceleration a of the star is given by

$$a = \frac{v^2}{r}, \tag{1.15}$$

directed towards the center of the galaxy. If the acceleration is provided by the gravitational attraction of the galaxy, the acceleration equals

$$a = G\frac{m(r)}{r^2}, \tag{1.16}$$

where $m(r)$ is is the mass contained in the sphere of radius r around the center of the galaxy, and G is the gravitational constant. We are here assuming the distribution of the mass is spherically symmetric. When we consider spiral galaxies we only get a small correction to this expression for the gravitational acceleration. Eqs. (1.15) and (1.16) then gives us the relation between orbital speed v and the mass $m(r)$,

$$v = \sqrt{G\frac{m(r)}{r}}. \tag{1.17}$$

Now, the surface brightness I of the disk of a spiral galaxy typically falls off exponentially with the distance from the center

$$I(r) = I(0)\exp\left(-\frac{r}{r_s}\right), \tag{1.18}$$

where the scale length r_s typically is a few kiloparsecs. Hence, a few scale lengths from the center, the mass of the stars inside r becomes approximately constant. If stars contributed all or most of the mass in a galaxy, the orbital speed of the stars would fall as

$$v \propto \frac{1}{\sqrt{r}} \tag{1.19}$$

at large radii ($r \gtrsim 3r_s$), cf. (1.17). The relation (1.19) between orbital speed and radius is referred to as "Keplerian rotation" (as Kepler found for the solar system, since its mass is strongly concentrated toward the center).

In contrast, measurements of orbital speeds of stars in spiral galaxies show that the orbital speed is not decreasing in a Keplerian way, but is approximately constant at large radii, see Fig. 1.1. Since the orbital speed of stars and gas at large radii is greater than it would be if stars and gas was the only matter present, it is consistent to propose the presence of a dark halo within which the visible stellar disk is embedded.

1.1.2 Dark matter in galaxy clusters

Suppose that a cluster of galaxies consists of N galaxies, and that each of them can be approximated as a point mass m_i, with position $\vec{x}$ and velocity $\dot{\vec{x}}$. Galaxy clusters are gravitationally bound objects, not expanding with the Hubble law. They are small compared to the horizon size $d_{\rm hor}$ (the largest distance a photon can travel during the age of the universe); the radius of the Coma cluster, e.g. , is ≈ 3 Mpc $\approx 0.0002 d_{\rm hor}$. The galaxies within a cluster are moving at nonrelativistic speeds; the velocity dispersion (the range of velocities about the mean velocity) within the Coma cluster is $\sigma_{\rm Coma} \approx 900\,{\rm km\ s^{-1}} \approx 0.003c$. Hence we can treat the dynamics of the Coma cluster, and other clusters, in a Newtonian manner.

The potential energy of the cluster can be written

$$W = -\alpha\frac{GM^2}{r_h}, \tag{1.20}$$

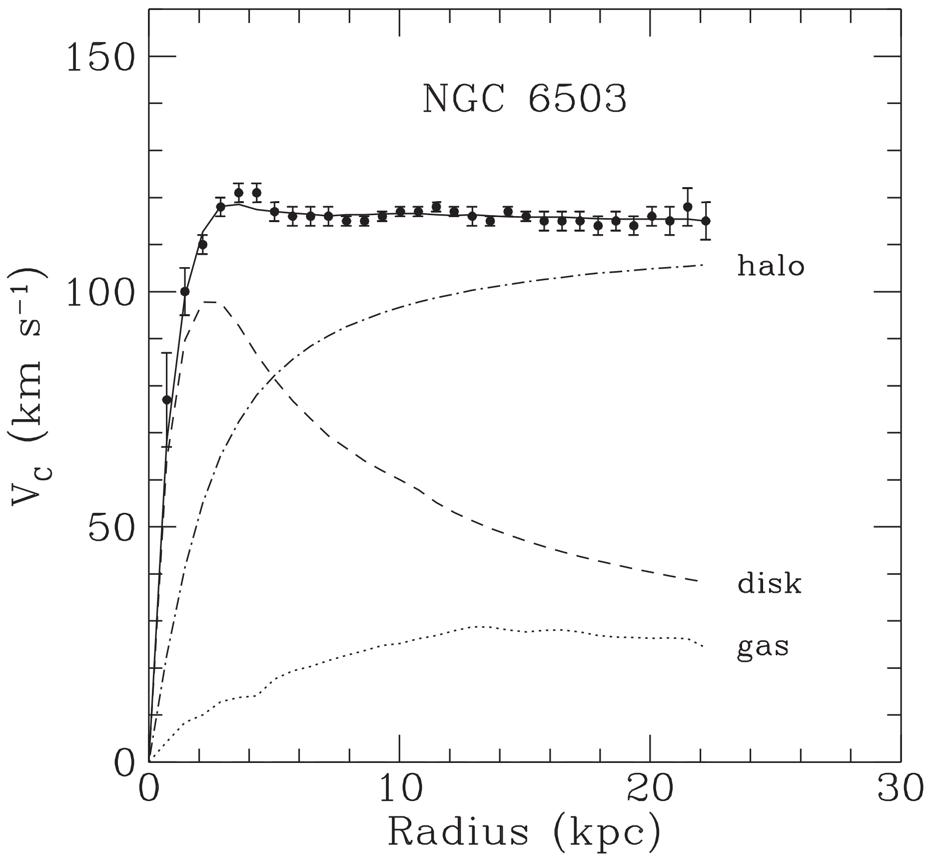


Figure 1.1: Rotation curve for the spiral galaxy NGC6503. The dashed and dotted graphs are the contribution to the orbital speed due to the observed disk and gas, respectively, and the dot-dashed curve is the contribution attributed to the dark halo [10].

where $M = \sum m_i$ is the total mass of all the galaxies in the cluster, α is a numerical factor of order unity that depends on the density profile of the cluster, and r_h is the half-mass radius of the cluster, that is the radius of a sphere centered on the cluster's center of mass, containing a mass $M/2$. For observed clusters of galaxies, it is found that $\alpha \approx 0.4$ for a typical density profile of a cluster [1].

The kinetic energy associated with the relative motion of the galaxies in the cluster can be expressed as

$$K = \tfrac{1}{2} M \langle v^2 \rangle, \tag{1.21}$$

where

$$\langle v^2 \rangle = \frac{1}{M} \sum_i m_i |\dot{\vec{x}}_i|^2 \tag{1.22}$$

is the mean square velocity of all the galaxies in the cluster.

The moment of inertia of the cluster is defined as

$$I = \sum_i m_i |\vec{x}_i|^2. \tag{1.23}$$

Then we can derive

$$\ddot{I} = 2W + 4K, \tag{1.24}$$

a result which is known as the virial theorem. When I = constant, the steady-state virial theorem is

$$K = -\frac{W}{2}. \tag{1.25}$$

To ensure the moment of inertia is constant, we have to assume that we are using a coordinate system were the center of mass of the cluster is at rest, in addition to the assumption that the cluster does not expand or contract. From this we derive

$$\frac{M}{2} \langle v^2 \rangle = \frac{\alpha}{2} \frac{GM^2}{r_h}, \tag{1.26}$$

and then we can use the virial theorem to estimate the mass of a cluster of galaxies, or any other self-gravitating steady-state system:

$$M = \frac{\langle v^2 \rangle r_h}{\alpha G}. \tag{1.27}$$

Assuming that the velocity dispersion in a cluster is isotropic, redshift measurements of hundreds of galaxies in the Coma cluster dictates its mean square velocity to be $\langle v^2 \rangle = 2.32 \times 10^{12}\,\mathrm{m^2\,s^{-2}}$. Moreover, assuming that the mass-to-light ratio is constant with radius, the sphere containing half the mass of the cluster will be the same as the sphere containing half the luminosity of the cluster. If we additionally assume the cluster is spherical, the observed distribution of galaxies within the Coma cluster indicates a half-mass radius $r_h \approx 1.5$ Mpc $\approx 4.6 \times 10^{22}\,\mathrm{m}$ [1]. Inserting these numbers in eq. (1.27) yields

$$M_{\mathrm{Coma}} \approx 4 \times 10^{45}\mathrm{kg} \approx 2 \times 10^{15} M_\odot. \tag{1.28}$$

Hence, observations show that less than two percent of the Coma cluster consists of stars,

$$M_{\text{Coma},\bigstar} \approx 3 \times 10^{13} M_{\odot}, \tag{1.29}$$

and only ten percent consists of hot, intracluster gas,

$$M_{\text{Coma, gas}} \approx 2 \times 10^{14} M_{\odot}, \tag{1.30}$$

which suggests the presence of almost 90% dark matter. This concludes Fritz Zwicky's argument from 1933 for the existence of dark matter in galaxy clusters [2].

The presence of dark matter in galaxy clusters is confirmed by the observations of hot, x-ray emitting intracluster gas (i.e. gas between the galaxies in the cluster). If there were no dark matter to anchor the gas gravitationally, the hot gas would have expanded beyond the cluster on time scales much shorter than the Hubble time. If the gas is supported by its own pressure against gravitational infall, it must obey the equation of hydrostatic equilibrium,

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{1.31}$$

where p is the pressure and ρ the density of the gas, while $M(r)$ is the total mass inside a sphere of radius r. The pressure p is given by the perfect gas law

$$p = \frac{\rho k T}{m_g} \tag{1.32}$$

where T is the temperature of the gas and m_g is the (effective) mass of the particle species constituting the gas. Combining eqs. (1.31) and (1.32) gives us an expression for the total mass of the cluster, as a function of radius,

$$M(r) = \frac{-kT(r)r}{Gm_g}\left(\frac{d\log\rho}{d\log r} + \frac{d\log T}{d\log r}\right). \tag{1.33}$$

Equation (1.33) gives an alternative way of calculating a cluster's total mass, which is in general consistent with cluster masses calculated by eq. (1.27) above. Moreover, gravitational lensing observations of galaxy clusters allow direct estimates of the gravitational mass based on its effect on light from background galaxies. Cluster masses calculated by the means of gravitational lensing are also in general agreement with the masses found by applying the virial theorem to the velocities of the galaxies in the clusters [1].

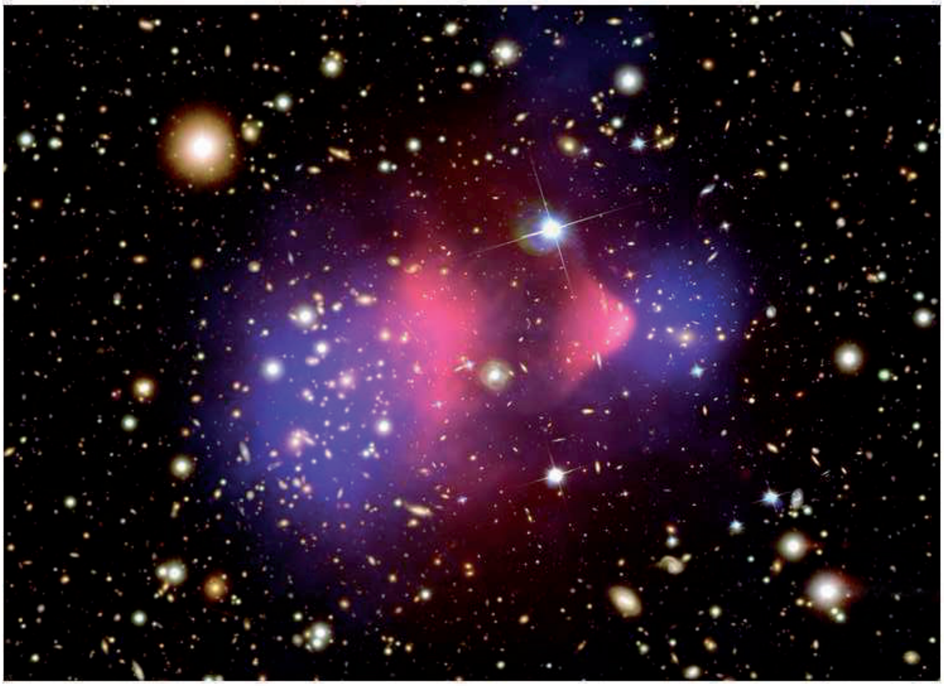


Figure 1.2: Hot gas detected in x-rays (by the Chandra satellite) is seen as two pink clumps in the image and contains most of the "normal" (baryonic) matter in the two clusters. The bullet-shaped clump on the right is the hot gas from one cluster, which passed through the hot gas from the other larger cluster during the collision. An optical image (from Magellan and the Hubble Space Telescope) shows the galaxies in orange and white. The blue areas show where most of the mass in the clusters is found by gravitational lensing. Most of the matter in the clusters (blue) is clearly separated from the baryonic matter (pink), giving direct evidence that nearly all of the matter in the clusters is dark [4].

Moreover, in the Bullet Cluster, lensing observations show that much of the lensing mass is separated from the x-ray emitting baryonic mass [3], as seen in fig. 1.2. The hot gas in each cluster is here seen to be slowed by the friction between the two gas clouds, during the collision. In contrast, the dark matter is not slowed by the impact because it does (approximately, at least) not interact directly with itself or the gas except through gravity, i.e. it is collisionless. Therefore, during the collision, the dark matter clumps from the two clusters moved ahead of the hot gas, producing the separation of the dark and normal matter seen in the image. If hot gas was the most massive component in the clusters, such an effect would not have been seen. Instead, this result shows that dark matter is required.

1.1.3 Dark matter in structure formation

Finally, we will consider a few more theoretical (and more model-dependent) arguments for the existence of dark matter.

If the Universe consisted solely of baryonic matter, then the observed anisotropies in the cosmic microwave background radiation are too small to explain the observed structure of the present universe: Baryonic structure formation via gravitational collapse can only begin when photons decouple from the baryonic matter at $z_{\rm dec} \approx 1100$ (when protons and electrons merge to neutral hydrogen), while the DM can collapse from matter-radiation equality on, i.e. at $z_{\rm eq} \approx 10000$: The CMB anisotropy is relatively small, of the magnitude $\delta T/T \sim 10^{-5}$, and this should be reflected by a similar fractional variation in density $\delta\rho/\rho$ in a universe with only baryonic matter, $\delta T/T = 3\delta\rho/\rho$ for adiabatic fluctuations. On the other hand, $\delta\rho/\rho$ should be of the size $\delta\rho/\rho \sim 10^{-4}$ in a universe with only baryonic matter (on angular scales corresponding to galaxies or galaxy clusters), to be consistent with observed baryonic structure. A solution of this discrepancy is, as indicated above, that a non-interacting dark matter fluid could start to collapse at the time of matter-radiation equality ($z_{\rm eq} \approx 10000$) instead of at the time of CMB decoupling ($z_{\rm dec} \approx 1100$). At $z_{\rm dec}$ the dark matter fluctuations can have grown to the necessary $\delta\rho/\rho \sim 10^{-4}$, while the CMB temperature fluctuations are stuck at 10^{-5} as observed. This is so because photon-baryon fluid density fluctuations are held at the value 10^{-5} until decoupling, since the photon-baryon fluid is not collisionless like the dark matter. Instead of freely falling into the DM gravitational potential wells, the photon-baryon fluid oscillates, due to the pressure gradient generated when the fluid starts to fall into the potential wells. This phenomenon is called baryon acoustic oscillations (BAO), since they represent a type of standing sound wave in the photon-baryon fluid. After decoupling the neutral baryonic matter, being pressureless, is free to collapse into the potential wells of the DM.

Moreover, performing a detailed analysis of the observed anisotropies in the CMB, the fractional contribution of baryons to the present critical density ρ_c should be [12]

$$\Omega_b h^2 = (0.02273 \pm 0.00062), \tag{1.34}$$

where h is a parametrization of the Hubble constant, $H_0 = h \times 100$ km s^{-1} Mpc^{-1}, so $h \approx 0.7$. On the other hand, the corresponding total fractional contribution of matter to ρ_c is, again from the CMB anisotropies, found to be

$$\Omega_m h^2 = 0.1326 \pm 0.0063 \tag{1.35}$$

both results at 68% confidence levels (CL). The discrepancy between eqs. (1.34) and (1.35) suggests the presence of non-baryonic matter. The amount of cold, non-baryonic dark matter is from the CMB anisotropies found to equal [12]

$$\Omega_{\text{CDM}} h^2 = 0.1099 \pm 0.0062, \tag{1.36}$$

at 68% CL. The value of Ω_b from (1.34) confirmes the estimate (that is, theoretical prediction and observed light element abundance combined) from standard Big-Bang nucleosynthesis (BBN) [13]:

$$0.017 \leq \Omega_b h^2 \leq 0.024, \tag{1.37}$$

at a 95% confidence level. Hence, baryons alone cannot ensure $\Omega = \Omega_{\text{total}} \approx 1$, which is the value predicted by inflation models and the value indicated by a combination of supernova data and the CMB anisotropies [13]. More specific, baryons alone cannot account for the value of Ω_m consistent with the mentioned supernova data and the CMB anisotropies, so Ω_m should have a significant non-baryonic component, see fig. 1.3.

Altogether, the existence of dark matter seems to be confirmed by a range of both astrophysical observations and theoretical considerations.

1.2 The freeze-out of dark matter

At a sufficiently early moment the universe was hot enough that matter was dissociated into its most basic constituents - the elementary particles. The abundance of the various kinds of elementary particles where, for all known particles, set by the conditions of thermal equilibrium. (Hypothetical particles like axions and superheavy dark matter are examples of particles which, if they exist, have never been in thermal equilibrium).

We will now, as an example, consider the lightest supersymmetric particle (LSP) as the dark matter. Supersymmetry (SUSY) is a proposed symmetry between bosons and fermions, such that for each boson (or more precise each degree of bosonic freedom) there is a corresponding fermion (fermionic degree of freedom) with the same properties as the boson, and vice versa. Since e.g. no scalar electrons are found in nature, the symmetry somehow has to be broken, inferring that the supersymmetric partners of the standard model particles have failed to be discovered because of their heavy masses. One motivation for (weak scale) SUSY is that it stabilizes the Higgs mass from radiative corrections that are quadratically divergent in the Standard Model (SM), and hence solves the so-called hierarchy problem. The minimal supersymmetric model (MSSM) is a minimal variant of SUSY (minimal in the sense that it adds a minimum of new particles to the SM).

The neutralinos χ_i, $i = 1, \ldots, 4$ are linear combinations of the supersymmetric partners of the two Higgses necessary to construct the MSSM, in addition to the photon and the Z-boson. Hence the neutralinos are neutral, as the name indicates. In this section we take the lightest neutralino χ to be the LSP and the dark matter. In the MSSM the LSP is stable, due to the imposed R-parity of the model. R-parity is a $\mathbb{Z}_2$-symmetry acting on the MSSM fields, and can be defined as

$$R = (-1)^{2s+3B+L}, \tag{1.38}$$

where s is spin, B is baryon number, and L is lepton number. All SM particles have R-parity 1 while supersymmetric particles have R-parity -1. The R-parity ensures that a SUSY particle cannot decay into SM particles only, and by this the LSP becomes stable. Moreover, R-parity also ensures the longevity of the proton, which otherwise could decay too fast through supersymmetric channels.

Then, in the early universe the lightest neutralino χ and e.g. neutrinos would interact through reactions like

$$\chi\chi \longleftrightarrow \nu\bar{\nu} \tag{1.39}$$

and

$$\nu\chi \longleftrightarrow \nu\chi. \tag{1.40}$$

Both processes are mediated by heavy messengers, namely the Z-boson and sneutrinos, and hence the interactions have a limited range. In thermal equilibrium the number density of these particles is such that these reaction rates of, e.g. , creation and annihilation rates balance each other. Reactions of the type (1.39) change the number density n_χ of neutralinos, while reactions of the type (1.40) only exchange energy between them and the thermal plasma.

As the universe expands, the temperature drops according to

$$T(t) = T(t_0)\left[\frac{a(t_0)}{a(t)}\right], \tag{1.41}$$

where $a(t)$ is the scale factor of the universe, and t_0 is the present age of the universe. The number of reactions per unit time, Γ, decreases with temperature for processes such as (1.39) and (1.40), because both the thermally averaged cross-sections $\langle \sigma v \rangle$ and n_χ decrease with T. If the rate Γ drops below the expansion rate of the universe,

$$\Gamma(t) < H(t) \equiv \frac{\dot{a}(t)}{a(t)}, \tag{1.42}$$

where $H(t)$ is the Hubble parameter, the reaction is not longer fast enough to maintain thermal equilibrium: When condition (1.42) is fulfilled, the specific reaction freezes out. This is a rule of thumb. The freeze-out of the number-changing reaction (1.39) is called chemical freeze-out and the freeze-out of energy-exchanging reaction (1.40) is called kinetic freeze-out. The interaction rate Γ of eq. (1.42) for the reaction (1.39) can be expressed as

$$\Gamma = n_\chi \langle \sigma v \rangle, \tag{1.43}$$

and where n_χ is the number density (per volume) of the neutralino, and $\langle \sigma v \rangle$ is the annihilation cross-section of neutralinos to neutrinos (or some other particle interacting with the neutralino) times relative velocity v of the annihilating particles, thermally averaged. This averaging is necessary, since the annihilating particles have random thermal velocities and directions. We will come back to this thermal averaging below.

The criterion (1.42) for freeze-out can be seen to be plausible by regarding a (rather realistic) interaction rate $\Gamma \propto T^n$: Then the number of interactions a species has from the time t onward is

$$N = \int_t^\infty \Gamma(x) dx. \tag{1.44}$$

Now $T \propto a^{-1}(t)$, and if we take the universe to be radiation dominated, the scale factor $a(t) \propto t^{1/2}$. By this $T^n \propto t^{-n/2}$ and hence $N = 2t^{-n/2+1}/(n-2)$, hence

$$N = \frac{\Gamma(t)}{H(t)(n-2)}, \tag{1.45}$$

and we see that for $n > 2$ a particle interacts less than one time subsequent to the time when $\Gamma = H$. Now $\Gamma < H$ is not a sufficient condition for a departure from thermal equilibrium: A massless, non-interacting species once in thermal equilibrium will forever maintain an equilibrium distribution with $T \propto a^{-1}(t)$ [15]. For instance, the photons of the cosmic background radiation are neither in chemical nor kinetic equilibrium with any particle species, still the radiation is in thermal equilibrium.

1.2.1 The chemical freeze-out

After the chemical freeze-out (1.39), the neutralino abundance $\sim n_\chi/n_\gamma$ is fixed, while it would be exponentially suppressed if neutralinos would stay in chemical equilibrium,

$$\frac{n_\chi}{n_\gamma} \propto T^{-3/2} e^{-m_\chi/T} . \tag{1.46}$$

Hence, we have a surviving abundance, a so-called relic abundance, of the neutralino because of the freeze-out. When the neutrlino is stable, as it is in the minimal supersymmetric model (MSSM), there will be a relic abundance around today, which could provide an explaination of the dark matter. The same would be true for any stable, neutral weakly interacting particle (WIMP) [16]. In fig. 1.4 we see how relic abundances arise at chemical freeze-out.

To calculate relic abundances, we have to study how the number density n of the regarded particle species evolves with time. Considering the example of the neutralino, the rate of the reaction (1.39) is the same in both directions under exact thermal equilibrium, and the number density $n_\chi(T)$ will equal the equilibrium value, $n_\chi^{\mathrm{EQ}}(T)$. When the actual number density $n_\chi(T)$ is larger than the equilibrium density, the reaction will go faster to the right, and the neutralinos will annihilate faster than they are created. The annihilation rate of χ should be proportional to $\langle\sigma_{\chi\chi\to\nu\bar\nu} v\rangle n_\chi^2$, but the χ's are also created by the inverse process at a rate proportional to $(n_\chi^{\mathrm{EQ}}(T))^2$. The resulting equation governing the evolution of the number density n_χ is

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma_{\chi\chi\to\nu\bar\nu} v\rangle \Big(n_\chi^2 - (n_\chi^{\mathrm{EQ}}(T))^2 \Big). \tag{1.47}$$

The left-hand side comes from $(1/a^3)(d/dt)(n_\chi a^3)$, and the term proportional to $3H$ expresses the dilution effect of the expansion of the universe. Equation (1.47) was introduced in [17]. A derivation of eq. (1.47) from the Boltzmann transport equation in an expanding background can be found in [18] or [15].

The thermally averaged annihilation cross-section times relative velocity $\langle\sigma v\rangle$ should be defined [18, 19]

$$\langle\sigma v\rangle = \frac{1}{n^{\mathrm{EQ}}} \int d^3p_1 d^3p_2 f(\omega_1) f(\omega_2)\, \sigma v_{\mathrm{rel}}, \tag{1.48}$$

where σv_{rel} is just the usual annihilation cross-sections times relative velocity of the two annihilating particles, see e.g. [21]. The function $f(\omega)$, $\omega = \sqrt{\vec{p}^2 + m^2}$, is the energy distribution of the annihilating particles, in the case of the neutralino (a fermion)

$$f_\chi(\omega) = \frac{1}{e^{\frac{\omega}{T}} + 1} \tag{1.49}$$

where the chemical potential $\mu = 0$ since the neutralino is a Majorana particle [20]. Moreover,

$$n_\chi^{\mathrm{EQ}} = \frac{g_\chi}{(2\pi)^3} \int d^3 p f_\chi(\omega), \tag{1.50}$$

where $g_\chi = 2$ represents the two polarization degrees of freedom of the neutralino. When the freeze-out temperature T_f is much less than the mass of the dark matter, $T_f \ll m_\chi$, the DM is non-relativistic at freeze-out. Then the Fermi distribution (1.49) can be well approximated by an ordinary Boltzmann distribution $e^{-\frac{\omega}{T}}$. We call DM which is non-relativistic at freeze-out cold dark matter (CDM). The typical WIMP χ freezes out at $T_t \simeq m_\chi/20$ [5, 19], and is hence an example of a CDM particle. This suggests that the total annihilation cross section times relative velocity can be expanded in terms of $x = T/M$ (or $\langle v_{\mathrm{rel}}^2 \rangle$). The first terms of the expansion are given by [19]

$$\begin{aligned} \langle \sigma v \rangle = \frac{1}{m_\chi^2} \Big(& w - \frac{3}{2}(2w - w')x + \frac{3}{8}(16w - 8w' + 5w'')x^2 \\ & - \frac{5}{16}(30w - 15w' + 3w'' - 7w''')x^2 + \mathcal{O}(x^4) \Big)_{s/4m_\chi^2 = 1}, \end{aligned} \tag{1.51}$$

where primes denote derivatives with respect to $s/4m_\chi^2$ (rather than s itself), and w and its derivatives are all to be evaluated at $s/4m_\chi^2 = 1$. The function w is given by

$$w = \omega_1 \, \omega_2 \, \sigma \, v_{\mathrm{rel}}, \tag{1.52}$$

where ω_i, $i = 1, 2$ are the total energies of the two annihilating particles. Since v_{rel} is relative velocity each χ move with the speed $v_{\mathrm{rel}}/2$ in the center-of-mass frame. Moreover, the leading (x^0) term can be identified with s-wave annihilation, the term proportional to x with p-wave annihilation and so on. In case s-wave annihilation is unsuppressed it is enough to regard the leading term: In the center-of-mass frame the differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CoM}} = \frac{1}{64\pi^2(\omega_1 + \omega_2)^2} \frac{|\vec{p_3}|}{|\vec{p_1}|} |\mathcal{M}|^2, \tag{1.53}$$

which for two identical, non-relativistic particles χ annihilating becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm CoM} = \frac{1}{256\pi^2 m_\chi^2} \frac{|\vec{p_3}|}{m_\chi v_{\rm rel}/2} |\mathcal{M}|^2, \tag{1.54}$$

since $(\omega_1 + \omega_2)^2 = s = 4m_\chi^2$, all equalities in this case of annihilating χs being exact. In the center-of-mass frame the outgoing momenta $\vec{p_3} = -\vec{p_4}$, and for non-relativistic neutralinos we get

$$\beta\, m_\chi \equiv |\vec{p_3}| = \sqrt{1 + \frac{(m_3^2 - m_4^2)^{\,2} - 8\,(m_3^2 + m_4^2)\, m_\chi^2}{16 m_\chi^4}}\; m_\chi, \tag{1.55}$$

which in the case the outgoing particles have identical mass becomes

$$\beta = \sqrt{1 - \frac{m_3^2}{m_\chi^2}}. \tag{1.56}$$

We now assume that $|\mathcal{M}|^2$ is independent of the scattering angle, like it is (to leading order) in the annihilations in part C of paper I. Integrating eq. (1.54) over the solid angle Ω and multiplying by $v_{\rm rel}$ then gives us

$$\sigma v_{\rm rel} = \frac{\beta}{32\pi m_\chi^2} |\mathcal{M}|^2. \tag{1.57}$$

Moreover, inserting into the leading term of eq. (1.51) yields

$$\langle \sigma v \rangle = \frac{\beta}{32\pi m_\chi^2} |\mathcal{M}|^2 + \mathcal{O}(x), \tag{1.58}$$

where v on the left hand side refers to $v_{\rm rel}$ as usual.

