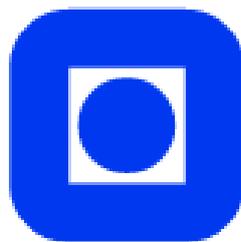


# Energy spectra and scaling relations in numerical turbulence with laboratory and astrophysical applications

by

Nils Erland Leinebø Haugen

Thesis submitted in partial fulfillment of the requirements for  
the Norwegian academic degree of *Doktor Scientarium*



Department of Physics  
Norwegian University of Science and Technology  
Trondheim, Norway



# Abstract

In this work we have focused on the statistical properties of turbulence. This has been done in two different settings; one with neutral gas (the first four papers) and the other with ionized gas (the last four papers).

Regarding the work on the neutral gas, we have looked at four different aspects;

1. Is the mean energy dissipation rate,  $C_\epsilon$ , independent of Reynolds number for large Reynolds numbers? This is one of the fundamental questions in turbulence, and one believe the answer will be yes, but this is as yet not conclusive. In Paper 1 we demonstrate that the value of  $C_\epsilon$  is highly sensitive to the method used to measure it. This might explain the discrepancies in the values of  $C_\epsilon$  found by previous authors. We also show how one can find  $C_\epsilon$  for a spread of Reynolds numbers from a single simulation.
2. Is there a “bottleneck” in the energy spectrum between the inertial range and the dissipative range? Such a bottleneck is extremely weak - or totally absent, in wind tunnel experiments. In large numerical simulations however the bottleneck is pretty clear. In Paper 2 we show that this discrepancy is due to the physical nature of the one-dimensional energy spectra found in wind tunnels and the three dimensional energy spectra found in numerical simulations.
3. In order to achieve larger Reynolds numbers we investigate the possible errors introduced by using hyper viscosity instead of normal viscosity in Paper 3. Our conclusion is that while hyper viscosity increase the hight of the bottleneck and shortens the dissipative range, it does not otherwise have any significant effect on the energy spectrum, or the structure functions. The inertial range and the large scales are the same both with normal viscosity and hyper viscosity.
4. In decaying turbulence one can find relations under which the Navier-Stokes equations are scale invariant. Using these relations it has recently been suggested by Ditlevsen et al.[1] that the energy spectrum for decaying hydrodynamical turbulence can be described by a scaling function with only two arguments. This has previously been shown both analytically and experimentally, and in Paper 4 we also confirm this in numerical experiments.

For the ionized gas we have focused on five different aspects;

1. What does the large Reynolds number energy spectra look like? Are the kinetic and magnetic energy spectra similar? The results are as yet not conclusive because the Reynolds numbers are still too small, but it seems that what at first looked like a  $k^{-3/2}$  inertial range is actually the bottleneck in a  $k^{-5/3}$  inertial range. Furthermore we have in Paper 5 found that the peak of the magnetic energy spectrum is *not* proportional to the resistive scale, but to the forcing scale.
2. As intermittency is still an unresolved topic we have looked at the different structure functions of the MHD dynamo. In Paper 6 the longitudinal structure functions based on the Elsasser variables are found to scale like in the model of She & Leveque[2], and the magnetic field is more intermittent than the velocity field. The Elsasser variables have been shown to have a linear scaling of the third order structure function. We do not, however, find the same linear scaling for the individual structure functions of the magnetic and the kinetic field.
3. In Paper 6 we investigate the growth rate of the magnetic field as a function of magnetic Reynolds number, and we find the critical magnetic Reynolds number as a function of magnetic Prandtl number.
4. How is the dynamo altered when one imposes an external large scale magnetic field? In Paper 7 we find that an imposed field tend to suppress the dynamo activity on all scales if the field is large enough. For an imposed magnetic field of the same size as the rms velocity field equipartition is found between magnetic and kinetic energy spectra.
5. Will there be dynamos in supersonic media? One could envisage that the supersonic shock swept up and dissipated the magnetic fields before they got time to grow. Numerical simulations in Paper 8 seem to show that as one increases the Mach number toward unity the critical magnetic Reynolds number increases, but as the Mach number grows even more the critical magnetic Reynolds number stays approximately constant.

# Acknowledgment

First of all I want to express my gratitude to my supervisor Professor Axel Brandenburg. Axel is one of those great people that is always motivating and helpful, both in research and otherwise in life. He has taught me basically all I know of turbulence, and has been an invaluable inspiration and support in all of my scientific work.

As Axel is not at NTNU but at NORDITA in Copenhagen, Alex Hansen has been my formal supervisor here at NTNU, and I am grateful for his support and help. I also would like to thank NORDITA for its great hospitality, I have really appreciated my many stays in Copenhagen.

As there is not much MHD going on at NTNU the value of the many interesting discussions and good cooperation with Tarek A. Yousef can not be exaggerated.

Regarding financial support for my many trips to Denmark I am in dept to the Norwegian Research Council (NRC), and to Professor Bo-Sture Skagerstam who, in a troubled time, was willing to be responsible for my NRC application. I am also in dept to the department of physics at NTNU and to NORDITA who payed for many of my trips.

Furthermore I will like to thank all of the people at Theoretical Physics at NTNU, in particular Terje Røsten for teaching me about Linux and for helping me with all of my daily computer problems, Erlend Østgaard for introducing me to Astrophysics and inspiring me to continue with a PhD, and Tommy Øvergård and Jan Myrheim for interesting astrophysical discussions.

Finally I will like to thank for all the support, understanding and interest shown me by my dear wife Ragnhild and my parents Solbjørg and Ola. Without their encouragement and help I would never have got this far!

Trondheim, August 2004

Nils Erland Leinebø Haugen



# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The PENCIL CODE</b>	<b>4</b>
<b>3</b>	<b>Statistics</b>	<b>7</b>
3.1	Energy spectrum . . . . .	7
3.2	Structure functions . . . . .	8
<b>4</b>	<b>Analytical models</b>	<b>10</b>
4.1	Kolmogorov theory . . . . .	10
4.2	Iroshnikov-Kraichnan (IK) . . . . .	12
4.3	Goldreich-Sridhar (GS) . . . . .	13
<b>5</b>	<b>What we have learned</b>	<b>15</b>
<b>6</b>	<b>What next?</b>	<b>19</b>



# List of scientific papers

**Paper 1:** B. R. Pearson, T. A. Yousef, N. E. L. Haugen, A. Brandenburg, and P. Å. Krogstad

“Delayed correlation between turbulent energy injection and dissipation”  
*Phys. Rev. E* **70** 056301 (2004).

