Department of Structural Engineering
NTNU

TKT4222
Concrete Structures 3

Compendium

CONTENT

STRUCTURAL ANALYSIS
- Analysis methods
- Geometric imperfections
- Redistribution

DESIGN OF CONCRETE DEEP BEAMS AND WALLS
- Strut and tie models
- Compression field theory

DESIGN OF CONCRETE SLABS
- Theory of elasticity
- Yield line theory
- Strip method
- Flat slabs

PUNCHING OF CONCRETE SLABS
- Punching shear resistance
- Design for punching in flat slabs
- Punching resistance of column bases

DESIGN OF CONCRETE SHELLS
- Finite element analysis
- Axisymmetric shells – classical theory
- Membrane method
- Sandwich model
- Iteration method

DESIGN OF FOUNDATIONS
- Required size of foundations
- Wall foundations
- Column foundations

CHAPTER 1

STRUCTURAL ANALYSIS

Jan Arve Øverli

1.1 General

Structural analysis is performed to establish the response of a structure. The responses are typical forces, moments, stresses, strains, curvatures, rotations and displacements. Design of concrete structures is based on verification of the resistance of the cross sections. In design codes the resistance is normally given based on forces and moments. Thus, the result of the structural analysis must also be forces and moments.

The first task in a structural analysis is to decide the structural model. Depending on the geometry of the structure, this can be straightforward or sometimes an engineering judgement must be employed. The structural model can cover the whole or a part of the structure. Figure 1.1.1 illustrates how a braced frame can be analysed as a whole structure or by partition the structure into components. For unbraced structures involving lateral forces the whole structure must normally be considered.

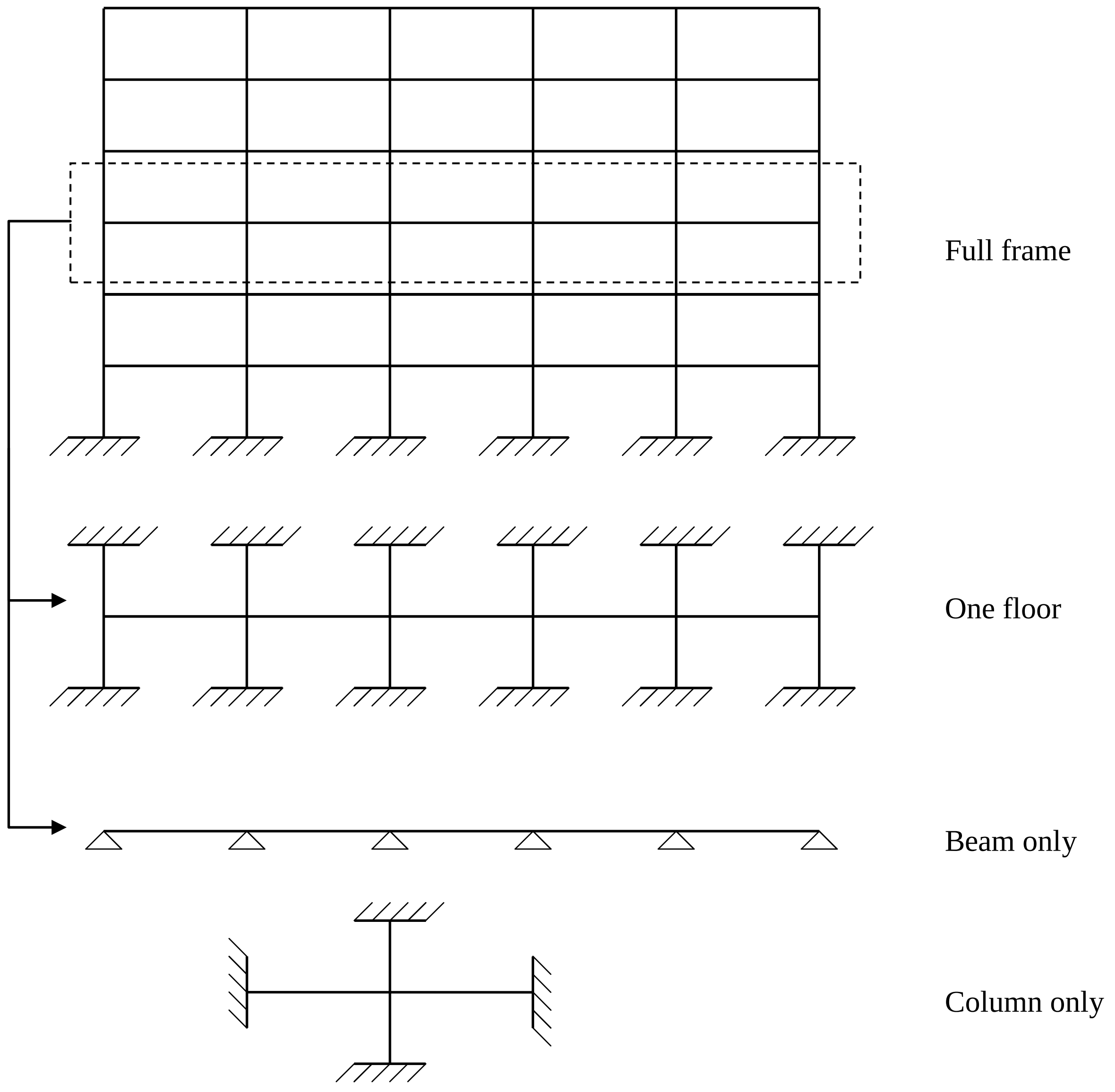


Figure 1.1.1 Different structural models for a braced frame

The frame in Figure 1.1.1 is a global 2-dimensional model of a structure. Additional local analyses must be performed in areas where the assumption of linear strain distribution is not valid. Typical areas are:

- Close to supports
- Around concentrated loads
- Beam-column intersections
- Changes in cross-sections

With today's computer facilities it is possible to model the complete structure with a 3D finite element model. Figure 1.1.2 is an example of such a model including slabs, beams, columns, shear walls, basement and foundations. It is also possible to extend the model by including the soil to find the interaction between soil pressures and foundations and basements walls.

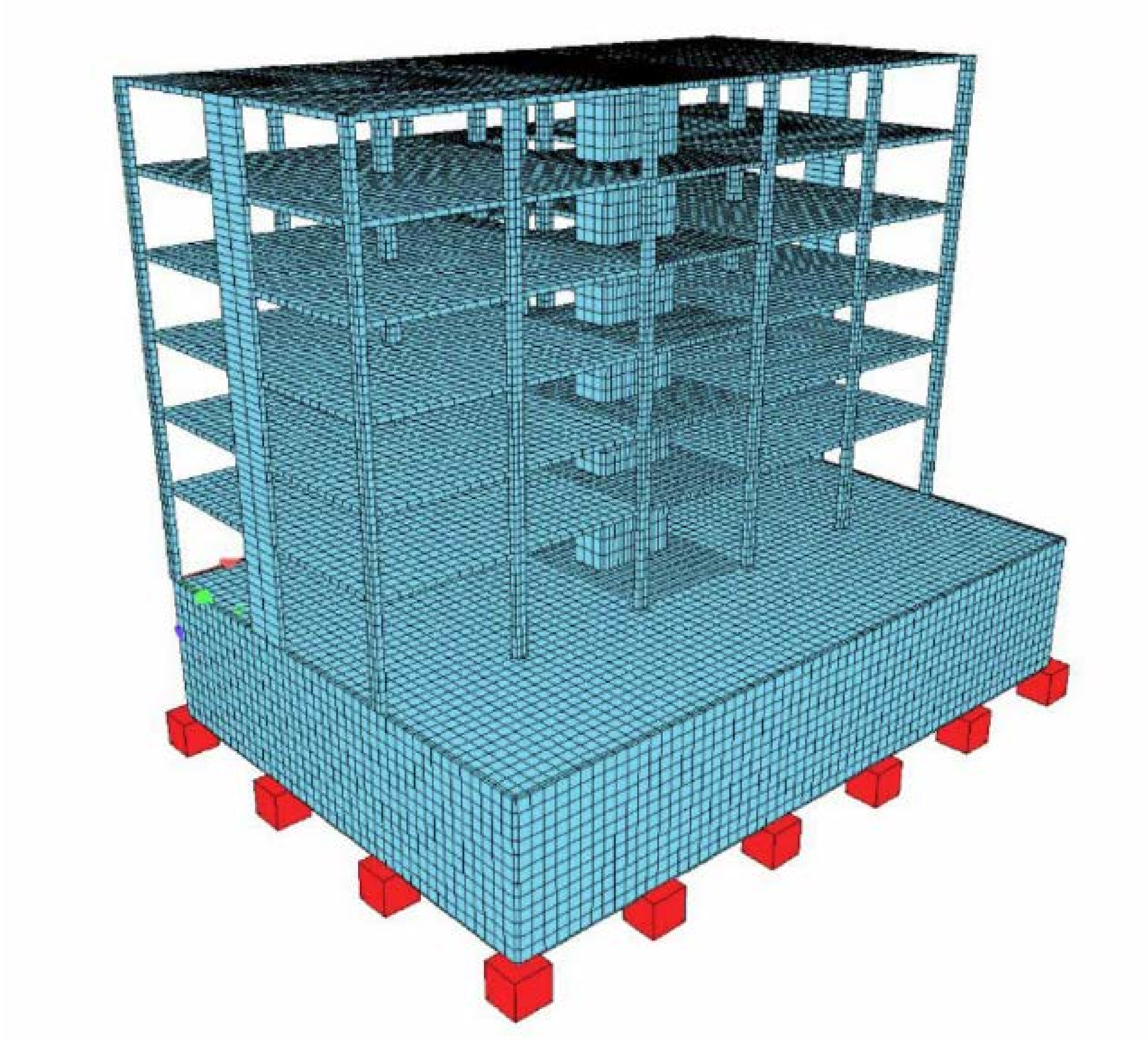


Figure 1.1.2 3D finite element model of a building

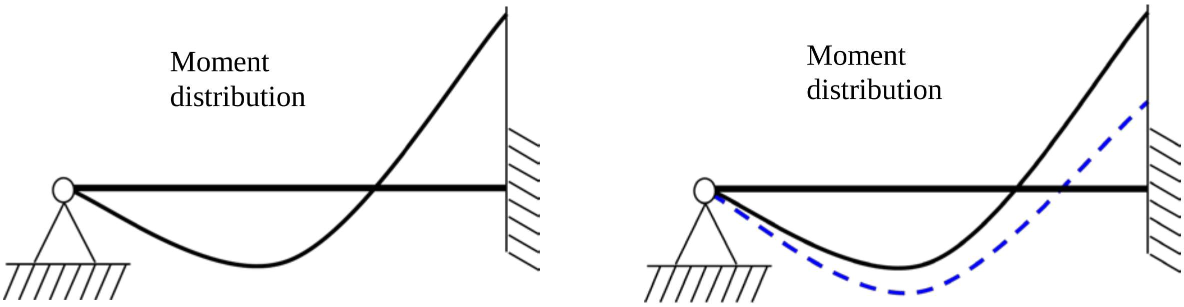
It is always necessary to assess complexity against simplicity when making a structural model. The structural response is normally best described with a 3D model, but making the model can be very time consuming with respect to both geometry and load combinations. In addition there is an enormous amount of results from a 3D analysis. The interpretation of results to find for example the governing load combinations in different limit states is not necessary straightforward. Computer programs for design do this automatically. Still the structural engineer must validate the results to ensure that the most unfavourable load combination is taken into account. Even the most sophisticated software cannot include all structural effects in the analysis. Geometrical imperfections are as an example not easy to take into account in a consistent manner even in a 3D model.

Simple structural models together with engineering judgement can sometimes be the best and fastest solution.

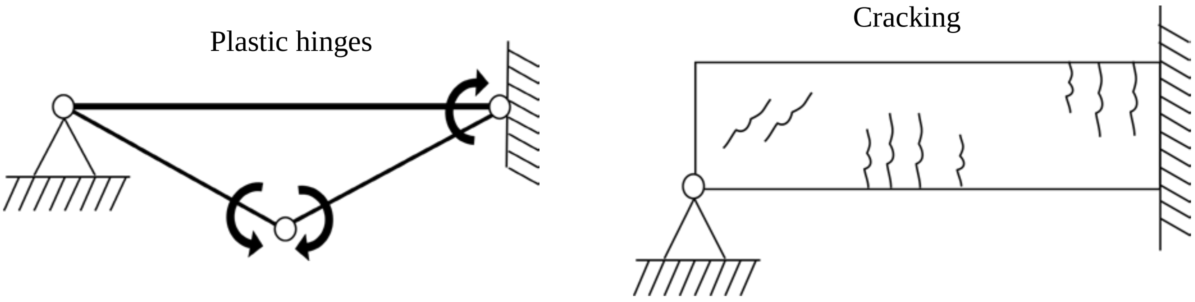
In the analysis both the geometry and the behaviour of the structure must be idealised. The behaviour of the structure can be idealised using:

- Linear elastic analysis
- Linear elastic analysis with limited redistribution
- Plastic analysis
- Non-linear analysis

Figure 1.1.3 shows the different analysis methods for a simple beam, clamped at one support and with a distributed load. In design of structures linear elastic analysis, assuming un-cracked cross sections, are normally performed to calculate forces and moments. In a concrete structure this is not true since the stiffness varies along the beam axis due different reinforcement amounts and degree of cracking. However, it is very convenient since the reinforcement does not need to be known before the structural analysis and the principle of superposition is valid. Hence, results from different load cases can be added.



a) Linear analysis **b) Linear analysis with redistribution**



c) Plastic analysis **d) Non-linear analysis**

Figure 1.1.3 Different analyses methods

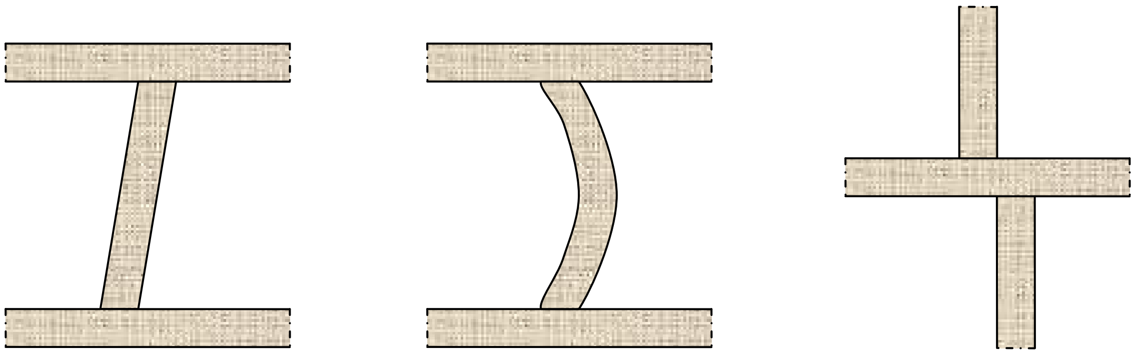
Linear elastic analysis with limited redistribution will be described later in this chapter.

Both the upper and the lower bound theorem of plasticity can be used to analyse concrete structures. An upper bound method is the yield line theory for slabs, see chapter 3.2. Lower bounds methods are the strip method for slabs, see chapter 3.3, and strut and tie models for discontinue regions, see chapter 2.1.

Non-linear finite element analysis takes into account the non-linear material properties in concrete and reinforcement, like cracking and yielding, and the influence of changing geometry on the response. Hence it is the most accurate prediction of the response in a structure. However, in practical design of concrete structures it is impractical to use non-linear analysis. The analysis are very complex and too time consuming. In addition the results are valid only for one load combination. Separate analysis must be performed for every design load combination.

1.2 Geometric imperfections

No structure is perfect. Geometric imperfection describes the deviation of the exact geometry from the drawings that occur during construction. Construction can only be executed to certain tolerances. Load bearing elements may be out of plumb and loads may be applied eccentric. Figure 1.2.1 shows typical examples of imperfections for vertical members in a building.



a) Inclination **b) Curvature** **c) Deviation between centres**

Figure 1.2.1 Examples of deviations for walls and columns

In design of structures there are different types of geometrical imperfections and uncertainties. How they are taken into account in design depends on the type of structural material. For concrete structures and steel structures the requirements are given in Eurocode 2 (EC2) /1.1/ and Eurocode 3 /1.2/ respectively.

In concrete structures the tolerances in cross-section dimensions are accounted for in the material factors. For design of cross-sections in compression, a minimum eccentricity according to EC2 6.2(4), e_0=h/30 or not less than 20mm, is required. This eccentricity is not part of the structural analysis.

Unfavourable effects of possible deviations, like the inclination in Figure 1.2.1a, must be considered in the analysis of members and structures. Only Ultimate Limit State (ULS) needs to take into account the imperfections. In a structural analysis the uncertainties in geometry and position of axial loads are defined as an additional first order effect.

The effect of imperfections has two important consequences in the analysis of structures with vertical loading:

- Additional moments in axial loaded members due to eccentricity.
- Horizontal component of the vertical loads.

In EC2, the building structures are considered to have an arbitrary vertical inclination defined by a rotation θ_i in radians as:

$$\theta_i = (1/200) \cdot \alpha_h \cdot \alpha_m \tag{1.2.1}$$
$$\alpha_h = 2/\sqrt{l}\ ;\ \ 2/3 \le \alpha_h \le 1$$
$$\alpha_m = \sqrt{0,5(1+1/m)}$$

where α_h and α_m are reduction factors for length and numbers of elements respectively. The factors *l* and *m* depends on the effect considered, and recognizes that imperfections are unlikely to be the same in all members:

- On isolated members (e.g. one column), *l* is the actual length and *m*=1.
- On bracing system (e.g. shear wall), *l* is the building height and *m* is the number of vertical members contributing to the horizontal force on the bracing system.
- On floors distributing horizontal loads, *l* is the storey height and *m* is the number of vertical members contributing to the horizontal force on the floor.

An isolated member is defined as a geometric stand alone member or members in a structure which in design can be treated as being isolated. A bracing system contributes to the overall horizontal stability of the structure.

The geometric imperfections in the structural analysis must be linked to tolerances during construction of the building. The numeric values in Eq. (1.2.1) are related to normal executions deviations according to Class 1 in NS-EN 13670 *Execution of concrete structures* /1.3/. With use of other deviations, the values must be adjusted accordingly.

The effect of imperfection can in an analysis either be applied directly or replaced by horizontal forces. For a braced member in a structure a global inclination is the worst case scenario. Figure 1.2.2 illustrates the effect from a frame braced to a shear wall. On each floor level the shear wall is loaded with a transverse force H_i given by:

$$H_i = \theta_i \left(N_b - N_a \right) \tag{1.2.2}$$

where N_b and N_a are the columns loads above and below the level being considered. This horizontal action must be added to other horizontal actions, such as wind loads.

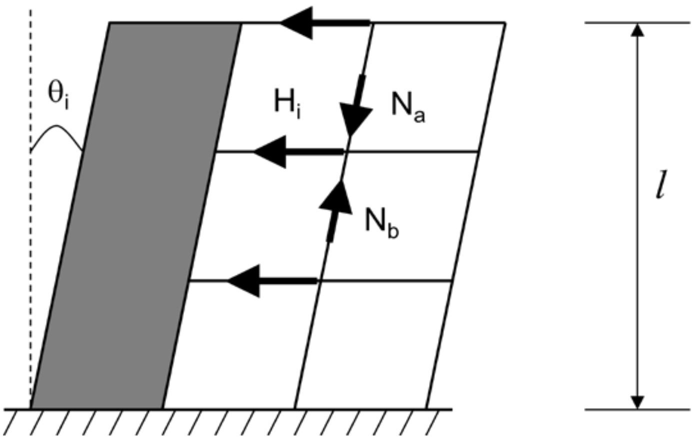


Figure 1.2.2 Geometrical imperfection on a braced structure

For members transferring forces to bracing systems, like floor and roof diaphragms, a local inclination at the diaphragm level gives the largest horizontal forces, see Figure 1.2.3. By using static equilibrium the transverse forces can be calculated to:

$$H_i = \theta_i \left(N_b + N_a\right)/2 \qquad \text{on floor diaphragm} \qquad (1.2.3)$$
$$H_i = \theta_i \cdot N_a \qquad \text{on roof diaphragm}$$

These forces must be taken into account when designing the diaphragms, but not in the design of the bracing element.

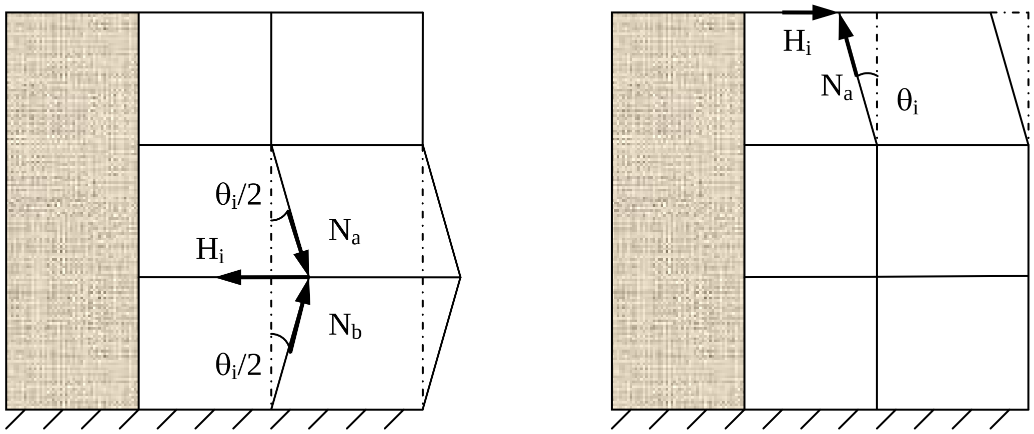


Figure 1.2.3 Effect of geometric imperfection on floor and roof diaphragms

For isolated members the effect of imperfection can either be modelled directly in the structural system with eccentricities or by replacing them with equivalent forces. Figure 1.2.4 illustrates the effect with eccentric axial force or lateral force on a pin-ended column and a cantilever. Lateral forces can be useful since the same model can be employed to model different eccentricities.

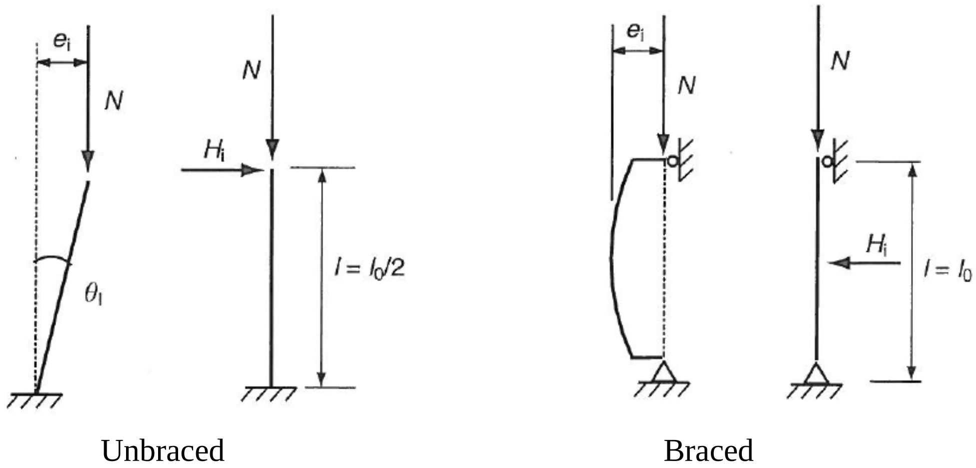


Figure 1.2.4 Effect of geometric imperfection on unbraced and braced isolated members /1.5/

Imperfection defined as an eccentricity is given as:

$$e_i = \theta_i \cdot l_0 / 2 \qquad (1.2.4)$$

where l_0 is the effective (buckling) length and θ_i defined according to Eq. (1.2.1). As a simplification $e_i = l_0/400$ can be used for walls and isolated columns in braced systems. In a braced system the columns do not contribute to the overall stability of the structure.