We will also calculate a typical (approximate) relic abundance of a WIMP χ. Using the Hubble-expansion rate for the radiation dominated early universe,

$$H(T) = 1.66\,\sqrt{N_{\rm eff}}\frac{T^2}{m_{\rm Pl}}, \tag{1.59}$$

where $N_{\rm eff}$ is the effective number of relativistic degrees of freedom and $m_{\rm Pl} = \sqrt{\hbar c/G} \approx 1.22 \times 10^{19}\,{\rm GeV}$ is the Planck mass. Moreover, if we disregard the possibility of exotic, entropy-producing phenomena, the entropy per comoving volume in the universe is constant, so that n_χ/s remains constant, where $s \simeq 0.4 N_{\rm eff} T^3$. Combining this with eq. (1.59) and the freeze-out condition $n_\chi\langle\sigma v\rangle = \Gamma_{\text{chemical}} = H$, we get

$$\left(\frac{n_\chi}{s}\right)_0 = \left(\frac{n_\chi}{s}\right)_f \simeq \frac{80}{m_\chi m_{\text{Pl}}\sqrt{N_{\text{eff}}}\langle\sigma v\rangle} \simeq \frac{10^{-8}}{\frac{m_\chi}{\text{GeV}}\frac{\langle\sigma v\rangle}{10^{-27}\text{cm}^3\text{s}^{-1}}}, \tag{1.60}$$

where the subscripts 0 and f denotes "today" and "freeze-out", respectively. Here the unit GeV^2 has to be converted to cm^{-2} and a factor c has to be inserted to get the expression in metric units (n_χ/s is dimensionless). The current entropy density is $s_0 \simeq 4000$ cm^{-3} while the critical energy density today is $\rho_c \simeq 10^{-5}h^2$ GeV cm^{-3}. Hence the present mass density of the WIMP χ in units of the critical density is

$$\Omega_\chi h^2 = \frac{m_\chi n_\chi}{\rho_c/h^2} \simeq \left(\frac{3\times 10^{-27}\text{cm}^3\text{s}^{-1}}{\langle\sigma v\rangle}\right). \tag{1.61}$$

The parameter h is included to be able to correct expressions when better measurements are available (from current WMAP data $h = 0.735 \pm 0.032$ while the best estimate combined with other cosmological data is $h = 0.708 \pm 0.016$, according to NASA's webpage). Now we will make an estimate of the WIMP cross-section and hence calculate an estimate for the relic WIMP abundance: $\langle\sigma v\rangle \sim \alpha^2/s \sim 10^{-25}\text{cm}^3\text{s}^{-1}$ for $\alpha \sim 10^{-2}$ (the fine structure constant) and $s \sim 100\text{GeV}^2$ for a WIMP with mass at the weak scale (defined by the vacuum expectation value (VEV) v of the Higgs field, $v = 246$ GeV). Inserting for $\langle\sigma v\rangle$ in eq. (1.61), gives us $\Omega_\chi \simeq 0.06$ which is remarkebly close to the value required to account the DM in the universe ($\Omega_\chi \simeq 0.2$), remembering there is no apriori reason for a weak-scale interaction to have anything to do with a cosmological mass density [5]. This is the so-called WIMP miracle.

1.2.2 The kinetic freeze-out

The kinetic freeze-out generally appears some time later than the chemical one. Before the kinetic freeze-out the reaction (1.40) is still efficient, and the neutralinos and the neutrinos stay at the same temperature. After kinetic freeze-out, the two particle species will evolve with different temperatures. The inverse of the number of reactions per unit time for the reactions of the type (1.40), Γ_{kinetic} is called the energy relaxation time,

$$\tau_{\text{rel}} = \Gamma^{-1}_{\text{kinetic}}(t). \tag{1.62}$$

Hence, from (1.42), the DM particles goes out of kinetic equilibrium, when $\tau_{\rm rel}$ becomes larger than the Hubble time $H^{-1}(t)$. Afterwards, the neutralinos and neutrinos effectively evolve separately and can have different temperatures: While relativistic particles like neutrinos have an equilibrium distribution with $T_\nu \sim T_\gamma \sim 1/a(t)$, the energy of non-relativistic particles as cold dark matter (CDM) scales as $E \propto p^2 \propto (1/a(t))^2$. We define t_d, the decoupling time, as the time neutralinos go out of kinetic equilibrium, i.e. the time when eq. (1.62) is satisfied.

We could have calculated the evolution of the neutralinos distribution function by studying a Boltzmann equation analogous to eq. (1.47). But we will instead use the criterion (1.42), i.e. $\tau_{\rm rel} > H^{-1}(t)$, and assume that the neutralinos are in kinetic equilibrium until $t = t_d$, while they evolve undisturbed after $t = t_d$. The energy relaxation time can be defined as $\tau_{\rm rel}^{-1} = \Gamma = |(\Delta E_k/\Delta t)/E_k|$ where $(\Delta E_k/\Delta t)$ is the mean change of energy in one scattering divided by the corresponding time between two scatterings, and $E_k = (3/2)T$ is the mean kinetic energy of the non-relativistic neutralinos. Hence we are not only taking the frequency of interactions into consideration, but also the effect of the interactions, in terms of gained kinetic energy. Then the enery relaxation time can be written

$$\tau_{\rm rel}^{-1} = \frac{N_{\rm eff}}{2E_k m_\chi} \int d\Omega \int d\omega n_0(\omega) \left(\frac{d\sigma}{d\Omega}\right)_{\nu\chi} (\delta\vec{p})^2 \tag{1.63}$$

where $(\delta\vec{p})^2$ is the neutralino momentum obtained in one scattering, $n_0(\omega)$ is the number density of relativistic fermions with one polarization and energy ω and $N_{\rm eff}$ is the relevant relativistic degrees of freedom, weighted with the relative size of their cross-section compared to a neutrino, cf. eqs. (26) to (28) of paper I. Also note that the expression $(\delta\vec{p})^2/(2m_\chi)$ is the change of kinetic energy in one scattering for a non-relativistic neutralino.

Now, the free streaming length is the characteristic scale a weakly interacting massive particle (WIMP) - i.e. the neutralino - is able to stream in the time it takes for structure to form. The neutralinos will first after the time of decoupling, t_d, be able to stream free. The length scale is hence defined as

$$\lambda_{\rm fs} = a(t_0) \int_{t_d}^{t_0} \frac{v(t)\, dt}{a(t)}, \tag{1.64}$$

a formula which both takes both the distance due to the particles velocity and the distance due to expansion of the universe, into account.

If the free streaming length is larger than the considered length scale of the density fluctuations (e.g. a galactic DM halo), then the free streaming

WIMPs will tend to erase structures. Then the self-gravity of the density fluctuation will be to small to form structure [7].

The large streaming lengths of Hot Dark Matter (HDM) models, have made it hard to explain how structures like galaxies might have formed, and these theories have been abandoned by most cosmologists in favor of the CDM alternative [8].

Small-scale self-gravitating dark matter clumps (DMCs) may have been formed in the early universe due to several mechanisms [6]. These fluctuations in mass density, can explain structure formation. Nevertheless, these density fluctuations are partially washed out by two processes: In the case of neutralinos as the DM particle, the first is the neutralino diffusion due to the scattering off neutrinos, electrons and positrons (and possibly also weak gauge bosons, cf. eq. (5) in paper III). This process is effective as long as the neutralinos are in kinetic equilibrium with the cosmological plasma. Up to the moment of decoupling t_d all perturbations with mass

$$M < M_{\rm diff} \tag{1.65}$$

are washed out. The second process is neutralino free streaming. Starting later, at $t > t_d$ it washes out the larger perturbations with $M \leq M_{\rm fs}$, where the free streaming cut-off mass $M_{\rm fs}$ [9, 6] is defined in terms of the free streaming length (1.64),

$$M_{\rm fs}(t) = \frac{4\pi}{3}\rho_\chi(t)\lambda_{\rm fs}^3(t), \tag{1.66}$$

where $\rho_\chi(t) = \rho_{\rm eq} a_{\rm eq}^3/a^3(t)$. Then $M_{\rm fs}$ determines $M_{\rm min}$, the minimum clump mass at the present,

$$M_{\rm min} = M_{\rm fs}(t_0), \tag{1.67}$$

where t_0 is the present. $M_{\rm min}$ is hence the smallest mass of a DM clump, and $M_{\rm min}$ is a cut-off mass in the sense that it is the smallest of all possible clump masses.

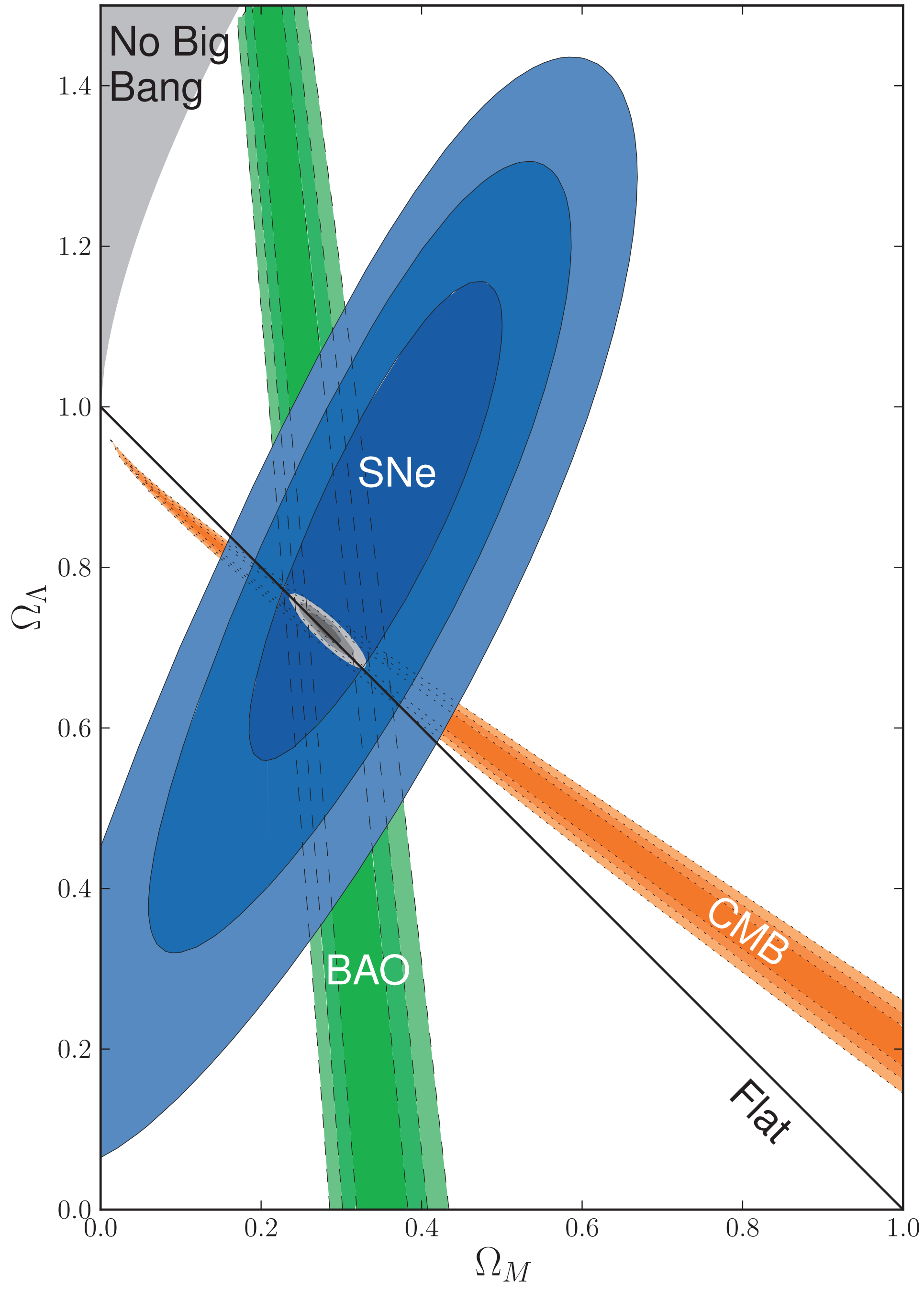


Figure 1.3: 68%, 95%, and 99.7% confidence regions in the $(\Omega_m, \Omega_\Lambda)$ plane from type Ia supernovae (SNe Ia) with systematic errors, combined with the constraints from baryon acoustic oscillations (BAO) and CMB. Cosmological constant dark energy $(w = -1)$ has been assumed. From [14].

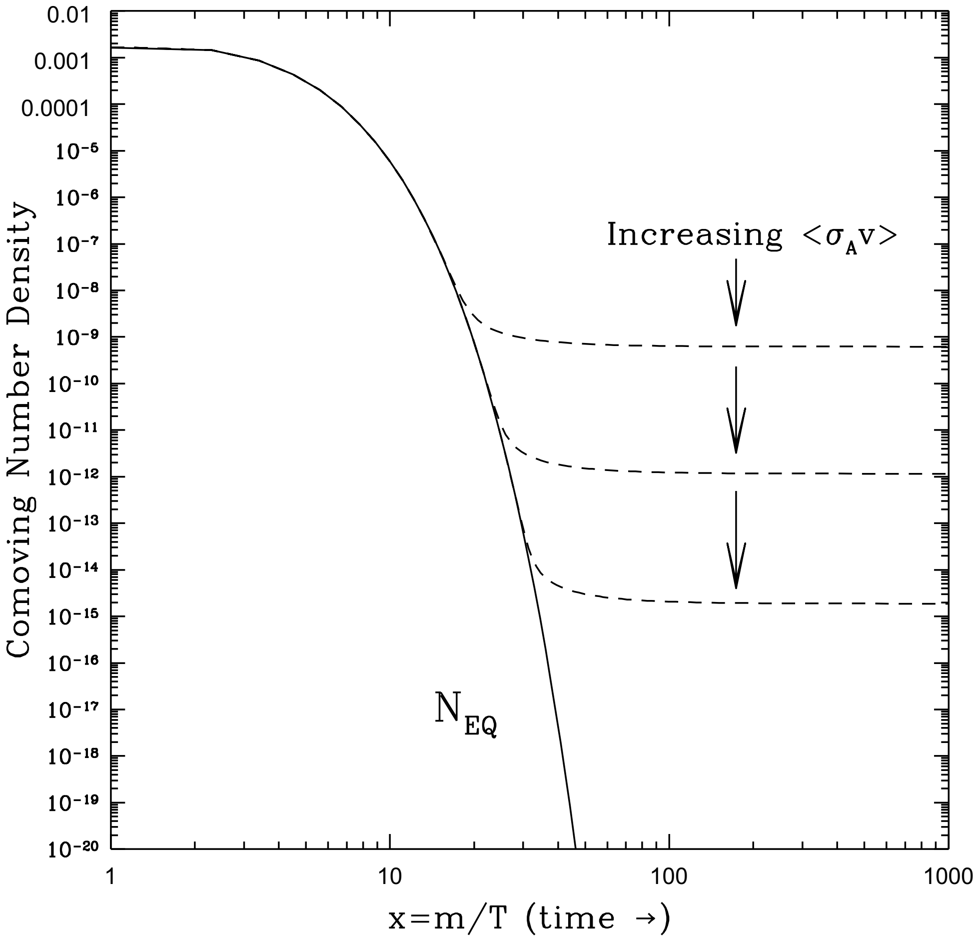


Figure 1.4: Equilibrium (solid curve) and relic abundance (dashed curves) of the neutralino (or any other stable, neutral WIMP particle). The greater the thermally averaged annihilation cross-section times velocity $\langle \sigma_A v \rangle$ is, the later the neutralino depart from (chemical) equilibrium, and hence the smaller the relic abundance becomes. From [5].

Chapter 2

Superheavy dark matter

2.1 Production of SHDM by gravitational mechanisms

In this section we will, following [11], argue how superheavy dark matter (SHDM) could be produced by gravitational mechanisms in the expanding universe, a production mechanism referred to in papers I-III.

For simplicity, consider a scalar field χ corresponding to particles of mass m_χ in the expanding universe. Let η denote conformal time, and $a(\eta)$ the time dependence of the expansion scale factor. Also assume for simplicity that the universe is (spatially) flat. The scalar field can be expanded in spatial Fourier modes as

$$\chi(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^{3/2}a(\eta)} \left(a_k h_k(\eta) e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger h_k^*(\eta) e^{-i\vec{k}\cdot\vec{x}} \right). \tag{2.1}$$

Here a_k and $a_k^\dagger$ are creation operators, and $h_k(\eta)$ are mode functions that satisfy the (i) normalization condition

$$h_k h_k'^* - h_k' h_k^* = i, \tag{2.2}$$

where a prime indicates a derivative with respect to conformal time, and (ii) the mode equation

$$h_k''(\eta) + \omega_k^2(\eta) h_k(\eta) = 0, \tag{2.3}$$

where

$$\omega_k^2(\eta) = k^2 + m_\chi^2 a^2 (6\xi - 1)\frac{a''}{a}. \tag{2.4}$$

The parameter ξ is $\xi = 0$ for a minimally coupled field and $\xi = 1/6$ for a conformally coupled field. For a given complete set of positive-frequency solutions $h_k(\eta)$, the vacuum of the field with χ, i.e. the state with no χ particles, is defined as the state that satisfies $a_k|0_h\rangle$ for all k. Since eq. (2.3) is a second order equation and the frequency depends on time, the normalization condition is in general not sufficient to specify the positive-frequency modes uniquely, contrary to the case of constant frequency ω_0, for which $h_k^0(\eta) = e^{-i\omega_0\eta}/\sqrt{2\omega_0}$. Different boundary conditions for the solutions $h_k(\eta)$ define in general different creation and annihilation operators a_k and $a_k^\dagger$, and thus in general different vacua. For instance, solutions which satisfy the condition of having only positive frequencies in the distant past,

$$h(\eta) \sim \alpha_k e^{-i\omega_k^+\eta} + \beta_k e^{+i\omega_k^+\eta} \quad \text{for } \eta \to +\infty. \tag{2.5}$$

Here $\omega_k^\pm = \lim_{\eta\to\pm\infty} \omega_k(\eta)$. As a consequence, an initial vacuum state is no longer a vacuum state at later times, which means particles are created, because of the expansion.

The number density of χ particles is given in terms of the Bogolubov coefficient β_k in eq. (2.5) by

$$n_\chi = \frac{1}{(2\pi a)^3} \int d^3k|\beta_k|^2. \tag{2.6}$$

These ideas have been applied to gravitational particle creation at the end of inflation by [23] and [24]. Particles with masses m_χ of the order of the Hubble parameter at the end of inflation, $H_I \approx 10^{-6} M_{\rm Pl} \approx 10^{13}$ GeV, may have been created with a density which today may be comparable with the critical density. Fig. 2.1 shows the relic density $\Omega_\chi h^2$ of this SHDM as a function of the mass m_χ in units of H_I. Curves are shown for inflation models that have a smooth transition to a radiation-dominated epoch (dashed line) and a matter-dominated epoch (solid line). The third curve (dotted line) shows the thermal particle density at temperature $T_I = H_I/2\pi$. Also shown in the figure is the region where the SHDM are thermal relics. It is clear that it is possible for dark matter to be in the form of SHDM generated gravitationally at the end of inflation, since the relic abundance $\Omega_\chi h^2$ is generically of order one when $m_\chi \approx H_I$.

2.2 Additional SHDM formulas

Now we will present some Feynman amplitudes, couplings and eigenvectors of the neutralino mass matrix relevant for our SHDM scenario, but that are not

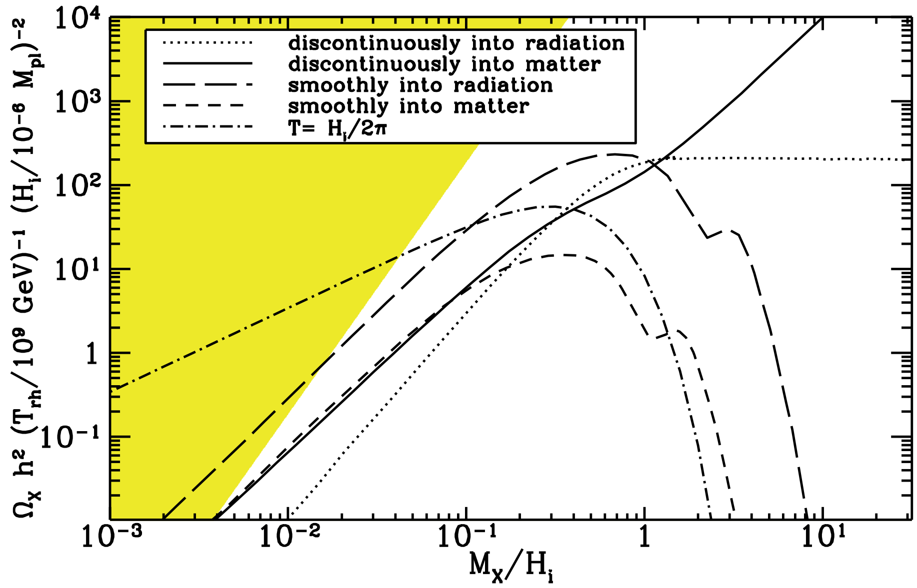


Figure 2.1: The DM abundance today shown as a function of the particle mass for various models. H_I is the Hubble parameter at the end of inflation, $T_{\rm rh}$ is the reheating temperature and $M_{\rm Pl} \approx 3\times 10^{19}$ GeV is the Planck mass. The long- and short-dashed lines correspond to inflationary models that respectively end in a radiation- or matter-dominated epoch. The dotted and solid lines are models that discontinously end in radiation- or matter-dominated epochs. The dash-dotted line is a thermal distribution at the Gibbons-Hawking temperature $T = H_I/2\pi$, while outside the shaded "thermalized" region the SHDM cannot reach thermal equilibrium. Taken from [23].

included in papers I and III. Most of the Feynman amplitudes were omitted since they are subdominant to other processes.

First, the eigenvectors of the neutralino mass-matrix M_χ, eq. (3) in paper I, is in the limit $m_Z \ll M_{\rm SUSY}$ given by

$$|M_1\rangle^T = \begin{pmatrix} 1 & 0 & \frac{m_Z s_W\left(c_\beta M_1+\mu s_\beta\right)}{\mu^2-M_1^2} & -\frac{m_Z s_W\left(\mu c_\beta+M_1 s_\beta\right)}{\mu^2-M_1^2} \end{pmatrix} \tag{2.7}$$

$$|M_2\rangle^T = \begin{pmatrix} 0 & 1 & -\frac{c_W m_Z\left(c_\beta M_2+\mu s_\beta\right)}{\mu^2-M_2^2} & \frac{c_W m_Z\left(\mu c_\beta+M_2 s_\beta\right)}{\mu^2-M_2^2} \end{pmatrix} \tag{2.8}$$

$$|\mu\rangle^T = \begin{pmatrix} \frac{m_Z s_W\left(c_\beta+s_\beta\right)}{\sqrt{2}(\mu-M_1)} & -\frac{c_W m_Z\left(c_\beta+s_\beta\right)}{\sqrt{2}(\mu-M_2)} & -\frac{1}{\sqrt{2}}+a\,m_Z^2 & \frac{1}{\sqrt{2}}+b\,m_Z^2 \end{pmatrix} \tag{2.9}$$

$$|-\mu\rangle^T = \begin{pmatrix} \frac{m_Z s_W\left(c_\beta-s_\beta\right)}{\sqrt{2}(\mu+M_1)} & \frac{c_W m_Z\left(s_\beta-c_\beta\right)}{\sqrt{2}(\mu+M_2)} & \frac{1}{\sqrt{2}}+c\,m_Z^2 & \frac{1}{\sqrt{2}}+d\,m_Z^2 \end{pmatrix}, \tag{2.10}$$

where

$$a = \frac{(c_\beta + s_\beta)\left(\frac{(c_\beta+s_\beta)c_W^2}{(\mu-M_2)^2} - \frac{(-M_1 c_W^2 - M_2 s_W^2 + \mu)(c_\beta - s_\beta)}{\mu(\mu-M_1)(\mu-M_2)} + \frac{s_W^2(c_\beta+s_\beta)}{(\mu-M_1)^2}\right)}{4\sqrt{2}} \tag{2.11}$$

$$b = \frac{(c_\beta + s_\beta)\left(-\frac{(c_\beta+s_\beta)c_W^2}{(\mu-M_2)^2} - \frac{(-M_1 c_W^2 - M_2 s_W^2 + \mu)(c_\beta - s_\beta)}{\mu(\mu-M_1)(\mu-M_2)} - \frac{s_W^2(c_\beta+s_\beta)}{(\mu-M_1)^2}\right)}{4\sqrt{2}} \tag{2.12}$$

$$c = \frac{\frac{(s_{2\beta}-1)c_W^2}{(\mu+M_2)^2} + \frac{c_{2\beta}(M_1 c_W^2 + M_2 s_W^2 + \mu)}{\mu(\mu+M_1)(\mu+M_2)} + \frac{s_W^2(s_{2\beta}-1)}{(\mu+M_1)^2}}{4\sqrt{2}} \tag{2.13}$$

$$d = \frac{\frac{(s_{2\beta}-1)c_W^2}{(\mu+M_2)^2} - \frac{c_{2\beta}(M_1 c_W^2 + M_2 s_W^2 + \mu)}{\mu(\mu+M_1)(\mu+M_2)} + \frac{s_W^2(s_{2\beta}-1)}{(\mu+M_1)^2}}{4\sqrt{2}} \tag{2.14}$$

The eigenvectors are normalized, to the actual order of accuracy: In the Bino/Wino cases the eigenvectors are the normalized eigenvectors modulo $\mathcal{O}(m_z/M_{\text{SUSY}})^2$, and in the Higgsino cases they are the normalized eigenvectors modulo $\mathcal{O}(m_z/M_{\text{SUSY}})^3$, although the latter perhaps does not look that way since $1/\sqrt{1+km_Z^2} \approx 1 - km_Z^2/2$, and hence we would expect a second order contribution in every element. But due to a cancellation, we do not get any second order contribution in m_Z in the first two elements of the Higgsino eigenvectors.