**Paper 2:** W. Dobler, N. E. L. Haugen, T. A. Yousef, and A. Brandenburg

“Bottleneck effect in three-dimensional turbulence simulations”  
*Phys. Rev. E* **68** 026304 (2003).

**Paper 3:** N. E. L. Haugen and A. Brandenburg

“Inertial range scaling in numerical turbulence with hyperviscosity”  
*Phys. Rev. E* **70** 026405 (2004)

**Paper 4:** T. A. Yousef, N. E. L. Haugen, and A. Brandenburg

“Self-similar scaling in decaying numerical turbulence”  
*Phys. Rev. E* **69** 056303 (2004).

**Paper 5:** N. E. L. Haugen, A. Brandenburg, and W. Dobler

“Is nonhelical hydromagnetic turbulence peaked at small scales?”  
*Astroph. J.* **597** L141 (2003).

**Paper 6:** N. E. L. Haugen, A. Brandenburg, and W. Dobler

“Simulations of nonhelical hydromagnetic turbulence”  
*Phys. Rev. E* **70** 016308 (2004).

**Paper 7:** N. E. L. Haugen and A. Brandenburg

“Suppression of small scale dynamo action by an imposed magnetic field”  
*Phys. Rev. E* **70** 036408 (2004).

**Paper 8:** N. E. L. Haugen, A. Brandenburg, and A. J. Mee

“Mach number dependence of the onset of dynamo action”  
*Month. Not. Roy. Ast. Soc.* **353** 947 (2004).



# Chapter 1

## Introduction

The foundation of fluid mechanics is the Navier-Stokes equation, which appeared in its current version for the first time in 1827, when Navier introduced the viscous term in the Euler equation. Later on, Stokes got his name associated with the famous equation by showing theoretically why there must be an additional viscous term to the Euler equations. The Navier-Stokes equations consist of the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \nabla \cdot (2\nu \rho \mathbf{S}), \quad (1.1)$$

but since any fluid generally is compressional the momentum equation has to be augmented by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1.2)$$

where  $\mathbf{u}$  is velocity,  $\rho$  is density,  $P$  is pressure,  $\nu$  is viscosity and  $\mathbf{S}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\delta_{ij}u_{k,k}$  is the traceless rate of strain tensor.

Even though these equations look relatively simple at first glance, they still, after all the years since Navier first put them forth in 1827, generally remain an unresolved problem. The reason is that in spite of their simplicity they have incredibly complex solutions which are not yet mathematically understood.

The last term in Eq. (1.1) is a dissipative term converting small scale kinetic energy into heat, while the other term on the right hand side is the pressure term which tends to smooth out pressure variations. The non-linear term on the left hand side is the advection term. The nature of a certain flow is characterized by the ratio between the advective term and the dissipative term. This ratio is called the Reynolds number;

$$\text{Re} = \frac{(U^2/L)}{(\nu U/L^2)} = \frac{UL}{\nu}, \quad (1.3)$$

where  $U$  is a typical velocity and  $L$  is a typical length scale. The Reynolds number is interesting in that all flows with the same Reynolds number have the same physical properties. There are two different branches of fluid flows; if  $\text{Re} < 1$  the flow is laminar, and if  $\text{Re} \gg 1$  it is turbulent if there is an instability. In a laminar flow all fluid elements move in paths in the direction of the fluid mean flow. In a turbulent flow there is irregular random motions of fluid particles in directions transverse to the mean flow. Turbulence is encountered in many applications, such as in engineering and in nature, both on earth and in astrophysics. A turbulent flow is by its nature irregular and random, this makes a deterministic approach to turbulence impossible. Instead one has to rely on statistical methods in describing the turbulence. In all of the following we shall concentrate on the turbulent case.

In many contexts one chooses to work with the incompressible Navier-Stokes equations. Assuming incompressibility is for instance very popular when working analytically, but it is also often used in direct numerical simulations - especially in so called spectral codes. In a spectral code the spatial derivatives are calculated in Fourier space instead of in real space. In all of the following we will *not* assume incompressibility. In most aspects, however, we choose the Mach number, which is defined as the ratio of the fluid velocity to the speed of sound, to be small enough ( $\approx 0.13$ ) so that it is practically incompressible.

In this work we have focused on two different aspects of turbulence; turbulence in a neutral fluid, as described by the Navier-Stokes equations, and turbulence in an ionized fluid (a plasma).

In astrophysics we often find that a gas is partly, or fully, ionized, e.g. in the sun, in the interstellar medium, in the geodynamo, in the inner parts of accretion discs or in the cores of galaxy clusters. In such plasmas we must add the Lorentz force to the Navier-Stokes equations, and solve the Maxwell equations (here in SI units):

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= \rho_e / \epsilon_0 && \text{Gauss' law} \\
 \nabla \cdot \mathbf{B} &= 0 && \text{Gauss' law in magnetism} \\
 \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{Faraday's law} \\
 \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} && \text{Ampere-Maxwell's law}
 \end{aligned} \tag{1.4}$$

where  $\mathbf{B}$  is magnetic field,  $\mathbf{E}$  is electric field,  $\mathbf{J}$  is current density,  $\rho_e$  is charge density,  $\epsilon_0$  is the permittivity of free space,  $\mu_0$  is the magnetic permeability of free space and  $c$  is the speed of light. With the standard Ohm's law, for a fixed frame of reference,

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{1.5}$$

where  $\sigma$  is the electric conductivity, we can formulate the so called induction equation from Faraday's law;

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{J}), \quad (1.6)$$

where we have introduced the magnetic resistivity  $\eta = 1/(\sigma \mu_0)$ .

