Dark energy from quantum gravity

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Contents

Preface

The thesis consists of two parts, where the last one tries to model the evolution of the universe by using a scalar field. Within this class of models there is a huge amount of freedom and few simple restrictions. Often only examples are studied which are just tested against a subset of the requirements. In a try of being more systematic I developed a simulation for proposing and testing different evolutions of the universe more comprehensively.

A large subset of the proposed models pass the tests, and singling out models against observed requirements is like trying to start from the answer. For this reason I began to look for an underlying reason for the expansion of the universe. All conventional scalar field models are based on classical fields. Could the observed accelerated expansion of the universe emerge as a quantum gravitational phenomenon? The goal was a bit farfetched, but would provide a natural explanation for the expansion and could add insight to the field of quantum gravity.

After a longer period of unsuccessful tries, it was about time to write down the first draft of the thesis. It consisted of a review of dark energy models and quantum gravitational theory. Shortly after finishing the second revision I found a new way of attacking the problem. After a week of calculations I came to the conclusion that the new approach made sense and started over again, discarding the old draft.

The first part, which I regard to be the most important one, contains the new ideas and calculations. Here I studied the effect of uncertainties in the measurement of parallel transported four-vectors. This way of modeling quantum mechanical effects leads to a term which can cancel the vacuum energy contribution. The expansion of space results in an imperfect cancellation leading to a term equal to the observed dark energy. Though the results are not fully rigorous, since they aren't based on a fully developed theory of quantum gravity, they are surprising and give a reasonable basis for further studies.

I am grateful to my supervisor Kåre Olaussen for his encouragement, patience and liberal attitude when advising students. Big thanks goes to S. D. Odintsov and E. Elizalde for the hospitality during my stay at IEEC, Spain. And last, I want to thank Monique Pruss for pointing out a subtle mathematical flaw in the draft.

Chapter 1

Dark energy from quantum gravity

1.1 Introduction

What is dark energy? In 1998 a group of researchers observed an accelerated expansion of space [15]. By conventional gravitational theory, the acceleration can either be due to a previously unknown energy, the vacuum energy, or a constant in the equations of general relativity. In this thesis, the two terms will be used interchangeably.

The cosmological constant first was introduced by Einstein. He believed in a static universe and introduced a negative cosmological constant. This solution was instable and Hubble's discovery [19] of an expanding universe lead Einstein to remove the cosmological constant. In 1998 the universe was shown not only to expand, but the expansion also was accelerating. The acceleration can be explained by a small and positive cosmological constant. Unfortunately vacuum energy can provide an explanation for the cosmological constant, except that the answer differs with embarrassing 120 orders of magnitude.

Einsteins general relativity relates space-time geometry with the matter energy and stress in the universe. Two obvious tries are to either modify gravity itself or add new fields to the universe. These fields introduce large degrees of freedom to the models (see chapter 2) and there is no scarcity of models. Compared with observational data and weighted based on the number of free parameters, a model with a cosmological constant ranks highest. The fifth year Wilkinson Microwave Anisotropy Probe (WMAP) data strongly indicate a dark energy component acting like a cosmological constant. Combining WMAP data with observations of baryon acoustic oscillations (BAO) and Type Ia supernova's (SN) increases the accuracy and supports a cosmological constant. We accept these measurements and aim for a more fundamental theory of dark energy.

The second class of models modifies gravitation. By changing the Hilbert-Einstein Lagrangian, new versions of gravity arise. The modifications often contain no great justification beyond introducing more freedom which possibly can reproduce one or more effects. In this thesis modification of gravity will play a central role, but in the form of quantum gravity.

In the beginning of the 20th century two new directions started to emerge. The unified framework of classical mechanics split in a series of developments into two fundamental theories of nature. Einsteins general theory of relativity introduced a radical revision of gravity, space and time. Space and time form one entity which is changed by gravitational interactions. After observing light bend, the theory was put into contact with observation. Later the theory was studied intensively and remarkable effects were predicted. A large dying star can collapse and form a black hole. No light escapes a black hole and the black hole entropy numbers the amount of states within the black hole.

Quantum mechanics or the improved version, quantum field theory, is the second fundamental theory of physics. The theory has deep conceptual issues, but makes impressively good predictions. Electromagnetism, electroweak and the strong force are all quantum theories and combined form the standard model. Therefore quantum theory is tried to be given different interpretations which agree on the basic postulates up to small modifications. The quality of the interpretations and the number of them point to an unsolved problem. What is so special with nature on small scales? However philosophical, the question likely needs more insight before finding a theory of quantum gravity. In section 1.4 we investigate information in quantum measurements and conclude that limited information plays a central role in quantum mechanics.

General relativity is based on a continuous space-time, and measurements done by comparing two four-vectors are possible to an infinite good precision. Starting out from the vacuum energy and entropy of a black hole, we find a fundamental length in the universe. When treating time under the same standard, we find a seemingly negative form of energy. This expression cancels the vacuum exacting for a static universe. Spatial expansion naturally leads to an imperfect cancellation. The correction is an energy density that is exactly equal to the observed dark energy, but with higher order corrections.

1.2 Cosmology based on observations

The effects of quantum gravity are often thought of as only arising in regions with extreme gravitational effects like the early universe. Probing nature on the Planck scale is normally considered impossible, except by some brave experimentalists [4]. Like huge water tanks make proton decays experiments possible [27], the universe itself might magnify the effects of quantum gravity. Inflation, dark matter and dark energy are three possible candidates. The effects can be modelled mathematically, but require the introduction of new fields.

In the current section we review the observational status. Fluctuations in the cosmic microwave background (CMB) spectrum from the fifth year WMAP data set is the main source [18]. For improving the accuracy, these are combined with both baryon acoustic oscillations (BAO) and Type Ia supernova measurements (SN). The accuracy then becomes quite good and puts more speculative theories into trouble [38].

Cosmology starts with the cosmic principle [23] stating that the universe on large scale looks the same in all directions (isotropic) and that matter is evenly distributed (homogeneous). A spatially homogeneous and isotropic universe corresponds to the Friedmann-Lemaître-Robertson-Walker (FLRW) model.

Definition 1.2.1 (LFRW metric)**.** *The metric for the FLRW in spherical* *coordinates* (t, r, θ, ϕ) *is [21]*

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + \ r^2 \sin^2(\theta) d\phi^2. \qquad (1.1)$$

Theories that assume the general relativity

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \ \Lambda_{\mathrm{Grav}} g_{\mu\nu} \ = 8\pi G T_{\mu\nu} \qquad (1.2)$$

and aim to provide an explanation for dark energy, dark matter and/or inflation often introduce one or more fields. The simplest example is an ideal perfect fluid as the source, with the corresponding stress-energy-momentum tensor

$$T_{\mu\nu} = \mathrm{Diag}(-\rho, p, p, p). \qquad (1.3)$$

Three common perfect fluids are radiation, matter and the vacuum energy. They are described by

- Radiation - $p = \frac{1}{3}\rho \rightarrow \rho \propto a^{-4}$
- Matter - $p = 0 \rightarrow \rho \propto a^{-3}$
- Vacuum energy - $p = -\rho \rightarrow \rho \propto \mathrm{const}$

where a is the expansion factor in the FLRW metric 1.1. In cosmology, the equation of state (EOS) is defined as

$$\omega \equiv \frac{p}{\rho}. \qquad (1.4)$$

A dark energy component from vacuum energy has equation of state $\omega = -1$. The WMAP5 data [18]

Table 1.1: Total density and the equation of state

Description	Symbol	WMAP-only	WMAP+BAO+SN
Total density	Ω_{tot}	$1.099^{+0.100}_{-0.085}$	$1.0050^{+0.0060}_{-0.0061}$
Equation of state	ω	$-1.06^{+0.41}_{-0.42}$	$-0.992^{+0.061}_{-0.062}$

For a cosmological constant as dark energy, $\omega = -1$. From 1.1 The observed value $\omega = -0.992^{+0.061}_{-0.062}$ from WMAP+BAO+SN data strongly indicates a dark energy component behaving like a cosmological constant.

Energy densities Ω are measured relative to the critical density. At the critical density our universe is spatially flat. The different contributions to the total density are baryons (Ω_b), dark matter (Ω_d) and dark energy (Ω_Λ) [1] The total density

$$\Omega_{\mathrm{tot}} = \Omega_{\mathrm{b}} + \Omega_{\mathrm{c}} + \Omega_\Lambda \tag{1.5}$$

is Ωtot $= 1.0050$ for the WMAP+BAO+SN data 1.1. A spatially flat model is therefore justified. Further interpretations of measurements are based on a ΛCDM model and the reader is refereed to [18] for the justifications of a cold dark matter (CDM) component.

Table 1.2: The Hubble constant and energy densities

Description	Symbol	WMAP-only	WMAP+BAO+SN
Hubble constant	H_0	$71.9^{+2.6}_{-2.7}$ km/s/Mpc	70.5 ± 1.3 km/s/Mpc
Baryon density	Ω_b	0.0441 ± 0.0030	0.0456 ± 0.0015
Dark matter density	Ω_c	0.214 ± 0.027	0.228 ± 0.013
Dark energy density	Ω_Λ	0.742 ± 0.030	0.726 ± 0.015

The correspondence between a cosmological constant and the energy and momentum densities for a dark energy component is

$$\rho = \frac{\Lambda}{8\pi G} \tag{1.6}$$

$$p = \frac{-\Lambda}{8\pi G} \tag{1.7}$$

A FLRW model with constant acceleration from the cosmological constant is a de Sitter space. The metric is given by

$$ds^2 = -dt^2 + \exp(Ht)(dx^2 + dy^2 + dz^2) \tag{1.8}$$

Under the assumption of a ΛCDM model, the cosmological constant is related to the Hubble expansion H by

$$H = \sqrt{\Lambda/3}. \tag{1.9}$$

So, under the assumption of a ΛCDM model, the observed cosmological constant $\Lambda_{\mathrm{Observed}}$ using WMAP5, BAO and SN data [18] is

$$\Lambda_{\mathrm{Observed}} = 3 \cdot (70.5 \text{ m/s/Mpc})^2 \tag{1.10}$$

$$= 1.75 \cdot 10^{-35} \text{ s}^{-2}. \tag{1.11}$$

[1] The results are model independent and the notation for dark matter and dark energy simply is the conventional choice.