Applying the imperfection as a transverse force is defined by:

$$H_i = \theta_i \cdot N \quad \text{for unbraced members} \qquad (1.2.5)$$
$$H_i = 2\theta_i \cdot N \quad \text{for braced members}$$

where N is the axial load. The force must be placed in the position that gives maximum moment. The forces H_i can be substituted by some other equivalent transverse load. The numeric values of the eccentricity and the lateral forces in Eqs. (1.2.4) and (1.2.5), gives the same first order maximum moment. However, the distribution of th moment can be different.

The geometric imperfection is particular important in slender columns where the response is sensitive to second-order effects. However, imperfections are defined as a first-order property and must therefore always be included.

1.3 Idealisation of the structure

In design of structures it is useful to classify the components of a structure by considering their function and nature. Typically they are defined as beams, columns, slabs, membranes, shells, arches etc.

In EC2 the following provisions are applicable for buildings:

- A beam is a member for which the span is not less than 3 times the overall section depth. Otherwise is must be considered as a deep beam.
- A slab is a member for which the dimension is not less than 5 times the overall slab thickness.
- A slab subjected to dominantly distributed loads is considered to be a one-way slab if either
  - It possesses two unsupported and sensibly parallel edges
  - A rectangular slab supported on four edges with a ratio of the longer to shorter span greater than 2.
- A column is a member for which the section depth does not exceed 4 times its width and the height is at least 3 times the section depth. Otherwise it must be considered as a wall.

The definitions are useful in defining the detailing and analysis requirements for the components. For example distinction between a beam and a deep beam is necessary in determining appropriate verification model. A beam assumes linear strain distribution, and designed for moment, shear and torsion. In design of a deep beam strut and tie models are employed. Regarding detailing rules, like minimum reinforcement and placement of reinforcement, it can be appropriate to treat a beam, such as the web in a box girder, as a wall or a column.

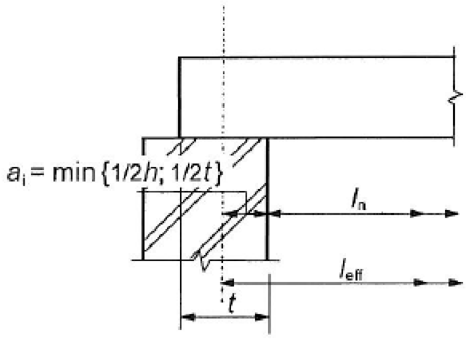
In verification models for beams and slabs, the effective (theoretical) spans are often the same as the distance between system lines. For wide supports this is rather conservative since the moment gradients over supports are large. Hence, in EC2 when considering a beam or a slab as a member analysis, the effective span, l_{eff}, is defined as:

$$l_{eff} = l_n + a_1 + a_2 \tag{1.3.1}$$

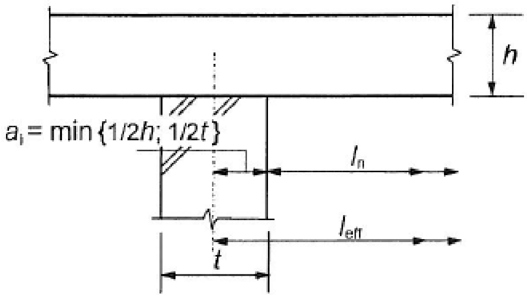
where l_n is the clear distance between the faces of the support and a_i is defined as the minimum of half the thickness of the beam/slab and half the width of the support, as illustrated in Figure 1.3.1. In this way the centre of reaction is kept in a realistic position for wide supports. For beams on bearings, Figure 1.3.1(e), the effective length is measured between centres of bearings. The effective length in Eq. (1.3.1) is not necessarily appropriate if the beams are modelled as a part of a frame analysis.

Continuous beams and slabs are normally analysed on the assumptions that the supports provide no rotational restraints.

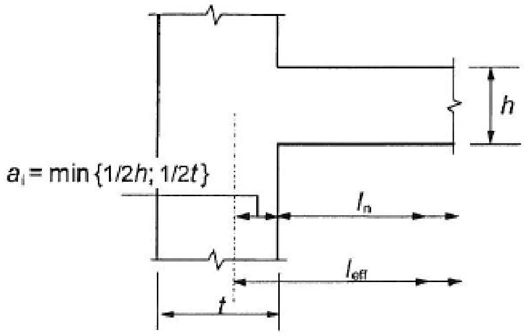
In cases where a slab or beam is monolithic with its supports, e.g. the beam-column connection in Figure 1.3.1(c) or the cantilever slab in Figure 1.3.1(e), the design moment at the support can be taken at the face of the support. In Figure 1.3.2 this is section A. However, in EC2 the moment at the face of the support must not be less than 65% of the fully fixed end moment.



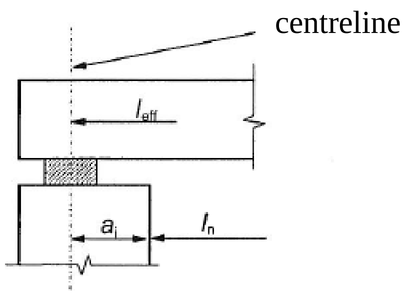
(a) Non-continous members



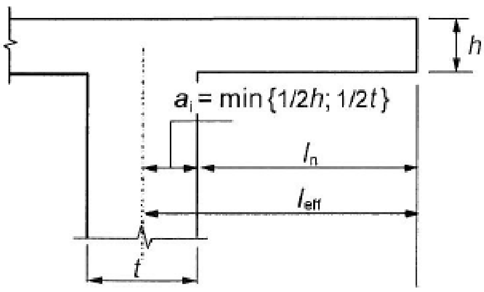
(b) Continous member



(c) Supports considered fully restrained



(e) Bearing provided



(d) Cantilever

Figure 1.3.1 Effective span for different supports /1.1/

Where a beam or slab is continuous over a support and the support provides no restraint to the rotation, the support moments at the centreline of the support can be reduced, section B in Figure 1.3.2. The reduction is due to the distributed pressure from the support reaction. Assuming the design support reaction, $F_{Ed,sup}$, is uniformly distributed, q, over the breadth of the support, t, the reduction can be calculated as:

$$\Delta M = q \cdot \frac{t}{2} \cdot \frac{t}{4} = \frac{F_{Ed,\mathrm{sup}}}{t} \cdot \frac{t}{8} = F_{Ed,\mathrm{sup}} \cdot \frac{t}{8} \qquad (1.3.2)$$

This reduction of the support moment is only valid if the effective spans are based on the system lines of the structure, and the analysis assumes point support. Eq. (1.3.2) is for a rectangular supported area. For a circular support with diameter D, the reduced moment should be calculated based on the centroid of half of a circular area.

$$\Delta M = \frac{F_{Ed,\sup}}{2} \cdot \frac{4 \cdot D/2}{3\pi} = F_{Ed,\sup} \cdot \frac{D}{3\pi} \tag{1.3.3}$$

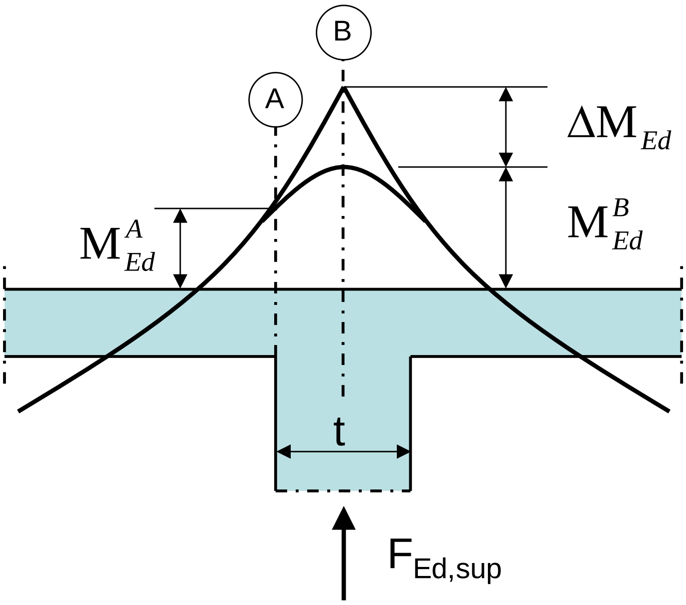


Figure 1.3.2 Design section and reduced design moment at centreline of the support

Example

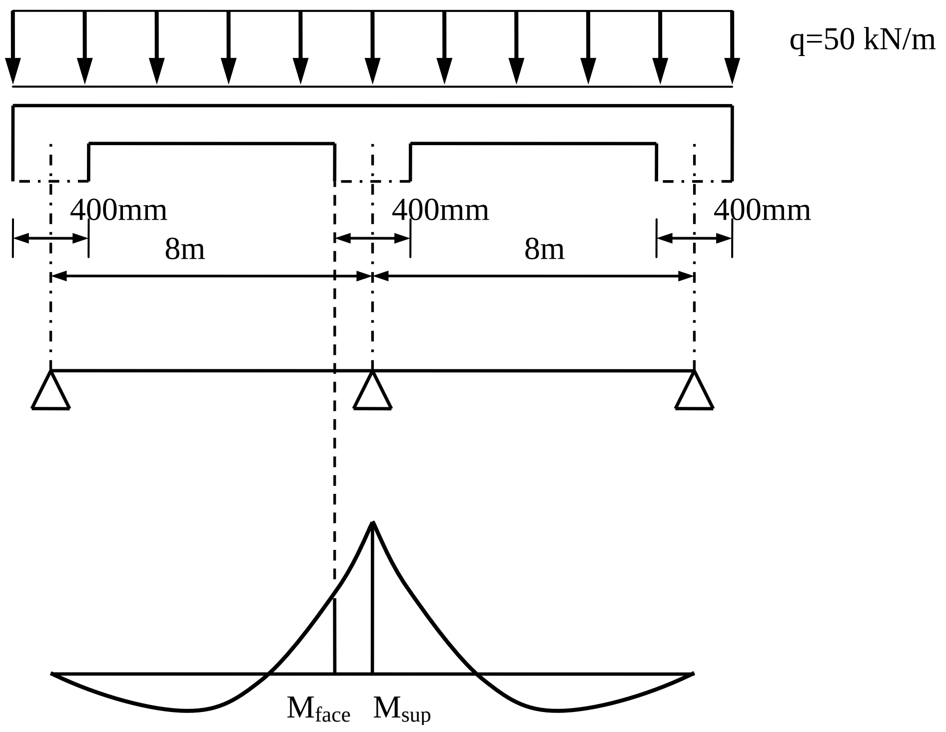


Figure 1.3.3 Reduction of bending moments in a two-span beam

The two-span beam in Figure 1.3.3 is used as an example of the reduction of the design support moment when taking into account the width of the support. Assuming the beam and support are monolithic, the support moment and the moment at the face of the support is given as:

$$M_{\sup} = \frac{q \cdot l^2}{8} = \frac{50 \cdot 8^2}{8} = 400 \text{ kNm}$$

$$M_{face} = -\frac{3}{8} q \cdot l \cdot (l - 0,2) = +\frac{q \cdot (l - 0,2)^2}{2} = 351 \text{ kNm}$$

$$\frac{M_{face}}{M_{\sup}} = \frac{351}{400} = 0,876$$

The reduction of the design moment is approximately 12%. Consequently the required longitudinal reinforcement will also be reduced in ultimate limit state.

Assuming the support provides no restraint to the rotation, the reduced support moment can be calculated according to Eq. (1.3.2) for a rectangular support.

$$F_{Ed,\sup} = 2 \cdot \frac{5}{8} \cdot q \cdot l = 2 \cdot \frac{5}{8} \cdot 50 \cdot 8 = 500 \text{ kN}$$

$$\Delta M = F_{Ed,\sup} \cdot \frac{t}{8} = 500 \cdot \frac{0,4}{8} = 25 \text{ kN}$$

$$M_{\sup}^{red} = 400 - 25 = 375 \text{ kN}$$

$$\frac{M_{\sup}^{red}}{M_{\sup}} = \frac{375}{400} = 0,938$$

Hence, the design moment is reduced with 6% by applying one simple equation.

1.4 Linear elastic analysis with limited redistribution

Elastic analyses, based on un-cracked cross-sections, are normally used to calculate forces and moments in concrete structures. When the load approaches ultimate capacity, sections will behave plastic. Figure 1.4.1 gives the elastic bending moment distribution for a fixed beam, and the distribution assuming plastic hinges at the supports and in the middle of the span. If the beam has a constant ultimate bending strength (M_u), which is typical for steel beams, the load can increase before reaching the capacity of the beam. For a fixed beam the increase is q/3. Hence, the beam has an inherent safety if the design is based on the elastic distribution of the moments.

By comparing the elastic and plastic moments in Figure 1.4.1, the support moments have been reduced (redistributed) from $(1/12)\cdot q\cdot L^2$ to $(1/16)\cdot q\cdot L^2$, which is 25%. The span moment is increased by 50%. The point of contra-flexure (zero bending moment) also changes, which will influence the reinforcement layout in a concrete beam.

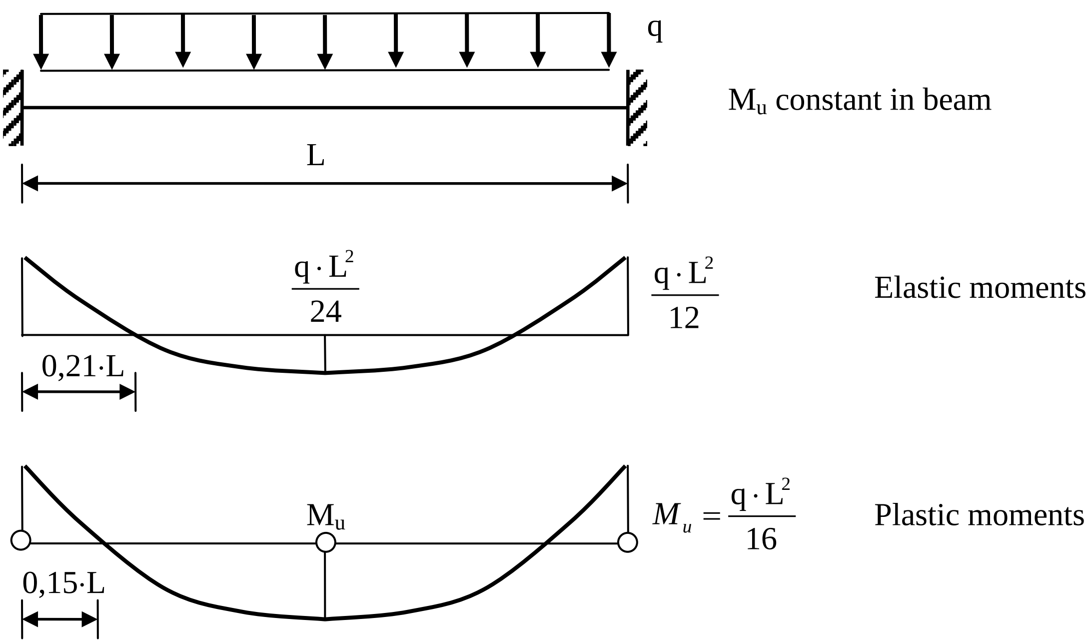


Figure 1.4.1 Moment redistribution for a fixed beam

In a concrete beam the bending strength is different at the support and in the span due to different reinforcement amounts. If the bending strengths are designed by using elastic moments, there is no increased capacity with a plastic analysis for a one-span beam since the plastic hinges at the support and in the span forms at the same load level. Still the concept with plastic hinges is useful in design of concrete structures. At supports the elastic bending moments are locally very high. By allowing plastic hinges to form at the support at a load level lower than the corresponding elastic distribution of moments, the support moment can be reduced. The reduced support moment requires an increased span moment to maintain static equilibrium of the structure. This is called redistribution of moments. In order to achieve redistribution, the cross-section must be designed so that plastic hinges can form, which require the reinforcement to yield. Thus, the design must be based on what is called under-reinforced. Figure 1.4.2 shows the typical moment-curvature diagram for an under-reinforced cross-section. After yielding of the reinforcement, the response is plastic until the concrete fails in compression. The result is a ductile structure which has a gradual failure in ultimate limit state. Thus, it is the rotation capacity of the hinge which governs the capacity. A long plateau after M_u is reached implies a large rotation capacity.

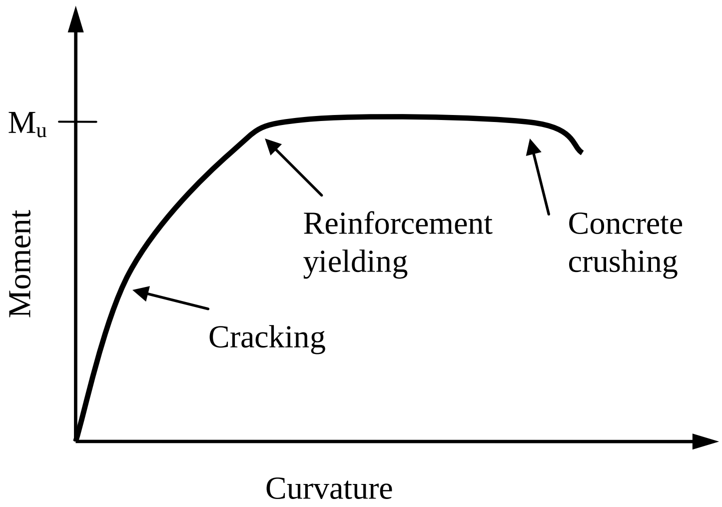


Figure 1.4.2   Typical moment-curvature diagram for reinforced concrete

In practice it can be possible to reduce supports moment in continuous beams without increasing the span moments due to the load combinations, as seen in Figure 1.4.3. This depends on the ratio between permanent and live loads. Maximum support moment arises from live load in both spans adjacent to the support, and maximum span moment arises from live load in one span only.

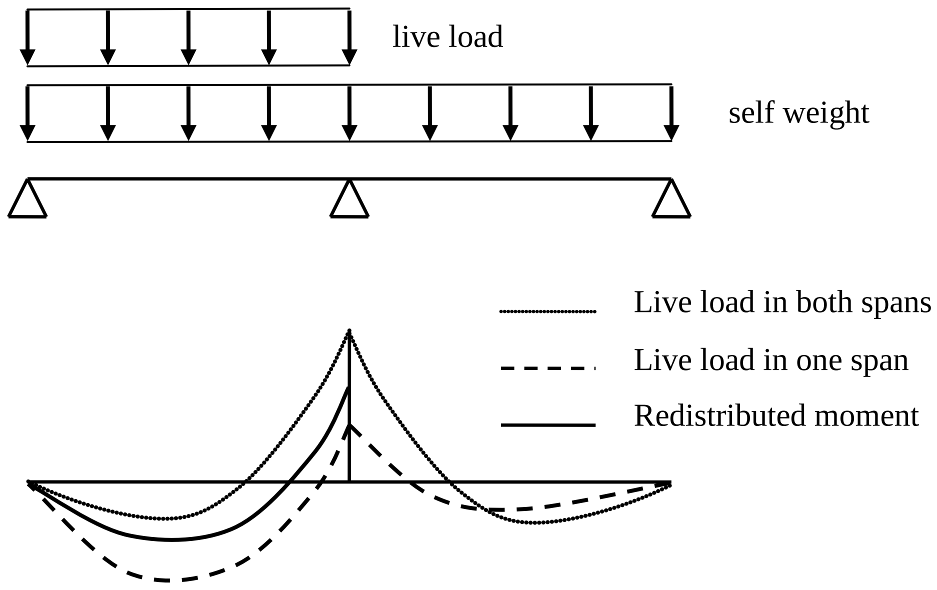


Figure 1.4.3 Moment distribution with redistribution for a two span beam

In EC2 there is a simplified approach where redistribution of bending moments can be carried out without explicit check of the rotation capacity for continuous beams and slabs. Provided that the structure is predominately subjected to flexure and the ratio of the lengths of adjacent spans are between 0,5 and 2, the redistribution can be calculated for $f_{ck} \leq 50$ MPa and Class B and Class C reinforcement as:

$$\delta \geq k_1 + k_2 \cdot \frac{x_u}{d} = 0,44 + 1,25\left(0,6 + 0,0014/\varepsilon_{cu2}\right) \cdot \frac{x_u}{d} \tag{1.4.1}$$

$$\delta \geq k_5 = 0,7$$

$$\delta = \frac{\text{redistributed moment}}{\text{elastic moment}}$$

where x_u is the depth of the compression zone at the ultimate limit state after redistribution, and ε_{cu2} the ultimate compressive strain. k_1, k_2 and k_5 are Nationally Dependent Parameters in EC2 and taken from the Norwegian Annex. From Eq. (1.4.1) it can be seen that at most 30% of a bending moment can be redistributed without check of the rotation capacity. The limitation on the compressive zone ensures sufficient ductility to be able to redistribute moments. In addition it helps the performance in serviceability limit state where the response may be close to un-cracked elastic analysis. For different compression zones the maximum allowable redistribution is given in Figure 1.4.4. For maximum redistribution of 30% the limit of x_u/d=0,21, while x_u/d=0,44 permits no redistribution.

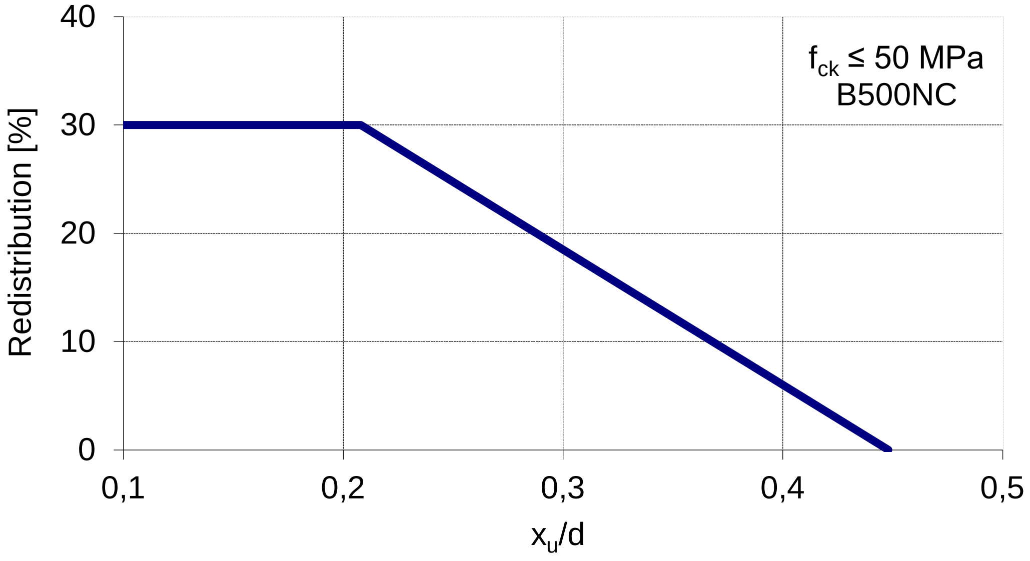


Figure 1.4.4 Limitations on compression zone when employing redistribution

The simplified approach is a design method for bending moments in ULS for continuous beams and slabs. The design of columns must be carried out based on the elastic moments from frame actions without any redistribution. For SLS verifications of beams and slabs the moments must not be redistributed.

