NTNU

Norwegian University of
Science and Technology

Remaining Useful Lifetime Modeling of a Compressor System

Markus Heimdal

Reliability, Availability, Maintainability and Safety (RAMS)
Submission date: June 2017
Supervisor: Anne Barros, MTP

Norwegian University of Science and Technology
Department of Mechanical and Industrial Engineering

NTNU – Trondheim
Norwegian University of
Science and Technology

Statoil

RAMS
Reliability, Availability
Maintainability and Safety

Remaining Useful Lifetime Modeling of a Compressor Drive System

Markus Heimdal

Trondheim, June 2017

TPK4950 – RAMS, Master's Thesis

Department of Mechanical and Industrial Engineering
Norwegian University of Science and Technology

Supervisor: Anne Barros (Professor in Subsea Reliability at MTP, NTNU)
Co-supervisor: Erling Lunde (Researcher at Statoil)

Preface

This master thesis is conducted as a part of the 2-year international Master of Science (MSc) study program in Reliability, Maintainability, Availability and Safety (RAMS) at Norwegian University of Science and Technology (NTNU). The thesis was written during the spring semester 2017 and has the weighting of 30 credits.

The Department of Mechanical and Industrial Engineering (MTP) presented in the spring of 2016 suggestions for topics related to the master thesis. I selected the task remaining useful lifetime prediction of a compressor drive system, which fitted well for my study direction of lifetime analysis. During the autumn semester of 2016 a pre-study for the upcoming master thesis was written, which established the fundamental for the master thesis.

The master thesis is written in cooperation with the energy company Statoil. The fact that the topic of this thesis is of interest for Statoil gave me motivation during the writing.

Trondheim, 2017-06-11

Markus Heimdal

Markus Heimdal

Acknowledgement

With this, I want to express my sincere gratitude to all those who has been involved with this master thesis. First of all I want to thank NTNU, the MTP department and Statoil for giving me the opportunity to write the thesis.

A special thanks goes to my supervisor, professor Anne Barros, for all the valuable advices, significant theory inputs and helpful guidance that she has provided during the whole thesis. I would also like to express me deep appreciation to Anne for the encouragement and support she as given me for the completion. Anne also deserves a thanks for the time spent on our weekly discussions, which has considerably improved the quality of the end result of the thesis.

I am also grateful for my co-supervisor, Erling Lunde, who presented the problem itself in a clear and precise way. Erling has provided answers to my questions, whenever needed, which has been to great help during the writing.

Lastly, I would like to thank my classmates at the RAMS department. Those who had the same topic as me deserve an extra gratitude for all the interesting conversations. A huge thanks is also well deservedly given to my family and to those who I have shared an office with during the thesis. Their valuable moral support and encouragement is deeply appreciated.

Summary

The master thesis has the aim of predicting the remaining useful lifetime (RUL) for an aging compressor drive system (CDS). Several objectives where established to achieve the aim. Among them are a review of the aging process, aging of the CDS, selection of a degradation model, prediction given condition and parameter estimation.

The CDS is located at the process plant Kollsnes, which is an onshore center for treatment of gas. Kollsnes supplies 8% of the European gas demand, and the fact that the CDS is aging makes the modeling of RUL of interest. The CDS at Kollsnes consist of 4 main components, where a synchronous electrical motor has been the main focus in this thesis.

Aging is a process that will gradually lead the motor towards an unacceptable condition due time and operation. The most relevant aging process for this thesis is functional degradation, which is a physical process that reduces the motors ability to function and/or perform as required. Aging stresses will cause degradation mechanisms to occur, which eventually will result in a motor failure. Based on several motors studies the insulation is identified as the part that limits the motors lifetime. A reason for this is its lack of maintainability, in addition to its vulnerability to the degradation mechanisms. Partial discharge (PD) are closely linked to these degradation mechanisms, where PD is the best variable to represent the aging process. The PD level is therefore used as a health indicator for the insulation condition, thus the motor condition.

The prediction of the RUL of the motor is based on its current condition and how the condition of similar motors has developed historically. There are several probabilistic models that can be used for this prediction. For this thesis a Markov process is selected, due to its ability of modeling degraded states with increasing failure rates, while it in addition gives a good overview of the degradation process. The Markov process is modeled with fictional failure rates since no real failure data were available. The degraded model is presented with several survival measures, like the RUL, and plots. The motor as a single component is the main focus of the modeling, however models on a system level considering season differences is presented in addition

The model used in the thesis is based on parameters, which is why a method for parameter estimation from interval censored data is presented. Actions to reduce the estimation error are identified, where they are related to inspection frequency and the number of samples included in the study.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

Chapter 1 introduces the master thesis and starts with a presentation of the background, which explains the importance of the topic. The aim and objectives are then defined, which together with the problem description and the limitation, will denote the scope of the thesis. Further, the actors that are involved are presented, followed by the methodology that the author used to write the thesis and lastly the structure of the thesis.

1.1 Background

Norway is the third largest gas exporter in the world, only beaten by Russia and Qatar [1]. Nearly all the gas that is produced in Norway is sold on the European market, where Norway supplies over 20% of the demand. Around 67% of the expected natural gas resources in Norway are still not produced. This thesis will focus on the gas plant Kollsnes, which constitutes 40% of the total gas delivery from Norway [2]. This means that Kollsnes supplies 8% of the European marked demand.

Kollsnes process plant is an onshore center for treatment of gas from the fields of Fram, Visund, Kvitebjørn and Troll [2]. Statoil has been the technical service provider at Kollsnes since 1996, while Gassco is the operator. At Kollsnes, the gas is cleaned, dried and compressed before it sent out as dry gas to Europe thorough the export pipelines Zeepipe IIA and IIB [2], [3]. The secreted wet gas that is cleaned out is sent to Mongstad where it is fractionated to propane, butane and naphtha [2].

The Norwegian continental shelf, including the systems at Kollsnes, is aging [4]. The aging process decreases a components ability to perform its indented function [5]. If the ability to process the gas at Kollsnes is reduces, it will lead to losses if the demand of the market cannot be supplied. Degradation can therefore result in production losses and thus unplanned maintenance cost.

With accurate predictions, the potential losses could be avoided by maintenance implementations, to archive a minimal maintenance cost and a maximum lifetime of a component [6]. A type of prediction is to estimate the time to a failure from the current time, given current condition based on history [7]. The length of this time period, from the current time to the end of the life, is known as the remaining useful lifetime (RUL). Therefore, RUL estimations has attracted more and more attention in recent years [6]

1.2 Aim and Objectives

The aim of the master thesis is to model the remaining useful lifetime (RUL) of an aging compressor drive system (CDS). The model should predict a future condition, given current condition and history- (or expert) based parameters. Survival measures and plots should illustrate the predicted RUL. A classification of the CDS condition and a presentation of method for estimating historical parameters is an important step of the aim. The overall aim is achieved by considering the following objectives:

- Make a brief introduction of the CDS
- Provide a literature review of the aging process
- Relate the aging theory to the CDS and define a condition as representation of aging
- Describe the characteristic of survival modeling and select a model for the RUL
- Model the RUL
- Present a method for parameter estimation and how to deal with incomplete datasets
- Discuss the results and propose further work

1.3 Problem Description

The topic of the master thesis is prepared by professor Anne Barros, from NTNU, and researcher Eling Lunde, from Statoil.

The problem of this thesis is related to six aging CDS trains, which are crucial for the gas exportation out of Norway. Since the CDS are aging there will be an issue with degradation, which eventually will cause the system to fail. It is therefore of interest to model the RUL, which is about predicting a future condition based on the systems current state.

1.4 Limitations

The prior knowledge and education, of the author, within electrical theory was significantly low. For this reason a lot of time was spent on researching to get a better understanding of this field. It is assumed that the reader has some prior knowledge of aging-, reliability- and preferably electrical motor studies. This saves time and effort for the author so the focus could be the aim without explaining every part in detail.

The thesis is written on the basis of theory from open sources, and literature available through NTNUs licenses. A limitation is thereby the availability of literature, where some relevant data could be rare to find and some information was not available at all. There are for example no available data and information about the exact type of motor that is at Kollsnes.

Due to the time issue the focus has been on the electrical motor, not the system as a whole. There are also differences in s summer- and winter- operation, which not have not been studied into depth. Elements like, cost, operators, maintenance strategies etc. were not covered or described in detailed. External factor that can cause failure to the system is not considered. Due to scope and time limitation only a few selected survival models were discussed, where a probabilistic model of type Markov processes was the main focus.

1.5 Actors Involved

1.5.1 NTNU

The master thesis is written in cooperation with NTNU and the Department of Mechanical and Industrial Engineering (MTP). NTNU is one of the leading research institutions in Norway and has its processional standing within technology and natural science [8]. NTNU was established in 1996 after a merge of several institutions. In 2016 NTNU included two more collages in their system, and is with almost 40 000 students and 6500 employees the largest university in Norway.

At MTP there exist several research groups, where the RAMS group is one of them [9]. The main topic of the RAMS group is risk- and reliability assessment of complex systems. The overall objectives of the group is to contribute to a safer, more productive and more sustainable industry, transport and society. This is accomplished by providing RAMS research results and education at the highest possible level of quality.

Professor Anne Barros from MTP at NTNU and a member of the RAMS group has been the main supervisor for this master thesis.

1.5.2 Statoil

Statoil is an international energy company stationed in Norway [10]. Spread around the world, the company has approximately 20 500 employees. Statoil was founded in 1972 under the name "Det Norske Stats Oljeselskap ASA. Based on more then 40 years of experience from oil- and gas- production on the Norwegian shelf, Statoil is using technology and creating innovative business solutions to meet the world's energy needs in a responsible manner [10].

Statoil has absolute requirements for health, safety and environment (HSE) [10]. Their goal is to cover the energy demand that is needed for economic and social development. Statoil also wants to act responsibly towards the environment and works actively to reduce the global climate change. The main priority of Statoil is to operate safely and efficiently [10]. They have received widespread recognition for their monitoring system of technical security and their safety programs. They believe that all accidents can be avoided and they aim at zero men to be injured.

Researcher Erling Lunde from Statoil has been the co-supervisor for this master thesis.

1.6 Methodology

The master thesis is written on the fundamental of a pre-study from the autumn semester 2016. The method for the pre-study was mainly of a qualitative approach with discussion. This pre-study research is finalized in the master thesis and it is used as a basis for a more quantitative approach.

The literature covered was gathered from books, published articles and lectures. The database Oria was used to collect general and specified information and articles. Programing codes for the quantitative modeling and the simulation were mainly made by the help of professor Anne Barros, in addition to the help function in MATLAB. Weekly meetings with the main supervisor has been held, where advices for the thesis was provided. Brief interviews with both supervisors were conducted regularly, where topics were discussed from predetermined- and impulsive questions. The author also had several presentations of the thesis status with the aim of getting feedback.

Throughout the whole thesis the author also had regular contact with the supervisor and co-supervisor by mail. Meetings and statues presentations were planned, and specific questions were answered.

1.7 Structure of the Master Thesis

Chapter 1 is the introduction of the specialization project. Here the project aim and objectives is stated, followed by a general problem description and project limitations. The actors involved, the methodology approach and the structure of the project is also presented here.

Chapter 2 presents an overview of the problem of issue for the master thesis. It has the intension to give the reader an additional understanding of the situation and systems at Kollsens. First an overview of the CDS is presented. Further, a literature review of the components of the CDS is displayed. The chapter ends with an introduction and formalism of the RUL.

Chapter 3 is a literature of the aging process. Here the aging process is classified in three categories, where the most relevant aging process is described closely, Chapter 4 is linking the aging review from chapter 3 to the CDS. A in depth study of the aging process are here presented, where the CDS weak parts and condition indicators are identified.

Chapter 5 is a description of the modeling relevant for the thesis. Here model theory are presented and described in the sense of aging studies. The structure of this chapter is more general in the beginning, before it gradually gets more specific.

Chapter 6 is the modeling chapter. The modeling of the RUL of the aging CDS is conducted here. The chapter explains how the model is built up, the modeling is performed and the results are displayed. The focus of the modeling is on a component level, however season differences are also considered. Chapter 6 is in addition presenting and conducting a method of parameter estimation.

Chapter 7 is a discussion based on the results from the previous chapters. The structure of this chapter is therefore following the structure of the chapters.

Chapter 5 is the conclusion of the project aim and objectives, where a suggestion for further work is presented at the end.

Chapter 2
Overview of the Problem

Chapter 2 shall contribute to a basic understanding of the problem. First a description of the compressor drive system is presented, where winter- and summer operation is included, in addition to a short update on the age of the system. A literature review follows, where the components of the compressor drive system is described individually. Lastly an introduction of reaming useful lifetime is presented, where a definition and formalism is displayed.

2.1 Overview of the Compressor Drive System

At Kollsnes there is six parallel compressor drive system (CDS) trains, which are assumed identical. This indicates that each of the trains contribute one sixth of the total production capacity. Figure 1 illustrates one CDS train, which consist of, from left, a variable speed drive (VDS), a motor, a gearbox and a compressor. The capacity requirement for the CDS trains varies, where all trains might not be needed depending on the season. A more detailed presentation of the CDS trails is displayed in appendix II.

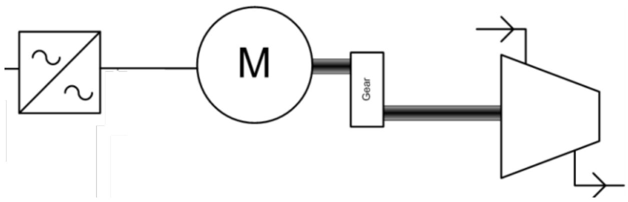


Figure 1: Compressor drive system at Kollsnes

2.1.1 Winter Operation

During the winter a full capacity is required and all the six CDS trains have to be in operation. An illustration of the winter situation is presented as a fault tree in figure 2. A fault tree is a logical system that presents all possible combinations of potential failures that may lead to a system failure [11]. In the fault tree there is only OR-gates, which means that the top event, CDS failure, will occur if one of the components fails. This can also be illustrated by the use of minimal cut sets. A cut set is a set of basic events that will cause the top event to occur, if the events happen at the same time [11]. A cut set is minimal if the set cannot be reduces without loosing its status as cut set. Since a single basic event in our case will lead to the top event the minimal cut sets are as followed:

{1VSDF}, {1MF}, {1GF}, {1CF}, {2VSDF}, {2MF}, {2GF}, {2CF}, {3VSDF}, {3MF}, {3GF}, {3CF}, {4VSDF}, {4MF}, {4GF}, {4CF}, {5VSDF}, {5MF}, {5GF}, {5CF}, {6VSDF}, {6MF}, {6GF}, {6CF}.

Where VSDF = variable speed drive fails, MF = motor fails, GB = gearbox fails, CF = compressor fails.

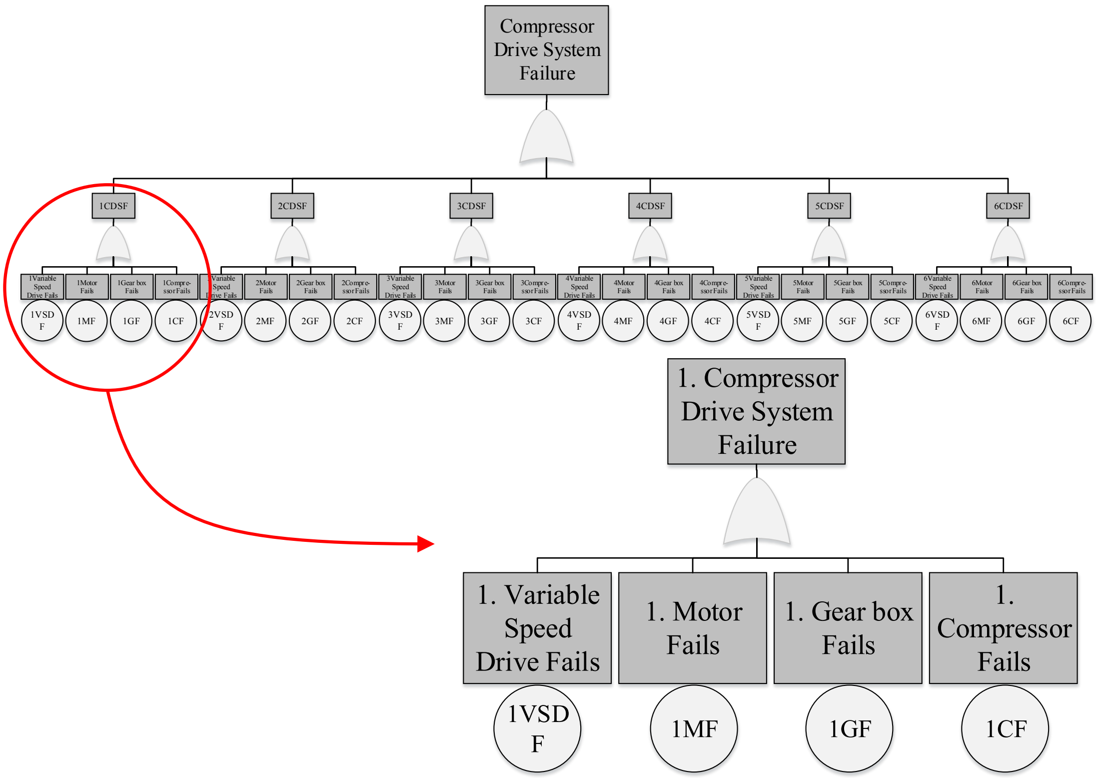


Figure 2: Fault tree for the compressor drive system, winter operation

2.1.2 Summer Operation

For the summer only parts of the total capacity are required, where it is assumed that minimum 3 out of 6 CDS trains has to work to at the same time. A fault tree for this situation is presented in figure 3. The OR-gate before the top event is changed to a “4 out of 6” AND-gate. This means that the system will fail if four or more CDS trains fail.

Minimal cut sets can also be found for the summer situation. Since we need four CDS trains to fail to get a system failure, the cut sets will not be made out of basic event this time. However, each minimal cut set will now consist of four CDS train failures. A basic event will still lead to a CDS failure (OR-gate), so it is not necessary to consider each individually. The minimal cut sets are as followed:

{1CDSF, 2CDSF, 3CDSF, 4CDSF}, {1CDSF, 2CDSF, 3CDSF, 5CDSF}, {1CDSF, 2CDSF, 3CDSF, 6CDSF}, {1CDSF, 2CDSF, 4CDSF, 5CDSF}, {1CDSF, 2CDSF, 4CDSF, 6CDSF}, {1CDSF, 2CDSF, 5CDSF, 6CDSF}, {1CDSF, 3CDSF, 4CDSF, 5CDSF}, {1CDSF, 3CDSF, 4CDSF, 6CDSF}, {1CDSF, 3CDSF, 5CDSF, 6CDSF}, {1CDSF, 4CDSF, 5CDSF, 6CDSF}, {2CDSF, 3CDSF, 4CDSF, 5CDSF}, {2CDSF, 3CDSF, 4CDSF, 6CDSF}, {2CDSF, 3CDSF, 5CDSF, 6CDSF}, {2CDSF, 4CDSF, 5CDSF, 6CDSF}, {3CDSF, 4CDSF, 5CDSF, 6CDSF}.

Where CDSF = compressor drive system failure

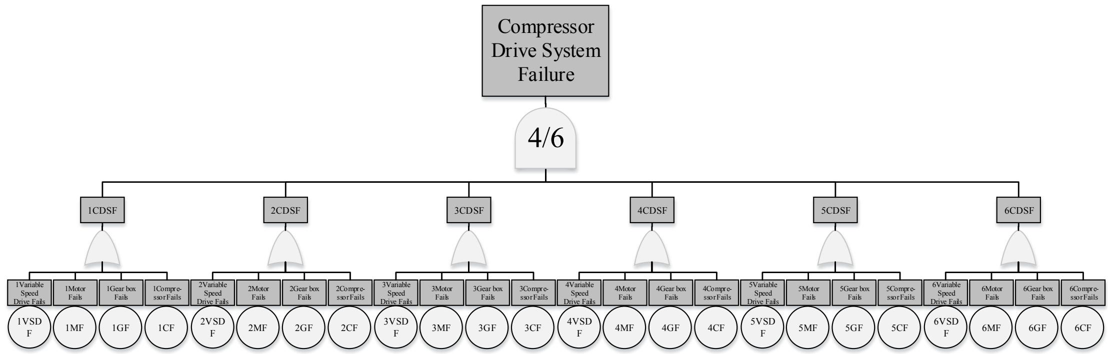


Figure 3: Fault tree for the Compressor Drive System during summer

2.1.3 Aging System

Statoil has some concerns about the age of the CDS trains. They are fully aware that aging processes gradually is happening as time goes. This can lead to system failure and it is therefore of Statoil's interest to model the remaining useful lifetime (RUL). The electrical motor is especially exposed for aging, which will be further explained in section 2.2.3 and chapter 4. It is also worth mentioning that the motors at Kollsnes are "one of a kind", so the exact same type of motors are not believed to exist.

Figure 4 below illustrates one of the compressors at Kollsnes. By looking at the workers in the figure it is possible to get an understanding of the size of the compressors that is in the CDS trains.

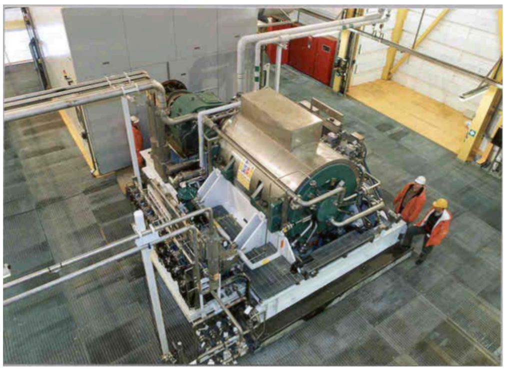


Figure 4: Compressors at Kollsens [12]

2.2 Review of the Components in the Compressor Drive System

2.2.1 Variable Speed Drive

A variable speed drive (VSD) is an equipment that regulates the speed and the rotational (torque) force output of an electric motor [13]. There are several reasons for using a VSD, where some motors cannot run without them, while others are using them to save energy [14]. In general a VSD are used to; match the speed of a drive to the process requirements, match the torque of a drive to the process requirements, save energy and improve efficiency.

2.2.1 Gearbox

A gearbox is a component that provides power transmission [15]. It consists of pairs of gears that are placed on shared axles as illustrated in figure 5 [16]. Most gearboxes have two or three gear pairs, were the gears are used to transfer a rotational torque from an axel to another. The axels can be parallel or be placed in 90° to each other A gearbox is normally conducted with an inbound and an outbound axel. The transmission occurs due to gears in different sizes [15], which leads to a different number of revolutions and a different driving force in the inbound and outbound axel.

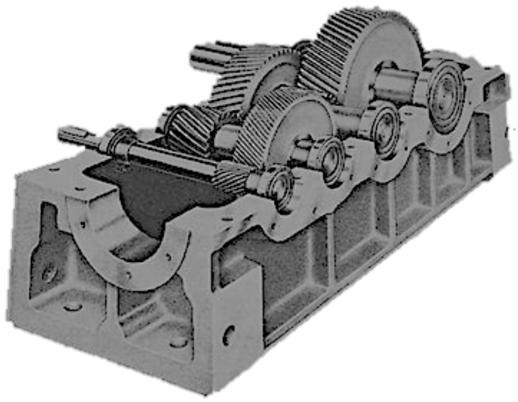


Figure 5: A big gearbox, were the upper part is removed [16]

2.2.2 Gas Compressor

A compressor is a machine that changes the pressure in a gas from a lower- to a higher-pressure level [17]. This is done by mechanically increasing the pressure around the gas, which leads to a reduction of the volume of the gas [12]. The process of compressing the gas also increases the gas temperature. Gas compressors are used in various applications and can be useful in for example transportation of gas in pipelines [18]. The lower gas volume gives a lower velocity of the gas, which gives less friction between the gas and the pipeline surface. This results in a lower energy requirement for transporting the gas.

Compressors are divided into two main categories, which are displacement compressors and dynamic compressors [17]. A centrifugal compressor is of the dynamic compressor type and is used for large/medium flow and for high pressure [18]. This makes it a good fit for a process plant like Kollsnes.

Figure 6 illustrates a Rolls-Royce gas compressor that has a centrifugal technology [19]. The machine is driven by an electrical motor or a gas turbine, and is found in natural gas compression applications onshore and offshore.

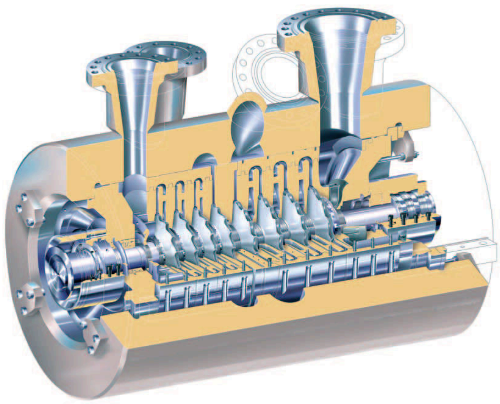


Figure 6: Rolls-Royce barrel centrifugal compressor [19]

2.2.3 Electrical Motor

A motor is a machine that converts a hydraulic-, mechanical- or electrical driving force into a mechanical force [29]. There are different types of electrical motors, however, all of them works under the laws of electromagnetism [21]. The mechanical output power comes from current carrying conductors, which are placed in a magnetic field. The conductors will experience a force, which the motor mechanism is exploiting to produce rotation. This principle is illustrated in figure 7. The motor that this report will focus on is a synchronous electrical motor, which is the type that is being used at Kollstad.

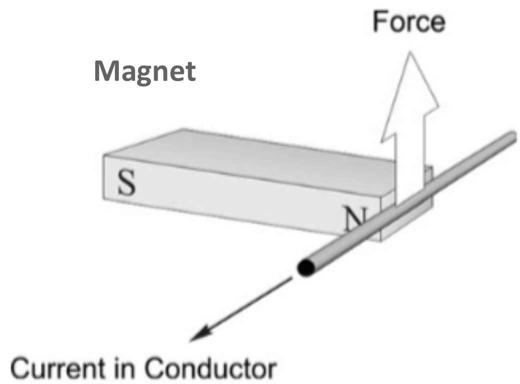


Figure 7: Mechanical force produced on a conductor in a magnetic field [21]

2.2.3.1 Synchronous Electrical Motor

A Synchronous motor is a rotating-field motor [22]. Of practical reasons it is build with a fixed stator and a rotating system called rotor [23]. At the stator there is placed conductors, called windings, which are current carrying and will therefore produce a rotating magnetic field. The rotor system acts like a permanent magnet with minus and plus poles. The poles at the rotor will attach to anchor points at the rotating magnetic fields created by the stator. This way the rotor will rotate at a synchronous speed together with the magnetic field, herby the name synchronous motor. Due to this the rotation speed is locked and not dependent on the load applied to the motor [22]. Figure 8 illustrates a typical synchronous motor with its main parts, stator- and rotor- core in addition to windings.