1.3 Vacuum energy contribution

In classical systems zero energy in a system is allowed, while in a quantum system, the ground state will always have energy above zero. A classical harmonic oscillator can have any energy, while the energy levels in quantum mechanical version [26]

$$E = (n + \frac{1}{2})\hbar\omega \tag{1.12}$$

have $n \in \mathbb{N}$. The ground state $E_0 = \frac{1}{2}$ corresponds to $n = 0$. A quantum field consists of a set of oscillators. Their zero-point energy is measurable and with e.g. the boundary condition of plates it gives rise to the Casimir effect [17]. In quantum fields theories the energy zero point energy can often be defined away by choosing a different value for zero energy. We are interested in one exception, how the vacuum energy affects gravitation. The vacuum energy density in a volume Vol is

$$\rho_{\text{Vac}} = \frac{1}{\text{Vol}} \frac{\hbar}{2} \sum_i \omega_i. \tag{1.13}$$

Corresponding to a vacuum energy ρ is the stress-energy-momentum tensor

$$T_{\mu\nu} = \text{Diag}(-\rho, \rho, \rho, \rho) \tag{1.14}$$

and this term can be moved to the other side of Einsteins field equation and interpreted as a cosmological constant Λ_{Vac}. The vacuum energy and the real cosmological constant Λ_{Grav} contribute on an equal footing.

Any serious attempt of explaining the value of the cosmological constant should take both contributions into account. Many researchers "forget" to mention this part when introducing new fields that can contribute to a small and positive cosmological constant. The energy density for the two terms is

$$\rho_{\text{Vac}} = \frac{1}{\text{Vol}} \frac{\hbar}{2} \sum_i \omega_i \tag{1.15}$$

$$\rho_{\text{Grav}} = \frac{\Lambda_{\text{Grav}}}{8\pi G} \tag{1.16}$$

For convenience we define Λ_{Vac} as the constant corresponding to a vacuum energy ρ_{Vac}.

$$\Lambda_{\text{Vac}} \equiv 8\pi G \rho_{\text{Vac}}. \tag{1.17}$$

The effective cosmological constant Λ_{Eff} is then

$$\Lambda_{\text{Eff}} \equiv \Lambda_{\text{Grav}} + \Lambda_{\text{Vac}} \tag{1.18}$$

The vacuum energy is calculated from summing up contributions from zero modes within a volume (1.13). Without a cutoff in frequency, the zero point modes sum to infinity. In the section about quantum measurements in gravity 1.6 we find a specific lower length in the universe. But most of this thesis simply uses

- l_f - Fundamental length
- t_f - Fundamental time

related by $l_f = t_f$ as the lowest entity of length and time. For a fundamental lower length l_f, then $\lambda \geq l_f$. Using a cubical volume l_f^3 as the lower volume and a mode with wavelength $2l_f$ just fitting, the vacuum energy is

$$\rho_{\text{Vac}} = \frac{1}{l_f^3}\frac{\hbar}{2}\frac{1}{2l_f} \tag{1.19}$$

$$= \frac{\hbar}{4l_f^4} \tag{1.20}$$

The cosmological constant Λ_{Vac} corresponding to a contribution from the vacuum energy is then

$$\Lambda_{\text{Vac}} = 8\pi G\rho_{\text{Vac}} \tag{1.21}$$

$$= \frac{2\pi G\hbar}{l_f^4}. \tag{1.22}$$

Conventionally the Planck length or Planck energy is used as the cutoff. Setting $l_f = 2l_P$ results in the following numerical value of the vacuum energy contribution.

$$\Lambda_{\text{Vac}} = \frac{1}{4t_P^2} \tag{1.23}$$

$$= \frac{1}{4(5.39 \cdot 10^{-44}\ \text{s})^2} \tag{1.24}$$

$$= 1.35 \cdot 10^{86}\ \text{s}^{-2} \tag{1.25}$$

The ratio between the cosmological constant with a Planck scale cutoff and the observed value is

$$\frac{\Lambda_{\text{Vac}}}{\Lambda_{\text{Observed}}} = \frac{1.35 \cdot 10^{86}\ \text{s}^{-2}}{1.77 \cdot 10^{-35}\ \text{s}^{-2}} \tag{1.26}$$

$$= 7.64 \cdot 10^{120}\ \text{s}^{-2}. \tag{1.27}$$

which is wrong with 120 orders of magnitude. Normally cosmologists are not concerned about the exact number, since it is horrible wrong anyway. We will therefore not discuss the usage of a cubic volume or the choice of the fundamental length.

A simple theory with only vacuum energy gives this embarrassing prediction. To explain the smallness we should find a mechanism for almost cancelling out the vacuum energy. By introducing a new field, such a result is easily derived, but would include an unscientific fine tuning of 120 orders of magnitude.

In a later section 1.6 we find a term perfectly cancelling ρ_{Vac} with little tuning. The small cosmological constant then arises as a correction. These results build on considering the role of measurement in quantum mechanics and general relativity. At first glance the next two sections might seem like a digression, but are the essential motivation for the argument to come and should not be skipped.

1.4 Finite information in physics

Space-time in general relativity is a manifold. The phase space M in classical mechanics is a differential manifold with observables as smooth functions, $C^\infty(M)$, over M. Quantum mechanics introduces an operator $\hat{f}$ to each classical observable f. A Hilbert space H became the physical state space and the foundation of quantum mechanics rests on four postulates. One of these is the measurement postulate included below [17].

Postulate 1.4.1. *The only possible measurements for a observable quantity F are the eigenvalues f_n of the quantum mechanical operator $\hat{F}$. In other words*

$$\hat{F}\,|\psi\rangle = f_n\,|\psi\rangle$$

where $|\psi\rangle \in H$.

A quantum mechanical measurement can therefore at times only give a discrete set of values. In classical physics a continuum of values is always allowed, except in the case of a constant. The information stored in one quantum mechanical state is in quantum information theory linked to the entropy. First, when classically (no quantum mechanics) sending information, the Shannon's theorem of noiseless encoding tells the average bits per symbol or the average information gained when doing a measurement.

Theorem 1.4.2 (Shannon's noiseless encoding theorem)**.** *[36] Let $X = (x_1, \cdots, x_n)$ be a set of random variables. For this probability the associated* Shannon entropy *with the distribution is*

$$H(X) \equiv H(p_1, \cdots, p_n) = \sum_i p_i \log(p_i). \tag{1.28}$$

Let there be a source *which has probability distribution X for different outcomes. For sending this information lossless over a classical communication channel, the optimal encoding needs $H(X)$ bits per source symbol.*

In quantum information theory, an information measure is defined for each state, the von Neumann entropy [26].

Definition 1.4.3 (von Neumann entropy)**.** *The* von Neumann entropy *for a state ρ in density operator notation is*

$$S(\rho) \equiv -\mathrm{Tr}(\rho \log(\rho)). \tag{1.29}$$

Another result shows how the von Neumann entropy can be calculated from the eigenvalues. If λ_x is the set of eigenvalues for ρ, then [26]

$$S(\rho) = -\sum_i \lambda_i \log(\lambda_i). \tag{1.30}$$

We are going to find an expression for the information gained when doing a specific measurement and a representation for a measurement operator.

Definition 1.4.4 (Diagonal representation)**.** *An operator $\hat{F}$ on a vector space H is in the* diagonal representation

$$\hat{F} = \sum_i \lambda_i |i\rangle \langle i| . \tag{1.31}$$

Here the vectors $|i\rangle$ are an orthonormal set of eigenvectors for $\hat{F}$ with eigenvalues λ_i.

The measurement postulate (1.4.1) tells that when measuring F you can only measure the eigenvalues λ_i for the physical observable F.

$$\hat{F} |j\rangle \langle j| = \lambda_j |j\rangle \langle j| \tag{1.32}$$

Postulate 1.4.5 (Expectation value)**.** *The expectation value for an observable quantity F in the state $|\psi\rangle$ is*

$$\langle F \rangle = \langle \psi | \hat{F} | \psi \rangle \tag{1.33}$$

When measuring F, the probability p_i of finding the system in $|i\rangle$ is

$$p_i = |i\rangle \langle i| \hat{F} |i\rangle \langle i| \tag{1.34}$$
$$= |i\rangle \langle i| (\sum_j \lambda_j |j\rangle \langle j|) |i\rangle \langle i| \tag{1.35}$$
$$= \lambda_i. \tag{1.36}$$

Let $|i\rangle$ correspond to the random variable $x_i \in X$ with probability λ_i. For this distribution, we can apply Shannon's classical theorem of noiseless communication. Instead of the von Neumann entropy for information in a state, we can define an entropy related to the measurements.

Definition 1.4.6 (Entropy of a measurement)**.** *For a measurement with an operator $\hat{F}$ with an orthonormal eigenvector basis $|i\rangle$ and eigenvalues λ_i. Define the* measurement entropy H_m *of F by*

$$H_m(F) \equiv -\sum_i \lambda_i \log(\lambda_i). \tag{1.37}$$

What does this mean? Let us use $\hat{F}$ to measure on the state

$$I/d = \sum_1^d |i\rangle \langle i| \tag{1.38}$$

Then H_m is the information gained when measuring F. Before using this insight, we need the following lemma.

Lemma 1.4.7 (Maximal entropy)**.** *Suppose X is a random variable with d distinct outcomes. Then $H(X) \leq \log(d)$ with equality if and only if X is uniformly distributed.*

The average information gained, given by the measurement entropy $H_m(F)$, has an upper bound $\log d$. In a continuum, the number of distinguishable states is infinite. General relativity use a continuous space-time, so in theory, two points can be distinguished below the Planck length. The exception comes from the formation of Black holes.

Over a critical density, a mass distribution would collapses into a black hole. This puts a limit on how good you can measure in general relativity. The wavelength λ of a single light quanta with energy E is

$$\lambda = \frac{h}{E}. \tag{1.39}$$

If E_f is the value for which the photon collapse into the black hole, then a minimal size for measurement is

$$l_f = \frac{\pi\hbar}{E}. \tag{1.40}$$

These considerations might seem overly theoretical and without direct relevance for dark energy and a quantum theory of gravity. On the contrary, it will found the basis for the next section 1.5. We use the effect of parallel transportation over a loop combined together with the difficulties of imperfect measurements. These considerations will play a central role in the last section 1.6 where we study how dark energy might arise from imperfect measurements.