Redistribution of support moments does not necessary reduce the total amount of flexural reinforcement. Due to restriction on the depth of the compression zone, additional compression steel may be required to maintain sufficient bending strength. The main advantages of employing redistribution are:

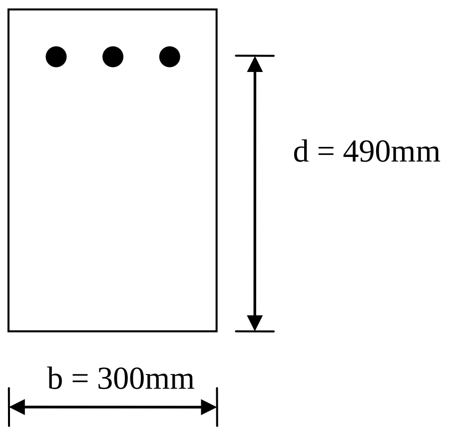
- Lower cross-section depths because the bending moments are smaller.
- With less flexural steel it is easier to fulfil requirements for bar spacing such that concrete can be properly compacted.

Drawbacks that may restrict the use of redistributing moments are:

- Reduced shear resistance due to less longitudinal reinforcement and lower cross-sections. This also requires more shear reinforcement.
- In SLS the deflections and cracking may increase in the span as the continuity at the internal supports is less effective.

Example

The cross section in Figure 1.4.5 is subjected to a bending moment at the support of a continuous beam of M_{Ed} = 288 kNm. Assuming 20% redistribution, it is possible to employ the simplified approach, Eq. (1.4.1), to calculate moment resistance and required longitudinal reinforcement.



B25→ $\varepsilon_{cu2}=3{,}5 \cdot 10^{-3}$
B500NC
M_{Ed}=288 kNm

Figure 1.4.5 Design example with moment redistribution

$$\delta = 0{,}8 \Rightarrow M_{Ed}^{red} = 0{,}8 \cdot 288 = 230 \text{ kNm}$$

$$\delta = 0{,}44 + 1{,}25(0{,}6 + 0{,}0014 / \varepsilon_{cu2}) \cdot x_u / d = 0{,}8$$

$$x_u = 141 \text{ mm}; \quad \Rightarrow \alpha = \frac{x_u}{d} = \frac{141}{490} = 0{,}288$$

$$M_{Rd} = 0{,}8\alpha(1 - 0{,}4\alpha) f_{cd} b d^2 = 0{,}8 \cdot 0{,}288 \cdot (1 - 0{,}4 \cdot 0{,}288) \cdot 14{,}2 \cdot 260 \cdot 490^2 = \underline{\underline{181 \text{ kNm}}}$$

M_{Ed} = 230 kNm > M_{Rd} → compression steel is required.

Compression reinforcement together with redistribution of moments is often a good solution to reduce cross-section heights. Support moments are very local and should not dictate the dimension of cross-section for the entire length of a beam. However, if compression steel is not applicable it is possible to use Eq. (1.4.1) to find how much of the moment can be redistributed without requiring compression steel. In Figure 1.4.6 the obtained design moment and resistance moment for different distribution factors are plotted. As seen approximately 8% of the bending moment can be redistributed in this example without utilizing compression reinforcement.

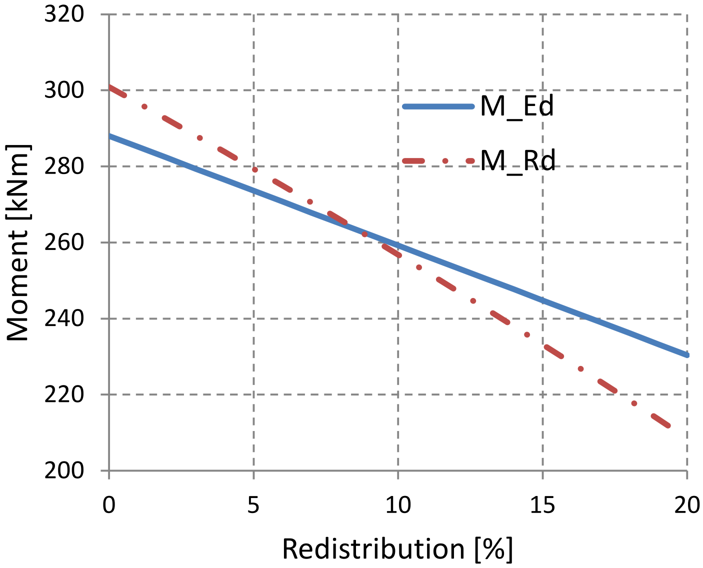


Figure 1.4.6 Design and resistance moment for different redistribution

1.5 References

/1.1/ Standard Norge, *NS-EN 1992-1-1:2004+NA:2008, Eurocode 2: Design of concrete structures. Part 1-1: General rules and rules for buildings.* 2008.

/1.2/ Standard Norge, *NS-EN 1993-1-1:2005+NA:2008, Eurocode 3: Design of steel structures. Part 1-1: General rules and rules for buildings.* 2008.

/1.3/ Standard Norge, *NS-EN 13670:2009+NA:2010, Execution of concrete structures.* 2010.

/1.4/ C. R. Hendy and D. A Smith, *Designers' guide to EN 1992-2*, Thomas Telford, 2007.

/1.5/ R. S Narayanan and A. Beeby, *Designers' guide to EN 1992-1-1 and EN 1992-1-2*, Thomas Telford, 2005.

CHAPTER 2

DESIGN OF CONCRETE DEEP BEAMS AND WALLS

Svein I Sørensen

2.1 Analysis and design of wall beams based on strut-and-tie models

2.1.1 Discontinuity regions

Strut-and-tie models can be used for analysis and design of so-called discontinuity regions, or ”D-regions”.

Deep beams or wall beams are examples of such D-regions, where Bernoulli’s hypothesis that plane sections remain plane and perpendicular to the beam axis in bending, and that elementary beam theory with linear bending stress distribution according to Navier’s formula are not valid.

This is shown in Figure 2.1.1 for a simply supported wall beam with span equal the wall depth and uniformly distributed load q at the top. The figure shows results from an analysis using two-dimensional theory of elasticity – bending stress distribution (σ_x) at midspan and principal stress trajectories (compression and tension).
The figure also shows how the load in principle can be carried by a simple strut and tie model with compression struts (dashed line) in concrete and a tension tie in the reinforcement, following the principal compression and tension trajectories.

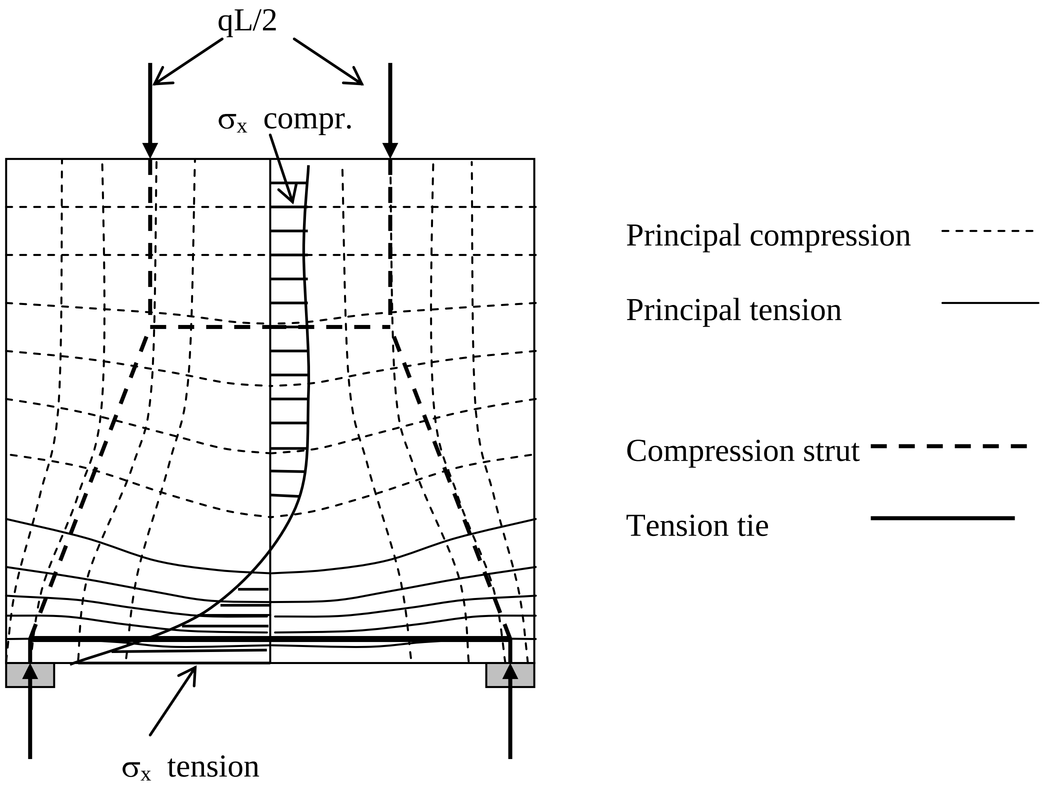


Figure 2.1.1 Strut-and-tie model for simply supported wall beam

In Eurocode 2 (EC2) /2.1.1/ strut-and-tie models are described in clauses 5.6.4 and 6.5.

EC2, 5.6.4:
(1) Strut-and-tie models may be used for design in ULS of continuity regions (cracked state of beams and slabs, see 6.1 – 6.4), and for the design in ULS and detailing of discontinuity regions (see 6.5). In general these extend up to a distance h (section depth of member) from the discontinuity. Figure 2.1.2 shows examples of D-regions.

(2) Verifications in SLS may also be carried out using strut-and-tie models, e.g. verification of steel stresses and crack width control, if approximate compatibility for strut and tie models is ensured (in particular the position and direction of important struts should be oriented according to linear elastic theory).

(3) Strut-and-tie models consist of struts representing compressive stress fields, of ties representing the reinforcement, and of connecting nodes. The forces in the elements of a strut-and-tie model should be determined by maintaining the equilibrium with the applied loads in ULS. The elements of strut-and-tie models should be dimensioned according to rules given in 6.5.

(4) The ties og a strut-and-tie model should coincide in position and direction with the corresponding reinforcement.

(5) Possible means for developing suitable strut-and-tie models include the adoption of stress trajectories and distributions from linear elastic theory or the load path method.
All strut-and-tie models may be optimised by energy criteria.

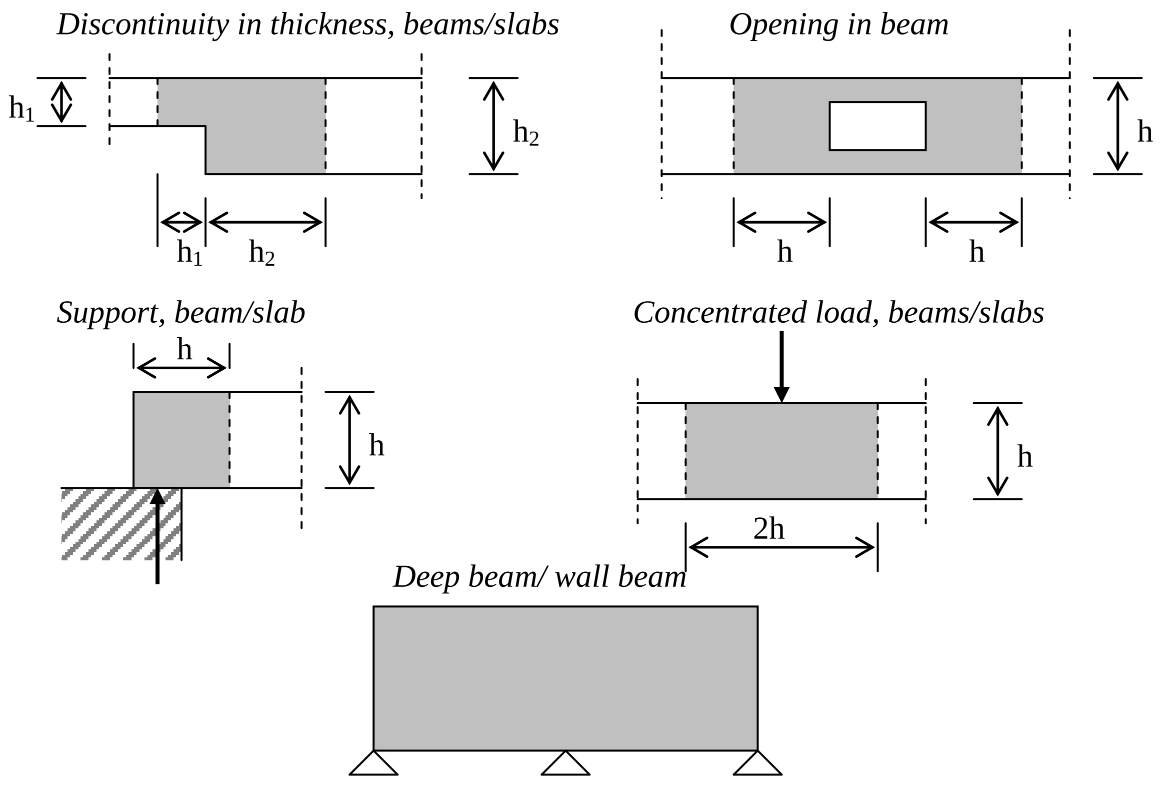


Figure 2.1.2 D-region examples

2.1.2 Simply supported wall beams

Here, only wall beams will be considered since other D-regions, e.g. nodes in precast structures, are covered in the course Concrete structures 2.

Figure 2.1.3 shows stress distributions, stress resultants and internal lever arms at midspan for simply supported wall beams loaded at the upper edge, for various span/depth-ratios L/h = 4, 2, 1 and < 1.

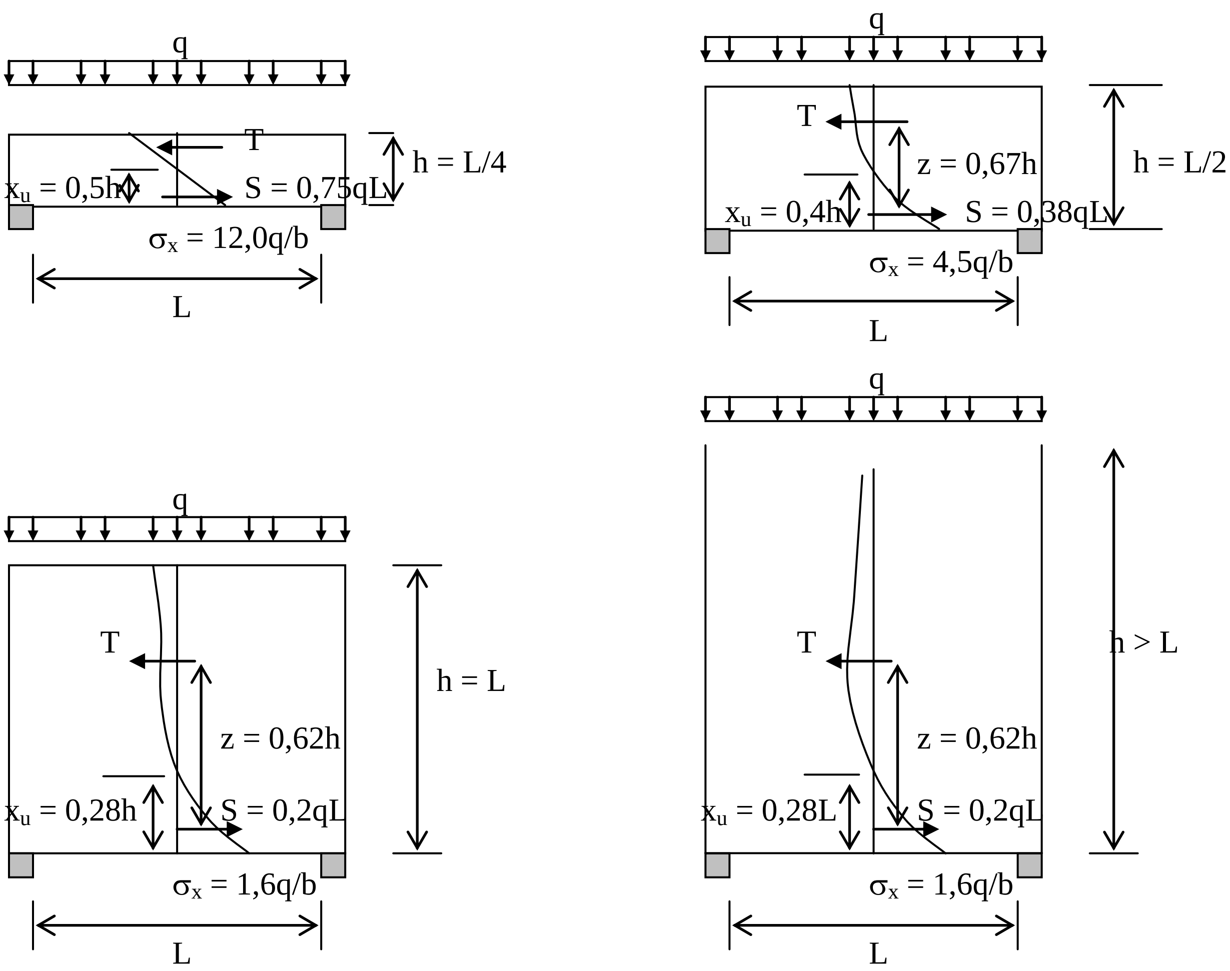
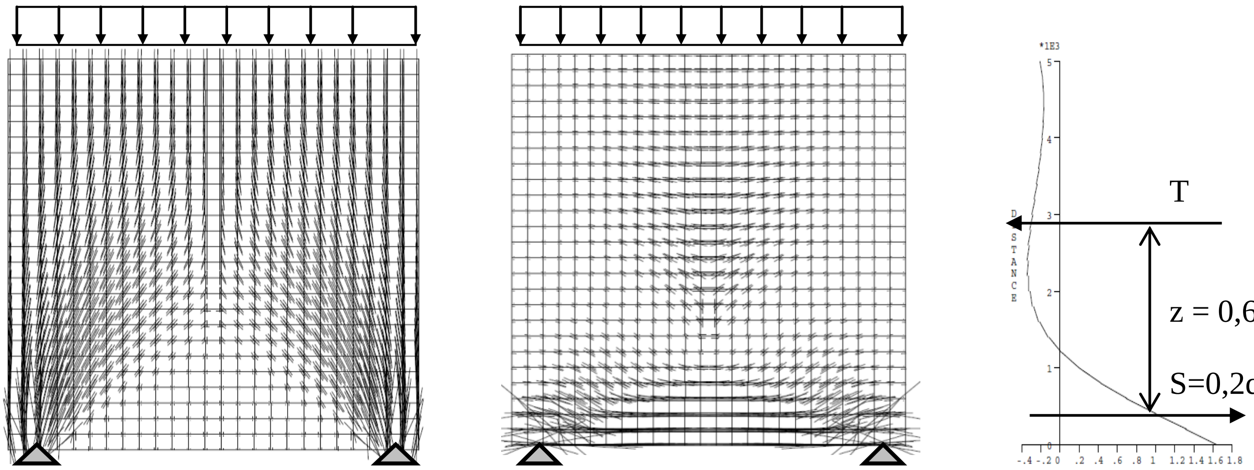


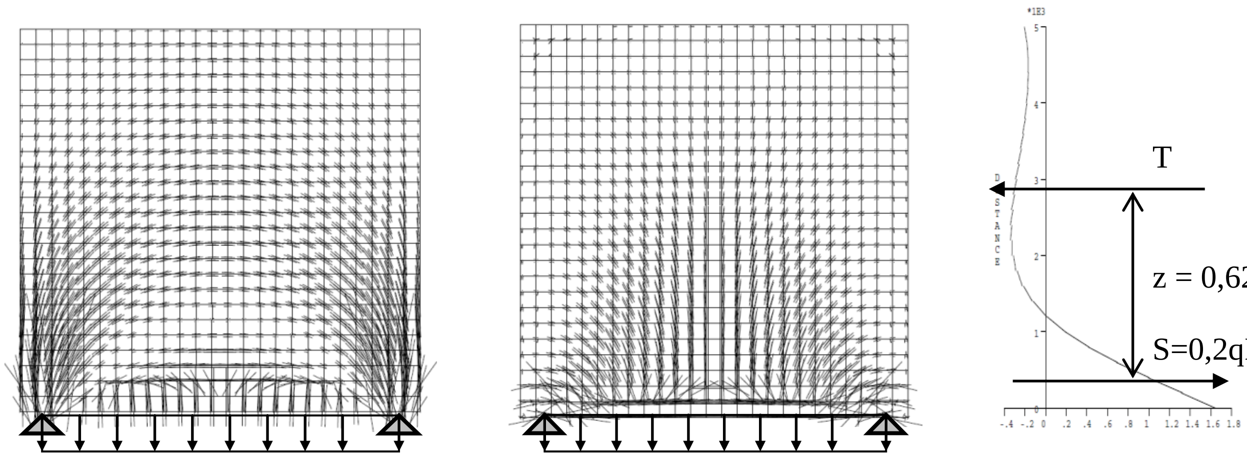
Figure 2.1.3 Stresses, forces and internal lever arm at midspan of simply supported wall beams. Uniformly distributed load at upper edge.

If the load is acting at the lower edge, stress distribution, forces and lever arms are similar to the results in Figure 2.1.3, but the principal stress distributions are different for the two load situations.

This is illustrated by results from FEM-analyses using the program DIANA /2.1.2/ in Figure 2.1.4.



Principal compressive stress Principal tensile stress Stresses at midspan



Principal compressive stress Principal tensile stress Stresses at midspan

Figure 2.1.4 Simply supported wall beams loaded at upper and lower edge. Principal stresses and stress distribution at midspan.

For both load cases the horizontal tension resultants (S) are equal. Hence, the required reinforcement is equal.

The strut-and tie models for the two cases will be different. For load at upper edge the actual strut-and –tie model is shown in Figure 2.1.1, while for load at lower edge the load has to be transferred to a compression arc by vertical tension ties, requiring vertical reinforcement from the lower edge.

Both strut-and-tie models are shown in Figure 2.1.5. Calculated results for the two models can be superposed.

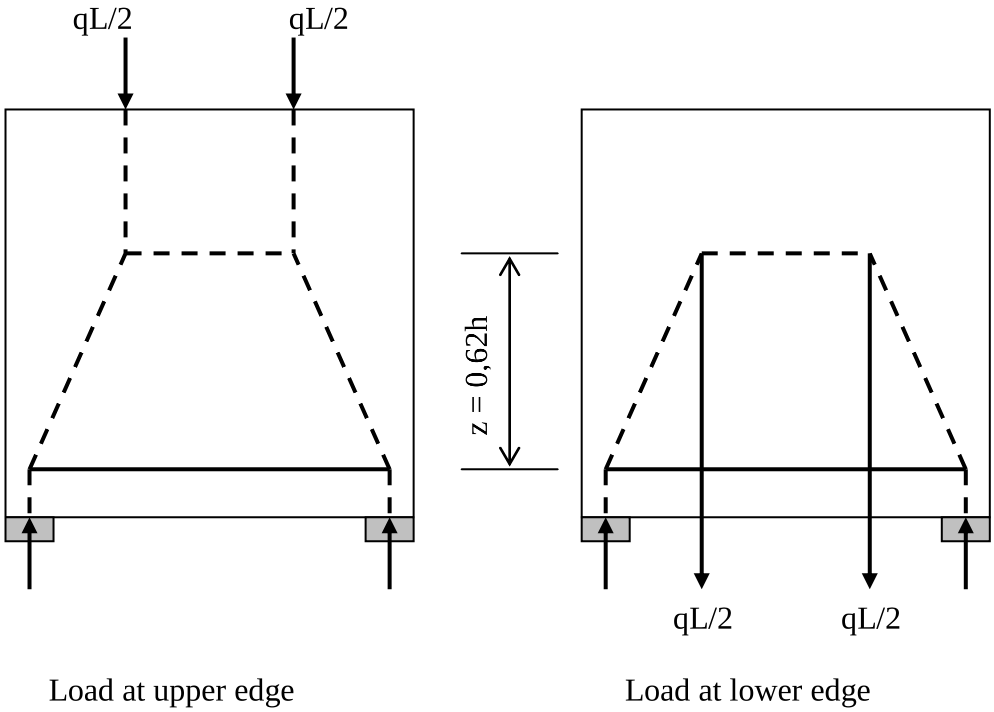


Figure 2.1.5 Strut-and tie models for different load cases

Rules for dimensioning by strut-and tie models are given in EC2, 6.5.