2.2.1 Couplings

We are now going to calculate the couplings of processes involving the lightest neutralino, which we assume to be the LSP. The couplings are evaluated in the limit $m_\chi \gg m_Z$, and the definitions of the couplings are as in paper [5].

$Z^0\chi_1^0\chi_1^0$:

The interaction between Z^0 and the neutralinos are given by the interaction Lagrangian (cf. [5] p.244)

$$\mathcal{L}_{Z\chi^0\chi^0} = \frac{g}{2c_W} Z_\mu [\bar{\chi}_n^0 \gamma^\mu (O''^L_{nm} P_L + O''^R_{nm} P_R)\chi_m^0], \tag{2.15}$$

where $m = n = 1$ yields the interaction between Z^0 and the LSP's. Moreover, we have that

$$O''^L_{nm} = -O''^{R*}_{nm} = \tfrac{1}{2}(-N_{3n}N_{3m}^* + N_{4n}N_{4m}^*), \tag{2.16}$$

which in the case $m = n = 1$ gives (since N is real)

$$O''^{L}_{11} = \begin{cases} \frac{\cos(2\beta)\sin^2(\theta_W)m_Z^2}{2\left(\mu^2-M_1^2\right)} & \text{if } M_1 < M_2, \mu; \\ \frac{\cos(2\beta)\cos^2(\theta_W)m_Z^2}{2\left(\mu^2-M_2^2\right)} & \text{if } M_2 < M_1, \mu; \\ -\frac{\cos(2\beta)m_Z^2\left(M_1\cos^2(\theta_W)+\mu+\sin^2(\theta_W)M_2\right)}{4\mu(\mu+M_1)(\mu+M_2)} & \text{if } 0 < -\mu < M_1, M_2; \\ \frac{\cos(2\beta)m_Z^2\left(M_1\cos^2(\theta_W)-\mu+\sin^2(\theta_W)M_2\right)}{4\mu(\mu-M_1)(\mu-M_2)} & \text{if } 0 < \mu < M_1, M_2,, \end{cases} \tag{2.17}$$

since N_{31} corresponds to the third component of the eigenvector of the least eigenvalue of M_χ etc.

$H^0\chi_1^0\chi_1^0$:

The interaction between H^0 and the neutralinos are given by the interaction Lagrangian (cf. [5] p.245)

$$\mathcal{L}_{H^0\chi^0\chi^0} = \frac{g}{2}H^0\bar{\chi}_n^0[T_{Hnm}^*P_L + T_{Hnm}P_R)]\chi_m^0, \tag{2.18}$$

where

$$T_{Hnm} = -\cos\alpha Q''_{nm} + \sin\alpha S''_{nm}, \tag{2.19}$$

where the mixing angle α from the two-Higgs-doublet model (2HDM), (which the Higgs sector of the MSSM is a special case of) satisfies (at tree-level)

$$\begin{aligned} \sin 2\alpha &= -(\sin 2\beta)(\frac{m_A^2 + m_Z^2}{m_H^2 - m_h^2}), \\ \cos 2\alpha &= -(\cos 2\beta)(\frac{m_A^2 - m_Z^2}{m_H^2 - m_h^2}). \end{aligned} \tag{2.20}$$

With heavy neutralinos, we will have that

$$m_H, m_A \gg m_h, m_Z, \tag{2.21}$$

and since we also in the 2HDM have (at tree-level) $m_A^2 + m_Z^2 = m_H^2 + m_h^2$ eq. (2.20) yields $\sin 2\alpha = -(\sin 2\beta)$ and $\cos 2\alpha = -(\cos 2\beta)$ modulo $(m_Z/M_{\text{SUSY}})^2$. Hence,

$$\alpha = \beta + \pi(\frac{1}{2} + n) \tag{2.22}$$

where α traditionally is chosen such that [22]

$$\alpha = \beta - \frac{\pi}{2}, \tag{2.23}$$

which we will use to simplify the calculations which involve the mixing angle α.

By convention, v_u, v_d are taken to be real and positive [22], and hence we can define

$$\tan\beta = v_u/v_d \geq 0 \tag{2.24}$$

with

$$\beta \in [0, \pi/2], \tag{2.25}$$

and hence

$$\alpha \in [-\frac{\pi}{2}, 0]. \tag{2.26}$$

We now turn back to the expressions

$$Q''_{nm} = \tfrac{1}{2} N_{3n}(N_{2m} - \tan\theta_W N_{1m}) + (n \leftrightarrow m), \tag{2.27}$$

$$S''_{nm} = \tfrac{1}{2} N_{4n}(N_{2m} - \tan\theta_W N_{1m}) + (n \leftrightarrow m), \tag{2.28}$$

and then the coupling (2.19) becomes

$$T_{H11} = \begin{cases} -\frac{\mu\cos(2\beta)\sin(\theta_W)m_Z\tan(\theta_W)}{\mu^2-M_1^2} & \text{if } M_1 < M_2, \mu; \\ -\frac{\mu\cos(2\beta)\cos(\theta_W)m_Z}{\mu^2-M_2^2} & \text{if } M_2 < M_1, \mu; \\ \frac{\cos(2\beta)\sec(\theta_W)m_Z\left(-M_1\cos^2(\theta_W)+\mu-\sin^2(\theta_W)M_2\right)}{2(\mu-M_1)(\mu-M_2)} & \text{if } 0 < \mu < M_1, M_2 \\ \frac{\cos(2\beta)\sec(\theta_W)m_Z\left(M_1\cos^2(\theta_W)+\mu+\sin^2(\theta_W)M_2\right)}{2(\mu+M_1)(\mu+M_2)} & \text{if } 0 < -\mu < M_1, M_2. \end{cases} \tag{2.29}$$

$h^0\chi_1^0\chi_1^0$:

The interaction between h^0 and the neutralinos are given by the interaction Lagrangian

$$\mathcal{L}_{h^0\chi^0\chi^0} = \frac{g}{2} h^0 \bar{\chi}_n^0 [T_{hnm}P_L + T_{hnm}P_R]\chi_m^0, \tag{2.30}$$

where

$$T_{hnm} = \sin\alpha Q''_{nm} + \cos\alpha S''_{nm}, \tag{2.31}$$

so

$$T_{h11} = \begin{cases} \frac{\sin(\theta_W)m_Z(\mu\sin(2\beta)+M_1)\tan(\theta_W)}{\mu^2-M_1^2} & \text{if } M_1 < M_2, \mu; \\ \frac{\cos(\theta_W)m_Z(\mu\sin(2\beta)+M_2)}{\mu^2-M_2^2} & \text{if } M_2 < M_1, \mu; \\ -\frac{\sec(\theta_W)(\sin(2\beta)+1)m_Z\left(-M_1\cos^2(\theta_W)+\mu-\sin^2(\theta_W)M_2\right)}{2(\mu-M_1)(\mu-M_2)} & \text{if } 0 < \mu < M_1, M_2. \\ -\frac{\sec(\theta_W)(\sin(2\beta)-1)m_Z\left(M_1\cos^2(\theta_W)+\mu+\sin^2(\theta_W)M_2\right)}{2(\mu+M_1)(\mu+M_2)} & \text{if } 0 < -\mu < M_1, M_2. \end{cases} \tag{2.32}$$

$A^0\chi_1^0\chi_1^0$:

The interaction between A^0 and the neutralinos are given by the interaction Lagrangian

$$\mathcal{L}_{h^0\chi^0\chi^0} = \frac{ig}{2} A^0 \bar{\chi}_n^0 [-T_{Anm} P_L + T_{Anm} P_R] \chi_m^0, \tag{2.33}$$

where

$$T_{Anm} = -\sin\beta Q_{nm}^{''*} + \cos\beta S_{nm}^{''*}, \tag{2.34}$$

so, since $Q_{nm}^{''}, S_{nm}^{''}$ are real,

$$T_{A11} = \begin{cases} \frac{\sin(\theta_W) m_Z(\mu+\sin(2\beta)M_1)\tan(\theta_W)}{\mu^2-M_1^2} & \text{if } M_1 < M_2, \mu; \\ \frac{\cos(\theta_W) m_Z(\mu+\sin(2\beta)M_2)}{\mu^2-M_2^2} & \text{if } M_2 < M_1, \mu; \\ -\frac{\sec(\theta_W)(\sin(2\beta)+1)m_Z\left(-M_1\cos^2(\theta_W)+\mu-\sin^2(\theta_W)M_2\right)}{2(\mu-M_1)(\mu-M_2)} & \text{if } 0 < \mu < M_1, M_2. \\ \frac{\sec(\theta_W)(\sin(2\beta)-1)m_Z\left(M_1\cos^2(\theta_W)+\mu+\sin^2(\theta_W)M_2\right)}{2(\mu+M_1)(\mu+M_2)} & \text{if } 0 < -\mu < M_1, M_2. \end{cases} \tag{2.35}$$

2.2.2 The Chargino mixing matrices

In this section we derive Chargino mixing matrices in the superheavy limit. These mixing matrices are used deriving Feynman amplitudes for neutralino annihilations.

The Bino as the LSP

The Chargino mass matrix is given by [5]

$$M_{\text{Charginos}} = \begin{pmatrix} M_2 & \sqrt{2}\sin(\beta)m_W \\ \sqrt{2}\cos(\beta)m_W & \mu \end{pmatrix}, \tag{2.36}$$

and this matrix is bidiagonalized by the matrices

$$\text{Umix} = \tag{2.37}$$

$$\begin{pmatrix} \frac{m_W^2(\mu\sin(\beta)+\cos(\beta)M_2)^2-\left(\mu^2-M_2^2\right)^2}{\left(\mu^2-M_2^2\right)^2} & \frac{\sqrt{2}m_W(\mu\sin(\beta)+\cos(\beta)M_2)}{\mu^2-M_2^2} \\ \frac{\sqrt{2}m_W(\mu\sin(\beta)+\cos(\beta)M_2)}{\mu^2-M_2^2} & 1-\frac{m_W^2(\mu\sin(\beta)+\cos(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} \end{pmatrix},$$

and

$$\text{Vmix} = \tag{2.38}$$

$$\begin{pmatrix} \frac{m_W^2(\mu\cos(\beta)+\sin(\beta)M_2)^2-\left(\mu^2-M_2^2\right)^2}{\left(\mu^2-M_2^2\right)^2} & \frac{\sqrt{2}m_W(\mu\cos(\beta)+\sin(\beta)M_2)}{\mu^2-M_2^2} \\ \frac{\sqrt{2}m_W(\mu\cos(\beta)+\sin(\beta)M_2)}{\mu^2-M_2^2} & 1-\frac{m_W^2(\mu\cos(\beta)+\sin(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} \end{pmatrix},$$

such that

$$\text{Umix}\, M_{\text{Charginos}}\, \text{Vmix}^T = \tag{2.39}$$

$$\begin{pmatrix} -\frac{M_2^3-\mu^2M_2+m_W^2(\mu\sin(2\beta)+M_2)}{\mu^2-M_2^2} & 0 \\ 0 & \frac{\mu^3-M_2^2\mu+m_W^2(\mu+\sin(2\beta)M_2)}{\mu^2-M_2^2} \end{pmatrix} + \mathcal{O}(\frac{m_W^3}{M_{\text{Susy}}^2}),$$

$$= \begin{pmatrix} M_2 & 0 \\ 0 & \mu \end{pmatrix} + \mathcal{O}(\frac{m_W^2}{M_{\text{Susy}}}), \tag{2.40}$$

and we write Vmix^T instead of $\text{Vmix}^\dagger$ since it is real.

The Higgsino as the LSP

The Chargino mass matrix (2.36) is in the Higgsino case bidiagonalized by the matrices (expanded to an error of order $O\left(m_W^3\right)$)

$$\text{Umix} = \tag{2.41}$$

$$\begin{pmatrix} \frac{\sqrt{2}m_W(\mu\sin(\beta)+\cos(\beta)M_2)}{\mu^2-M_2^2} & 1-\frac{m_W^2(\mu\sin(\beta)+\cos(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} \\ \frac{\left(\mu^2-M_2^2\right)^2-m_W^2(\mu\sin(\beta)+\cos(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} & -\frac{\sqrt{2}m_W(\mu\sin(\beta)+\cos(\beta)M_2)}{\mu^2-M_2^2} \end{pmatrix},$$

and

$$\text{Vmix} = \tag{2.42}$$

$$\begin{pmatrix} \frac{\sqrt{2}m_W(\mu\cos(\beta)+\sin(\beta)M_2)}{\mu^2-M_2^2} & 1-\frac{m_W^2(\mu\cos(\beta)+\sin(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} \\ \frac{\left(\mu^2-M_2^2\right)^2-m_W^2(\mu\cos(\beta)+\sin(\beta)M_2)^2}{\left(\mu^2-M_2^2\right)^2} & \frac{\sqrt{2}m_W(\mu\cos(\beta)+\sin(\beta)M_2)}{M_2^2-\mu^2} \end{pmatrix},$$

such that

$$\text{Umix}\, M_{\text{Charginos}}\, \text{Vmix}^T = \tag{2.43}$$

$$\begin{pmatrix} \mu+\frac{(\mu+\sin(2\beta)M_2)m_W^2}{\mu^2-M_2^2} & 0 \\ 0 & M_2-\frac{(\mu\sin(2\beta)+M_2)m_W^2}{\mu^2-M_2^2} \end{pmatrix} + O\left(m_W^3\right),$$

$$= \begin{pmatrix} \mu & 0 \\ 0 & M_2 \end{pmatrix} + \mathcal{O}(\frac{m_W^2}{M_{\text{Susy}}}), \tag{2.44}$$

and we write Vmix^T instead of $\text{Vmix}^\dagger$ since it is real.

Note that this bidiagonalization puts the diagonalized masses in increasing order, following the Les Houches accord [25], as for the Bino case. The mixing matrices in the Higgsino case become different from the mixing matrices in the Bino case, since M_2 and μ there came in a different order, and hence the Higgsino bidiagonalization matrices have to flip these two mass parameters.

2.2.3 Neutralino scattering, $T \sim \omega \gg m_Z$

In this section we assume that

$$m_Z \ll \omega \ll M_{\text{SUSY}}. \tag{2.45}$$

The Bino as the LSP

In the process $\chi\nu \leftrightarrow \chi\nu$ the s and u channels contribute, the t-channel (alone) is suppressed by a factor $\mathcal{O}(m_Z^4/\omega^4)$, relative to the leading contribution and is of order $\mathcal{O}(m_Z^4/(\omega^2 M_{\text{SUSY}}^2))$. The total matrix element is of order $\mathcal{O}(\omega^2/M_{\text{SUSY}}^2)$:

$$|\mathcal{M}|^2 = \frac{e^4\omega^2(3-\cos(\theta))M_1^2}{2c_W^4\left(M_{\tilde{\nu}}^2 - M_1^2\right)^2}. \tag{2.46}$$

Energy relaxation time From eq. (2.46) we get a contribution to the energy relaxation time,

$$\tau = \frac{\pi^3 c_W^4 M_1 \left(M_{\tilde{\nu}}{}^2 - M_1^2\right)^2}{25 e^4 T^6 N_{\text{eff}}}. \tag{2.47}$$

As we will see, neutralinos scattering on weak bosons will yield the dominant contribution to the energy relaxation time.

The Higgsino as the LSP

In the process $\chi\nu \leftrightarrow \chi\nu$ the t-channel is of order $\mathcal{O}(m_Z^4/(\omega^2 M_{\text{SUSY}}^2))$ here dominates, while the other channels are of order $\mathcal{O}(\omega^2 m_Z^4/M_{\text{SUSY}}^6)$ or higher. The dominating channel yields

$$|\mathcal{M}|^2 = |\mathcal{M}_t|^2 = \frac{e^4\left(4\omega^2 - t\right)\cos^2(2\beta)\csc^4\left(2\theta_W\right)m_Z^4\left(M_1\cos^2\left(\theta_W\right) - \mu + \sin^2\left(\theta_W\right)M_2\right)^2}{\left(m_Z^2 - t\right)^2\left(\mu - M_1\right)^2\left(\mu - M_2\right)^2}, \tag{2.48}$$

where

$$t = 2\omega^2(\cos(\theta) - 1). \tag{2.49}$$

Energy relaxation time The subdominant contribution to the energy relaxation time corresponding to the amplitude (2.48) can now be calculated as

$$\frac{1}{\tau_{\mathrm{rel}}}=\frac{N_{\mathrm{eff}}}{2E_k m_{\chi_1^0}}\int_0^{\infty}\frac{1}{2\omega^2}\int_{t=-4\omega^2}^{0}\int_0^{2\pi}d\phi\, n_0(\omega)\left(\frac{d\sigma_{\mathrm{el}}}{d\Omega}\right)_{f_L\chi_1^0}(-t)\,dt\,d\omega, \quad (2.50)$$

where $d\theta = dt/(-2\omega^2\sin\theta)$, which yields the factor $1/2\omega^2$ in eq. (2.50). The integrand in eq. (2.50) contains a factor

$$\frac{16\omega^4\log\left(\frac{4\omega^2}{m_Z^2}+1\right)}{4\omega^2+m_Z^2}, \quad (2.51)$$

which under our present conditions (2.45), is simplified to

$$4\omega^2\log\left(\frac{4\omega^2}{m_Z^2}\right), \quad (2.52)$$

which again yields an integral

$$\int_0^{\infty}e^{-\frac{\omega}{T}}\left(4\omega^2\log\left(\frac{4\omega^2}{m_Z^2}\right)\right)d\omega \quad (2.53)$$

in eq. (2.50). By using Maple or Mathematica 7 this integral becomes

$$8T^3\left(\log\left(\frac{4T^2}{m_Z^2}\right)-2\gamma+3\right). \quad (2.54)$$

Then the energy relaxation time, using a Maxwell-Boltzmann distribution, becomes

$$\tau=\frac{48\pi^3\mu^3\,(\mu-M_1)^2\,(\mu-M_2)^2 s_{2W}^4}{e^4T^2\left(\log\left(\frac{4T^2}{m_Z^2}\right)-2\gamma+3\right)c_{2\beta}^2 m_Z^4 N_{\mathrm{eff}}\,(M_1c_W^2+M_2s_W^2-\mu)^2}. \quad (2.55)$$

Also in the case we are consideing now, the Higgsino case, neutralinos scattering on gauge bosons and light Higgs will give the dominating contribution to the energy relaxation time.

Taking scattering on gauge bosons and light Higgs into account

Since we assume that $T \gg m_{\mathrm{weak}}$ at freeze-out of the neutralinos, the weak gauge bosons are present in the thermal bath, and we have to take scattering on these into account to get the leading contribution to the energy relaxation times. In addition we have to consider scattering on the light Higgs. In the case $T \gg m_{\text{weak}}$, electroweak symmetry is restored, and the weak bosons are massless. But they will have an effective mass, which will be a small fraction of the temperature, hence we will still use the weak boson masses in our calculations.

We will see that the scattering on weak bosons give $\mathcal{O}(1)$ contributions, so in the SHDM scenario we can disregard neutrino scattering completely, when we want to calculate energy relaxation times. However, in models with a smaller gap between M_{SUSY} and the electroweak scale, the contributions from neutrino scattering could be relevant.

Feynman amplitudes for weak boson scatterings In the process $\chi Z \to \chi Z$ only light Higgs exchange contributes. The squared amplitude is

$$|\mathcal{M}_{\chi Z \to \chi Z}|^2 = \frac{e^4 \sec^4(\theta_W)\, M_1^2\, (\mu \sin(2\beta) + M_1)^2}{3\,(\mu^2 - M_1^2)^2}, \tag{2.56}$$

which equals eq. (5) in paper III.

We get exactly the same result for W^+ scattering, $\chi W^+ \to \chi W^+$

$$|\mathcal{M}_{\chi W \to \chi W}|^2 = \frac{e^4 \sec^4(\theta_W)\, M_1^2\, (\mu \sin(2\beta) + M_1)^2}{3\,(\mu^2 - M_1^2)^2}, \tag{2.57}$$

and the same formula is of course valid for W^- scattering.

In the case of light Higgs scattering, $\chi_1^0 h \to \chi_1^0 h$, the four amplitudes involving the (in this section Higgsino-) mediators χ_3^0 and χ_4^0 give leading order contributions.

The χ_3^0 annihilation (s-channel) squared amplitude equals the χ_3^0 exchange (u-channel) squared amplitude, i.e.

$$|\mathcal{M}_{\chi_3^0 s}|^2 = |\mathcal{M}_{\chi_3^0 u}|^2. \tag{2.58}$$

Moreover, each of these squared amplitudes equals one fourth of the total squared amplitude for χ_3^0,

$$|\mathcal{M}_{\chi_3^0 s} + \mathcal{M}_{\chi_3^0 u}|^2 = 4|\mathcal{M}_{\chi_3^0 s}|^2 = 4|\mathcal{M}_{\chi_3^0 u}|^2. \tag{2.59}$$

Complex algebra then forces

$$\mathcal{M}_{\chi_3^0 s} = \mathcal{M}_{\chi_3^0 u}. \tag{2.60}$$

In a totally analogous manner we get that

$$\mathcal{M}_{\chi_4^0 s} = \mathcal{M}_{\chi_4^0 u}. \tag{2.61}$$

Then the total squared amplitude

$$|\mathcal{M}_{\chi h\to\chi h}| = |\mathcal{M}_{\chi_3^0 s} + \mathcal{M}_{\chi_3^0 u} + \mathcal{M}_{\chi_4^0 s} + \mathcal{M}_{\chi_4^0 u}|^2 = 4|\mathcal{M}_{\chi_3^0 s} + \mathcal{M}_{\chi_4^0 s}|^2, \quad (2.62)$$

where $\chi = \chi_1^0$, and where we calculate the latter expression of eq. (2.62) in CalcHEP, and hence get

$$|\mathcal{M}_{\chi h\to\chi h}| = \frac{e^4 \sec^4(\theta_W)\, M_1^2\, (\mu\sin(2\beta) + M_1)^2}{(\mu^2 - M_1^2)^2}. \quad (2.63)$$

Hence, the total weak boson scattering amplitude becomes

$$|\mathcal{M}_{\rm wb}|^2 = g_Z|\mathcal{M}_{\chi Z\to\chi Z}|^2 + 2g_W|\mathcal{M}_{\chi W\to\chi W}|^2 + g_h|\mathcal{M}_{\chi h\to\chi h}|^2 \quad (2.64)$$

$$= (\frac{g_Z}{3} + 2\frac{g_W}{3} + g_h)\frac{e^4 \sec^4(\theta_W)\, M_1^2\, (\mu\sin(2\beta) + M_1)^2}{(\mu^2 - M_1^2)^2}, \quad (2.65)$$

where the polarization degrees of freedom equal $g_Z = g_W = 3$ and $g_h = 1$, so

$$|\mathcal{M}_{\rm wb}|^2 = \frac{4e^4 \sec^4(\theta_W)\, M_1^2\, (\mu\sin(2\beta) + M_1)^2}{(\mu^2 - M_1^2)^2}. \quad (2.66)$$

This is the dominating processes which yield the energy relaxation time given by eq. (6) in paper III.

Higgsino case This case is dominated by the processes as $\chi_1^0\nu_e \leftrightarrow \chi_1^+ e^-$, see eqs. (7)-(9) in paper III. We will here include the Feynman amplitudes of some subdominant processes.

The process $\chi Z \to \chi Z$: Only the following channels (referring to unitary gauge, as usual) give leading contributions of order $\mathcal{O}(M_{\rm SUSY}/m_Z)^4$:

$$|\mathcal{M}_{\chi_2 s}|^2 = |\mathcal{M}_{\chi_2 u}|^2 = \frac{4e^4\mu^2\omega^2 \csc^4(2\theta_W)}{3m_Z^4}, \quad (2.67)$$

but it turns out that the leading order contributions of the χ_2 s- and u-channels cancel, so the total $O(1)$ squared amplitude becomes in the limit $\mu \ll M_1, M_2$

$$|\mathcal{M}_{\chi Z\to\chi_2\to\chi Z}|^2 = \frac{2}{3}e^4(\cos(2\theta) + 3)\csc^4(2\theta_W). \quad (2.68)$$

The cancellation of $\mathcal{O}((M_{\rm SUSY}/m_Z)^4$ is exact for all models, not only our scenario. This has to do with the fact that gauge bosons do not couple strongly in the superheavy limit, as we have seen in earlier sections.

The only other leading order contribution comes from light Higgs exchange, $\chi Z \to h \to \chi Z$

$$|\mathcal{M}_{\chi Z\to h\to\chi Z}|^2 = \frac{4e^4\mu^2\,(M_1 c_W^2 + M_2 s_W^2 - \mu)^2\,(c_\beta + s_\beta)^4}{3\,(\mu - M_1)^2\,(\mu - M_2)^2\, s_{2W}^4}. \quad (2.69)$$

The process $\chi W^+ \to \chi W^+$: In this process only the mediators h (light Higgs) and perhaps also χ_1^+ (lightest chargino, both s and u channels) give leading order contributions. We do not calculate the latter, since it later will be evident it is subdominant anyway. The total squared h-exhchange amplitude becomes

$$|\mathcal{M}_{\chi W^+ \to h \to \chi W^+}|^2 = \frac{4e^4\mu^2\,(M_1 c_W^2 + M_2 s_W^2 - \mu)^{\,2}\,(c_\beta + s_\beta)^{\,4}}{3\,(\mu - M_1)^{\,2}\,(\mu - M_2)^{\,2} s_{2W}^4}, \qquad (2.70)$$

which is the same as for the case of h-mediated scattering on Z's.

The two diagrams $\chi W^+ \to \chi_1^+ \to \chi W^+$: The amplitudes yielded by the mediator χ_1^+ in s- and u channels cancel each other exactly in orders $1/m_Z^4$, $1/m_Z^3$ and $1/m_Z^2$. (To calculate correct coefficients for higher orders in m_Z, we have to specify some of the masses and mixing matrices to a higher degree in the perturbation m_Z.)

But we can convince ourselves of that we cannot have a non-zero term of order $1/m_Z$: We can write

$$|\mathcal{M}|^2 = \frac{1}{m_Z^4}(\mathcal{M}'_s + \mathcal{M}'_u)(\mathcal{M}'_s + \mathcal{M}'_u)^* = \frac{FF^*}{m_Z^4} = \frac{P}{m_Z^4}, \qquad (2.71)$$

where $F = (\mathcal{M}'_s + \mathcal{M}'_u)$ is a polynomial in m_Z with no negative orders of m_Z. Now, if $|\mathcal{M}|^2$ contains a non-zero element propotional to $1/m_Z$, then we either have a non-zero term $\propto m_Z^3$ or m_Z^2 in F and a non-zero term $\propto m_Z^0$ or m_Z^1 in F^*. But then we have either a term $\propto m_Z^0$ or m_Z^1 in both F^* and its complex conjugation, since complex conjugation does not change the presence of terms. In case there were a term $\propto m_Z^0$ present in both the factors F and F^*, there would be a term $\propto 1/m_Z^4$ in $|\mathcal{M}|^2$, which there is not. If there is a term term proportional to m_Z^1 present in the factors F and F^*, then there would be term propopoportional to $1/m_Z^2$ in $|\mathcal{M}|^2$, which there is not, unless this term was canceled by another term $\propto m_Z^2 \in P$. But there cannot be such a term, because it would have to come from a term $\propto m_Z^0$ in one factor and a term $\propto m_Z^2$ in the other, but we saw that there could not be any non-zero term $\propto m_Z^0$ in the factors F or F^*.