1.5 Measurements in general relativity

In the last section we mention that general relativity is defined by using a manifold as the underlying space time. A manifold M is mathematically a topological space satisfying the following three conditions [37].

Definition 1.5.1 (Hausdorff)**.** *Let X be a topological space and $x, y \in X$. A space is* Hausdorff *if for any choice of x and y can find two open sets $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.*

Definition 1.5.2 (Second countable)**.** *A topological space X is* second countable *if X has a countable basis.*

Definition 1.5.3 (Locally homeomorphic)**.** *For each $x \in M$ there exist an open set $x \in U$ and an integer $n \geq 0$ such that U is homeomorphic to $\mathbb{R}^n$.*

We are not concerned with other technical details and how to define local charts. A volume within a continuous space-time can be split into an infinite amount of sets. Does space itself consist of a continuum of states? Or is only measuring them problematic? A remarkable result from general relativity is the Bekenstein formulae [6] for the entropy in a black hole. For a Schwarzschild black hole

$$ds^2 = (1 - \frac{r_s}{r})c^2 dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \tag{1.41}$$

light does not escape outside the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}. \tag{1.42}$$

The surface A of the sphere in spatial space with radius r_s is proportional to the degrees of freedom within the black hole volume. This result by Bekenstein is one of the most remarkable theorems in theoretical physics. A fundamental theory of quantum gravity should provide the same answer and both string theory and loop quantum gravity (LQG) [34] have expressions for the black hole entropy.

Theorem 1.5.4 (Black hole entropy)**.** *The entropy to a Schwarzschild black hole with horizon area A is*

$$S_{BH} = k_B \frac{A}{4l_P^2}. \tag{1.43}$$

In the last section 1.4 we discussed information in quantum mechanical systems. The result above is related to quantum mechanics through the Hawking radiation. A black hole radiates due to quantum effects, but since nothing except the radius, mass and charge (Kerr black hole) is known for an outside observer, we choose to view Bekenstein entropy as a pure space-time effect. This interpretation is not rigorous, but a starting point for studying quantum gravitational effects without matter fields.

What is measurement in general relativity? A general manifold has no concepts of length alone. A Riemannian manifold is a manifold with a bilinear form $g(\cdot, \cdot)$. In general relativity this form gives the inner product and induces a length on the manifold. For convenience we define the following notation

$$\langle v_1 | v_2 \rangle = g(v_1, v_2) \tag{1.44}$$

which is more used in mathematics.

The special theory of relativity compares the length of four-vectors by the Lorentz transformations. These transformations form the mathematically formulated predictions and follow from postulating a constant speed of light.

Theorem 1.5.5 (The Lorentz transformations)**.** *Let O_1 and O_2 be two observers in respectable coordinate systems (t, x, y, z) and $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ and observer O_2 moving in direction x with a speed V in relation to O_1. The coordinate systems are related by*

$$\tilde{t} = \gamma(t - \frac{vx}{c^2}) \tag{1.45}$$

$$\tilde{x} = \gamma(x - vt) \tag{1.46}$$

$$\tilde{y} = y \tag{1.47}$$

$$\tilde{z} = z. \tag{1.48}$$

General relativity further complicates the measurement of four-vectors. The theory relates how space-time curves with the density of energy, momentum and stress by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \tag{1.49}$$

where the left side contains geometrical terms from the curvature. Matter fields give, through variational calculations, a way for defining general relativity from a field, rise to the stress-momentum-energy tensor $T_{\mu\nu}$. It is defined by

Definition 1.5.6 (Stress-momentum energy tensor).

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}}. \tag{1.50}$$

The geometrical part, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ contains the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the metric $g_{\mu\nu}$. From the metric $g_{\mu\nu}$ the Riemann tensor, R^l_{ijk} a multi linear beast, is given by [16]

Definition 1.5.7 (Riemann tensor).

$$R^{\alpha}_{\beta\gamma\delta} = \frac{1}{2}(\frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\gamma\epsilon}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\epsilon}\Gamma^{\epsilon}_{\beta\gamma} \tag{1.51}$$

where Γ^m_{ij} is the Christoffel symbols

Definition 1.5.8 (Christoffel symbols).

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}). \tag{1.52}$$

These definitions are assumed known and briefly stating them in an unpedagogical way should be of no loss to the reader. To start with a given metric and then compute the tensors above is a common way of studying general relativity. A physically more intuitive understanding comes from the holonomy. The holonomy $H(\gamma, D)$ gives how each component of a vector will transform when being parallel transported over the path γ [5].

Definition 1.5.9 (Holonomy around a loop). *In the special case when parallel transporting over a loop 1.1 the holonomy is*

$$H(\gamma, D)^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - \epsilon^2 R^{\alpha}_{\mu\nu\beta}. \tag{1.53}$$

How can you determine if the space-time within a loop is flat? Let $\epsilon > 0$ be the side length of the loop. One way of determining the curvature is to parallel transport vector v around two different paths and compare the results. We will study the effect of comparing to light signals. The figure 1.2
shows the two different paths through space-time. In the lower path in the diagram a light signal is first sent, received and the observer waits for the

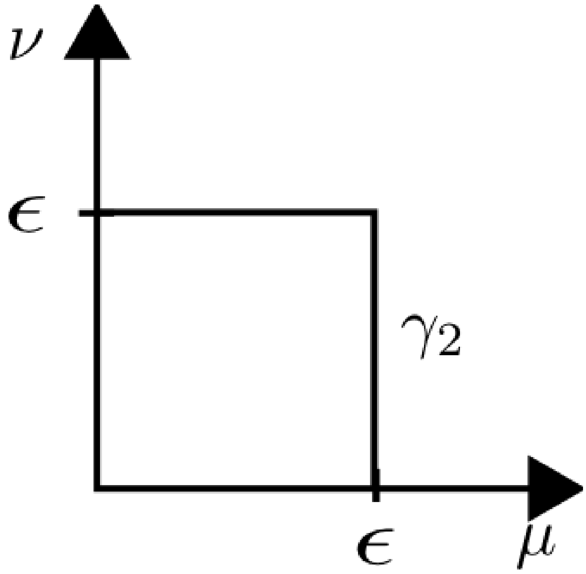

Figure 1.1: Holonomy around a spatial loop

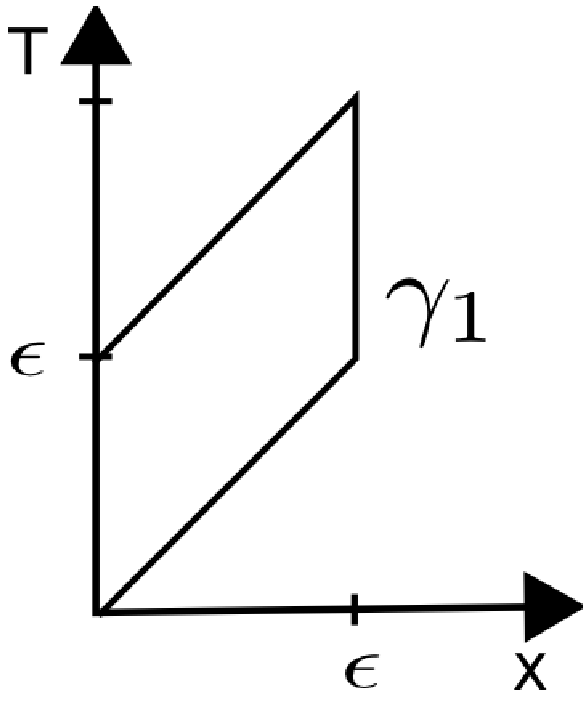

Figure 1.2: Holonomy around a time-light loop

next signal to arrive. The upper path is the world line of the second signal sent out a time ϵ after the first signal.

When parallel transporting a vector v around the loop 1.2 γ_2, the resulting vector $\tilde{v}$ is

$$\tilde{v} = H(\gamma_2, D)v. \tag{1.54}$$

Here, the the component v_i is transformed to $\tilde{v}_j$ by

$$\tilde{v}_j = v_j - \epsilon^2 R^i_j v_i. \tag{1.55}$$

where repeated indexes *do not* indicate summation. Before using the holonomy to perform calculations, we need the definition of holonomy along piecewise smooth curves, the holonomy around 1.2 and some identities.

Definition 1.5.10 (Holonomy of a piecewise smooth path). *[5] Let $\gamma : [0, T] \to M$ be a piecewise smooth path from p to q, $p, q \in M$. A piecewise smooth path can be decomposed into smooth pieces $\gamma_i : [t_i, t_{i+1}] \to M$ where $1 \leq i \leq n$ and $\gamma = \gamma_n \circ \cdots \gamma_1$. Here $\circ$ denote path compositions. The holonomy is then given by*

$$H(\gamma, D) = H(\gamma_n, D) \cdots H(\gamma_1, D) \tag{1.56}$$

Theorem 1.5.11. *The Holonomy around the space-time loop 1.2 γ_2 is equal to the space-space loop 1.1 $gamma_1$.*

Proof. The group element ϕ_l for the light transport and its inverse ϕ_{-l} is

$$\phi_l = \exp(\epsilon(\nabla_t + \nabla_x)) \tag{1.57}$$
$$\phi_{-l} = \exp(-\epsilon\nabla_x - \epsilon\nabla_t). \tag{1.58}$$

The holonomy around the loops γ_1 and γ_2 is defined by

$$H(\gamma_1, D) = \phi_{-t} \circ \phi_{-x} \circ \phi_t \circ \phi_x \tag{1.59}$$
$$H(\gamma_2, D) = \phi_{-t} \circ \phi_{-l} \circ \phi_t \circ \phi_l. \tag{1.60}$$

Then the effect of parallel transporting around the loop γ_2 can be shown equal to parallel transport around γ_1.

$$\begin{aligned} \gamma_2 &= \phi_{-t} \circ \phi_{-l} \circ \phi_t \circ \phi_l && (1.61)\\ &= \phi_{-t} \exp(-\epsilon\nabla_x - \epsilon\nabla_t)\ \exp(\epsilon\nabla_t) \exp(\epsilon\nabla_t + \epsilon\nabla_x) && (1.62)\\ &= \phi_{-t} \circ \phi_{-x} \circ \phi_t \circ \phi_x && (1.63)\\ &= \gamma_1 && (1.64) \end{aligned}$$

We have thereby shown that the holonomy around the loops γ_1 and γ_2 is equal. □

Now we state some simple identities for the holonomy. For a path α from p to q,

$$H(\alpha, D)^{-1} = H(\alpha^{-1}, D) \tag{1.65}$$
$$H(1_P\alpha, D) = H(\alpha, D) \tag{1.66}$$
$$H(\alpha 1_P, D) = H(\alpha, D) \tag{1.67}$$
$$H(1_P, D) = 1 \tag{1.68}$$

where 1_P is a path $1_P(t) = p$ for all t and $\alpha^{-1}\alpha = 1_P$.