In all situations considered here we can safely neglect the displacement current, such that Ampere-Maxwell's law reduces to Ampere's law

$$\mathbf{J} = \nabla \times \mathbf{B} / \mu_0. \quad (1.7)$$

Since the gas is ionized the magnetic field will exert a body force on the gas due to the Lorentz force, we therefore rewrite the momentum equation (1.1) to read

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{\mathbf{J} \times \mathbf{B}}{\rho} + \frac{1}{\rho} \nabla \cdot (2\nu \rho \mathbf{S}). \quad (1.8)$$

In the same manner as we defined a kinematic Reynolds number in Eq. (1.3) we can now define a magnetic Reynolds number as

$$\text{Re}_M = \frac{UL}{\eta}. \quad (1.9)$$

Furthermore we define the magnetic Prandtl number as

$$\text{Pr}_M = \frac{\text{Re}_M}{\text{Re}} = \frac{\nu}{\eta}. \quad (1.10)$$

Empirically it is found that for turbulent flows with unit magnetic Prandtl number one gets dynamo action, i.e. exponential growth of magnetic energy, for  $\text{Re}_M \gtrsim 30$ . That is; the critical magnetic Reynolds number,  $\text{Re}_{M,c}$ , is  $\approx 30$ . The dependence of  $\text{Re}_{M,c}$  on  $\text{Pr}_M$  is covered in Paper 6.

# Chapter 2

## The PENCIL CODE

Since it is practically impossible to solve the turbulent Navier-Stokes for all kinds of applications analytically, or even experimentally, one must for many applications rely on numerical methods. In engineering it is popular to use methods which make certain assumptions on the flow, such that the numerical costs are kept at a minimum, at the same time as the interesting physics is not lost. Such methods are for example the Reynolds Averaged Navier Stokes (RANS) or the Large Eddy Simulations (LES). In a RANS simulation one average over time, there will therefore be no time dependence. A LES simulation is however basically the same as a Direct Numerical Simulation (DNS), i.e. it is time dependent, except that one do some kind of sub grid scale modeling such that one does not need to resolve the smallest scales.

As RANS and LES only consider the largest scales, and average out or model the small scales, one can not be sure that they are correct under all circumstances. Since conditions in astrophysics normally are very different from engineering it is not necessarily a valid assumptions that methods used in engineering are fine also in an astrophysical context. Furthermore, in astrophysics one can not test the validity of the methods in experiments, as one to some extent can in engineering. In many situations one is therefore forced to resolve all scales consistently, i.e. not to average out or model the smallest scales, by doing a much more costly DNS.

In a DNS it is crucial that all scales are resolved. One possibility is to make  $k_{\max}/k_d = \text{const} \geq 1$  for all Reynolds numbers. Here  $k_{\max} = N_x/2$  is the maximum resolved wave number where  $N_x$  is the number of meshpoints in each spatial direction,  $k_d = (\epsilon/\nu^3)^{1/4}$  is the Kolmogorov scale, and  $\epsilon$  is the energy dissipation rate. Together with Eq. (1.3) this implies that the total number of meshpoints  $N \propto \text{Re}^{9/4}$ . Another even more stringent (and for a finite difference code probably also more correct) constrain on the Reynolds number is to limit the grid Reynolds number,  $\text{Re}_{\text{grid}} = U\Delta x/\nu < \text{Re}_{\text{grid},0}$ ,

where  $Re_{\text{grid},0} \approx 5$ , and  $\Delta x$  is the grid spacing, from which it follows that  $N \propto Re^3$ . It is evident from these numbers that a large Reynolds number three-dimensional simulation is computationally very expensive. The primary aim is therefore to find a computer code that is as fast as possible, but that at the same time is accurate enough.

Until now the most commonly used codes are the pseudo spectral ones. As a pseudo spectral code solve the derivatives in wave number space, where they are just multiplications, it is very accurate, the accuracy is only limited by the discretization error. The problem is however that they are not very fast due to the fact that they have to perform a lot of Fourier transforms. We have therefore chosen to use the PENCIL CODE [3], which is a finite difference code, which should be faster than a pseudo spectral code, but not as accurate. In order to make it accurate enough we use sixth order spatial derivatives in space and third order Runge-Kutta in time. The coordinate system is Cartesian, and for the time stepping the 2N-RK3 scheme of Williamson[4] is used.

The code is constructed such that at every instant it only works on a single one-dimensional array, a 'pencil', which is why the code was named the PENCIL CODE. Working just on the 'pencil', and not on the much larger full domain, ensures that the information currently being used by the CPU always fits in cache.

The PENCIL CODE is highly modular, i.e. it has separate modules for the different equations such that one need to solve only the equations that are indeed necessary for a specific problem. For the problems we have concentrated on in this work we have solved the continuity equation and the momentum equation for the simulations of a neutral gas, while in the cases where the gas were ionized we have also solved the induction equation. We have avoided solving the entropy equation by assuming the gas to be isothermal, i.e.  $P = K\rho$  where  $K$  is some constant. In addition to the modules we have used there are also modules for entropy, passive scalars, radiation and dust.

Since the PENCIL CODE use sixth order central differences it need three points on every side in order to do a derivative. Due to this there is a three meshpoint thick layer, a "ghost zone", around the full domain of every processor. When communicating between processors we only communicate the "ghost zone". This is one of the great advantages of a finite difference code compared to a spectral code, in which one must communicate the full domain. In order to make the code portable to all kinds of computers we use Message Passing Interface (MPI) for communication between processors.

The initial condition is uniform density and zero velocity. When the gas is ionized we start with a weak random magnetic field.

For all simulations, except the decaying ones in Paper 4, we add a forcing function  $\mathbf{f}$  on the right hand side of Eq. (1.8). In this way we can start with zero initial velocity, and see the velocity grow with time until it reach a stationary state; when the energy input through the forcing equals the energy lost through the diffusion. The forcing function  $\mathbf{f}$  consists of random isotropic non-helical transversal waves;

$$\mathbf{f}(\mathbf{x}, t) = \text{Re}\{N \mathbf{f}_{\mathbf{k}(t)} \exp[i\mathbf{k}(t) \cdot \mathbf{x} + i\phi(t)]\}, \quad (2.1)$$

where  $\mathbf{x}$  is the position vector. The wave vector  $\mathbf{k}(t)$  and the random phase  $-\pi < \phi(t) \leq \pi$  change at every time step, so  $\mathbf{f}(\mathbf{x}, t)$  is  $\delta$ -correlated in time, and  $N = f_0 c_s (|\mathbf{k}| c_s / \delta t)^{1/2}$ , where  $f_0$  is a nondimensional forcing amplitude, and  $\delta t$  is the time step. We choose the wavevectors to be within a narrow band around a forcing wavenumber  $k_f$  in order to have an as small as possible portion of wavenumber space 'lost' in the forcing.