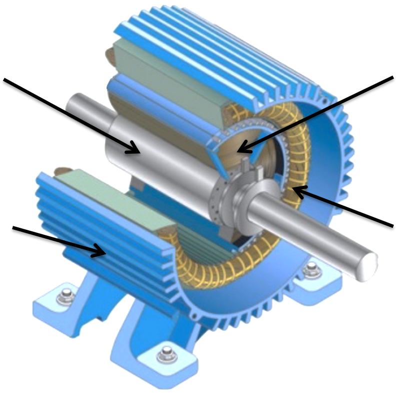


Figure 8: Exploded Synchronous electrical motor [24]

The electrical motors at Kollsenes have been identified as the most critical components of the CDS trains, due to their current age, condition and maintainability. Most electrical motors in industrial and utility applications have an expected life of at least 20-40 years before a rewind is needed [25]. A failure of a motor will lead to a CDS failure, which can result high production losses (especially during winter). The electrical motor will therefore be the component of focus in this thesis.

2.3 Introduction of Remaining Useful Lifetime

In section 2.1 and 2.2 an overview of the CDS that the master thesis is considering was presented. It was denoted that the system is aging, which is why it is of interest to model the RUL of the system. An explanation of the RUL is thereby in its plays.

The total lifetime of a system is referred to as service life and can be defined as the product´s total life in use from the point of sale to the point of discard [26]. It can also be described as a sequence of phases providing a logical path from specification thorough design, development, commissioning, operation, maintenance and at the end decommission [27].

Since the CDS system has been operating for several years, it is therefore of interest to find out what is left of the total lifetime. This remaining lifetime is the RUL, which known under several different names; among them are forward recurrence time and excess life [11]. It is defined by Rausand et al. as the time to the next failure measured from an arbitrary point of time t.

RUL can be presented as a stochastic process, where RUL(t) is a random variable corresponding to the RUL at time t [11]. An stochastic process $\{Y(t), t \in \Theta\}$ is a collection of random variables, where the state or condition (y(t)) is called the process at time t for the index set (Θ).

2.4 Formalism of Remaining Useful Lifetime

Supervisor Anne Barros has suggested a formalism of the RUL based on a comparison of different definition of prognostic. The stochastic process RUL(t) can than be defined by the following formula:

$$RUL(t) = \ inf\{h: Y(t+h) \in S_L \,|\, Y(t), Y(t) \notin S_L\} \tag{1}$$

Where: inf = infinum (the greatest lower bound of a set), t = current time, h = time after t, Y(t) = current condition, Y(t+h) = future condition at any time after t, S_L = set of failed- or unacceptable conditions

In words, the formalism predicts the RUL(t) by considering the current and the future condition. For this process it is assumed that the current condition, Y(t), not is in a failed- or unacceptable condition (S_L). The infinum of the future condition is the lowest bound of a condition to be in the set S_L. The RUL(t) will therefore be the time after the current time t where an failed- or unacceptable condition is observed, here h.

Chapter 3

Review of the Aging Process

Chapter 3 is a literature review of the aging process. First the total aging is described and classified into three categories, where each category is briefly discussed. Further the most relevant aging category, functional degradation, is displayed in a more detailed review, where degradation mechanisms, failure mode and effect- and monitoring of functional degradation are discussed.

A system will as a result of time and operation be exposed to degradation processes thorough out its whole design life. Degradation processes is referred to as aging and will increase towards the end of the design life and further in a potential life extension [5]. The design life of a system is the time a system is constructed to last with respect to parameters as fatigue, corrosion etc. [28]. Design life is defined on the basis of the various property, operation and maintenance of a system [5].

Aging is the result of the physical properties of a system as well as changes in the company that owns the system and the environment around the company [5]. The total aging is a combination of functional degradation, obsolescence and organizational issues, as illustrated in figure 9 [29], [30]. Aging systems should not be exposed to higher risks when we want to extend its lifetime [5].

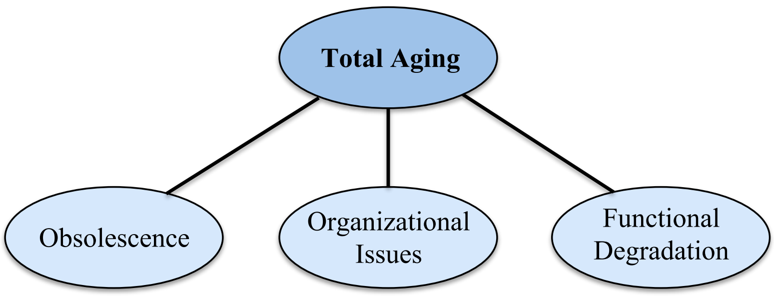


Figure 9: Total aging based on three functions [29], [30]

3.1 Obsolescence

Aging of a component can occur when the technology changes, and new technology challenges the old [5] This can for example be a new emission requirement from the government, new requirement due to a new company strategy and challenges with performing maintenance etc.

Obsolescence can cause aging due to [29], [31]

- Equipment becomes “out of date”, leading to for example non-available spare parts, services, etc.
- New needs and new types of operations requiring new technology or giving other operational/environmental conditions
- Design changes due to new technologies
- New requirements (emission-, new strategy- requirements)

3.2 Organizational Issues

Organizational issues deals with the importance of having clear responsibilities, maintaining expertise, for example transferring knowledge from retiring personnel and revising documents [31].

Aging as a result of organizational issues can be due to [29], [31]

- Reorganization
- Aging of facility personnel and transfer of knowledge
- New operations that require changes in the organization
- Changes in required competence
- Increase in work load (for example due to increased maintenance)

3.3 Functional Degradation

Functional degradation occurs when the function- and performance- ability of a component decreases as a result of physical aging [5]. Examples on physical aging are corrosion, erosion, fatigue and wear.

Aging as a result of functional degradation can be due to [29], [31]

- Material properties
- Operational condition (and changes in operational conditions)
- Environmental condition (and changes in environmental conditions)
- Maintenance practices

Functional degradation is the most relevant aging process for this report, and a more detailed description will therefor follow in the upcoming sections.

3.3.1 Degradation Mechanisms

To establish a reliable life study a good knowledge of the degradation mechanisms of the component is important [32]. A degradation mechanism is a specific process that gradually changes the characteristics of a system, structure or a component [5]. Degradation mechanisms contribute to challenges with maintaining the desired performance level, and are affected by the equipment design and process- and operation- parameters such as temperature, pressure, operation frequency and load. In section 4.3 a review of some significant parameters, called stressors, is presented. The most relevant degradation mechanisms related to aging in general are displayed in table 1.

Table 1: Degradation mechanisms due to aging [31]

Degradation mechanisms	Importance (frequency)	Description
Blockage	III	Blocked (partly or complete) equipment due to scaling, fouling or build up of corrosion products
Corrosion	I	Material degradation as a result of interaction the environment
Creep	III	Continuous permanent deformation of material due to a long term load that are below the yield stress
Flow induced metal loss	I	Removal of material from a surface as a result of impact from fluids. Ex: erosion and cavitation
Fatigue	I	Cracking under the influence of fluctuating stresses du to variation in pressure, temperature or applied load
Hydrogen related cracking	I	Cracking due to the availability of atomic hydrogen in a metal
Material deterioration	I	Loss of material properties due to exposure period, temperature, environment and load pattern
Overload	II	The actual load on the system is higher than according to the design
Physical damage	II	Damage as a result of dropped objects or maintenance
Temp./Thermal embrittlement	III	Embrittlement of alloy steels caused by holding within, or cooling slowly bellow the transformation range
Wear	II	Loss of material due to friction between moving parts
Temperature expansion and contraction	II	Variation in the temperature can cause volume changes of the component, which can cause damage on the actual component or connected equipment
Quick pressure change	III	Pressure changes can cause rupture in a component, for example if we get pressure buildup between layers
Accumulated plastic deformation	III	Or ratcheting caused by cyclic load leading to increased diameter in for example a pipe

3.3.2 Failure Mode

A failure mode is a state that a component can be in, where a function cannot be performed as required [29]. The main failure modes related to aging are cracking and fracture, physical deformation, burst, collapse, leakage, wall thinning, delamination and malfunction [31]. The degradation mechanisms displayed in table 1 can cause or induce different types of failure modes. Table 2 below illustrates the relation between failure modes and degradation mechanisms.

Delamination is a typical failure mode for motor insulation, and is related to the degradation mechanisms of corrosion, material deterioration and quick pressure change. Delamination is separation of layers due to loss of bonding strength and/or impregnating compound [25]. A more specific study of degradation mechanisms and failure modes for windings are presented in chapter 4.

Table 2: Relation between failure modes and degradation mechanisms [31]

Description	Blockage	Corrosjon	Creep	Metal loss	Fatigue	Hyd.rel-cracking	M.deterioation	Overload	Phys.damage	Temp.embrittlem	Wear	Temp.expans	Q-press.change	Accum.plastic deformation
Cracking and failure	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Physical deformation	x		x					x	x			x		x
Burst	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Collapse		x	x	x	x		x	x	x			x	x	x
Leakage		x		x	x	x		x	x		x			
Wall-thinning		x	x	x			x				x			
Delamination		x					x						x	
Malfunction	x	x		x							x	x		

3.3.1 Effect of Functional Degradation

A failure of the function of a component can be classified in three classes; critical, degraded and incipient [32]. A critical failure causes immediate cessation of the ability to perform a required function. A degraded failure is an event that not ceases the fundamental function(s) of the component, however compromises on other function(s). A degraded failure means that the component has a degraded stat, which indicates that maintenance are required to avoid that the state develops further into a critical state [31]. An incipient failure is an imperfection in the state (or condition) of a component such that a degraded or critical failure might develop if no actions are taken.

The effect of functional degradation can be considered on the basis of this failure classification. Aging stressors will lead to degradation mechanisms, which can be viewed an incipient- or a degraded- failure (depending on the level). When the degradation mechanisms are on a level where the fundamental function cannot be performed, the component has reached the state of failure mode. This state is classified as a state where a critical failure is present.

In this report a component will be classified in the critical failure category if the fundamental function cannot be performed, due to a failure as well as an unacceptable condition.

3.3.1 Monitoring of Functional Degradation

Monitoring can be used to detect the technical condition of a component [5]. The monitoring should provide a measurable variable that are correlated to degradation or loss of performance [11]. The effect that the functional degradation has on a component may be detected by a change in these variables [31].

Periodic monitoring is a possible detection strategy, which provides intermittent information [31]. Some information can be missing with this strategy, however, degradation due to normal aging is generally relatively slow, and periodic monitoring is therefore suitable as long as the interval lengths are not excessive. Examples of monitoring technics are sampling of vibration, corrosion, thickness and temperature measures.

Chapter 4

Aging of Electrical Motors

Chapter 4 starts with a presentation of studies on electrical motor failure. The most critical part of the motor is identified and described in more detail. To place the aging subject into context, a presentation of aging stresses, degradation mechanisms and failure mode specific for the motor is displayed.

4.1 Studies on Electrical Motors Failures

There have been performed many studies on electrical motors. The results varies a lot, which is understandable based on the fact that tis exists different types and sizes of motors, as well as differences in the usage sector, age, maintenance strategies and more. An article written by Maintenance technology analyses what different studies is saying about large electrical motor reliability [33].

10 presents a comparison of three different studies on motor failure. The Y-axis displays how many percentages of the motors that have failed due to a specific motor part, which is in the X-axis. All three studies found that a majority of faults were a result of bearing failures. The windings have the second highest rate of failure. The "other" category comes as a close number three. While the rotor is the motor part that fails least frequent out of the four categories.

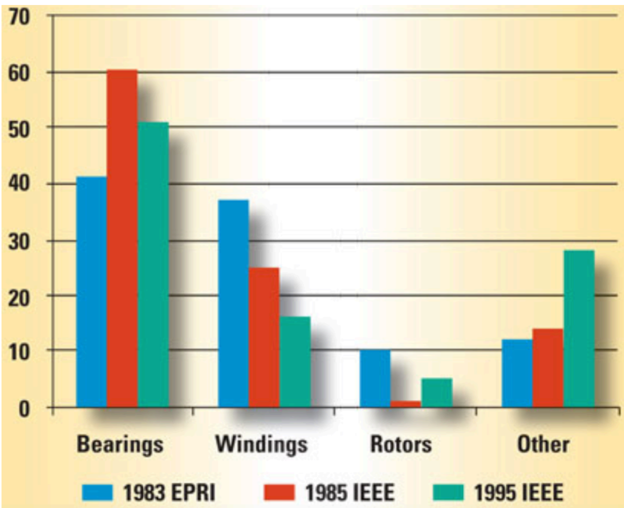


Figure 10: Comparison of ERPI and IEEE surveys of electrical motor failures [33]

The survey from IEEE (1985) did not only look into general failures like the one from EPRI does [33]. It also broke out the importance of service factor, speed and maintenance, which IEEE (1993) further modifies by identifying size and voltage. The results indicated that larger motors had a higher failure rate than smaller motors, slower speed gave a higher failure rate and maintenance gave a lower failure rate. According to IEEE researches, a failure rate can be reduced by two-thirds when condition monitoring are used.

Although most studies on electrical motors indicates that the bearings is most critical part, failures of windings is what this paper will focus on. This is due to the following reasons (Lunde, E. 2016)

- The type of bearings on these machines are quite robust (few incidents)
- Bearing condition is taken care of by vibration monitoring.
- Bearings are easy to replace.
- Stator windings are getting old and winding failure will cause lost production for several months (not off-the-shelf motors)

For this reason a further literature search of winding was conducted, where the failure distribution illustrated in table 3 was found. The table is based on the EPRI study from 1983. We can see that Stator related failures are more frequent than rotor related failures. It is also clear that most insulation failure is the main root cause for the stator.

Table 3: Failure percentages related to rotor and stator, based on EPRI [34]

Stator Related	36%	Rotor Related	9%
Ground Insulation	22%	Cage	4%
Turn Insulation	4%	Shaft	2%
Bracing	3%	Core	1%
Wedges	1%	Balance Weight	1%
Frame	1%	Other	1%
Cable	1%		
Connections	1%		
Other	3%		

4.2 Electrical Motor Winding

The stator- and the rotor part of a synchronous electrical motor can both have windings on them. A winding consist of several parts, each with its own function [25].

The windings that are used on the motors at Kollsnes are of the coil type. There are two main types, which are form-wound coil and random-wound coils [35]. The winding is square- or rectangular shaped for form-wound coil, while the winding is round- or diamond shaped for random-wound coils.

4.2.1 Stator Winding

The three main components of the stator are the conductors (often copper), the stator core and the insulation [25]. The cobber is a conduit for the stator winding current. As previous mentioned the current introduced in the conductors is creating a magnetic field that forces the rotor to move. The current can cause overheating if the cross section of the conductors is too small. Thermal stress will be covered in section 4.3.1.

The stator core consists of thin sheets of magnetic steel, which concentrates the magnetic field from the conductors on the stator [25]. This makes a path for the magnetic field from the stator to the rotor, and in addition it prevents most of the magnetic field from “escaping” as current into adjacent conductive material.

The final part of a stator winding is the insulation. Insulation is, as apposed to the conductors and the core, a passive component [25]. This means that it does not produce a magnetic field or guide its path. The function of the insulation is a combination of providing electrical isolation, mechanical support, heat dissipation, energy storage and personnel safety [36]. The insulations primary contains organic material, which in general softens at lower temperatures and has a lower mechanical strength than cobber and steel [25]. For this reason the insulation often limits the lifetime of a stator winding, rather than the conductor or the core. Thus, stator winding maintenance and testing almost always refers to actions regarding the insulation.

4.2.2 Rotor Windings

The rotor winding has in general the same components as the stator, yet there are important differences [25]. Also here conductors are present to act as conduit for the current. The current flowing thorough the conductors in the rotor are steady state and usually of DC type for synchronous motors. The need for a magnetic steel rotor core is less critical due to this current, which has a lower frequency. For large and high speed motors the centrifugal forces can be high, which means that the magnetic steel core needs to be forged and not laminated in layers so the stress can be tolerated.

The material of the rotor winding insulation is also of the organic type [25]. Thus it has poor thermal and mechanical properties compared to the material of the conductor. For this reason the insulation often determines the expected life of a rotor winding.

4.3 Aging Stresses of Windings

Failure of insulation in stator and rotor winding is due to presence of degrading stresses, also called aging processes [36]. This is stress such as thermal, electrical and mechanical stress in addition to environmental factors. The stresses can be constant, they can be present for only a brief time and they can act together as multiple stresses [25]. When a constant stress is present, the time to failure is proportional to the number of operating hours of the motor. Brief stresses, like those that occurs as the motor starts, has a time to failure that is proportional to the number of transients the motor experiences. In addition, radiation stresses can cause aging.

4.3.1 Thermal Stress

The thermal stress in closely linked to the lifetime of a motor, which means that the components of a motor can be evaluated from a thermal point of view [22] This stress is probably the most recognized cause of the aging process [25]. Windings must therefore be evaluated for its capability to be under thermal stress. The aging processes due to thermal stress are caused by high temperature environment, resistive losses or chemical instability [36].

For an electrical motor to operate, transformation of energy is required [22] Transformation of energy generates resistive losses in the form of heat. Usually, the winding is the largest heat source of the motor. Organic materials, like the winding insulation, are sensitive against heat stress. The same is the windings themselves, as well as other metal parts, lubrication and permanent magnets.

The high temperature causes a chemical reaction when the motor operates above the threshold temperature [25]. This is an oxidation process that makes the insulation brittle and/or causes delamination. An approximation for the life of the insulation related to the temperature is as followed

$$L = Ae^{\frac{B}{T}} \tag{2}$$

This formula is based on the Arrhenius rate law, which governs the rate of reaction for the oxidation process [25], [37]. Where L is the lifetime in hours, T is temperature in Kelvin and A and B is assumed to be constants. Based on the approximation, the lifetime of the winding will decrease by 50% for every 10°C increase in temperature. This is only valid at higher operating temperatures, due to no thermal aging will occur below a threshold, which is different for each insulation. More than one chemical reaction can also occur at the same time. Even though the formula is not strictly valid, there is little desire to make it more accurate since it is firmly entrenched in standards.

The standard estimates the thermal life of a winding on the basis of the formula (accelerated aging tests), and classifies insulation into thermal classes [25]. With a specific thermal class, the manufacturer guaranties that (under rated conditions) the described heat sources will not impermissibly stress the motor, so that the specific product life expectancy can be reached [22] For instance, a “thermal class A” insulation will withstand a maximum winding temperature of 105°C.

The conductors will expand axially if we have a temperature rise [25]. If this temperature change occurs quickly a shear stress between the conductor and the insulation will be present, due to metal conductors grows more quickly than organic insulation. This thermal stress can cause the bond between the insulation and the conductor to break.

Temperature can have an effect on other failure processes that are not strictly thermal, and high temperatures can also have a positive consequence [25]. However, this will not be covered in this thesis.

4.3.2 Electrical Stress

The electrical stress is the rated power frequency voltage (V, in kilovolts) divided by the insulation thickness (d, in mm), as presented in the following equation [25]

$$E = \frac{V}{d}\ (kV/mm) \tag{3}$$

The electric stress has a small impact on the aging process of insulation if the winding is rated at around 1000 V or less. [25]. This is due to the required thickness of the insulation is primarily determined by mechanical stress. For windings rated above about 1000 V, the thickness of the insulation is primarily determined by the electrical stress.

Power frequency voltage can lead to aging of the insulation if partial discharge (PD) is presents [25]. PD can be described as an electric- pulse or discharge that occurs within air pockets (voids) in the insulation or at a dielectric surface on the insulation system [38]. These discharges can occur in any void, which may be located internal in the insulation itself, between the outer insulation wall and the grounded frame, between the insulation and the conductor or along the surface of the insulation. This is illustrated in figure 11.

The electric pulses contain electrons and ions that bombard the insulation [25]. Organic metals such as the insulation degrades under this bombardment due to breaking of chemical bonds. After enough time, the PD will erode a hole through the organic parts, which will lead to failure. Insulation tracking can also occur, usually on the insulation surface [38]. This area of PD can bride the potential gradient between the applied voltage and surface by cracks or paths. Cracks are the most critical insulation condition due to PD, followed by (in decreasing order) void between conductor and insulation, void between insulation and core and void internal to insulation.

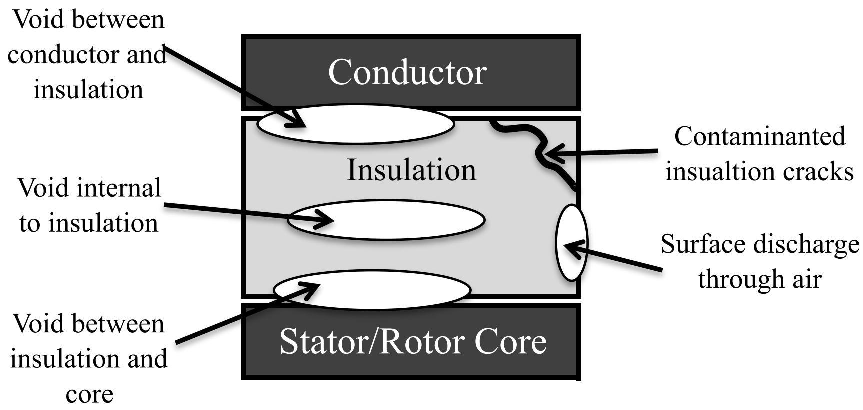


Figure 11: Surface and internal Partial Discharges, based on [38].

Another way that electric stress can age the insulation is when many repetitive voltage surges are impressed across the turn insulation [25]. These surges can impose relative high voltages across the first turn in a stator winding. This process involves, with every voltage surge, emission of electrons from surface imperfections on the conductor into the insulation, which can lead to puncture.

The Lifetime of the insulation (L in hours) can be represented by the inverse power model displayed bellow [25], [39]

$$L = cE^{-n} \tag{4}$$

The model is based on the effect of the stress level E (in kilovolt/millimeter) and can be applied if PDs are present [25]. The c is a constant and n is called the power law constant. Also for electrical stress there is no effect of aging bellow a threshold. This threshold is the PD extinction voltage, and can be included in the formula over by replacing E with $E - E_0$, where E_0 is the threshold. The power law constant is normally between 9 and 12 for insulation systems. If we assume n to be 10, a two-time increase in electricity stress will reduce the life drastically by about 1000 times.

4.3.3 Mechanical Stress

The insulation on an electrical motor is exposed for three main sources of mechanical stress. These are centrifugal force, magnetic force and stress caused by transients [25]. The centrifugal force is a non-vibrating force that can crush or distort the insulation. There are not much aging related to this, however some material may creep, which eventually may lead to failure. Creep is when a material slowly creeps away from a high stress area.

The magnetic force is a result of the power frequency current, which gives a mechanical stress oscillating at twice the power frequency [25]. If the winding coil is loose in the stator slots, the force causes the coil to vibrate, which can lead to the groundwall insulation being abraded.

The third common mechanical stress is caused by transients, and can occur while switching-on of motors, or if a synchronous motor is out of phase [25]. Both gives rise to a large transient power current that might be five times, or more, higher than the current in normal operation. This means that if the motor is out of phase during startup, the magnetically induced mechanical force is 25 or more times stronger than under normal conditions. This transition force tends to bend the coils in the stator winding, which can lead to insulation cracks it the force cannot be withstood. If a motor is frequently started and stopped, many transients may occur, which can result in the end winding to gradually loosen over time.

Cygan et al. specifies that the aging process due to mechanical stress also is caused by the forces that occur under thermal expansion, etc. [36]. This is explained in the section 4.3.1 about thermal stress.

Unlike Thermal and electrical stresses, there are no well-accepted life models related to mechanical stresses. There exist models on this, however none have become the basis for standards. Anyhow, a model that is used is displayed as bellow [40]

$$L = K_m \times S^{-m} \tag{5}$$

The S represent the mechanical stress that the insulation is subjected to, while the K_m and m are constants.

4.3.4 Environmental Stress

The aging process due to environmental stress is caused by conditions in the environment of the insulation, such as chemical reaction, oxidation, moisture, radiation, etc. [36]. Stone et al. refers to this as ambient stress, and list factors as humidity, oil, abrasive particles, dirt and debris in addition to those derived by Cygan et al. [25].

Each of the stresses can affect the insulation in different ways [25]. Sometimes these factors in themselves do not cause aging, however, if combined with another stress aging can occur. For instance can oil and/or moisture combined with dirt, etc., can lead to a partly conductive film over the insulation, which can cause surface currents and electrical tracking. The same combinations can also get stuck in ventilation passages and between coins, which can block cooling airflow and cause thermal determination. Low humidity can for instance lead to greater PD activity.

For the factors, it is often not possible to relate the level of stress directly to the rate of deterioration [25]. The factors are usually either present or not when aging testes are performed.

4.3.5 Multiple Stress

In many cases two or more stresses need to interact before degradation can occur [25]. Many degradation mechanisms do therefore not arise due to a single stress. However, they are often a result of a combination of stresses, here referred to as multiple stresses. This is illustrated in figure 12. When two or more of the single stresses are present, the failure process would be much faster than if only one single stress where present, if the single stress would ever lead to failure by itself.

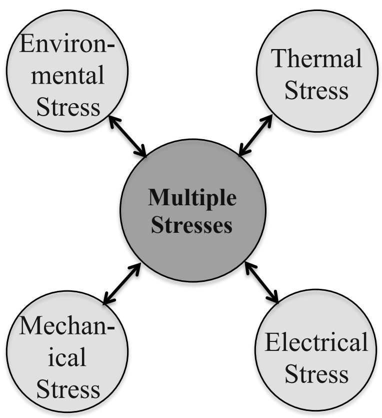


Figure 12: Combination of multiple aging stresses

In real applications, one of the most frequently encountered situations is electrical- and thermal- stresses acting simultaneously [36]. Analyses have been performed for motor insulations, where different types of stresses, and combinations of them, have been applied. The aging result obtained was quite different, for stresses applied separately and simultaneously. Hence, it is not strange that there is an interest in modeling with multiple stresses.