<u>Struts :</u>
The design strength for a concrete strut in a region with transverse compressive stress or no transverse stress is, according to EC2, 6.5.2(1):

$$\sigma_{Rd,max} = f_{cd} \tag{2.1.1}$$

For concrete struts in cracked compression fields a reduced design strength according to EC2, 6.5.2(2) may be used:

$$\sigma_{Rd,max} = 0,6\nu' f_{cd} \tag{2.1.2}$$

where

$$\nu' = 1 - f_{ck}/250 \tag{2.1.3}$$

<u>Ties :</u>
Force in horizontal reinforcement may be determined as

$$S_h = \frac{M_{Ed}}{z} \tag{2.1.4}$$

where M_{Ed} is the maximum moment in the span and z is the internal lever arm.

For a quadratic wall beam (h = L) is

$$S_h = \frac{qL^2/8}{0,62L} = 0,2qL \tag{2.1.5}$$

which corresponds to the result shown in Figure 2.1.3.

The required cross section area of the horizontal reinforcement is

$$A_{sh} = \frac{S_h}{f_{yd}} \tag{2.1.6}$$

The centre of gravity of the reinforcement should be located in the lower third of the tension zone according to elasticity theory.

For load at lower edge the total force in the vertical reinforcement is

$$S_v = qL \tag{2.1.7}$$

The required cross section area of the vertical reinforcement is

$$A_{sv} = \frac{S_v}{f_{yd}} \tag{2.1.8}$$

This vertical reinforcement is distributed along the entire span.

<u>Nodes :</u>
EC2, 6.5.4(2)P :
The forces acting at nodes shall be in equilibrium (of course). Transverse tensile forces perpendicular to an in-plane node shall be considered.

EC2, NA.6.5.4(4) :
Design strength at nodes is

a) In compression nodes where no ties are anchored at the node

$$\sigma_{Rd,max} = \nu' f_{cd} \tag{2.1.9}$$

b) In compression-tension nodes with anchored ties in one direction

$$\sigma_{Rd,max} = 0,85\nu' f_{cd} \tag{2.1.10}$$

c) In compression-tension nodes with anchored ties in more than one direction

$$\sigma_{Rd,max} = 0,75\nu' f_{cd} \tag{2.1.11}$$

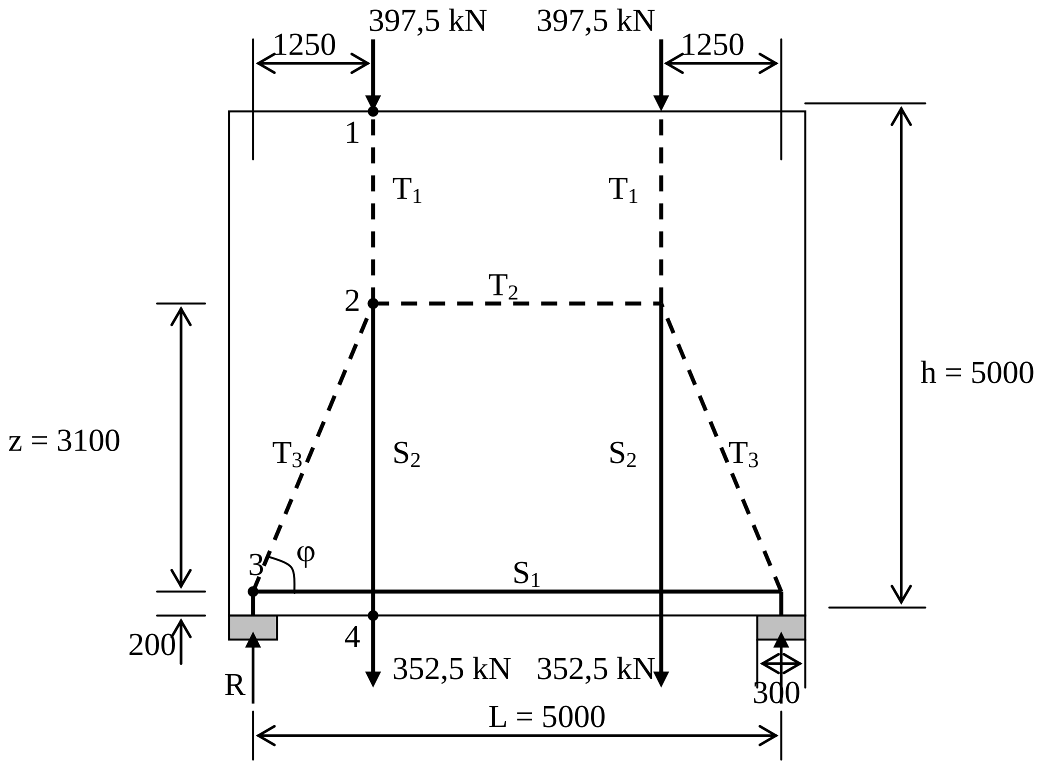
2.1.3 Example - Design of simply supported wall beam

Given a wall beam with uniformly distributed load in ULS at upper and lower edge, q_{Ed} = 150 kN/m.
The actual strut-and-tie model is a combination of the models in Figure 2.1.3.
The wall thickness is 250 mm, and the support length is 300mm.

Materials: Concrete B30 ; $f_{cd} = \frac{0,85}{1,5} \cdot 30 = 17 \mathrm{N/mm^2}$

Reinforcement B500NC; $f_{yd} = \frac{500}{1,15} = 434 \mathrm{N/mm^2}$

Distribution of horizontal reinforcement is chosen over 400mm from the lower edge, and the internal lever arm is z = 0,62h = 3100mm.



The inclination angle of struts T_3 is: $\varphi = \arctan(\frac{3100}{1250}) = 68^\circ$

Concentrated loads at upper edge: $\frac{q_{Ed} L_{total}}{2} = \frac{150 \cdot 5,3}{2} = 397,5 \mathrm{kN}$

Concentrated loads at lower edge: $\frac{q_{Ed}(L-0,3)}{2} = \frac{150 \cdot 4,7}{2} = 352,5 \mathrm{kN}$

Strut and tie forced determined by equilibrium conditions:

<u>Node 3:</u>

Support reaction : $R = \frac{2qL_{total}}{2} = 150 \cdot 5,3 = 795\text{kN}$

Strut 3 : $T_3 = \frac{R}{\sin\varphi} = \frac{795}{\sin 68} = 857,4\text{kN}$

Tie 1 : $S_1 = T_3 \cos\varphi = 857,4 \cdot \cos 68 = 321,2\text{kN}$

($0,2 \cdot 2q \cdot L_{total} = 318\text{kN}$, according to Figure 2.1.3)

<u>Node 4 :</u>

Tie 2 : $S_2 = \frac{q(L-0,3)}{2} = \frac{150 \cdot 4,7}{2} = 352,5\text{kN}$

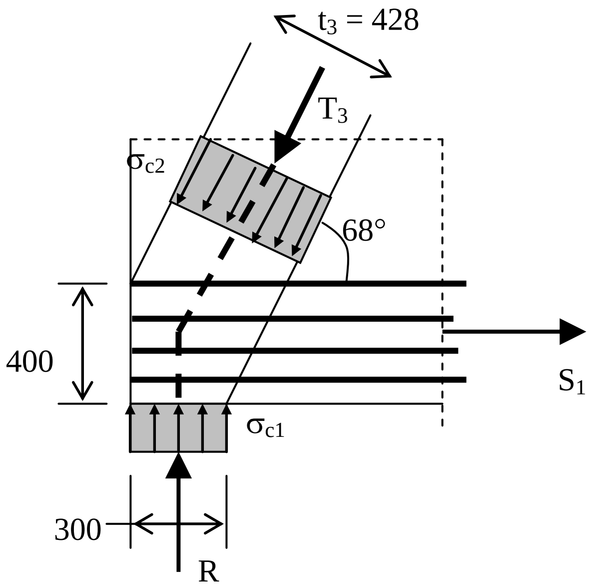
Further is $T_1 = 397,5\text{kN}$ and $T_2 = S_1 = 321,2\text{kN}$

Required reinforcement:

Horizontally: $A_{sh} = \frac{S_1}{f_{yd}} = \frac{321200}{434} = 740\text{mm}^2$. Choose 4ø16→ A_{sh} = 804mm^2

Vertically: $A_{sv} = \frac{2S_2}{f_{yd}} = \frac{2 \cdot 352500}{434} = 1624\text{mm}^2$. ø12s325 along 4,7m → A_{sv}=1636mm^2

Control of node 3 at the support:



Width of strut T_3 :

$$\frac{t_1}{300} = \sin 68 \Rightarrow t_1 = 278{,}2\text{mm}$$

$$\frac{t_2}{400} = \cos 68 \Rightarrow t_2 = 149{,}8\text{mm}$$

$$t = t_1 + t_2 = 428\text{mm}$$

Reduced design strength of concrete, reinforcement anchored in one direction, according to Eq. (2.1.10) :

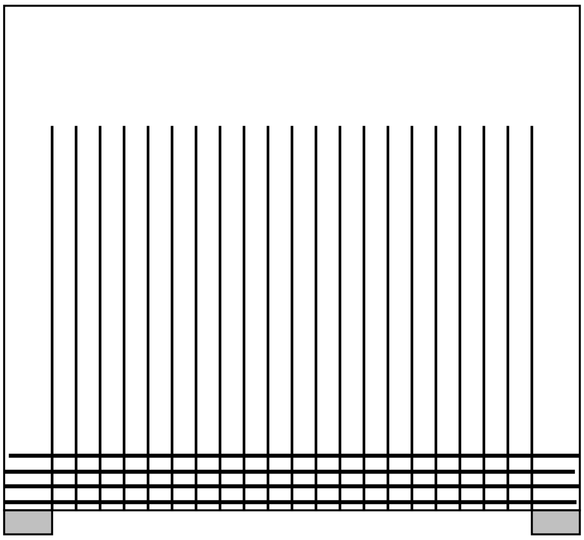
$$\sigma_{Rd,max} = 0{,}85 \cdot (1 - \frac{30}{250}) \cdot 17 = 12{,}7\text{N}/\text{mm}^2$$

Concrete compressive stresses:

Strut T_3 : $\sigma_{c2} = \frac{T_3}{bt} = \frac{857400}{250 \cdot 428} = 8{,}0\text{N}/\text{mm}^2 < \sigma_{Rd,max}$ → OK!

Support: $\sigma_{c1} = \frac{R}{b \cdot 300} = \frac{795000}{250 \cdot 300} = 10{,}6\text{N}/\text{mm}^2 < \sigma_{Rd,max}$ → OK!

The results of the strut-and-tie calculations are shown in the figure:



Horizontal reinforcement, bottom
4ϕ16 → A_{sh} = 804 mm^2

Vertical reinf., distributed
ϕ12c325 → A_{sv} = 348 mm^2/m

Max compressive stress in concrete
$\sigma_{c,max}$ = 10,6 N/mm^2

Further, a nonlinear FEM-analysis with DIANA is carried out to check the capacity of the wall beam.
Figure 2.1.6 shows the finite element model with pinned support.

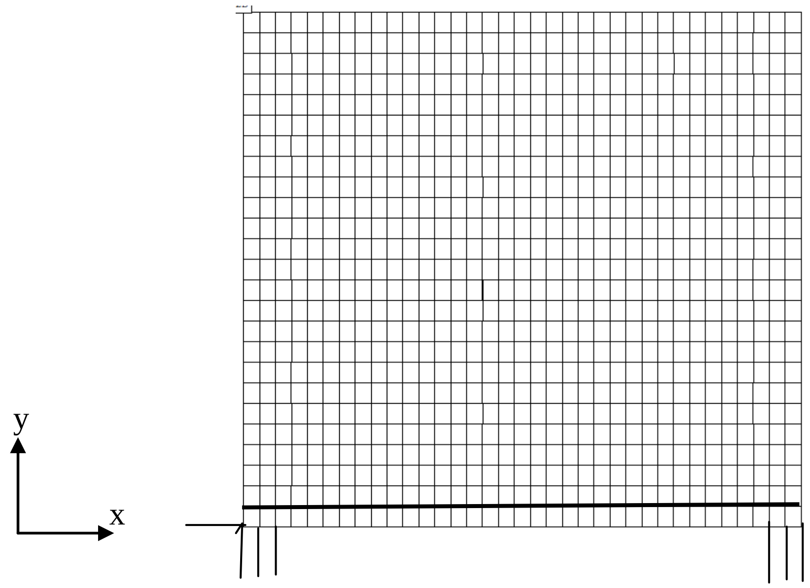


Figure 2.1.6 Finite element model for analysis of wall beam

The horizontal reinforcement is defined by ”reinforcement bar” as shown in the figure, while the vertical reinforcement is define by ”reinforcement grid” vertically in the entire wall.

Material data:

Reinforcement elastic-ideal plastic with E_s = 200000 N/mm^2 and yield strength 434 N/mm^2.

The assumed stress-strain behaviour of concrete is shown in Figure 2.1.7.

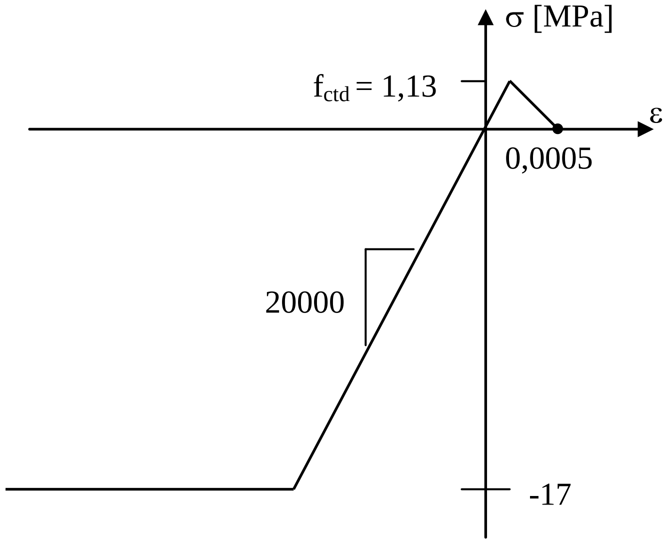


Figure 2.1.7 Stress-strain curve for concrete in the DIANA-analysis

The reference load q_{Ed} = 150 kN/m is applied at upper and lower edge.
The analysis is performed using rotating crack model for concrete, successive load incrementation and a numerical solution technique with arc length control.

Figure 2.1.8 shows the load-deflection curve at midspan at the lower edge.

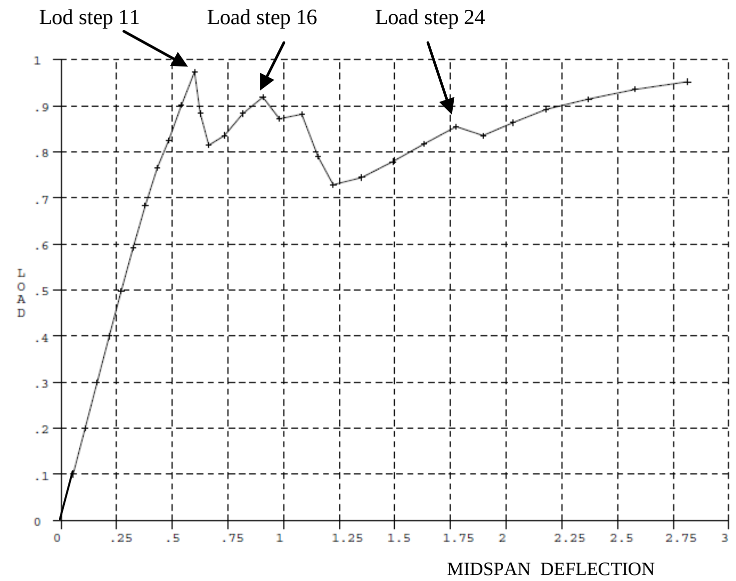
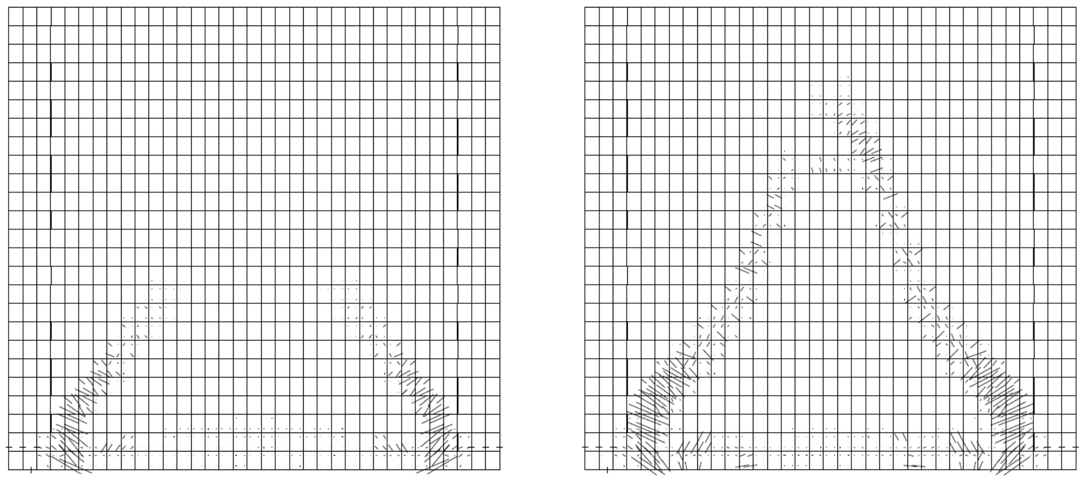


Figure 2.1.8 Load-deflection at midspan from DIANA-analysis

The ordinate axis ”load” gives a load factor, where 1,0 corresponds to the reference load (150 kN/m). The results indicate a load capacity of the wall beam approximately as $0{,}97q_{Ed}$, or ca 145 kN/m, hence closed to the design load used in the strut-and-tie calculations.

By studying the DIANA results for the first peak of the load-deflection curve (load step 11), it is obvious that the reinforcement stresses are well below the design strength, while maximum compressive stress at the supports is 10,2 N/mm^2, i.e. approximately the same as calculated in the strut-and-tie model. The concrete design strength is 17 N/mm^2, hence neither reinforcement nor concrete stresses should cause capacity limitation shown in the computed load-deflection diagram.

A possible reason for failure may be formation of shear cracks.
Figure 2.1.9 shows strains perpendicular to cracks at load steps 16 and 24 (see Figure 2.1.8), and illustrates clearly how skew shear cracks develop. At both these load steps the stress in the vertical reinforcement has reached the design strength in the cracks.



Load step 16 Load step 24

Figure 2.1.9 Strains perpendicular to cracks showing localised shear cracking

Minimum reinforcement in walls

This kind of unintended failure in wall beams can be avoided by reinforcement grids that satisfy minimum requirements for vertical and horizontal reinforcement.

Rules for this are given in EC2, 9.6 and NA.

EC2, NA.9.6.2 specifies minimum required vertical reinforcement: $A_{s,vmin} = 0{,}002A_c$, and from EC2, NA.9.6.3 minimum horizontal reinforcement as $A_{s,hmin} = \max\{0{,}25A_{sv}\ ;\ 0{,}3A_c f_{ctm}/f_{yk}\}$.

This means for the example:

$$A_{s,v\,min} = 0{,}002 \cdot 250 \cdot 1000 = 500\text{mm}^2/\text{m}$$

$$A_{s,h\,min} = \max\{0{,}25 \cdot 500; 0{,}3 \cdot 250000 \cdot 2{,}9/500\} = 435\text{mm}^2/\text{m}$$

With ”reinforcement grid” in the DIANA-example with these reinforcement cross sections, the computed load-deflection curve is shown in Figure 2.1.10.
The load capacity is found at load step 13 with a load factor 1,13, i.e. a load approximately 170 kN/m.

The load-deflection curve can be followed up to load step 30 with a load factor ca 1,28 (q = 190), but his is fictitious, since the convergence criteria are not satisfied beyond load step 13. Hence, the correct load capacity is 170 kN/m, and design by the strut-and-tie model is to the safe side with the minimum reinforcement grid.

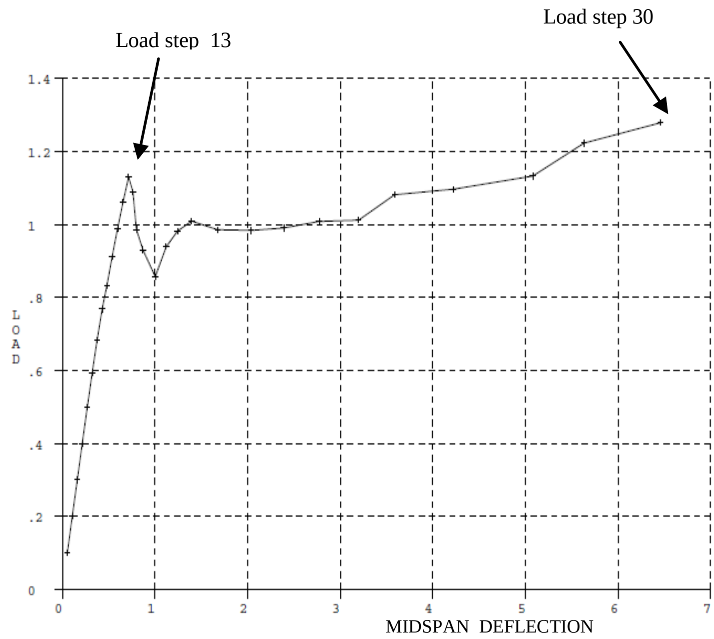


Figure 2.1.10 Load-deflection at midspan from DIANA-analysis with minimum wall reinforcement according to EC2

2.1.4 Continuous wall beams

In order to illustrate the behaviour of continuous wall beams, consider the two-span wall beam in Figure 2.1.11.