When evolving the magnetic field we choose to evolve the magnetic vector potential  $\mathbf{A}$  instead of the magnetic field itself. This is done in order to ensure that  $\nabla \cdot \mathbf{B} = 0$ . Furthermore we choose a gauge such that  $\nabla \phi = 0$ , and the induction equation therefore becomes

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}, \quad (2.2)$$

and the magnetic field is found from  $\mathbf{B} = \nabla \times \mathbf{A}$ .

# Chapter 3

## Statistics

For our largest simulation with  $1024^3$  meshpoints, and seven (velocity, density and magnetic field) unknowns per meshpoint, there is a total of  $7.5 \times 10^9$  variables per time step. Analyzing such enormous amounts of data directly is, for all practical purposes, impossible, or at least very difficult and possibly subjective. Our objective is therefore to look at various kinds of statistics in order to describe the true nature of the flow. There are many kinds of possible statistics which might be analyzed, all depending on one's interests. Here we concentrate on the energy spectrum and the structure functions.

### 3.1 Energy spectrum

Let us define the correlation tensor as

$$R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x})u_j(\mathbf{x} + \mathbf{r}) \rangle, \quad (3.1)$$

where  $u_i$  is the velocity component in the spatial directions  $i$ , and angular brackets denote ensemble averaging. Such a correlation tensor can tell something about the size of the turbulent eddies. But one can also, by calculating the Fourier transform of the correlation tensor, find the kinetic energy spectrum.

If one experimentally measures turbulent velocity in, say, a wind tunnel, one get the variation along one spatial direction only. Let the direction along which the variations are measured be the  $x$ -direction, one then finds

$$R_{11}(r) = \langle u_1(x)u_1(x + r) \rangle, \quad (3.2)$$

and, by taking the Fourier transform of this, the one dimensional energy spectrum;

$$E^{1D}(k) = \mathcal{F} \left[ \frac{1}{2} R_{11}(r) \right] = \frac{1}{2} \tilde{u}_1^{1D}(k) \tilde{u}_1^{1D*}(k), \quad (3.3)$$

where superscript 1D indicate that the Fourier transform is done in only one spatial direction, and the asterisk means complex conjugate.

Assume now that we have the full correlation tensor  $R_{ij}(\mathbf{r})$ , or at least the diagonal terms, at hand, calculated for example in a numerical simulation. The three dimensional energy spectrum is then

$$E(\mathbf{k}) = \frac{1}{2} \mathcal{F} [R_{11}(\mathbf{r}) + R_{22}(\mathbf{r}) + R_{33}(\mathbf{r})], \quad (3.4)$$

which, by performing the Fourier transform, yields;

$$E(\mathbf{k}) = \frac{1}{2} \sum_{i=1,3} \tilde{u}_i(\mathbf{k}) \tilde{u}_i^*(\mathbf{k}), \quad (3.5)$$

where  $\tilde{u}_i(\mathbf{k})$  is the three dimensional Fourier transform of the velocity in the  $i$ 'th direction. By assuming isotropy and homogeneity we can integrate out all directional information. This is done by integrating over spherical shells;

$$E(k) = \frac{1}{2} \sum_{k < |\mathbf{k}'| < k+dk} E(\mathbf{k}'). \quad (3.6)$$

One major difference between the three dimensional and the one dimensional energy spectrum is that the latter suffers from aliasing effects [5]. Furthermore one dimensional energy spectra show only weakly the bottleneck between the inertial range and the dissipative range [Paper 2]. If at all feasible one should therefore always use three dimensional energy spectra.

## 3.2 Structure functions

Originally turbulence was associated with self-similarity, but later it has been shown that turbulence is not really self-similar but intermittent. In other words, small structures are not evenly distributed in space, but rather clumped together in small volumes.

In order to investigate intermittency one needs to analyze the structure functions. Consider the velocity increment  $\delta \mathbf{u}(\mathbf{x}, \mathbf{l}) = \mathbf{u}(\mathbf{x} + \mathbf{l}) - \mathbf{u}(\mathbf{x})$  between two points separated by  $\mathbf{l}$ . The longitudinal component is  $\delta u_{\parallel} = \delta \mathbf{u}(\mathbf{x}, \mathbf{l}) \cdot \mathbf{l} / l$  such that we can define the  $p$ -th order longitudinal velocity structure function as

$$S^p(l) = \langle |\delta u_{\parallel}|^p \rangle. \quad (3.7)$$

One finds that in the inertial range the structure functions follow a power law;

$$S^p(l) = a_p l^{\xi_p}, \quad (3.8)$$

where the  $\xi_p$  are the scaling exponents and  $a_p$  are some constants. In order for the turbulence to be self similar it is required that the state at some scale can be mapped onto the state at some other scale by a simple linear scaling relation, i.e.

$$\xi_p = cp, \tag{3.9}$$

where  $c$  is some constant.

Trying to find structure functions and scaling exponents from experiments or numerical simulations turns out to be very hard if the Reynolds number is not extremely large. By exploiting the fact that we know from Kolmogorov's 4/5 law [6] that  $S^3$  scales linearly, this problem can, to a surprisingly large extent, be overcome by plotting

$$S^p = (a_p/a_3^{\xi_p})[S^3]^{\xi_p} = c_p[S^3]^{\xi_p}. \tag{3.10}$$

Alternatively one often plot the logarithmic derivative of  $S^p$  with respect to  $S^3$ ;

$$\frac{d \log S^p}{d \log S^3} = \xi_p \tag{3.11}$$

which gives a straight line for most of the length of  $S^3$ . This is known as the extended self similarity (ESS) and was first used by Benzi et al. in 1993 [7]. By using this method one finds the scaling exponents to good accuracy even at rather small Reynolds numbers. One should be aware, however, that using ESS on other than the velocity statistics is not necessarily OK since  $S^3$  is known to be linear in  $l$  only for the velocity.