The resulting four-vectors after parallel transporting a vector v over two paths γ_1 and γ_2 is respectably v_1 and v_2. In terms of the holonomy, the parallel transported vectors are

$$v_1 = H(\gamma_1, D) \tag{1.69}$$
$$v_2 = H(\gamma_2, D). \tag{1.70}$$

For comparing two four-vectors in the same point, you can only use the vector space properties and the inner product. The holonomy preserves the metric, hence

$$\langle v_1, v_2 \rangle = \langle H(\gamma_1, D) | H(\gamma_2, D) v \rangle \tag{1.71}$$
$$= \langle v | H(\gamma_1^{-1} \circ \gamma_2, D) v \rangle \tag{1.72}$$
$$= \langle v | H(\gamma, D) v \rangle \tag{1.73}$$

by (1.66) and defining $\gamma \equiv \gamma_1^{-1} \circ \gamma_2$.

Simply measuring the inner product between two parallel transported vectors is a natural first try. For simplifying the result we use the following symmetry of the Riemann tensor.

$$R^a_{bcd} = -R^a_{bdc}. \tag{1.74}$$

The Ricci tensor $R_{\mu\nu}$ is formed by contracting two indexes in the Riemann tensor.

Definition 1.5.12 (Ricci tensor)**.**

$$R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu} \tag{1.75}$$

Further, the Ricci scalar R comes from contracting the remaining two indexes in the Ricci tensor.

Definition 1.5.13 (Ricci scalar).

$$R \equiv R^{\alpha}_{\alpha} \tag{1.76}$$

The physics in figure 1.2 is defined more easily by introducing a new set of coordinates

$$u = t - x \tag{1.77}$$
$$v = t - y \tag{1.78}$$
$$w = t - z. \tag{1.79}$$

In these coordinates the metric for the FLRW (1.1) model is

$$ds^2 = (3a^2(t) - 1)dt^2 + a^2(t) \cdot [2dt(du + dv + dw) + \ (du^2 + dv^2 + dw^2)] \tag{1.80}$$

The curvature will give a difference in parallel transportation. Starting from (1.73) we find

$$\langle v_1 | v_2 \rangle = \langle v | v - \epsilon^2 R_{0i} v \rangle \tag{1.81}$$
$$= \langle v | v \rangle - \epsilon^2 g^{\beta\lambda} R^{\alpha}_{0i\beta}\ v_\alpha v_\lambda \tag{1.82}$$
$$= \langle v | v \rangle - \frac{1}{2}\epsilon^2 g^{\beta\lambda}\ (g^{\alpha}_{\beta} g^{\beta}_{\alpha})\ R^{\alpha}_{0i\beta} v_\alpha v_\lambda \tag{1.83}$$
$$= \langle v | v \rangle - \frac{1}{2}\epsilon^2 R^{\alpha}_{0i\alpha} \langle v, v \rangle \tag{1.84}$$
$$= \langle v | v \rangle + \frac{1}{2}\epsilon^2 R_{0i} \langle v, v \rangle \tag{1.85}$$

Let the inner product between the two parallel transported vectors v_1 and v_2 be expressed by

$$\langle v_1 | v_2 \rangle = \kappa \langle v | v \rangle. \tag{1.86}$$

The curvature can then be expressed in terms of κ and ϵ

$$R_{0i} = 2\frac{\kappa - 1}{\epsilon^2}. \tag{1.87}$$

For $\kappa = 1$, the curvature component R_{0i} is flat. An observer measuring κ with an uncertainty, can not determine if space is flat. When measuring $1 - \mu \leq \kappa \leq 1 + \mu$, then

$$-2\frac{\mu}{\epsilon^2} \leq R_{0i} \leq 2\frac{\mu}{\epsilon^2}. \tag{1.88}$$

Uncertainties in measurements put restrictions on how well you can measure, but are those limits only theoretical? A fundamental uncertainty in measuring μ is likely quite small and ϵ can always be larger by using a larger loop. But any uncertainty affects how gravity works and the effects can be magnified on a cosmological scale. In the next section we will study such effects.

The holonomy has the nice property of being independent of the gauge. In a physical system, there might be several mathematical descriptions giving rise to the same physical system. The concept is deeply related to symmetries and we will first introduce continuous symmetries groups, the Lie groups [37].

Definition 1.5.14 (Lie group). *A* Lie group *is a group G which is also a manifold with a C^∞ (smooth) structure such that*

$$(x, y) \mapsto xy \tag{1.89}$$

$$x \mapsto x^{-1}. \tag{1.90}$$

are C^∞ functions.

For a physical system, the Lie groups show which states that are considered equal by symmetry. This idea is best explained by considering an object in three dimensional space. Imagine rotating the object a bit? Is the rotated object different? Yes, but we are often considering objects which can be rotated into each other as equal. Mathematically, a length preserving rotating in a three dimensional (vector) space is corresponding to the Lie group $SO(3)$ [32].

Definition 1.5.15 (The rotation group SO(3)). *The Lie group SO(3) consists of all linear transformations of a three-dimensional vector space E that preserve a positive-definite inner product and with determinant $+1$.*

$$SO(3) = \{a \in M_3(\mathbb{R}) | a^* a = 1, \det(a) = +1\} \tag{1.91}$$

where $*$ denotes the adjoint with respect to the inner product.

Let S be all three dimensional objects (e.g. manifolds) and $G = SO(3)$. Two physical objects s_1 and s_2 in S can be considered equal if there exists a $g \in G$ such that $s_2 = gs_1$. To relax equalities is often done both in physics and in daily life almost automatically. Two herds of sheep can superficially be considered equal if the number of animals is equal. This concept might seem very trivial, but not treating equality correctly can lead to problems which are hard to discover. Soon we will see on example, but we need to introduce the concept of an equivalence relation.

Definition 1.5.16 (Equivalence relation)**.** *Let A be a set and $\sim$ a binary relation. The binary relation $\sim$ is called an equivalence relation if and only if the three following properties hold*

- *Reflexivity* $a \sim a$
- *Symmetry* $a \sim b \rightarrow b \sim a$
- *Transitivity* $a \sim b \quad \text{and} \quad b \sim c \rightarrow a \sim c$

Lemma 1.5.17. *For a group G and set A, sim_g is an equivalence relation defined by: Let $a \sim_g b \quad \forall \quad a, b \in A$ if there exist a $g \in G$ so $a = gb$.*

A theory describing time dependence of a system with gauge, should not treat states equal up to gauge different. For not introducing all the notation of classical physics, let us consider a simplified example. Let S denote a set of states and G a gauge group. Two states s_1 and s_2 are considered equal up to gauge ($s_1 \sim s_2$) if there exists a $g_1 \in G$ such that $s_1 = g_1 s_2$. Now, let $f : S \mapsto S$ be the time evolution of states in S between the times t_0 and t_1. If there does not exist a $g_2 \in G$ such that $g_2 f(s_1) = f(s_2)$, then the time evolution does not treat two physically similar states in the same manner.

General relativity can mathematically be given a gauge. A formal definition of a *G-connection* which is used for formulating gauge in general relativity builds on a formal definition of vector bundles and connections. Since we are only going to state one theorem proven elsewhere [5], we skip stating the formal definitions [37] [5].

Definition 1.5.18 (Wilson loop)**.** *The* Wilson loop

$$W(\gamma, D) = Tr(H(\gamma, D)) \tag{1.92}$$

is the trace of the holonomy around the loop γ.

Theorem 1.5.19. *The Wilson loop is gauge invariant.*

Proof. See [5]. □

Physical observables should not be gauge dependent and this is one reason for us to use the inner product of the parallel transported vectors (1.73).

1.6 Quantum measurements of gravity

In the last section we studied measurement of four-vectors after parallel transporting around a loop. When using the loop 1.2 a parameter κ was introduced to give the uncertainty when measuring the inner product of the parallel transported vectors. So far we only stated that imprecise measurement from e.g. quantum effects, would lead to $\kappa \neq 1$. In this section we will introduce and study a specific model for κ.

The uncertainty on measurements κ is generally a function $\kappa(t, x, y, z)$, where the presence of matter fields or any other physical effect is indirectly included by the coordinates. For simplifying the problem, we study the effect in deep space away from galaxies where the baryonic and dark matter is located. Only the vacuum energy can safely be considered to influence the measurements.

A κ varying with the presence of matter can possibly give measurable effect. That is not necessarily contradicting observation. Besides dark energy, for explaining the rotational speed of galaxies, dark matter has been introduced. Introducing new scalar fields can explain the presence of dark matter, but these fields are as mentioned ad hoc postulated after the observations.

How do we model a non-zero κ? In our model we assume κ is a constant in absence of matter, otherwise the uncertainty would be coordinate dependent. Further we study the parallel transportation around a small loops. This justified by considering a $\kappa \neq 0$. The curvature around a loop in the limit of zero area is

$$\lim_{\epsilon \to 0} R_{0i} = \lim_{\epsilon \to 0} 2\frac{\kappa - 1}{\epsilon^2} \tag{1.93}$$

$$= \infty \tag{1.94}$$

Later we will find $R_{0i} = \Lambda$ and space-time being a de Sitter space. An infinite Λ would lead to a not observed infinite spatial acceleration. A natural try is to propose that there is a fundamental length or area which the loops move around. We therefore continue to use l_f and the corresponding time t_f as a lower measure for respectable length and time in our universe.