Multiple stress modeling is of great significant in aging studies on electrical insulation [36]. Normally, models are limited by the electrical- and thermal- stresses. This is due to the reason that the presence of these two stresses is almost unavoidable. There has therefore been made several different models for lifetime calculation with these stresses. Based on previous work the following formula was derived [41]

$$L_{TE} = L\times e^{-BcT}\times\left(\frac{E_0}{E_1}\right)^{-\left(\frac{\log(E_1)}{\log(E_2)}-1\times cT\right)} \tag{6}$$

Where: $L = T_{log}\times 0.5^{\frac{T_1-T}{B}}$, $cT = \frac{1}{K_R}-\frac{1}{K_E}$, cT = the conventional thermal stress, K_E = the absolute temperature evaluated in Kelvin, K_R = the absolute rom temperature in Kelvin, L = the thermal life at elevated temperature, T_{log} = average life from thermal endurance testing in hours, T_1 = the temperature of interest, T = the temperature of the thermal life study, B = the slope of the corresponding Arrhenius curve, E_0 = voltage applied to windings, E_1 = voltage applied during the studies, E_2 = the additional stress based upon the initial electrical endurance of the winding insulation system

There exist models where mechanical stress is added inn addition to thermal- and electrical- stress, however, this will not be covered in this thesis.

4.4 Degradation Mechanisms of Stator Windings

The thermal- electrical- mechanical and environmental- aging stresses can lead to degradation mechanisms of a synchronous electrical motor. The main degradation mechanisms due to aging of the stator windings are presented in table 4.1 and 4.2 below [25]. A similar table for rotor windings will not be presented in this report, due to the fact that the stator windings is most exposed to degradation. Many of the degradation mechanisms for stator winding will also be relevant for rotor windings.

The coil type is significant for which degradation mechanisms that can be present. In table 4.1 and 4.2, degradations mechanisms relevant for form-wound coils are marked with I, while it is marked with II if it is relevant for random-wound coils. Parenthesis around the type indicates that it is unlikely that this type can experience the applicable degradation mechanisms. The description section covers the topics of general info (I), general process (P) and root cause (R). Symptoms and remedy is not deliberated in the table.

Table 4.1: Degradation mechanisms for stator windings part 1 [25]

Degradation mechanisms	**Coil type**	**Description**
1. Thermal Deterioration	I, II	I: Caused by operation at high temperature. Not common for water-cooled windings, however one of the most common reasons for failure for air-cooled motors. P: Can lead to two mechanisms; The conductors are not held tightly together and PD occurrence in the delaminated insulation. R: Overload operation, poor design, poor manufacture, insufficient time between motor starts, high harmonic currents from drivers, negative sequence currents from voltage imbalance, dirt, plugging, debris/copper oxide, loose coils, operation under-excited and to many dips and bakes.
2. Thermal Cycling	I, (II)	I: Most likely to occur in motors with long stator cores and when rapid starts and stops are rapidly changing loads. Direct water-cooled motors are unlikely to experience this due to conductor temperatures changes slowly. P: Mechanisms such as girth cracking/tape separation, conventional thermoset determination and vacuum pressure impregnation. R: A combination of too fast load change, operation at to high temperature and inadequate design of the insulation system.
3. Inadequate Resin Impregnation	(I), II	I: Resin impregnation is also called dipping. It seals the windings against dirt and moisture, improves heat transfer, and reduces vibrations. Dipping can reduce voids that would lead to PD. P: Poorly dipping will cause imperfection in insulation, as pinholes or cracks, which can lead to higher operation temperatures. R: too thin/thick resin, resin chemistry, to high bake temp, not following supplier's recommendations and inadequate covering.
4. Loose Coil in the Slot	I, (II)	I: Sometimes called slot discharge. Most common in thermoset-type coils, and not likely for thermoplastic windings. P: Vibrations between insulation and stator core. Can cause insulation to be rubbed way, if 30% is rubbed away we will have a failure. Has two stages, where PD will occur in stage 2. R: Shrinkage of insulation, ripple springs ability loss and wedge loosening

Table 4.2: Degradation mechanisms for stator windings part 2 [25]

5. Coating Failure	I	I: Coating is present to prevent PD. It is most likely to occur for very high voltage impulses and for air-cooled machines (since involving PDs). P: determination of the coating of the coils in the absence of coil vibration. R: Poor maintenance, to high surface resistance, local heating, and switching voltage frequencies.
6. High intensity Slot Discharge	I, (II)	I: A discharge where coil are not loose in the slots. P: Semiconductive coating becoming isolated from the stator core for the entire length of the slot. Since gap between the semicon- ductive coating and the stator core is discharged when contact is made, a very large discharge results. T: Poor design, Poor manufacturing
7. Vibration Sparking	I, (II)	I: Also called spark erosion. Related to loose coil problems. A rare problem brought on by poor design. Not a PD mechanism. P: A spark can be formed if semiconductive coating losses contact with core and the current is large enough. R: Poor design
8. Repetitive Voltage Surges Due to Drives	(I), II	I: Gradual aging, if the voltage surges from drive are of high magnitude. Smaller motor is most likely to have this problem. P: Gradual deterioration of insulation and coating due to repetitive voltage surge. R: A fast surge rise-time, mismatch between motor and cable, air pockets and presence many surges per second.
9. Contamination	I, II	I: Leads to faster thermal deterioration, chemical attack and electrical tracking. Electrical tracking occurs most often in dirty and wet motors with high operation voltage. P: Enables current to flow over the surface of the insulation. These currents degrades and eventually causes the groundwall insulation to fail. R: partly conductive contamination, prior thermal deterioration or winding vibration.
10. Chemical Attack	I, II	I/P: Deterioration of the insulation that occurs when exposed to an environment in which chemical such as acids, solvents, oil and water are present. R: Presents of chemicals and bad maintenance
11. Inadequate End Winding Spacing	I	I: In large stators, space is left between coisl in the end winding to ensure cooling. Unlikely to occur in high-pressure, water-cooled motors. P: To small space between coils in the end winding will lead to PD given sufficient time. R: Insufficient space between coils, thin groundwall design, inconsistent coil shape, poor installation, insufficient blocking, poor and chemical control.
12. End Winding Vibration	I, (II)	I: If the end windings are not adequately supported, the coils will vibrate. One of the most common degradation mechanism of large gas ant turbine generators. P: Current flowing through coil creating large magnetic forces that leads to vibrations that gradually abrades the insulation. R: Inadequate initial design, poor manufacturing procedures, prior out-of-phase synchronization, long-term operation at high temperatures, and excessive oil in the end winding.
13. Stator Coolant Water Leaks	I, II	I: Water leaks from a water-cooling system onto the insulation. A ground fault can immediately occur. P: Water will reduce the mechanical- and electrical- strength of the insulation. A slow deterioration process that can eventually lead to corrosion. R: Poor brazing, poor workmanship and marginal plumbing connections.
14. Poor Electrical Connections	I, (II)	I: Windings consist of hundreds if not thousands of electrical connections. P: If the resistance of the connections is too high, overheating thermally degrades the insulation. Over time the connections between coils may fatigue-crack. R: Poor workmanship, poor brazing, poor soldering and inadequately- tightened or designed bolted connections.

4.5 Failure Mode of Stator Windings

Degradation mechanisms will eventually cause the windings to reach a failure mode. Recall that a failure mode is a state where the condition of the winding not is acceptable. The modeling of the winding RUL is closely related to this state. Therefore, it is of interest to identify a variable that can represent the aging process towards a failure mode.

4.5.1 Health Indicator of the Windings

Co-supervisor E. Lunde has explained that the PD is the best known measurable variable that can represent the aging process. In table 4.1 and 4.2 from section 4.4, it can be seen that PD is closely related to the degradation mechanisms that affects the insulation. Insulation was, earlier in this chapter, identified as a part that limits the lifetime of the windings. By this, it can be concluded that PD is a good health indicator for the insulation condition, and can therefore be formalized as in the equation 7 bellow. The degradation at time t, Y(t), is determined by the partial discharge at time t, PD(t). PD(t) is measured in nanocoulombs [nC].

$$Y(t) = PD(t) \tag{7}$$

4.5.1 Failure Mode based on Partial Discharge

Equation 7 indicates that a high value of PD equals a high level of degradation. Note that a failure can occur at every value of PD, however, as the condition gets worse the frequency of failure should increase. Different types of failures can occur, like for example an insulation crack or a rubbed away coating. Since a failure can occur at any time a failure might not be a good way to represent the aging process. The failure mode considered in this thesis is therefore the level of PD that indicates an unacceptable insulation condition. This PD level is displayed in table 5 in section 5.1.

4.5.1 Monitoring of Partial Discharge

There exist several different ways to detect and monitor the PD [42]. Most detection system has elements related to sensors to detect pulses, electronics to convert the pulse signal to digital form and a signaling process that reduces the information to manageable quantities. The monitoring process can be conducted on-line while a motor is still operating, or the PD can also be recorded off-line. The PD can be measured both continuously and at periodic inspections.

Chapter 5
Model Description

Chapter 5 is presenting classifications, assumptions, limitations and model theory that are of interest for the modeling of the remaining useful lifetime of a motor. First a classification of the motor condition will be displayed, which will be the basis of the final model. Then a presentation of the importance of data will follow. Further a general model description is presented, which will develop into a more specific study of the model of interest. Several equations and a estimation method is also presented.

5.1 Classification of Motor Condition

Partial discharges (PD) have a significant importance for the aging process of insulations. PD will gradually lead to degradation mechanisms such as presented in chapter 4. It was denoted that the level of PD could represent the condition of the insulation, hence the motor itself. The modeling of the RUL can therefore be represented by the development of the PD, and a motor condition calcification based on the PD is therefore in its place

Karsten Moholt has presented a table that displays the insulation condition based on specific PD value intervals. This is illustrated in table 5, where each condition interval has been assigned to a state number. The best condition is called excellent and will in this thesis be referred to as state 5. The worst condition is state 0, where the insulation is said to be unreliable. This will be thought of as the failed state, however the insulation is not necessary failed were an unacceptable condition is possible. This means that once a state 0 condition is observed the motor should be taken out of operation, and thus it is considered as failed.

Table 5: Insulation quality measured in PD (nC), based on [43]

Insulation condition	Partial Discharge [nC]	State
Excellent	<2	5
Good	>2<4	4
Average	>4<10	3
Still Acceptable	>10<15	2
Inspection Recommended	>15<25	1
Unreliable	>25	0

In chapter 6 this classification of the insulation condition will be used for the modeling of the remaining useful lifetime (RUL) of the motor. To do this it is of interest to investigate historical data to se how the PD develops over time. This data is therefore important for which model that should be used.

5.2 Importance of Data

The selection of the best survival model is, as mentioned to the left, dependent on the available data. The quality of these data is an important factor, which also means that the study of the data that are under investigation should be considered [44]. For example should the uncertainty- and the distribution of the data be of interest when selecting a model, as well should the credibility- and the size of the study not be overlooked. Sometimes we also have to deal with an issue related to censored data, which will be presented in section 5.8. There exist several examples that illustrate the importance of data. For example it is significant that the data represent a real situation and not special cases, such that the predictions is as accurate as possible. Also if the data available indicates that the time spent in a state does not have an exponential behavior, a model that is based on an exponential distribution, like Markov, would not be a good choice of model.

For this thesis there where no available data that could be used to model the RUL. A model could therefore not be selected based on data. The model to select should therefore be chosen based on the properties of the model itself. For example how good the model can present the aging process of the motor. It is therefore of interest to take a closer look at the theory and properties behind models.

5.3 Model Introduction

In the chapter above it was explained that data is important for the selection of a model. Before the selection of a model it is also essential to know the properties of a model. A model is an abstract representation of a functional/mathematical relationship, in the form of a graph, chart or equation [45]. These mathematical relationships are build on variables and parameters, where a parameter is a known constant value and a variable represents a symbol that can take any value. This can be used to introduce two different models types, called deterministic- and probabilistic- models.

5.3.1 Deterministic Modeling

If all parameters in a model are known with certainty, the result that the model presents is assumed to be certain [45]. Models of this type are called deterministic models. A given input variable in such models will always give the same output result without variations. The inverse power model presented in equation 4 in section 4.3.2 is an example on a deterministic model. The relationship between the parameters and variables gives the opportunity to make lifetime predictions, and to see how variables affect each other.

5.3.2 Probabilistic Modeling

In contrast, a probabilistic model contains uncertainty, which for example can be a future condition [45]. This means that more than one outcome of a model is possible, and that there is some doubt about which outcome that will occur. A probabilistic model can be defined as a representation of a random event within a sample space [46].

5.3.3 Selection of Model

Since the deterministic models do not include uncertainty, their relation to survival studies can be a bit unclear. A way they can be useful is in the decision making phase. The models can give us an indication of the effect the different stresses has on the aging process. For example it might be possible to use the inverse power model (equation 2) to se how the PD level behaves as the voltage increases. It is clear that the motor lifetime will decrease as the voltage increase. This can for example be an indication that the motor voltage (or applied load) should be reduced if the PD level is in a critical state. Deterministic models can also be used for parameter estimation, for example by performing accelerated life tests of the motor. Accelerated life testing is about speeding up the degradation process by increasing stresses, like for example the temperature [47]. It is known that an Arrhenius relationship exists between the temperature and the failure time (equation 2). The aim is therefore to extrapolate the result from the accelerated life test to normal operation conditions.

A probabilistic model seems to be a good fit for reliability issues, due to its uncertainty and randomness perspective. This is also concluded in an SINTEF report [31]. In this report Hokstad et al. classifies probabilistic models into four methods, which is lifetime models, rate of occurrence of failure (ROCOF) models, Markov and probability of failure (PoF) models. For this thesis, Markov model has been selected, where an explanation for this choice will follow in the following section.

5.4 Introduction to Markov Modeling

A Markov model is a probabilistic method used to model systems with several states, and the transitions between the states [31]. This method considers a random process through discrete states and through a finite number of possible sates. The Markov method can be used to calculate several survival measures, in addition it is a good tool to get an overview of a system.

Markov models are a special type of a stochastic process $\{X(t), t > 0\}$ [11]. A stochastic process $\{X(t), t \in \Theta\}$ is a collection of random variables, where $X(t)$ is a random variable that denotes the state of the process at index t, where Θ is the index set. The index t is often represented as time. The collection of the possible states is called state space, and is denoted by $\boldsymbol{\mathcal{X}} = \{0, 1, 2, \dots, r\}$. A stochastic process with a Markov property is memoryless in the sense that the future states are given by the present state and are independent of past states. Such a process is called Markov chain.

5.4.1 Markov Chain

A stochastic process with a finite number of possible states will be considered. The index t is assumed to be nonnegative values. If X(t) = i, the process is at state i at time t. When the process in state i we have a fixed probability P_{ij} that a transition to state j will occur. That is

$$
\begin{aligned}
&P\{X(t+1) = j \mid X(t) = i, X(t-1) = i(t-1), \dots, X(1) \\
&= i(1), X(0) = i(0)\} = P_{ij}
\end{aligned} \tag{8}
$$

for all states j, i, i(0), i(1), … i(t - 1) and for t ≥ 0, the stochastic process is called a Markov chain. Equation 8 illustrates that the conditional distribution of future state, X(t + 1), depends and the present state X(t), and is independent of the past states,
X(0), X(1), … , X(t - 1). P_{ij} will be further presented in section 5.4.2 and 5.4.4, and more theory about Markov chain can be read in Ross`s book [48].

5.4.2 Markov Process

When dealing with a Markov model the time can be either discrete (taking distinct values {0, 1, 2, …}), or continuous [11]. If the time is discrete the model is called a discrete-time Markov chain, and if the time is continuous the model is known as a continuous-time Markov chain. A continuous-time Markov chain is also called a Markov process.

Consider a Markov process for all times s, t ≥ 0, and nonnegative i, j, x(u), 0 ≤ u < s [48]. Lets assume that the state of the process at time s is X(s) = i, we can say that the conditional probability to be in state j at time t + s is [11]

$$
P(X(t+s) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < s) \tag{9}
$$

If equation 9 is equal to $P(X(t+s) = j \mid X(s) = i)$, the stochastic process has Markov property, and thus it is a Markov process [11]. In addition, if a transition between two states is independent of s, the Markov process is said to have a homogenous transition probability [48]. A process with this property is called a time-homogeneous process, and is valid if $(X(t+s) = j \mid X(s) = i) = P(X(t) = j \mid X(0) = i)$ [11]. This means that

$$P(X(t) = j \mid X(0) = i) = \ P_{ij} \qquad for\ all\ i, j\ in\ the\ state\ space \tag{10}$$

It has been established that the Markov process has Markov property. This means that the amount of time the process stays in state i, $T_{i,}$ is memoryless and must thus be exponentially distributed, $T_i \sim\lambda_i e^{-\lambda_i t}$ [48]. In fact, a Markov process has the properties that when it enters a new state

- The time spent in the state before a transition to a new state is exponentially distributed with a rate, lets say λ_i.
- When the process leaves the state, it will enter state j with some probability, P_{ij}, where $\sum_{j\neq i} p_{ij} = 1$.

5.4.3 State Transition Diagram

A Markov process can be visually presented as a state transition diagram, also known as a Markov diagram [11]. In the diagram circles are used to represent states, and discrete arrows are used to represent transitions between the states. The transitions are given as rates, α_i, which can be a repair rate, μ_i, (a transition to a better state) or a failure rate, λ_i, (a transition to a worse state). For this thesis the state transition diagram will have a sequential structure, and will appear in the similar style as figure 13 illustrates.

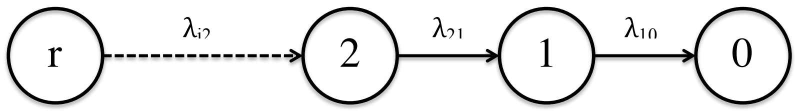


Figure 13: State transition diagram, sequential structure

5.4.4 Transition Rate Matrix and Chapman-Kolmogorov Equation

Consider the probability, P_{ij}, presented in equation 10. This probability can also be presented in a matrix form as displayed in equation 11 [11]. When the motor is in sate i at time 0, it most either be in state i at time t or made a transition to a different state. The sum of each row in the matrix is therefore equal to 1.

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{00}(t) & P_{01}(t) & \cdots & P_{0r}(t) \\ P_{10}(t) & P_{11}(t) & \cdots & P_{1r}(t) \\ \cdots & \cdots & \ddots & \cdots \\ P_{r0}(t) & P_{r1}(t) & \cdots & P_{rr}(t) \end{bmatrix} \tag{11}$$

From the Markov property defined in section 5.4.2 it is known that the time spent in sate i is exponentially distributed with rate α_i, and a probability, P_{ij} to enter a new state j. The relationship between α_i and P_{ij} can be represented as followed

$$\alpha_{ij} = \alpha_i \times \mathrm{P}_{ij} \qquad for\ all\ i \neq j \tag{12}$$

Since α_i is the rate at which the motor leaves state i, and P_{ij} is the probability that it moves to state j, it means that α_{ij} is the rate at which the motor moves from i to j.

The transition rate α_{ij} can be presented in a matrix, **A**, as presented bellow, called transition rate matrix. Opposite from matrix 11 the sum of the rows is equal to 0. This is due to the notation of the diagonal rate, α_{ii}, which is set to be equal to $-\alpha_i$.

$$\boldsymbol{A} = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0r} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1r} \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_{r0} & \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix} \tag{13}$$

A compact version of Chapman-Kolmogorov forward equation can be presented as [11]

$$\boldsymbol{P}(t) \times \boldsymbol{A} = \dot{\boldsymbol{P}}(t) \tag{14}$$

The equation is based on the Markov property and the law of total probability, where $\dot{\boldsymbol{P}}(t)$ represent the time derivative of $\boldsymbol{P}(t)$, and is applicable when $\alpha_{ii} = -\alpha_i$. In matrix form this can be presented as

$$[P_0(t), \dots, P_r(t)] \times \begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0r} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1r} \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_{r0} & \alpha_{r1} & \cdots & \alpha_{rr} \end{bmatrix} = [\dot{P_0}(t), \dots, \dot{P_r}(t)] \tag{15}$$

More interpretation of the transition rate matrix and the Chapman-Kolmogorov equation(s) can be found in the book of Rausand et al. [11].

5.5 Survival Measures

There exist several survival measures, which is of interest considering the Markov model. Some of the most common measures are presented bellow [11].

5.5.1 The Reliability Function

Reliability is a probability and is defined as the ability of a component to perform a required function under given conditions for a given time interval [32]. From this the reliability can be expressed as a function of time. The reliability function (or survival function) can be defined as

$$R(t) = 1 - F(t) = pr(T > t) \qquad for\ t > 0 \tag{16}$$

Here, t is the time, T is the time to failure and F(t) is the cumulative distribution function (CDF). The CDF denotes the probability that the component fails within the time interval (0,t]. The derivative of F(t) displays the probability density function (PDF), which presents how the probability of failure is distributed. Formalisms and diagrams of the PDF and the CDF can be seen in the book of Rausand et al. [11].

5.5.2 The Transition Rate Function

Transition rate (or failure rate) is the conditional probability per unit of time that the component fails between t and t + Δt, given that it has been working over [0, t]. This probability is defined as

$$\Pr(t < T \leq t + \Delta t |\ T > t = \frac{\Pr\,(t < T \leq t + \Delta t)}{\Pr\,(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)} \tag{17}$$

To get the transition rate function z(t) we divide this probability by the length of the time interval, Δt, and by letting Δt → 0, z(t) is

$$z(t) = \lim_{\Delta t \to 0} \frac{\Pr(t < T \leq t + \Delta t | T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathrm{F(t + \Delta t) - F(t)}}{\Delta t} \frac{1}{R(t)} = \frac{f(t)}{R(t)} \tag{18}$$

5.5.3 Mean Time To Failure

Mean Time To Failure (MTTF) is the expected time before a component fails, where it can be defined as

$$MTTF = -\int_0^\infty tR'(t)\,dt = -[tR(t)]_0^\infty + \int_0^\infty R(t)\,dt = \int_0^\infty R(t)\,dt \tag{19}$$

Where it can be shown that $[tR(t)]_0^\infty$=0 if MTTF is less than infinite.

5.5.4 Sojourn Time

In a model with several states we say that $0 = S_0 \leq S_1 \leq S_2 \leq \ldots$ is times when transitions occurs. S_0 is the initial state; hence the transition time of S_0 is 0. We can then define $T_i = S_{i+1} - S_i$ to be the ith interoccurrence, or the sojourn time, for i = 0, 1, 2, …. This means that the sojourn time is the time spent in a state before a transition occurs.

For a Markov model it is assumed that the time, T_i, spent in state i is exponentially distributed with a rate, like for example α_i. Some probability to enter the next state will therefore be present. The mean sojourn time can be estimated as followed

$$E(T_i) = \frac{1}{\alpha_i} \tag{20}$$

If α_i = 0 we know that state i is an absorbing stat. That means that since we have no transition out of the state, we will never leave the state.

5.6 Exponential Distribution

The exponential distribution is one if several different types of survival distributions. Survival distributions can be used to model the lifetime of components [11] Here, only the exponential distributions will be described, since that is the distribution that is applicable for a Markov model.

A continuous random variable, as for example time to failure T, has an exponential distribution with parameter $\lambda > 0$ if the PDF is in following form [44]

$$f(t) = \begin{cases} \lambda\, e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{21}$$

The exponential distribution is also denoted T ~ exp(λ). It follow that T has the reliability function as [11]

$$R(t) = \Pr(T > t) = \int_t^{\infty} f(u)du = e^{-\lambda t} \quad for\ t > 0 \tag{22}$$

The mean time to failure for exponential distribution is

$$MTTF = \int_0^{\infty} R(t)\, dt = \int_0^{\infty} e^{-\lambda t}\, dt = \frac{1}{\lambda} \tag{23}$$

The variance of t for exponential distribution is

$$var(t) = \frac{1}{\lambda^2} \tag{24}$$

The standard devotion of t for exponential distribution is

$$SD(t) = \sqrt{var(t)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} \quad (25)$$

The hazard rate function for exponential distribution is

$$SD(t) = \sqrt{var(t)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda} \quad (26)$$

The conditional survivor function is presented in equation 27 bellow. Here it is of interest to find the reliability of a component at the additional time point x, when we know it is in a functioning condition at time t. The equation implies that the exponential distribution has no memory of the past since the conditional survivor function is R(x). This is presented with an example in section 6.5.4.

$$\begin{aligned} R(x|t) &= \Pr(T > t + x \mid T > t) = \frac{Pr(T > t + x\,)}{\Pr(T > t)} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} \\ &= e^{-\lambda x} = \Pr(T > x) = R(x) \end{aligned} \quad (27)$$

5.7 Statistical Inputs

5.7.1 Mean, Variance and Standard Deviation

The mean is an average value of set samples or of a random variable X [49]. This average value describes where the sample or probability distribution of X is centered. Statisticians refer to this mean as the mathematically expectation or the value we expect a random value X to take. It is often denoted E(X), however, it can also be presented in other ways, like for example μ, λ_{avg} and MTTF.

By itself, the mean does not given an adequate description of the shape of the probability distribution [49]. The variance of the sample or the random variable X should also be considered. It represent the variability/uncertainty in the distribution and is often denoted as Var(X) or σ^2.

The positive square rot of the variance result in a measure called standard deviation of X, denoted as σ or SD(X). This is the deviation of an observation from the mean. If a set of values is close to the mean, the standard deviation will be smaller than if the values are further away from the mean.

The equations for mean, variance and standard deviation for exponential distribution was presented section 5.6. If a sample set is available then the following formulas can be used [49]:

$$\hat{x} = \sum_{i=1}^{n} \frac{x_i}{n} \qquad \sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \hat{x})^2}{n-1} \qquad \sigma = \sqrt{\sigma^2} \tag{28}$$

5.7.2 Confidence Interval

Confidence intervals (CI) are used to quantify the variability in the parameter estimation [27]. It is a random interval that is estimated, where the true value of the parameter lies with some probability. For example, a 90%-CI denotes that we are 90% certain that the true value of the parameter lies within the interval.

There exist several different ways to calculate CI, where the choice of method is often dependent on the available data and on which parameter that is of interest. In this thesis an exponential distribution is assumed and a standard 95%-CI for positive parameters will be considered. The following approximation is therefore used [50].

$$P\left(\hat{\mu} \times e^{-1.96 \times \frac{\widehat{SD(\hat{\mu})}}{\hat{\mu}}} \leq \mu \leq \hat{\mu} \times e^{1.96 \times \frac{\widehat{SD(\hat{\mu})}}{\hat{\mu}}}\right) \approx 0.95 \tag{29}$$

Here $\hat{\mu}$ is an estimator for the mean, μ is the exact value for mean and $SD(\hat{\mu})$ is an estimator for the standard deviation. The variable 1.96 denotes that we have a 95%-CI, where 1.96 is the z-value, $z_{\varepsilon/2}$ found in the table presented appendix III. $z_{\varepsilon/2}$ is a value that leaves an area of $\varepsilon/2$ to the right of the CI, where ε is the percentiles outside the CI (5%). Note that the CI is no better than the quality of the point estimator, in this case the $SD(\hat{\mu})$ [49].

5.7.3 Error

An error is a discrepancy between an estimated, observed or measured value and the true value [11]. By considering the error of a parameter estimation, it might be possible to determine if an estimation good or not. This error can for example be explained by the use of a percentage error or a standard error.