# Chapter 4

## Analytical models

### 4.1 Kolmogorov theory

The Kolmogorov (1941)[6] theory is one of the most famous theories of turbulence. Assume the turbulence to be homogeneous and isotropic, and let the kinetic energy injection rate  $\epsilon_i$  be injected into the system on the large scales,  $L$ . Following Richardson (1922) [8] the energy will then cascade down the system to the dissipative scales  $l_d \sim 1/k_d$ , where it will be transformed into heat at a rate  $\epsilon_d$ . In statistically stationary turbulence what goes into the system must also go out of it, giving the equality  $\epsilon_i = \epsilon_d = \epsilon$ . Assuming that the energy input rate determines the large scale velocities, which again determines  $u_{\text{rms}}$ , it is, following the above equality, possible to find the dimensionless number

$$C_\epsilon = 3^{3/2} \epsilon L / u_{\text{rms}}^3 \quad (4.1)$$

which is supposed to be constant when the scale separation is large enough. The factor  $3^{3/2}$  in the above equation comes from the fact that the equation was originally defined with the one dimensional velocity  $u_{\text{rms}}^{\text{1D}} = 3^{-1/2} u_{\text{rms}}$ . This is the topic of Paper 1, where we find  $C_\epsilon \approx 0.5$ , but this value is still a matter of debate.

The cascade of energy to smaller scales takes place through the break up of large eddies into smaller ones. An eddy survives for approximately one eddy turnover time,  $\tau_l = l/u_l$ , before it breaks up. If  $\text{Re} \rightarrow \infty$  there is a large scale separation between the energy input scale and the dissipative scale, there will therefore be a wide range of scales which are independent of both the large scale geometry and the viscosity. This range, which is far from both the large and the small scales, is called the inertial range. In the inertial range the energy flux from one wavenumber to a smaller wavenumber

is supposed to be constant such that

$$\epsilon \sim \frac{u_l^2}{\tau_l} = \frac{u_l^3}{l} = \text{constant} \quad (4.2)$$

where the energy at scale  $l$  is  $\sim u_l^2$ . From the above equation we get the Kolmogorov scaling:

$$u_l \sim (\epsilon l)^{1/3}. \quad (4.3)$$

When  $E(k)$  is the kinetic energy spectrum,  $C_K$  is the Kolmogorov constant and the wavenumber  $k \sim 1/l$ , we find

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}. \quad (4.4)$$

The value of  $C_K$  has to be found empirically, and is around 1.5.

From Kolmogorov's famous 4/5 law [6], which is one of the few exact laws in turbulence, we know that for homogeneous, isotropic, steady, incompressible hydrodynamical turbulence the third order longitudinal structure function scale linearly with separation, i.e.  $S^3(l) = -4/5\epsilon l$ . Assuming self similarity Kolmogorov then found, using Eq. (3.9),  $\xi_p = p/3$ , but this is now known not to be correct (for all  $p \neq 3$ ) since turbulent flows are not self similar.

She & Leveque [2] found, by assuming log-Poisson statistics, the scaling relation;

$$\xi_p^{SL} = \frac{p}{9} + 2 \left( 1 - \left( \frac{2}{3} \right)^{p/3} \right), \quad (4.5)$$

which reproduce experimental data to very high accuracy, in addition to  $S^3(l) \propto l$ , and is currently the most popular model of intermittency.

It can easily be shown that there exist a relation between the second order scaling exponent,  $\xi_2$ , and the energy spectrum;

$$E(k) \propto k^{-(1+\xi_2)} \quad (4.6)$$

which shows that if there were no intermittency, i.e.  $\xi_2 = 2/3$ , then  $E(k) \propto k^{-5/3}$ . But as we know that all turbulent flows are intermittent it is obvious that the classical Kolmogorov scaling is not correct. Using the intermittency model of She & Leveque (Eq. (4.5)) implies  $E(k) \propto k^{-(1+\xi_2^{SL})} \approx k^{-1.696}$ .

Around the Kolmogorov wavenumber  $k_d = (\epsilon/\nu^3)^{1/4}$  the energy spectrum changes from a power law with a  $k^{-5/3}$  slope in the inertial range to an exponential cut off in the dissipative range. Due to the sudden lack of interacting wavenumbers in the very steep dissipative range, the energy at the lower end of the inertial range has problems fulfilling the cascade to smaller scales [9]. The result is the build up of the so called bottleneck around the Kolmogorov scale; see Paper 2. When using hyper viscosity the dissipative range is even steeper, resulting in a stronger bottleneck; see Paper 3.

## 4.2 Iroshnikov-Kraichnan (IK)

If it was a hard task to describe hydrodynamical turbulence it is even harder to describe *magneto*-hydrodynamical (MHD) turbulence. This difficult task was however attempted solved by R. S. Iroshnikov in 1963 [10] and R. H. Kraichnan in 1965 [11], which resulted in the Iroshnikov-Kraichnan (IK) theory.

Let a uniform external magnetic field  $B_0$  be present. If the media is incompressible then all perturbations to the field will propagate along the field lines as Alfvén waves with a velocity  $V_A = B_0/\sqrt{4\pi\rho}$ . In order to get an energy cascade going the IK theory let two oppositely traveling wave packets, of size  $l$ , collide. The collision lasts for a time  $\Delta t \sim l/V_A$ , and the energy loss to smaller scales is

$$\Delta E \sim \frac{dE(l)}{dt} \Delta t, \quad (4.7)$$

when  $E(l) \sim v_l^2$  is the energy of an eddy with size  $l$ . This yields

$$\Delta E \sim v_l \dot{v}_l \Delta t \sim v_l \frac{v_l^2}{l} \Delta t \sim \frac{v_l^3}{l} \frac{l}{V_A} \sim v_l^2 \frac{v_l}{V_A}, \quad (4.8)$$

when

$$\dot{v}_l = \frac{dv_l}{dt} \sim \frac{v_l}{\tau} \sim \frac{v_l^2}{\pi l} \sim \frac{v_l^2}{l}, \quad (4.9)$$

and  $\tau = 2\pi l/v_l$  is the eddy turnover time. The energy change per collision is  $(v_l^3/V_A)$  which is much smaller than  $v_l^2$  for  $V_A \ll v_l$ . We therefore need a lot of collisions in order to fulfill the cascade to smaller eddies. Naively one would think that  $N = v_l^2/\Delta E$  collisions were required in order to fulfill the cascade to smaller scales, but since this is a random walk process one need  $N = (v_l^2/\Delta E)^2$  collisions. The energy cascade time is then

$$t_{\text{cas}} = N \Delta t \sim \left( \frac{v_l^2}{\Delta E} \right)^2 \Delta t \sim \frac{l V_A}{v_l^2}. \quad (4.10)$$