When $\kappa = -1$, two parallel transported vectors $|v_1\rangle$ and $|v_2\rangle$ has the inner product

$$\langle v_2|v_1\rangle = -\langle v|v\rangle. \tag{1.95}$$

where $|v\rangle$ is the vector before parallel transportation. In this limit measurement of parallel transport of four vectors completely breaks down. Let $|v_1\rangle = (a, b, 0, 0)$ be the four vector parallel transported in one direction with a suitable basis. For $\kappa = -1$, the four vector transported around the other path yields $|v_2\rangle = (b, a, 0, 0)$. The space and time component is switched for the two paths. In the following we will use $\kappa = -1$.

For $\epsilon = l_f$ and $\kappa = -1$, the resulting curvature component R_{0i} is

$$R_{0i} = \frac{-4}{l_f^2}. \tag{1.96}$$

1.6.1 Black hole entropy

What is the best measurement possible of the interior state of a black hole? Imagine being able to form a sphere of detectors near the surface. A black hole radiates, but where on the surface are you most likely to observe a photon? No light and information escapes the black hole, so we do not have any knowledge of the internals of a black hole. With no knowledge of the internals, the probability distribution for measuring a particle is uniform over the sphere of detectors.

Divide the surface S into a set of cells (a_i) so $S = \cup_i a_i$ and $a_i \cap a_j = \emptyset$ when $i =\neq j$. Let f be the constant probability density of measuring a particle. Then $p_i \equiv f a_i$ is the probability of measuring a particle in the area a_i. Classically, the information gained from measuring the black hole is

$$H = -\sum_i f a_i \log(f a_i). \tag{1.97}$$

An observer maximizes the gained information from a black hole by using $a_i = a_j$ for all i, j. For $b \equiv a_i$, then

$$\frac{A}{b} \tag{1.98}$$

is the number of distinct outcomes the observer on the outside can measure.

How is that related to the interior physics of a black hole? Let S_1 and S_2 be two finite sets where $f : S_1 \to S_2$ is 1-1 and onto S_2. The number of members in S_1 and S_2 is then equal. Our observer measures a finite set of outcomes S_2. A physical law f relates the internal states of a black hole (S_1) to the observed outcome (S_2).

The exact form of f does not matter, the observer should only think such a law exists. For each observed outcome y, she infers that the there exists a state $x \in S_1$ such that $f(x) = y$ (onto). If she also thinks there is no hidden state (1-1), then the outside observer will assume the number of states inside the black hole equals the number of states observed.

For those conditions, an observer measuring in area units of b on the surface of a black Hole estimate

$$\frac{A}{b} \tag{1.99}$$

degrees of freedom within the black hole. And relating these internal degrees of freedom to entropy yields

$$S_{\text{Black hole}} = \frac{A}{b}. \tag{1.100}$$

What is the lowest area an observer could measure? Let l_f be the fundamental length. The fundamental area is then l_f^2 and the black hole entropy is

$$S_{\text{Black hole}} = k_B \frac{A}{l_f^2}. \tag{1.101}$$

When setting the fundamental length $l_f = 2l_P$, the expression

$$S_{\text{Black hole}} = k_B \frac{A}{4l_P^2}. \tag{1.102}$$

reduces to the Bekenstein entropy. While this argument is not rigorous, it motivates setting $l_f = 2l_P$ in the expression of R_{0i}.

1.6.2 Dark energy

When the fundamental length equals to two Planck length, then

$$R_{0i} = \frac{-4}{l_f^2} = \frac{-1}{l_P^2}. \tag{1.103}$$

How does this term relate to the vacuum energy and the cosmological constant? In section 1.2 we mentioned that the LFRW model (1.1) is when assuming the cosmological principle the most general spatially flat cosmological model. A LFRW model in the coordinates (1.79), has the following non-zero components (see appendix A)

$$R_{00} = \frac{3}{a}[2a\dot{a}^2 + \ddot{a}a^2 - \ddot{a}] \tag{1.104}$$

$$R_{0i} = 2\dot{a}^2 + a\ddot{a}. \tag{1.105}$$

where $a = a(t)$, $\dot{a} = \frac{da(t)}{dt}$ and $\ddot{a}(t) = \frac{d^2}{dt^2}a(t)$. Define $K \equiv R_{0i}$. Then $K = \Lambda$ is a constant since R_{0i} is a constant. This leads to

$$R_{00} = 3(2\dot{a}^2 + \ddot{a}a - \frac{\ddot{a}}{a}) \tag{1.106}$$

$$= 3(K - \frac{\ddot{a}}{a}) \tag{1.107}$$

Let ϕ_t be the line through space-time moving a length t in the time direction. If $R_{00} \neq 0$, then $\phi_t \circ \phi_t \neq \phi_t \circ \phi_t$. We therefore demand R_{00}. The acceleration for the spatial expansion is then

$$\frac{\ddot{a}}{a} = \text{const.} \tag{1.108}$$

This corresponds to a de Sitter space. For a de Sitter metric 1.8, the R_{0i} component is related to a cosmological constant by

$$R_{0i} = 3H^2a^2(t) = \Lambda. \tag{1.109}$$

We therefore have a contribution Λ_{Grav} to the cosmological constant from ρ_{Grav}, and

$$\Lambda_{\text{Grav}} = \frac{-1}{l_P t_P} \tag{1.110}$$

The value of the vacuum energy was of the same order. By choosing a different type of volume, the differing factors of 2s and π can be eliminated. The space-time geometry is quite unknown territory and for avoiding a high level of numerology, we use

$$\Lambda_{\text{Vac}} = -\Lambda_{\text{Grav}}. \tag{1.111}$$

The combined effect Λ_{Eff} 1.18 from Λ_{Grav} 1.110 and Λ_{Vac} 1.22 with no spatially expansion is then

$$\Lambda_{\text{Eff}} \equiv \Lambda_{\text{Grav}} + \Lambda_{\text{Vac}} \tag{1.112}$$
$$= 0 \tag{1.113}$$

Compared to other approaches, the amount of tuning is quite innocent. The important part is how an exact cancellation for a static universe would change when the universe is expanding.

Let $\tilde{H}\,\frac{\dot{a}(t)}{a(t)}$ denote the actual Hubble constant which affects the loop sizes. Only the spatial lengths are expanding and not the time dimension. For a time T and the actual Hubble rate $\tilde{H}$, a spatial length L changes with a length $\delta L = \tilde{H}\,TL$. If the observer is moving in the spatial direction with the speed of light, the length expands with $\delta L = \frac{1}{2}\,\tilde{H}\,TL$ by a simple geometric consideration.

Gravitational and vacuum contributions come from physics involving a different number of spatial dimensions. The gravitational effect resulted from a minimum uncertainty when sending a signal. The loop in space-time was lying in a slice with one time and one spatial dimension. For the vacuum energy, the result was dependent upon a spatial volume, i.e. three spatial dimensions.

Let us imagine an empty universe. Is rotating the physical system observable? According to Machs principle, it is not. Rotating a four-vector in space-time should then preserve the length. Let $v_1 = (3x, 3x, 3x, 0)$ be a four-vector used when measuring the contribution of the space-time loop and $v_2 = (0, x, x, x)$ for the volume. These two four-vectors have the same length since

$$
\begin{aligned}
|v_1| &= -3x^2 + 3x^2 + 3x^2 && (1.114)\\
&= 3x^2 && (1.115)\\
|v_2| &= x^2 + x^2 + x^2 && (1.116)\\
&= 3x^2. && (1.117)
\end{aligned}
$$

The spatial length expansion in Λ_{Grav} is therefore 3 times as big as the ones in Λ_{Vac}. We observe in the three volume the expansion

$$
\begin{aligned}
x &= \delta l && (1.118)\\
&= \frac{1}{2}\,\tilde{H}\,l_f t_f && (1.119)
\end{aligned}
$$

so the spatial length in the light-time loop expands with $3x = 3\,\tilde{H}\,l_f t_f$. The effective cosmological constant in a spatially expanding space with conserved energy is then

$$
\begin{aligned}
\Lambda_{\mathrm{Eff}} = &\ -\frac{4}{t_f l_f (1+\frac{3\delta l}{l_f})} && (1.120)\\
= &-\frac{4}{t_f l_f (1+\frac{3\delta l}{l_f})} + \frac{4 l_f}{l_f^3 (1+(\frac{\delta l}{l_f})^3} && (1.121)\\
= &\ \frac{4}{l_f^2}[-(1+\frac{3\delta l}{l_f})^{-1} + \ (1+\frac{\delta l}{l_f})^{-3}] && (1.122)\\
& && (1.123)
\end{aligned}
$$

From Taylor expansion we find the following term

$$
-\frac{1}{1+3x} + \frac{1}{(1+x)^3} = -3x^2 + 17x^3 + O(x^4) \qquad (1.124)
$$

The effective cosmological constant, which both includes contributions from gravity and vacuum, is then

$$
\begin{aligned}
\Lambda_{\mathrm{Eff}} = &\ -3(\frac{\delta l}{l_f^2})^2 + 17(\frac{\delta l}{l_f^2})^3 && (1.125)\\
= &-3\,\tilde{H}^2 + 17\,\tilde{H}^3. && (1.126)
\end{aligned}
$$

Here both the vacuum energy density and the uncertainty decay when space is expanding. Physically we know the vacuum energy density is constant and if the uncertainty κ is unchanged, then the density of the gravitational contribution should also be a constant. Now, waving the hands a bit and consider the situation from the viewpoint of the observers moving over the loop. If the loop contains more energy due to the expansion, while the observer is considering the loop size fixed, the energy density would grow so

$$\begin{aligned} \Lambda_{\mathrm{Eff}} &= \frac{4}{l_f^2}[-(1+3x)+(1+x)^3] && (1.127)\\ &= \frac{4}{l_f^2}[3\frac{1}{4}\,\tilde{H}^2\,l_f^2] && (1.128)\\ &= 3\,\tilde{H}^2\,. && (1.129) \end{aligned}$$

With this argument, the two contributions which cancel each other in a static space, give rise to a small cosmological term when space is expanding. The result depends on the actual expansion $\tilde{H}$ and little tuning of the parameters.

1.7 Conclusion

Dark energy is the result of an imperfect cancellation due to spatial expansion.