The process $\chi h \to \chi h$: The total squared amplitude for neutralino scattering on light Higgs is

$$|\mathcal{M}_{\chi h \to \chi h}|^2 = \frac{4e^4\mu^2\,(M_1 c_W^2 + M_2 s_W^2 - \mu)^{\,2}\,(c_\beta + s_\beta)^{\,4}}{(\mu - M_1)^{\,2}\,(\mu - M_2)^{\,2} s_{2W}^4}. \qquad (2.72)$$

We do not include any energy relaxation time for Higgsinos, $\omega \gg m_Z$, for scattering on gauge bosons, since these processes will be subdominant to the (in this case energetically possible) process $\chi, \nu \to \chi_1^+, e^-$. The latter process is also possible in the Wino case, but not in the Bino case, since there is no chargino with mass comparable with M_1.

2.2.4 Sommerfeld enhancements in neutralino annihilations

We will now decide which neutralino annihilation channels could be Sommerfeld enhanced: A Sommerfeld enhancement of the annihilation cross-section can occur when the two particles may annihilate through (relatively) long ranged messengers (i.e. relatively light messengers) at low velocities. Because of the low relative velocity and the (relatively) long ranged interaction the interaction effectively becomes too strong for the assumptions underlying ordinary perturbation theory to be valid. Hence the cross-sections from ordinary perturbation theory are enhanced. In the context of SHDM the possible relatively light messengers will be the weak gauge bosons $Z, W^\pm$ and the light Higgs h. See also the discussion in chapter IV B in paper III.

In this section we assume, as indicated above, that the kinetic energy ω of the neutralinos is small,

$$\omega \ll m_Z \ll M_{\text{SUSY}}, \tag{2.73}$$

i.e. we are considering the $v \to 0$ limit (the neutralinos have a typical velocity of no more than 1ms^{-1} when annihilating in the clumps, see sec. III C of paper III.)

Sommerfeld enhancements of the leading Bino annihilations

The Bino does not couple to the Z or the $W^\pm$, so we do not get any Sommerfeld enhancements by light mediators in these cases. But the Bino couples to the light Higgs, and hence the annihilation into W^+H^- could be enhanced. But the contribution from h-annihilation is not leading (it is proportional to ω).

Sommerfeld enhancements of the leading Higgsino annihilations

Sommerfeld enhancements (of orders up to 10^{10}) of some of the leading annihilations, will make these enhanced leading annihilation dominate over the others. Sommerfeld enhancements require the mediating particle to be light (i.e. not superheavy). The actual particles are $Z, W^\pm, h$. The light and superheavy sectors must be gauge invariant each by itself, although the differently mediated annihilations are not generally gauge invariant.

The two dominating Higgsino annihilations that are Sommerfeld enhanced are

$$|\mathcal{M}_{Z\to Zh}|^2 = \frac{2e^4\mu^2 c_{2\beta}^2\,(M_1 c_W^2 + M_2 s_W^2 - \mu)^{\,2}}{(\mu - M_1)^{\,2}\,(\mu - M_2)^{\,2} s_{2W}^4}, \tag{2.74}$$

through Z-annihilation, as indicated, which also is the only diagram for neutralino annihilation into Z, h with a light mediator. We see that this differs from the previous annihilation amplitude into Z, h eq. (64) of paper I, so Z-annihilation was not the only contributing diagram there.

The second Sommerfeld enhanced dominating Higgsino annihilation is

$$|\mathcal{M}_{Z\to A^0 H}|^2 = \frac{e^4 c_{2\beta}^2\,(M_A^2 - M_H^2)^{\,2}\,(M_1 c_W^2 + M_2 s_W^2 - \mu)^{\,2}}{8\mu^2\,(\mu - M_1)^{\,2}\,(\mu - M_2)^{\,2} s_{2W}^4}, \tag{2.75}$$

through Z-annihilation, as indicated, which also is the only diagram with a light mediator. We see that this differs from the previous annihilation amplitude into A, H eq. (76) of paper I, so Z-annihilation was not the only contributing diagram there either.

Chapter 3

Electroweak bremsstrahlung

3.1 The PAMELA excess

PAMELA (Payload for Antimatter Matter Exploration and Light-nuclei Astrophysics) is an cosmic ray (CR) research module attatched to an earth orbiting satellite. PAMELA was launched 15 June 2006 and is a satellite-based experiment dedicated to the detection of cosmic radiation, with particular focus on its antimatter component in the form of positrons and antiprotons.

The flux of antimatter is generally expected to fall relative to the flux of matter as one moves to higher energies. This is so because "secondary production" is believed to be the dominant production mechanism of cosmic antimatter rays: Cosmic ray nuclei and interstellar matter interact, generating the antimatter component consisting of positrons and antiprotons. The higher the velocity of the CR nuclei is, the shorter the traversed path before leaving the galaxy becomes. Then the probability that a high energy CR interacts and produces antimatter is less than the probability that a low energy CR interacts and produces antimatter. Thus the ratio of antimatter CR from secondary production and matter CR from primary/secondary production falls with increasing energy. Hence, if secondary production dominates, the positron fraction

$$\frac{\phi(e^+)}{\phi(e^+) + \phi(e^-)}, \tag{3.1}$$

where $\phi(x)$ indicate the flux of a particle species x, is expected to fall as a smooth function of increasing energy. Nevertheless, the PAMELA experiment shows that the positron fraction rises by a factor 2.5 from 10.2 to 82.6 GeV [26], see fig. 3.1. On the other hand, no corresponding surplus of antiprotons is found, see fig. 3.2. This surplus (compared to the estimated

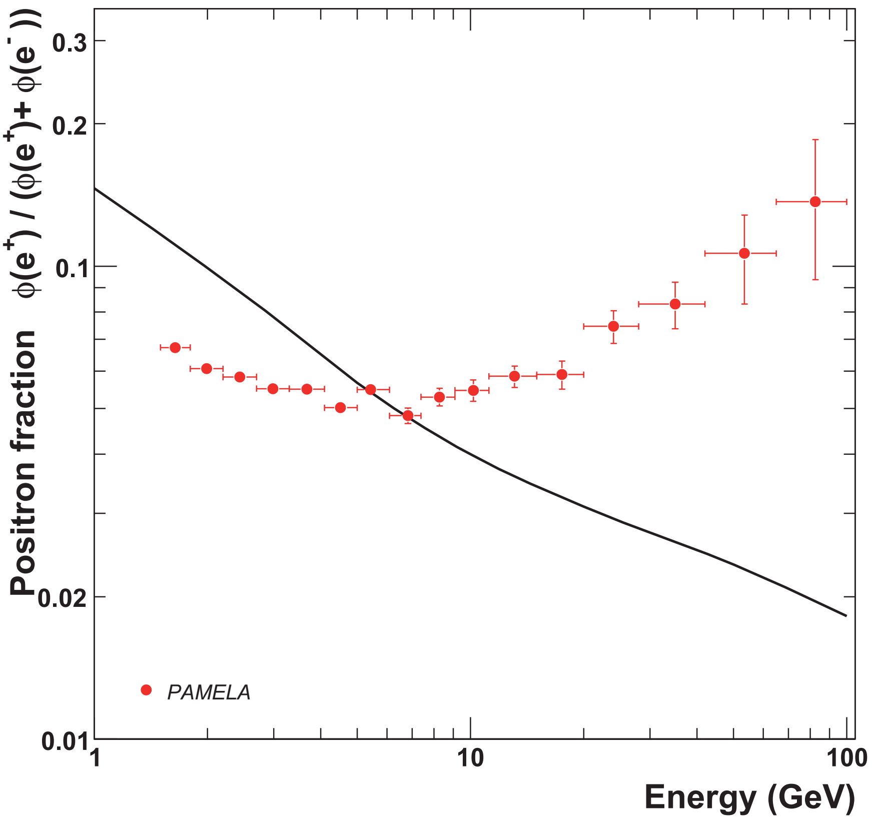


Figure 3.1: Observed positron fraction from the PAMELA experiment, contra an expectation of the background (with positrons from secondary production only) calculated in [28].

background) of positrons in the range $10-90\,\text{GeV}$ is known as the PAMELA excess. This has, as we remark in paper IV, prompted several models trying to explain the data by engineering relatively heavy ($m_\chi \gtrsim 1\,\text{TeV}$) DM candidates with large annihilation cross sections (or decay rates) and small or vanishing (tree-level) branching ratios into hadronic final states (i.e. an explaination of the positron surplus by primary production from DM annihilations or decays). In these models, the corrections from Z- and W-bremsstrahlung have generally been ignored, although these electroweak bremsstrahlung corrections generically turn out to be substantial, see paper IV. Another possible explaination of the PAMELA excess, in terms of primary production, is the possible production of positrons by nearby astrophysical sources, such as

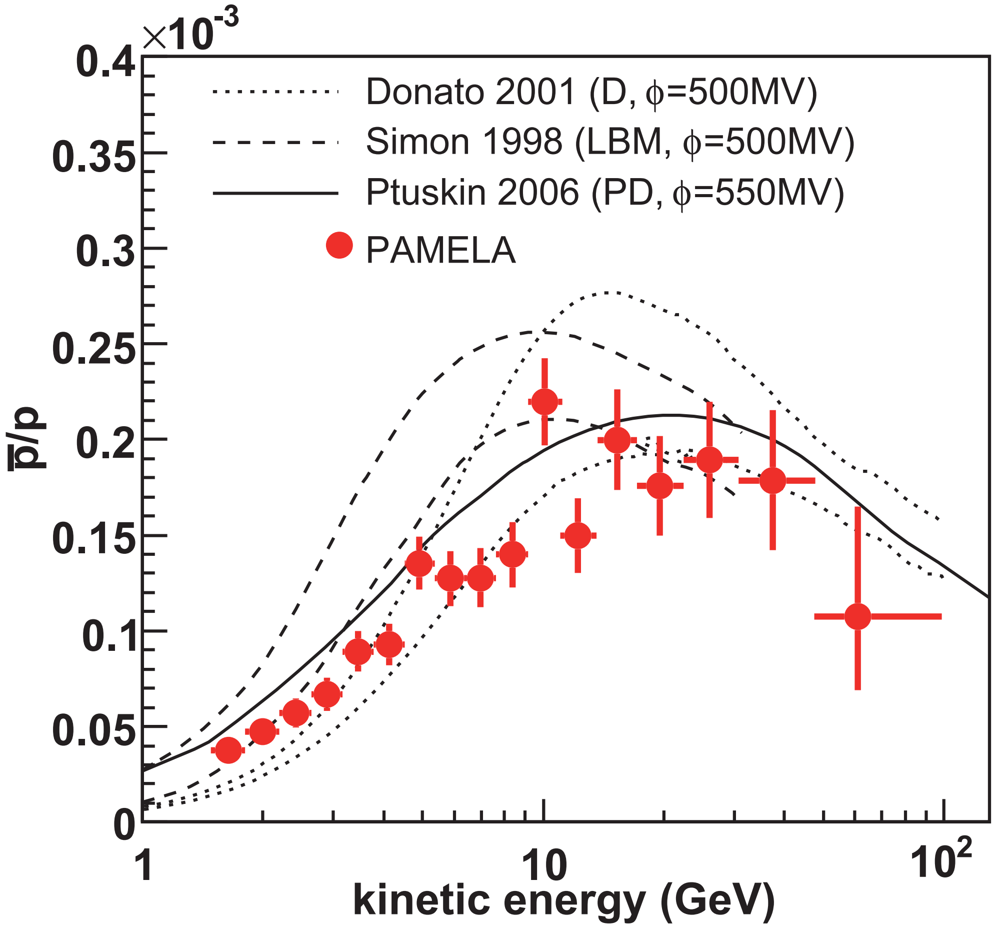


Figure 3.2: Antiproton-to-proton flux ratio observed by the PAMELA experiment [27], versus different theoretical backgrounds calculated regarding a pure secondary production of antiprotons.

pulsars [29, 30] and microquasars [31].

3.2 Additional energy spectrum plots

Now we will present some plots of energy fragmentation spectra dN/dx for values of m_χ not considered in paper IV: Figure 3.3 corresponds to Fig. 3 ("neutrinos only" in tree-level annihilations) and Fig. 4 ("electrons only" in tree-level annihilations) in paper IV. While we in paper IV only considered masses 300 GeV and 3 TeV, we here include plots for masses 100 GeV, 1 TeV and 10 TeV. We here also report the monochromatic spikes at $x = 1$ from the tree-level annihilations.

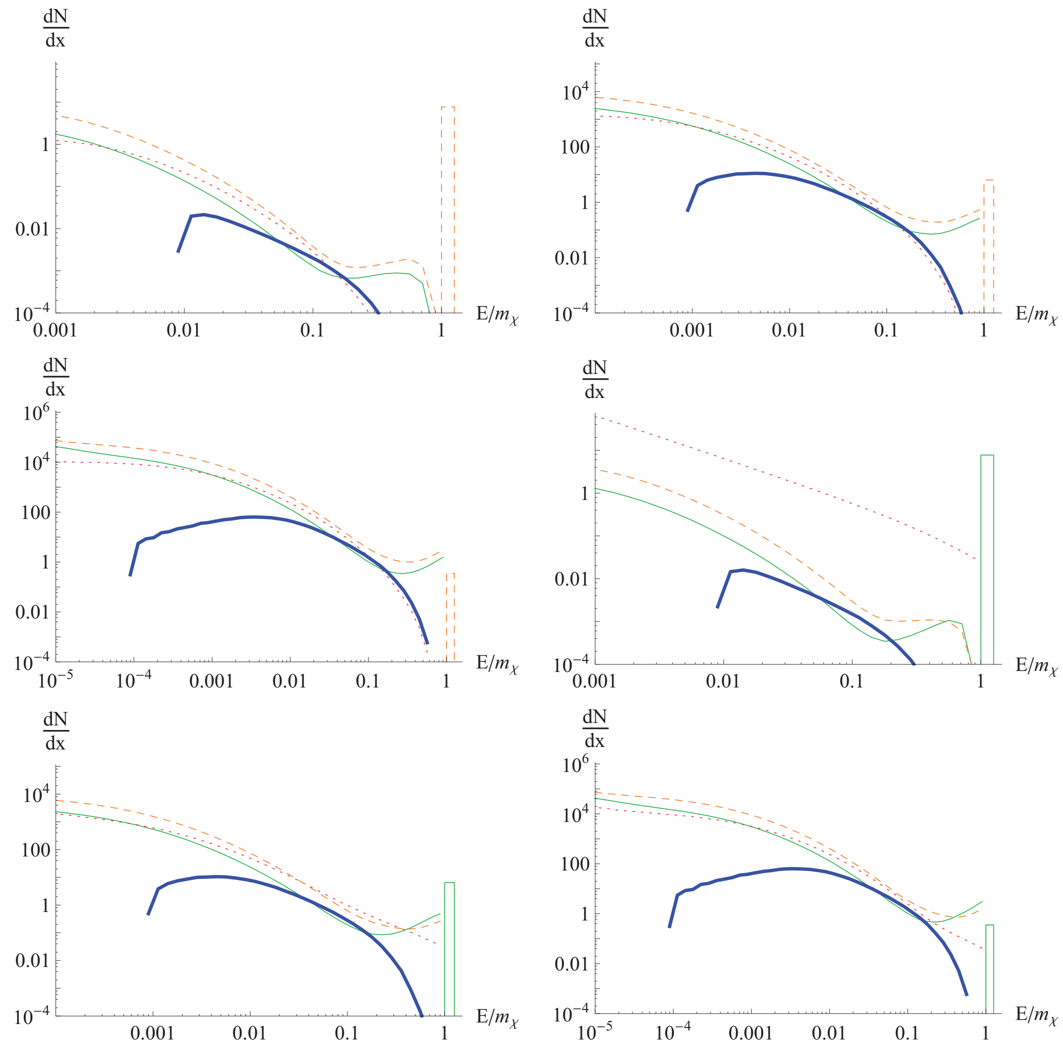


Figure 3.3: The fragmentation spectra dN/dx of electrons (solid green), photons (dotted red), neutrinos (dashed orange), protons (thick blue) in “only neutrino” annihilations for $m_X = 100\,\mathrm{GeV}$ (upper left), $m_X = 1\,\mathrm{TeV}$ (upper right) and $m_X = 10\,\mathrm{TeV}$ (middle left). The three last panels show the same fragmentation spectra in "only electrons" annihilations, for $m_X = 100\,\mathrm{GeV}$ (middle right), $m_X = 1\,\mathrm{TeV}$ (lower left) and $m_X = 10\,\mathrm{TeV}$ (lower right).

3.3 The ratio R_Z of the process $h \to e^+e^-Z$

3.3.1 Introduction

In this section we calculate the exact ratio R_Z of the decay rates of the processes

$$h \to e^+e^-Z, \qquad h \to e^+e^- \tag{3.2}$$

considering final state Z-radiation (FSR) only in the former process. In paper IV we calculated R_W corresponding to the processes $h \to e^+e^-Z$ $h \to \nu\bar{\nu}Z$, and R_Z corresponding to the latter process, see eq. (9) in paper IV. In other words, we will here give a formula for the ratio R_Z corresponding to the Z-strahlung process not considered in that paper.

We compare our result with numerical results for the ratio R_Z of the cross-sections of the processes

$$\chi\chi \to h \to e^+e^-Z\,(\mathrm{FSR}) \qquad \chi\chi \to h \to e^+e^- \tag{3.3}$$

and also with numerical results for R_Z correponding to the processes $\chi\chi \to e^+e^-Z$ (FSR and selectron mediation only) and $e^+e^- \to \chi\chi Z$ (initial state radiation (ISR) and selectron mediation only). For the R_Z from (3.3) we find an excellent agreement with the R_Z from (3.2), while for the two latter processes the agreement is not that good.

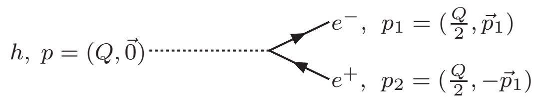
The ratio R_Z of (3.2) and the R_Z of the processes (3.3) should coincide, due to a factorization of the cross-sections: The Feynman amplitude squared of the process $\chi\chi \to h \to e^+e^-Z$ is proportional to $|\mathcal{M}|^2 \propto AB$, where A and B are traces from spin sums of the ingoing and outgoing particles respectively. Since the mediating Higgs is a scalar, there are no Lorentz indices connecting the two factors. Hence the ratios of the cross-sections of the annihilation processes (3.3) equals (in the center of mass frame)

$$\frac{\sigma_3}{\sigma_2} = \frac{\int d\Omega\, A(\chi,\chi)B(e^+,e^-,Z)}{\int d\Omega\, A(\chi,\chi)B(e^+,e^-)} = \frac{\int d\Omega\, B(e^+,e^-,Z)}{\int d\Omega\, B(e^+,e^-)} = \frac{\Gamma_Z}{\Gamma_{\mathrm{tree}}}, \tag{3.4}$$

the second equality since $A(\chi,\chi)$ is independent of the solid angle Ω. Here the magnitudes Γ_Z and Γ_{tree} respectively denote the Z-strahlung and tree-level decay rates of a Higgs with mass equaling the center of mass energy of the two annihilating neutralinos, i.e. their ratio is the R_Z corresponding to the processes (3.2). The factorization described above is confirmed by fig. 3.5.

3.3.2 Tree-level decay rate

The Feynman diagram of the tree-level decay of Higgs to electrons can be written



Tree level Higgs decay

Then the squared Feynman amplitude is,

$$\mathcal{M}\mathcal{M}^{\dagger} = \frac{2e^2Q^2m_e^2\csc^2(2\theta_W)}{m_Z^2}. \tag{3.5}$$

In the rest frame of the Higgs the differential decay can be written

$$d\Gamma = \frac{1}{32\pi^2}|\mathcal{M}|^2\frac{|\vec{p}_1|}{Q^2}\,d\Omega, \tag{3.6}$$

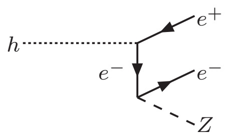
and hence the tree-level decay rate becomes

$$\Gamma(h \to e^- e^+) = \Gamma_{\text{Tree}} = \frac{Qm_e^2}{8\pi v^2} = \frac{e^2Qm_e^2\csc^2(2\theta_W)}{8\pi m_Z^2}. \tag{3.7}$$

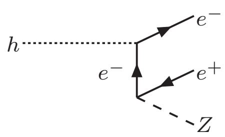
3.3.3 FSR Z-strahlung Feynman amplitudes

Electrons couple to the Z proportional to $(c_L P_L + c_R P_R)$, in contrast to neutrinos that couple proportional to $P_L = (1-\gamma^5)/2$. Because of this more complicated coupling, the cross terms of the decay we are studying are proportional to m_e^0 (or m_e^2 if we also regard the Higgs coupling). Hence the cross terms can not be disregarded, in contrast to the case where the Higgs is decaying into neutrinos.

The FSR Z-strahlung Feynman diagrams are



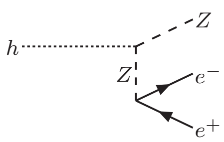
diagr.1



diagr.2

We are excluding the ZZh-coupling diagram, diagram 3, since this is not electroweak FSR diagram.

If $Q \neq m_h$, the Higgs is virtual, and we can assume we are considering a part of the diagram for two particles annihilating through the Higgs channel, in the c.m. frame of the two colliding particles.



diagr.3

The squared Feynman amplitude for the diagrams 1 and 2 then becomes

$$|\mathcal{M}|^2 = -\frac{2e^4 m_e^2 \csc^4(2\theta_W)\left(Am_Z^2 + Bm_Z^4 + Cm_Z^6 + Dm_Z^8\right)}{m_{23}^4 m_Z^4 \left(m_Z^2 - m_{12}^2 - m_{23}^2 + Q^2\right)^2}, \tag{3.8}$$

where

$$\begin{aligned}
A &= m_{23}^2\left(m_{12}^2 + m_{23}^2 - Q^2\right)\Bigg(\left(c_{4W} - 2c_{2W}\right)\left(m_{12}^4 + Q^4\right) \\
&\quad + 2\left(-\left(m_{12}^2 + m_{23}^2\right)Q^2 + \left(m_{12}^2 + m_{23}^2\right)^2 + Q^4\right)\Bigg) \\
&\quad - m_{12}^2 m_{23}^4\left(m_{12}^2 + m_{23}^2 - Q^2\right)^2 \\
B &= m_{12}^4\left(-2c_{2W} + c_{4W} + 2\right)\left(Q^2 - 3m_{23}^2\right) \\
&\quad + m_{12}^2\left(4c_{2W}\left(Q^2 - m_{23}^2\right)^2 - 2c_{4W}\left(Q^2 - m_{23}^2\right)^2 - \left(2Q^2 - 3m_{23}^2\right)^2\right) \\
&\quad + \left(Q - m_{23}\right)\left(m_{23} + Q\right)\left(Q^2 - 2m_{23}^2\right)\left(Q^2\left(-2c_{2W} + c_{4W} + 2\right) - 2m_{23}^2\right) \\
C &= \left(c_{4W} - 2c_{2W}\right)\left(-2m_{12}^2 Q^2 + 3m_{23}^2\left(m_{12}^2 - Q^2\right) + m_{23}^4 + 2Q^4\right) \\
&\quad + m_{12}^2\left(6m_{23}^2 - 4Q^2\right) + 4\left(Q^2 - m_{23}^2\right)^2 \\
D &= \left(-2c_{2W} + c_{4W} + 2\right)\left(Q^2 - m_{23}^2\right)
\end{aligned} \tag{3.9}$$

3.3.4 Decay rate

Now the differential decay rate is (in the standard form for the Dalitz plot, see eq. (6) of paper IV)

$$d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32Q^3}|\mathcal{M}|^2 dm_{12}^2 dm_{23}^2. \tag{3.10}$$

We now want to evaluate the integral, cf. the Dalitz plot (see discussion in paper IV or [13]).

$$\text{Int} = \int_{m_{12}^2=0}^{(Q-m_Z)^2} \int_{(m_{23}^2)_{\min}}^{(m_{23}^2)_{\max}} |\mathcal{M}|^2 \, dm_{23}^2 \, dm_{12}^2. \tag{3.11}$$

As we will see, this integral has an exact closed form solution, using both the exact integrand and integration limits (but with m_e set to zero, of course).

Nevertheless, we will first only regard the lowest order terms in m_Z, which gives us the leading order approximation (in Q) Γ_{Z0} of the decay rate,

$$\Gamma_{Z0} = \frac{e^4 m_e^2 Q^3 \csc^4(2\theta_W)}{768\pi^3 m_Z^4}, \tag{3.12}$$

and $R_{Z0} = \Gamma_Z/\Gamma_{\text{tree}}$ then becomes

$$R_{Z0} = \frac{e^2 Q^2 \csc^2(2\theta_W)}{96\pi^2 m_Z^2}, \tag{3.13}$$

which is a good approximation when $Q \gg m_Z$.

3.3.5 Exact expression for R_Z

Integrating over m_{23}^2 (with exact integration limits as in eq. (7) of paper IV) we get the result

$$\begin{aligned} I_1 &= \int_{(m_{23}^2)_{\min}}^{(m_{23}^2)_{\max}} |\mathcal{M}|^2 \, dm_{23}^2 \\ &= -\frac{2e^4 m_e^2 \csc^4(2\theta_W)\left(A m_Z^2 + B m_Z^4 + C m_Z^6 + K m_{12}^2 (m_{12}^2 - Q^2)\right)}{m_Z^4 (m_Z^2 - m_{12}^2 + Q^2)} \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} A =& -\left(\cos(4\theta_W) - 2\cos(2\theta_W)\right)\left(2K\left(m_{12}^2 - Q^2\right) - L\left(m_{12}^4 + Q^4\right)\right) \\ & + K\left(6Q^2 - 7m_{12}^2\right) + 2L\left(-m_{12}^2 Q^2 + m_{12}^4 + Q^4\right) \\ B =& 2\left(\left(K - L m_{12}^2\right)\left(\cos(4\theta_W) - 2\cos(2\theta_W)\right) + 3K + L\left(Q^2 - 2m_{12}^2\right)\right) \\ C =& L\left(-2\cos(2\theta_W) + \cos(4\theta_W) + 2\right) \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} K =& \sqrt{-2\left(m_{12}^2 + Q^2\right) m_Z^2 + \left(m_{12}^2 - Q^2\right)^2 + m_Z^4} \\ L =& \log\left(\frac{\left(\sqrt{-2\left(m_{12}^2 + Q^2\right) m_Z^2 + \left(m_{12}^2 - Q^2\right)^2 + m_Z^4} - m_Z^2 + m_{12}^2 - Q^2\right)^2}{\left(\sqrt{-2\left(m_{12}^2 + Q^2\right) m_Z^2 + \left(m_{12}^2 - Q^2\right)^2 + m_Z^4} + m_Z^2 - m_{12}^2 + Q^2\right)^2}\right) \end{aligned} \tag{3.16}$$

When evaluating the integral

$$\text{Int} = \int_0^{(Q - m_Z)^2} I_1 \, dm_{12}^2, \tag{3.17}$$

the logarithm L turns out to be problematic. Although L itself has an antiderivative, expressions like

$$\frac{L}{\left(m_{12}^2 - Q^2\right) m_Z^4} \tag{3.18}$$

do not, according to Mathematica at least. Mathematica is neither able to calculate the total integral (3.14).