The percentage error denotes how many percentage the estimated parameter deviates from the true value. If the original parameters, let say transition rates λ_{ij}, are known, it is possible to calculate the error with equation 30 [51]

$$error = \frac{|\lambda_{ij} - \tilde{\lambda}_{ij}|}{\lambda_{ij}} \times 100 \tag{30}$$

The standard error of an estimator is its standard deviation [49]. Thus, the standard error explains the deviation of the estimator from its true value. A small standard error should in theory mean that the estimator is close to the true value, while a high standard error indicates that estimate might deviate significantly from the true value. The standard error can be presented as in equation 31 [49]

$$SE = \widehat{SD(\sigma)} = \frac{\sigma}{\sqrt{n}} \tag{31}$$

For an exponential distribution we know that $MTTF = \hat{\mu} = \frac{1}{\lambda}$ and that SD(T) = $\sqrt{Var(T)} = \sigma = \frac{1}{\lambda}$. Thus, the standard deviation is equal to the mean for an exponential distribution. Therefore, the standard error for an exponential distribution can be presented as in equation 32 [50]

$$SE = \widehat{SD(\hat{\mu})} = \frac{\hat{\mu}}{\sqrt{n}} \tag{32}$$

According to the equation 32 over, the largest possible standard error is SE = $\hat{\mu}$, which will be the case if n = 1 sample. This means that if a standard error (calculated based on samples with equation 31) of a parameter is larger than the estimated parameter itself, the estimated parameter is not applicable for an exponential model.

5.7.4 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a way to estimate the parameters of a distribution [11]. Consider the likelihood function, $L(\theta, \boldsymbol{x})$, which expresses the probability of observing values from distribution $\boldsymbol{x}$ given the parameter θ. The maximum likelihood principle is to estimate the value of $\hat{\theta}$ that is most likely to produce the values of $\boldsymbol{x}$, thus

$$L\big(\hat{\theta}(\boldsymbol{x}); \boldsymbol{x}\big) \geq L(\theta; \boldsymbol{x}) \quad for\ \boldsymbol{\theta} \in \Theta \tag{33}$$

5.8 Maximum Likelihood Estimation with Censored Data

PD has been identified as the data of interest considering the aging process of the motor. At Kollsens, PD has been collected on-line, with opportunistic inspections. These inspections reveal a specific motor state at the time of the inspection. For this reason the exact transition times are unknown, which means we have to deal with an issue called censored data. Since the Markov model has been selected for this thesis, it is of interest to see how censored data can be used to estimate the transition rate parameters of a Markov model.

5.8.1 Censored Data

Incomplete data sets can lead to challenges for the modeling of a process [11]. This might be a result of an expensive or impractical waiting process to get the complete data, individual components can be "lost", or and event or events can occur at unknown times within an interval of the components lifetime. Thus, there are several different types of censored data. The three most common are right- left- and interval censored data.

Right censored data is if an event occurs later then the time of censoring (observation), in which case the true lifetime is "to the right" of the censored time [52]. Left censoring is if the event of interest has already occurred before the component is observed in the study. Interval censored data is when the event is only known to occur within an interval.

Table 6 bellow illustrates right- left- and interval censored data, which are collected with inspections. L denotes the left side of the time interval and R denotes the right side of the time interval. For sample 1 an inspection was made at time 2 and no transition was observed, which means that the time of transition, t^i, is occurring at an unknown time after time 2. This is called right censoring. For sample 2 the inspection at time 5 is the first observation of its condition. At this time it is observed that a transition to a more degraded stat had occurred sometimes before the recordings had started. Such data is called left censored data. The last sample, number 3, indicates that the t^i has occurred sometimes between the inspection intervals at time 3 and 6. This is reffered to as interval censoring.

Table 6: Example of censoring types

Sample nr	L	R	Transition observed
1	2	NA	No
2	NA	5	Yes
3	3	6	Yes
...	...	...	...

5.8.2 Maximum Likelihood Estimation based on Periodic Inspection

The monitoring of the motor condition at Kollsnes is opportunistic inspected, which in this thesis is assumed to be inspections at a periodic time, Δt. This means that at each inspection, it can be observed if the motor is in the same state as it were at the last inspection, or it can be observed if a transition to a more degraded state has taken place. Thus, it is not possible to observe the exact time of the transition, and therefor the exact time spent in state i is unknown. This type of data is typically interval censored and can be handled as followed.

The available information, about censored date collected at a periodic time, can be mathematically presented as [51]

$$T_{l_k^i} < t_k^i \le T_{l_k^i} + \Delta T \tag{34}$$

Where k is a specific sample (k = 1, … , n), n is the total number of samples, i is the observed sample state, $T_{l_k^i}$ and $T_{l_k^i} + \Delta T$ are the inspection dates of sample k between which a transition from i is present and t_k^i is the unknown transition time of sample k from state i.

Based on the observations at the inspection dates, $T_{l_k^i}$, it is possible to estimate transition parameters. Barros et al. presents the following maximum likelihood estimation for parameter λ_{ij}, where it is assume that the distribution has a Markov property [51]

$$\Pr\left(T_{l_k^i} < t_k^i \le T_{l_k^i} + \Delta T\right) = \left(1 - e^{-\lambda_{ij}\Delta T}\right) \times e^{-\lambda_{01}\Delta T l_k^i} \tag{35}$$

Here l_k^i is the number of times a state i condition is observed for sample k. When a transition from state i has been present, the inspection data $T_{l_k^i}$ are “reset”, such that $T_{l_k^i} = \Delta T\ l_k^i$ is true for all states in the state space. The likelihood function of l_k^i given λ_{ij} for all n samples can then be calculated as

$$\begin{aligned} L\left(\lambda_{ij,} l_k^i\right) &= \prod_{k=1}^{n} \Pr\left(T_{l_k^i} < t_k^i \le T_{l_k^i} + \Delta T\right) \\ &= \left(1 - e^{-\lambda_{ij}\Delta T}\right)^n \times e^{-\lambda_{ij}\Delta T \sum_{k=1}^{n} l_k^i} \end{aligned} \tag{36}$$

By maximizing $L\left(\lambda_{ij,} l_k^i\right)$ the parameters $\tilde{\lambda}_{ij}$ can be determined:

$$\tilde{\lambda}_{ij} = argmax_{\lambda_{ij}} L\left(\lambda_{ij,} l_k^i\right) = \frac{1}{\Delta T} \ln\left(\frac{n + \sum_{k=1}^{n} l_k^i}{\sum_{k=1}^{n} l_k^i}\right) \tag{37}$$

Chapter 6

Modeling of the Remaining Useful Lifetime

Chapter 6 is where the modeling of the remaining useful lifetime of the motor will be conduced. First the model selection is presented, followed by the making of the model and its fictional transition rates. Further the modeling is conducted with subsequent results, where several methods-, tests- and component levels are considered. A method for parameter estimation based on censored data is also presented, where the estimation quality is studied.

6.1 Model Selection

The modeling of the motor RUL will be conducted with a Markov model. The reason for- and the approach for the selection of this model were presented in chapter 5. The main reason is the ability of the Markov to model a degradation process, where degraded states can be used with increasing transition rates. The fact no data of the motor lifetime were available where also a contributing factor the selection.

The model used is a Markov chain with continuous time (Markov process) and discrete states. The Markov model is memoryless and it is assumed to be time homogenous were the time spent in each state is exponentially distributed. No repair is considered and only one transition can occur at the same time.

6.2 Markov States

6.2.1 Markov Transition Diagram

In section 5.1 the motor condition was classified into the state space $\boldsymbol{\mathcal{X}} = \{5, 4, \dots, 0\}$, where state 5 is the best condition and state 0 is the worst. Based on these six state conditions it is possible to make a Markov state transition diagram, which is displayed in figure 14. The figure illustrates the states and how the transitions can occur between them. The transitions are illustrated with failure rates, λ_{ij}, where i is the state the system move from and j is the state the system move towards. It is assumed that the PD level cannot be better, so no transition to a better state is possible, and it is also assumed that it is not possible to skip at state by for example jumping from state 5 directly to state 2. Thus, only one transition can occur at the time.

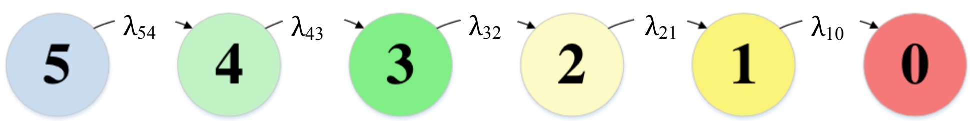


Figure 14: Markov state transition diagram

6.2.2 Markov Transition Matrix

Section 5.4.4 describes how transition matrixes are derived. Recall that this matrix can be used to calculate the probability to be in each state at the given time, by considering the Chapman-Kolmogorov equation. The Chapman-Kolmogorov equation in matrix form will be as expressed in figure 15, where the center matrix is the transition matrix of the motor states.

$$\begin{bmatrix} P_0(t), P_1(t), P_2(t), P_3(t), P_4(t), P_5(t) \end{bmatrix} \text{ X } \begin{array}{cc} & \begin{array}{cccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{10} & -(\lambda_{10}) & 0 & 0 & 0 & 0 \\ 0 & \lambda_{21} & -(\lambda_{21}) & 0 & 0 & 0 \\ 0 & 0 & \lambda_{32} & -(\lambda_{32}) & 0 & 0 \\ 0 & 0 & 0 & \lambda_{43} & -(\lambda_{43}) & 0 \\ 0 & 0 & 0 & 0 & \lambda_{54} & -\lambda_{54} \end{bmatrix} \end{array} = \begin{bmatrix} \dot{P}_0(t), \dot{P}_1(t), \dot{P}_2(t), \dot{P}_3(t), \dot{P}_4(t), \dot{P}_5(t) \end{bmatrix}$$

Figure 15: Chapman-Kolmogorov equation in matrix form

6.3 Estimation of Transition Rates

For the transition matrix to be useful it is essential to have a value for each of the transition rate parameters, λ_{ij}. A possibility way to obtain these is by estimating based on historical data. Since there are no available data, an alternative method for the estimation has been used. Here it is assumed that the expected remaining lifetime is 5 years. This assumption gives the follow failure rate per hour

$$\frac{1}{\lambda_{tot}} = \frac{5}{365 * 24} => \lambda_{tot} = 0.000571 \tag{38}$$

The failure rate, λ_{tot}, is denoting the rate from the current condition into a failed state. Since a classification of 6 states has been made, the current state should be determined based on the motor condition at the current time. For this thesis a condition of state 5 will be assumed as the initial condition. This means that five state transitions should be present before the failed state 0 is observed. The average of these five failure rates can be calculated as followed

$$\lambda_{avg} = \frac{\lambda_{tot}}{5} = \frac{0.000571}{5} = 0.000114 \tag{39}$$

Recall that the failure rate should be increasing as the motor condition degrades. Since the λ_{avg} is constant, MINITAB was used to present increasing parameters based on some predetermined restrictions. The MINITAB function *random data* was used, where exponential distribution was selected to derive increasing parameters. MINITAB was also asked to print 5 values with a mean equal to the λ_{avg}. The result is presented in table 7 bellow.

Table 7: Transition rates

$\lambda_{10} = 0.0003517$
$\lambda_{21} = 0.0001423$
$\lambda_{32} = 0.0000876$
$\lambda_{43} = 0.0000470$
$\lambda_{54} = 0.0000378$

6.4 Markov Calculations by hand

6.4.1 Markov State Equations

From the Chapman-Kolmogorov equation in matrix form presented in figure 15, the following equations are derived:

$$\begin{gathered} P_1(t)\times\lambda_{10} = \dot{P_0}(t) \\ P_1(t)\times(-\lambda_{10}) + P_2(t)\times\lambda_{21} = \dot{P_1}(t) \\ P_2(t)\times(-\lambda_{21}) + P_3(t)\times\lambda_{32} = \dot{P_2}(t) \\ P_3(t)\times(-\lambda_{32}) + P_4(t)\times\lambda_{43} = \dot{P_3}(t) \\ P_4(t)\times(-\lambda_{43}) + P_5(t)\times\lambda_{54} = \dot{P_4}(t) \\ P_5(t)\times(-\lambda_{54}) = \dot{P_5}(t) \\ P_0(0) = 1,\ P_1(0) = 0,\ P_2(0) = 0, \dots, P_5(0) = 0 \end{gathered} \tag{40}$$

By solving the equations from set 40, based on $P_i(t)$ it is possible to derive equations that are used to find the probability to be in state i. The calculations can be viewed in appendix IV, and the final state equation are as followed:

State 5

$$\boldsymbol{P_5(t) = e^{-\lambda_{54}t}} \tag{41}$$

State 4

$$\boldsymbol{P_4(t) = \frac{\lambda_{54}}{\lambda_{43} - \lambda_{54}}(e^{-\lambda_{54}t} - e^{-\lambda_{43}t})} \tag{42}$$

State 3

$$\boldsymbol{\begin{aligned} P_3(t) = {} & \frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43} - \lambda_{54})\times(\lambda_{32} - \lambda_{54})}(e^{-\lambda_{54}t} - e^{-\lambda_{32}t}) \\ & + \frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43} - \lambda_{54})\times(\lambda_{32} - \lambda_{43})}\times(e^{-\lambda_{32}t} - e^{-\lambda_{43}t}) \end{aligned}} \tag{43}$$

State 2

$$\begin{aligned}P_2(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times(e^{-\lambda_{54}t}-e^{-\lambda_{21}t}) \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times(e^{-\lambda_{21}t}-e^{-\lambda_{43}t}) \\ & +\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times(e^{-\lambda_{32}t}-e^{-\lambda_{21}t})\end{aligned} \tag{44}$$

State 1

$$\begin{aligned}P_1(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times(e^{-\lambda_{54}t}-e^{-\lambda_{10}t})+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times(e^{-\lambda_{10}t} \\ & -e^{-\lambda_{43}t})+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\times(e^{-\lambda_{32}t}-e^{-\lambda_{10}t}) \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\times(e^{-\lambda_{10}t}-e^{-\lambda_{32}t}) \\ & +(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})})\times(e^{-\lambda_{10}t} \\ & -e^{-\lambda_{21}t})+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})})\times(e^{-\lambda_{21}t}-e^{-\lambda_{10}t})\end{aligned} \tag{45}$$

State 0

$$\begin{aligned}P_0(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})}\times(e^{-\lambda_{54}t}-1)+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})}\times(1-e^{-\lambda_{43}t}) \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\times(e^{-\lambda_{32}t}-1) \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\times(1-e^{-\lambda_{32}t})+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})})\times(e^{-\lambda_{21}t}-1)+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})})\times(1-e^{-\lambda_{21}t})+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})})\times(e^{-\lambda_{10}t}-1)+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})})\times(1-e^{-\lambda_{10}t})+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})})\times(1-e^{-\lambda_{10}t})+(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\ & +\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})})\times(e^{-\lambda_{10}t}-1)\end{aligned} \tag{46}$$

6.4.2 State Probabilities with Excel

To test the equations with the transition rates an excel calculator has been made. The calculator is displayed in figure 16. The green cells are for input values and the red cells provide the output probabilities to be in each state.

	λ_{10}	λ_{21}	λ_{32}	λ_{43}	λ_{54}
Transition rates					
Time (Hours)					
State 0	State 1	State 2	State 3	State 4	State 5

Figure 16: Markov state probability calculator, Excel

For a time period of 1 year (8760 hours) the calculator, including transition rates, will look as figure 17 illustrates. It is displayed that the probability of the motor to be in state 5 after 1 year is 71.81%, and the probability of being in an unacceptable state (state 0) is 0.14%.

	λ_{10}	λ_{21}	λ_{32}	λ_{43}	λ_{54}
Transition rates	0.0003517	0.0001423	0.0000876	0.0000470	0.0000378
Time (Hours)	8760				
State 0	State 1	State 2	State 3	State 4	State 5
0.0014	0.0018	0.0089	0.0414	0.2285	0.7181

Figure 17: Test of Markov state probability calculator, Excel

The temporary transition rates and the calculator seems to work correctly, however it could be interesting to see if a model in MATLAB provides the same result as excel. In MATLAB there is also better options for plotting, as well as for more possibilities for other reliability measures.

6.5 Markov Calculations with MATLAB

To make the model in MATLAB a code has been made, which is displayed in appendix V part I. The same transition rates as for the excel calculator was used, however, several different time tests has been conducted.

6.5.1 Markov Test 1 – Time Period of 1 Year

The inputs values for test 1 are the same as those used in the Excel calculator. The result is presented in 18, where the probabilities obtained with MATLAB are the same as those derived in Excel.

```
State 0   State 1   State 2   State 3   State 4   State 5
0.0014    0.0018    0.0089    0.0414    0.2285    0.7181
```

Figure 18: Test 1 - MATLAB output, state probabilities

In addition to the state probability output, it is of interest to extend the MATLAB model by including MTTF, reliability and different plots. Figure 19 displays the MTTF and the reliability.

The reliability is presented to be 0.9986 and the MTTF is 1 year. This indicates that the motor is reliable, and it should not fail a time period of 1 year when the initial condition is state 5.

```
--------------------------------------------------------------
For time t = 1 Year(s)
--------------------------------------------------------------
- The reliability, R(t), is 0.9986
- The Mean Time To Failure, MTTF, is 1.00 Years
- The state probabilities are:

    State 0   State 1   State 2   State 3   State 4   State 5
    0.0014    0.0018    0.0089    0.0414    0.2285    0.7181
```

Figure 19: Test 1 - MATLAB output, MTTF, R(t) and P(t)

Figure 20 illustrates two different plots for the time period of 1 year. In both plots the x-axis is the time and the y-axis is the probability.

The plot on the top illustrates the probability for the motor to be in each state, in the time interval of 0 to 1 years. Each of the six states has been assigned to a color, where all are not viewable at this plot due to the low time period. The light blue line is assigned to state 5, which is the best state. At time 0, it is displayed that the probability of being in state 5 is 100%, while the probability of being in state 0-4 is naturally 0%. When time t increases, the probability of being in state 5 decreases, which can be seen in the plot by an decreasing light blue line. In the plot in the middle the reliability is represented. It is illustrated that the reliability is decreasing as time t increases. After 1 year the reliability still is high and not far from 1. The last plot displays the PDF. The density that is viewable for 1 year period is low, which indicates that the probability of failure is low for this time period.

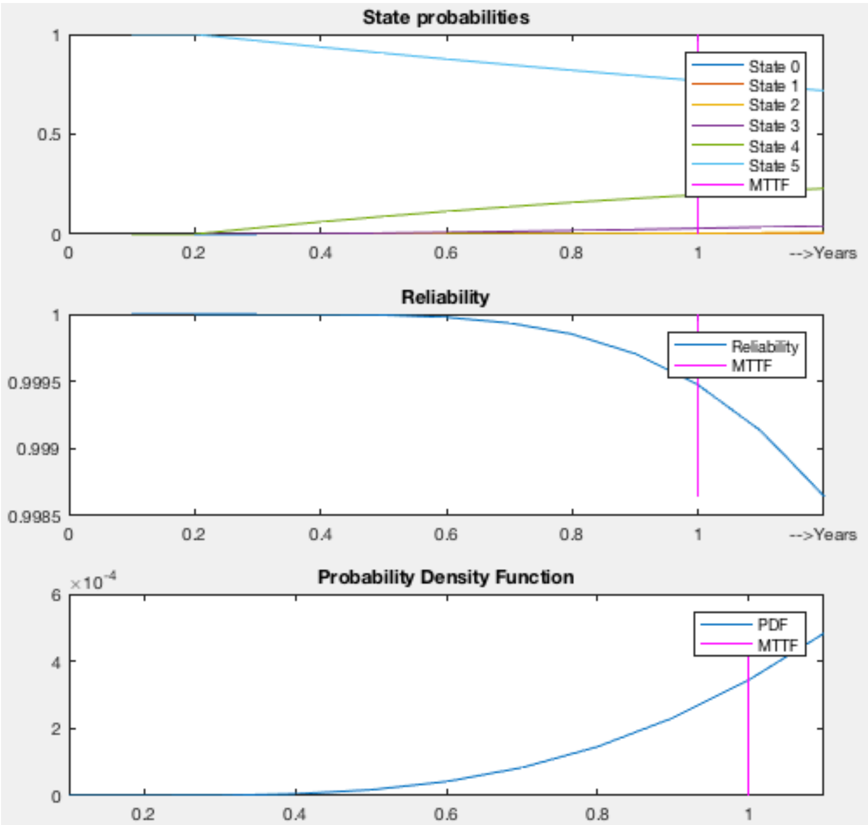


Figure 20: Test 1 - MATLAB output, plots

6.5.2 Markov Test 2 – Time Period of 20 Years

The model with a time period of 1 year did not give a good indicator for the final motor RUL. For this reason the time period was increased to 20 years, so the whole picture could be seen. The output result is displayed in figure 21.

The probability to be in the unacceptable state 0 after 20 years is 98.58%. Hence, the reliability is 0.0142 and the MTTF is 7.83 years. Based on this, it seams like the motor not will work after 20 years, and that the expected time of a motor failure will be around 7.83 years.

```
--------------------------------------------------------------
For time t = 20 Year(s)
--------------------------------------------------------------
- The reliability, R(t), is 0.0142
- The Mean Time To Failure, MTTF, is 7.83 Years
- The state probabilities are:

    State 0    State 1    State 2    State 3    State 4    State 5
    0.9858     0.0014     0.0032     0.0039     0.0044     0.0013
```

Figure 21: Test 2 - MATLAB output, MTTF, R(t) and P(t)

The plots for the time period of 20 years are illustrated in figure 22. The figure now gives a clearer view of what is happening, where it can be seen how time affects the probability to be in each state. It is displayed that state 5 decreases all the time, and it approaches 0 at approximately 16 years. State 4 increases in the start, before it turns around at around year 3. State 0 is increasing all the time and seams to approach 0 at year 20.

The reliability decreases as time goes, where line first has a concave shape, before it turns around slightly before the MTTF and goes into a convex shape. This means that the reliability is decreasing exponentially until we reach the time where the motor is expected to fail.

The PDF plot indicates a large variability for a motor lifetime. The low top point and the density, which is spread out over a long period of time, illustrate this. A precise prediction of the RUL is therefore though to provide.

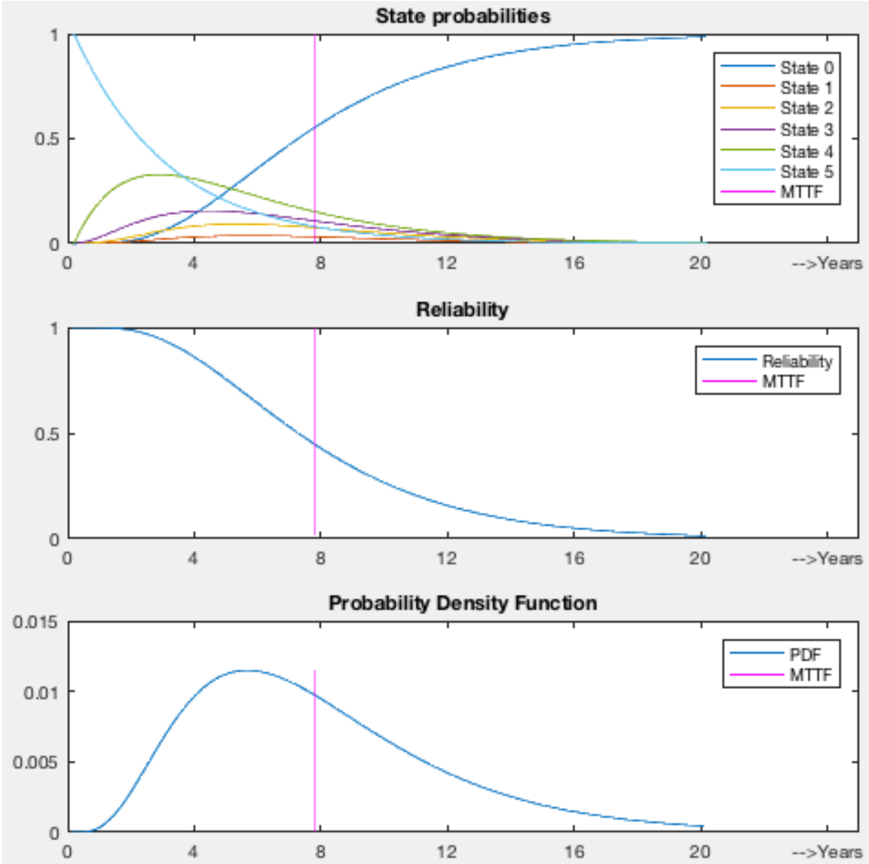


Figure 22: Test 2 - MATLAB output, plots

6.5.3 Markov Test 3 – Time Period of 20 Years and Inspection after 3 Years

It is of interest to see what happens to the reliability when the motor condition changes. By inspecting the motor condition, the observed situation can be quite different from the original prognosis. The inspection can revile a worse or a better motor condition than expected, which will be an important factor for the true RUL. Therefore, conditional reliability has been included in the MATLAB code as presented in appendix V part II.

A total time period of 20 years is still considered for test 3, however, an inspection after 3 years is assumed to revile a state 4 condition. The output is presented in figure 23, where the probability to be in state 5 now is 0. This is due to no repair and since a condition of state 4 has already been observed. The probability to be in the unacceptable state 0 after 20 years is 99.66%, which is higher then the situation without conditional reliability (98.58%). The reliability is now 0.0034 (before 0.0142) and the total MTTF is 7.85 year (before 7.83 year). The MTTF in the figure is measured from the time of inspection, so the MTTF from time 0 is therefore 3 + 4.85 years. Based on this it is concluded that the motor most likely not is useful after 20 years, and that time of failure is expected to be around 4.85 years after the inspection.

```
----------------------------------------------------------------
For a total time of t = 20 Year(s),
and a condition of state 4 at time t = 3 Year(s)
----------------------------------------------------------------
- The reliability, R(t), is 0.0034
- When the motor is at state 4 after 3.00 years,
  The Mean Time To Failure, MTTF, is 4.85 Years.
- The state probabilities are:

    State 0    State 1    State 2    State 3    State 4    State 5
    0.9966     0.0005     0.0010     0.0011     0.0009          0
```

Figure 23: Test 3 - MATLAB output, MTTF, R(t) and P(t)

The conditional reliability indicates a slightly worse condition then initially predicted, since the reliability now is lower. However, the MTTF seems to be slightly better after the inspection was conducted. This can be perceived as a bit strange since the observed condition is degraded. A reason for the increased MTTF is that the motor is 100% certain not failed at the time of inspection. Where, at the starting point at time 0, a worse- or even a failed motor state was a possibility for time 3. This means that the reliability is better then first predicted in the first period after the inspection, while it gets worse as time goes.