In the IK theory one makes the same assumption as in the Kolmogorov theory that the energy cascades to smaller scales with a constant speed, i.e.

$$\frac{v_l^2}{t_{\text{cas}}} = \frac{v_l^4}{l V_A} = \text{constant}, \quad (4.11)$$

and henceforth  $v_l \propto l^{1/4}$ . Knowing that  $E(k)k \propto v_l^2$  we find

$$E(k) \propto k^{-3/2}, \quad (4.12)$$

which we contradict in Paper 5, Paper 6 and Paper 7.

Another prediction of the IK theory is that  $\xi_4 = 1$ , which is found not to be correct in Paper 5. The fact that both  $E(k)$  and  $\xi_4$  are found not to be correct in the IK theory helped abandoning this theory, which is now mostly interesting for historical reasons. What made the greatest contribution to its bad reputation, however, came about already before we had empirical values for  $E(k)$  and  $\xi_4$ , namely the fact that the IK theory assumes isotropy at the smallest scales. This assumption followed from the widely accepted Kolmogorov's hypothesis of local isotropy, which states that in large Reynolds number *hydrodynamical* flows the scales much smaller than the energy carrying scales are isotropic. This is however not true in *magneto-hydrodynamical* flows, since the magnetic field introduces anisotropy at *all* scales.

### 4.3 Goldreich-Sridhar (GS)

Due to the obvious flaw of assuming isotropy in the IK theory, P. Goldreich and S. Sridhar developed a new and improved theory of MHD turbulence in 1995 [12]. In their theory they let the turbulence be anisotropic corresponding to the direction of the local magnetic field. This is done by introducing the concept of scale dependent anisotropy, which states that the turbulent eddies get more and more elongated as they get smaller and smaller.

Their theory is based on the fact that magnetic fields are easy to mix but hard to bend. Indeed they point out that the magnetic field lines mix, in the direction perpendicular to the magnetic field, due to turbulent *eddies*, just like in the pure hydrodynamical case. The fluid motion in the parallel direction is however dominated by the *wave*-like motions of the Alfvén waves.

A perturbation in the direction *perpendicular* to the magnetic field line is associated with a time scale  $\tau_{\text{per}} = l_{\perp}/u_k = 1/(k_{\perp}u_k)$ , where  $u_k$  is the typical velocity on a scale  $l = 1/k$ , and  $k_{\perp}$  is the wavenumber perpendicular to the local mean magnetic field. A perturbation in the perpendicular direction will also cause a perturbation in the direction *parallel* to the magnetic field with the time scale  $\tau_{\text{par}}$ . The typical velocity in the parallel direction is the Alfvén velocity  $V_A$ , we therefore find  $\tau_{\text{par}} = 1/(k_{\parallel}V_A)$  when  $k_{\parallel}$  is the wavenumber in the direction parallel to the magnetic field. Since  $\tau_{\text{per}}$  is obviously equal to  $\tau_{\text{par}}$  one finds that the mixing in the perpendicular direction is coupled to the wave-like motions in the parallel direction through the condition of critical balance;

$$k_{\parallel}V_A \sim k_{\perp}u_k. \quad (4.13)$$

One can think of the condition of critical balance as being kept in balance by the tension of the local magnetic field; if the left hand side of the above

equation becomes too small, i.e.  $k_{\parallel}$  is too small, the tension of the magnetic field have no effect and the parallel cascade acts like if there were no magnetic field, that is through hydrodynamic turbulence. In this way  $k_{\parallel}$  is quickly increased. If  $k_{\parallel}$  becomes too large the magnetic tension will be too strong for any parallel cascade to take place, and all the cascade will be in the perpendicular direction, until balance is again achieved.

Assume now, as we did in both the Kolmogorov and the IK theory, that the energy cascade rate (we concentrate only on the cascade in the perpendicular direction since this is the dominant one)  $E_l/\tau_{cas} = u_l^3/l = u_k^3 k_{\perp} =$  constant. This together with the condition of critical balance yields

$$k_{\parallel} \propto k_{\perp}^{2/3}, \quad (4.14)$$

which describes the scale dependent anisotropy. We see that as we go to smaller and smaller scales the turbulent eddies become more and more anisotropic. Since the magnetic field is harder to bend the further down in scale you go, and the small scale kinetic motions are less energetic than the large scale ones, this is just as one would suspect. From the above assumption of constant energy cascade, together with the assumption that the dominating cascade direction is the perpendicular one, it is easily found that

$$E(k) \propto k_{\perp}^{-5/3}. \quad (4.15)$$

From Paper 7 we see that although the GS theory is the best we have to date, it is not perfect. Explaining, for example, the suppression of the magnetic energy spectrum as the mean magnetic field is increased is still an unresolved puzzle. Furthermore the GS theory is only applicable when there is a mean magnetic field. If both the mean magnetic field and the magnetic helicity is zero, as in Paper 5, Paper 6 and Paper 7, there is currently no theory for the non-linear regime.