We could try to series expand in m_Z, obtaining

$$L = 4\log\left(\frac{Qm_Z}{Q^2 - m_{12}^2}\right) + \frac{4m_{12}^2 m_Z^2}{(m_{12}^2 - Q^2)^2} + \frac{2m_{12}^2 (m_{12}^2 + 2Q^2) m_Z^4}{(m_{12}^2 - Q^2)^4} + O(m_Z^5), \tag{3.19}$$

but inserting the endpoint of the integration Int, $m_{12}^2 = (Q - m_Z)^2$, we see the series become (at least superficially) non-convergent,

$$\begin{aligned} 0 =& L(m_{12}^2 = (Q - m_Z)^2) \\ =& 4\log\left(\frac{Q}{2Q - m_Z}\right) + \frac{4(Q - m_Z)^2}{(m_Z - 2Q)^2} + \frac{2(-2Qm_Z + m_Z^2 + 3Q^2)(Q - m_Z)^2}{(m_Z - 2Q)^4} + \ldots \\ =& \mathcal{O}(1) + \mathcal{O}(1) + \mathcal{O}(1) + \ldots . \end{aligned} \tag{3.20}$$

On the other hand, the approximation R_{Z0} works well for $Q \gg m_Z$, since the problematic logarithm L does not show up in the lowest order of m_Z, even when we take into consideration that factors like $(m_{12}^2 - Q^2)$ are of order $\mathcal{O}(m_Z)$ (or really $\mathcal{O}(m_Z Q)$) when m_{12}^2 is close to the endpoint $(Q - m_Z)^2$.

Moreover, expanding the integrand in powers of m_Z and then integrate over m_{12}^2 give a fairly good convergence, as shown in Fig. 3.4, in spite of the bad approximation (3.20) when expanding in powers of m_Z.

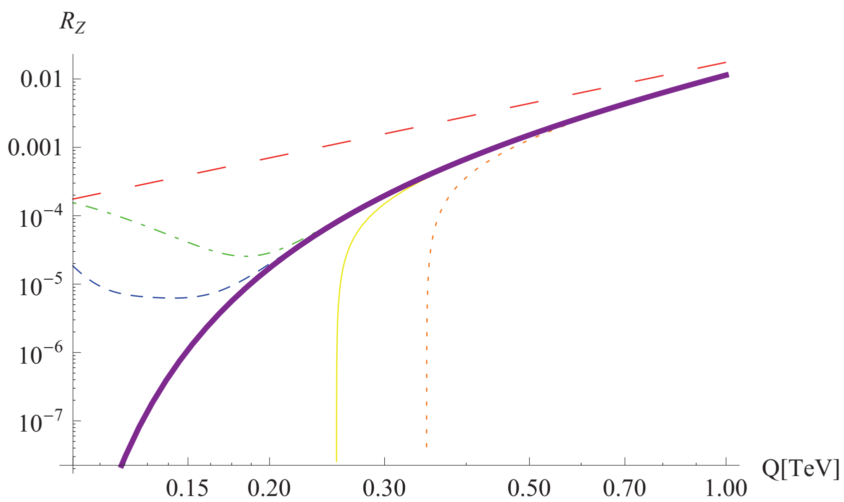


Figure 3.4: The exact R_Z compared with approximations produced by expanding in powers of m_Z before integrating over m_{12}^2.

Nevertheless, by doing the substitution

$$m_{12}^2 = -2\sqrt{\frac{Q^2 m_Z^2}{1-x^2}} + m_Z^2 + Q^2, \tag{3.21}$$

the logarithm L becomes

$$L = \log\left(\frac{(x-1)^2}{(x+1)^2}\right). \tag{3.22}$$

Now L can be convergently expanded in powers of x (converges for $x^2 < 1$), and for the relevant m_{12}^2 ($0 \leq m_{12}^2 \leq (Q-m_Z)^2$) we have

$$0 \leq x \leq \frac{Q^2 - m_Z^2}{Q^2 + m_Z^2} < 1, \tag{3.23}$$

so L always converges when expanded in powers of x. Hence, x is the right parameter to expand in. On the other hand, an expansion in m_Z (implicitly divided by other massful parameters) does not necessarily converge, since kQm_Z/m_{12}^2 is not necessarily less than zero when m_{12}^2 is close to $(Q-m_Z)^2$ and k is of order $\mathcal{O}(1)$.

Furthermore, changing integration variable in the integral Int (3.17) from m_{12}^2 to x, simplifies the integrand, and makes Mathematica able to solve the integral exactly. We then get the corresponding decay rate for the Z-strahlung process,

$$\Gamma_Z = \frac{1}{(2\pi)^3}\frac{1}{32Q^3}\,\mathrm{Int} \;=\; \frac{1}{(2\pi)^3}\frac{1}{32Q^3}\int_{x=\frac{Q^2-m_Z^2}{Q^2+m_Z^2}}^{0} I_1(x)\frac{dm_{12}^2}{dx}\,dx, \tag{3.24}$$

and then the exact ratio $R_Z = \Gamma_Z/\Gamma_{\mathrm{tree}}$ is given by

$$\begin{aligned} R_Z =& \frac{e^2}{96\pi^2 Q^4 m_Z^2 s_{2W}^2}\Bigg(Q^6 - Q^4 m_Z^2\Big(12\left(c_{4W} - 2c_{2W}\right)\left(\mathrm{Li}_2(X) - \mathrm{Li}_2(W)\right) + M + N\Big) \\ & - 3Q^2 m_Z^4\Big(-16(Y+2)c_{2W} + 8(Y+2)c_{4W} - 4Y + 19\Big) \\ & - m_Z^6\Big(24(Y+1)c_{2W} - 12(Y+1)c_{4W} - 24Y - 29\Big)\Bigg) \end{aligned} \tag{3.25}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function, and

$$
\begin{aligned}
M =& -12c_{2W}\left(\log^2\left(\frac{m_Z^2}{Q^2}+1\right)+6\log\left(\frac{Q}{m_Z}\right)-\log^2(W)-6\right)\\
&+24\mathrm{Li}_2\left(-\frac{m_Z^2}{Q^2}\right)+24\log\left(\frac{Q^3}{Z^2}\right)\log\left(\frac{Q}{m_Z}\right)\\
&+12\left(\log\left(m_Z^2\right)+3\right)\log\left(\frac{Q}{m_Z}\right)+2\pi^2-27,\\
N =& 6c_{4W}\Bigg(\log\left(\frac{Q}{m_Z}\right)\left(\log\left(\frac{256Q^8m_Z^8}{Z^8}\right)+6\right)+\log(W)\log\left(4Q^4m_Z^2\right)\\
&+\log\left(\frac{1}{Q}\right)\log\left(\frac{256Q^4m_Z^{16}}{Z^8}\right)+\log(Z)\log\left(\frac{16Q^4m_Z^2}{Z^2}\right)\\
&+\log\left(\frac{1}{\left(\frac{Q^2}{m_Z^2}+1\right)^2}\right)\log\left(\frac{2Q^2}{m_Z^2+Q^2}\right)-6\Bigg),
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
W &= \frac{m_Z^2}{m_Z^2+Q^2},\\
X &= \frac{Q^2}{m_Z^2+Q^2},\\
Y &= \log\left(\frac{Q}{m_Z}\right),\\
Z &= m_Z^2+Q^2.
\end{aligned}
\tag{3.27}
$$

We see that the leading behaviour of R_Z in eq. (3.25) is proportional to Q^2, and equals R_{Z0} given in eq. (3.13), as it should.

We also see in figure 3.5 that factorization works excellent for the s-channel (FSR only) DM annihilation process

$$
\chi\chi \to h \to e^+e^-Z, \tag{3.28}
$$

since R_Z from eq. (3.25) equals the ratio R_Z of the process (3.28). In the numerical calculation of R_Z of the process (3.28) we used Madgraph. We found that the numerical results for R_Z only depends on the neutralino center of mass energy, and not on how much of the energy that is carried by the mass. (The results are also independent of the neutralino couplings, since these are divided away in the ratio R_Z).

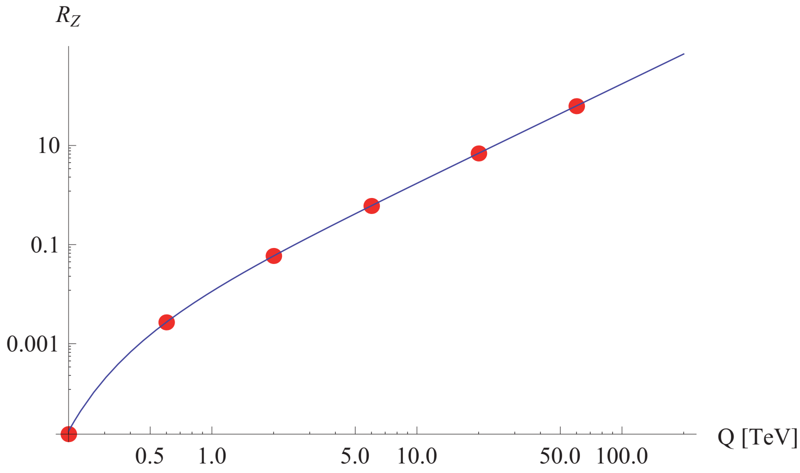


Figure 3.5: The exact R_Z, given in eq. (3.25), from the process $h \to e^+e^-$ (with Z emitted as final state radiation (FSR) only, compared with R_Z from the process $\chi\chi \to h \to e^+e^-Z$ (FSR Z-emission only), computed numerically by Madgraph (red points).

3.3.6 Comparison with R_Z from other processes

In Fig. 3.6 we see that R_Z unfortunately does not give us a good approximation of the R_Z corresponding to the selectron mediated processes

$$e^+e^- \to \chi\chi Z \tag{3.29}$$

or

$$\chi\chi \to e^+e^-Z \tag{3.30}$$

considering respectively initial and final state Z-emission only. The neutralino χ was here in Madgraph taken to be a light almost pure bino. Both right and left selectrons were included (these diagrams are dominant in the sense that they are few but as big as the size of all the other diagrams together–including all diagrams will on the other hand give us a much smaller cross section (by cancellations), since the cross sections in the MSSM goes roughly like $1/m^2$ and hence preserves unitarity.) Including only the right handed electron (which couple to the Bino only) we get the same magnitude for the cross sections (and R_Z) for this parameter point.

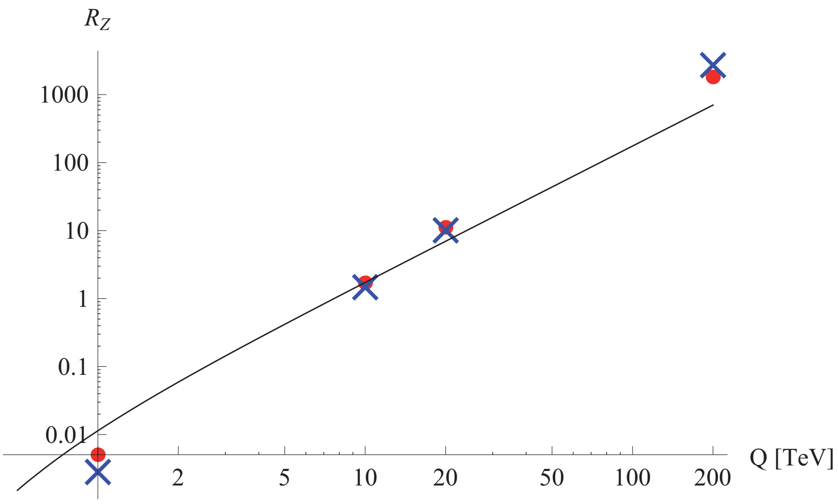


Figure 3.6: The exact ratio R_Z from the process $h \to e^+e^-$ (with Z emitted as final state radiation (FSR) only (i.e. emitted from the electrons)) compared with R_Z from the process $e^+e^- \to \chi\chi$ (marked by blue crosses), mediated by selectrons only $\tilde{e}_{L,R}$ (in t and u channels), with a light almost pure Bino as the neutralino χ, and where the Z is emitted as ISR only. Here we have disregarded all diagrams but the selectron diagrams (in Madgraph). The red dots are the R_Z's for the process $\chi\chi \to e^+e^-$, with Z emitted as final state radiation (FSR) only, other conditions the same as for the case of the "inverse" process (which was indicated by crosses.)

Chapter 4

Multi-Higgs-doublet models

4.1 The inert doublet model of dark matter

In its simplest form, the inert doublet model (IDM) of dark matter is the SM with one additional Higgs doublet Φ_2 [32, 33, 34]. Unlike the SM Higgs doublet Φ_1, the extra "inert" Higgs doublet Φ_2 does not couple directly to fermions, and has no vacuum expectation value (VEV). This is ensured by imposing a $\mathbb{Z}_2$ symmetry on the Lagrangian, where Φ_2 is odd under the symmetry, while all other fields are even,

$$\Phi_1 \to \Phi_1, \quad \Phi_2 \to -\Phi_2. \tag{4.1}$$

Hence, the inert doublet cannot couple to fermions through Yukawa couplings. When we deny the inert doublet a VEV, the inert particles cannot decay into gauge bosons or SM Higgs exclusively either, with the result that the lightest inert particle is stable, since it has no lighter particles to decay into. Since it is stable, it could be a good dark matter candidate. Moreover, the influence of the inert particles on electroweak precision tests (EWPT) makes it consistent to have a natural (i.e. with not too much fine tuning) SM Higgs mass of up to $400 - 600\,\text{GeV}$ [35]. In contrast, assuming that no new physics influences the EWPT, indicate that the SM Higgs is light, $m_h < 186\,\text{GeV}$ at 95% CL [36], with a central value considerably below the lower bound of $114.4\,\text{GeV} < m_h$ from direct searches [13]. Also DM constraints could be satisfied in the IDM model of [35], choosing masses of scalar and pseudoscalar inert Higgses around $80\,\text{GeV}$.

Nevertheless, the IDM is too restricted by the $\mathbb{Z}_2$-symmetry, to allow for CP-violation in the Higgs potential [37]. To accomodate CP-violation in the Higgs potential, [37] suggests combining the standard two-Higgs-doublet model (2HDM) with one inert Higgs doublet Φ_3 (the model is denoted IDM2).

In other words, the authors of [37] are considering a constrained three-Higgs-doublet model (3HDM) as a model for DM and CP-violation. Hence, both IDM and IDM2 are N-Higgs-doublet models (NHDM) which provides a dark matter candidate.

4.2 The approximate $SO(4)$ symmetry of the SM electroweak Lagrangian

Let the SM Higgs doublet be written

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{4.2}$$

then the global $SO(4) = SU(2) \times SU(2)$ so-called custodial symmetry [38, 39] of the Higgs potential can be made manifest by rewriting the Higgs field as a matrix (bidoublet)

$$\tilde{\Phi} = \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^{+*} & \phi^0 \end{pmatrix} = \begin{pmatrix} \eta - i\phi_3 & \phi_1 + i\phi_2 \\ -\phi_1 + i\phi_2 & \eta + i\phi_3 \end{pmatrix}. \tag{4.3}$$

Here the Higgs potential will be a function of $\mathrm{Tr}[\tilde{\Phi}^\dagger \tilde{\Phi}]$, and is hence invariant under the global transformation

$$\tilde{\Phi} \to U_L \tilde{\Phi} U_R, \tag{4.4}$$

where U_L and U_R are $SU(2)$ matrices. Moreover, U_L represents the usual gauged $SU(2)_L$ invariance. On the other hand, U_R represents an ordinary global transformation (where the gauge fields does not transform parallely with the Higgs doublet).

The electroweak Lagrangian can then be written

$$\mathcal{L} = \frac{1}{2}\left(\mathrm{Tr}[(D_\mu \tilde{\Phi})^\dagger D^\mu \tilde{\Phi}] + \mu^2 \mathrm{Tr}[\tilde{\Phi}^\dagger \tilde{\Phi}] - \lambda \mathrm{Tr}[\tilde{\Phi}^\dagger \tilde{\Phi} \tilde{\Phi}^\dagger \tilde{\Phi}]\right), \tag{4.5}$$

where the covariant derivative in the present notation is

$$D_\mu \tilde{\Phi} = \partial_\mu \tilde{\Phi} + \frac{1}{2} i g W_{i\mu} \sigma_i \tilde{\Phi} - \frac{1}{2} i g' B_\mu \tilde{\Phi} \sigma_3, \tag{4.6}$$

where σ_i are the Pauli matrices. We see that the last term breaks the $SU(2)_R$ symmetry because of the factor σ_3. However, in the limit $g' \to 0$ the whole SM Lagrangian has the full $SU(2)_L \times SU(2)_R$ symmetry, when the W fields transform as a triplet under (the gauged) $SU(2)_L$ and as a singlet under the global $SU(2)_R$ symmetry. (Eqs. (4.3) to (4.6) are taken from [40]).

4.3 The operators $\widehat{B}$ and $\widehat{C}$ in the bidoublet formulation $\tilde{\Phi}$

Consider the operators $\widehat{B}$ and $\widehat{C}$ defined in eq. (2.3) of paper V as

$$\begin{aligned}\widehat{B}_{mn} = \widehat{B}(\Phi_m, \Phi_n) &= \tfrac{1}{2}(\Phi_m^\dagger \Phi_n + \Phi_n^\dagger \Phi_m) \\ \widehat{C}_{mn} = \widehat{C}(\Phi_m, \Phi_n) &= -\tfrac{i}{2}(\Phi_m^\dagger \Phi_n - \Phi_n^\dagger \Phi_m),\end{aligned} \tag{4.7}$$

where Φ_i refers to a doublet of the form (4.2) where every scalar field has an additional index i. (See section 2 of paper V for a full definition of the NHDM Lagrangian.)

In the same manner as in the last section, we will now show that the operator $\widehat{C}$ does not share the $SO(4)$ symmetry held by the rest of the NHDM potential: Let $\tilde{\Phi}_i$ refer to a bidoublet of the form (4.3), where every scalar field has an additional index i. A simple calculation shows that

$$\widehat{B}(\Phi_m, \Phi_n) = \frac{1}{2}\mathrm{Tr}(\widehat{B}(\tilde{\Phi}_m, \tilde{\Phi}_n)), \tag{4.8}$$

while

$$\widehat{C}(\Phi_m, \Phi_n) = -\frac{1}{2}\mathrm{Tr}(\widehat{C}(\tilde{\Phi}_m, \tilde{\Phi}_n)\sigma_3) = -\frac{1}{2}\mathrm{Tr}(\sigma_3\widehat{C}(\tilde{\Phi}_m, \tilde{\Phi}_n)). \tag{4.9}$$

The latter confirms that the operator $\widehat{C}$ does not have the $SU(2)_R$ (and hence neither the $SO(4)$-) symmetry, in contrast to $\widehat{B}$, since the presence of the factor σ_3 hinders us from utilizing the cyclic property of the trace. We also derive from the equality

$$\begin{aligned}\widehat{C}(\Phi_m, \Phi_n)\widehat{C}(\Phi_{m'}, \Phi_{n'}) &= -\frac{1}{2}\mathrm{Tr}(\widehat{C}(\tilde{\Phi}_m, \tilde{\Phi}_n)\sigma_3\widehat{C}(\tilde{\Phi}_{m'}, \tilde{\Phi}_{n'})\sigma_3) \\ &= -\frac{1}{2}\mathrm{Tr}(\sigma_3\widehat{C}(\tilde{\Phi}_m, \tilde{\Phi}_n)\sigma_3\widehat{C}(\tilde{\Phi}_{m'}, \tilde{\Phi}_{n'})),\end{aligned} \tag{4.10}$$

that operators of the type $\widehat{C}^2$ in the same way do not share the $SO(4)$ symmetry of the rest of the NHDM potential (which is build up by the operators $\widehat{B}$). This is also shown in section 2.2 of paper V, but in a different manner.

4.4 Symmetries of the kinetic terms

We now turn to the (global) symmetries of the kinetic terms of the NHDM Lagrangian,

$$K = \sum_{n=1}^{N} [(\partial^\mu + G^\mu)\Phi_n(x)]^\dagger [(\partial_\mu + G_\mu)\Phi_n(x)], \tag{4.11}$$

with

$$G^\mu = igT_i W_i^\mu + ig' Y B^\mu. \tag{4.12}$$

Let K_i denote the terms of the i'th order in the gauge fields.

4.4.1 The approximate $SO(4)$ symmetry $\mathcal{O}$ of the kinetic terms

We will now explore the possibility of discrete or continous symmetries beyond the global $U(2)$ gauge symmetry. We do this by writing the electroweak NHDM Lagrangian on real form, i.e. we write the complex Higgs doublets as real quadruplets. We will show the kinetic terms does not allow for a bigger symmetry than the usual $U(2) \times C$, where $C = \{I, \mathcal{C}\}$, $\mathcal{C}$ being the charge conjugation operator.

Consider the kinetic terms not involving gauge fields, K_0, of the Higgs Lagrangian

$$K_0 = \sum_{n=1}^{N} \partial^\mu \Phi_n^T \partial_\mu \Phi_n \tag{4.13}$$

where $\Phi_n = \Psi_n + i\Theta_n$ is written on real form,

$$\Phi_n = \begin{pmatrix} \Psi_n \\ \Theta_n, \end{pmatrix}. \tag{4.14}$$

As in section 2.2 of paper V, we regard Φ_n (4.14) as a real $2k$-plet. Hence we for generality are considering a gauge group $SU(k) \times U(1)$. We see that we can assign these terms a $O(2k)$ symmetry by

$$\Phi_n \to O\Phi_n, \tag{4.15}$$

with $O \in O(2k)$ and Φ_m given by eq. (4.14). The terms K_0 are invariant under the transformation (4.15) since $O^T O = I$ for $O \in O(2k)$.

Consider the transformation of the kinetic terms linear in the gauge fields, K_1, under the map ρ defined in appendix B of paper V. We can then write[1]

$$\begin{aligned} K_1 &= \sum_{n=1}^{N} \partial^\mu(\Phi_n)_k^\dagger G_\mu(\Phi_n)_k + (\Phi_n)_k^\dagger G^{\mu\dagger}\partial_\mu(\Phi_n)_k \\ &= \sum_{n=1}^{N} \rho(\partial^\mu(\Phi_n)_k^\dagger)\rho(G_\mu)\rho((\Phi_n)_k) + \rho((\Phi_n)_k^\dagger)\rho(G^{\mu\dagger})\rho(\partial_\mu(\Phi_n)_k) \\ &= \sum_{n=1}^{N} \partial^\mu\Phi_n^T\mathcal{T}_\mu\Phi_n + \Phi_n^T(-\mathcal{T}^\mu)\partial_\mu\Phi_n \end{aligned} \tag{4.16}$$

where the subscript k in $(\Phi_n)_k$ indicates this is the usual complex Higgs k-plet (in the case $k = 2$ the usual complex Higgs doublet), while Φ_n is the $2k$ dimensional real vector (4.14), and where we also use eqs. (B.2), (B.3) and (B.6).

We then see that the kinetic terms K_1 are apparently invariant under the $O(2k)$ transformation (4.15) if we let the matrix $\mathcal{T}^\mu$ transform

$$\mathcal{T}^\mu \to O\mathcal{T}^\mu O^T \tag{4.17}$$

simultaneously with (4.15). We denote the combined transformations (4.15) and (4.17) as $\mathcal{O}$. The only problem is that the transformation (4.17) might not be well-defined for choices of O beyond the global gauge group $U(2)$. The transformation is only well-defined when it induces consistent transformations of each of the fields, i.e. it is well-defined when it makes each of the fields transform in the same manner everywhere in the matrix $\mathcal{T}$.

Third, we consider the kinetic terms quadratic in the gauge fields,

$$\begin{aligned} K_2 &= \sum_{n=1}^{N} (\Phi_n)_k^\dagger G^{\mu\dagger} G_\mu(\Phi_n)_k \\ &= \sum_{n=1}^{N} \rho((\Phi_n)_k^\dagger)\rho(G^{\mu\dagger})\rho(G_\mu)\rho((\Phi_n)_k) \\ &= -\sum_{n=1}^{N} \Phi_n^T\mathcal{T}^2\Phi_n, \end{aligned} \tag{4.18}$$

which obviously are invariant under the $O(2k)$ symmetry given by eqs. (4.15) and (4.17), when the transformation is well-defined.

[1] Disregarding so-called Schwinger terms—here terms proportional to $i[\partial_\mu\phi(x), \phi(x)]$ for a scalar field ϕ—or, alternatively, reasoning classically.

4.4.2 The gauge field Lagrangian

Finally we want to show that the transformations (4.15) and (4.17) are symmetries also of the kinetic part of the gauge field Lagrangian.

Consider the kinetic terms for the gauge fields formulated as the trace of the commutator of two covariant derivatives (see eq. (2.2) in paper V), on complex form,

$$\mathcal{L}_{GB} = -\frac{1}{2}\mathrm{Tr}\left(\left(\frac{i}{g}[D_\mu, D_\nu]\right)^2\right)\Big|_{g'Y\to g}. \tag{4.19}$$

A general relation is

$$\mathrm{Tr}(\rho(X)) = 2\mathrm{Re}(\mathrm{Tr}(X)) = 2\mathrm{Tr}(X), \tag{4.20}$$

where ρ was defined in eq. (B.1) in paper V, and the last equality valid when the trace is real, as it is in eq. (4.19).

Then

$$\begin{aligned}\mathcal{L}_{GB} &= -\frac{1}{4}\mathrm{Tr}\left(\rho\left(\left(\frac{i}{g}[D_\mu, D_\nu]\right)^2\right)\right)\Big|_{g'Y\to g}\\ &= \frac{1}{4g^2}\mathrm{Tr}\left([\rho(D_\mu), \rho(D_\nu)]^2\right)\Big|_{g'Y\to g}\\ &= \frac{1}{4g^2}\mathrm{Tr}\left([\mathcal{T}_\mu, \mathcal{T}_\nu]^2\right)\Big|_{g'Y\to g},\end{aligned} \tag{4.21}$$

since ρ preserves matrix multiplication and addition, and since the partial derivatives commute. Eq. (4.21) is invariant under $\mathcal{O}$, cf. (4.17), by the cyclic property of the trace.

4.5 The adjoint representation of $O(4)$

To decide which transformations beyond $U(2)$ makes the transformation (4.17) well-defined, we have to consider the adjoint representation of $O(4)$. We know the standard model gauge fields transform as the adjont representation of $U(2)$, which is, written on real form, a sub-representation of the adjoint representation of $O(4)$. Hopefully, we at least could find some matrix $d \in O(4)$ such that (4.17) is well-defined for $O = d$, and hence d would generate a discrete symmetry of the kinetic terms of the electroweak Lagrangian. Furthermore, if we could find such a $d \in O(4) - P(2, \mathbb{R})$ (where $P(2, \mathbb{R})$ is the symmetry group of the operators of the type $\widehat{C}^2$ defined in eq. (2.17) in paper

V) we could impose this discrete symmetry on the electroweak NHDM Lagrangian and hence get rid of terms proportional to $O(4)$-violating operators $\widehat{C}$ and $\widehat{C}^2$. Then we, because of the discrete symmetry, would not have to worry about terms of these kinds, even though having been set to zero in the first place, coming back as counterterms in the renormalization procedure. If they show up as counterterms, although they in the beginning are set to zero, they will violate the $O(3)$ symmetry between the charged and CP-odd sectors of the potential, in addition to the kinetic terms proportional to the parameter g'.