Figure 24 bellow illustrates the graphs for a condition of state 4 after 3 years. For the upper plot, the green line, which represent state 4, is 100%, after 3 years, while the other states is at 0%. The probability of being in state 5 drops at time 3 at stays at 0% for the rest of the time period. States 0, 1, 2 and 3 increases right after time 3, while state 4 decreases. States 1, 2 and 3 will increase until they reach a top point before they decreases towards 0%, while the failed state 0 will increase towards 100%.

The reliability can be seen in the middle plot, where it is 1 after 3 years. It is this high since the motor is known to work at this point. As time continue to pass beyond 3 years, the reliability decreases and reaches 0 at around 16 years.

In the last plot the PDF is displayed, where the uncertainty still seams significant. However, it is slightly less compared to test 2, where spread now is smaller and the top point is hence higher (approximately 0.018 vs. 0.012 before). The reason for this improved uncertainty is that more information is known for test 3 due to the inspection. There are now four possible transitions left, while it was five before.

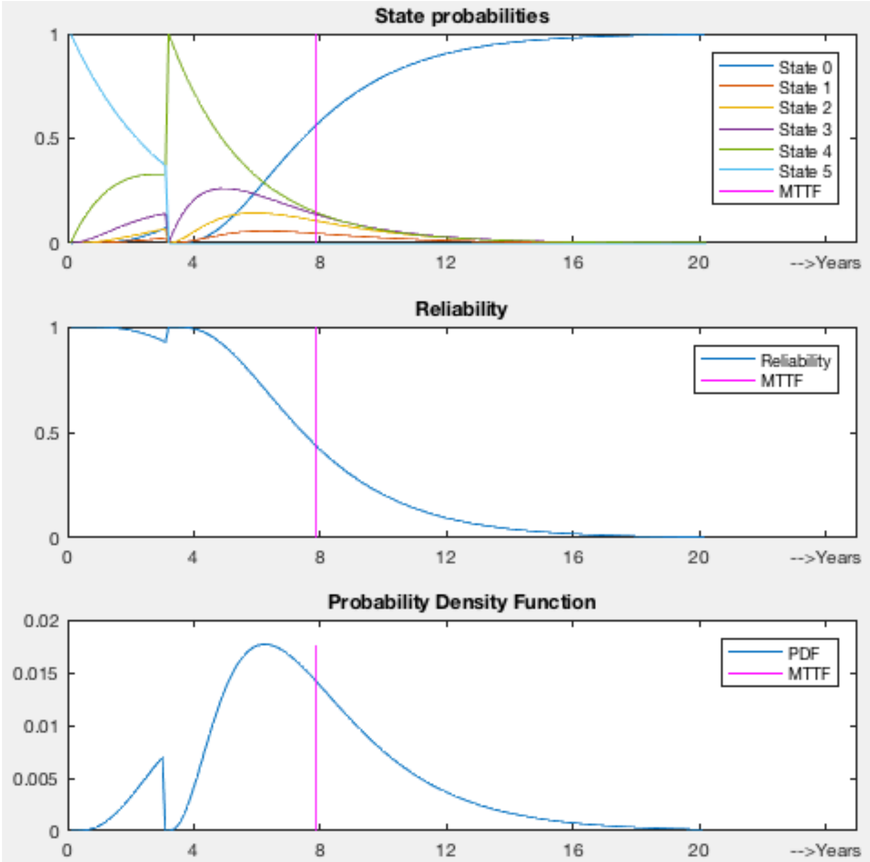


Figure 24: Test 3 - MATLAB output, plots

6.5.4 Markov Test 4 – Importance of Time

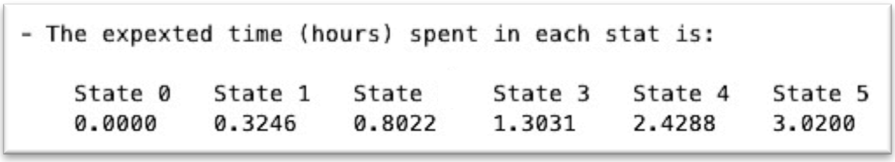
Test 4 is conducted to test how the time affects the MTTF and the reliability. The test is conducted in MINITAB, where some small changes were made to the input value. For test 4 a state 4 motor condition is observed at several different inspection times. This means that inspections at time 2, 3 and 6 years all reveled a state 4 condition.

The result was a MTTF of 4.85 and a reliability of 0.0034 for all the three inspections. This proves, as explained in section 5.6, that time has no impact on the Markov model, which means that what happened before the inspection should not have an significant effect on the future condition. (Note, since the MATLAB code integrates the reliability when calculating the MTTF, the total time period after the inspections had to be the same for all cases (or very large). The total time period was therefore increased by the same amount as the increasing in inspection time).

6.5.5 Sojourn Times with MATLAB

Recall from section 5.4.4 that the sojourn time is the expected time spent in a state. The sojourn times for the six degraded states are presented in figure 25, where the initial condition is assumed to be state 5. It is displayed that state 0 has a time equal to zero, which is since the state is an absorbing state. While state 0 is present, the condition will not change since no repair is considered, and the state is unacceptable so the motor should not be in operation.

The state where the motor is expected to be in for the longest time is state 5. The lower the state, the less time is expected to be spend in the state. This is natural since the transition rates are increasing when the motor moves to a more degraded state. The sum of the all to expected sojourn time is equal to the MTTF.



```
- The expexted time (hours) spent in each stat is:

    State 0    State 1    State      State 3    State 4    State 5
    0.0000     0.3246     0.8022     1.3031     2.4288     3.0200
```

Figure 25: Expected sojourn time for each state

6.5.6 Confidence Interval of the remaining lifetime

It is possible to find the confidence intervals for the MTTF be using equation 29 from section 5.7.2. To do this an estimation of the standard deviation (standard error) of the MTTF needs to be derived. Since no samples are used for the derivation of the MTTF, equation 32 from section 5.7.3, which are applicable for exponential distribution, are been used. First test 2 will be considered, where the condition of the motor is assumed to be state 5.

Thus, MTTF = $\frac{1}{\lambda}$ = 7.83 => $\lambda = 0.128$, and $SD\left(\widehat{MTTF}\right) = \sqrt{\frac{1}{\lambda^2}} = \sqrt{\frac{1}{0.128^2}} = 7.83$

The z-value is set to 1.96 so the confidence of the interval is 95%. The 95%-CI is therefore calculated as: $\left[7.83 \times e^{-1.96 \times \frac{7.83}{7.83}}, 7.83 \times e^{1.96 \times \frac{7.83}{7.83}}\right]$ that is $[1.10, 55.59]$.

The 95%-CI is quite large, where the lowest bound is 1.10 year and to highest is 55.59 years. Such a large interval was expected since the standard error of the MTTF has the same value as the MTTF itself. This indicates uncertainty of the RUL of the motor is large when we now that the motor has a state 5 condition.

For test 3 a condition of state 4 where observed, which gave a MTTF of 4.85 years. The 95%-CI of test 3 can then be calculated in the same way as for test 4, where the result is $[0.68, 34.43]$.

The spread is reduces for the 95%-CI of test 3 compared for test 2, which indicates that the uncertainty is reduced when the degradation process of the motor is increased. However, the uncertainty is still quite significant where the lowest bound is 0.68 year and the highest bound is 34.43 years.

6.6 Censored Data Application

The transition rates that has been used for the Markov models until now was estimated based on simple assumptions (section 6.3). These parameters are therefore probably not an accurate representation of a real situation. Another way to derive such transition rates is by estimating them based on a data set, which are collected with inspections as explained in section 5.8.

6.6.1 Simulation of Inspection Dates

In this section a data set collected with periodic inspections will be considered. These inspections reveal specific motor states at the given time, and for this reason the exact transition times are unknown. This section will present a set of censored data and use the known information to estimate transition parameters.

Supervisor Anne Barros made a MATLAB code that simulates the degradation process of the motor. The code is based on the transition rates presented in table 7 in section 6.3, and is attached in appendix V part III. The simulation provides the exact dates of transitions towards more degraded states. To get more samples the code was run 20 times, and the result is presented in table 8. A sample is here considered as the lifetime of a motor. For sample 1 the first transition, from state 5 to 4, occurred at time 2.1795 years. The second transition was at time 4.2294 and so on. The same goes for all the samples.

Table 8: Transition dates, 20 samples

State observed	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Sample 7	Sample 8	Sample 9	Sample 10
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	2.1795	1.8884	1.8424	0.7566	0.1099	0.4590	2.7774	2.7648	3.3130	0.6823
3	4.2294	3.6084	1.9298	3.6632	1.3927	3.3724	6.2383	3.1591	3.9048	1.3035
2	4.5300	4.1066	3.1600	4.4410	3.6484	3.6069	7.8159	3.7706	4.2275	1.3758
1	4.6955	4.3479	3.5365	4.6926	4.9849	4.5996	8.1563	4.1911	4.9067	1.3933
0	5.1724	4.4280	3.9621	4.7254	5.3707	4.6205	8.3690	4.2157	5.0676	1.3953

State observed	Sample 11	Sample 12	Sample 13	Sample 14	Sample 15	Sample 16	Sample 17	Sample 18	Sample 19	Sample 20
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5399	4.8123	1.3427	2.1093	0.5063	0.5921	0.3724	2.2240	5.2767	0.4205
3	0.6600	5.3023	4.1846	7.3839	1.8239	0.6004	5.6315	2.4232	5.5747	1.4306
2	0.7463	6.6348	4.6692	9.0667	1.8283	1.2139	6.6781	4.3691	6.1972	2.6259
1	1.0577	7.0829	4.9308	9.1304	3.6194	1.2318	7.6251	5.3055	6.6175	3.0946
0	1.0886	7.5941	5.0133	9.6653	3.8511	1.2375	7.6885	5.8535	7.1666	3.3539

In reality these transition rates is what we would have observed if continuous monitoring were conducted. Since the inspections frequency at Kollsnes has been assumed to be periodic, it is not possible to observe the exact times that the table presents. The exact transition times will be hidden behind censored data. The data has therefore been rewritten as presented in table 9, where only the information that are observable with inspection are displayed. The lifetime of the samples begins at time 0, where a state 5 condition is assumed. This observation is called inspection number 1. After one year the second inspection will follow and the condition observed will be reported. For sample 1, a state 4 condition was first observed at the 4th inspection, which indicates that the transition from state 5 to 4 happened at an unknown time between year 2 and 3. The exact time of the transition was 2.1795, however this is hidden.

Table 9: Censored transition dates, Δ 1 year, 20 samples

Inspection nr	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Sample 7	Sample 8	Sample 9	Sample 10
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	4	5	4	5	5	5	4
3	5	4	3	4	3	4	5	5	5	0
4	4	4	3	4	3	4	4	4	5	
5	4	3	0	3	2	2	4	2	3	
6	1	0		0	1	0	4	0	1	
7	0				0		4		0	
8							3			
9							2			
10							0			
11										

Inspection nr	Sample 11	Sample 12	Sample 13	Sample 14	Sample 15	Sample 16	Sample 17	Sample 18	Sample 19	Sample 20
1	5	5	5	5	5	5	5	5	5	5
2	2	5	5	5	4	3	4	5	5	4
3	0	5	4	5	2	0	4	5	5	3
4		5	4	4	2		4	3	5	2
5		5	4	4	0		4	3	5	0
6		4	1	4			4	2	5	
7		3	0	4			3	0	3	
8		2		4			2		1	
9		0		3			0		0	
10				3						
11				0						

Each of the samples can be presented as a timeline, such as sample 3 in figure 26. The timeline gives a clear overview of which state that is observed at the inspections. In addition to display in which time interval a transition(s) has been present, the timeline can also indicate the range of the sojourn times in the six states. This is

$$1 < T_5 < 2, \quad 0 < T_4 < 1, \quad 1 < T_3 < 3 \quad 0 < T_2 < 1, \\ 0 < T_1 < 1, \quad 0 < T_0 < 1, \tag{47}$$

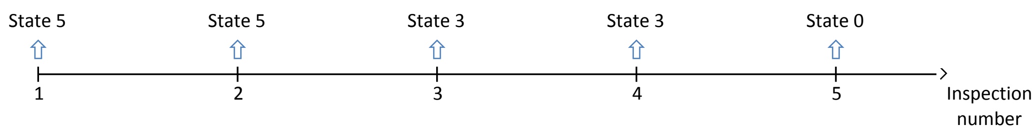


Figure 26: Timeline for sample 3

6.6.2 Study 1 - Parameter Estimation Based on Censored Data

In chapter 5 it was explained how censored data similar to table 9 could be used to estimate the parameters $\tilde{\lambda}_{ij}$ with equation 37. For study 1, the inspection time ΔT has been set to 1 year, the numbers of samples n to 20 and the observations in state i to $l_k^5 = 46$, $l_k^4 = 32$, $l_k^3 = 17$, $l_k^2 = 17$ and $l_k^1 = 11$. Examples of calculations are

$$\tilde{\lambda}_{54} = \frac{1}{1}\ln\left(\frac{20+46}{46}\right) = 0.0000412 \quad \dots \quad \tilde{\lambda}_{10} = \frac{1}{1}\ln\left(\frac{20+46}{46}\right) = 0.0001837 \tag{48}$$

Table 10 bellow, displays all the transition parameters that are obtained with the estimation

Table 10: Study 1 - Estimated transition rates

$\tilde{\lambda}_{10}$ = 0.0001837
$\tilde{\lambda}_{21}$= 0.0001183
$\tilde{\lambda}_{32}$= 0.0000888
$\tilde{\lambda}_{43}$= 0.0000554
$\tilde{\lambda}_{54}$= 0.0000412

6.6.3 Error and Standard Error of Estimated Parameters

It is of interest to find out how good the parameter estimation of study 1 is. Since the original transition rates, λ_{ij}, is known, it is possible to calculate the percentage error with equation 30 from section 5.7.3. The result is displayed in table 11, where the best estimate by percentage is $\tilde{\lambda}_{32}$ with an error of 1.37% and the worst is $\tilde{\lambda}_{10}$ with an error of 47.98%.

Table 11: Study 1 - Percentage error of parameter estimation

Original parameter	Estimated parameter	Error
λ_{10}= 0.0003517	$\tilde{\lambda}_{10}$ = 0.0001837	47.76%
λ_{21} = 0.0001423	$\tilde{\lambda}_{21}$= 0.0001183	16.88%
λ_{32} = 0.0000876	$\tilde{\lambda}_{32}$= 0.0000888	1.35%
λ_{43} = 0.0000470	$\tilde{\lambda}_{43}$= 0.0000554	17.92%
λ_{54} = 0.0000378	$\tilde{\lambda}_{54}$= 0.0000412	9.903%

An accepted error percentage limit has not been presented in this thesis, due to the reason that no fixed standard was found and the limit would vary from situation to situation. However, some of the error seams to significantly deviate from the true value, so a better set of parameters could be estimated.

First, the standard error of the estimated parameters of study 1 will be considered. The standard error is based on the sample size and the standard deviation of the estimated parameters. Since the parameters is estimated based on interval censored data, the standard error can be calculated by, $\widehat{SD(\tilde{\lambda}_{ij})}/\sqrt{n}$, where the standard deviation, $\widehat{SD(\tilde{\lambda}_{ij})}$, will be calculated by using equation 28 from section 5.7.1. Since the exact time is interval censored, the worst possible situation will be assumed, where all transition within an interval occurs the second after the inspection has been conducted. An illustration of the standard deviation calculation for parameter $\tilde{\lambda}_{54}$ is:

$$SD(\tilde{\lambda}_{54}) = \frac{\sum_{i=1}^{n} \frac{(x_i - \hat{x})^2}{n-1}}{356 * 24} = \frac{\frac{(2-1.3)^2}{20-1} + \frac{(1-1.3)^2}{20-1} + \cdots + \frac{(0-1.3)^2}{20-1}}{356 * 24} \tag{49}$$
$$= 0.000166$$

The standard errors for the estimated parameters are displayed in table 12. All the standard errors are lower than the parameters, which means that they are applicable for an exponential distribution. The standard error for $\tilde{\lambda}_{54}$ is however quite close to the estimated parameter. As the motor gets more degraded the standard error decreases, and thus error margin increases. The reason for this is that the expected time spent in a state gets lower. A higher standard error might therefore be accepted for a lower sojourn time, while it would not be acceptable for a larger sojourn time. For example for $\tilde{\lambda}_{43}$, the standard error of 0.0000434 is accepted, however if the standard error of $\tilde{\lambda}_{54}$ were 0.0000434 it would not have been applicable.

Table 12: Study 1 - Standard error of parameter estimation

Estimated parameter	Standard error
$\tilde{\lambda}_{10}$ = 0.0001837	0.0000113
$\tilde{\lambda}_{21}$= 0.0001183	0.0000154
$\tilde{\lambda}_{32}$= 0.0000888	0.0000190
$\tilde{\lambda}_{43}$= 0.0000554	0.0000434
$\tilde{\lambda}_{54}$= 0.0000412	0.0000371

6.6.4 Study 1 - Confidence Interval based on Censored Sample Data

In the previous section it was explained that the estimated parameters were applicable for an exponential model. This can also be seen be considering confidence intervals for each of the estimated parameters. To test this, equation 29 from section 5.7.2 is used, which calculates the 95%-CI for the parameters. The equation also takes the standard errors from table 12 into consideration.

The result is presented in table 13, where it is displayed that all the estimated parameters are within the 95%-CI. This confirms that the estimated parameters are applicable for an exponential model. It is also illustrated that the lengths of confidence intervals are decreasing as the motor gets more degraded.

Table 13: Study 3 - 95%-CI for estimated parameters

Estimated parameter	95%-CI
$\tilde{\lambda}_{10}$ = 0.0001837	[0.0001621, 0.0002066]
$\tilde{\lambda}_{21}$= 0.0001183	[0.0000916, 0.0001621]
$\tilde{\lambda}_{32}$= 0.0000888	[0.0000584, 0.0001351]
$\tilde{\lambda}_{43}$= 0.0000554	[0.0000115, 0.0002591]
$\tilde{\lambda}_{54}$= 0.0000412	[0.0000070, 0.0002410]

6.6.5 Study 2 and 3 – Improvement of Estimation

With the aim of reducing the percentage error and the standard error, some changes in the input variables were implemented. For study 2 the inspection frequency was increased to inspections every 6 months, and the same 20 samples are considered. The censoring table is attached in appendix VI part I, and the result is presented in the table bellow. The percentage error is more evenly spread out among the parameters, and the standard errors are slight better or the same. Thus the estimation of the new parameter seams slightly better now that the inspection frequency is increased.

Table 14: Study 2 - Estimated parameters and their errors

Original parameter	Estimated parameter	Error	Standard error
λ_{10} = 0.0003517	$\tilde{\lambda}_{10}$ = 0.0002860	18.68%	0.0000064
λ_{21} = 0.0001423	$\tilde{\lambda}_{21}$= 0.0001528	7.34%	0.0000127
λ_{32} = 0.0000876	$\tilde{\lambda}_{32}$= 0.0001009	15.15%	0.0000188
λ_{43} = 0.0000470	$\tilde{\lambda}_{43}$= 0.0000621	32.10%	0.0000435
λ_{54} = 0.0000378	$\tilde{\lambda}_{54}$= 0.0000504	33.28%	0.0000369

For study 3 an inspection frequency of 6 months was still considered, while the sample size was increased to 40. The exact transition times of the 40 samples and the censored data table can be found in appendix VI part II and III. The estimated parameters with errors and standard errors are presented in table 15 bellow. The percentage error does not seam to have a significant improvement, however, the standard error has decreased for all parameters.

Table 15: Study 3 - Estimated parameters and their errors

Original parameter	Estimated parameter	Error	Standard error
λ_{10} = 0.0003517	$\tilde{\lambda}_{10}$ = 0.0002182	37.97%	0.0000064
λ_{21} = 0.0001423	$\tilde{\lambda}_{21}$= 0.0001612	9.25%	0.0000083
λ_{32} = 0.0000876	$\tilde{\lambda}_{32}$= 0.0001020	16.46%	0.0000122
λ_{43} = 0.0000470	$\tilde{\lambda}_{43}$= 0.0000625	33.01%	0.0000284
λ_{54} = 0.0000378	$\tilde{\lambda}_{54}$= 0.0000449	18.81%	0.0000301

6.6.6 Study 3 - Confidence Interval based on Censored Sample Data

In section 6.6.4 it was illustrated that the estimated parameters should be in the 95%-CI if the standard error denoted that the estimated parameters were applicable for an exponential model. This is also valid for study 2 and 3. Since study 2 is based in the same 20 sample sets as for study 2 a confidence interval for the estimates of study 2 will not be provided. Study 3 is considering 40 samples, which is why its confidence interval will follow.

Equation 29 from section 5.7.2 is used to calculate the 95%-CI, where the standard errors from table 15 is considered. The result is displayed in table 16, where all the estimated parameters areas suspected within the 95%-CI. It is also displayed that the lengths of confidence intervals are shorter then what they were for study 1. This is a good indicator that the estimations of study 3 are better then for study 1, since less uncertainty is involved.

Table 16: Study 3 - 95%-CI for estimated parameters

Estimated parameter	95%-CI
$\tilde{\lambda}_{10}$ = 0.0002182	[0.0002060, 0.0002310]
$\tilde{\lambda}_{21}$= 0.0001612	[0.0001457, 0.0001783]
$\tilde{\lambda}_{32}$= 0.0001020	[0.0000807, 0.0001289]
$\tilde{\lambda}_{43}$= 0.0000625	[0.0000257, 0.0001522]
$\tilde{\lambda}_{54}$= 0.0000449	[0.0000121, 0.0001670]

6.6.7 Study 1, 2 and 3 - Interpretation of Estimation

Three sets of transition rates were estimated in the section above, where three different combinations of input values were used. According to the standard error, all the three estimations where applicable from an exponential point of view.

When the numbers of inspections and the number samples were increased, the standard error decreased. This indicates that the estimated parameters should get closer to the real value, hence the estimations is better with more samples and more frequent inspections. This was also illustrated with 95%-CI, where the intervals decreased when the sample size increased.

However, the error in percentage did not have the same improvement as the standard error, while the sample size was increased and inspections were doubled. Solutions, that might be able to reduce the error, are to increase the sample size or the inspection frequency even more. Maybe another estimation formula for the parameters could have been more accurate as well.

The issue can also be a result of errors in the simulated data. Consider the exact transition dates presented in table 8 in section 6.6.1, where the sample size is 20. A quick analysis of the data displays that the average time spent in state 5 is 1.75 years. Recall from section 6.5.1 that the expected time spent in state 5 was 3.02, which makes the simulated data that is used for estimation quite pessimistic. The same analysis was performed for all states and for simulated data with 40 samples, where the result is presented in table 17. The average time spent in each state is quite similar for both sample sizes (with some deviation), where the sojourn times are lower than what was expected.

Table 17: Expected sojourn times, true values and 20- and 40 samples

State i	Expected time spent in state i	Average time spent in state i, 20 simulation	Average time spent in state i, 40 simulation
5	3.02	1.75	2.02
4	2.43	1.64	1.62
3	1.30	0.84	0.84
2	0.80	0.52	0.51
1	0.32	0.23	0.29

The error percentage of the sojourn time is presented in table 18 bellow. It seams like the error is slightly better for 40 simulations compared to 20 samples. Overall, for both sample sizes, the error is quite significant, so it is not strange that the estimated transition rates also are a bit off.

Table 18: Expected sojourn time error in percentage for 20 and 40 samples

State i	Error percentage, 20 simulations	Error percentage, 40 simulations
5	42.05	33.11
4	32.51	33.33
3	35.38	35.38
2	35.00	36.25
1	28.12	9.34

The reason why the simulated values have an error percentage this significant can be explained by the probability density function (PDF). A PDF graph is illustrated is figure 22, where it is displayed that the density is spread out over a long time period of almost 20 years. The top point is as low as 0.012, which means that the variability of the lifetime is quite significant. This can also be illustrated by the 95%-CI for the remaining lifetime presented in section 6.5.2. Where, the motor lifetime is predicted to be between 1.10 and 55.59 years.

6.6.8 Study 4 – Parameter Estimation with Optimal Observations

The average sojourn times (presented in table 17) for 40 samples can be used to find the optimal number of observations based on the inspection frequency. For example, the expected sojourn time for state 5 is 2.02 years. With inspections every 6th month, a total number of 4 state 5 observation would have been optimal for 1 sample. The sojourn times can also be used to find failure rates, where λ = 1/(sojourn time). A calculator that estimates transition rates based on the optimal number of observations were made in excel, and is attached with viewable formulas in appendix VII. The error in percentage is also displayed since it is a good measure for comparing estimates with the true value. The standard error is not included due to its indication of applicable estimates for study 1 2 and 3.

Figure 27 displays an example of parameter estimation with optimal observations, where the green cells are input values. The inspection times are set to every 6th month (4272 hours), the "old parameters" are the transition rates that generated the observations and the sample size are set to 40. In this case the old parameters are found by considering the 40 original simulations. Since the number of observations is calculated to be optimal, the sample size will not affect the estimated transition rates. This means tat the estimated transition rates will be the same for all values of n in this calculator. The error in percentage is also included.

The result illustrates that the optimal number of observation of a state 5 condition is 248, for state 4 it is 199 and so on. The error percentage is smaller than those obtained earlier in this thesis. However, they are still significant considering that the observations are optimal. The reason for this is that interval censoring is still an issue. There is only one way to make more information available, and that is by increasing the inspection frequency.

Parameter Estimaiton Based on Optimal Observations

Inspection frequency: 4272 Number of samples : 40

Transition of Interst	Optimal # of Observations	Estimated Parameter	Old Parameter	Error Percentage
5 -> 4	248	0.000035	0.000038	7.30%
4 -> 3	199	0.000043	0.000047	8.87%
3 -> 2	107	0.000074	0.000088	15.05%
2 -> 1	66	0.000111	0.000142	21.87%
1 -> 0	27	0.000215	0.000352	38.97%

Figure 27: Study 4 - Optimal parameter estimation, Δt = 6 months

The inspection frequency was therefore increased to 3rd month and the result of the estimation is as displayed in figure 28. The error percentage is now lower for all the estimated parameters. Note that the error percentage is increasing as the parameters are increasing. This is always the case in theory when optimal observation are considered, which indicates that estimated parameters are closer to their true value when lower parameters are considered.

Parameter Estimaiton Based on Optimal Observations

Inspection frequency: 2136 Number of samples : 40

Transition of Interst	Optimal # of Observations	Estimated Parameter	Old Parameter	Error Percentage
5 -> 4	495	0.000036	0.000038	3.83%
4 -> 3	398	0.000045	0.000047	4.71%
3 -> 2	214	0.000080	0.000088	8.33%
2 -> 1	132	0.000124	0.000142	12.68%
1 -> 0	53	0.000262	0.000352	25.43%

Figure 28: Study 4 - optimal parameter estimation, Δt = 3 months

If the maximum acceptable percentage error is set to be 10%, the goal seek tool in excel can be used. This can be done the following way; set the cell with the highest error percentage (H13) to 0.0999 by changing the cell with the inspection frequency (E5). The result are presented in figure 29, where it is displayed that a estimation will then be accepted if inspections are performed every 653 hour (≈ 27 days) or more often.