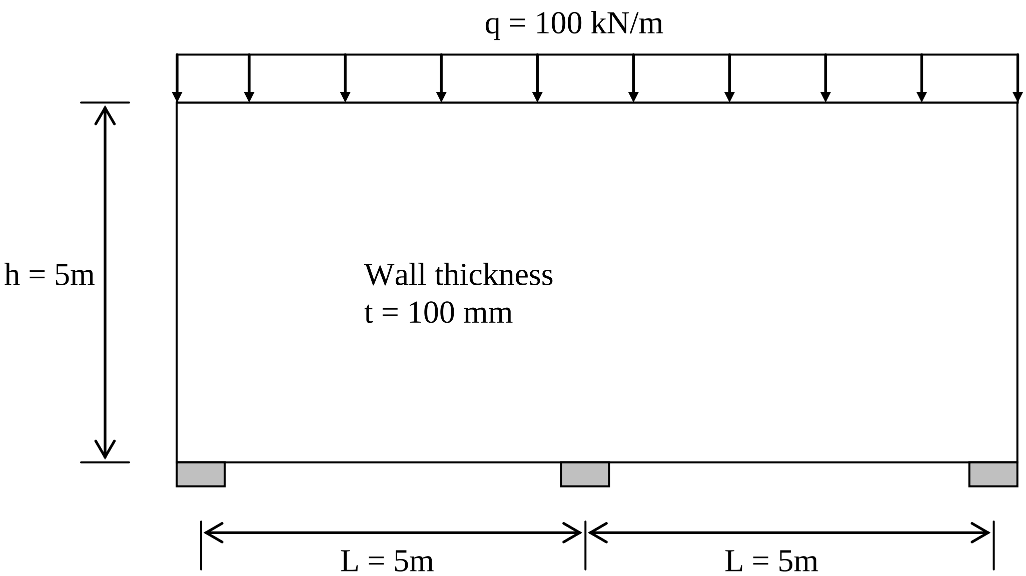
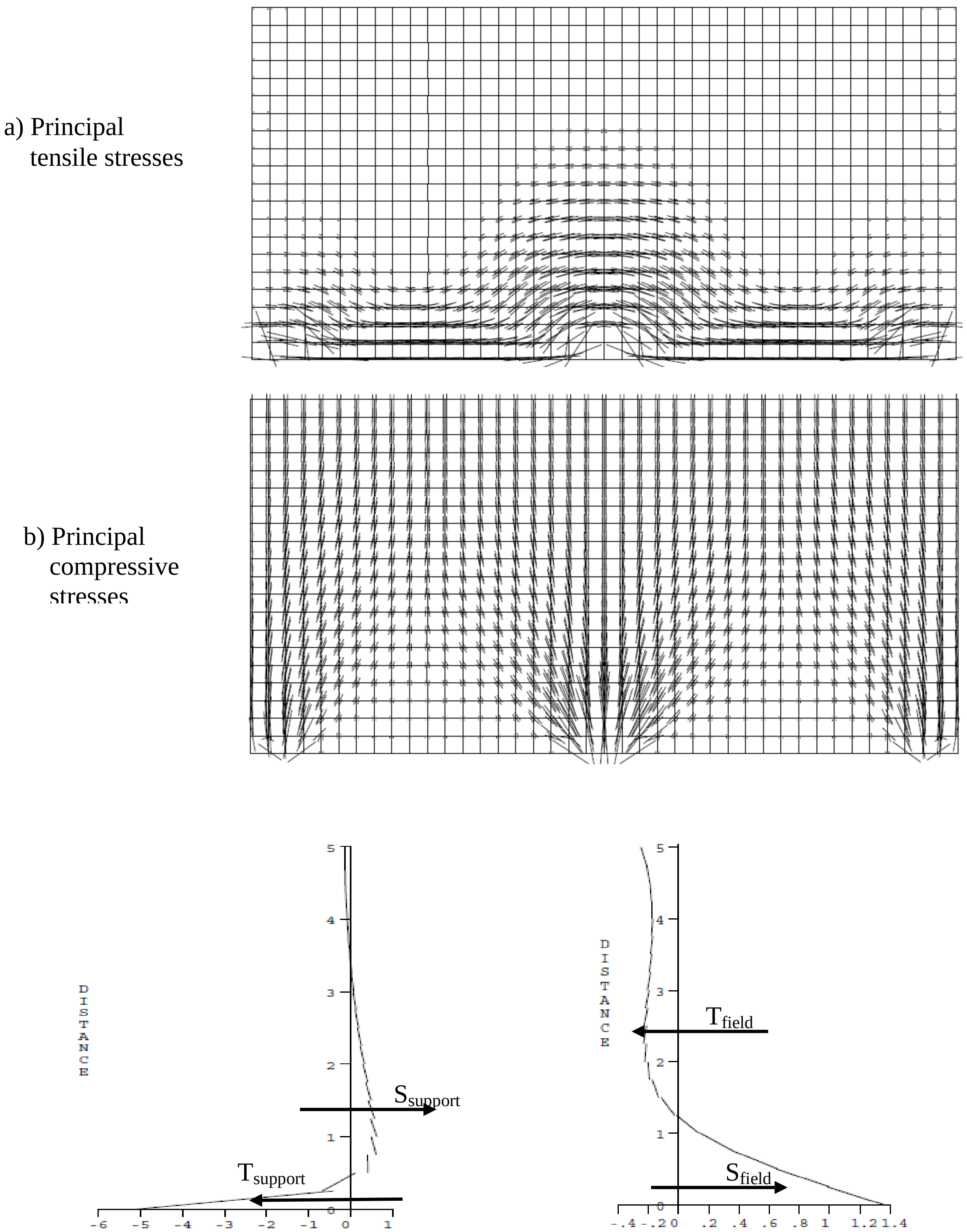


Figure 2.1.11 Two-span wall beam loaded at upper edge

Figure 2.1.12 shows results from a linear DIANA-analysis for the two-span wall beam.



c) Stress at mid-support

d) Stress at midspan

Figure 2.1.12 Results from linear DIANA-analysis

Based on the principal stress distribution and the stress resultants in Figure 2.1.12, a possible simplified strut-and-tie model is shown in Figure 2.1.13.

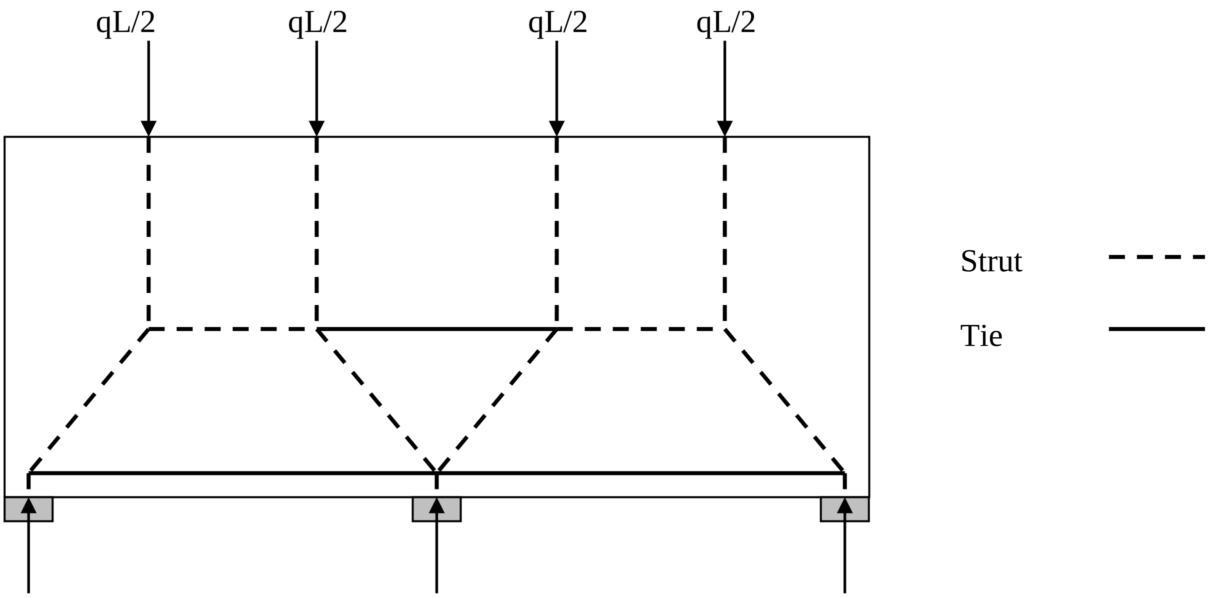


Figure 2.1.13 Simplified strut-and-tie model for continuous wall beam

Strut and tie forces are determined by equilibrium conditions as in example 2.1.1, and concrete compressive stress is controlled at the nodes.
In addition minimum wall reinforcement grid according to EC2 has to be included.

The practical reinforcement layout will be half of the reinforcement at each wall surface, i.e. double reinforcement layer.

Figure 2.1.14 shows an example of reinforcement layout in a two-span wall beam.
The figure is taken from refs. /2.1.3/ and /2.1.4/. In these references one may find more about strut-and-tie models for wall beams, and references to other literature on the subject.

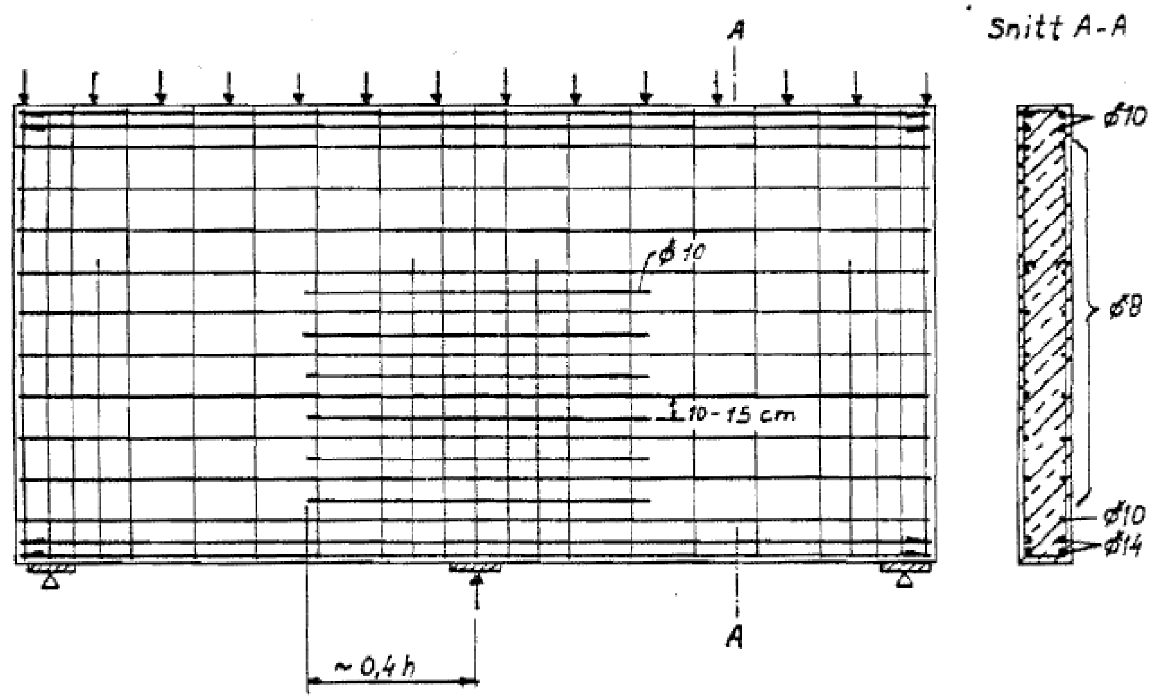


Figure 2.1.14 Example of reinforcement layout in a two-span wall beam from refs. /2.1.3/ and /2.1.4/

2.1.5 References

/2.1.1/ NS-EN 1992-1-1:2004+NA:2008 (Eurocode 2)

/2.1.2/ DIANA Finite Element Analysis, TNO Delft.

/2.1.3/ Hagberg, Thore: Dimensjonering av veggskiver i bruddgrensetilstanden, NIF-kurs, januar 1990, NTH. (in Norwegian)

/2.1.4/ Leonhardt, F.: Vorlesungen über Massivbau, Springer Verlag 1975. (in German)

2.2 Analysis and design of deep beams based on compression field theory

2.2.1 Fundamentals

By linear elastic finite element analyses of deep beams (wall beams), the in-plane orthogonal stresses σ_x , σ_y , τ_{xy} are determined. The stress resultants (in-plane forces) N_x , N_y og N_{xy} are obtained by multiplying with the beam thickness t .

Actual problems in the Ultimate Limit State (ULS):

1) Determine required beam thickness and reinforcement for given N_x , N_y og N_{xy} .
2) Check capacity of a reinforced deep beam for given N_x , N_y og N_{xy} .

Theoretically, it is possible that the reinforcement layout mirrors the principal force trajectories N_1 and N_2, and hence all the reinforcement is fully utilized when the concrete cracks. However, generally the principal forces vary in both magnitude and direction in the beam, which makes a principal force trajectories reinforcement layout impossible.

Here, only orthogonal reinforcement meshes will be considered.

In ULS design it has to be decided whether or not the tensile strength of the concrete should be included. The tensile strength may be reduced due to several causes:

1) Tensile stresses due to restraining of temperature and/or shrinkage strains may result in cracking.
2) The tensile strength is reduced due to permanent and repeated loads.
3) Beam tests indicate no significant influence to the failure load from the concrete in tension.

The tensile strength is therefore assumed to be zero in the ULS design.
This assumption will, as will be shown later, result in a simple design procedure with closed solutions for the unknown quantities.

Based on this assumption design methods proposed by Baumann /2.1.1/ and Hagberg /2.1.2/ and /2.1.3/ will be presented in the following.

2.2.2 Forces in concrete and reinforcement

Equilibrium equations for a cracked, reinforced element give relations between external forces in ULS N_x, N_y, N_{xy} and internal forces in reinforcement and concrete F_{sx}, F_{sy} og F_c. Figure 2.2.1 shows external forces from an elasticity theory analysis (normally from a finite element analysis) and internal forces in a cracked element.

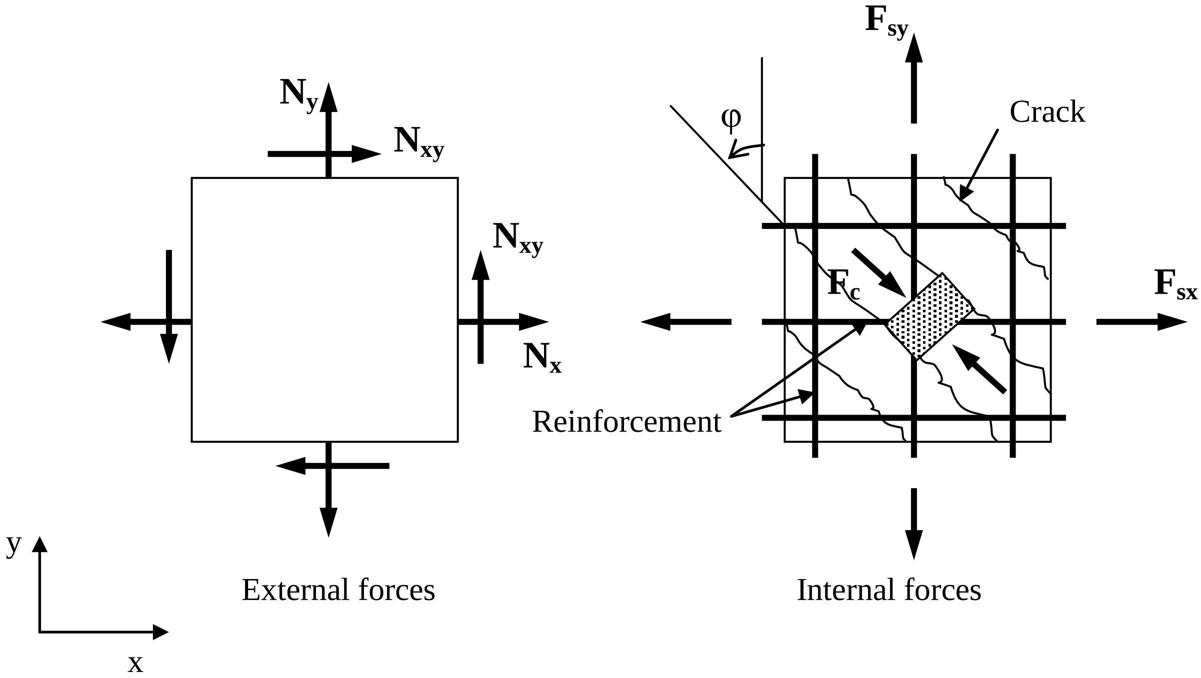


Figure 2.2.1 External and internal forces in a cracked element

With wall thickness t and reinforcement per unit length in x- and y-directions respectively, A_{sx}, A_{sy}, the internal forces can be expressed by the stresses in concrete and reinforcement as:

Force in the concrete compression field: $F_c = \sigma_c \cdot t$ (positive in compression)

Forces in reinforcement:

$$\left.\begin{aligned} F_{sx} &= \sigma_{sx} \cdot A_{sx} \\ F_{sy} &= \sigma_{sy} \cdot A_{sy} \end{aligned}\right\} \text{(positive in tension)}$$

The angle between cracks and the y-axis is φ as shown in the figure, and the concrete between the cracks represents the so-called compression field.

Figure 2.2.2 shows a section parallel to the y- and x-axes at a crack

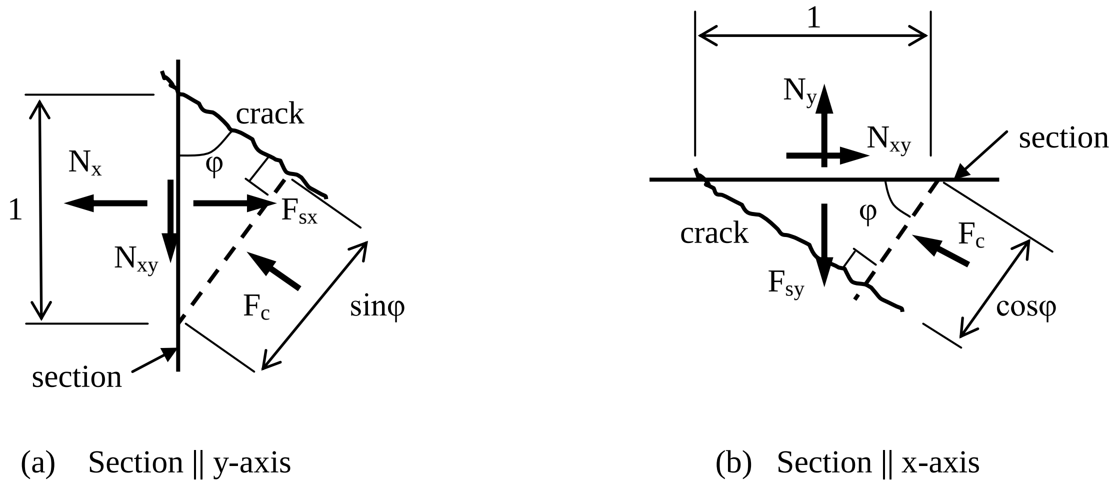


Figure 2.2.2 External and internal forces in sections parallel to the axes

Equilibrium in y-direction in (a) or x-direction in (b) gives

$$N_{xy} = F_c \cdot \sin\varphi\cos\varphi$$

or

$$\mathbf{F_c = N_{xy} / \sin\varphi\cos\varphi} \qquad (2.2.1)$$

$$F_{sx} = N_x + F_c \cdot \sin^2\varphi$$

With F_c from Eq.(2.2.1)

$$\mathbf{F_{sx} = N_x + N_{xy} \cdot \tan\varphi} \qquad (2.2.2)$$

Equilibrium in y-direction in (b) gives

$$F_{sy} = N_y + F_c \cdot \cos^2\varphi$$

or

$$\mathbf{F_{sy} = N_y + N_{xy} \cdot \cot\varphi} \qquad (2.2.3)$$

Eqs. (2.2.1) - (2.2.3) give the internal forces in concrete and reinforcement expressed by the external actions (in-plane forces) and the crack angle φ.

As soon as the crack angle is found, the equations can be used to calculate required reinforcement in both directions, and to check the stress in the compression field compared to the design compressive strength of the concrete.

2.2.3 Crack angle

The crack angle φ can be found in different ways:

Alternative1:

The principal tension direction is calculated based on homogeneous, isotropic and linear elastic material. Cracks are assumed to develop perpendicular to this direction.

The maximum principal tensile force is

$$N_1 = \frac{N_x + N_y}{2} + \sqrt{\frac{\left(N_x - N_y\right)^2}{4} + N_{xy}^2}$$

The crack angle is found from

$$\tan\varphi = \frac{N_{xy}}{N_1 - N_y}$$

This method results in equal reinforcement forces in x- and y-direction

$$F_{sx} = F_{sy} = N_1$$

Therefore, the method requires equal reinforcement amounts in the two directions, and represents a conservative approach.

Laboratory tests show that the first cracks correspond well with the crack angle in this approach, and represent the transition between an uncracked (State I) and a cracked state (State II). By further loading, or repeated loading, new cracks develop in different directions than the first, and these close. This is due to the stiffness change when the cracks occur, and is particularly pronounced with significant difference between reinforcement and principal tension directions.

Alternative 2:

A better alternative is to calculate the crack angle corresponding to yielding of reinforcement in both directions. This implies full utilization of the reinforcement, and hence an economical approach.
With design strength in reinforcement in ULS f_{yd} , the internal forces in the reinforcement are expressed as:

$$F_{sx} = A_{sx} \cdot f_{yd} \qquad \text{og} \qquad F_{sy} = A_{sy} \cdot f_{yd}$$

Therefore

$$f_{yd} = F_{sx} / A_{sx} = F_{sy} / A_{sy}$$

With F_{sx} and F_{sy} from Eqs.(2.2.1) and (2.2.3):

$$\frac{1}{A_{sx}}\left(N_x + N_{xy}\tan\varphi\right) = \frac{1}{A_{sy}}\left(N_y + N_{xy}\cdot\frac{1}{\tan\varphi}\right)$$

Multiplying this equation with $A_{sx}\tan\varphi / N_{xy}$ results in a simple 2nd degree equation in $\tan\varphi$.

$$\tan^2\phi + \left(\frac{N_x}{N_{xy}} - \frac{N_y}{N_{xy}}\cdot\frac{A_{sx}}{A_{sy}}\right)\cdot\tan\phi - \frac{A_{sx}}{A_{sy}} = 0 \qquad (2.2.4)$$

Eq. (2.2.4) has two roots for $\tan\varphi$ and the crack angle φ (one positive and one negative). The correct root is illustrated in Figure 2.2.3.

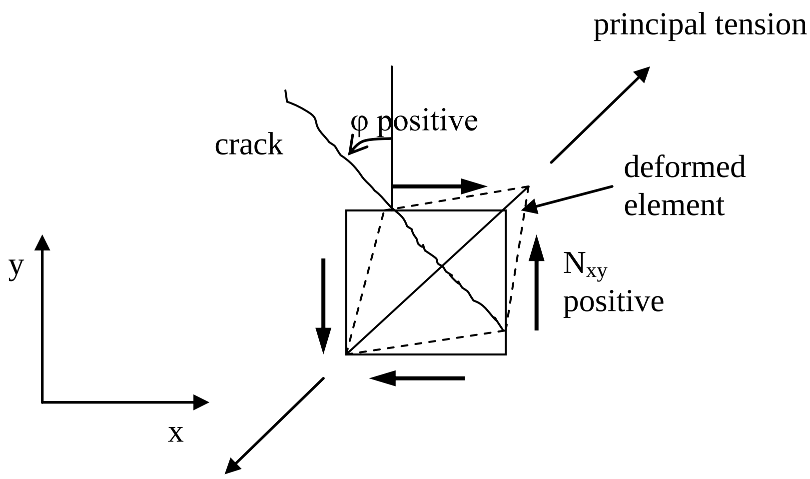


Figure 2.2.3 Choice of correct root for the crack angle

The figure shows that the crack angle is chosen with the same sign as the in-plane shear force, positive N_{xy} corresponds to positive φ , negative N_{xy} corresponds to negative φ .

Alternative 3:

A third alternative approach is based on the assumption of linear elastic reinforcement and concrete in compression, and no tensile strength of concrete, i.e. a typical "State II – situation".