Generally, the adjoint representation of a Lie group G is a homomorphism

$$Ad : G \to GL(\mathfrak{g}, NR), \tag{4.22}$$

where $\mathfrak{g}$ is the Lie algebra of G and

$$Ad(g)(X) = gXg^{-1}, \tag{4.23}$$

for $g \in G$ and $X \in \mathfrak{g}$. We see that for a infinitesimal group element $g = I + \epsilon Y$ with $Y \in \mathfrak{g}$, $Ad(g)(X) = X + \epsilon[Y, X] \in \mathfrak{g}$, so Ad is a mapping on $\mathfrak{g}$. This is also the case for non-infinitesimal group elements g, since Ad is the derivative at the identity of the map $\Psi : G \to Aut(G)$ where $\Psi(g)(h) = ghg^{-1}$: Since $\Psi(g)$ already is linear, its linearization $Ad(g)$ is of the same form as $\Psi(g)$.

Moreover, denote the image of the adjoint representation Ad_G, where $Ad_G \subset GL(\mathfrak{g}, NR)$. If G is connected, the kernel of the adjoint representation coincides with the kernel of Ψ which is just the center of G, $Z(G)$,

$$Z(G) = \{g \in G | \forall x \in G(xg = gx)\}. \tag{4.24}$$

Therefore the adjoint representation of a connected Lie group G is faithful if and only if G is centerless. More generally, if G is not connected, then the kernel of the adjoint map is the centralizer of the identity component G_0 of G, $C_G(G_0)$,

$$C_G(G_0) = \{g \in G | \forall x \in G_0(xg = gx)\}. \tag{4.25}$$

By the first isomorphism theorem we have

$$Ad_G \cong G/C_G(G_0), \tag{4.26}$$

and if G is connected, $G = G_0$ and $G/C_G(G) = G/Z(G)$.

In the case of $SO(4)$, which is connected, $Z(SO(4)) = \{\pm I\}$. Hence

$$Ad_{SO(4)} = SO(4)/\{\pm I\}. \tag{4.27}$$

Now the group $SO(4)$ is not simple, in contrast to $SO(N)$ for $N = 3$ and for $N \geq 5$, and can modulo its center $\{\pm I\}$ be written as a direct product [41]

$$SO(4)/\{\pm I\} \cong SO(3) \times SO(3). \tag{4.28}$$

Hence, we have that

$$Ad_{SO(4)} \cong SO(3) \times SO(3), \tag{4.29}$$

which we also will see explicitly below.

4.5.1 The effect of the adjoint action

We will now explicitly consider the effect of the adjoint action of $O(4)$ on its Lie algebra $so(4)$ to see if it permits any symmetries beyond the (global) $U(2) = SU(2)_L \times U(1)_Y$ symmetry of the SM. We regard the real variant of $U(2)$ embedded in $SO(4)$ by the map ρ, and try to see if the adjoint action (4.17) can be well-defined for any matrices $O \in O(4) - U(2)$ when we restrict us to the sub-Lie algebra $u(2) \subset so(4)$, i.e. we do not want any additional gauge fields, only the W and the B fields from the SM.

Let $\{X_1, \ldots, X_n\}$ be a certain basis for the n-dimensional Lie algebra $\mathfrak{g}$, and let G be the Lie group generated through exponentiation of this Lie-algebra. Then we know that the set

$$U = \{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_n X_n} \mid \vec{t} \in \mathbb{R}^n\} \subseteq G, \tag{4.30}$$

with a possible equality.

Now regard the basis of $so(4)$ given by

$$\begin{aligned} &\{X_i\} = \\ &\{\tfrac{1}{2}(J_2 + J_5), \tfrac{1}{2}(J_1 + J_4), \tfrac{1}{2}(J_6 - J_3), \tfrac{1}{2}(J_6 + J_3), \tfrac{1}{2}(J_2 - J_5), \tfrac{1}{2}(J_1 - J_4)\}, \end{aligned} \tag{4.31}$$

where the first four are the real forms of $(i/2)\vec{\sigma}$ and $(i/2)I_2$, i.e. the generators of the $SU(2)_L \times U(1)_Y$ gauge group. Now we will exponentiate the generators X_i, and see which effect each of them has on the Lie algebra $so(4)$ [each of them will yield a linear transformation on so(4), according to (4.22)].

Let $P_{ij}(\theta)$ denote the 6×6 matrix with elements

$$\begin{aligned} p_{ii} &= p_{jj} = \cos(\theta) \\ p_{ij} &= -p_{ji} = \sin(\theta) \\ p_{kk} &= 1 \quad k \neq i, j, \end{aligned} \tag{4.32}$$

with all other elements equaling zero. For instance, $P_{24}(\theta)$ is then given by

$$P_{24}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.33}$$

The effects of each $e^{t_i X_i}$ (no sum) on $so(4)$ by the adjoint action is then

$$\begin{aligned} E_1 = P_{23}(t_1), E_2 = P_{31}(t_2), E_3 = P_{12}(t_3) \\ E_4 = P_{56}(t_4), E_5 = P_{64}(t_5), E_6 = P_{45}(t_6), \end{aligned} \tag{4.34}$$

and where the effect of the reflection

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.35}$$

is

$$E_r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.36}$$

Here $so(4)$ is expressed by the basis (4.31), and also the X_i's below refer to this basis. For instance,

$$e^{t_1 X_1} \sum_{i=1}^{6} s_i X_i e^{-t_1 X_1} = e^{t_1 \frac{1}{2}(J_2+J_5)} \sum_{i=1}^{6} s_i X_i e^{-t_1 \frac{1}{2}(J_2+J_5)} = E_1 \vec{s}, \tag{4.37}$$

where $\vec{s}\vec{X} = \sum_{i=1}^{6} s_i X_i$.

The effect of the general element of U from (4.30) is then given by

$$\vec{s'}\vec{X} = U\vec{s}\vec{X}U^{-1} \Leftrightarrow \vec{s'} = E\vec{s} \tag{4.38}$$

where

$$E = E_1 E_2 \cdots E_6. \tag{4.39}$$

Now a calculation shows

$$E = \begin{pmatrix} A(t_1, t_2, t_3) & 0_3 \\ 0_3 & A(t_4, t_5, t_6) \end{pmatrix}, \tag{4.40}$$

where

$$A(x, y, z) = \begin{pmatrix} c_y c_z & c_y s_z & -s_y \\ c_z s_x s_y - c_x s_z & c_x c_z + s_x s_y s_z & c_y s_x \\ c_x c_z s_y + s_x s_z & c_x s_y s_z - c_z s_x & c_x c_y \end{pmatrix}, \tag{4.41}$$

which is just the general element of $SO(3)$, written in the so-called "pitch-roll-yaw" convention. Hence the effect E of the general element U is just $SO(3) \times SO(3)$, since eq. (4.40) yields two independent copies of $SO(3)$. Since we already know that $Ad_{SO(4)} = SO(3) \times SO(3)$, we know that the parametrization U covers the whole of $SO(4)$, at least modulo multiplication by $\pm I$. What is important here is that we in eq. (4.40) have parametrized the entire $Ad_{SO(4)}$.

Since we want to set the fields not corresponding to any SM gauge boson, i.e. the parameters s_5, s_5', s_6 and s_6', to zero (we want them to equal zero both before and after application of the adjoint action), we have to demand

$$E_{i1} = E_{i2} = E_{i3} = E_{i4} = 0, \; i = 5, 6. \tag{4.42}$$

The only non-trivial constraints $E_{54} = E_{64} = 0$ then infer, conferring eqs. (4.40) and (4.41), that

$$\begin{aligned} \sin(t_5) &= \tan(t_6)/\tan(t_4), \\ \sin(t_5) &= -\tan(t_6)\tan(t_4), \end{aligned} \tag{4.43}$$

and hence, since the two expressions for $\sin(t_5)$ have opposite sign,

$$\sin(t_5) = 0, \tag{4.44}$$

and then also

$$\sin(t_6) = 0, \tag{4.45}$$

with t_4 still being arbitrary. So we have

$$t_5, t_6 = n\pi, \quad n \in \mathbb{Z}, \tag{4.46}$$

not necessarily with the same n for both t_5 and t_6. Calculating the most general $SO(4)$ matrix

$$O = e^{t_1 X_1} e^{t_2 X_2} e^{t_3 X_3} e^{t_4 X_4} e^{t_5 X_5} e^{t_6 X_6}, \tag{4.47}$$

we find that it depends on angles $t_i/2$ for all i. Then we have to regard the cases where

$$t_5, t_6 \in \{0, \pi, 2\pi, 3\pi\}. \tag{4.48}$$

We then find that

$$\cos(t_5)\cos(t_6) = 1 \Rightarrow O \in SO(4) \cap SP(2,\mathbb{R}) = U(2) \tag{4.49}$$

but when

$$\cos(t_5)\cos(t_6) = -1 \Rightarrow O \in SO(4) \cap P^-(2,\mathbb{R}), \tag{4.50}$$

by comparing the form of O with respectively the forms (A.2) and (A.4) given in Appendix A. (Because of its size, we do not write out the matrix O explicitly here, and we easily do the investigation of the forms of O by Mathematica.) I.e. we get symmetries outside the gauge group $U(2)$ whenever we choose the field B_μ to change sign, $B_\mu \to -B_\mu$, as in eq. (4.50). Moreover, the symmetries (4.50) are exactly the ones

$$O \in U(2)\mathcal{C} \tag{4.51}$$

since we know that charge conjugation is a symmetry of the kinetic terms, and since $U(2)\mathcal{C} = (SO(4) \cap SP(2,\mathbb{R}))\mathcal{C} = SO(4) \cap (SP(2,\mathbb{R})\mathcal{C}) = SO(4) \cap P^-(2,\mathbb{R})$, where we defined $\mathcal{C}$ as

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.52}$$

Still, we find no well-defined $O(4)$ transformation beyond the symmetry group $P(2,\mathbb{R})$ of the operator $\widehat{C}^2$, so there is no discrete symmetry that we can impose on the electroweak Lagrangian to exclude the terms of the type $\widehat{C}^2$.

On the other hand, if we also set $g' = 0$ (i.e. $s_4 = 0$), then we see from eq. (4.40) that all symmetries are well-defined, and hence the whole $SO(4)$ is a symmetry group of the SM electroweak Lagrangian. This is the custodial symmetry from section 4.2, demonstrated in an alternative manner.

We also want to consider the adjoint action of elements of $O^-(4)$, the orthogonal matrices with determinant -1. The general effect under the adjoin representation of an element of $O^-(4)$ can be written[2]

$$E^- = E_r E_1 E_2 \cdots E_6, \tag{4.53}$$

[2] $O^-(n) = SO(n)R = RSO(n)$ for any $R \in O^-(n)$: We have $RSO(n) \subset O^-(n)$ since $\det(R)\det(S) = -1$ for $S \in SO(n)$, and $O^-(n) \subset RSO(n)$ since for $O \in O^-(n)$, $\det(R^T O) = 1$, hence $R^T O \in SO(n)$, and then $O = R(R^T O) \in RSO(n)$.

cf. (4.39), where E_r is given in eq. (4.36), and we calculate it to be of the form

$$E^- = \begin{pmatrix} 0_3 & D(t_4,t_5,t_6) \\ C(t_1,t_2,t_3) & 0_3 \end{pmatrix}, \tag{4.54}$$

with

$$D = \begin{pmatrix} c_{t_6}s_{t_4}s_{t_5} - c_{t_4}s_{t_6} & c_{t_4}c_{t_6} + s_{t_4}s_{t_5}s_{t_6} & c_{t_5}s_{t_4} \\ c_{t_4}c_{t_6}s_{t_5} + s_{t_4}s_{t_6} & c_{t_4}s_{t_5}s_{t_6} - c_{t_6}s_{t_4} & c_{t_4}c_{t_5} \\ c_{t_5}c_{t_6} & c_{t_5}s_{t_6} & -s_{t_5} \end{pmatrix} \tag{4.55}$$

and

$$C = \begin{pmatrix} c_{t_1}c_{t_3}s_{t_2} + s_{t_1}s_{t_3} & c_{t_1}s_{t_2}s_{t_3} - c_{t_3}s_{t_1} & c_{t_1}c_{t_2} \\ c_{t_2}c_{t_3} & c_{t_2}s_{t_3} & -s_{t_2} \\ c_{t_3}s_{t_1}s_{t_2} - c_{t_1}s_{t_3} & c_{t_1}c_{t_3} + s_{t_1}s_{t_2}s_{t_3} & c_{t_2}s_{t_1}, \end{pmatrix} \tag{4.56}$$

but then the constraints (4.42) are inconsistent: $E^-_{53} = \sin(t_2) = 0$ give $t_2 = k_2\pi$, $E^-_{63} = \cos(t_2)\sin(t_1) = 0$ then gives $\sin(t_1) = 0$, i.e. $t_1 = k_1\pi$, and $E^-_{52} = \cos(t_2)\sin(t_3) = 0$ give $\sin(t_3) = 0$, i.e. $t_3 = k_3\pi$, but then $E^-_{53} = \cos(t_2)\cos(t_3) = \pm 1 \neq 0$. Hence $O^-(4)$ does not provide any new symmetries of the kinetic terms of the electroweak Lagrangian.

Appendix A

The form of some $O(2k)$ matrices

Let $S \in O(2k)$. Then the condition $S^T \mathcal{J} S = \mathcal{J}$ (i.e. $S \in Sp(k, \mathbb{R})$ can be written

$$\begin{aligned} \mathcal{J} S &= (S^T)^{-1} \mathcal{J}, \\ \mathcal{J} S &= S \mathcal{J}, \end{aligned} \tag{A.1}$$

which forces the solutions of eq. (A.1) to be exactly the matrices of the form

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{A.2}$$

for arbitrary $k \times k$ matrices A, B.

Again, let $S \in O(2k)$. Still, regarding finite (i.e. not infinitesimal) solutions of the equation the condition $S^T \mathcal{J} S = -\mathcal{J}$ (which is the definition of $P^-(k, \mathbb{R})$), can be written

$$\mathcal{J} S + S \mathcal{J} = 0_{2k}, \tag{A.3}$$

which forces the matrices S to be exactly the ones of the form

$$S = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \tag{A.4}$$

for arbitrary $k \times k$ matrices A, B.

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Symmetry and Mass Degeneration in Multi-Higgs-Doublet Models

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Abstract: We investigate possible symmetry properties of the scalar sector of Multi-Higgs-Doublet Models, and, to some extent, the generalization of such models to gauge groups other than $SU(2)_L \times U(1)_Y$. In models where the $\mathcal{C}$ (charge conjugation) violating operator $\widehat{C}$ is not present, the scalar potential is invariant under a group larger than the gauge group, $O(4)$ when the Higgs fields are doublets. If the Higgs fields develop aligned vacuum expectation values, this symmetry will break to an $O(3)$ subgroup, which in general is further broken by loop corrections involving the gauge bosons. Assuming such corrections are small, the physical properties of the Higgs sector will approximately organize into representations of $SO(3)$. If the vacuum expectation values of the Higgs fields are aligned in the direction of the $\mathcal{C}$ even fields, the mass spectra of the charged and $\mathcal{C}$ odd sectors will be degenerate. Moreover, if the Higgs fields develop a pair of non-aligned vacuum expectations values, so that the charge conjugation symmetry is spontaneously broken (but not the $U(1)$ electromagnetic gauge invariance), a pair of light charged Higgs bosons should appear.

Keywords: Quantum field theory, Gauge field theories, Symmetries.

Contents

1. Introduction

Future discoveries at the LHC may reveal a sector of scalar particles that is much richer than that of the Standard Model (SM). Some of the scalars may be responsible for generating the masses of fermions and the electroweak bosons [1], whereas others may be responsible for the dark matter [2, 3, 4, 5, 6, 7, 8].

It is natural to classify such scalars according to their properties under the $SU(2)$ associated with the electroweak sector of the Standard Model. In order to be compatible with electroweak precision data, one usually considers only $SU(2)$ doublets and singlets. Even these representations are severely constrained by the data [9, 10, 11, 12].

Going beyond one or two doublets [13, 14], one immediately has to face models having a large number of parameters. The structure of such potentials has been studied in [15].

Different doublets could be distinguished via their couplings to fermion fields. This idea is exploited in the so-called Model II version of the two-Higgs-doublet model (2HDM) [13], where one doublet couples to up-type fermions, and the other couples to down-type fermions. Another version of this idea is the one considered in Ref. [7], where each fermion or each family has its own Higgs field.

We shall here consider instead the case when the different doublets can not be distinguished (since we are not considering couplings to the fermions). Thus, we shall here study the symmetries of models with N doublets—it turns out that by setting a certain $SO(4)$-violating parameter $\lambda^{(3)}$ to zero and assuming vacuum alignment, the spectrum simplifies considerably. In particular, a certain "custodial" $SO(3)$ symmetry [16] leads to a degeneracy between the mass matrix of the $\mathcal{C}$ odd (or equivalently, CP odd) and the charged Higgs bosons.

This possibility of a symmetry group of the scalar polynomial which is larger than required by gauge invariance was pointed out by Weinberg [17] many years ago. In the theories considered the extra symmetry was assumed to be a symmetry of any quartic (i.e. renormalizable) potential of the scalar sector.

The Standard Model with its single Higgs doublet is an example of a theory were the most general scalar potential has an extra $O(4)$ symmetry not contained in the $SU(2)\times U(1)$ gauge symmetry. On the other hand, an extension of the SM with an extra Higgs doublet (as required if we want to introduce supersymmetry) or more, and with $\mathcal{C}$ symmetry, induces a most general scalar potential where the extra $O(4)$ symmetry is broken (before spontaneous symmetry breaking). In our notation, this extra symmetry is broken by the parameter(s) $\lambda^{(3)}$, cf. eq. (2.8). Standard renormalizability instructs us to include into the Lagrangian, *all* terms allowed by the $SU(2) \times U(1)$ symmetry, hence it may be inconsistent to leave out terms proportional to $\lambda^{(3)}$. The presence of such parameters will in general lead to an order $\lambda^{(3)}$ tree-level breaking of the additional symmetry. We don't think this in principle is any different from having the symmetry broken by loop corrections. It becomes in any case a question of the magnitude of the perturbation.

However, for renormalization it is only necessary to study the theory with unbroken symmetry, since the renormalization is not changed when $v \neq 0$ [26]. Hence, only quadratic counterterms have to be considered in the case of mass relations. Thus, in this case counterterms of the type $\lambda^{(3)}$ will be irrelevant.

2. The NHDM potential and Lagrangian

We define the N-Higgs-doublet model, abbreviated NHDM, to be a system of N two-component complex scalar fields $\Phi_1, \Phi_2, \ldots, \Phi_N$, each with the same transformation property under $SU(2)_L \times U(1)_Y$ as the Higgs field of the Standard Model, and with dynamics defined by the Lagrangian density

$$\mathcal{L}(x) = \sum_m [D^\mu \Phi_m(x)]^\dagger [D_\mu \Phi_m(x)] - V(\Phi_1, \Phi_2, \ldots, \Phi_N), \tag{2.1}$$

where $V(\Phi_1, \Phi_2, \ldots, \Phi_N)$ is a potential that—in its most general form—is given by (2.5) below. The *covariant derivative* D^μ is defined as

$$D^\mu = \partial^\mu + igT_i W_i^\mu + ig' Y B^\mu, \tag{2.2}$$

where W_i^μ and B^μ are the $SU(2)_L$ and $U(1)_Y$ gauge fields, respectively, and $T_i = \frac{1}{2}\sigma_i$ are the generators of $SU(2)$ with $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices. Thus, our Higgs fields are labeled by two indices: The *row index* running from 1 to N is often written out explicitly as above, and an often hidden *group index* acted on by the gauge group. The latter are acted on by the matrices T_i in (2.2) (whose indices are also hidden). When written explicitly we shall use Greek letters from the beginning of the alphabet.

To write the most general gauge-invariant potential in a renormalizable NHDM in a compact way, we introduce a set of linearly independent[1] hermitian operators invariant under local $SU(2)_L \times U(1)_Y$ transformations (this is a generalization of the approach for the 2HDM in [18]):

$$\begin{aligned} \widehat{A}_m &= \Phi_m^\dagger \Phi_m, \\ \widehat{B}_{mn} &= \tfrac{1}{2}(\Phi_m^\dagger \Phi_n + \Phi_n^\dagger \Phi_m) = \mathrm{Re}(\Phi_m^\dagger \Phi_n) \equiv \widehat{B}_a, \\ \widehat{C}_{mn} &= -\tfrac{i}{2}(\Phi_m^\dagger \Phi_n - \Phi_n^\dagger \Phi_m) = \mathrm{Im}(\Phi_m^\dagger \Phi_n) \equiv \widehat{C}_a. \end{aligned} \tag{2.3}$$

Due to (anti-)symmetry under interchange of m and n we may impose the restriction that $1 \le m < n \le N$, and introduce indices $a, b, \ldots$ labeling such pairs. An explicit invertible encoding is

$$1 \le a = a(m,n) = m + \frac{1}{2}(n-2)(n-1) \le \frac{1}{2}N(N-1) \equiv \mathcal{N}. \tag{2.4}$$

We let $m(a), n(a)$ denote the inverse of this encoding. We will use the summation convention that repeated indices from the beginning of the alphabet are summed from 1 to $\mathcal{N}$, and repeated indices from the middle of the alphabet are summed from 1 to N. The most general potential in the NHDM thus becomes a linear combination of all different quadratic and quartic factors in the Φ_m (and $\Phi_m^\dagger$) which can be formed from $\widehat{A}_m$, $\widehat{B}_a$ and $\widehat{C}_a$:[2]

$$\begin{aligned} V_{\mathrm{g}}(\Phi_1, \ldots, \Phi_N) = {} & \mu_m^{(1)} \widehat{A}_m + \mu_a^{(2)} \widehat{B}_a + \mu_a^{(3)} \widehat{C}_a + \lambda_{mn}^{(1)} \widehat{A}_m \widehat{A}_n + \lambda_{ab}^{(2)} \widehat{B}_a \widehat{B}_b \\ & + \lambda_{ab}^{(3)} \widehat{C}_a \widehat{C}_b + \lambda_{ma}^{(4)} \widehat{A}_m \widehat{B}_a + \lambda_{ma}^{(5)} \widehat{A}_m \widehat{C}_a + \lambda_{ab}^{(6)} \widehat{B}_a \widehat{C}_b, \end{aligned} \tag{2.5}$$

where the "g" in V_{g} denotes "general". To avoid double counting we introduce the restriction $m \le n$ in the term involving $\lambda_{mn}^{(1)}$, and the restriction $a \le b$ in the terms involving $\lambda_{ab}^{(2)}$, $\lambda_{ab}^{(3)}$ and $\lambda_{ab}^{(6)}$. We will not consider terms of degree higher than four, because these would destroy the renormalizability of the model [22]. From the hermiticity of the potential V_{g} all parameters μ and λ in the expansion (2.5) must be real. Thus the number of free real parameters in (2.5) is

$$N_{\mathrm{tot}} = N + 2\mathcal{N} + \tfrac{1}{2}N(N+1) + \mathcal{N}(\mathcal{N}+1) + 2N\mathcal{N} + \mathcal{N}^2 = \tfrac{1}{2}N^2(N^2+3), \tag{2.6}$$

which for $N = 1$ gives us the 2 parameters of the Standard Model (μ^2 and λ). $N = 2$ gives us the usual 14 parameters for the 2HDM. There are 54 parameters for $N = 3$ and 152 parameters for $N = 4$.

This counting does not take into account the fact that we may make $SU(N)$ row transformations on the fields Φ_m to eliminate some terms in (2.5). One possible choice is

[1] There are no linear relations between the operators in (2.3). However, they are algebraically dependent when $N \ge 3$, being restricted by $(N-2)^2$ polynomial equations of 8'th order in the fields.

[2] Since $\left(\Phi_k^\dagger \sigma^j \Phi_\ell\right)\left(\Phi_m^\dagger \sigma^j \Phi_n\right) = -\left(\Phi_k^\dagger \Phi_\ell\right)\left(\Phi_m^\dagger \Phi_n\right) + 2\left(\Phi_k^\dagger \Phi_n\right)\left(\Phi_m^\dagger \Phi_\ell\right)$, other quartic invariants may be expressed by those chosen.

to transform the quadratic terms into a diagonal form, i.e. so that $\mu_a^{(2)} = \mu_a^{(3)} = 0$. This in general leaves a matrix of $N-1$ independent diagonal phase transformations (such that the determinant is unity). We may for instance use it to transform all $\lambda_{1a}^{(5)}$ with $n(a) = m(a)+1$ to zero. This reduces the number of parameters by N^2-1, i.e. to $N'_{\rm tot} = \frac{1}{2}\left(N^4+N^2+2\right)$, yielding 11 parameters for $N=2$ (in agreement with Barroso *et. al.* [14]), 46 parameters for $N=3$, and 137 parameters for $N=4$.

2.1 The most general $\mathcal{C}$-invariant NHDM-potential

The charge conjugation operator $\mathcal{C}$ is a linear operator which leaves complex constants unaltered, but maps fields onto their hermitian conjugate; $\mathcal{C}(z\Phi_m) = z\Phi_m^\dagger$, where z is a complex number.[3] Then $\mathcal{C}(\widehat{C}_a) = -\widehat{C}_a$, in contrast to $\mathcal{C}(\widehat{A}_m) = \widehat{A}_m$, and $\mathcal{C}(\widehat{B}_a) = \widehat{B}_a$. We obtain a $\mathcal{C}$-invariant potential by leaving out all terms which are odd in $\widehat{C}_a$, i.e., terms involving $\mu_a^{(3)}$, $\lambda_{ma}^{(5)}$, and $\lambda_{ab}^{(6)}$. There are $\mathcal{N} + N\mathcal{N} + \mathcal{N}^2 = \frac{1}{4}N(N-1)(N^2+N+2)$ such terms, leaving

$$N_{\mathcal{C}} = \frac{1}{4}N(N^3+5N+2) \tag{2.7}$$

free parameters for the general renormalizable $\mathcal{C}$-invariant potential,

$$\begin{aligned} V_{\mathcal{C}}(\Phi_1,\ldots,\Phi_N) = {}& \mu_m^{(1)}\widehat{A}_m + \mu_a^{(2)}\widehat{B}_a + \lambda_{mn}^{(1)}\widehat{A}_m\widehat{A}_n + \lambda_{ab}^{(2)}\widehat{B}_a\widehat{B}_b \\ & + \lambda_{ab}^{(3)}\widehat{C}_a\widehat{C}_b + \lambda_{ma}^{(4)}\widehat{A}_m\widehat{B}_a. \end{aligned} \tag{2.8}$$

For $N=1$ we get the usual 2 parameters of the standard model, the Higgs potential of which is automatically $\mathcal{C}$-invariant. For $N=2$ we get the usual (see e.g., [18]) 10 parameters. For $N=3$ we get 33 parameters, and for $N=4$ we get 86 parameters.

This counting does not take into account that we may make $O(N)$ transformations on the row of fields Φ_m to eliminate some terms in (2.8). A natural choice is to transform the quadratic terms to diagonal form, i.e. so that $\mu_a^{(2)} = 0$. This reduces the number of parameters by $\frac{1}{2}N(N-1)$, i.e. to $N'_{\mathcal{C}} = \frac{1}{4}N(N^3+3N+4)$. This gives 9 parameters for $N=2$ (in agreement with [14]), 30 parameters for $N=3$, and 80 parameters for $N=4$.