Parameter Estimaiton Based on Optimal Observations

Inspection frequency: 652.8621 Number of samples : 40

Transition of Interst	Optimal # of Observations	Estimated Parameter	Old Parameter	Error Percentage
5 -> 4	1621	0.000037	0.000038	1.21%
4 -> 3	1304	0.000046	0.000047	1.50%
3 -> 2	699	0.000085	0.000088	2.75%
2 -> 1	431	0.000136	0.000142	4.38%
1 -> 0	174	0.000317	0.000352	9.99%

Figure 29: Study 4 - optimal parameter estimation, goal seeking

If the inspection frequency were reduced to 1 hour, the monitoring process would be approximately continuous. Such a low inspection frequency would provide data that displays transition date close to the true value. This was tested in the Excel calculator and the result is provided in figure 30. The Percentage error is now close to 0 for all the estimated parameters, which means that the estimated parameters are approximately the same as the true parameters. The estimations are therefore accurate when all information is available.

Parameter Estimaiton Based on Optimal Observations

Inspection frequency: 1 Number of samples : 20

Transition of Interst	Optimal # of Observations	Estimated Parameter	Old Parameter	Error Percentage
5 -> 4	529101	0.000038	0.000038	0.00%
4 -> 3	425532	0.000047	0.000047	0.00%
3 -> 2	228311	0.000088	0.000088	0.00%
2 -> 1	140548	0.000142	0.000142	0.01%
1 -> 0	56818	0.000352	0.000352	0.02%

Figure 30: Study 4 - optimal parameter estimation, Δt = 1 hour

6.6.9 Markov Modeling with Estimated Parameters

The transition rates estimated from 40 simulations and an inspection frequency of every 6th month has been used for Markov modeling as figure 31 and 32 bellow illustrates.

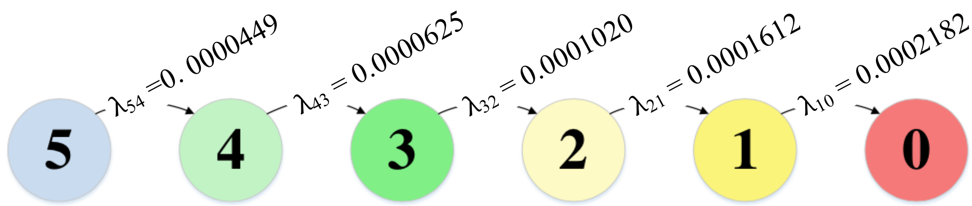


Figure 31: Markov transition diagram, censored data

$$
\begin{array}{c}
\begin{array}{cccccc} \quad\mathbf{0} & \qquad\mathbf{1} & \qquad\mathbf{2} & \qquad\mathbf{3} & \qquad\mathbf{4} & \qquad\mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0.0002182 & -0.0002182 & 0 & 0 & 0 & 0 \\
0 & 0.0001612 & -0.0001612 & 0 & 0 & 0 \\
0 & 0 & 0.0001020 & -0.0001020 & 0 & 0 \\
0 & 0 & 0 & 0.0000625 & -0.0000625 & 0 \\
0 & 0 & 0 & 0 & 0.0000449 & -0.0000449
\end{bmatrix}
\end{array}
$$

Figure 32: Markov transition rate matrix, censored data

To find reliability measures and plots for the estimated transition rates, they were places in the MATLB code. The output result is displayed in table 19, together with the original result (from test 2) for comparison. A time period of 20 is used for both cases.

It is illustrated that the new MTTF is shorter compared to the old one. The 95%-CI has moved to lower values in addition to a decreased interval length. The reliability at time 20 years is lower for the new model, however both are significantly low and the motor condition would most likely be in the unaccepted at this point. All in all the predictions of the new estimates are more pessimistic and a shorter RUL is predicted based on the new parameters.

Table 19: Comparison of MATLAB output, old vs. new model

	New model	Old model
MTTF	6.71	7.83
95%-CI for MTTF	[0.95, 47.64]	[1.10, 55.59]
Reliability at 20	0.0040	0.0142

Figure 33 bellow presents the plots for the new and the old model. They illustrate the same as the table above, where the RUL predicted are more pessimistic for the new model. In addition the PDF seams to have a smaller spread then before. The higher top point of the PDF in new model illustrates this, in addition it seams like its density is more compact around the MTTF. This indicates that the newer model predicts a lower variability then what the old model does.

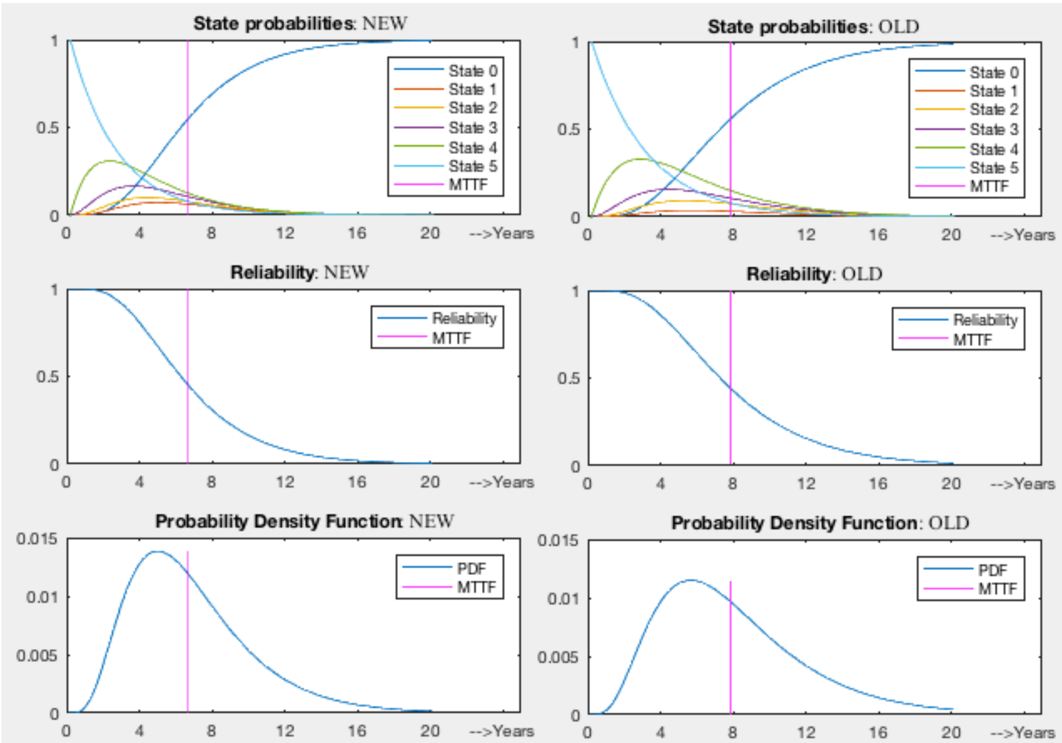


Figure 33: Comparison of MATLAB plots, old vs. new model

6.7 Markov Modeling based on season

Until now a single electrical motor has been considered for the modeling. In reality a CDS train consist of 3 additional components, which naturally will affect the RUL of the CDS. The 4 components in the CDS are in a series structure and will be simplified and represented by a merged state, as illustrated in figure 34.

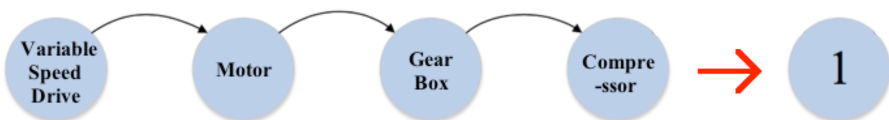


Figure 34: Merged compressor drive system state

Each CDS will be assumed to be in either a functioning- or a failed state. The transition rate will be established based on the fact that a series structure is at most as reliable as the least reliable component [11]. The motor is the least reliable component, where it was predicted from fictional transition rates that the RUL was 7.83 years. This will be used to calculate the failure rate for the full CDS train, where $\lambda = \frac{1}{7.83\times365\times24} = 0.00001458$. This is based on the assumption that the initial motor condition is excellent. The failure rate will be the same for all the CDS train, due to the assumption that each of them are identical to another.

There are also two different seasons which have different requirement for the number of operating CDS train. This is the winter season were it is required that all the 6 CDS train is operating, and the summer season were 3 out of 6 trains are required. No repair will be considered and the time spent in each sate is assumed exponentially distributed. It is also assumed that a failure is observed at the same time that the event occurs. No common cause failure will be considered, and thus only 1 train can fail at the time.

6.7.1 Markov Modeling Winter Operation

A Markov transition diagram for winter operation is presented in figure 35, where all 6 CDS train are required to operate. The 6 trains are represented with states 1 – 6, while a system failure is represented as sate 0.

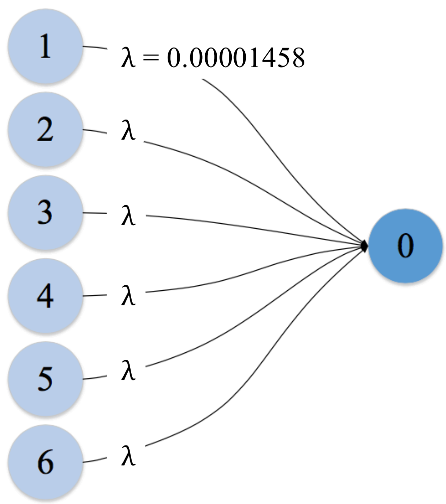


Figure 35: Markov transition diagram, winter operation

Since all the CDS trains are identical, it is possible to make the Markov transition diagram more simplified. This is illustrated in figure 36. The failure rates are merged into one, where it is calculated as $6\lambda = 6\times0.00001458 = 0.00008748$.

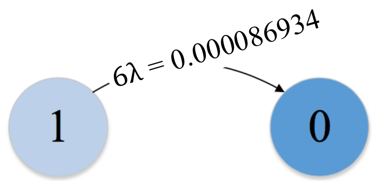


Figure 36: Simplified Markov transition diagram, winter operation

The Markov diagram was used to make a transition matrix illustrated in figure 37

$$
\begin{array}{c c}
 & \begin{array}{cc} \mathbf{0} & \mathbf{1} \end{array} \\
\begin{array}{c} \mathbf{0} \\ \mathbf{1} \end{array} & \begin{bmatrix} 0 & 0 \\ 0.000086934 & -0.000086934 \end{bmatrix}
\end{array}
$$

Figure 37: Markov transition matrix, winter operation

The result of the Markov modeling is presented in section 6.7.3.

6.7.2 Markov Modeling Summer Operation

During summer operation it is required that 3 out of the 6 CDS trains are operating. Several different operation strategies are therefore possible, where two will be considered in this thesis. The first strategy is displayed in figure 38, where all 6 CDS are initially operating. In state 4 there are 6 trains operating, in state 3 there are 5 trains operating and so on. If 4 trains has failed and 2 are operating it is assumed that the whole system has failed.

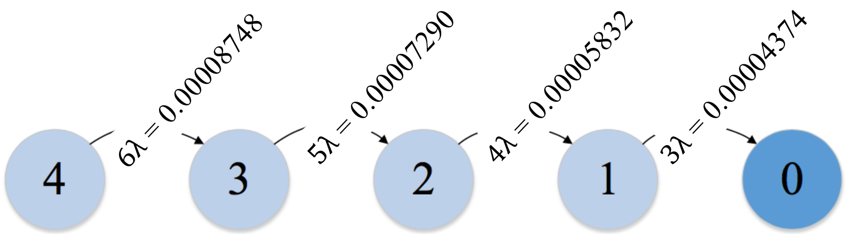


Figure 38: Markov transition diagram summer, operation, no redundancy

The transition matrix is as followed

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\
\begin{array}{c} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.00008748 & -0.00008748 & 0 & 0 & 0 \\ 0 & 0.00007290 & -0.00007290 & 0 & 0 \\ 0 & 0 & 0.00005832 & -0.00005832 & 0 \\ 0 & 0 & 0 & 0.00004374 & -0.00004374 \end{bmatrix}
\end{array}
$$

Figure 39: Markov transition matrix, summer operation, no redundancy

The result of the Markov modeling is presented in section 6.7.3.

Strategy number two is called passive redundancy, which means that once a motor fails, another will take its place. The minimum requirement of 3 operating trains will therefore be the initial strategy, where 3 trains are standby. It is assumed that the switching happens without any production losses. The Markov transition diagram with passive redundancy is presented in figure 40. Here, it is illustrated that exactly three motors is operating in the states 1 - 4, and the system is failed in state 0 where 4 motors not is operating (no passive redundancy motors left for replacement).

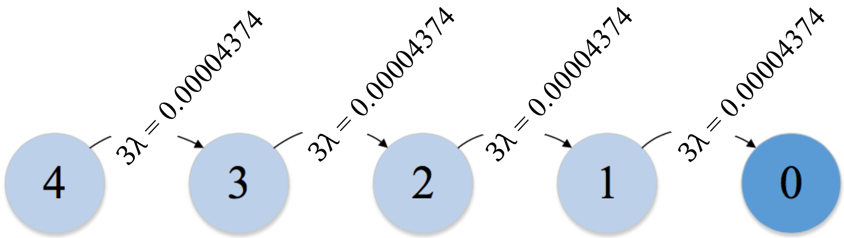


Figure 40: Markov transition diagram, summer operation, passive redundancy

The transition matrix is displayed bellow

$$\begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.00004374 & -0.00004374 & 0 & 0 & 0 \\ 0 & 0.00004374 & -0.00004374 & 0 & 0 \\ 0 & 0 & 0.00004374 & -0.00004374 & 0 \\ 0 & 0 & 0 & 0.00004374 & -0.00004374 \end{bmatrix} \end{array}$$

Figure 41: Markov Transition matrix for, summer operation, passive redundancy

The result of the Markov modeling is presented in section 6.7.3

6.7.3 Results of Markov modeling for Winter- and Summer

The MATLAB codes in appendix V part I was used to model the different types of CDS operations, where the matrix dimension in the code where change to fit the seasons. The result is presented in table 20 bellow where each of operation types are compared. Since 1 season last for 6 months the reliability is presented at this date. Both the two summer operations has a good reliability, while it is significantly lower during the winter. Thus, the MTTF is quite short for winter while it is longer for the summer, where the passive redundancy gave the best result. Compared to the MTTF of a single CDS, which is 7.83, the passive redundancy MTTF is longer, while it is slightly lower for the summer without redundancy. The confidence intervals are, as always for exponentially distribution, predicting that the RUL of a CDS will vary a lot. This goes for all three types, where the variability is lowest during the winter.

Table 20: Comparison of MATLAB output, winter vs. summer(s)

	Winter operation No redundancy	**Summer operation** No redundancy	**Summer operation** Passive redundancy
MTTF	1.33	7.41	10.44
95%-CI for MTTF	[0.19, 9.44]	[1.04, 52.61]	[1.47, 74.12]
Reliability at 0.5	0.68	0.99	1

Figure 42 bellow presents the plots for the three operational types. Since the predicted RUL of the CDS vary a lot from the types, the time period of the plots are different. The time periods considered are; 7 years for the winter, 20 years for the summer with no redundancy and 30 years for the summer with passive redundancy. A dotted line, which represents the first season, is displayed in each plot. It is clear that a failure of the CDS at the 6 month date is less and less expected as the operation type gets better. Thus, the plots give the same demotion as table 20 above. The probability of failure is largest during winter operation, while the passive redundancy is the best option for the summer operation.

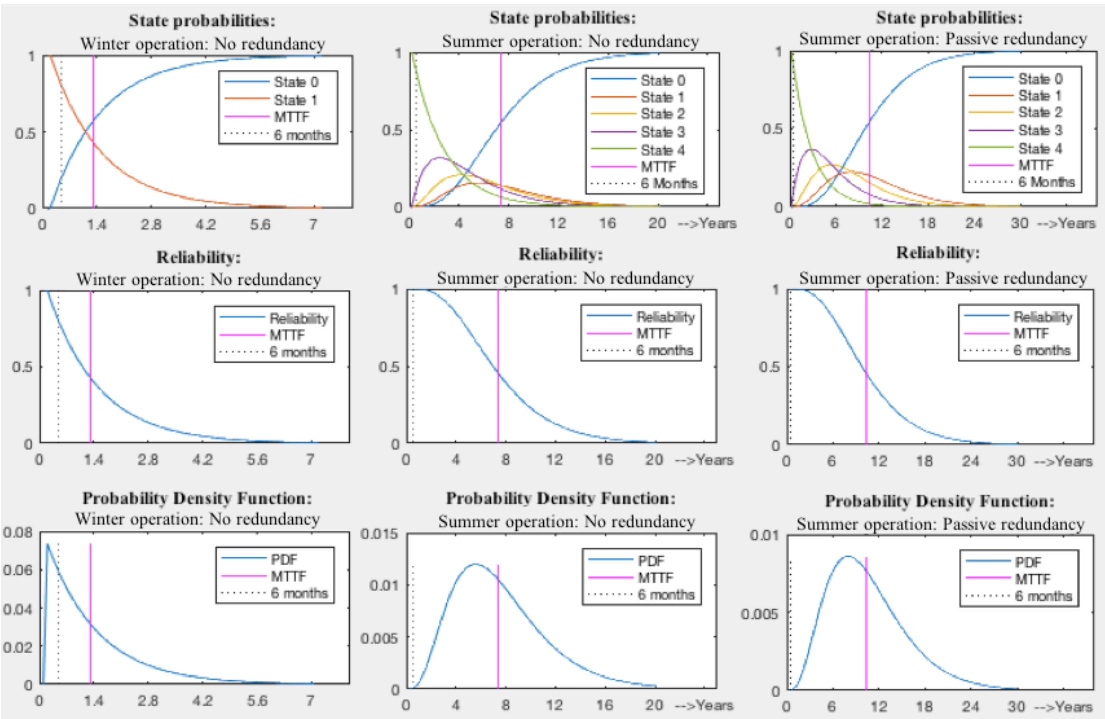


Figure 42: Comparison of MATLAB plots, winter vs. summer vs. summer w/redundancy

Chapter 7

Discussion

In this chapter the result of the thesis will be discussed. The chapter shall explain the relations between the main topics of the report, in addition to clarify what the topics contribute with towards the stated aim and objective. First the physical aging will be discussed and linked to the electrical motor. Further the selection of model will be discussed based on the aging process of the motor. Lastly the concept and result of the remaining useful lifetime modeling will be discussed.

7.1 Aging of Electrical Motor

The aging of a component consist of several accepts, where functional degradation is most relevant for the compressor drive system (CDS) at Kollsnes. This is due to the fact that the CDS has been operating for several years and a system will be exposed to aging stresses through its lifetime. The CDS consist of four components in a serial structure, where Statoil has identified the synchronous electrical motor as the most critical of them.

The synchronous electrical motor has therefore been the main focus of the thesis, where bearings and windings were found to be the motor parts that have most frequent failures. The motor bearings at Kollsenes are quite robust, they are condition monitored and are easily replaced. The windings are getting old and are not easy to replace, where a failure can cause production loses for several months. Insulation is the part of the winding that often limits the lifetime, due to the mechanical strength of its organic material. In this thesis, the insulation condition has therefore been selected to represent the synchronous electrical motor condition.

Through its lifetime the motor will be exposed to aging stresses that will cause degradation mechanisms and eventually a motor failure. The aging process of the motor was covered in chapter 4 and it was identified that partial discharges (PD) was significant for the motor lifetime. PD is related to several degradation mechanisms like for example thermal deterioration, inadequate resin impregnation, loose coil in the slots and semiconductive coating failure.

7.2 Selection of Model

The aim of the master thesis was to model the RUL of the aging CDS. The RUL modeling is about predicting a future motor condition considering the current condition. The current condition will change as the system gets older, and it is therefore of interest to identify a variable that can represent the aging process. This variable would be the basis of the model.

Since PD has a significantly important for the aging process of the electrical motor, it would also be a good health indicator for its condition. The PD level can be monitored online with no need to take the motor out of operation. The current condition of the imotor can therefore be observed without any production losses. By monitoring the PD it is possible to observe its development over time, which means that the motor health can be recorded as it get older. Such PD data seams to be the best option to consider in the making of a RUL model for the motor.

There exist several different options of models that can be used for the RUL prediction. For the aging stresses several well-accepted life approximations was presented in section 4.3. These are deterministic models, which means that no variability is included. The temperature is for example a parameter and not a variable, and the approximated life is not a function of time. This means that a deterministic model might not give a good representation of the PD development as the motor ages.

A probabilistic model seams to be suitable for the RUL predictions, due to its aspect of uncertainty. Among several different types of probabilistic models, there was no optimal choice since no data was available. In the end a continuous time Markov process with discrete states was selected. A Markov model was chosen due to the possibility to model degraded states with increasing failure rates, and since it gives an good overview of the situation.

Karsten Moholt presented a classification of the motor condition and it was adapted to represent the degraded states of a Markov transition diagram. A sequential structure was selected and no repairs were considered. Since no PD data was provided the transition rates was estimated based on the assumption of a remaining lifetime of 5 years. These rates are therefore not based on real life motor condition development, however they are perfect to illustrate how the RUL can be modeled with such parameters. Historical PD data can be used to estimate the transition rate parameters, which will be discussed later in section 7.3.

7.3 Remaining Useful Lifetime modeling

In this thesis the RUL modeling with Markov has been conducted with Excel and MATLAB. In Excel the probability to be in each state was calculated, where the formulas was derived by hand. The calculations where time consuming and required almost 10 pages of formulas. If the system was more complex, with more then 6 sates, repair rates and a possibility to skip states, the model will expand fast. The method with hand- and Excel calculations would not be efficient and it would be easy to make mistakes. The suggested method for RUL modeling is therefore MATLAB. In MATLAB Markov modeling is conducted by coding, and it is therefore possible to include many different survival measures and survival plots.

Several different tests for the Markov model were conducted. The first one was with a time period of 1 year. The probabilities that the motor was in one of the working states was high, and therefore the reliability was as high as 0.9986. This indicates that the motor condition is predicted to be acceptable after 1 year, which also can be seen by the MTTF of 1 year. The RUL should therefore be longer than 1 year, which is why the time period was increased to 20 years for test 2.

The result of test 2 displayed that the probability to be in the unacceptable state was 98.58% at the time of 20 years. A failure is therefore expected to have occurred after 20 years, thus, the RUL should be smaller than 20 years. The MTTF was found to be at 7.83 years, which means that the RUL is predicted to be 7.83 years since no repair of a failure is considered. The 95%-CI were calculated to be $[1.10, 55.59]$. This means that the true RUL can deviate significantly from the predicted estimate. The reason for this large uncertainty can be explained be the probability density function, where it denotes a wide spread of the density and a low top point.

The Markov modeling with test 2 was conducted without inspection of the condition in the time interval 0 to 20 years. If however a condition was observed at a time later than 0, an updated Markov model based on the new observed state should be made. This was done at test 3 where a state 4 condition was observed at time 3 years. The result denoted that the probability to be in the unacceptable state at time 20 still was high. The new MTTF was 4.85, which means that the RUL prediction from test 2 was accurate ($3 + 4.85 = 7.85 \approx 7.83$). A state 4 condition observed at time 3 is therefore an indication that the motor condition is developing close to the expected average. The PDF of test 3 indicated a slightly less uncertainty then for test 2, where the 95%-CI of $[0.68, 34.43]$ indicates the same. The reason for this improved uncertainty is that more information is known for test 3 due to the inspection. There are now four possible transitions left, while it was five before.

Test number 4 considered how the RUL was changing when inspections where conducted at different times, while the observed condition always was found to be state 4. The result displayed that the RUL was 4.85 no matter if the inspection was after 2, 3 or 6 years. This is the effect of the Markov property where the past has no impact on the future. However, since an observation of state 4 was on point at time 3, the observation at time 2 indicates a worse condition then expected and the observation at time 6 indicates a better condition then expected.

A Markov model is based on transition rates between the states. A way to estimate these parameters based on periodic inspection observation was therefore presented in the thesis. The data that are collected with periodic inspection will be of type interval censored, which means that a transition is known to have occurred at an unknown time in-between two inspections. The parameter estimation with the interval censored data was conducted with four studies. Several studies were conducted to see how the estimation quality changed by considering the percentage error and the standard error.

Study 1 was conducted with 20 samples with a time interval of 1 year, study 2 was with 20 samples and an interval of 6 months, study 3 was with 40 samples and an interval of 6 months, while test 4 was with perfect observations.

The percentage errors of study 1, 2 and 3 did not have a clear improvement when the number of samples and the inspection frequency was increased. In addition they seamed to deviate quite significant from the true transition parameters. A reason for this is that the simulated transition times not gave a good representation of the true parameters. This is also due to the probability density function of the true failure rates. An increased simulation with a large number of samples should however by able to represent the true transition times. Nonetheless, the standard error of study 1, 2 and 3 did denote that all the estimated parameters were applicable for an exponential distribution. Predictions based on the estimated transition rates would be quite pessimistic since most of the rates were estimated to be larger then the true value. The motor should therefore, on average, have a longer RUL compared to what the prediction from the estimates would indicate.

To compensate the bad simulations an Excel calculator was made, which was based on optimal observations according to the expected sojourn times. The result presented the importance of the interval censored data. With large inspection frequencies the error would be large, since more information about the true value would be unknown. A lower interval frequency therefore gave estimates closer to the true parameters. When the interval frequency was set to approximately continuous (1 hour) the error of the estimates was close to 0, which means that the estimates represented the true parameters almost perfectly.

The estimated parameters from study 4 were tested in a Markov model, where the result was compared to the original Markov model (test 2). The result was as suspected more pessimistic, and the RUL with parameters from study 4 was 6.71 years (test 2 was 7.83 years). Based on the PDF, it seamed like the uncertainty was a bit smaller for the study 4 model. The reason for this is the reduced RUL, where the 95%-CI indicates a lower upper bound [0.95, 47.64].

The last modeling section considered the full CDS in addition to season differences. Each season was modeled individually, which clearly presents the differences in the two seasons reliabilities. Due to complexity and uncertainties it is often of interest to make simplified versions of the Markov processes. For this reason the four components of the CDS were merged into one state.

First winter season was considered, where 6 out of 6 CDS trains was required to operate. The RUL for the full system was significantly reduces compared to the RUL for a single CDS, respectively 1.33 and 7.83 years. For the summer season it is required that 3 out of 6 trains is operating. Two strategies were tested, where the first was to operate all from from the start, while the second was to operate 3 in the start with the remaining 3 in standby (passive redundancy). The result was respectively 7.41 and 14.44 years.