Chapter 2

Reconstructing the evolution of the universe

2.1 Introduction

The current accelerated expansion of space caught the physics community with surprise when discovered in 1998[15]. The common belief at that time was an evolution where space first expanded rapidly in an early phase, big bang or inflation [20], the universe then turned into a decelerated phase such that the expansion would eventually stop. The acceleration leads to the introduction of a hypothetical form of energy, dark energy, responsible for the current acceleration of the expansion. Furthermore, another form of matter, dark matter, was introduced to explain rotational speed of galaxies. Recent data suggests that our universe consists of 72.1% dark energy, 23.3% dark matter and 4.6% baryonic matter [18].

Great efforts have been taken to find viable explanations for the nature of dark matter and dark energy. Several theories/models have been introduced (see [11] and [14] for two recent reviews). Quintessence introduces a scalar field. The only known effects of dark energy are gravitational and the effect does only minimally interact with gravity. The action can be described by an action

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2}(\nabla\phi)^2 - V(\phi)] \tag{2.1}$$

where $V(\phi)$ is the potential of the field. This potential can lead to a late time acceleration. The expansion rate of space is often characterized by the Hubble parameter, which is the relation $H \equiv \frac{\dot{a}(t)}{a(t)}$ where $a(t)$ is the time dependent space expansion

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{2.2}$$

The Hubble parameter and expansion are then

$$H^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \tag{2.3}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}[\dot{\phi}^2 - V(\phi)] \tag{2.4}$$

which can provide different rates of expansion by changing the form of the scalar fields potential $V(\phi)$.

A different way is K-essence, which instead of modeling the expansion by different potentials, uses a non-standard form of kinetic energy. In other models, like the quintessence (2.1), the kinetic energy is given by a separate term $\frac{1}{2}\nabla\phi^2$ in the action. Instead, in the most general models the variable X is used for the kinetic energy. The action is then given by

$$S = \int d^4x\sqrt{-g}p(\phi, X) \tag{2.5}$$

where the variable name p is chosen, since the Lagrangian density can be interpreted as a pressure.

Another theory is phantom fields, or ghost fields. A relation between the pressure and energy can be defined by $\omega = \frac{p}{\rho}$. Observation predicts a value of ω lying around -1. Phantom fields have, unlike e.g. the quintessence models $\omega < -1$. These models can either arise directly from a scalar field with negative energy or the Brans-Dicke scalar-tensor theory, which is an alternative gravitational theory.

Further, there are brane models [39], Chaplygin gas and tachyon fields. In other words, there is no scarcity of models in the field of cosmology. The available observational data is currently not good enough to rule out many of the suggested dark matter and dark energy theories. In fact, one of the simplest models, which combines the cosmological constant Λ and cold dark matter (ΛCDM), is still not ruled out by observations [18]. Lacking data, people even started to use the Bayesian "Akaike information criterion" (AIC) and BIC information criterias for model selection, to rank the different theories [38].

Several theories operate with the assumption that Einstein's general theory of relativity is correct. Einsteins general relativity, including a cosmological constant, is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.6}$$

The first class of models, like scalar field theories introduced a new form of matter that is only introduced to provide an explanation for the expansion and is not motivated by particle physics. Another class of theories suggests modification to general relativity (GR) by allowing a more general form for the Einstein-Hilbert action [28]

$$S = \int d^4x \sqrt{-g} f(R). \tag{2.7}$$

Using $f(R)$ gravitation theories, theorists have constructed theories for unified matter dominated and accelerated phases. The study of the bullet cluster (1E 0657-56) consisting of two colliding clusters of galaxies provide evidence for dark matter [10] [24].

In this paper, we study models for unifying inflation and late time acceleration [9]. One possibility is a scalar field with varying equation of state ω (EOS), being able to account for both phenomenons. This field must predict a decelerated phase for formation of large scale structures and behave like a cosmological constant today. Here only models with one scalar-field are studied. One known problem with these is the known instability when crossing between phantom ($\omega < -1$) and non-phantom phases.

The aim of our work is to construct a simpler and more systematic method to find models satisfying a set of requirements. The theories can be characterized by the Hubble parameter $\frac{\dot{a}}{a}$ or a. Using symbolic computations, we have created a program for testing models given the Hubble parameter against a set of requirements. These requirements arise from our knowledge of the universe. Further, another program provides a way for randomly generating possible evolutions of the universe. Together these two are used for suggesting models governing dark matter and dark energy.

In the present paper we develop the general bounds for unified inflation and dark energy models. Models satisfying these requirements are then presented. Section 2.2 describes scalar-fields with evolving EOS uncoupled to matter. Then in section 2.3 we discuss restrictions on these models from observational traits. Further in section 2.4 a program for automatically generating acceptable solutions is introduced and finally in section 2.5 example models are presented.

2.2 Scalar-tensor theory

Scalar-tensor theory can be stated in the form of the action [30]

$$S = \int d^4x\sqrt{-g}\{\frac{R}{2\kappa^2} - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\} + S_m \tag{2.8}$$

where $V(\phi)$ and $\omega(\phi)$ are functions of the scalar field. The matter term S_m is not coupled to dark matter.

In a homogeneous and isotropic universe, under conditions that hold on a large scale, the Friedmann-Lemaître-Robertson-Walker (LFRW) metric (2.6).

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-\sigma r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{2.9}$$

is a solution in general relativity. The constant σ describe the geometry and a close, flat and open universe corresponding to $\sigma = +1, 0, -1$. For a FLRW metric, we can use Einsteins equations of gravity

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu = 8\pi G T^\mu_\nu \tag{2.10}$$

to find the Friedmann equations. We use the R^0_0 curvature term

$$R^0_0 = 3\frac{\ddot{a}}{a} \tag{2.11}$$

and consider an ideal perfect fluid as the source, so the T^μ_ν term is given by

$$T^\mu_\nu = \mathrm{Diag}(-\rho, p, p, p). \tag{2.12}$$

In this paper we will assume spatially-flat LFRW metric

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3} dx_i{}^2. \tag{2.13}$$

since the latest Wilkinson Microwave Anisotropy Probe (WMAP) five-year data set predicts $-0.0175 < \Omega_k < 0.0085$ [18]. The two resulting independent equations for $\sigma = 0$ are the two Friedmann equations

$$\frac{3}{\kappa^2}H^2 = \rho, \quad -\frac{2}{\kappa^2}\dot{H} = p + \rho \tag{2.14}$$

and conservation for energy

$$\dot{\rho_m} + 3H(\rho_m + p_m) = 0. \tag{2.15}$$

Further, the energy and pressure for the scalar field ϕ is

$$\rho = \frac{1}{2}M(\phi)\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}M(\phi)\dot{\phi}^2 - V(\phi) \tag{2.16}$$

By using the Friedmann equations (2.14), we find an expression for EOS in terms of the Hubble parameter

$$\omega \equiv \frac{p}{\rho} = \frac{-\frac{a}{\kappa^2}\dot{H} - \frac{3}{\kappa^2}H^2}{\frac{3}{\kappa^2}H^2} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}. \tag{2.17}$$

Now we can use the results from [30] to express $\omega(\phi)$ and $V(\phi)$ in terms of the single function $f(\phi)$

$$M(\phi) = -\frac{2}{\kappa^2}f'(\phi), \quad V(\phi) = \frac{1}{\kappa^2}(3f(\phi)^2 + f'(\phi)). \tag{2.18}$$

By varying the field ϕ, the result is the scalar-field equation

$$\rho = M(\phi)\ddot{\phi} + \frac{1}{2}M'(\phi)\dot{\phi}^2 + 3HM(\phi)\dot{\phi}^2 + V'(\phi). \tag{2.19}$$

Inserting $\phi = t$ and (2.18) in (2.19), shows that [30]

$$\phi = t, \quad H = f(t) \tag{2.20}$$

is a possible solution. A scalar-tensor theory without matter interactions is therefore described by a single function $f(t)$.

2.3 Constraints on the Hubble function

Below follows a list of 6 requirements that we use to evaluate if a Hubble function is a candidate to provide the observed expansion of our universe.

Requirement 1 and 2 - The universe accelerated in the start For our universe to have evolved quickly, two requirements are put on the acceleration at the beginning of time. The first one

$$\lim_{t \to 0} \frac{\ddot{a}(t)}{a(t)} > 0 \tag{2.21}$$

only requires a positive acceleration, while the second requirement

$$\lim_{t \to 0} \dot{H} > 0, \quad \text{i.e.} \ \frac{\ddot{a}(t)}{a(t)} > \frac{\dot{a}^2}{a} \tag{2.22}$$

requires even more acceleration. Theories that predict too little expansion on space can be discarded, since they do not fit with our estimates for the size of the universe.

Requirement 3 - The equation of state ω equals -1 for late times Recent WMAP5 observations predict $\omega = -1.60^{+0.41}_{-0.42}$ for current times with a 95% confidence level[18]. Combining WMAP5 data with Super nova Ia and baryonic oscillation in galaxies increases the accuracy to $\omega = -0.972^{+0.061}_{-0.060}$. We therefore use the requirement

$$\lim_{t \to \infty} \omega = -1. \tag{2.23}$$

which does not necessarily hold true. Alternatively our current phase of seemingly constant acceleration is temporary or the universe can be cyclic [39]. A cyclic universe will repeatedly go through expansion and contraction phases, but the nature of those models is speculative.

In a non-matter coupled scalar-tensor theory, the requirement (2.23) can with (2.17) be written in terms of the Hubble parameter

$$\lim_{t \to \infty} \frac{\dot{H}}{H^2} = 0. \tag{2.24}$$

Requirement 4 - The Hubble constant is positive A negative H corresponds to a contracting universe. Our Universe might go through a contracting phase in the future, but until now, no such phase has been observed. A natural choice is therefore to introduce the requirement that $H > 0$ for all $t > 0$.

Requirement 5 and 6 - There exists a decelerated and a phantom phase For a decelerated phase to exist, $\ddot{a}$ must be negative for some value of t. If the universe is in a phantom phase, then $\omega < -1$ or $\dot{H} > 0$ for an interval of t values. It follows from the definition of H that

$$\dot{H} = \frac{\ddot{a}}{a^2} + (\frac{\dot{a}}{a})^2 \tag{2.25}$$

and a further requirement for deceleration from the phantom phase. A well known problem is the instability of the system when crossing between phantom and non-phantom phases, $\omega = -1$, as discussed in [7].