# Chapter 5

## What we have learned

In recent years the discovery of a bottleneck at the end of the inertial range, just before the dissipative subrange, in the hydrodynamic energy spectrum has become accepted in the community. In Paper 2 we show the relation between one dimensional and three dimensional energy spectra, and explain why the bottleneck is almost absent in the one dimensional energy spectrum while it is relatively strong in the three dimensional energy spectrum. This is also the reason why the bottleneck is a relatively newly discovered phenomenon; previously one could reach Reynolds numbers large enough to see bottlenecks only in wind tunnel, and similar, experiments. In such experiments the deduced energy spectra were, however, just one dimensional, and therefore had none, or very weak, bottlenecks. It is only in recent years that computers have become sufficiently fast to achieve Reynolds numbers large enough to see bottlenecks in numerical three dimensional DNS, where we can calculate the three dimensional energy spectrum. As we have argued in Paper 2 the bottleneck effect is a - wavenumber space only - effect, the bottleneck does not affect the structure functions. Since there is a relation between the second order structure function scaling exponent, and the slope of the energy spectrum, we suggest to use structure functions in order to find the slope of the energy spectrum in simulations where the Reynolds numbers are too small to resolve both the bottleneck and the rest of the inertial range. We use this method in Papers 5 and 6, and find that the kinetic and magnetic energy spectra in a non-helical MHD simulation with  $1024^3$  meshpoints actually have a Kolmogorov like slope of the energy spectrum, even though the three dimensional energy spectrum shows a slope that is readily compatible with a  $k^{-3/2}$  slope. The notion of a Kolmogorov like energy spectrum is also supported by the one dimensional energy spectrum, which, as explained before, is not notably affected by the bottleneck. The conclusion is therefore that the MHD energy spectrum follow a  $k^{-5/3}$  slope, and that the shallower

slope in the three dimensional spectrum is due to the bottleneck.

In Papers 5 and 6 we find the magnetic energy to be in super equipartition at all scales smaller than  $\sim 5k_f$ . This is in contrast to what is found in previous simulations with imposed magnetic fields (e.g. Cho & Vishniac (2000) [13] and Maron & Goldreich (2001) [14]), where equipartition between magnetic and kinetic energy spectra is found. In Paper 7 we do a parameter scan of different imposed magnetic field strengths. Our result is that the imposed magnetic field suppresses dynamo action, such that the stronger the imposed field is the less magnetic energy is present on scales smaller than the forcing scale. For imposed fields of the same size as the rms velocity we find equipartition between magnetic and kinetic energy, which explains the equipartition of Refs [13, 14], who indeed had an imposed field of the same order as the rms velocity.

We now return to the bottleneck issue; Falkovich [9] explain the bottleneck as being due to a steep dissipative range, the dissipative range being so steep that there is not enough wave vectors to interact with in the area around the lower end of the inertial range. In Paper 3 we introduce hyper viscosity instead of the ordinary viscosity, which make the dissipative sub-range even steeper. Following Falkovich's arguments this should result in a stronger bottleneck - which is indeed what we find. But even though the bottleneck is found to be stronger, i.e. higher, it is not wider. This enables us to use simulations with hyper viscosity and  $512^3$  meshpoints to reach the same Reynolds numbers as with  $2048^3$  meshpoints and ordinary viscosity. We were therefore able to confirm the slope of  $k^{-1.77}$  in the inertial range as found first by Kaneda et al. [15]. This slope is steeper than the slope of  $k^{-1.67}$  as suggested by Kolmogorov, but also steeper than  $k^{-1.70}$  as suggested by She & Leveque [2]. The fact that the inertial range seems to be steeper than what is suggested by SL is explained by Tsuji (2004) [16], who find that in very large experiments ( $\text{Re} \sim \mathcal{O}(10^4)$ ) the inertial range has two different slopes. One of these slopes are consistent with  $k^{-1.77}$  while the other one is consistent with SL.

In our simulations we find the Elsasser variables to be slightly more intermittent than the SL intermittency model. This is however in agreement with the intermittency model for compressible MHD turbulence of Padoan et al. (2003) [17]. Regarding the intermittency of the individual fields we find the velocity field to be less intermittent, and the magnetic field to be more intermittent than the Elsasser variables.

Previously the belief has been that in non-helical MHD turbulence the peak of the magnetic energy spectrum would be proportional to the resistive scale [18]. If this was correct there would not be any significant amount of large scale magnetic fields from dynamo action if there were no kinetic

helicity present. In many astrophysical applications there are considerable amounts of helicity, but there are exceptions from this rule; e.g. in the very hot intergalactic gas in the center of some galactic clusters we expect the kinetic helicity to be negligible. Following the results of Maron & Blackman (2002) [18] one would therefore expect the scale of the magnetic field in this gas to be very small. In Paper 5 we show, however, that the peak of the magnetic field is *not* at the resistive scale, but rather at the forcing scale. This suggests that some amount of large scale magnetic fields may be present also in applications where the kinetic helicity is negligible.

As there are also many other aspects than the energy spectra to bother about in turbulence we now leave them for a while. One interesting and very crucial question is whether the small scale dissipation is independent of the largest scales? A dimensionless number  $C_\epsilon$  has been constructed in order to answer this question. Previous authors have agreed that  $C_\epsilon \sim \mathcal{O}(1)$ , but a more exact number is not yet found. In order to find an exact value of  $C_\epsilon$  it is of course important to have a large enough scale separation, i.e. to have large enough Reynolds numbers, but even simulations with very large Reynolds numbers do not agree upon the value of  $C_\epsilon$ . In Paper 1 we show that it is crucial how one measures  $C_\epsilon$ . Indeed we are able to produce  $C_\epsilon$  for a large spread of Reynolds numbers from just one simulation. The reason is that the energy put into the system,  $\epsilon$ , is not constant, i.e. there are small perturbations on  $\epsilon$ . It will take some time for a given energy perturbation to cascade down the system to the dissipative scales, such that it will not give a correct result  $C_\epsilon$  if one measures both the large and the small scale diagnostics at one identical time, which is often done (see e.g. Ref [15]). One must therefore either average all statistics over a long time interval, or, alternatively measure the small scale statistics at a certain time later than the large scale statistics. This explains, we believe, the discrepancy between previous authors, and we follow up by suggesting a value of  $C_\epsilon = 0.5$ .