The difference $N_{\rm phases} = N'_{\rm tot} - N'_{\mathcal{C}} = \frac{1}{4}N^2(N^2-1)+1-N$ counts the number of genuine $\mathcal{C}$-violating parameters in $V_{\rm g}$ (in agreement with Branco *et al.* [19]).

2.2 Symmetries of $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{C}^2$

For generality we here consider k (rather than 2)-component fields, i.e. with $SU(k)\times U(1)$ as gauge group. To make it simpler to explore all possible symmetries we express the field Φ_m in terms of its independent real (hermitian) components, $\Phi_m = \Psi_m + {\rm i}\Theta_m$. Define $2k\times 2k$ matrices

$$\mathcal{I} = \begin{pmatrix} I_k & 0_k \\ 0_k & I_k \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0_k & I_k \\ -I_k & 0_k \end{pmatrix}, \tag{2.9}$$

[3]This definition assumes that we for some reasons have decided on a decomposition of all fields into their real and imaginary parts. It is not invariant under complex transformations of the fields, see e.g. [19].

where the subscript k indicates the linear dimension of the submatrix involved. The complex scalar product between two fields Φ_m and Φ_n, invariant under unitary ($U(k)$) transformations, can be expressed in terms of two real bilinear forms[4]

$$\mathrm{Re}(\Phi_m^\dagger \Phi_n) = \widehat{B}_{mn} = \left(\Psi_m^T, \Theta_m^T\right) \mathcal{I} \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} = \Psi_m^T \Psi_n + \Theta_m^T \Theta_n, \tag{2.10a}$$

$$\mathrm{Im}(\Phi_m^\dagger \Phi_n) = \widehat{C}_{mn} = \left(\Psi_m^T, \Theta_m^T\right) \mathcal{J} \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} = \Psi_m^T \Theta_n - \Theta_m^T \Psi_n. \tag{2.10b}$$

The first is the Euclidean dot product between $2k$-component real vectors, the second is the Poisson bracket (symplectic form) of the same quantities viewed as coordinates and momenta of $2k$-dimensional phase space. The quantities in (2.10) are individually invariant under transformation groups larger than $U(k)$. The first form $\widehat{B}$ (with $\widehat{A}$ as a special case) is invariant under the $O(2k)$ group of real orthogonal transformations,

$$\begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} \to O \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix}, \quad O^T O = \mathcal{I}, \tag{2.11}$$

the second form $\widehat{C}$ is invariant under the $Sp(k, R)$ group of real symplectic transformations,

$$\begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} \to S \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix}, \quad S^T \mathcal{J} S = \mathcal{J}. \tag{2.12}$$

In this formulation the charge conjugation operator $\mathcal{C}$ discussed above can be represented as a particular $O(2k)$ transformation when acting on the fields Ψ_n and Θ_n

$$\mathcal{C} = \begin{pmatrix} I_k & 0_k \\ 0_k & -I_k \end{pmatrix}. \tag{2.13}$$

Considering infinitesimal transformations, $O = \mathcal{I} + \epsilon L + \mathcal{O}(\epsilon^2)$, $S = \mathcal{I} + \epsilon M + \mathcal{O}(\epsilon^2)$, the conditions (2.11) and (2.12) become

$$L^T \mathcal{I} + \mathcal{I} L = 0_{2k}, \quad M^T \mathcal{J} + \mathcal{J} M = 0_{2k}. \tag{2.14}$$

Thus L must be a $2k \times 2k$ antisymmetric real matrix; there is a set (Lie algebra) $so(2k)$ of $2k^2 - k$ linearly independent such matrices. Writing out the condition for M in terms of $k \times k$ submatrices we find that it must have the form

$$M = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \quad B = B^T, \ C = C^T. \tag{2.15}$$

There is a set $sp(k)$ of $k^2 + k(k+1) = 2k^2 + k$ linearly independent such matrices. The infinitesimal transformations of the original $U(k)$ are the intersection of the sets $so(2k)$ and $sp(k)$. I.e., the matrices of the form $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$, with $A = -A^T$ and $B^T = B$. There are $\frac{1}{2}k(k-1) + \frac{1}{2}k(k+1) = k^2$ such linearly independent matrices.

[4] $\langle \Phi_m, \Phi_n \rangle = \Phi_m^* \cdot \Phi_n = \widehat{B}_{mn} + i\widehat{C}_{mn}$.

The symmetries of $\widehat{C}^2$ The form $\widehat{C}^2$ (or more precisely $\widehat{C}_{mn}\widehat{C}_{m'n'}$) has a bigger symmetry group than the form $\widehat{C}$. Still, we will see that such operators (forms) will violate the full $O(4)$ symmetry we can assign the rest of the Lagrangian. In analogy with (2.12), $\widehat{C}^2$ symmetries are given by

$$\begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} \to S \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix}, \quad S^T \mathcal{J} S = \pm \mathcal{J}, \tag{2.16}$$

which can be collected in a set

$$P(k, \mathbb{R}) = \{S \in GL_{2k}(\mathbb{R}) | S^T \mathcal{J} S = \pm \mathcal{J}\}, \tag{2.17}$$

which we in Appendix A show is a Lie group.

The component

$$P^-(k, \mathbb{R}) = \{S \in GL_{2k}(\mathbb{R}) | S^T \mathcal{J} S = -\mathcal{J}\}, \tag{2.18}$$

consists of matrices with determinant

$$\det(P^-(k, \mathbb{R})) = (-1)^k, \tag{2.19}$$

as shown in appendix A. The group $P(k, \mathbb{R}) = Sp(k, \mathbb{R}) \cup P^-(k, \mathbb{R})$ will have the same Lie algebra as $Sp(k, \mathbb{R})$, since the new component $P^-(k, \mathbb{R})$ is not connected with the identity. This is manifested by the equation corresponding to eq. (2.14),

$$\mathcal{J} + \epsilon(M^T \mathcal{J} + \mathcal{J} M) = \pm \mathcal{J}. \tag{2.20}$$

not having any solution for the case of a $-\mathcal{J}$ on the right side, see appendix A for a proof.

The most general $O(2k)$-symmetric potential We can conclude that the most general $O(2k)$-invariant potential can be written

$$\begin{aligned} V_{O(2k)}(\Phi_1, \ldots, \Phi_N) = \mu^{(1)}_m \widehat{A}_m + \mu^{(2)}_a \widehat{B}_a + \lambda^{(1)}_{mn} \widehat{A}_m \widehat{A}_n \\ + \lambda^{(2)}_{ab} \widehat{B}_a \widehat{B}_b + \lambda^{(4)}_{ma} \widehat{A}_m \widehat{B}_a, \end{aligned} \tag{2.21}$$

since we have seen that operators not containing any factor $\widehat{C}$ are $O(2k)$-invariant. We obtain a $O(2k)$-invariant potential by leaving out terms proportional to $\lambda^{(3)}_{ab}$ from the $\mathcal{C}$ invariant potential $V_{\mathcal{C}}$ (2.8). The number of terms in $V_{O(2k)}$ is then $N_{O(2k)} = N_{\mathcal{C}} - \mathcal{N}^2 = \frac{1}{4}N(N^3 + 5N + 2) - \frac{1}{4}N^2(N-1)^2$, giving

$$N_{O(2k)} = \frac{1}{2}N(N+1)^2 \tag{2.22}$$

free parameters for the general renormalizable $O(2k)$-invariant potential $V_{O(2k)}$. For $N = 1$ we get the usual 2 parameters of the standard model, the Higgs potential being automatically $O(2k)$-invariant. For $N = 2$ we get the usual 9 parameters, one parameter less than for the $\mathcal{C}$-invariant potential. For $N = 3$ we get 24 parameters, and for $N = 4$ we get 50 parameters.

This counting does again not take into account that we may make $O(N)$ transformations on the row of fields Φ_m to eliminate some terms in (2.21). We may transform the quadratic terms to diagonal form, so that $\mu^{(2)}_a = 0$. This reduces the number of parameters by $\frac{1}{2}N(N-1)$, i.e. to $N'_{O(2k)} = \frac{1}{2}N(N^2 + N + 2)$. This gives 8 parameters for $N = 2$, 21 parameters for $N = 3$, and 44 parameters for $N = 4$.

2.3 Symmetries of the NHDM potential

Since the NHDM-potential $V_{\rm g}$ is constructed from the invariants (2.3) the symmetries of the latter are reflected in the symmetries of the former, but in a manner depending on details of the construction:

1. If $V_{\rm g}$ depends only on the $\widehat{A}_m$'s, i.e. if only the parameters $\mu_m^{(1)}$ and $\lambda_{mn}^{(1)}$ are nonzero, then the symmetry group of $V_{\rm g}$ is at least[5] $\bigotimes_{m=1}^{N} O(2k)$, since we can make independent transformations on each Φ_m.

2. If $V_{\rm g}$ depends only on the $\widehat{A}_m$'s and the $\widehat{B}_a$'s, i.e. for a $\mathcal{C}$-invariant theory (2.8), where in addition the parameters $\lambda_{ab}^{(3)} = 0$, then the symmetry group of $V_{\rm g}$ is at least $O(2k)$. It may contain several such factors if some of the parameters $\mu_a^{(2)}$ and $\lambda_{ab}^{(2)}$ vanish. To analyze this we partition the Φ_m's into sets: If a parameter $\mu_a^{(2)}$ is nonzero, then the fields $\Phi_{m(a)}$ and $\Phi_{n(a)}$ belong to the same set, with $m(a)$ and $n(a)$ denoting that m and n are contained in a. If a parameter $\lambda_{ab}^{(2)}$ is nonzero, then the fields $\Phi_{m(a)}$ and $\Phi_{n(a)}$ belong to the same set, and the fields $\Phi_{m(b)}$ and $\Phi_{n(b)}$ belong to the same set. With this partitioning into a maximal number of sets we may make one independent $O(2k)$ transformation for each set.

3. If $V_{\rm g}$ depends only on the $\widehat{C}_a$'s, i.e. with only the parameters $\mu_a^{(3)}$ and $\lambda_{ab}^{(3)}$ being nonzero, then the symmetry group of $V_{\rm g}$ is at least $Sp(k, \mathbb{R})$. If we (in the same manner as above) can partition the fields into several sets, then we may make independent $Sp(k, \mathbb{R})$ transformations on fields belonging to different sets. However, since the additional symmetries in this case fail to be symmetries of even the zero'th order kinetic term (2.30), their significance is uncertain.

4. With all parameters arbitrary the symmetry group of $V_{\rm g}$ is just the original $SU(k) \times U(1)$ gauge symmetry.

In this work we will pay special attention to the second scenario, with $k = 2$.

2.4 Symmetries of the kinetic terms

We now turn to the (global) symmetries of the kinetic terms of the Lagrangian,

$$K = \sum_{n=1}^{N} [(\partial^\mu + G^\mu)\Phi_n(x)]^\dagger [(\partial_\mu + G_\mu)\Phi_n(x)], \tag{2.23}$$

with

$$G^\mu = igT_i W_i^\mu + ig' Y B^\mu. \tag{2.24}$$

Let K_i denote the terms of the i'th order in the gauge fields.

[5]It could possibly be larger, since there might be additional row symmetries transforming fields Φ_m with different m into each other; such symmetries would require special relations among the parameters $\mu_m^{(1)}$ and $\lambda_{mn}^{(1)}$.

Consider the transformation of the kinetic terms linear in the gauge fields, K_1, under the map ρ defined in appendix B. We can then write[6]

$$
\begin{aligned}
K_1 &= \sum_{n=1}^{N} \partial^\mu (\Phi_n)_k^\dagger G_\mu (\Phi_n)_k + (\Phi_n)_k^\dagger G^{\mu\dagger} \partial_\mu (\Phi_n)_k \\
&= \sum_{n=1}^{N} \rho(\partial^\mu (\Phi_n)_k^\dagger) \rho(G_\mu) \rho((\Phi_n)_k) + \rho((\Phi_n)_k^\dagger) \rho(G^{\mu\dagger}) \rho(\partial_\mu (\Phi_n)_k) \\
&= \sum_{n=1}^{N} \partial^\mu \Phi_n^T \mathcal{T}_\mu \Phi_n + \Phi_n^T (-\mathcal{T}^\mu) \partial_\mu \Phi_n
\end{aligned}
\tag{2.25}
$$

where the subscript k in $(\Phi_n)_k$ indicates this is the usual complex Higgs k-plet (in the case $k = 2$ the usual complex Higgs doublet), while Φ_n is the $2k$ dimensional real vector (2.26),

$$
\Phi_n = \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix}, \tag{2.26}
$$

where $(\Phi_n)_k = \Psi_n + i\Theta_n$, and where we also use eqs. (B.2), (B.3) and (B.6).

In eq. (2.25) we have applied the transformation ρ on the gauge terms G^μ defined in eq. (2.24),

$$
\rho\,(G^\mu) = \mathcal{T}^\mu, \quad \rho\,(G^{\mu\dagger}) = -\mathcal{T}^\mu, \tag{2.27}
$$

where $\mathcal{T}$ then reads

$$
\mathcal{T}^\mu = \begin{pmatrix} gW_I^\mu & -gW_R^\mu - g' Y B^\mu \\ gW_R^\mu + g' Y B^\mu & gW_I^\mu \end{pmatrix}, \tag{2.28}
$$

with $W_R^\mu = \sum_i' W_i^\mu T_i^{\mathrm{s}}$, summed over the set of real symmetric generators T_i^s of $SU(k)$, and $W_I^\mu = \mathrm{i} \sum_i' W_i^\mu T_i^{\mathrm{a}}$, summed over the set of imaginary antisymmetric generators T_i^a. For $k = 2$ the two sets are respectively $\left\{\frac{1}{2}\sigma^1, \frac{1}{2}\sigma^3\right\}$ and $\left\{\frac{1}{2}\sigma^2\right\}$.

Third, we consider the kinetic terms quadratic in the gauge fields,

$$
\begin{aligned}
K_2 &= \sum_{n=1}^{N} (\Phi_n)_k^\dagger G^{\mu\dagger} G_\mu (\Phi_n)_k \\
&= \sum_{n=1}^{N} \rho((\Phi_n)_k^\dagger) \rho(G^{\mu\dagger}) \rho(G_\mu) \rho((\Phi_n)_k) \\
&= -\sum_{n=1}^{N} \Phi_n^T \mathcal{T}^2 \Phi_n.
\end{aligned}
\tag{2.29}
$$

[6]Disregarding so-called Schwinger terms—here terms proportional to $i[\partial_\mu \phi(x), \phi(x)]$ for a scalar field ϕ—or, alternatively, reasoning classically.

The symmetries of K_0 When we first ignore couplings to the gauge fields the remaining terms can be written

$$K_0 = \sum_{n=1}^{N} \sum_{\alpha=1}^{k} \left(\partial^\mu \Psi_{n\alpha} \, \partial_\mu \Psi_{n\alpha} + \partial^\mu \Theta_{n\alpha} \, \partial_\mu \Theta_{n\alpha} \right), \tag{2.30}$$

where the group index α labels the k components of $\Phi_n = \Psi_n + \mathrm{i}\Theta_n$. This term is invariant under rotation of *all* components $\{\Psi_{n\alpha}, \Theta_{n\alpha}\}$ into each other. I.e., the symmetry group of K_0 is $O(2kN)$. The connected part of this group is $SO(2kN)$, whose generators are all real antisymmetric matrices, $L_{mn,\alpha\beta} = -L_{nm,\beta\alpha}$ (i.e. $L^T = -L$, where transposition refers to both sets of indices).

The symmetries of K_0 *and* K_1: Next, consider the terms linear in the gauge fields again, cf. eq. (2.25),

$$K_1 = \sum_{n=1}^{N} \left(\left(\partial_\mu \left(\Psi_n^T, \Theta_n^T \right) \right) \mathcal{T} \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} - \left(\Psi_n^T, \Theta_n^T \right) \mathcal{T} \partial_\mu \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix} \right), \tag{2.31}$$

with $\mathcal{T}$ given in eq. (2.28).

Consider now an infinitesimal transformation $\delta\Phi_{m,\alpha} = L_{mn,\alpha\beta} \, \Phi_{n,\beta}$, and $\mathcal{T}$ denoting the $2k \times 2k$ antisymmetric matrix in equation (2.28) (in group indices α, β — in addition it is proportional to the $N \times N$ unit matrix in row indices). The requirement that this is an infinitesimal symmetry transformation for K_1 is that $L^T \mathcal{T} + \mathcal{T} L = 0$. Or, when we restrict L to be antisymmetric so that it also is an infinitesimal symmetry transformation for K_0,

$$L_{mn,\alpha\beta} \, \mathcal{T}_{\beta\gamma} - \mathcal{T}_{\alpha\beta} \, L_{mn,\beta\gamma} = 0. \tag{2.32}$$

In order to determine the allowed structure of L, we expand these matrices into terms of definite symmetries ($L^{(\mathrm{s})}$ symmetric, and $L^{(\mathrm{a})}$ antisymmetric) in the mn indices:

$$L_{mn,\alpha\beta} = \sum \left(S_{\alpha\beta} L^{(\mathrm{a})}_{mn} + A_{\alpha\beta} L^{(\mathrm{s})}_{mn} \right), \tag{2.33}$$

with S (symmetric) and A (antisymmetric) restricted by the constraint (2.32). The sum runs over all possible combinations of allowed matrices[7]. We next note that the antisymmetric matrices $\mathcal{T}$ can be expanded in the set

$$\widehat{\mathcal{T}} = \{ T_i^{\mathrm{a}} \, \mathcal{I}, \, T_i^{\mathrm{s}} \, \mathcal{J}, \, \mathcal{J} \} = \left\{ \begin{pmatrix} T_i^{\mathrm{a}} & 0_k \\ 0_k & T_i^{\mathrm{a}} \end{pmatrix}, \begin{pmatrix} 0_k & T_i^{\mathrm{s}} \\ -T_i^{\mathrm{s}} & 0_k \end{pmatrix}, \begin{pmatrix} 0_k & 1_k \\ -1_k & 0_k \end{pmatrix} \right\}. \tag{2.34}$$

By substituting (2.33) into (2.32) we are led to search for the set of $2k \times 2k$ real matrices S and A which commute with $\mathcal{T}$ for arbitrary values of the fields W_i^μ and B^μ. It is sufficient to verify that this property holds for all elements of the set $\widehat{\mathcal{T}}$. Let

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad X = S, \quad \text{or} \quad X = A. \tag{2.35}$$

[7]Without the restriction (2.32) there would be $\frac{1}{2}k(2k+1)N(N-1) + \frac{1}{2}k(2k-1)N(N+1) = kN(2kN-1)$ independent terms, equal to the number of generators of $SO(2kN)$.

Requiring commutativity [see eqs. (2.32) and (2.33)] with the three types of matrices in $\widehat{\mathcal{T}}$ we obtain the conditions

$$X_{jk}T_i^{\rm a} = T_i^{\rm a}X_{jk}, \tag{2.36a}$$

$$X_{11}T_i^{\rm s} = T_i^{\rm s}X_{22}, \quad X_{22}T_i^{\rm s} = T_i^{\rm s}X_{11}, \quad X_{12}T_i^{\rm s} = -T_i^{\rm s}X_{21}, \quad X_{21}T_i^{\rm s} = -T_i^{\rm s}X_{12}, \tag{2.36b}$$

$$X_{11} = X_{22}, \quad X_{12} = -X_{21}. \tag{2.36c}$$

Using (2.36c) we find that X_{11} and X_{12} must commute with all matrices $T_i^{\rm a}$, $T_i^{\rm s}$ (assumed to form an irreducible representation). By Schur's lemma they must then be proportional to the $k \times k$ unit matrix, so that $S \propto \mathcal{I}$ and $A \propto \mathcal{J}$. Thus, the Lie algebra of the symmetry group of K_0 and K_1 consists of elements of the form

$$L_{mn} = \mathcal{I}\, L_{mn}^{(\rm a)} + \mathcal{J}\, L_{mn}^{(\rm s)}. \tag{2.37}$$

This is the Lie algebra of $U(N)$ written in real variables.

The symmetries of K_0 *and* K_1 in the limit $g' \to 0$: A more interesting situation arises if we remove $\mathcal{J}$ from the set $\widehat{\mathcal{T}}$, as would apply to the limit $g' \to 0$. Then we still find that $X_{11} + X_{22}$ and $X_{12} - X_{21}$ must commute with all matrices $T_i^{\rm a}$, $T_i^{\rm s}$, and hence must be proportional to the unit matrix. Further, $X_{11} - X_{22}$ and $X_{12} + X_{21}$ must commute with all $T_i^{\rm a}$, but anticommute with all $T_i^{\rm s}$. For $k = 2$, i.e. for the gauge group $SU(2)_L \times U(1)_Y$ in the limit $g' \to 0$, we find that nonzero solutions of (2.32),

$$X_{11} - X_{22} \propto \varepsilon \equiv {\rm i}\,\sigma^2, \quad X_{12} + X_{21} \propto \varepsilon, \tag{2.38}$$

are possible [see eqs. (2.33) and (2.35)]. This means that the possible antisymmetric matrices A may be any linear combination of matrices from the set

$$\mathcal{G} = \left\{ \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \mathcal{J} \right\}, \tag{2.39}$$

where the 2×2 matrix ε was defined in eq. (2.38). The set $\mathcal{G}$ is a basis of generators for $SU(2)$. Thus, the Lie algebra of symmetry generators for K_0 and K_1 in this case consists of elements of the form

$$L_{mn} = \mathcal{I}\, L_{mn}^{(\rm a)} + \sum_{A \in \mathcal{G}} A\, L_{mn}^{(\rm s)}, \tag{2.40}$$

allowing all possible symmetric $N \times N$ matrix $L^{(\rm s)}$ for each A. There are $\frac{1}{2}N(N-1) + \frac{3}{2}N(N+1) = 2N^2 + N$ independent terms, equal to the number of generators of the $N \times N$ quaternionic symplectic group $Sp(N)$. The generators (2.40) generate $Sp(N)$, where the elements of $\mathcal{G}$ act as the quaternions i, j and k.

The results above were again found under the assumption that the fields W_i^μ are arbitrary, and kept constant under the transformation. Combined $SU(k)$ transformations of the W_i^μ and the Φ_m fields still remain a symmetry. For $k = 2$ this symmetry is enhanced to (at least) $Sp(N) \times SO(4)$ as $g' \to 0$.[8] (In the case $g' \neq 0$ it is $U(N) \times SU(k)$.) The generators for $SO(4)$ are the 3 generators in $\mathcal{G}$ plus the 3 generators for the $SU(2)_L$ gauge group (written in real form).

[8]The $SO(4)$ symmetry cannot be extended to a $O(4)$ symmetry [27].

The symmetries of K_2 Next, consider the terms quadratic in the gauge fields cf. eq. (2.29),

$$K_2 = -\sum_{n=1}^{N} \left(\Psi_n^T, \Theta_n^T\right) \mathcal{T}^2 \begin{pmatrix} \Psi_n \\ \Theta_n \end{pmatrix}. \tag{2.41}$$

As in the symmetry analysis of K_1 we want to find all matrices X such that $X\mathcal{T}^2 = \mathcal{T}^2 X$. All matrices X which commute with $\widehat{\mathcal{T}}$ will fulfill this criterion (since $\mathcal{T}^2$ can be expanded in a set which consists of products of all possible *pairs* of matrices from $\widehat{\mathcal{T}}$). Therefore, the symmetries of K_1 are also symmetries of K_2.

3. Spontaneous symmetry breakdown

In this section we return to the case of $k = 2$, i.e. with $SU(2)_L \times U(1)_Y$ as the gauge group and a row of N Higgs doublets Φ_m. Note however that many of our considerations have straightforward generalizations to $k > 2$.

As for the Standard Model, the potential $V_{\rm g}$ of equation (2.5), or $V_{\mathcal{C}}$ of equation (2.8), may acquire its minimum at nonzero values of the scalar fields, $\langle\Phi\rangle_0 = \Phi^{(0)}$, where Φ (without a lower index) refers to the whole set of fields Φ_m. This point, $\Phi^{(0)}$, will belong to one or more manifolds of equivalent minima related by the symmetries of the potential. One may use these symmetries to transform $\Phi^{(0)}$ to a particular form. A possible one is to require for $\Phi_1^{(0)}$ that only its lowest real component is nonzero. This can always be achieved by an $SU(2)_L \times U(1)_Y$ gauge transformation. Next, the upper component of $\Phi_2^{(0)}$ can be made real by the remaining $U(1)$ gauge transformation which keeps $\Phi_1^{(0)}$ unchanged. Then one has no gauge freedom left to change $\Phi_n^{(0)}$ for $n \geq 3$. However, it was shown by Barroso et al. [23] that a sequence of unitary row transformations can shift the vacuum expectation values to the first two fields of the row only[9], for instance (when written in complex form)

$$\Phi_1^{(0)} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \Phi_2^{(0)} = \begin{pmatrix} u_2 \\ v_2 e^{i\delta} \end{pmatrix}, \quad \Phi_n^{(0)} = 0 \text{ for } n \geq 3, \tag{3.1}$$

with v_1, u_2, v_2 and δ real. The special case $u_2 = \delta = 0$ is usually referred to as *vacuum alignment*, in which case we may also transform v_2 to zero by an orthogonal row transformation involving only Φ_1 and Φ_2. This is known as the *Higgs basis* [24]. However, for other purposes it may be more convenient to adopt a "democratic" basis in which the lower component of all (or most) fields Φ_m have a nonzero real expectation value. It is related to the Higgs basis by an orthogonal row transformation which preserves the form of $\mathcal{C}$ and $U(1)$ electromagnetic gauge transformations (the latter preserving the definition of electric charge).

[9] One may collect the quantities $\Phi_{m\alpha}^{(0)}$ $(\alpha = 1, 2)$ into two N-component complex vectors $\tilde{\Phi}^{(1)}$ and $\tilde{\Phi}^{(2)}$. By a $U(N)$ row transformation one may first rotate $\tilde{\Phi}_m^{(1)}$ so that only the component $\tilde{\Phi}_1^{(1)}$ is nonzero, with $\tilde{\Phi}_1^{(1)}$ real. There is a group of $U(N-1)$ transformations preserving this condition; this may be used to transform $\tilde{\Phi}_m^{(2)}$ so that only the components $\tilde{\Phi}_1^{(2)}$, $\tilde{\Phi}_2^{(2)}$ are nonzero, with $\tilde{\Phi}_2^{(2)}$ real. One cannot do better due to the existence of four real $U(N)$ invariant parameters in $||\tilde{\Phi}^{(1)}||$, $||\tilde{\Phi}^{(2)}||$, and $\tilde{\Phi}^{(1)\dagger}\,\tilde{\Phi}^{(2)}$. But there remains a $U(N-2)$ group of transformations preserving this condition which can be used for other purposes. For a $SU(k) \times U(1)$ gauge group one may generalize this procedure to k vectors $\tilde{\Phi}^{(j)}$, $j = 1 \ldots k$.