2.4 Reconstruction theory

Reconstruction of the Hubble parameter can be done in different ways. Experimental cosmology has provided significantly better observations during the last two decades. Among these are supernovae data, which were the first to show an accelerating expansion of the universe. Researchers have tried to determine $H(z)$, the Hubble functions as a function of the red shift, using statistical tools [35]. Their approach used the Gold and SNLS as datasets and reconstructed $h(z)$ and $\omega(z)$ independently of dark energy model. A number of attempts have been made [13], including usage of other data sets like CMB data [22].

Other researchers propose models based on a theoretical framework with many limitations. Toy models are presented as examples and account for one or more desirable traits, and undesirable effects are often not discussed [12]. Ideally a realistic model is tested simultaneously against the set of all requirements. One undesirable fact can discard the model as unrealistic.

2.4.1 Function generation

To find a model that accounts for all known phenomenons connected to the expansion of space is not straightforward. The theoretical analysis or intuition leading to suggest one model often originates from the wish of satisfying a subset of known requirements. A model made with one requirement in mind often ends up violating the other requirements, and the other way around. Our approach, instead of this guess work, is to take a brute force approach to the problem. A method based on brute force looses the advantage of prior theoretical hints, but it is more efficient and systematic in the end.

A model with non-coupled dark matter can equivalently be stated by $a(t)$, $h(t) = f(t)$ or $\omega(t)$. We choose to generate and study models specified by the Hubble parameter. The function generating part is written in Python [33], a modern high-level and dynamically typed language suitable for rapid development. Generated evolutions of the Hubble parameter are written to a file in Maple syntax and each model is evaluated in the next step.

The focus here is on the set of analytic functions that are combinations of simple functions. They are generated from combining the functions cos, sin and exp, the operators *, +, -, / and pow and four variables t, n, $k1$, $k2$. The values of n are restricted to integer values to reduce the amount of possible tuning in models involving n. A function needs one argument and operators two. To generate a function, one starts out with picking one of the already mentioned functions or operators. Now the argument is generated by another random pick that also includes the variable. Naturally this part of the code is created with a recursive function. Models can contain different numbers of terms, based on a random decision of when to stop the recursion, but an upper limit is set to four levels.

2.4.2 Discarding models

After the function generation, the Maple program checks the validity of the different models by comparing them with the stated requirements 2.3. For each function all requirements are tested in serial order, and discarded when one of them does not hold. The resulting output will be functions simultaneously satisfying all requirements.

Our models for the Hubble parameter often include several tunable parameters. One choice of values for the parameter set might satisfy one requirement, but violate another. By changing the values the previously failing requirement might be fulfilled, but now the first requirement is failing. One example is the Hubble parameter

$$H = \exp(-\alpha t), \tag{2.26}$$

which can individually fulfill the two requirements

$$\lim_{t\to\infty} \dot{H} = 0 \tag{2.27}$$

$$\lim_{t\to 0} \dot{H} > 0. \tag{2.28}$$

The first requirement holds for $\alpha > 0$ and the second for $\alpha < 0$. They hold separately, but no choice of parameter simultaneously satisfies both. A set of combined requirements is more restrictive than the requirements alone. The work of checking if a model can satisfy all requirements simultaneously, is often time consuming and filled with guess work.

Maple [2] can handle symbolic calculations. Our manipulations of functions involve derivatives and simple forms of limits. That functionality, at least derivation, can easily be created from scratch in another language [3]. The reason for basing this work on Maple instead of other packages for symbolic calculations is the *assume* facility. The assume facility is able to find possible intervals of the functions based on assumptions about the equations. A simple example are to requirement $a + b = 0$ and $a > 0$. The following Maple code shows that Maple concludes that b is negative.

```
> assume(a::real,b::real);
> additionally(a+b = 0);
> additionally(a > 0);
> coulditbe(b<0);
                                    true

> coulditbe(b>0);
                                   false
```

2.4.3 Problems when testing several requirements

Each requirement results in a subset (possible empty) of the parameter space, where the requirement holds. As we saw, comparing these subsets poses further restrictions on our model. Each requirement is ideally not checked alone, but together, to further restrict the allowed parameter space. A set of n requirements gives rise to $n(n+1)/2$ checks. When $n = 6$, like in our case, then the number of different tests is 21. It should be noted that 2 of our requirements are slightly redundant. A super accelerated start (requirement 2) is naturally also accelerated (requirement 1) and a phantom phase (requirement 6) implies a decelerated phase (requirement 5).

There are two types of requirements, those that include a time parameter and time independent ones. Expressions that include a limit of t, e.g.

$$\lim_{t \to 0} \dot{H} > 0 \tag{2.29}$$

are time independent, while $H > 0$ for $t > 0$ is time dependent. The difference is crucial, because Maple has problems with time dependent ones. For example, assuming $k \cos(t) + c > 0$ in Maple, is not interpreted as $c > b$. This assumption also affects the time variable, which is physically unacceptable, since we are looking for a Hubble parameter which is acceptable for all $t > 0$.

In this paper, the problem is solved by relaxing the requirements. For requirement 5 and 6 we only test these against existing requirements, but not how they affect each other. Since a phantom phase implies a decelerated phase, this is currently not a problem. Further the positive Hubble parameter requirement is dropped. Instead we check for this manually afterwards. Maple also sometimes struggles with satisfying requirements involving intricate functions. These models are dropped without any considerable loss. The computational time indicates a large amount of fine tuning and the models are therefore considered unusable.

2.5 The number of possible reconstructions

Approaching the reconstruction problem from testing a set of functions instead of hand calculations, gives another interesting perspective. How many of the generated scalar-field evolutions satisfy our requirements? Instead of only studying the known amount of models in the literature, we can use the ratio

$$\mathrm{r} = \frac{\text{Usable models}}{\text{Tested models}} \tag{2.30}$$

to determine how restrictive the constraints 2.3 are. If the ratio is high, the resulting models are less likely to be the right unification of inflation and late-time acceleration. They are just mathematical formulas fitting weak requirements. Further work should then try to incorporate additional restrictions coming from e.g. primordial perturbations or the number of e-folding from inflation.

The ratio depends both on which requirements are used to restrict and which functions are tested. Currently the algorithm generates functions that are typographically different, but mathematically equivalent. For example the two functions

$$H(t) = t + kt \tag{2.31}$$
$$H(t) = kt \tag{2.32}$$

are equal when redefining k in one of them. If more passing formulas are double generated, or vice versa, this will affect the ratio.

Also, our choice of functions, operators and variables is not unique. Harmonic functions were included to allow for oscillation [29] universes, but e.g. *arcsin*, *arccos* and *arctan* were excluded. By changing these, the ratio will vary. Studies of how the restrictions work, should therefore be done with a fixed set of functions, operators and variables.

2.5.1 Sample scalar-field models

Let us now consider an example of a rapidly expanding universe, with the Hubble given by:

$$H(t) = t^{kt} \tag{2.33}$$

where k is a constant. Using (2.17) and (2.33) we find the corresponding equation of state:

$$\omega(t) = -1 - \frac{\frac{2}{3}k\ln(t) + k}{t^{kt}} \tag{2.34}$$

From $\ddot{a}/a = \dot{H} + H^2$ we find the acceleration of the spatial component to be

$$\frac{\ddot{a}(t)}{a(t)} = k^{kt}(k\ln(t) + k) + (t^{kt})^2. \tag{2.35}$$

The universe is in a decelerated phase when

$$k\ln(t) + k < -t^{kt}. \tag{2.36}$$

Further it has phantom epochs, since the Hubble parameter derivative is:

$$\dot{H}(t) = t^{kt}(k\ln(t) + k) \tag{2.37}$$

which gives a phantom epoch when $t < \frac{1}{e}$.

As a second example we consider a Hubble function given by:

$$H(t) = (kt + s)^p \tag{2.38}$$

here k, s and p are constants. This evolution has the corresponding equation of state:

$$\omega = -1 - \frac{2}{3}(kt + s)^{-p-1}pk. \tag{2.39}$$

For $p+1 > 0$ and equal signs of k and s, a singularity will occur for $t_s = -\frac{s}{k}$. Then the equation of state will be infinite because $|p| \to \infty$, $|\rho| \to 0$ or both. Choosing $p + 1 < 0$ will avoid this singularity [31] [8], but only behave like a cosmological constant close to $t_s = -\frac{s}{k}$. We therefore choose equal signs for k and s and avoid the singularity and in addition get $\omega = -1$ for late times. This universe passes through different phases, since the acceleration

$$\frac{\ddot{a}(t)}{a(t)} = (kt + s)^{p-1}pk + (kt + s)^{2p} \tag{2.40}$$

switches sign. The universe is in a decelerated epoch for

$$t < \frac{1}{k}[\exp\frac{\ln pk}{p+1} - s], \quad t > -\frac{k}{s} \tag{2.41}$$

$$t > \frac{1}{k}[\exp\frac{\ln pk}{p+1} - s], \quad t < -\frac{k}{s}. \tag{2.42}$$

Further the system can not both avoid a singularity and undergo a transition between a phantom and a non-phantom epoch, since the derivative of the Hubble parameter:

$$\dot{H}(t) = (kt + s)^{p-1}pk \tag{2.43}$$

changes sign for $t = -\frac{s}{k}$. This model therefore gives rise to both accelerated and decelerated epochs, but no phantom eras. Also it is interesting that this scalar field behaves like a cosmological constant for late times.

2.6 Conclusion

We have proposed a method for finding possible scalar-field theories. This was not equally effective as originally expected. An improved work should both include more requirements and create a better mechanism for discarding functions. The original idea was to study the models using numerical simulations. But analytical requirements are computationally less intensive.

A direction of future research can be to combine analytical and numerical requirements in a hybrid framework for studying a general set of cosmological evolutions of the universe. In the first step a large set can be discarded based on simple tests, and the promising ones can be tested using eg. predictions for irregularities in the CMB spectrum.

It should be noted that the method of generating functions and testing them against a set of requirements also applies to other fields than cosmology. Instead of suggesting methods and checking their validity by working out all the details by hand, the program can instead suggest a list of possible models obeying the requirements.

Appendix A

GRTensorII

Calculating quantities in GR like the Christoffel symbols (1.5.8), Riemann tensor (1.5.7), Ricci tensor (1.5.12) and the Ricci scalar (1.5.13) is often tedious and error prone. Under follows an example of how to use GRTensor II [25] to calculate the Ricci tensor from the FRW metric (2.9).