For a linear elastic material the work done per unit volume of the material is:

$$dA_i = \frac{1}{2}\sigma\varepsilon dV = \frac{\sigma^2}{2E}dV$$

The internal work in reinforcement in x- and y-direction of a reinforced element with volume t·1·1, is:

$$A_{isx} = \int_V dA_{isx} = \int_V \frac{\sigma_{sx}^2}{2E_s} dV = \frac{1}{2}\left(\frac{F_{sx}}{A_{sx}}\right)^2 \frac{A_{sx}}{E_s}$$

and

$$A_{isy} = \int_V dA_{isy} = \int_V \frac{\sigma_{sy}^2}{2E_s} dV = \frac{1}{2}\left(\frac{F_{sy}}{A_{sy}}\right)^2 \frac{A_{sy}}{E_s}$$

It can be shown that the internal work in the concrete is small compared to the reinforcement. The total internal work is therefore approximated as:

$$A_i = A_{isx} + A_{isy} = \frac{1}{2E_s}\left[\frac{\left(N_x + N_{xy}\tan\varphi\right)^2}{A_{sx}} + \frac{\left(N_y + N_{xy}\cot\varphi\right)^2}{A_{sy}}\right]$$

A crack will develop in the direction corresponding to minimum internal work, hence:

$$\frac{dA_i}{d\varphi} = 0$$

By carrying out the differentiation a 4th degree equation in tanφ is obtained:

$$\tan^4\phi + \frac{N_x}{N_{xy}} \cdot \tan^3\phi - \frac{N_y}{N_{xy}} \cdot \frac{A_{sx}}{A_{sy}} \cdot \tan\phi - \frac{A_{sx}}{A_{sy}} = 0 \qquad (2.2.5)$$

Design according to Eq. (2.2.5) corresponds to initial yielding in one reinforcement direction. This implies that this approach ensures a certain control of deformations and crack widths. In the Serviceability Limit State (SLS), this approach should be used.

2.2.4 Control of concrete stress in the compression field

According to EC2, 6.5.2(2), a reduced design compressive strength for concrete in the compression field has to be used:

$$\sigma_{Rd,max} = 0,6\nu' f_{cd} \qquad (2.2.6)$$

where $\nu' = 1 - f_{ck}/250$

2.2.5 Example - Design based on compression field theory

From a linear FE analysis of a deep beam with thickness t = 100mm, the state of stress in an element is:

σ_x = 5,0 N/mm^2 (tension) ; σ_y = - 2,0 N/mm^2 (compression) ; τ_{xy} = 2,5 N/mm^2

Hence, in-plane forces are: N_x = 500 N/mm ; N_y = - 200 N/mm ; N_{xy} = 250 N/mm

Materials:
Concrete C30 → f_{cd} = 17 N/mm^2 ; Reinforcement B500NC→f_{yd} = 434 N/mm^2

Calculate required reinforcement and check the concrete stress in the compression field for all three alternative crack angle approaches.

Alt.1: Crack angle based on homogeneous, isotropic and linear elastic material

Max principal force : $N_1 = \dfrac{500-200}{2} + \sqrt{\dfrac{(500+200)^2}{4} + 250^2} = 580\,\mathrm{N/mm}$

Crack angle : $\tan\varphi = \dfrac{250}{580-(-200)} = 0{,}3205$ → $\varphi = 17{,}8^\circ$

Forces in reinf.: $F_{sx} = F_{sy} = N_1$ = 580 N/mm

Required reinf.: $A_{sx} = A_{sy}$ = 580000/434 = 1336 mm^2/m

Compressive force in concrete : $F_c = \dfrac{250}{\sin 17{,}8 \cdot \cos 17{,}8} = 859\,\mathrm{N/mm}$

Compressive concrete stress : $\sigma_c = \dfrac{F_c}{t} = \dfrac{859}{100} = 8{,}59\,\mathrm{N/mm^2}$

Reduced concrete strength : $\sigma_{Rd,max} = 0{,}6 \cdot \left(1 - \dfrac{30}{250}\right) \cdot 17 = 8{,}98\,\mathrm{N/mm^2}$

Hence $\sigma_c < \sigma_{Rd,max}$ → OK

Alt.2: Crack angle based on yielding of reinforcement in both directions

When N_y is compressive, it is reasonable to assume more reinforcement in x-direction than in y-direction.
Eurocode 2 requires both vertical and horizontal wall reinforcement.

Assume $A_{sx}/A_{sy} = 3$

Eq. (2.2.4):

$$\tan^2\varphi + \left(\frac{500}{250} + \frac{200}{250}\cdot 3\right)\tan\varphi - 3 = 0$$

$$\tan^2\varphi + 4{,}4\tan\varphi - 3 = 0 \quad \rightarrow \quad \text{Solution} \quad \tan\varphi = 0{,}6 \quad \rightarrow \quad \varphi = 31^\circ$$

Reinforcement forces:

$$F_{sx} = 500 + 250\cdot 0{,}6 = 650\,\text{N/mm} \quad ; \quad F_{sy} = -200 + 250\cdot\frac{1}{0{,}6} = 217\,\text{N/mm}$$

Required reinforcement: $A_{sx} = \dfrac{F_{sx}}{f_{yd}} = \dfrac{650000}{434} = 1500\,\text{mm}^2/\text{m}$

$$A_{sy} = \frac{F_{sy}}{f_{yd}} = \frac{217000}{434} = 500\,\text{mm}^2/\text{m}$$

(The reinforcement ratio equals 3 as presumed)

Compressive force in concrete: $F_c = \dfrac{250}{\sin 31\cdot\cos 31} = 566\,\text{N/mm}$

Compressive concrete stress: $\sigma_c = \dfrac{F_c}{t} = \dfrac{566}{100} = 5{,}66\,\text{N/mm}^2$

Hence $\sigma_c < \sigma_{Rd,max}$ → OK

Alt.3: Crack angle based on linear elastic reinforcement

Assume the same reinforcement ratio as Alt.2: $A_{sx}/A_{sy} = 3$

Eq. (2.2.5):

$$\tan^4\varphi + \frac{500}{250}\cdot\tan^3\varphi - \frac{-200}{250}\cdot 3\cdot\tan\varphi - 3 = 0$$

or $\quad f(\varphi) = \tan^4\varphi + 2\cdot\tan^3\varphi + 2{,}4\cdot\tan\varphi - 3 = 0$

This equation may be solved using an advanced calculator, but not with the calculators allowed for exams at NTNU. Therefore it has to be solved by iteration or trial and error.

A simple graphical method:

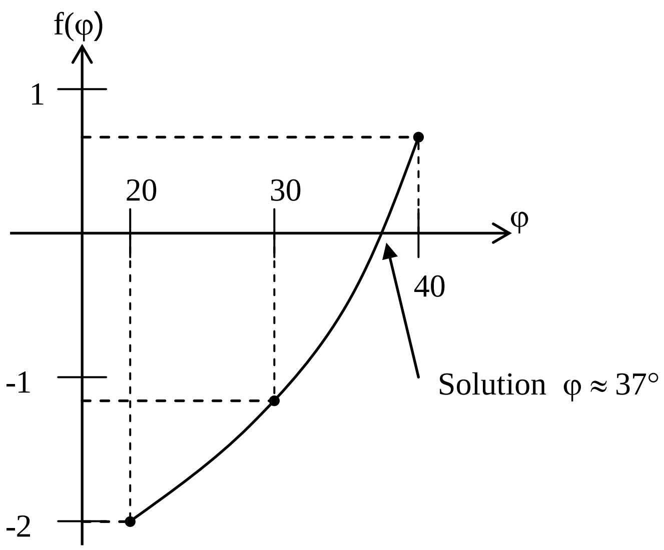
Choose values for φ and calculate f(φ). Solution when f(φ) changes sign:

$\varphi = 20°$: $\quad f(\varphi) = 0{,}0175 + 0{,}0964 + 0{,}8735 - 3 = -2{,}01$

$\varphi = 30°$: $\quad f(\varphi) = 0{,}111 + 0{,}3849 + 1{,}3856 - 3 = -1{,}12$

$\varphi = 40°$: $\quad f(\varphi) = 0{,}496 + 01{,}180 + 2{,}014 - 3 = 0{,}69$

Plot result:



Control by inserting $\varphi = 37°$:

$f(\varphi) = 0{,}322 + 0{,}856 + 1{,}810 - 3 = -0{,}01$; dvs $\approx 0 \rightarrow$ OK

$\tan 37° = 0{,}75$

Reinforcement forces:

$$F_{sx} = 500 + 250 \cdot 0,75 = 688\,N/mm \quad ; \quad F_{sy} = -200 + 250 \cdot \frac{1}{0,75} = 133\,N/mm$$

Required reinforcement: $A_{sx} = \frac{F_{sx}}{f_{yd}} = \frac{688000}{434} = 1585\,mm^2/m$

$$A_{sy} = \frac{F_{sy}}{f_{yd}} = \frac{133000}{434} = 306\,mm^2/m$$

Here $A_{sy} < A_{sx}/3$,as was presumed.

Hence, required y-reinforcement: $A_{sy} = \frac{A_{sx}}{3} = \frac{1585}{3} = 528\,mm^2/m$

This means that reinforcement in y-direction is not fully utilized.

The stress is $\sigma_{sy} = \frac{F_{sy}}{f_{yd}} = \frac{133000}{434} = 306\,N/mm^2$

This approach is based on linear elastic reinforcement. Here the x-direction reinforcement reaches initial yield, while strains are smaller in y-direction.
This means that one has some control of the strains, in contradiction to Alt.2, where the strains are unlimited. SLS requirements for crack widths are therefore more likely to be satisfied when design is based on Alt.3.

Compressive force in concrete: $F_c = \frac{250}{\sin 37 \cdot \cos 37} = 520\,N/mm$

Compressive concrete stress: $\sigma_c = \frac{F_c}{t} = \frac{520}{100} = 5,2\,N/mm^2$

Hence $\sigma_c < \sigma_{Rd,max}$ → OK

Comparing total reinforcement amount ($A_{s,tot} = A_{sx} + A_{sy}$) :

Alt.1 : $A_{s,tot} = 2 \cdot 1336 = 2672\ mm^2/m$

Alt.2 : $A_{s,tot} = 1500 + 500 = 2000\ mm^2/m$

Alt.3 : $A_{s,tot} = 1585 + 528 = 2113\ mm^2/m$

This shows that design based on Alt.1 requires 26% and 33% more reinforcement than Alt.3 and Alt.2, respectively.

2.2.6 Practical design/capacity control using compression field theory

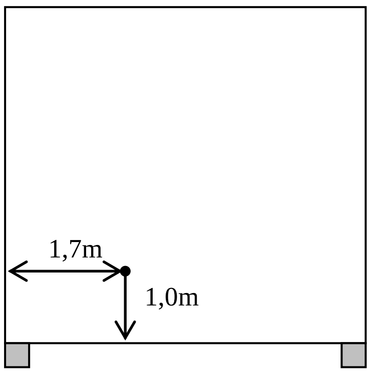
The example in chapter 2.2.5 shows how a single point in a deep beam can be designed by compression field theory. It is obviously not suited for a complete design of the entire beam, and definitively not by hand calculations.

Most consulting companies have their own post-processing programs with interface to some FE program. The stress resultants from a linear FE analysis are treated in the post-processor as in the example in chapter 2.2.5, and results in varying reinforcement all over the deep beam. Reinforcement layout is chosen based on these results. The post-processor is then used to check the capacity in all elements with the chosen reinforcement. The results are presented as utilization ratios for reinforcement and concrete.

2.2.7 Example - Capacity control for specified reinforcement

A linear analysis of the deep beam designed in the example in chapter 2.1.3 is carried out. The figure shows a chosen point with the following stresses:

$\sigma_x = 0{,}14\ \text{N/mm}^2$; $\sigma_y = 0{,}49\ \text{N/mm}^2$; $\tau_{xy} = -\ 0{,}265\ \text{N/mm}^2$



In-plane forces when the wall thickness is t = 250 mm:

$N_x = 35\ \text{N/mm}$; $N_y = 123\ \text{N/mm}$; $N_{xy} = -\ 66\ \text{N/mm}$

Minimum reinforcement according to EC2 was chosen as an orthogonal mesh with

$A_{sx} = 435\ \text{mm}^2/\text{m}$ og $A_{sy} = 500\ \text{mm}^2/\text{m}$ $\rightarrow$ $A_{sx}/A_{sy} = 0{,}87$

It is reasonable to believe that the reinforcement stresses are smaller than the design strength, i.e. in the linear elastic region. Therefore, Alt.3 is chosen to calculate the crack angle.

The following 4th degree equation has to be solved:

$$\tan^4\varphi + \frac{35}{-66}\cdot\tan^3\varphi - \frac{123}{-66}\cdot 0{,}87\cdot\tan\varphi - 0{,}87 = 0$$

or $$\tan^4\varphi - 0{,}53\cdot\tan^3\varphi + 1{,}62\cdot 0{,}87\cdot\tan\varphi - 0{,}87 = 0$$

Solution: $\tan\varphi = -1{,}171$ and $\varphi = -49{,}5°$

Forces in reinforcement:

$$F_{sx} = 35 + (-66)\cdot(-1{,}171) = 112\,\text{N/mm}$$

$$F_{sy} = 123 + \frac{-66}{-1{,}171} = 179\,\text{N/mm}$$

Reinforcement stresses and utilization ratios:

$$\sigma_{sx} = \frac{112000}{435} = 257\,\text{N/mm}^2 \rightarrow \eta_{sx} = \frac{\sigma_{sx}}{f_{yd}} = \frac{257}{434} = 0{,}59$$

$$\sigma_{sy} = \frac{179000}{500} = 358\,\text{N/mm}^2 \rightarrow \eta_{sy} = \frac{\sigma_{sy}}{f_{yd}} = \frac{358}{434} = 0{,}82$$

The nonlinear DIANA-analysis in the Example in section 2.1.3 resulted in a load capacity of 170 kN/m, while the deep beam was designed for a load of 150 kN/m.
Hence, the utilization ratio at the reference load was 150/170 = 0,88.
Since the point considered here not necessarily is the most stressed, one may conclude that it is good correspondence in utilization ratios from the DIANA analysis and according to the compression field theory.

2.2.8 References

/2.1.1/ Baumann: ”Tragwirkung orthogonaler Bewehrungsnetze beliebiger Richtung in Flächentragwerken”, Deutscher Ausschuss für Stahlbeton, Heft 217, 1972, (in German)

/2.1.2/ Hagberg: ”Dimensjonering for skjærkrefter utenom bøyningsplan”, Nordisk betong, nr. 1, 1974. (in Norwegian)

/2.1.3/ Hagberg: ”Dimensjonering av vilkårlige skiver i bruddgrensetilstanden – skiveutsnitt”, NIF-kurs Dimensjonering av skiver og plater i betong, NTH, Jan. 1990. (in Norwegian)

CHAPTER 3

DESIGN OF CONCRETE SLABS

Svein I Sørensen

3.1 Two-way slabs – analysis and design base on theory of elasticity

3.1.1 Introduction

In two-way slabs the loads (usually self-weight and live load) are carried in two directions, resulting in bending moments in two directions, M_x and M_y , and also torsion moment M_{xy}.

How much load that is carried in the two directions depends on the span ratio.
Figure 3.1.1 shows a quadratic slab with span ratio $L_y/L_x = 1,0$, and a rectangular slab with span ratio $L_y/L_x = 2,0$. The slabs are simply supported along all four edges.

The figure shows deflection lines in x- and y-direction. Because the deflection in the middle of the slab is equal for both directions it is obvious that the curvature and the bending moment in the shortest span direction (x-direction) are largest.
This means that the major share of the load is carried in the shortest span direction. For the quadratic slab the two directions are identical, and half the load is carried in each direction.

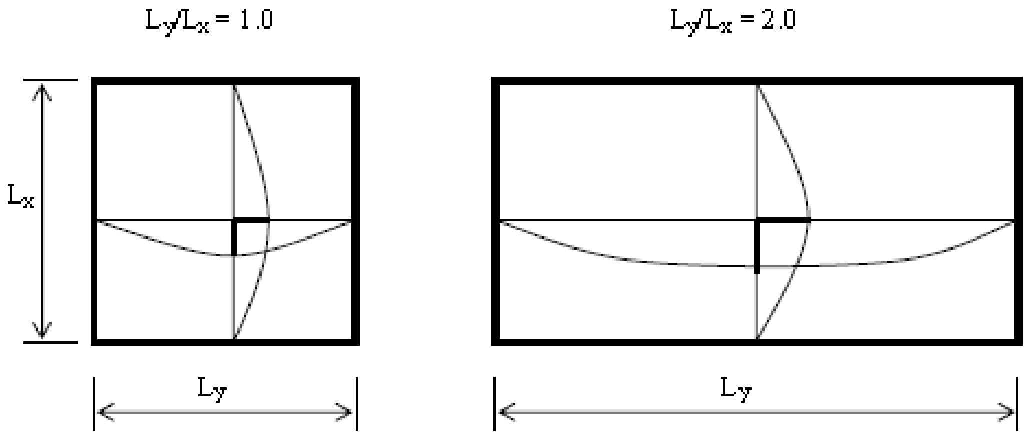


Figure 3.1.1 Deflection lines for slabs with different span ratios

Common practice is that a slab with span ratio $L_y/L_x > 2,0$ is considered as a one-way slab where all load is carried in the x-direction, i.e. the shortest span direction.

In this case x-direction is reinforced for the bending moment $M_{Ed} = q_{Ed}L_x^2/8$ (moment per unit length in y-direction), while minimum reinforcement according to EC2, NA.9.2.1.1(1) is required in y-direction.

The torsion effect is visualized in Figure 3.1.2. The deflection lines at a section close to the support edge (1-1) and at midspan (2-2) are shown in the same figure.

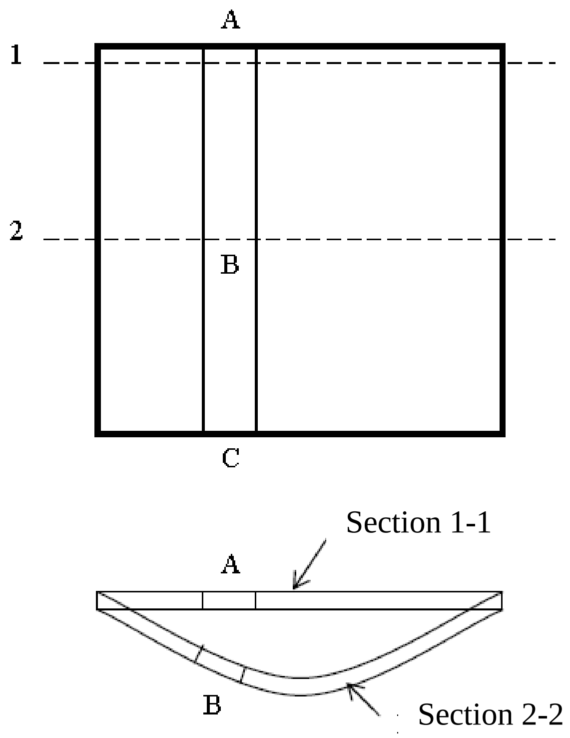


Figure 3.1.2 Visualization of the torsion effect in a two-way slab

The figure shows that the strip ABC is rotated at B but not at A. This results in the torsion moment M_{xy}.

A popular simplification for analysis of two-way slabs is to neglect the torsion effects, and the loads are carried by pure bending of strips in x- and y-direction. In this approach the bending moments will be larger than if the torsion stiffness is included. Hence, the method results in reinforcement quantities to the safe side. The method is called “Strip method”, and is described in detail in Chapter 3.3.

A complete solution including both bending and torsion moments requires analysis according to the theory of elasticity.
For rectangular slabs with distributed loads and ideal boundary conditions, a large number of handbooks with tabulated moments and deflections are available.
For general slab geometries, loads and boundary conditions, analysis by finite element programs, e.g. DIANA /3.2.1/, is appropriate.

Here, only rectangular two-way slabs will be considered.

3.1.2 Theory of elasticity for plates

The classical theory of elasticity for plates loaded perpendicular to the plate plane was derived early 1800.
Figure 3.1.3 shows an infinitesimal plate element in xy-plane, with bending moments M_x and M_y, torsion moments M_{xy} and M_{yx} and shear forces V_x og V_y.
The plate is subjected to a distributed load in z-direction, q.

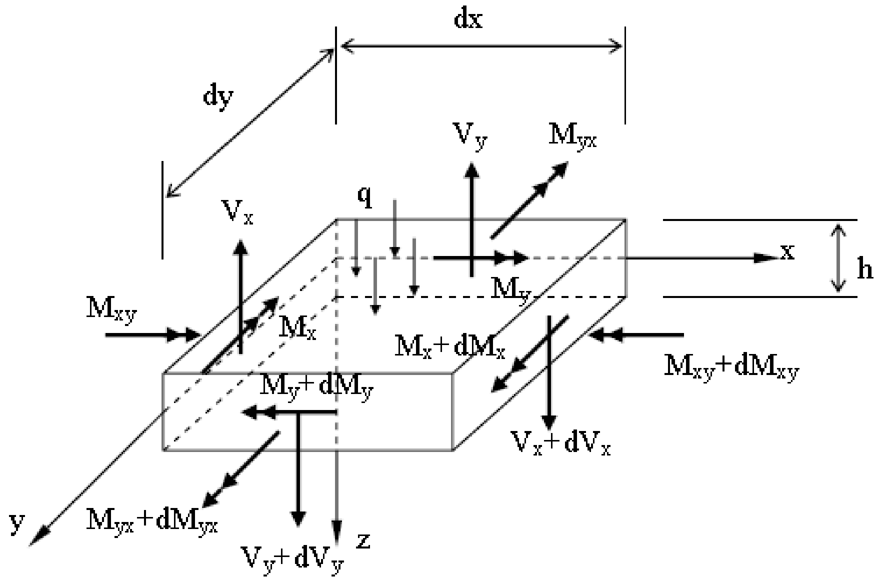


Figure 3.1.3 Infinitesimal plate element

Equilibrium conditions result in the equilibrium equation of the plate:

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{3.1.1}$$

Here $M_{xy} = M_{yx}$ due to shear stresses occur in pairs, i.e. $\tau_{xy} = \tau_{yx}$.

Eq. (3.1.1) is a pure equilibrium equation, independent if the material is elastic or plastic, and independent of Poisson's effect and if the plate is isotropic or anisotropic.

Further assumptions:

1. The material is linear elastic and isotropic, hence, Hooke's law is valid.
2. Deflections are small compared to the thickness of the plate.
3. Kirchhoff's hypothesis is valid (similar to Navier/Bernoulli for beams).
4. Plane stress state in xy-plane.

Using these assumptions as basis for strain compatibility results in the following expressions for bending and torsion moments :

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{3.1.2}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{3.1.3}$$

$$M_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \cdot D \cdot (1-\nu) \tag{3.1.4}$$

Here :

w = deflection of the plate (positive in positive z-direction)

ν = Poisson's ratio

$D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ = plate flexural stiffness

Eqs. (3.1.2)-(3.1.4) into Eq.(3.1.1) results in a fourth order partial differential equation for elastic bending of isotropic plates. The equation is called the Lagrange-equation:

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{3.1.5}$$

Eq. (3.1.5) is valid for ”medium thick plates”. In practice these are plates sufficiently thin so the shear deformations can be neglected, but also sufficiently thick so that the effects of membrane forces can be neglected. This is normally satisfied if $h \le L_x/8$ and $w \le h/2$.

Reinforced concrete slabs will normally be within these limits for slab thickness.

Eq. (3.1.5) can be solved analytically for rectangular plates with ideal boundary conditions. Detailed derivation of the theory, and solutions for a number of cases for rectangular plates can be found in the book ”Theory of Plates and Shells” by Timoshenko and Woinowsky-Krieger /3.2.2/.

Tables for moments and deflections for various support conditions and span ratios can be found in the German handbook ”Betonkalender” /3.2.3/ and in the Swedish handbook ”Bygg” /3.2.4/.
Ref. /3.2.3/ gives results for Poisson’s ratio $\nu = 0$, while ref. /3.2.4/ gives results for $\nu = 0$ and $\nu = 0,3$.

Table 3.1.1 shows bending moments and deflections at midspan in a simply supported, rectangular plate subjected to uniformly distributed load, q, for span ratios L_x/L_y ranging from 1,0 to ∞. Poisson’s ratio $\nu = 0$.
M_x is the moment in the shortest span direction (x-direction).