In this thesis we deal mostly with weakly compressible turbulence, but since one often finds supersonic turbulence in possible astrophysical dynamos, an interesting question is whether one can have dynamo action together with shocks. The argument against is that a shock sweeping through the medium would diffuse away the magnetic field in the very sharp shock front, such that the magnetic energy would always be negligible. In Paper 8 we find however that this argument is not correct. It is indeed true that the presence of shocks makes it harder to get dynamo action, but not very much. We find that as the rms Mach number is increased from very small values up towards  $\sim 0.5$  the critical magnetic Reynolds number for dynamo action,  $\text{Re}_{M,c}$ , stays fairly constant. As the Mach number is increased beyond  $\sim 0.5$  we find that  $\text{Re}_{M,c}$  increases, but flattens out again around  $\sim 1.0$ . We therefore conclude

that the presence of shocks does indeed hinder dynamo action, but that the hindering is rather small, and more or less independent of Mach number for Mach numbers larger than  $\sim 1.0$ .

# Chapter 6

## What next?

The topic of turbulence in general, and MHD turbulence in particular, is still far from resolved. There are a large number of possible pathways to follow in future research.

First of all the asymptotic MHD energy spectra (without imposed fields) are still not known. Both the shape of the spectra and some theory explaining the saturation process is missing. Currently the only MHD theory available is the Goldreich-Sridhar (GS) theory for turbulence with imposed fields, but from Paper 7 we see that even this theory is still not complete. Regarding the asymptotic MHD energy spectra without imposed fields I think the way to go is to use numerical simulations with some sort of artificial viscosity in order to increase the Reynolds number without too much computer costs. We have done some preliminary tests which indicate that large eddy simulations (LES) are particularly promising when it comes to increasing Reynolds numbers without disturbing the energy spectra.

It is currently not settled how the growth of the magnetic energy through dynamo action works in general. It is for example no consensus on how the critical magnetic Reynolds number for dynamo action depend on the magnetic Prandtl number. There are basically two possibilities; either there exist a critical magnetic Prandtl number below which no dynamo action is possible [19], or there exist a magnetic Prandtl number below which the critical magnetic Reynolds number is constant [20, 19]. If we are lucky, this question will be settled rather soon by the use of numerical simulations. We must be aware however that since it is not known how small the magnetic Prandtl number must be in order to see the asymptotic behavior, the result might still be far into the future.

There are currently, in addition to our results in Paper 6, several analytical and numerical results on MHD intermittency [17, 22, 23, 24]. Theses results are however obtained with respect to different variables (velocity or

Elsasser), for either decaying or forced turbulence, or they are measured either with respect to the local magnetic field or with respect to the coordinate axis. As intermittency is found to be very important in MHD turbulence, it would therefore be of great importance to find a general MHD intermittency theory. The only exact analytical result currently known in MHD turbulence is that some mix of the Elsasser variables shall scale linearly [21]. This has however not yet been shown numerically. Therefore; firstly the linear scaling must be confirmed in numerical simulations, and then it should be laid as a ground stone for a more fundamental MHD intermittency theory.

In pure hydrodynamics it is believed that there exist a dimensionless number  $C_\epsilon = 3^{3/2}\epsilon L/u_{\text{rms}}^3$  which is constant for  $\text{Re} \rightarrow \infty$  (see; Paper 1). There might exist a dimensionless number with the same behavior also in MHD turbulence. This has to our knowledge not yet been investigated, and might be worth pursuing.

# Bibliography

- [1] P. D. Ditlevsen, M. H. Jensen, and P. Olesen, [arXiv:nlin.CD/0205055](https://arxiv.org/abs/nlin.CD/0205055).
- [2] Z. S. She & E. Leveque, *Phys. Rev. Lett.* **72**, 336 (1994).
- [3] <http://www.nordita.dk/data/brandenb/pencil-code>.
- [4] J. H. Williamson, *J. Comp. Phys.* **15**, L21 (1980).
- [5] H. Tennekes and J. L. Lumley, *A first course in turbulence* (The MIT Press, Cambridge, Massachusetts, 1972).
- [6] A. N. Kolmogorov, *Dokl. Akad. Nauk. SSSR* **30**, 299 (1941).
- [7] R. Benzi, S. Ciliberto, R. Tripicciono, C. Baudet, F. Massaioli, & S. Succi, *Phys. Rev.* **E 48**, R29 (1993).
- [8] L. F. Richardson, *Weather predicition by numerical process* (Cambridge University Press, Cambridge, 1922).
- [9] G. Falkovich, *Phys. Fluids* **6**, 1411 (1994).
- [10] R. S. Iroshnikov, *Sov. Astron.* **7**, 566 (1963).
- [11] R. H. Kraichnan, *Phys. Fluids* **8**, 1385 (1965).
- [12] P. Goldreich and S. Sridhar, *Astrophys. J.* **438**, 763 (1995).
- [13] J. Cho and E. T. Vishniac, *Astrophys. J.* **539**, 273 (2000).
- [14] J. Maron and P. Goldreich, *Astrophys. J.* **554**, 1175 (2001).
- [15] Y. Kaneda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, *Phys. Fluids* **15**, L21 (2003).
- [16] Y. Tsuji, *Phys. Fluids* **16**, L43 (2004).

- [17] P. Padoan, R. Jimenez, Å. Nordlund, and S. Boldyrev, Phys. Rev. Lett. **92**, 191102 (2004).
- [18] J. Maron, & E. G. Blackman, Astrophys. J. **566**, L41 (2002).
- [19] A. A. Schekochihin, S. C. Cowley, J. L. Maron and J. C. McWilliams, Phys. Rev. Lett. **92**, 054502 (2004).
- [20] S. Boldyrev & F. Cattaneo, Phys. Rev. Lett. **92**, 144501 (2004).
- [21] H. Politano & A. Pouquet, Geophys. Res. Lett. **25**, 273 (1998).
- [22] D. Biskamp and W.-C. Müller, Phys. Plasmas **7**, 4889 (2000).
- [23] J. Cho, A. Lazarian, and E. T. Vishniac, Astrophys. J. **595**, 812 (2003)..
- [24] S. Boldyrev, Astrophys. J. **569**, 841 (2002).

Papers are not included due to copyright.