Based on the season modeling the winter is clearly the bottleneck, where the RUL is significantly lower then for both the summer strategies. Considering reliability the best summer strategy is passive redundancy. However, if it is possible to "saved up" gas from operating all 6 trains in the summer, a reduced winter production could be accepted since the savings could be used to cover the market demand. This is not considered in the models or in this thesis, hence passive redundancy is the recommended summer strategy. During the summer season there are an opportunity to conduct maintenance activities to prepare the system for the winter. If it is wanted to maximize the systems lifetime and to avoid potential losses, this is the season to implement actions. However, as mentioned in section 4.3, startups and shutdowns of components (like the synchronous motor) can lead to additional stresses.

Chapter 8
Summary

Chapter 8 presents a summery of the content of the thesis and a conclusion of the RUL predictions. Suggestions for further work are given as well

8.1 Summary and Conclusions

The master thesis had the aim of predicting the remaining useful lifetime (RUL) for an aging compressor drive system (CDS). Objectives established to achieve this were related to studies of the CDS, aging processes, degradation modeling, condition classification and parameter estimation.

Due to limitation, the focus of the master thesis has been on the electrical motor, which is the most critical component of the CDS. Based on motors studies the insulation was identified as the part that limits the motors lifetime. A reason for this is its lack of maintainability and its vulnerability to several degradation mechanisms. Partial discharge (PD) are closely linked to these degradation mechanisms, and it has been presented as the variable that is the best representation of the aging process. The PD level is therefore concluded to be a good health indicator of the insulation condition, thus the motor condition.

The prediction of RUL is based on the current condition of the motor and how the condition of similar motors has developed historically. There are several probabilistic models that can be used for this prediction. A Markov process has been selected for this thesis, mainly due to its ability to model degraded states with an increasing failure rates.

Fictional transition rates were used for the Markov process modeling since no real failure data was available. The RUL was estimated to be 7.83 years, thus one single motor should on average function for a long time. However this average had a spread out probability density, and a large variance for the true motor lifetime is therefore expected. This can also be seen in the 95%-CI, which denotes a lifetime between 1.10 and 55.59 years. Frequent inspections of the PD are therefore recommended, since the development of the motor condition can deviate significantly from the predictions.

The Markov model is considering transition rates, which can be estimated based on censored data. A method for performing such estimations were presented and conducted, where the quality of the estimations were tested. The result indicated that a large inspection interval presented estimates far from the true value, while a smaller interval gave parameters that are closer to the true value. The importance of the quality of the inspected samples was tested as well. A sample set that represented the true average lifetime of a motor did naturally present the best estimates. In theory, a large number of samples should represent this true average lifetime. In practice it can be impossible to consider such a sufficient number of samples, due to the variance of the motor RUL in addition to the complexity and rarity of the motor. The quality of the estimated parameter can be concluded to be best for larger sample sizes and for frequent inspection intervals.

The main focus of the master thesis was on the motor as a single component. In addition a model at a system level was conducted, where all the CDS trains and the differences in winter- and summer operation were considered. The winter operation was concluded to be the critical season by far, where the system reliability was significantly reduces compared to a single component. The reliability during summer operation was therefore better, where the best strategy with passive redundancy gave a higher RUL then it was for a single motor. Activities to expend the system RUL should therefore by conducted during the summer,

To sum it all up the RUL for the motor were predicted to be 7.83 years. There are large uncertainty related to this estimate, witch among other things can be explained by the 95%-CI of [1.10, 55.59]. Due to this uncertainty the inspections should be performed as often as possible. This way the predictions of the RUL can be regulated as the condition develops. This can reduce to possibility of unexpected failures and thus losses. Frequent inspections can also be useful in the future, where the observed data can be used for predictions of the RUL of new motors. Actions to extend the RUL should be conducted during the summer, since the CDS is at is most reliable during this season.

8.2 Suggestion for Further Work

From the discussion chapter it was identified several accept of the modeling of the RUL that could be of interest to investigate on a deeper level. Also based on the thesis limitations a suggestion for future work is as followed:

- Take a closer look at all the components of the CDS,
- Model the RUL of the whole CDR, considering all four components, for both summer and winter operations (multiphase Markov).
- Look more into the deterministic life approximation models, in the sense of accelerated life testing
- Describe symptoms and remedy for degradation mechanisms
- Include more failure modes, repair rates (if component is repairable) and common cause failures for the modeling
- Model with other types of probabilistic models
- Suggest maintenance strategies for the summer, with the aim of expanding the RUL.

References

[1] Berthelsen, O. 2016. *Gas exports from the Norwegian shelf*. https://www.regjeringen.no/en/topics/energy/oil-and-gas/Gas-exports-from-the-Norwegian-shelf/id766092/, (Downloaded 08.09.16)

[2] Gassco & Statoil. 2009. *Kollsnes prosessanlegg*. https://www.gassco.no/globalassets/nedlastbart/informasjonsmateriell/kollsnes_nor.pdf, (Downloaded 26.08.16)

[3] Statoil. 2007. *Troll-feltet*. http://www.statoil.com/no/OurOperations/ExplorationProd/ncs/troll/Pages/default.aspx, (Downloaded 26.08.16)

[4] Larsen, M. I., Eeftink, E. 2015. *Sustainability report*. http://www.statoil.com/no/InvestorCentre/AnnualReport/AnnualReport2014/Documents/DownloadCentreFiles/01_KeyDownloads/Sustainability_report_2014.pdf, (Downloaded 28.10.2016)

[5] Meland, O., Schjølberg, P., Øien, K. 2008. *Vedlikeholf for aldrende innretninger - en utredning*. http://www.ptil.no/getfile.php/PDF/Rapport%20Vedlikehold%20for%20aldrende%20innretninger%20-%20en%20utredning.pdf, (Downloaded 07.02.15)

[6] Chen, J., Wang, X., Shou, W., Zhang, L., Liu. 2016. *Multiple Health Phases Based Remaining Useful Lifetime Prediction on Bearings*. New York. Advances in Data Mining Applications and Theoretical Aspects, 16th Industrial Conference. Springer

[7] Jardine, A. K., Lin, D., Banjevic, D. 2006. *A review on machinery diagnostics and prognostics implementing condition-based maintenance*. Toronto, Canada. Mechanical systems and signal processing, vol 20, issue 7. Elsevier Ltd

[8] Hansen, T. I. 2017. *NTNU*. http://www.snl.no/NTNU (Downloaded 18.05.2017)

[9] NTNU RAMS Group. *About us*. https://www.ntnu.edu/ross/rams (Downloaded 18.05.2017)

[10] Statoil. 2014. *Kort om Statoil*. http://www.statoil.com/no/About/InBrief/Pages/default.aspx, (Downloaded 26.08.16)

[11] Rausand, M., Høyland, A. 2004. *System Reliability Theory – Models, Statistical Methods, and Applications*. Second Edition. United States of America. John Wiley & Sons, Inc.

[12] Evisdom. 2014. *Hva er en gasskompressor*? http://www.evisdom.com/industri/2014/10/Hva-er-en-gasskompressor.html, (Downloaded 18.09.2016)

[13] ABB. 2008. *What is a variable speed drive*? http://www.abb.com/cawp/db0003db002698/a5bd0fc25708f141c12571f10040fd37.aspx, Downloaded 07.10.16)

[14] Barnes, M. 2003. *Practical Variable Speed Driver and Power Electronics*. Great Britain: Elsevier Ltd, pp 1

[15] Toldnes, B. 2012. *Gir - Tannhjulskonstruksjoner*. https://snl.no/gir%2Ftannhjulskonstruksjon, (Downloaded 13.10.2016)

[16] Johannessen, K. 2013. *Konstruksjonsteknikk*. Rev 2. Bergen, Norway, Fagbokforlaget. pp 283, 312 & 313

[17] Lorentzen, G. 2009. *Kompressor*. https://snl.no/kompressor, (Downloaded 18.09.2016)

[18] Eidsmoen, Ø. 2008. *Gasskompressorer*. http://www.dynatrend.no/uploads/articles/lysarkeidsmoen2005.pdf, (Downloaded 18.09.2016)

[19] Direct Industry. 2016. *Gas compressor / centrifugal / stationary / lubricated*. http://www.directindustry.com/prod/rolls-royce/product-22649-664845.html, (Downloaded 18.09.2016)

[20] Brautaset, K. 2013. *Innføring i oljehydraulikk*, vol 9. Oslo, Norge: Gyldendal Norsk Forlag. pp 8 & 87

[21] Hughes, A., Drury, B. 2013. *Electric Motors and Drives: Fundamentals, Types and Applications*, 4th ed. United Kingdom: Elsevier Ltd, pp. 10-35.

[22] Stölting, H. D., Kallenbach, E., Amrhein, W. 2008. *Handbook of Fractional-Horsepower Drivers. Berlin: Springer Science & Business Media*, pp 514, 515 518,

[23] Høyte, F. 2015. *Elektrisk maskin*. https://snl.no/elektrisk_maskin#Maskintyper, (Downloaded 08.09.16)

[24] Bitlanders. 2014. *Synchronous motor*. http://www.bitlanders.com/blogs/synchronous-motor/201606, (Downloaded 80.09.16)

[25] Stone, G. C., Culbert, I., Boulter, E. A., Dhirani, H. 2014. *Electrical Insulation for Rotating Machines – Design, Evaluation, Aging, Testing, and Repair*. Second Edition. New Jersey and Canada. IEEE press. John Wiley & Sons, Inc. ch 1, 2, 7 & 8

[26] Cooper, T. 1994. *Beyond Recycling – The longer life option*. London. New Economics Foundation. pp 5

[27] Rausand, M., Lundteigen, M. A. 2014. *Reliability of Safety-Critical Systems – Theory and Applications*. Rev 2. Preliminary extension to draft. United States of America. John Wiley & Sons, Inc.

[28] Petroleumstilsynet. 2008. *Ord og uttrykk i petroleumsvirksomheten*. http://www.ptil.no/ord-og-uttrykk/category38.html, (Downloaded 29.09.16)

[29] Petterson, L., Simola, K. 2006. *Ageing of Components and Systems*. An ESReDA Working Group Report. Det Norske Veritas. pp 112-127

[30] Heien, Ø., 1995. *Restlevetid og utskiftning av teknisk utstyr*. Prosjektoppgave ved Institutt for produksjons- og kvalitetsteknikk ved NTH, Trondheim.

[31] Hokstad, P., Håbrekke, S., Johnsen, R., Sangesland, S. 2010. *Ageing and life extension for offshore facilities in general and specific systems*. http://www.ptil.no/getfile.php/

PDF/Ageing%20and%20Life%20Extension%20Report%202%20final%20%28Sintef%202010%29.pdf, (Downloaded 29.09.16)

[32] International Standards Office. 2016. *Petroleum, petrochemical and natural gas industries – Collection and exchange of reliability and maintenance data for equipment*. Geneva: ISO/FDIS 14224:2016(E).

[33] Maintenance technology. 2012. *Large Electrical Motor Reliability: what Did The Studies Really Say?* http://www.maintenancetechnology.com/2012/03/large-electric-motor-reliability-what-did-the-studies-really-say/, (Downloaded 28.10.2016)

[34] Albrecht, P.F., Appiarius, J.C., McCoy, R. M. 1986. *Assessment of the reliability of motors in utility applications – uptated*. Vol. EC-1 No 1. New york. IEEE Transaction on Energy Conversion

[35] Kato Engineering inc. 2001. *Engineer Report: Form-Wound Coils, Random-Wound Coils*. http://www.emersonindustrial.com/en-en/documentcenter/electric-power-generation/KatoEngineering/Literature/FWRW_BK_PDF_0701.pdf, (Downloaded 13.12.2016)

[36] Cygan, P., Laghari, J. R. 1990. *Models for Insulation Aging under Electrical and Thermal Multistress*. Vol.25 No 5. IEEE Transaction on Electrical Insulation

[37] Dakin, T. W. 1948. *Electrical Insulation Deterioration Treated as a Chemical Rate Phenomenon*, A1EE Trans., Part I. 67. pp 113–122

[38] Gabe Paoletti, G. E. 1999. *Partial Discgarge Theory and Applications to Electrical Systems. Seattle*, WA. Presented at the IEEE IAS Pulp and Paper Industry Conference.

[39] IEEE Standard 930. 1987. *Guide for the Statistical Analysis of Electrical Insulation Voltage Endurance Test Data*. Also published as IEC 62539.

[40] Srinivas, M. B., Ramu, T. S. 1992. *Multifactor Aging of HV Generator Stator Insulation Including Mechanical Vibrations*. Vol 27, No. 5. IEEE Transactions on Electrical Insulation. Indian Institute of Science, Bangalore, India.

[41] Penrose, W. H. 2014. *Evaluating Reliability of Insulation Systems for Electrical Machines*. IEEE. Electrical Insulation Conference, Philadelphia, Pennsylvania, USA, 8 to 11 June 2014.

[42] Stone, G. C. 2013. *Condition Monitoring and Diagnostics of Motor and Stator Windings – A Review.* Vol 20, No. 6. IEEE Transactions on Dielectrics and Electrical Insulation.

[43] Karsten Moholt. *Insulation quality measured in partial discharge.* Internal presentation 2016 at NTNU, Trondheim

[44] Hout, v.d. A. 2017. *Multi-State Survival Models for Interval- Censored Data.* Boca Raton, London, New York. Taylor & Francis Group

[45] Taylor, W. B. 2013. *Introduction to Management Scienc.* 11 Ed. United States of America. Pearson Education, Inc.

[46] Yale University. 1997. *Probabilistic Models.* Connecticut. http://www.stat.yale.edu/Courses/1997-98/101/probint.htm, (Downloaded 20.01.2017)

[47] Lindqvist, B. 2015. *TMA4275 Lifetime Analysis, Slides 13Introduction to parametric inference in lifetime models*. NTNU, Trondheim. http://www.folk.ntnu.no/bo/TMA4275/2015v/TMA4275-Slides13-2015.pdf, (Downloaded 02.06.2017)

[48] Ross, M. S. 1996. *Stochastic Processes.* Second Edition. United States of America. John Wiley & Sons, Inc.

[49] Walpole, R. E., Myers, R. H., Myers, S. L., Ye, K. 2007. *Probability & Statistics for Engineers & Scientists.* Eight Edition. New Jersey. Pearson Education, Inc

[50] Lindqvist, B. 2015. *TMA4275 Lifetime Analysis, Slides 8: Parametric inference for the exponential model*. NTNU, Trondheim. http://www.folk.ntnu.no/bo/TMA4275/2015v/TMA4275-Slides8-2015.pdf, (Downloaded 22.05.2017)

[51] Vu, C. H., Barros, A., Lundteigen, M. A. 2016. *Markov models for deteriation modeling of the girder surface of Norhordalands bridge.* Trondheim, Norway. Internal report NTNU – Ferry free E39 project.

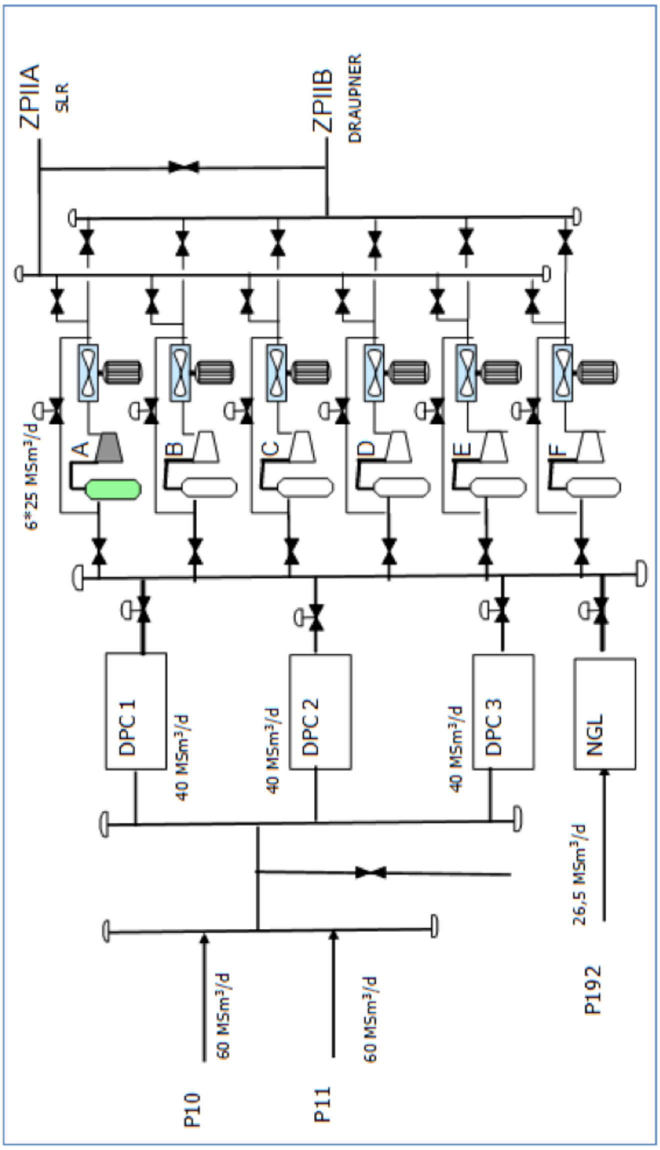
[52] Lindqvist, B. 2015. *TMA4275 Lifetime Analysis, Slides 7 Introduction to parametric inference in lifetime models*. NTNU, Trondheim. http://www.folk.ntnu.no/bo/TMA4275/2015v/TMA4275-Slides7-2015.pdf, (Downloaded 20.01.2017)

Appendix

Appendix I: Abbreviations

MSc	Master of Science
RAMS	Reliability, Availability, Maintainability and Safety
NTNU	Norwegian University of Science and Technology
MTP	Department of Mechanical and Industrial Engineering
RUL	Remaining useful lifetime
CDS	Compressor drive system
PD	Partial Discharge
HSE	Health, safety and environment
VDS	Variable speed drive
VDSF	Variable speed drive fails
CF	Compressor Fails
GF	Gearbox Fails
MF	Motor Fail
CDSF	Compressor drive system fails
ERPI	Electric Power Research Institute
IEEE	Institute of Electrical and Electronics Engineers
nC	nanocoulombs
ROCOF	Rate of occurrence of failure
PoF	Probability of failure
CDF	Cumulative distribution function
PDF	Probability density function
MTTF	Meant Time To Failure
CI	Confidence interval
MLE	Maximum likelihood estimation
VSD	Variable Speed Drive

Appendix II: Process plan at Kollsnes



H

Appendix III: Table for z-value

Table F.1 The Cumulative Standard Normal Distribution

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.500	.504	.508	.512	.516	.520	.524	.528	.532	.536
0.1	.540	.544	.548	.552	.556	.560	.564	.567	.571	.575
0.2	.579	.583	.587	.591	.595	.599	.603	.606	.610	.614
0.3	.618	.622	.626	.629	.633	.637	.641	.644	.648	.652
0.4	.655	.659	.663	.666	.670	.674	.677	.681	.684	.688
0.5	.691	.695	.698	.702	.705	.709	.712	.716	.719	.722
0.6	.726	.729	.732	.736	.739	.742	.745	.749	.752	.755
0.7	.758	.761	.764	.767	.770	.773	.776	.779	.782	.785
0.8	.788	.791	.794	.797	.800	.802	.805	.808	.811	.813
0.9	.816	.819	.821	.824	.826	.829	.831	.834	.836	.839
1.0	.841	.844	.846	.849	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.893	.894	.896	.898	.900	.901
1.3	.903	.905	.907	.908	.910	.911	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.926	.928	.929	.931	.932
1.5	.933	.934	.936	.937	.938	.939	.941	.942	.943	.944
1.6	.945	.946	.947	.948	.949	.951	.952	.953	.954	.954
1.7	.955	.956	.957	.958	.959	.960	.961	.962	.962	.963
1.8	.964	.965	.966	.966	.967	.968	.969	.969	.970	.971
1.9	.971	.972	.973	.973	.974	.974	.975	.976	.976	.977
2.0	.977	.978	.978	.979	.979	.980	.980	.981	.981	.982
2.1	.982	.983	.983	.983	.984	.984	.985	.985	.985	.986
2.2	.986	.986	.987	.987	.987	.988	.988	.988	.989	.989
2.3	.989	.990	.990	.990	.990	.991	.991	.991	.991	.992
2.4	.992	.992	.992	.992	.993	.993	.993	.993	.993	.994
2.5	.994	.994	.994	.994	.994	.995	.995	.995	.995	.995
2.6	.995	.995	.996	.996	.996	.996	.996	.996	.996	.996
2.7	.997	.997	.997	.997	.997	.997	.997	.997	.997	.997
2.8	.997	.998	.998	.998	.998	.998	.998	.998	.998	.998
2.9	.998	.998	.998	.998	.998	.998	.999	.999	.999	.999
3.0	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999

$\Phi(-z) = 1 - \Phi(z)$

[37]

Appendix IV: Hand Calculations of State Probabilities

State 5 (Eq6.)

$$P_5(t)\times(-\lambda_{54}) = P_5\dot{(t)}$$

$$P_5\dot{(t)} - P_5(t)\times(-\lambda_{54}) = 0 \qquad \Big| \times e^{\lambda_{54}t}$$

$$e^{\lambda_{54}t}\times[P_5\dot{(t)} + P_5(t)\times\lambda_{54}] = 0$$

$$\frac{d}{dt}[e^{\lambda_{54}t}\times P_5(t)] = 0 \qquad \Big| \text{Integral}$$

$$e^{\lambda_{54}t}\times P_5(t) = C \qquad \Big| \times e^{-\lambda_{54}t}$$

$$P_5(t) = C\times e^{-\lambda_{54}t}$$

$$\boldsymbol{P_5(t) = e^{-\lambda_{54}t}}$$

Calculations for C

When we set t = 0 for the second last equation we know that:

$$P_5(0) = 1$$

Thus:

$$1 = C\times e^{-\lambda_{54}(0)}$$

$$1 = C\times 1$$

$$=> C = 1$$

State 4 (Eq5.)

$$P_4(t)\times(-\lambda_{43}) + P_5(t)\times\lambda_{54} = P_4\dot{(t)}$$

$$P_4(t)\times(-\lambda_{43}) + \lambda_{54}\times e^{-\lambda_{54}t} = P_4\dot{(t)}$$

$$P_4\dot{(t)} + \lambda_{43}\times P_4(t) = \lambda_{54}\times e^{-\lambda_{54}t} \qquad \Big| \times e^{\lambda_{43}t}$$

$$e^{\lambda_{43}t}\times[P_4\dot{(t)} + \lambda_{43}\times P_4(t)] = e^{\lambda_{43}t}\times[\lambda_{54}\times e^{-\lambda_{54}t}]$$

$$\frac{d}{dt}[e^{\lambda_{43}t}\times P_4(t)] = e^{\lambda_{43}t}\times[\lambda_{54}\times e^{-\lambda_{54}t}] \qquad \Big| \text{Integral}$$

$$e^{\lambda_{43}t}\times P_4(t) = \frac{\lambda_{54}}{\lambda_{43}-\lambda_{54}}\times e^{\lambda_{43}t}\times e^{-\lambda_{54}t} + C \qquad \Big| \times e^{-\lambda_{43}t}$$

$$P_4(t) = \frac{\lambda_{54}}{\lambda_{43}-\lambda_{54}}\times e^{-\lambda_{54}t} + C\times e^{-\lambda_{43}t}$$

$$P_4(t) = \frac{\lambda_{54}}{\lambda_{43}-\lambda_{54}}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}}{\lambda_{43}-\lambda_{54}}\times e^{-\lambda_{43}t}$$

$$\boldsymbol{P_4(t) = \frac{\lambda_{54}}{\lambda_{43}-\lambda_{54}}(e^{-\lambda_{54}t} - e^{-\lambda_{43}t})}$$

Calculations for C

When we set t = 0 for the second last equation we know that:

J

$$P_4(0) = 0$$

Thus:

$$P_4(t) = \frac{\lambda_{54}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{54}t} + C \times e^{-\lambda_{43}t}$$

$$0 = \frac{\lambda_{54}}{\lambda_{43} - \lambda_{54}} + C$$

$$C = \frac{-\lambda_{54}}{\lambda_{43} - \lambda_{54}}$$

State 3 (Eq4.)

$$P_3(t) \times (-\lambda_{32}) + P_4(t) \times \lambda_{43} = \dot{P_3}(t)$$

$$P_3(t) \times (-\lambda_{32}) + \left[\frac{\lambda_{54}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{54}t} - \frac{\lambda_{54}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{43}t}\right] \times \lambda_{43} = \dot{P_3}(t)$$

$$\dot{P_3}(t) + \lambda_{32} \times P_3(t) = \frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{43}t} \quad \Big| \times e^{\lambda_{32}t}$$

$$e^{\lambda_{32}t} \times [\dot{P_3}(t) + \lambda_{32} \times P_3(t)] = e^{\lambda_{32}t} \times \left[\frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{43}t}\right]$$

$$\frac{d}{dt}[e^{\lambda_{32}t} \times P_3(t)] = e^{\lambda_{32}t} \times \left[\frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{\lambda_{43} - \lambda_{54}} \times e^{-\lambda_{43}t}\right] \quad \Big| \text{ Integral}$$

$$e^{\lambda_{32}t} \times P_3(t) = e^{\lambda_{32}t} \times \left[\frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} \times e^{-\lambda_{43}t}\right] + C \quad \Big| \times e^{-\lambda_{32}t}$$

$$P_3(t) = \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} \times e^{-\lambda_{43}t} + C \times e^{-\lambda_{32}t}$$

$$P_3(t) = \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} \times e^{-\lambda_{43}t} + \left(\frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})}\right) \times e^{-\lambda_{32}t}$$

$$\boldsymbol{P_3(t) = \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} (e^{-\lambda_{54}t} - e^{-\lambda_{32}t}) + \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} \times (e^{-\lambda_{32}t} - e^{-\lambda_{43}t})}$$

Calculations for C

When we set t = 0 for the second last equation we know that:

$$P_3(0) = 0$$

Thus:

$$P_3(t) = \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} \times e^{-\lambda_{54}t} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} \times e^{-\lambda_{43}t} + C \times e^{-\lambda_{32}t}$$

$$0 = \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} - \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})} + C$$

$$C = -\frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{54})} + \frac{\lambda_{54} \times \lambda_{43}}{(\lambda_{43} - \lambda_{54}) \times (\lambda_{32} - \lambda_{43})}$$

State 2 (Eq3.)