Table 3.1.1 Bending moments and deflections of simply supported rectangular plate, ν =0

	Bending moment		Deflection
L_y/L_x	$M_x = qL_x^2/\alpha$	$M_y = qL_x^2/\beta$	$w = \gamma \cdot qL_x^4/Eh^3$
	α	β	γ
1.0	27.2	27.2	0.0485
1.1	22.4	27.8	0.058
1.2	19.1	29.0	0.0675
1.3	16.8	30.8	0.0765
1.4	15.0	32.3	0.0845
1.5	13.7	34.7	0.0925
1.6	12.7	36.1	0.0995
1.7	11.9	37.3	0.106
1.8	11.3	38.5	0.112
1.9	10.8	39.4	0.117
2.0	10.4	40.3	0.122
3.0	8.5	192	0.147
4.0	8.1	667	0.154
5.0	8	770	0.156
∞	8	∞	0.156

An average value of Poisson’s ratio for concrete is $\nu = 0,2$. In design of reinforced concrete slabs, bending moments for $\nu = 0$ directly from the table are often used. The reason for this is that a concrete slab will always crack. Tensile stresses then have to be carried by reinforcement in two directions. Reinforcement stresses in one direction can only influence the reinforcement stresses in the other direction transferred by the concrete between cracks. This transfer is not much effective in cracked concrete. The compressive stresses in the concrete will to a certain extent be influenced by the lateral contraction, but as most concrete slabs are under-reinforced with partly utilized concrete in the compression zone, this will not influence the moment distribution significantly.

If the bending moments for e.g. $\nu = 0,2$ are wanted, these can be found as :

$M_{x,\,\nu=0,2} = M_x + \nu M_y$; M_x and M_y from Table 3.1.1.

The table shows that for a span ratio of 3,0 the moment in x-direction is

$$M_x = \frac{qL_x^2}{8,5} = 0,118qL_x^2$$

For a one-way slab the moment is

$$M_x = \frac{qL_x^2}{8} = 0,125qL_x^2$$

This means that considering the slab as a one-way slab implies that the moment in the principal direction is approx. 6% larger than according to the theory of elasticity.

Deflection from Table 3.1.1 for the same plate is

$$w = 0,147 \cdot \frac{qL_x^4}{Eh^3}$$

For one-way slab the deflection of a simply supported strip with unit width is

$$w = \frac{5}{384} \cdot \frac{qL_x^4}{EI} = \frac{5}{384} \cdot \frac{qL_x^4}{E\frac{1 \cdot h^3}{12}} = 0,156 \cdot \frac{qL_x^4}{Eh^3}$$

Hence, the one-way assumption will over-estimate the deflection with approx. 6%.

Comparing the two-way and one-way assumption for $L_y/L_x = 2,0$ results in over-estimation of the moment approx. 30% and the deflection approx. 28% by considering the slab as one-way.

This implies that the common practice, assuming that a slab may be considered as a one-way slab if the span ratio is larger than 2,0, is rather conservative for span ratios between 2,0 and 3,0.
For $L_y/L_x = 2,5$ both moment and deflection will still be approx.15% larger for one-way assumption than according to the theory of elasticity..

It should also be mentioned that in ref. /3.2.2/, Timoshenko and Woinowsky-Krieger conclude that one-way assumption is relevant for span ratios larger than 3,0.

Conclusion:

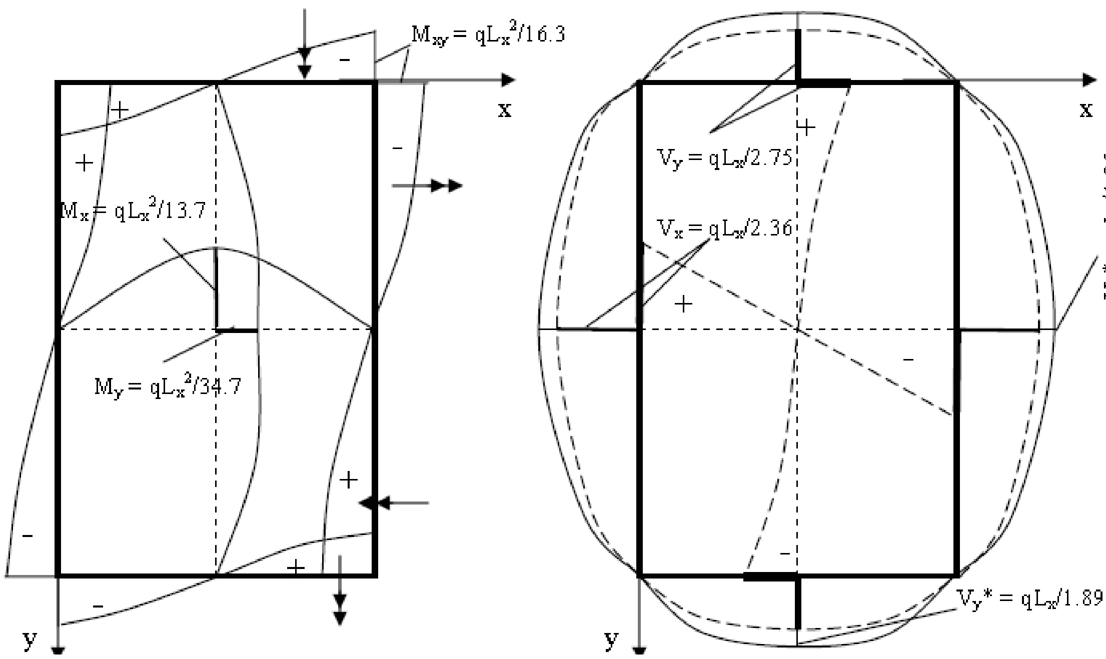
It is recommended to calculate a slab as two-way if $L_y/L_x < 3,0$, even if EC2, 5.3.1(5) says that with $L_y/L_x > 2,0$, it can be considered to be a one-way slab.

Table 3.1.2 gives values for maximum torsion moment M_{xy} (at slab corners), maximum shear forces V_x and V_y (at the middle of edges) and maximum support pressure V_x* and V_y* (at the middle of edges). The table is valid for $\nu = 0$.

Table 3.1.2 Torsion moments, shear forces and support pressure for simply supported rectangular plate

L_y/L_x	Torsion	Shear forces		Support pressure	
	$M_{xy} = qL_x^2/\phi$	$V_x = qL_x/\theta_x$	$V_y = qL_x/\theta_y$	$V_x^* = qL_x/\omega_x$	$V_y^* = qL_x/\omega_y$
	ϕ	θ_x	θ_y	ω_x	ω_y
1.0	21.6	2.96	2.96	2.19	2.19
1.1	19.7	2.78	2.89	2.11	2.09
1.2	18.4	2.64	2.84	2.04	2.02
1.3	17.5	2.52	2.80	2.00	1.96
1.4	16.8	2.43	2.76	1.97	1.92
1.5	16.3	2.36	2.75	1.95	1.89
1.6	15.9	2.30	2.73	1.93	1.87
1.7	15.6	2.25	2.73	1.92	1.85
1.8	15.4	2.21	2.72	1.92	1.83
1.9	15.3	2.18	2.71	1.92	1.82
2.0	15.1	2.15	2.70	1.92	1.82

Figure 3.1.4 shows bending moments, torsion moments, shear forces and support pressure for a plate with span ratio $L_x/L_y = 1,5$.

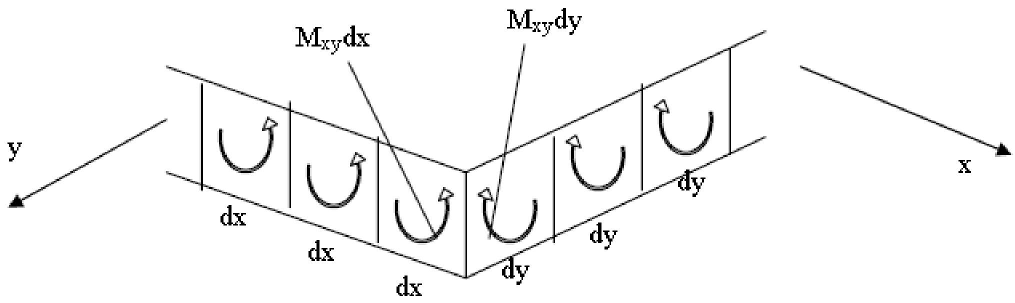


a) Moments b) Shear forces and support pressure

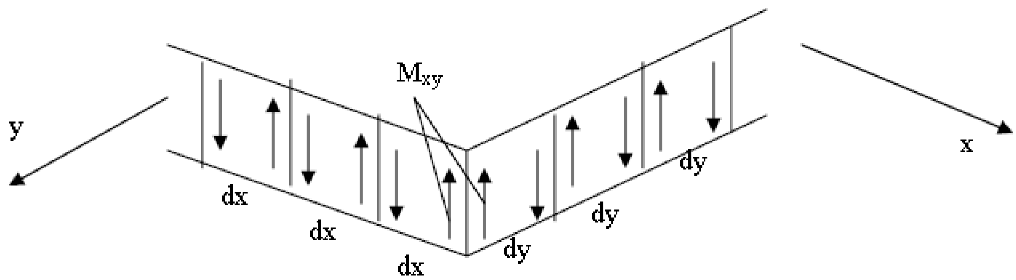
Figure 3.1.4 Section forces for simply supported plate with $L_y/L_x = 1,5$

The plate in Figure 3.1.4 is seen from top, i.e. in positive z- and q-direction referred to Figure 3.1.3.

Positive and negative directions of the torsion moments are shown by action symbols in Figure 3.1.4a. Considering the lower right corner of the plate in Figure 3.1.4a, the torsion moments may be visualized as shown in Figure 3.1.5.



a) Torsional moments near plate corner



b) Equivalent couples replacing the torsional moments

Figure 3.1.5 Effect of torsion at plate corner

Figure 3.1.5a shows torsion moments for lengths dx and dy along the x- and y-axis respectively (M_{xy} in the table is moment per unit length).
In Figure 3.1.5b these torsion moments are replaced by equivalent force couples. The equivalent forces counteract each other in the section borders along the entire plate edge.
In the corner an upwards force remains. The magnitude of the force is M_{xy} from both edges.

The internal stresses and curvatures in the plate results in an uplift force, R, in the corner:

$$R = 2 \cdot M_{xy} \tag{3.1.6}$$

If the plate is not anchored in the support for this force, the corners will lift.
Note that if the corners are free to lift, the section forces in the plate will change because these are determined based on boundary condition w = 0 along the entire length of all the support edges.

In Figure 3.1.4b, dashed lines show shear forces while solid lines represent the support pressure.

The reason for the difference between support forces and shear forces is that while the total shear forces are in equilibrium with the resultant of the plate load, $q \cdot L_x \cdot L_y$, the support forces in addition have to balance the four anchoring forces at the corners.

For rectangular plates rotationally fixed along all four edges, or simply supported along two parallel edges and fixed along the two other, see tables in refs. /3.2.3/ and /3.2.4/.

Tables for hydrostatic load (triangular load) can also be found in these handbooks.

3.1.3 Design of reinforced concrete two-way slabs

General requirements for slabs are given in Eurocode2.

<u>EC2, 5.4 :</u>

(1) Linear analysis of elements based on the theory of elasticity may be used for both the serviceability and ultimate limit states.
(2) For the determination of the action effects, linear analysis may be carried out assuming:
    (i) uncraccked cross sections
    (ii) linear stress-strain relationships and
    (iii) mean value of the modulus of elasticity.

For two-way slabs, in practice this means theory of elasticity as described in Chapter 3.1.2.

<u>EC2, 9.3 :</u>

(1) This section applies to one-way and two-way solid slabs for which b and L_{eff} are not less than 5h (see 5.3.1)

This section specifies requirements for principal reinforcement in the slab wit respect to minimum reinforcement, spacing, reinforcement close to supports, corner reinforcement and reinforcement at free edges.

In particular, 9.3.1.3(1) says that if the detailing arrangements at a support are such that lifting of the slab at a corner is restrained, suitable reinforcement should be provided.

Figure 3.1.6 shows the deflection of the diagonal in a quadratic slab with anchored corners.

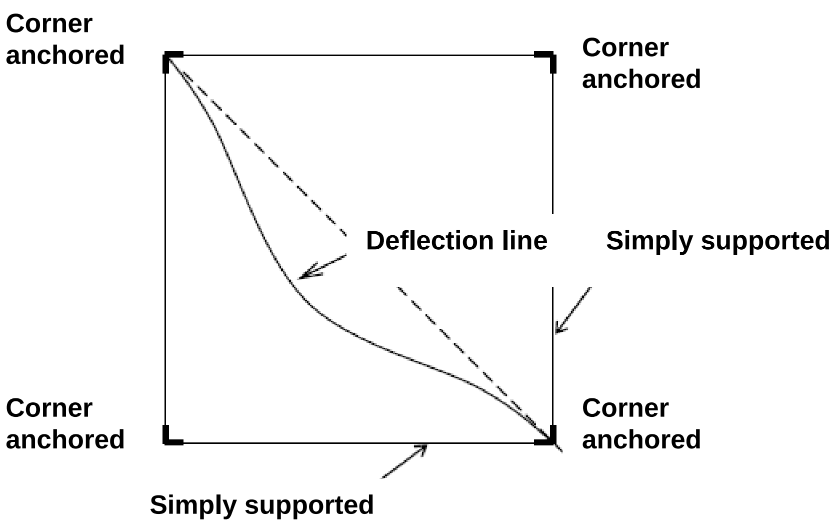


Figure 3.1.6 Deflection of diagonal in slab with anchored corners.

The figure shows tension at top surface at the corner regions. Distribution of cracks due to this tension is acceptable with top reinforcement according to EC2, 9.3.1.2(2).

Swedish handbooks recommend top reinforcement at the corners for a bending moment in the diagonal direction

$$M = \frac{q \cdot L_x \cdot \sqrt{L_x \cdot L_y}}{40} \tag{3.1.7}$$

__Ultimate limit state__

Dimensioning in ULS should be carried out separately for x- and y-direction

Reinforcement in x-direction (shortest span direction) should be given the largest effective depth d_x.

Reinforcement in y-direction will then have an effective depth $d_y = d_x - 0{,}5\phi_x - 0{,}5\phi_y$, where ϕ_x and ϕ_y are reinforcement diameters in x- and y-direction, respectively.

Serviceability limit state

Calculating deflections by the theory of elasticity requires that the material is homogenous and isotropic. None of these requirements are satisfied for a reinforced concrete slab which is cracked in SLS.

Therefore, some approximations have to be made in order to calculate deflections of a reinforced concrete slab by the theory of elasticity.

Figure 3.1.7 shows a typical crack pattern in ULS for a simply supported two-way slab (seen from bottom face).

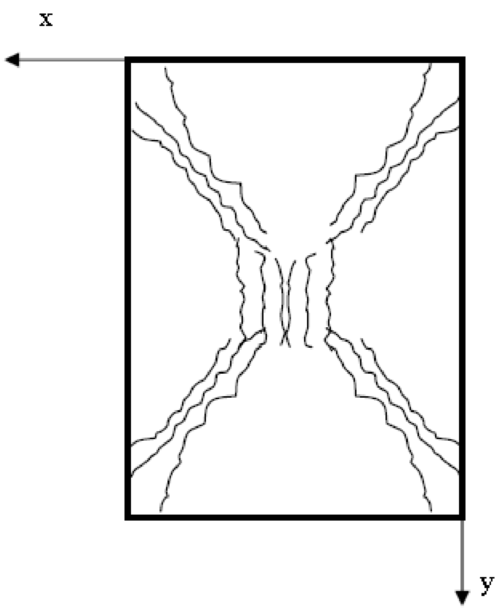


Figure 3.1.7 Typical crack pattern in simply supported two-way slab

Cracking in the middle of the slab starts perpendicular to the principal load-carrying direction. It is therefore reasonable to assume that the plate stiffness, D, is dominated by one-way acting plate strips in this direction (x-direction).

Reinforcement ratio in x-direction (A_{sx} in mm^2/m, d_x in mm):

$$\rho_x = \frac{A_{sx}}{10^3 \cdot d_x} \tag{3.1.8}$$

Elasticity modulus relation ($E_s = 2 \cdot 10^5$ N/mm^2 , $E_{c,mean}$ = mean E-modulus of concrete) :

$$\eta = \frac{E_s}{E_{c,mean}} \tag{3.1.9}$$

Compression zone depth factor in x-direction:

$$\alpha_x = \sqrt{(\eta\rho_x)^2 + 2\eta\rho_x} - \eta\rho_x \tag{3.1.10}$$

Second moment of area of 1 meter wide plate strip in x-direction (mm^4 for d_x in mm):

$$I_{cx} = \frac{1}{2} \cdot \alpha_x^2 \cdot \left(1 - \frac{\alpha_x}{3}\right) \cdot 10^3 \cdot d_x^3 \tag{3.1.11}$$

An effective thickness of a cracked plate strip is now obtained as:

$$I_{cx} = \frac{10^3 \cdot h_{eff}^3}{12} \tag{3.1.12}$$

$$h_{eff} = \sqrt[3]{\frac{12 \cdot I_{cx}}{10^3}} \qquad \text{(in mm)} \tag{3.1.13}$$

Deflection at midspan from Table 3.1.1 for the actual span ratio:

$$w = \frac{\gamma \cdot qL_x^4}{E_{c,mean} \cdot h_{eff}^3} \tag{3.1.14}$$

3.1.4 Example - Design of two-way slab

Dimensioning of a simply supported two-way slab in ULS. Furthermore, calculation of the deflection at midspan.

GIVEN DATA:
Spans : L_x = 5,0m ; L_y = 9,0m → L_y/L_x = 1,8
Concrete : B35 Reinforcement : B500NC
Live load (characteristic) : p = 5,0 kN/m^2
Mean E-modulus of concrete : $E_{c,mean}$ = 10000 N/mm^2 (chosen)
Exposure class : XC2
Slab thickness : h = 200mm

Self-weight : g = 0,2·25 = 5,0 kN/m^2

Design load in ULS :

q_{Ed} = 1,2g + 1,5p = 13,5 kN/m^2

Design bending moments according to Table 3.1.1 for $L_y/L_x = 1{,}8$:

$$M_{Edx} = \frac{q_{Ed}L_x^2}{11{,}3} = \frac{13{,}5 \cdot 5^2}{11{,}3} = 29{,}9 \text{kNm/m}$$

$$M_{Edy} = \frac{q_{Ed}L_x^2}{38{,}5} = \frac{13{,}5 \cdot 5^2}{38{,}5} = 8{,}8 \text{kNm/m}$$

Concrete cover required according to EC2, 4.4.1.1(1)P: $c_{nom} = c_{min} + \Delta c_{dev} = 35\text{mm}$

With reinforcement diameter 10mm:

Effective depth in x-direction: $d_x = h - c_{nom} - ø/2 = 200 - 35 - 5 = 160\text{mm}$

Effective depth in y-direction: $d_y = d_x - ø = 160 - 10 = 150\text{mm}$

Concrete design strength: $f_{cd} = \frac{0{,}85}{1{,}5} \cdot f_{ck} = 0{,}567 \cdot 35 = 19{,}8 \text{N/mm}^2$

Reinforcement design strength: $f_{yd} = \frac{f_{yk}}{1{,}15} = \frac{500}{1{,}15} = 434 \text{N/mm}^2$

Moment capacity of compression zone in x-direction (”normal reinforced”):

$$M_{Rdx} = 0{,}275 f_{cd} b d_x^2 = 0{,}275 \cdot 19{,}8 \cdot 10^3 \cdot 160^2 \cdot 10^{-6} = 139{,}4 \text{kNm/m}$$

Moment capacity of compression zone in y-direction (”normal reinforced”) :

$$M_{Rdy} = 0{,}275 f_{cd} b d_y^2 = 0{,}275 \cdot 19{,}8 \cdot 10^3 \cdot 150^2 \cdot 10^{-6} = 122{,}5 \text{kNm/m}$$

Hence, $M_{Rd} > M_{Ed}$ in both directions, and the compression zone is partly utilized.

Internal lever arms can be approximated as:

$$z_x = \left(1 - 0{,}17 \cdot \frac{M_{Edx}}{M_{Rdx}}\right) d_x = \left(1 - 0{,}17 \cdot \frac{29{,}9}{139{,}4}\right) d_x = 0{,}96 d_x$$

$$z_y = \left(1 - 0{,}17 \cdot \frac{M_{Edy}}{M_{Rdy}}\right) d_y = \left(1 - 0{,}17 \cdot \frac{8{,}8}{122{,}5}\right) d_y = 0{,}98 d_x$$

Do not allow $z > 0{,}95d$, therefore: $z_x = 0{,}95 \cdot 160 = 152\text{mm}$; $z_y = 0{,}95 \cdot 150 = 142\text{mm}$

Required reinforcement:

x-direction: $A_{sx} = \frac{M_{Edx}}{z_x f_{yd}} = \frac{29,9 \cdot 10^6}{152 \cdot 434} = 453\text{mm}^2/\text{m}$

y-direction: $A_{sy} = \frac{M_{Edy}}{z_y f_{yd}} = \frac{8,8 \cdot 10^6}{142 \cdot 434} = 143\text{mm}^2/\text{m}$

Minimum reinforcement according to EC2, NA.9.2.1.1(1) :

$$A_{s,min} = 0,26 \cdot \frac{f_{ctm}}{f_{yk}} \cdot b_t d \geq 0,0013 \cdot b_t d$$

For B35 , f_{ctm} = 3,2 N/mm^2 :

$$A_{sx,min} = 0,26 \cdot \frac{3,2}{500}.10^3 \cdot 152 = 253\text{mm}^2/\text{m}$$

$$A_{sy,min} = 0,26 \cdot \frac{3,2}{500}.10^3 \cdot 142 = 236\text{mm}^2/\text{m}$$

Reinforcement choice :

x-direction : ø10s170 → A_{sx} = 462 mm^2/m
y-direction : ø10s330 → A_{sy} = 238 mm^2/m

(A_{sy} satisfies requirement of $s_{max,slab}$ in EC2, NA.9.3.1.1(3))

Shear capacities are much larger than design shear forces. Therefore these controls are not carried out here.

Corner anchoring:

Anchoring force at each corner: $R_{Ed} = 2M_{Edxy}$

Torsion moments from Table 3.1.2:

$$M_{Edxy} = \frac{q_{Ed} L_x^2}{15,4} = \frac{13,5 \cdot 5^2}{15,4} = 21,9 \text{ kNm/m (or kN)}$$

$R_{Ed} = 2 \cdot 21,9 = 43,8$ kN

Required cross section area of anchoring reinforcement:

$$A_{anchoring} = R_{Ed}/f_{yd} = 43,8 \cdot 10^3 / 434 = 101\text{mm}^2$$

Choose anchoring bars 1ø12 in each corner, i.e. $A_{anchoring}$ = 113,1 mm^2

Deflection in SLS

Total load in SLS   q = g + p = 5 + 5 = 10 kN/m^2

$$\rho_x = \frac{A_{sx}}{bd_x} = \frac{462}{10^3 \cdot 152} = 0,003 \;;\quad \eta = \frac{E_s}{E_{c,middel}} = \frac{200000}{10000} = 20 \;;\quad \eta\rho_x = 0,06$$

$$\alpha_x = \sqrt{0,06^2 + 2 \cdot 0,06} - 0,06 = 0,292$$

$$I_{cx} = \frac{1}{2} \cdot 0,292^2 \cdot \left(1 - \frac{0,292}{3}\right) \cdot 10^3 \cdot 152^3 = 1,35 \cdot 10^8 \text{mm}^4$$

Effective depth from Eq. (3.1.13):

$$h_{eff} = \sqrt[3]{\frac{12 \cdot I_{cx}}{10^3}} = \sqrt[3]{\frac{12 \cdot 1,35 \cdot 10^8}{10^3}} = 117,4\text{mm}$$

Deflection at midspan from Eq. (3.1.14) with γ from Table 3.1.1:

$$w = \frac{\gamma \cdot q \cdot L_x^4}{E_{c,middel} \cdot h_{eff}^3} = \frac{0,112 \cdot 10 \cdot 10^{-3} \cdot 5000^4}{10000 \cdot 117,4^3} = 43\text{mm}$$

Compare with deflection for a one-way plate strip, width 1m:

$$w_{enveis} = \frac{5}{384} \cdot \frac{qL_x^4}{E_{c,middel} \cdot I_{cx}} = \frac{5 \cdot 10 \cdot 5000^4}{384 \cdot 10000 \cdot 1,35 \cdot 10^8} = 60\text{mm}$$

This results in approx. 40% larger deflection, and clearly demonstrates the advantage of applying the theory of elasticity for plates.