Assume now the case of vacuum alignment and a potential $V_{O(4)}$ which is $O(4)$ invariant. Then the existence of the vacuum expectation values $\Phi^{(0)}$ will break the (explicit) symmetry down to $O(3)$, with the consequence that the Higgs boson particle spectra and other physical properties will organize themselves into multiplets of $O(3)$ (broken by perturbative corrections in g'). The number of broken symmetry generators is 3 whether we consider the symmetry broken from $U(2)$ to $U(1)$ or from $O(4)$ to $O(3)$; this leads to the existence of 3 Higgs ghosts and no extra Goldstone bosons[10].

3.1 Mass-squared matrices

To make these statements slightly more explicit, as needed for calculation of the zero'th order (in g and g') particle masses, we expand the potential around $\Phi^{(0)}$ to second order. There are no first order terms since we are expanding around a minimum. The matrix of second derivatives is the mass-squared matrix $M^2_{mn\alpha\beta}$. It is restricted by symmetries to have a block diagonal form in the group indices α, β. We use coordinates where $\Phi_m = \Psi_m + \mathrm{i}\Theta_m$ is expressed in terms of four real fields,

$$\Phi_m = \Psi_m + \mathrm{i}\Theta_m = \begin{pmatrix} \phi_{m1} + \mathrm{i}\phi_{m2} \\ v_m + \eta_m + \mathrm{i}\phi_{m3} \end{pmatrix}, \quad \Phi_m^{(0)} = \begin{pmatrix} 0 \\ v_m \end{pmatrix}. \tag{3.2}$$

It is now convenient to represent these on real form as

$$\Phi_m = \begin{pmatrix} \Psi_m \\ \Theta_m \end{pmatrix} = \begin{pmatrix} \phi_{m1} \\ v_m + \eta_m \\ \phi_{m2} \\ \phi_{m3} \end{pmatrix}. \tag{3.3}$$

We have the expansion

$$V(\Phi^{(0)} + \Delta\Phi) = \langle V \rangle_0 + \frac{1}{2} \left\langle \frac{\partial^2 V}{\partial \Phi_{m\rho} \partial \Phi_{n\sigma}} \right\rangle_0 \Delta\Phi_{m\rho} \Delta\Phi_{n\sigma} + \mathcal{O}(\Delta\Phi^3), \tag{3.4}$$

where $\Phi_{m\rho}$ denotes one of the *four* possibilities ϕ_{m1}, η_m, ϕ_{m2}, ϕ_{m3}, and the subscript 0 indicates that a quantity is evaluated at $\Phi = \Phi^{(0)}$. Now a set of generators for $SO(4)$[11] is

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

[10] This remains true for general values of $k \geq 2$; a set of aligned vacuum expectation values will break $U(k)$ to $U(k-1)$ and $O(2k)$ to $O(2k-1)$. The number of broken generators is $2k-1$ in both cases. The situation is different if we have *two* broken real directions, as in equation (3.1) with $u_2 = 0$ but $\delta \neq 0$. Cf. section 3.3.

[11] Equivalently $O(4)$: $SO(4)$ and $O(4)$ have the same Lie algebra and hence the same generators.

where J_1, J_2, J_3 will transform the vacuum expectation value $\Phi^{(0)}$, while J_4, J_5, J_6 leave it unchanged, cf. Eq. (3.3). In terms of these, the broken generators of the $SU(2) \times U(1)_Y$ gauge group, written in real form by the transformation ρ defined in (B.1), are

$$\frac{\mathrm{i}}{2}\sigma^1 \to \frac{1}{2}(J_2 + J_5), \quad \frac{\mathrm{i}}{2}\sigma^2 \to \frac{1}{2}(J_1 + J_4), \quad \frac{\mathrm{i}}{2}(1 - \sigma^3) \to J_3, \tag{3.6}$$

and the unbroken $U(1)$ (electromagnetic gauge) generator is

$$\frac{\mathrm{i}}{2}(1 + \sigma^3) \to J_6. \tag{3.7}$$

If now V is invariant under

$$\Delta\phi_{m1} \to -\Delta\phi_{m1}, \quad \Delta\eta_m \to \Delta\eta_m, \quad \Delta\phi_{m2} \to -\Delta\phi_{m2}, \quad \Delta\phi_{m3} \to \Delta\phi_{m3},$$

there can be no terms in (3.4) mixing the sets $\{\Delta\phi_1, \Delta\phi_2\}$ and $\{\Delta\eta_m, \Delta\phi_{m3}\}$. If V in addition is invariant under

$$\Delta\phi_{m1} \to \Delta\phi_{m2}, \quad \Delta\eta_m \to \Delta\eta_m, \quad \Delta\phi_{m2} \to -\Delta\phi_{m1}, \quad \Delta\phi_{m3} \to \Delta\phi_{m3},$$

there can be no terms in (3.4) mixing $\Delta\phi_{m1}$ and $\Delta\phi_{m2}$, and we must have

$$\left\langle \frac{\partial^2 V}{\partial\phi_{m1}\,\partial\phi_{n1}} \right\rangle_0 = \left\langle \frac{\partial^2 V}{\partial\phi_{m2}\,\partial\phi_{n2}} \right\rangle_0 \equiv M^2_{\mathrm{ch},mn}. \tag{3.8}$$

We refer to this as the *charged* mass-squared matrix. The transformations considered generate a Z_4 subgroup of the $U(1)$ gauge group generated by J_6, assumed to be a symmetry of V. We have formulated it this way as a reminder that invariance under discrete subgroups may be sufficient to impose useful restrictions on the mass matrices.

If V is invariant under $\mathcal{C}$ transformations,

$$\Delta\eta_m \to \Delta\eta_m, \quad \Delta\phi_{m3} \to -\Delta\phi_{m3}, \tag{3.9}$$

(irrespective of how we define $\mathcal{C}$ to operate on the charged sector, f.i. $\Delta\phi_{m1} \to \Delta\phi_{m1}$, $\Delta\phi_{m2} \to -\Delta\phi_{m2}$) there can be no terms in (3.4) mixing $\Delta\eta_m$ and $\Delta\phi_{m3}$. Thus the neutral mass-squared matrix decomposes into two more blocks, a $\mathcal{C}$ *even* and a $\mathcal{C}$ *odd* one,

$$M^2_{\mathcal{C}+,mn} = \left\langle \frac{\partial^2 V}{\partial\eta_m\,\partial\eta_n} \right\rangle_0, \quad M^2_{\mathcal{C}-,mn} = \left\langle \frac{\partial^2 V}{\partial\phi_{m3}\,\partial\phi_{n3}} \right\rangle_0. \tag{3.10}$$

If V in addition is invariant under the transformations

$$\Delta\phi_{m1} \to \Delta\phi_{m1}, \quad \Delta\eta_m \to \Delta\eta_m, \quad \Delta\phi_{m2} \to \Delta\phi_{m3}, \quad \Delta\phi_{m3} \to \Delta\phi_{m2}, \tag{3.11}$$

which generate a Z_4 subgroup of the $SO(2)$ symmetry group generated by J_4, we obtain the relation

$$M^2_{\mathcal{C}-,mn} = M^2_{\mathrm{ch},mn}. \tag{3.12}$$

This explicitly demonstrates mass degeneracy between the charged and the $\mathcal{C}$ odd sectors [25]. Especially, if the potential is $O(4)$-invariant (2.21), that is, we have a $\mathcal{C}$-invariant theory where also the parameters $\lambda_{ab}^{(3)} = 0$ [the latter implies (3.11)][12], the above symmetry criteria for mass degeneracy are valid. Moreover, since the the renormalization is not changed when the Higgs fields aquire a vacuum expectation value [26], we won't get any mass renormalization counterterms from the quartic operators. So even though $O(4)$-violating quartic terms proportional to $\lambda_{ab}^{(3)}$ cannot alone be prohibited by any discrete symmetry imposed on the NHDM Lagrangian [27], they won't be showing up as counterterms when renormalizing the masses. Hence, the mass degeneration (3.12) will only be broken by loop corrections involving gauge bosons, since we get an exact $SO(3)$ symmetry when $g' = 0$. With $g' \neq 0$ and hence with an approximate $SO(3)$ symmetry, the mass differences of the charged and CP-odd sectors will be of order $\mathcal{O}(g'^4) \propto \mathcal{O}(e^4)$.

On the other hand, the $SO(3)$ symmetry between CP-odd and charged sectors could also be broken by counterterms of the type $\lambda_{ab}^{(3)} \widehat{C}^2$, even though these terms are set to zero in the original potential, if we are considering scattering processes and not mass relations.

3.2 The Higgs ghosts

Let $\Delta\Phi$ be chosen so that $\Phi^{(0)} + \epsilon\Delta\Phi + \mathcal{O}(\epsilon^2)$ is a family of minima related by the symmetry of the potential V,

$$\frac{\partial}{\partial \Phi_{m\alpha}} V(\Phi^{(0)} + \epsilon\Delta\Phi) = 0, \tag{3.13}$$

to first order in ϵ. By differentiating this relation with respect to ϵ and then setting $\epsilon = 0$ we find

$$\left\langle \frac{\partial^2 V}{\partial \Phi_{m\alpha}\, \partial \Phi_{n\beta}} \right\rangle_0 \Delta\Phi_{n\beta} = 0, \tag{3.14}$$

which reflects the fact that the matrix $M^2_{mn\alpha\beta}$ has zero eigenvalues with corresponding eigenvectors $\Delta\Phi_{n\beta}$. We may take the latter to be $\Delta\Phi^{(i)} \propto J_i \Phi^{(0)}$ for $i = 1, 2, 3$. Normalized,

$$\Delta\Phi_m^{(1)} = (v_m, 0, 0, 0)^T / a, \quad \Delta\Phi_m^{(2)} = (0, 0, v_m, 0)^T / a, \quad \Delta\Phi_m^{(3)} = (0, 0, 0, v_m)^T / a, \tag{3.15}$$

with $a^2 = \sum_m v_m^2$. The massless excitations in these directions correspond to a triplet of Higgs ghosts. There will be $N - 1$ additional $SO(3)$ triplets of excitations in directions orthogonal to the ghosts. They correspond to physical particles. There will also be N $SO(3)$ singlets, transforming evenly under $\mathcal{C}$, corresponding to physical particles. In the case of $N = 2$, the triplet is (H^+, H^-, A), whereas the singlets are h and H [13].

3.3 Non-aligned vacuum expectation values

We have assumed vacuum alignment in much of the previous discussions of this section. The phenomenologically most realistic deviation from this case is that we have a situation with *two* (real) broken directions, as in (3.1) with $u_2 = 0$ but $\delta \neq 0$. This corresponds to a situation which preserves the $U(1)$ electromagnetic gauge symmetry, and its corresponding definition of electric charge, but where the $\mathcal{C}$ symmetry is spontaneously broken. In this situation the $M^2_{\mathrm{ch},mn}$ mass-squared matrix remains in block form, but the $\mathcal{C}$ even and odd excitations may mix to give a $2N \times 2N$ mass matrix for the neutral particles. One of the

[12] For supersymmetric theories we typically have $\lambda_{ab}^{(3)} \neq 0$.

excitations will be massless, corresponding to a neutral Higgs ghost. We assume again that V is invariant under $O(4)$ transformations. The explicit symmetry is now broken down to $O(2) \simeq U(1)$, so that 5 generators are broken. As before, 3 of these will generate excitations which correspond to the Higgs ghosts; the remaining 2 will correspond to nearly massless charged pseudo-Goldstone[13] bosons (massless to zero'th order in g').

To analyze the situation we again write $\Phi = \Phi^{(0)} + \Phi'$ in terms of real fields,

$$\Phi_m = \Phi_m^{(0)} + (\phi_{m1}, \eta_m, \phi_{m2}, \chi_m)^T \text{ with } \Phi_m^{(0)} = (0, v_m, 0, w_m)^T .$$

J_4 and J_5 are now also broken by the vacuum expectation values. Acting with the broken generators on $\Phi^{(0)}$ one finds five eigenvectors of the mass matrix with zero eigenvalues, $\Delta\Phi^{(i)} \propto J_i\Phi^{(0)}$. After normalization

$$\begin{aligned}
\Delta\Phi_m^{(1)} &= (v_m, 0, 0, 0)^T / a, \\
\Delta\Phi_m^{(2)} &= (0, 0, -v_m, 0)^T / a, \\
\Delta\Phi_m^{(3)} &= (0, w_m, 0, -v_m)^T / \sqrt{a^2 + b^2}, \\
\Delta\Phi_m^{(4)} &= (0, 0, w_m, 0)^T / b, \\
\Delta\Phi_m^{(5)} &= (w_m, 0, 0, 0)^T / b,
\end{aligned} \tag{3.16}$$

where $a^2 = \sum_m v_m^2$ and $b^2 = \sum_m w_m^2$. These eigenvectors are normalized, but they are not necessarily orthogonal to each other. Their nonvanishing inner products are

$$\left(\Delta\Phi^{(1)}, \Delta\Phi^{(5)}\right) = -\left(\Delta\Phi^{(2)}, \Delta\Phi^{(4)}\right) = \frac{1}{ab}\sum_m v_m w_m \equiv \cos\vartheta.$$

Here $|\sin\vartheta| > 0$, since the vacuum expectation values by assumption are non-aligned. Thus, the orthonormalized eigenvectors corresponding to the Higgs ghosts can be written

$$\begin{aligned}
H_m^{(1)} &= \frac{1}{\sqrt{a^2+b^2}}\,(v_m, 0, w_m, 0)^T = \frac{a}{\sqrt{a^2+b^2}}\,\Delta\Phi_m^{(1)} + \frac{b}{\sqrt{a^2+b^2}}\,\Delta\Phi_m^{(4)}, \\
H_m^{(2)} &= \frac{1}{\sqrt{a^2+b^2}}\,(w_m, 0, -v_m, 0)^T = \frac{a}{\sqrt{a^2+b^2}}\,\Delta\Phi_m^{(2)} + \frac{b}{\sqrt{a^2+b^2}}\,\Delta\Phi_m^{(5)}, \\
H_m^{(3)} &= \frac{1}{\sqrt{a^2+b^2}}\,(0, w_m, 0, -v_m)^T = \Delta\Phi_m^{(3)},
\end{aligned} \tag{3.17}$$

where $H_m^{(i)} \propto G_i\Phi^{(0)}$, G_i denoting the $SU(2)$ generators as given by the map (3.6). The two eigenvectors corresponding to the Goldstone modes are orthogonal to those above,

$$\begin{aligned}
G^{(1)} &= \frac{1}{\sin\vartheta\,\sqrt{a^2+b^2}}\left[a\left(\Delta\Phi^{(4)} + \cos\vartheta\,\Delta\Phi^{(2)}\right) - b\left(\Delta\Phi^{(1)} - \cos\vartheta\,\Delta\Phi^{(5)}\right)\right], \\
G^{(2)} &= \frac{1}{\sin\vartheta\,\sqrt{a^2+b^2}}\left[-a\left(\Delta\Phi^{(5)} - \cos\vartheta\,\Delta\Phi^{(1)}\right) + b\left(\Delta\Phi^{(2)} + \cos\vartheta\,\Delta\Phi^{(4)}\right)\right].
\end{aligned} \tag{3.18}$$

[13] Pseudo-Goldstone bosons stems from broken generators of the extra $O(4)$ symmetry of the potential, while Higgs ghosts per definition is generated by the broken generators of the gauge symmetry (which of course is a symmetry of the whole lagrangian). The pseudo-Goldstone bosons acquire small masses from radiative corrections, and are hence not massless in all orders of perturbation theory, like true Goldstone bosons. True Goldstone bosons are, in contrast, generated by the spontaneous breaking of a symmetry of a total lagrangian, not only a potential.

They have been orthonormalized. We note that the normalization constant becomes infinite in the limit of aligned vacuum expectation values, $\sin\vartheta \to 0$. We recall that the set $\{H^{(1)}, H^{(2)}, H^{(3)}, G^{(1)}, G^{(2)}\}$ are just numerical eigenvectors of the mass-squared matrix. The corresponding zero mode fields are the quantum fields obtained by projecting Φ' on these eigenvectors,

$$\Phi^{H^{(i)}}_{m\alpha} = \left(H^{(i)}, \Phi'\right) H^{(i)}_{m\alpha}, \quad \Phi^{G^{(j)}}_{m\alpha} = \left(G^{(j)}, \Phi'\right) G^{(j)}_{m\alpha} \quad \text{for } i = 1,2,3 \text{ and } j = 1,2. \tag{3.19}$$

The field $\Phi^{H^{(3)}}$ is the neutral Higgs ghost field, while the fields $\Phi^{H^{(1)}}$ and $\Phi^{H^{(2)}}$ form the charged Higgs ghost field, and the fields $G^{(1)}$ and $G^{(2)}$ together form charged Goldstone boson fields.

If the vacuum expectation values broke the symmetry in even more directions, as in (3.1) with both $u_2 \neq 0$ and $\delta \neq 0$, the situation would be different: All 6 generators of $SO(4)$ would be broken, 4 of them corresponding to the 4 broken generators of the $U(2)$ gauge group. Thus, there would still be 2 pseudo-Goldstone bosons.

4. Concluding remarks

In this paper we have analyzed the additional (approximate) symmetries which may arise in multi-Higgs-doublet models, due to the fact that the scalar potential may have more symmetries than required by the imposed gauge invariance. Moreover, for the kinetic terms (as a whole) we found that the symmetry group was $U(N) \times SU(k)$. In the case $k = 2$ (i.e. scalar doublets) we found that the symmetry group of the kinetic terms, in the limit $g' \to 0$, is enhanced to $Sp(N) \times SO(4)$. The most general $\mathcal{C}$ invariant Higgs potential (2.8) has the same $SO(4)$ symmetry, only broken by the presence of the operator $\widehat{C}^2$, that is, terms proportional to $\lambda^{(3)}_{ab}$. In the case where $\lambda^{(3)}_{ab}$ is set to zero, we have a mass degeneration (3.12) (assuming vacuum alignment) between charged and $\mathcal{C}$ odd sectors in the limit $g' \to 0$. When we don't have vacuum alignment, but rather two broken (real) directions with the electromagnetic generator left unbroken, a pair of light, charged Higgs bosons should emerge (cf. section 3.3).

A. $P(k,\mathbb{R})$, the symmetry group of $\widehat{C}^2$

We will here show that the set

$$P(k,\mathbb{R}) = \{S \in GL_{2k}(\mathbb{R}) | S^T \mathcal{J} S = \pm \mathcal{J}\}, \tag{A.1}$$

given in eq. (2.17) is a Lie group: The associative law and the existence of the identity follow from $GL_{2k}(\mathbb{R})$ (the set of all invertible, real $2k \times 2k$ matrices) being a group. Define

$$P^-(k,\mathbb{R}) = \{S \in GL_{2k}(\mathbb{R}) | S^T \mathcal{J} S = -\mathcal{J}\}. \tag{A.2}$$

The other component of $P(k,\mathbb{R})$ (what we could call $P^+(k,\mathbb{R})$) is $Sp(k,\mathbb{R})$. Then, if $S^- \in P^-$ and $S^+ \in Sp(k,\mathbb{R})$, then we easily see by the definition that

$$\begin{aligned} &S^- S^+, S^+ S^- \in P^-(k,\mathbb{R}), \\ &S_1^+ S_2^+, S_1^- S_2^- \in Sp(k,\mathbb{R}). \end{aligned} \tag{A.3}$$

So the set $P(k,\mathbb{R})$ is closed under group multiplication. This set also includes the inverse of each element. We only have to show this for elements $S \in X^-$, since we already know $Sp(k,\mathbb{R})$ is a Lie group. Let $S^T \mathcal{J} S = -\mathcal{J}$. Then

$$(S^T)^{-1} S^T \mathcal{J} S S^{-1} = (S^T)^{-1}(-\mathcal{J})S^{-1}, \tag{A.4}$$

and since we generally have that $(A^T)^{-1} = (A^{-1})^T$,

$$-\mathcal{J} = (S^{-1})^T \mathcal{J} S^{-1}, \tag{A.5}$$

so $S^{-1} \in P^-$ too (still, P^- is not a group considered isolated, since it is not closed under group multiplication, and does not include the identity).

We have now derived that $P(k,\mathbb{R})$ is a group. To prove it is a Lie group, we must prove that it is a (topologically) closed subset of $GL_{2k}(\mathbb{R})$: $f(A) = A^T \mathcal{J} A$ is a continuous map, the set $\{\pm\mathcal{J}\}$ is closed in $GL_{2k}(\mathbb{R})$, and hence $P(k,\mathbb{R}) = f^{-1}[\{\pm\mathcal{J}\}]$ is closed in $GL_{2k}(\mathbb{R})$.

The determinant of $P^-(k,\mathbb{R})$ We will now show that the determinant of the matrices in the set $P^-(k,\mathbb{R})$, consisting of the real matrices with the property $S^T \mathcal{J} S = -\mathcal{J}$, is $(-1)^k$:

First, we claim the set $P^-(k,\mathbb{R})$ is given by

$$P^-(k,\mathbb{R}) = Sp(k,\mathbb{R})\,\mathcal{C} = \mathcal{C}\,Sp(k,\mathbb{R}), \tag{A.6}$$

with $\mathcal{C}$ defined in eq. (2.13). This is so because if $S' \in P^-(k,\mathbb{R})$, then $S'\mathcal{C} \in Sp(k,\mathbb{R})$ since

$$(S'\mathcal{C})^T \mathcal{J} (S'\mathcal{C}) = \mathcal{C}^T(-\mathcal{J})\mathcal{C} = \mathcal{J}, \tag{A.7}$$

and then $S' = S\mathcal{C}$ for $S = S'\mathcal{C} \in Sp(k,\mathbb{R})$, since $\mathcal{C}^2 = I$. Similarly with $\mathcal{C}\,Sp(k,\mathbb{R})$.

On the other hand, if $S \in Sp(k,\mathbb{R})$, then

$$(S\mathcal{C})^T \mathcal{J} (S\mathcal{C}) = \mathcal{C}^T \mathcal{J} \mathcal{C} = -\mathcal{J}, \tag{A.8}$$

so then $S\mathcal{C} \in P^-(k,\mathbb{R})$. Similarly, $\mathcal{C}S \in P^-(k,\mathbb{R})$.

Now we can evaluate the determinant of an arbitrary element in $S' \in P^-(k,\mathbb{R})$. Since $S' = S\mathcal{C}$ for an element $S \in Sp(k,\mathbb{R})$,

$$\det(S') = \det(S)\det(\mathcal{C}) = \det(\mathcal{C}), \tag{A.9}$$

since all matrices in $Sp(k,\mathbb{R})$ have determinant 1 [20]. The determinant of a $n \times n$ matrix A can be written (sum over repeated indices)

$$\det(A) = \epsilon^{i_1,\ldots,i_n} A_{1,i_1} \cdots A_{n,i_n} \tag{A.10}$$

(the Leibniz formula). Then there is only one non-zero term in this sum for the matrix $\mathcal{C}$, so the determinant is given by (no sum over k)

$$\det(\mathcal{C}) = \epsilon^{1,2,\ldots,2k} \mathcal{C}_{1,1}\,\mathcal{C}_{2,2} \cdots \mathcal{C}_{2k,\,2k} = 1^k(-1)^k = (-1)^k. \tag{A.11}$$

Hence by eqs. (A.9) and (A.11), the matrices of $P^-(k,\mathbb{R})$ have determinant $(-1)^k$.

$Sp(k,\mathbb{R})$ and $P^-(k,\mathbb{R})$ are not connected We want to show that $Sp(k,\mathbb{R})$ and $P^-(k,\mathbb{R})$ are two components of $P(k,\mathbb{R})$, i.e. they are not connected. Connected means the same as path connected for Lie groups. Assume that the two components are connected. Then there has to be a continuous path between e.g. $I \in Sp(k,\mathbb{R})$ and $R \in P^-(k,\mathbb{R})$. Let $X(t)$ be such a path, i.e. $X(0) = I$ and $X(1) = R$, where $X(t)$ is continuous. Consider the supremum

$$t_0 = \sup\{t \,|\, X^T(t)\mathcal{J}X(t) = +\mathcal{J}\}. \tag{A.12}$$

We know that $X(1)^T\mathcal{J}X(1) = -\mathcal{J}$. Moreover, consider the function

$$f(t) = \det(X^T(t)\mathcal{J}X(t) + \mathcal{J}), \tag{A.13}$$

which is continuous for continuous functions $X(t)$, since the determinant, matrix addition, multiplication and transposition are continuous. But $f(t)$ is discontinuous for $t = t_0$, since there in any open interval contining t_0 will be values t where $f(t) = 0$ and other values where $f(t) = \det(2\mathcal{J}) = 2^{2k}$, per definition of t_0. Hence our assumption that $X(t)$ is continuous must be wrong, and hence the sets $Sp(k,\mathbb{R})$ and $P^-(k,\mathbb{R})$ are not connected.

B. The map ρ

We will now introduce a map ρ which lets us easily translate between real and complex formulations of the kinetic terms we are studying. The map ρ preserves both matrix multiplication, addition and the identity.[14] We define ρ as a function from $M_k(\mathbb{C})$, the set of all $k \times k$ complex matrices, to $M_{2k}(\mathbb{R})$, the set of all $k \times k$ complex matrices by

$$\rho(X) = \begin{pmatrix} \mathrm{Re}(X) & -\mathrm{Im}(X) \\ \mathrm{Im}(X) & \mathrm{Re}(X) \end{pmatrix}. \tag{B.1}$$

With U a Lie group, ρ is a Lie group isomorphism from $U \subset M_k(\mathbb{C})$ to $\rho[U]$.

Now we want to show that the definition of ρ can be extended to vectors so that it preserve products of complex vectors and matrices: Let v be a complex $k \times 1$ vector, let $v = v_R + iv_I$, with v_R, v_I real and define

$$\rho(v) \equiv \begin{pmatrix} \mathrm{Re}(v) \\ \mathrm{Im}(v) \end{pmatrix} = \begin{pmatrix} v_R \\ v_I \end{pmatrix}, \tag{B.2}$$

and

$$\rho(v^\dagger) \equiv \begin{pmatrix} \mathrm{Re}(v^\dagger), & -\mathrm{Im}(v^\dagger) \end{pmatrix} = \begin{pmatrix} v_R^T, & v_I^T \end{pmatrix}. \tag{B.3}$$

Moreover, let A be a complex $k \times k$ matrix and let $A = (A_R + iA_I)$, with A_R, A_I real, then

$$\rho(Av) = \rho(A)\rho(v), \tag{B.4}$$

[14] ρ is an injective ring homomorphism [20]. On the other hand, the inclusion $\rho[U(2)] \subset SO(4)$ shows that ρ does not preserve the determinant, even though it is a ring (or group) isomorphism on its image.

since $\rho(Av) = \binom{(Av)_R}{(Av)_I} = \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} \binom{v_R}{v_I} = \rho(A)\rho(v)$. Furthermore, let u, v be complex $k \times 1$ vectors, then

$$\mathrm{Re}(u^\dagger A v) = \rho(u^\dagger)\rho(A)\rho(v), \tag{B.5}$$

since $\mathrm{Re}(u^\dagger A v) = \mathrm{Re}[(u_R^T - i u_I^T)(A_R + iA_I)(v_R + iv_I)] = (u_R^T\ u_I^T) \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} \binom{v_R}{v_I} = \rho(u^\dagger)\rho(A)\rho(v)$. Then,

$$u^\dagger A v + v^\dagger A^\dagger u = \rho(u^\dagger)\rho(A)\rho(v) + \rho(v^\dagger)\rho(A^\dagger)\rho(u), \tag{B.6}$$

since the left hand side of eq. (B.6) equals its real part.

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