$$P_2(t)\times(-\lambda_{21}) + P_3(t)\times\lambda_{32} = \dot{P_2}(t)$$

$$P_2(t)\times(-\lambda_{21}) + [\frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})}\times e^{-\lambda_{43}t} + (\frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})})\times e^{-\lambda_{32}t}]\times\lambda_{32} = \dot{P_2}(t)$$

$$\dot{P_2}(t) + \lambda_{21}\times P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})}\times e^{-\lambda_{43}t} + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})})\times e^{-\lambda_{32}t} \quad \Big| \times e^{\lambda_{21}t}$$

$$e^{\lambda_{21}t}\times[\dot{P_2}(t) + \lambda_{21}\times P_2(t)] =$$

$$e^{\lambda_{21}t}\times[\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})}\times e^{-\lambda_{43}t} + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})})\times e^{-\lambda_{32}t}]$$

$$\frac{d}{dt}[e^{\lambda_{21}t}\times P_2(t)] =$$

$$\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})}\times e^{-\lambda_{54}t}\times e^{\lambda_{21}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})}\times e^{-\lambda_{43}t}\times e^{\lambda_{21}t} + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})})\times e^{-\lambda_{32}t}\times e^{\lambda_{21}t} \quad \Big| \text{ Integral}$$

$$e^{\lambda_{21}t}\times P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t}\times e^{\lambda_{21}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t}\times e^{\lambda_{21}t} + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}\times e^{\lambda_{21}t} + C \quad \Big| \times e^{-\lambda_{21}t}$$

$$P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t} + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t} + C\times e^{-\lambda_{21}t}$$

$$P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t} + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t} + [-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}]\times e^{-\lambda_{21}t}$$

$$\boldsymbol{P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times(e^{-\lambda_{54}t} - e^{-\lambda_{21}t}) + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times(e^{-\lambda_{21}t} - e^{-\lambda_{43}t}) + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times(e^{-\lambda_{32}t} - e^{-\lambda_{21}t})}$$

Calculations for C

When we set t = 0 for the second last equation we know that:

L

$$P_2(0) = 0$$

Thus:

$$P_2(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+ C\times e^{-\lambda_{21}t}$$

$$0 = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right) + C$$

$$C = -\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}$$
$$- \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}$$

State 1 (Eq2.)

$$P_1(t)\times(-\lambda_{10}) + P_2(t)\times\lambda_{21} = \dot{P_1}(t)$$

$$P_1(t)\times(-\lambda_{10})$$
$$+\Big[\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+\Big(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}$$
$$- \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\Big)\times e^{-\lambda_{21}t}\Big]\times\lambda_{21}$$
$$= \dot{P_1}(t)$$

$$\dot{P_1}(t) + \lambda_{10}\times P_1(t)$$
$$= \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t} \quad \Big| \times e^{\lambda_{10}t}$$
$$+\Big(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}$$
$$- \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\Big)\times e^{-\lambda_{21}t}$$

$$e^{\lambda_{10}t}\times[\dot{P_1}(t)+\lambda_{10}\times P_1(t)]=$$

$$e^{\lambda_{10}t}\times[\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times e^{-\lambda_{43}t}+$$
$$\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}+$$
$$(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})}+$$
$$\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})})\times e^{-\lambda_{21}t}]$$

$$\frac{d}{dt}[e^{\lambda_{10}t}\times P_1(t)]=\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}\times e^{-\lambda_{54}t}\times e^{\lambda_{10}t}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}\times$$
$$e^{-\lambda_{43}t}\times e^{\lambda_{10}t}+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}\times e^{\lambda_{10}t}+$$
$$(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})}+$$
$$\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})})\times e^{-\lambda_{21}t}\times e^{\lambda_{10}t}$$

Integral

$$e^{\lambda_{10}t}\times P_1(t)=\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t}\times e^{\lambda_{10}t}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}\times e^{\lambda_{10}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$\times e^{\lambda_{10}t}+(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}$$
$$-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})})\times e^{-\lambda_{21}t}\times e^{\lambda_{10}t}+C$$

$\times e^{-\lambda_{10}t}$

$$P_1(t)=\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+\left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right.$$
$$\left.-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)\times e^{-\lambda_{21}t}$$
$$+C\times e^{-\lambda_{10}t}$$

$$P_1(t)=\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+\left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right.$$
$$\left.-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)\times e^{-\lambda_{21}t}$$
$$+\left[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\right.$$
$$-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})}-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}$$
$$\left.+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}x-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right]\times e^{-\lambda_{10}t}$$

N

$$
\begin{aligned}
P_1(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times(e^{-\lambda_{54}t}-e^{-\lambda_{10}t}) \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times(e^{-\lambda_{10}t}-e^{-\lambda_{43}t}) \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\times(e^{-\lambda_{32}t}-e^{-\lambda_{10}t}) \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\times(e^{-\lambda_{10}t}-e^{-\lambda_{32}t}) \\
& + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})})\times(e^{-\lambda_{10}t}-e^{-\lambda_{21}t}) \\
& + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})})\times(e^{-\lambda_{21}t}-e^{-\lambda_{10}t})
\end{aligned}
$$

Calculations for C

When we set t = 0 for the second last equation we know that:

$$P_1(0) = 0$$

Thus:

$$
\begin{aligned}
P_1(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t} \\
& + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t} \\
& + \left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right. \\
& \left. - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)\times e^{-\lambda_{21}t} + C\times e^{-\lambda_{10}t}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} + C
\end{aligned}
$$

$$
\begin{aligned}
C = {} & -\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}
\end{aligned}
$$

State 0 (Eq1.)

$$P_1(t)\times\lambda_{10} = P_0'(t)$$

$$\Big[\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)$$
$$\times e^{-\lambda_{32}t}$$
$$+\left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right.$$
$$\left.-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)$$
$$\times e^{-\lambda_{21}t}$$
$$+\Big[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}$$
$$-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\Big]$$
$$\times e^{-\lambda_{10}t}\Big]\times\lambda_{10} = P_0'(t)$$

$$P_0'(t) = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+\left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right.$$
$$\left.-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)\times e^{-\lambda_{21}t}$$
$$+\Big[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}$$
$$-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\Big]\times e^{-\lambda_{10}t}]\times e^{-\lambda_{10}t}$$

$$\frac{d}{dt}[P_0(t)] = \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}\times e^{-\lambda_{43}t}$$
$$+\left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}\right)\times e^{-\lambda_{32}t}$$
$$+\left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}\right.$$
$$\left.-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\right)\times e^{-\lambda_{21}t}$$
$$+\Big[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})}$$
$$-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})}$$
$$+\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})}\Big]\times e^{-\lambda_{10}t}]\times e^{-\lambda_{10}t}$$

Integral

P

$$
\begin{aligned}
P_0(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})}\times e^{-\lambda_{43}t} \\
& + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\right)\times e^{-\lambda_{32}t} \\
& + \left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right. \\
& \left. - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right)\times e^{-\lambda_{21}t} \\
& + \left[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})}\right. \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& \left. + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}\right]\times e^{-\lambda_{10}t} + C
\end{aligned}
$$

$$
\begin{aligned}
P_0(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})}\times e^{-\lambda_{43}t} \\
& + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\right)\times e^{-\lambda_{32}t} \\
& + \left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right. \\
& \left. - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right)\times e^{-\lambda_{21}t} \\
& + \left[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})}\right. \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& \left. + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}\right]\times e^{-\lambda_{10}t} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{P_0(t)} = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})}\times(e^{-\lambda_{54}t}-1) + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})}\times(1-e^{-\lambda_{43}t}) \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\times(e^{-\lambda_{32}t}-1) \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\times(1-e^{-\lambda_{32}t}) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})})\times(e^{-\lambda_{21}t}-1) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})})\times(1-e^{-\lambda_{21}t}) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})})\times(e^{-\lambda_{10}t}-1) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})})\times(1-e^{-\lambda_{10}t}) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})})\times(1-e^{-\lambda_{10}t}) + (\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})})\times(e^{-\lambda_{10}t}-1)
\end{aligned}
$$

Calculations for C

When we set t = 0 for the second last equation we know that:

Q

$$P_0(0) = 0$$

Thus:

$$\begin{aligned}
P_0(t) = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})}\times e^{-\lambda_{54}t} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})}\times e^{-\lambda_{43}t} \\
& + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\right)\times e^{-\lambda_{32}t} \\
& + \left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right. \\
& \left. - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right)\times e^{-\lambda_{21}t} \\
& + \left[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})}\right. \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& \left. + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}\right]\times e^{-\lambda_{10}t} + C
\end{aligned}$$

$$\begin{aligned}
0 = {} & \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})} \\
& + \left(\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})}\right) \\
& + \left(-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right. \\
& \left. - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})}\right) \\
& + \left[-\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})}\right. \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& \left. + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}\right] + C
\end{aligned}$$

$$\begin{aligned}
C = {} & -\frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{54})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{43})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{32})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{21})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{54})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{43})\times(-\lambda_{10})} \\
& + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{32})\times(-\lambda_{10})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{54})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{43})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} \\
& - \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{43})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})} + \frac{\lambda_{54}\times\lambda_{43}\times\lambda_{32}\times\lambda_{21}\times\lambda_{10}}{(\lambda_{43}-\lambda_{54})\times(\lambda_{32}-\lambda_{54})\times(\lambda_{21}-\lambda_{32})\times(\lambda_{10}-\lambda_{21})\times(-\lambda_{10})}
\end{aligned}$$

R

Appendix V: MATLAB Codes

Part I: Markov test 1 and test 2 (change matrix dimensions to fit the seasons)

```
function Markov(time);

y0=[0 0 0 0 0 1];                           % initial state
A=xlsread('Markov','Ark1','H34:M39');       % import transition matrix
SOJOURN=xlsread('Markov','Ark1','H55:M55');    % import sojourn times
time=24*365*1;                                 % *20 for test 2
delta=876;
y=[];

for i = 0:delta:time,
    ytempo=y0*expm(i*A);
    y = [y; ytempo];
end

%define space and seperators
E=[' '];
L=['------------------------------------------------------------- '];
disp(L);

%Print reliability
TimeText = ['For time t = ',num2str(time/(24*365),'%.0f'), ' Years'];
disp(TimeText);

disp(L);

X    =    ['-    The    reliability,    R(t),    is    ',num2str(1-
y((time/delta)+1,1),'%.4f')];
disp(X);

% Print MTTF
Int = (trapz(1-y(1:(time/delta)+1,1)));
MTTF = (Int*delta)/(24*365);
M = ['- The Mean Time To Failure, MTTF, is '...
    ,num2str(MTTF,'%.2f'),' Years'];
disp(M);

% Print state probabilities
```

```
StateText = '- The state probabilities are:';
disp(StateText);

disp(E);

StateText2 = ...
    '    State 0    State 1    State 2    State 3    State 4    State 5';
disp(StateText2);

disp(y((time/delta)+1,1:6));

% Print sojourn time
SojournText = '- The expexted time (years) spent in each stat is:';
disp(SojournText);

disp(E);

disp(StateText2);
disp(SOJOURN);

%Plots
CDF = y(1:(time/delta)+1,1);
PDF = diff(CDF);
MTTFx = [MTTF*(24*365/delta),MTTF*(24*365/delta)];        %for plot
MTTFy = [0,1];                                            %for plot
MTTFy2 = [min(1-CDF),1];                                  %for plot
MTTFy3 = [0,max(PDF)];                                    %for plot

% add first plot in 3 x 1 grid
subplot(3,1,1)
plot(y)
set(gca,'XTick',0:((time*1752)/(8760*delta)):... % set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5)) ...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
line(MTTFx,MTTFy,'Color',[1 0 1])
legend('State  0','State  1','State  2','State  3','State  4','State
5','MTTF')
title('State probabilities')

% add second plot in 3 x 1 grid
```

T

```
subplot(3,1,2)
plotR = plot(1-CDF,'Color',[0 0.447 0.741]);

set(gca,'XTick',0:((time*1752)/(8760*delta)):...% set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5)) ...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
plotMTTF = line(MTTFx,MTTFy2,'Color',[1 0 1]);
legend([plotR plotMTTF],{'Reliability','MTTF'});
title('Reliability')

% add second plot in 2 x 1 grid
subplot(3,1,3)
plot(PDF)
set(gca,'XTick',0:((time*1752)/(8760*delta)):... % set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5))...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
line(MTTFx,MTTFy3,'Color',[1 0 1])
legend('PDF','MTTF')
title('Probability Density Function')
end
```

Part I: Markov test 3 and test 4. Conditional reliability

```
function Markov(time);

y0=[0 0 0 0 0 1];                                  % initial state
A=xlsread('Markov','Ark1','H34:M39');         % import transition matrix
SOJOURN=xlsread('Markov','Ark1','H55:M55');     % import sojourn times
time=8760*20;                                     % *19*21*23 for test 4
delta=876;
y=[];

y2=[0 0 0 0 1 0];                                 % Observed state
timet = 8760*3;                                 % Inspection time,*2*4*6
timex = time - timet;

for i = 0:delta:timet,
    ytempo=y0*expm(i*A);
    y = [y; ytempo];
end
```

```
for i2 = 0:delta:timex,
    ytempo2=y2*expm(i2*A);
    y = [y; ytempo2];
end

% define state name
if y2 ==[1 0 0 0 0 0];
        S0 = 0;
else
    S0 = 0;
end
if y2 ==[0 1 0 0 0 0];
        S1 = 1;
else
    S1 = 0;
end
if y2 ==[0 0 1 0 0 0];
        S2 = 2;
else
    S2 = 0;
end
if y2 ==[0 0 0 1 0 0];
        S3 = 3;
else
    S3 = 0;
end
if y2 ==[0 0 0 0 1 0];
        S4 = 4;
else
    S4 = 0;
end
if y2 ==[0 0 0 0 0 1];
        S5 = 5;
else
    S5 = 0;
end
S6 = S0 + S1 + S2 + S3 +S4 +S5;

%define space and seperators
E=[' '];
% disp(E);
```

V

```
L=['----------------------------------------------------------- '];
disp(L);

%Print reliability
if timet == 0;
    TimeText = ['For time t = ',num2str(time/(24*365),'%.0f'), '
Year(s)'];
    disp(TimeText);
else timet > 0;
    TimeText2 = ['For a total time of t = '...
        ,num2str(time/(24*365),'%.0f'), ' Year(s),'];
    disp(TimeText2);
    TimeText3 = ['and a condition of state ' ,num2str(S6) ...
        ' at time t = ',num2str(timet/(24*365),'%.0f'), ' Year(s)'];
    disp(TimeText3)
end
disp(L);

X = ['- The reliability, R(t), is ',num2str(1-
y((time/delta)+2,1),'%.4f')];
disp(X);

% Print MTTF
if timet == 0
    Int = (trapz(1-y(((timet)/delta)+2:(time/delta)+2,1)));
    MTTF = (Int*delta)/(24*365);
    M = ['- The Mean Time To Failure, MTTF, is '...
        ,num2str(MTTF,'%.2f'),' Years'];
    disp(M);

    PDF = diff(y(1:(time/delta)+2,1));                        %for plot
    CDF = (y(1:(time/delta)+2,1));                            %for plot
    MTTFx = [MTTF*(24*365/delta),MTTF*(24*365/delta)];        %for plot
    MTTFy = [0,1];                                            %for plot
    MTTFy2 = [min(1-CDF),1];                                  %for plot
    MTTFy3 = [0,max(PDF)];                                    %for plot
else timet > 0;
    Int2 = trapz(1-y((timet/delta)+2:(time/delta)+2,1));
    MTTF2 = (Int2*delta)/(24*365);
    M2 = ['- When the motor is at state ' num2str(S6) ' after '...
```

```
        num2str(timet/(24*365),'%.2f') ...
        ' years,\n  The Mean Time To Failure, MTTF, is '...
        num2str(MTTF2,'%.2f'),' Years.\n'];
    fprintf(M2);

    PDF = diff(y(1:(time/delta)+2,1));                       %for plot
    CDF = (y(1:(time/delta)+2,1));                           %for plot
    MTTFx = [(MTTF2*(24*365/delta))+(timet/delta),...        %for plot
        (MTTF2*(24*365/delta))+(timet/delta)];               %for plot
    MTTFy = [0,1];                                           %for plot
    MTTFy2 = [min(1-CDF),1];                                 %for plot
    MTTFy3 = [0,max(PDF)];
end

% Print state probabilities
StateText = '- The state probabilities are:';
disp(StateText);

disp(E);

StateText2 = ...
    '     State 0    State 1    State 2    State 3    State 4    State 5';
disp(StateText2);

disp(y((time/delta)+2,1:6));

% Print sojourn time
SojournText = '- The expexted time (years) spent in each stat is:';
disp(SojournText);

disp(E);

disp(StateText2);
disp(SOJOURN);

%Plots
CDF = y(1:(time/delta)+2,1);
PDF = diff(CDF);

% add first plot in 3 x 1 grid
subplot(3,1,1)
```

x

```
plot(y)
set(gca,'XTick',0:((time*1752)/(8760*delta)):... % set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5)) ...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
line(MTTFx,MTTFy,'Color',[1 0 1])
legend('State  0','State  1','State  2','State  3','State  4','State
5','MTTF')
title('State probabilities')

% add second plot in 3 x 1 grid
subplot(3,1,2)
plotR = plot(1-CDF,'Color',[0 0.447 0.741]);

set(gca,'XTick',0:((time*1752)/(8760*delta)):... % set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5)) ...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
plotMTTF = line(MTTFx,MTTFy2,'Color',[1 0 1]);
legend([plotR plotMTTF],{'Reliability','MTTF'});
title('Reliability')

% add second plot in 2 x 1 grid
subplot(3,1,3)
plot(subplus(PDF))
set(gca,'XTick',0:((time*1752)/(8760*delta)):... % set xaxis to years
    (time),'XTickLabel',{'0',((time*1)/(8760*5)) ...
    ,((time*2)/(8760*5)),((time*3)/(8760*5)), ...
    ((time*4)/(8760*5)),time/8760,'-->Years'});
line(MTTFx,MTTFy3,'Color',[1 0 1])
legend('PDF','MTTF')
title('Probability Density Function')
end
```

Appendix VI: Censoring Table and Transition Dates

Part I: Censored transition dates, Δ6 months. 20 samples

Inspection nr	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Sample 7	Sample 8	Sample 9	Sample 10
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	4	4	5	5	5	5
3	5	5	5	4	4	4	5	5	5	5
4	5	5	5	4	3	4	5	5	5	0
5	5	4	3	4	3	4	5	5	5	
6	4	4	3	4	3	4	5	5	5	
7	4	4	3	4	3	4	4	4	5	
8	4	4	2	4	3	3	4	3	4	
9	4	3	0	3	2	2	4	2	3	
10	3	0		2	2	2	4	0	2	
11	1			0	1	0	4		1	
12	0				0		4		0	
13							4			
14							3			
15							3			
16							3			
17							2			
18							0			

Inspection nr	Sample 11	Sample 12	Sample 13	Sample 14	Sample 15	Sample 16	Sample 17	Sample 18	Sample 19	Sample 20
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	4	5	5	4
3	2	5	5	5	4	3	4	5	5	4
4	0	5	4	5	4	0	4	5	5	3
5		5	4	5	2		4	5	5	3
6		5	4	4	2		4	3	5	3
7		5	4	4	2		4	3	5	2
8		5	4	4	2		4	3	5	0
9		5	4	4	0		4	3	5	
10		5	3	4			4	2	5	
11		4	1	4			4	2	5	
12		3	0	4			4	1	4	
13		3		4			3	0	3	
14		3		4			3		2	
15		2		4			2		1	
16		1		3			2		0	
17		0		3			0			
18				3						
19				3						
20				1						
21				0						

Part II: Transition dates for 40 samples

State observed	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Sample 7	Sample 8	Sample 9	Sample 10
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	2.1795	1.8884	1.8424	0.7566	0.1099	0.4590	2.7774	2.7648	3.3130	0.6823
3	4.2294	3.6084	1.9298	3.6632	1.3927	3.3724	6.2383	3.1591	3.9048	1.3035
2	4.5300	4.1066	3.1600	4.4410	3.6484	3.6069	7.8159	3.7706	4.2275	1.3758
1	4.6955	4.3479	3.5365	4.6926	4.9849	4.5996	8.1563	4.1911	4.9067	1.3933
0	5.1724	4.4280	3.9621	4.7254	5.3707	4.6205	8.3690	4.2157	5.0676	1.3953

State observed	Sample 11	Sample 12	Sample 13	Sample 14	Sample 15	Sample 16	Sample 17	Sample 18	Sample 19	Sample 20
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5399	4.8123	1.3427	2.1093	0.5063	0.5921	0.3724	2.2240	5.2767	0.4205
3	0.6600	5.3023	4.1846	7.3839	1.8239	0.6004	5.6315	2.4232	5.5747	1.4306
2	0.7463	6.6348	4.6692	9.0667	1.8283	1.2139	6.6781	4.3691	6.1972	2.6259
1	1.0577	7.0829	4.9308	9.1304	3.6194	1.2318	7.6251	5.3055	6.6175	3.0946
0	1.0886	7.5941	5.0133	9.6653	3.8511	1.2375	7.6885	5.8535	7.1666	3.3539

State observed	Sample 21	Sample 22	Sample 23	Sample 24	Sample 25	Sample 26	Sample 27	Sample 28	Sample 29	Sample 30
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.5421	6.1574	4.8879	0.5139	0.7339	3.4055	2.3122	1.8912	0.6565	5.3576
3	0.7526	8.8773	4.9514	1.2885	1.3656	4.2269	8.1009	4.2009	2.6613	5.4846
2	3.1084	9.5662	5.0013	2.5996	2.4340	4.7098	8.8177	4.3211	4.2824	5.5358
1	3.1721	9.5967	5.5093	2.6355	2.7308	5.9862	8.8576	5.0212	4.9195	5.9244
0	3.3024	9.6069	5.5727	3.5933	3.2326	6.5915	9.0600	5.6457	5.5845	6.7254

State observed	Sample 31	Sample 32	Sample 33	Sample 34	Sample 35	Sample 36	Sample 37	Sample 38	Sample 39	Sample 40
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	1.5960	4.4854	0.1937	3.1297	1.1629	0.1665	4.1573	3.0890	0.0541	1.3740
3	4.1839	6.6096	0.7341	4.5686	3.2294	0.4483	6.7101	3.2586	1.8064	4.2221
2	4.5204	7.1450	1.5559	5.3355	3.4677	1.1305	7.5698	4.2215	4.3145	4.7998
1	5.6914	7.3194	2.1395	5.4770	3.9191	1.4636	8.6011	5.4080	5.2664	5.0393
0	5.7982	8.0335	2.3686	5.5423	4.2080	1.6150	8.6492	5.4364	5.5208	5.4676

Part III: Censored transition dates, Δ6 months. 40 samples

Inspection nr	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Sample 7	Sample 8	Sample 9	Sample 10
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	4	4	5	5	5	5
3	5	5	5	4	4	4	5	5	5	5
4	5	5	5	4	3	4	5	5	5	0
5	5	4	3	4	3	4	5	5	5	
6	4	4	3	4	3	4	5	5	5	
7	4	4	3	4	3	4	4	4	5	
8	4	4	2	4	3	3	4	3	4	
9	4	3	0	3	2	2	4	2	3	
10	3	0		2	2	2	4	0	2	
11	1			0	1	0	4		1	
12	0				0		4		0	
13							4			
14							3			
15							3			
16							3			
17							2			
18							0			

Inspection nr	Sample 11	Sample 12	Sample 13	Sample 14	Sample 15	Sample 16	Sample 17	Sample 18	Sample 19	Sample 20
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	4	5	5	4
3	2	5	5	5	4	3	4	5	5	4
4	0	5	4	5	4	0	4	5	5	3
5		5	4	5	2		4	5	5	3
6		5	4	4	2		4	3	5	3
7		5	4	4	2		4	3	5	2
8		5	4	4	2		4	3	5	0
9		5	4	4	0		4	3	5	
10		5	3	4			4	2	5	
11		4	1	4			4	2	5	
12		3	0	4			4	1	4	
13		3		4			3	0	3	
14		3		4			3		2	
15		2		4			2		1	
16		1		3			2		0	
17		0		3			0			
18				3						
19				3						
20				1						
21				0						

Inspection nr	Sample 21	Sample 22	Sample 23	Sample 24	Sample 25	Sample 26	Sample 27	Sample 28	Sample 29	Sample 30
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	5	5	5	5
3	4	5	5	4	4	5	5	5	4	5
4	3	5	5	3	3	5	5	5	4	5
5	3	5	5	3	3	5	5	4	4	5
6	3	5	5	3	2	5	4	4	4	5
7	3	5	5	1	1	5	4	4	3	5
8	0	5	5	1	0	4	4	4	3	5
9		5	5	0		4	4	4	3	5
10		5	5			3	4	2	2	5
11		5	3			2	4	2	1	5
12		5	2			2	4	1	1	3
13		5	0			1	4	0	0	1
14		4				1	4			1
15		4				0	4			0
16		4					4			
17		4					4			
18		4					3			
19		3					1			
20		3					0			
21		0								

Inspection nr	Sample 31	Sample 32	Sample 33	Sample 34	Sample 35	Sample 36	Sample 37	Sample 38	Sample 39	Sample 40
1	5	5	5	5	5	5	5	5	5	5
2	5	5	4	5	5	3	5	5	4	5
3	5	5	3	5	5	3	5	5	4	5
4	5	5	3	5	4	1	5	5	4	4
5	4	5	2	5	4	0	5	5	3	4
6	4	5	0	5	4		5	5	3	4
7	4	5		5	4		5	5	3	4
8	4	5		4	2		5	3	3	4
9	4	5		4	1		5	3	3	4
10	3	4		4	0		4	2	2	3
11	2	4		3			4	2	2	2
12	2	4		1			4	0	1	0
13	0	4		0			4		0	
14		4					4			
15		3					3			
16		1					3			
17		1					2			
18		0					2			
19							0			

Appendix VII: Parameter Estimation Calculator (Optimal)

Parameter Estimaiton Based on Optimal Observations

Inspection frequency: 652.862083191741 Number of samples : 20

Transition of Interst	Optimal # of Observations	Estimated Parameter	Old Parameter	Error Percentage
5 -> 4	=((1/G9)/$E$5)*$I$5	=(1/$E$5)*(LN(($I$5+C9	0.0000378	=(G9-E9)/G9
4 -> 3	=((1/G10)/$E$5)*$I$5	=(1/$E$5)*(LN(($I$5+C1	0.000047	=(G10-E10)/G10
3 -> 2	=((1/G11)/$E$5)*$I$5	=(1/$E$5)*(LN(($I$5+C1	0.0000876	=(G11-E11)/G11
2 -> 1	=((1/G12)/$E$5)*$I$5	=(1/$E$5)*(LN(($I$5+C1	0.0001423	=(G12-E12)/G12
1 -> 0	=((1/G13)/$E$5)*$I$5	=(1/$E$5)*(LN(($I$5+C1	0.000352	=(G13-(1/E5)*(LN(($I$5