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{A.1}$$

By choosing another set of coordinates, t, u, v, w

$$u = x - t \tag{A.2}$$
$$v = y - t \tag{A.3}$$
$$w = z - t \tag{A.4}$$

with t unchanged makes calculating on light rays easier. Transforming (A.1) to the new coordinates (A.4) yields

$$ds^2 = (3a^2(t) - 1)dt^2 + a^2(t)[(du^2 + dv^2 + dw^2) + 2dt(du + dv + dw)] \tag{A.5}$$

A.1 Metrics

GRTensor input file for the first metric (A.1)

```
Ndim_ := 4:
x1_ := t:
x2_ := x:
x3_ := y:
x4_ := z:

g11_ := -1:
g22_ := a(t)^2:
```

```
g33_ := a(t)^2:
g44_ := a(t)^2:
Info_:= ‘Spatially Flat Friedman-Robertson-Walker metric. \
         Perfect fluid‘:
```

and the metric with changed coordinates (A.5)

```
Ndim_ := 4:
x1_ := t:
x2_ := u:
x3_ := v:
x4_ := w:

g11_ := (3*a(t)^2-1):
g22_ := a(t)^2:
g33_ := a(t)^2:
g44_ := a(t)^2:
g12_ := a(t)^2:
g13_ := a(t)^2:
g14_ := a(t)^2:
g21_ := a(t)^2:
g31_ := a(t)^2:
g41_ := a(t)^2:
Info_:= ‘Spatially flat LFRW metric‘:
```

A.2 Maple program

```
# Load GRTensor package
grtw():

# Load metric from filename frw2.mpl
qload( frw2 ):

# Calculate and display the Ricci tensor
grcalc( R(dn,dn) ):
grdisplay( R(dn,dn) ):

# Calculate and display the Ricci scalar
grcalc( Ricciscalar ):
grdisplay( Ricciscalar ):
```

A.3 Maple session

```
# Load GRTensor package
```

```
> grtw():
                 GRTensorII Version 1.79 (R4)

                       6 February 2001

  Developed by Peter Musgrave, Denis Pollney and Kayll Lake

             Copyright 1994-2001 by the authors.

Latest version available from: http://grtensor.phy.queensu.ca/

>
# Load metric from filename frw2.mpl
> qload( frw2 ):
Warning: grOptionMetricPath has not been assigned.
Calculated ds for frw2 (0.000000 sec.)
                            Default spacetime = frw2

                            For the frw2 spacetime:

                                  Coordinates

                                     x(up)

                                  a
                               x    = [t, u, v, w]

                                  Line element

   2           2           2          2
 ds   = (3 a(t)   - 1)  d t   + 2 a(t)   d t  d  u

             2                     2
      + 2 a(t)   d t  d  v  + 2 a(t)   d t  d  w

           2     2        2     2        2     2
      + a(t)   d u   + a(t)   d v   + a(t)   d w

              Spatially flat LFRW metric

>
# Calculate and display the Ricci tensor
> grcalc( R(dn,dn) ):
Calculated detg for frw2 (0.000000 sec.)
```

```
Calculated g(up,up) for frw2 (0.004000 sec.)
Calculated g(dn,dn,pdn) for frw2 (0.000000 sec.)
Calculated Chr(dn,dn,dn) for frw2 (0.000000 sec.)
Calculated Chr(dn,dn,up) for frw2 (0.004000 sec.)
Calculated R(dn,dn) for frw2 (0.004000 sec.)
                               CPU Time  = 0.012

bytes used=2553916, alloc=1834672, time=0.08
> grdisplay( R(dn,dn) ):
                            For the frw2 spacetime:

                                Covariant Ricci

                                   R(dn, dn)

R [a]  [b] =

    [%1 , 0  , 0  ,  0 ]
    [                  ]
    [%2 , %2 , 0  ,  0 ]
    [                  ]
    [%2 , 0  , %2 ,  0 ]
    [                  ]
    [%2 , 0  , 0  ,  %2]



       /                   / 2       \         / 2       \\
       |       /d      \2  |d        |     2   |d        ||
%1 := 3|2 a(t) |-- a(t)|  + |--- a(t)| a(t) - |--- a(t)||/a(t)
       |       \dt     /   |  2      |         |  2      ||
       \                   \dt       /         \dt       //


                         / 2       \
        /d      \2       |d        |
%2 := 2 |-- a(t)|  + a(t) |--- a(t)|
        \dt     /        |  2      |
                         \dt       /

>
# Calculate and display the Ricci scalar
> grcalc( Ricciscalar ):
Calculated Ricciscalar for frw2 (0.000000 sec.)
```

```
                                 CPU Time  = 0.

bytes used=2652740, alloc=1834672, time=0.09
> grdisplay( Ricciscalar ):
                            For the frw2 spacetime:

                                  Ricci scalar

                            /     / 2      \             \
                            |     |d       |   /d      \2|
                          6 |a(t) |--- a(t)| + |-- a(t)| |
                            |     |  2     |   \dt     / |
                            \     \dt      /             /
                     R  = --------------------------------
                                           2
                                       a(t)

> quit
bytes used=2668760, alloc=1834672, time=0.09
```

Appendix B

Source code

The source code of the driver is written in Bash[1], the function generator in Python [33] and test of evolution requirements in Maple[2].

B.1 Driver

```
#!/usr/bin/env bash
# Automatically generate a scalar fields and then
# test against constraints.

# Generate function list
echo "Generate functions"
./gen_evolutions.py 1000 > functions

# Which functions statisfy all demands?
echo "Check functions"
maple -q test_evolutions.m
```

B.2 Generate evolutions

```
#!/usr/bin/python
# Generate n different functions.
import random
import sys

# Cos, +
sel = ['t','t','s','k1','k2','n','1/t']
two = ['*','+','-','/','^']
arg = ['cos','sin']

total = sel + two + arg
m = len(total) - 1

def compare(p1,p2):
    return (p1==p2)

def pick(level = 0):
    level = level + 1

    # Avoid reaching the Python maximum recursion
    # limit
    if level > 4:
        return sel[random.randint(0,len(sel)-1)]
```

```
    op = total[random.randint(0,m)]

    if op in arg:
        op = "%s(%s)" % (op,pick(level))
    elif op in two:
        l1 = pick(level)
        l2 = pick(level)

        if (compare(l1,l2) and op == "-"):
            raise Exception

        op = "(%s) %s (%s)" % (l1,op,l2)

    return op

def generate(nmax):
    """Generate nmax functions."""

    already = []
    i = 0
    while i<nmax:
        try:
            res = pick()
        except Exception:
            continue

        if not res in already:
            print "t -> %s" % res
            already.append(res)
            i = i + 1


def main():
    if len(sys.argv) == 1:
        print "Usage: %s number_of_functions" \
                % sys.argv[0]

        sys.exit(1)

    nfunct = int(sys.argv[1])
    generate(nfunct)

if __name__ == '__main__':
    main()
```

B.3 Test evolutions

```
dmsg := proc(msg)
  local debug:

  debug := false:

  if debug then
    print(msg):
  end if:
end proc:

read_file := proc(file_name)
   local functions_file, line, h_list:

   functions_file := fopen(file_name, READ, TEXT):
   line := readline(functions_file):

   h_list := []:
   while (line <> 0) do
      h_list := [op(h_list),parse(line)]:
      line := readline(functions_file):
   end do:

   close(functions_file):
   return h_list:
end proc:

wrapper := proc(file_name, ngood)
   global eps, descriptions, demands, h_deriv,
          a_twice, l1, l2, l3, ntests:
   local h_list:

   eps := 0.01:
   # Description of demands
   descriptions := ["Accelerated start",
                    "Super accel start",
                    "w=-1 for late times",
                    "Positive h(t)",
                    "Exist deaccel phase",
                    "Exist phantom phase"]:

   # Demands
```

```
   demands := [l1 < 0, l2 > 0, l3 = 0]:

   h_deriv := t-> diff(h(t),t):
   a_twice := t -> h(t)**2 + h_deriv(t):

   l1 := eval(a_twice(t),t=eps):
   l2 := eval(h_deriv(t), t=eps):
   l3 := limit(h_deriv(t)/h(t),t=infinity,'left'):

   # Evaluating the functions.
   ntests := nops(demands):
   check_functions(read_file(file_name),ngood):
end proc:

check_functions := proc(h_list,ngood)
   global h, n, t, k1, k2, s:
   local testnr, dem, results, nfits, nfound,
         ntot, res, skipit:

   results := [0,0,0,0,0,0]:
   nfound := 0:
   ntot := 0:

   assume(t::real,t>0):
   protect(t):
   for h in h_list do:
      ntot := ntot + 1:
      dmsg(h(t)):

      # Clear assumptions
      n,k1,k2,s := 'n','k1','k2','s':
      assume(n::integer,k1::real,k2::real,s::real):

      print(ntot):
      testnr := 0:
      skipit := false:
      for dem in demands do:
          testnr := testnr + 1:
          dmsg(cat("testnr ", testnr)):

          res := test_demand(dem):
          if (not res = 1) then
              skipit := true;
              break;
```

```
        end if;
    end do:

    # Avoid two loops
    if skipit then
        next;
    end if;

    dmsg("before 5"):
    e := solve([a_twice(t) = 0,t>0],t):
    if (length(e) = 1) then
        dmesg("found no a"):
        next:
    end if:

    dmsg("before 6"):
    f := solve([h_deriv(t) = 0,t>0],t):
    if (length(f) = 1) then
        dmesg("found no h_deriv"):
        next:
    end if:

    print(h(t));
    nfound := nfound + 1:
    print("tot: ",ntot, 'found', nfound):

    dmsg("end of loop"):
  end do:

end proc:

test_demand := proc(demand)
  try:
    dmsg(demand):
    additionally(evalf(demand)):

    return 1
  catch "cannot assume on a constant object":
    dmsg("testing a constant demand"):

    if evalf(demand) then
      return 1:
    else:
      return 0:
```

```
      end if:
   catch "contradictory assumptions":
      return 0:
   catch "attempting to assign to \
         `%1` which is protected":
      # The limit t->infinity does not give a
      # clear answer..
      return 0:
   end try:
end proc:

main := proc()
   local file_name,ngood:

   file_name := "functions":
   ngood := 2:

   wrapper(file_name,ngood):
end proc:

main():
```

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