3.1.5 References

/3.2.1/ DIANA Finite Element Analysis, TNO Delft.

/3.2.2/ Timoshenko, S og Woinowsky-Krieger, S : ”Theory of Plates and Shells”, 2nd edition, McGraw-Hill, 1959

/3.2.3/ ”Beton – Kalender”, German handbook

/3.2.4/ ”Bygg – Del 1 Allmänna Grunder”, Swedish handbook

3.2 Yield line theory for slabs

3.2.1 Introduction

The yield line theory for reinforced concrete slabs initiated by Ingerslev in 1923 was extended and advanced by K.W.Johansen /3.2.1/.
The method represents an upper bound approach with respect to load carrying capacity of slabs.

The load capacity is determined by assuming a collapse mechanism that is compatible with the boundary conditions. The moments at the plastic hinge lines are the ultimate moments of resistance of the sections, and the ultimate load is determined by the principle of virtual work or the equations of equilibrium. Being an upper bound approach the method gives an ultimate load for a given slab that is either correct or too high.

The regions of the slab between lines of plastic hinges are not examined to ensure that the moments there do not exceed the ultimate moments of resistance of the sections, because these will be exceeded only if an incorrect collapse mechanism is used. Thus, all the possible collapse mechanisms of the slab must be examined to ensure that the load-carrying capacity of the slab is not overestimated. The correct collapse mechanisms in nearly all common cases are well known, however, and therefore the designer is not often faced with the uncertainty of whether further alternatives exist.

It should be noted that yield line theory assumes a flexural collapse mode, that is, that the slab has sufficient shear strength to prevent a shear failure.

Deformation and stiffness requirements will often be decisive in SLS. This has to be examined separately, or accounted for e.g. by required span/thickness ratios.

Slabs are normally under-reinforced, and the reinforcement yields before the final failure load is reached. After the initial yielding of the reinforcing steel, the compressive resultant moves towards the compressed part of the section until final compressive failure occurs at a moment that is larger than at initial yielding (normally 5 – 10%). The additional rotation of the cross section is relatively large.

Redistribution of moments from the elastic state is necessary for the collapse mechanism to develop. The section has to be sufficiently ductile to allow the required plastic rotation when yield lines are developing in the entire slab until the assumed collapse mechanism is formed. This capability of the section is termed ”rotation capacity”.

3.2.2 Yield line theory and Eurocode2

EC2, 5.6 gives rules and requirements for use of plastic methods of analysis :

EC2, 5.6.1 General :

(1)P Methods based on plastic analysis shall only be used for the check in ULS.

(2)P The ductility of the critical sections shall be sufficient for the envisaged mechanism to be formed.

(3)P The plastic analysis should be based either on the lower bound (static) method or on the upper bound (kinematic) method.

Further, the effects of previous applications of loading (load history) may generally be ignored, and a monotonic increase of the intensity of the actions may be assumed.

With respect to clause (3)P, yield line theory is a kinematic method.

The next clause is important as it opens for use of yield line theory without checking the rotation capacity.

EC2, 5.6.2 Plastic analysis for beams, frames and slabs

(1)P Plastic analysis (here yield line theory) without any direct check of rotation capacity may be used for ULS if the conditions of 5.6.1(2)P are met.

(2) The required ductility may be deemed to be satisfied without explicit verification if all the following are fulfilled (hence, 5.6.1(2)P is satisfied) :

i) The area of tensile reinforcement is limited such that, at any section :
Compression zone at failure $\leq$ 0,25d for concrete strength classes $\leq$ B50
Compression zone at failure $\leq$ 0,15d for concrete strength classes $\geq$ B55

ii) Reinforcing steel is either class B or C

A third requirement is appropriate for e.g. continuous beams, and is not relevant here.

If the requirements in 5.6.2(2) are not satisfied, rotation capacity has to be verified according to EC2, 5.6.3. This is not considered necessary here.

3.2.3 Kinematic mechanisms

The yield line approach is based on development of collapse mechanisms in slabs with concentrated plastic hinges as straight lines (yield lines). The plastic deformations along the yield lines are much greater than the elastic deformations of the slab segments between the yield lines, and hence in the theory it is reasonable to assume that the segments between yield lines are plane. This means that once a mechanism has formed, all additional deformations occur as if each segment were a plane.

Examination of the geometry of the deformations gives basic rules for the determination of the yield line patterns:

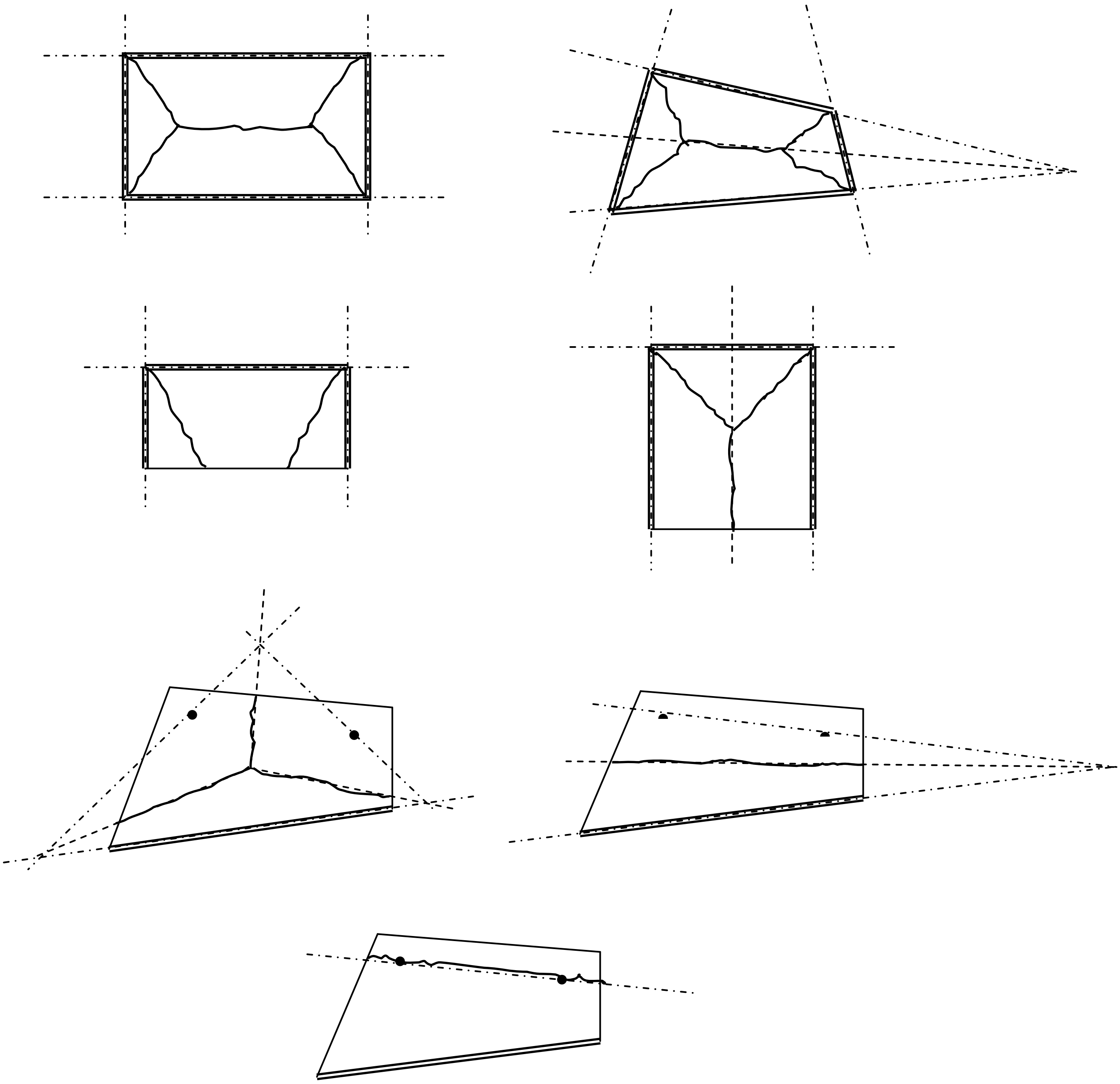
1. To act as plastic hinges of a collapse mechanism made up of plane segments, yield lines must be straight lines forming axes of rotation for the movement of the segments

2. The supports of the slab will act as axes of rotation. If an edge is fixed, a yield line may form along the support. An axis of rotation will pass over a column.

3. For compatibility of deformations, a yield line must pass through the intersection of the axes of rotation of the adjacent slab segments.

Figure 3.2.1 shows some examples of yield line patterns for uniformly loaded slabs of various shapes and boundary conditions. Note that for each slab there may be more than one family of possible yield line patterns (e.g. the column supported slab), any of which may be the critical pattern.

The ultimate load may be found from the yield line patterns using either the principle of virtual work or the equations of equilibrium. In general, the virtual work method is easier in principle than the equilibrium method. Therefore only the virtual work method will be considered here.



Symbols :

Free edge —————— Axis of rotation (along supports) -·-·-·-·-·-

Support edge ══════ Relative axis of rotation (between two adjacent segments) - - - - - - -

Yield line

Figure 3.2.1 Examples of yield line patterns for uniformly loaded slabs

3.2.4 Virtual work method

External virtual work per unit area

$$a_y = q \cdot w \qquad (3.2.1)$$

where q = load intensity, i.e. force per unit area
w = virtual deflection

The total external work is

$$A_y = \int_A q \cdot w dA \qquad (3.2.2)$$

For uniform load, i.e. q = constant :

$$A_y = q \int_A w dA = q \cdot V \qquad (3.2.3)$$

where V = ”deflection volume” or ”deformation volume”.

Internal virtual work is found as the product of the moment capacity m_d and the rotation θ along all the yield lines.

Figure 3.2.2 shows a yield line that is skew with respect to a support edge (axis of rotation).

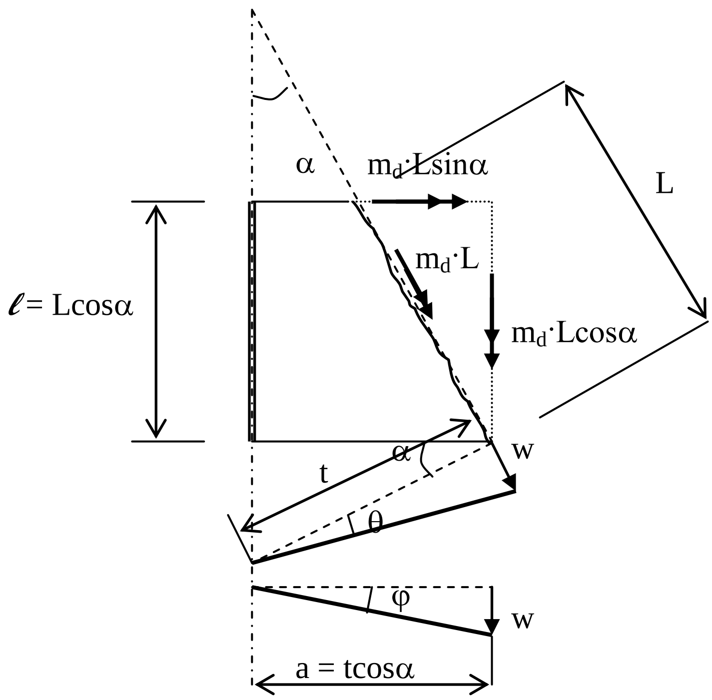


Figure 3.2.2 Skew yield line with regard to support axis of rotation

The internal work along the yield line is

$$A_i = m_d \cdot L \cdot \theta = m_d \cdot L \cdot w/t = m_d \cdot \ell \cdot w/t \cdot \cos\alpha = m_d \cdot \ell \cdot w/a = m_d \cdot \ell \cdot \varphi \qquad (3.2.4)$$

Note that here, φ is the slab segment's rotation about the support axis.

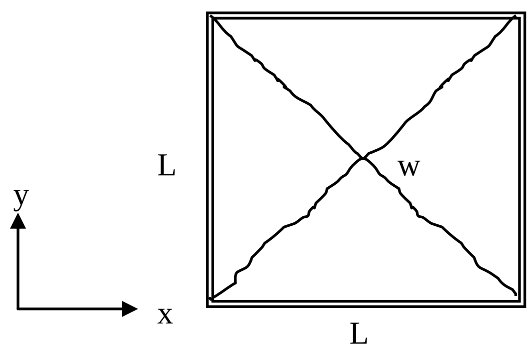
Eq. (3.2.4) shows that the internal work in a skew yield line is :

> <u>Product between the projection of the failure moment on the support axis and the slab segment's rotation about the same axis.</u>

The principle of virtual work requires that external work equals internal work, i.e.:

$$A_y = A_i \qquad (3.2.5)$$

3.2.5 Example - Simply supported quadratic slab with uniform load



A kinematic possible yield line pattern with virtual deflection w in the middle of the slab is shown in the figure.

The deflection volume is a pyramid with square base L^2 and height w:

$$V = \frac{L^2 \cdot w}{3}$$

External work : $A_y = q \cdot V = \frac{qL^2}{3} \cdot w$

With failure moment along the skew yield lines m_d , the internal work is :

$$A_i = 4 \cdot m_d \cdot L \cdot \frac{w}{L/2} = 8 \cdot m_d \cdot w$$

External work = internal work gives the moment capacity (if this is the critical yield pattern) :

$$m_d = \frac{qL^2}{24}$$

Required reinforcement in x- and y-direction is: $A_{sx} = A_{sy} = \frac{m_d}{z \cdot f_{yd}}$

3.2.6 Failure moments

In general, concrete slabs have different reinforcement in two directions. Here, only orthotropic reinforcement is considered, that is, different reinforcement in two perpendicular directions.

The largest reinforcement quantity is called ”primary reinforcement”, in the primary load-carrying direction, while the smaller reinforcement quantity is called ”secondary reinforcement”.

Failure moments (moment capacities) in the two directions:

$$\begin{aligned}&\text{Primary direction:} && m_{dp} = A_{sp} \cdot f_{yd} \cdot z_p \\ &\text{Secondary direction:} && m_{ds} = A_{ss} \cdot f_{yd} \cdot z_s\end{aligned} \tag{3.2.6}$$

where A_{sp} , A_{ss} = reinforcement in primary and secondary direction
z_p , z_s = internal lever arms for the two directions
f_{yd} = design strength of reinforcement steel

Define the relation:

$$\kappa = \frac{m_{ds}}{m_{dp}} \leq 1 \tag{3.2.7}$$

Generally, yield lines develop skew related to the reinforcement directions.
Figure 3.2.3 shows an element of a slab at a skew yield line. Equilibrium equations for this element can be used to determine the failure moment by rotation about the yield line, m_φ .

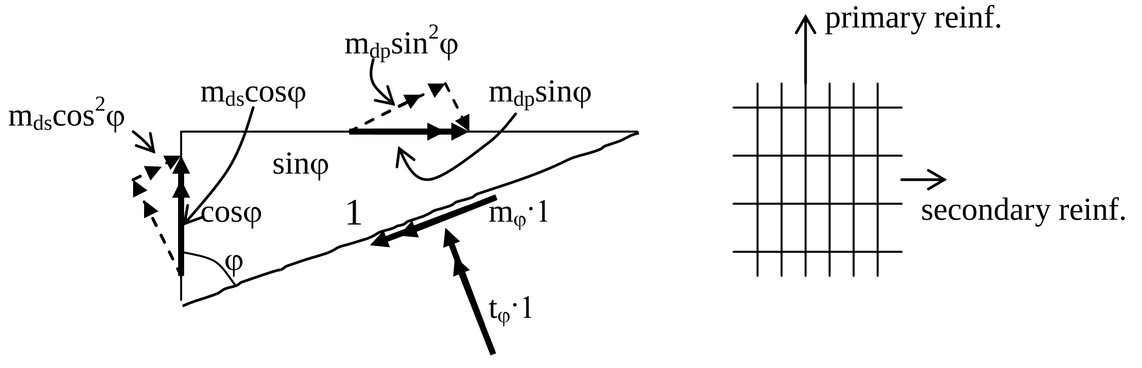


Figure 3.2.3 Element at skew yield line

Equilibrium about the yield line with unit length gives

$$m_{\varphi} \cdot 1 = m_{dp} \cdot \sin^2 \varphi + m_{ds} \cdot \cos^2 \varphi$$

or

$$m_{\varphi} = m_{dp} \cdot (\sin^2 \varphi + \kappa \cdot \cos^2 \varphi) \tag{3.2.8}$$

A torsion moment is acting perpendicularly to the yield line.
Equilibrium gives:

$$t_{\varphi} \cdot 1 = m_{dp} \cdot \sin\varphi \cos\varphi - m_{ds} \cdot \sin\varphi \cos\varphi$$

or

$$t_{\varphi} = m_{dp} \cdot (1 - \kappa) \sin\varphi \cos\varphi \tag{3.2.9}$$

For the special case with equal reinforcement in both directions (isotropic reinforcement):

$$m_{dp} = m_{ds} \quad \rightarrow \quad \kappa = 1 \tag{3.2.10}$$

Consequence:

$$m_{\varphi} = m_{dp} \cdot (\sin^2 \varphi + \cos^2 \varphi) = m_{dp} = m_{ds} \tag{3.2.11}$$

<u>Hence, equal failure moment in all directions</u>

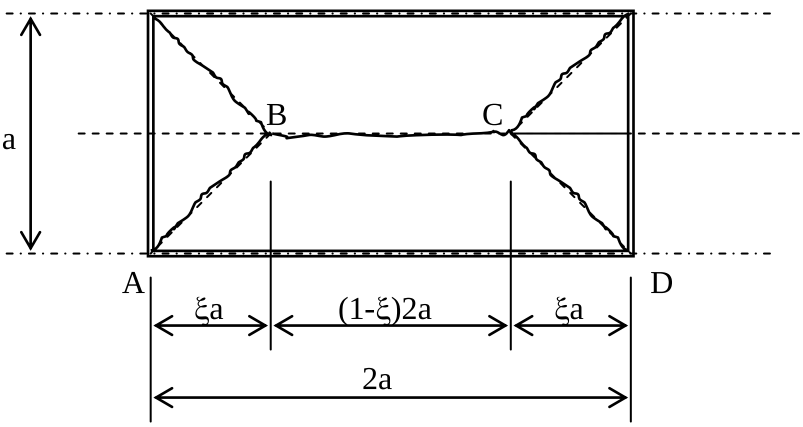
Further is $(1-\kappa) = 0$, and

$$t_{\varphi} = 0 \tag{3.2.12}$$

3.2.7 Example - Rectangular slab, $m_{dp} = m_{ds}$

The figure shows a simply supported slab with equal reinforcement in both directions. The span ratio is 2. The slab is subjected to uniform load q.

The figure also shows a possible yield line pattern which satisfies the rules for kinematic collapse mechanisms.
A constant virtual deflection w is assumed along yield line BC.



The deflection volume is:

$$V = \frac{1}{3} \cdot 2\xi a \cdot a \cdot w + \frac{1}{2} \cdot a \cdot w \cdot (1-\xi) \cdot 2a = \left(1 - \frac{\xi}{3}\right) a^2 \cdot w$$

External work: $A_y = q \cdot V = q \cdot \left(1 - \frac{\xi}{3}\right) a^2 \cdot w$

Internal work:

Along BC: $A_i^{BC} = 2 \cdot m_d \cdot (1-\xi) \cdot 2a \cdot \frac{w}{0,5a} = 8 \cdot m_d \cdot (1-\xi) \cdot w$

Along AB: $A_i^{AB} = m_d \cdot \xi a \cdot \frac{w}{0,5a} + m_d \cdot 0,5a \cdot \frac{w}{\xi a} = m_d \cdot \left(2\xi + \frac{0,5}{\xi}\right) \cdot w$

Total: $A_i = A_i^{BC} + 4 \cdot A_i^{AB} = m_d \cdot 2 \cdot \left(4 + \frac{1}{\xi}\right) \cdot w$

External work = Internal work: $q = \frac{2m_d}{a^2} \cdot \frac{4 + 1/\xi}{1 - \xi/3}$

The value of ξ that gives minimum q is determined by:

$$\frac{dq}{d\xi} = \frac{2m_d}{a^2} \cdot \frac{\left(-1/\xi^2\right)\cdot(1-\xi/3)-(4+1/\xi)\cdot(-1/3)}{(1-\xi/3)^2} = 0$$

or

$$\xi^2 + \xi/2 - 3/4 = 0 \qquad \text{with root} \quad \xi = 0,65$$

ξ = 0,65 in the expression for q gives :

$$q_{min} = \frac{2m_d}{a^2} \cdot \frac{4+1/0,65}{1-0.65/3} = 14,1 \cdot \frac{m_d}{a^2}$$

The value of q_{min} represents the failure load or load-carrying capacity for the chosen yield line pattern.

The sensibility of the solution for various ξ can be examined by varying the value of ξ in the expression for q:

ξ = 0,5 : $\quad q = 14,4 \cdot \frac{m_d}{a^2}$ (deviation of 2,1% from the load capacity)

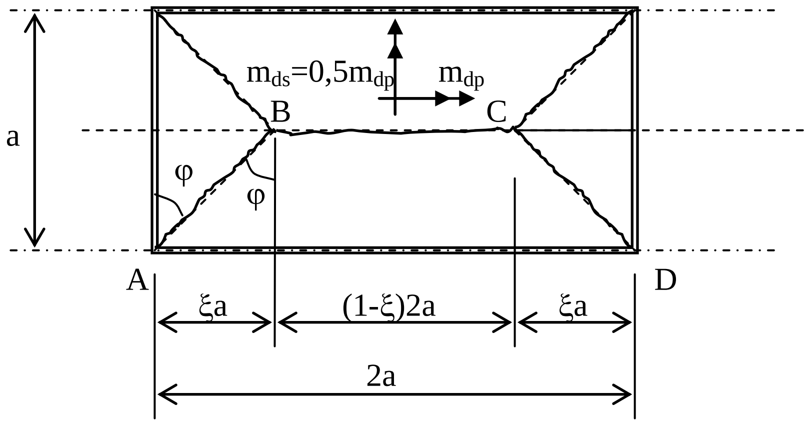
ξ = 1,0 : $\quad q = 15,0 \cdot \frac{m_d}{a^2}$ (deviation of 7,1% from the load capacity)

This shows that the calculated load capacity is not significantly influenced by small variations in ξ . Satisfactory accurate solutions can for many cases be obtained simply by choosing a reasonable value of ξ . This may be actual with yield line patterns that imply more complicated mathematical equations for exact calculation of q_{min}.

Trial and error techniques to determine an approximate value of q_{min} are also often used.

3.2.8 Example - Rectangular slab, $m_{ds} = 0{,}5m_{dp}$

The figure shows the same slab geometry as the example in chapter 3.2.7, but in this case the primary reinforcement is double of the secondary reinforcement.



The same external work as in the example in chapter 3.2.7:

$$A_y = q \cdot \left(1 - \frac{\xi}{3}\right) a^2 \cdot w$$

Length of skew yield lines:

$$AB = \sqrt{(\xi a)^2 + (a/2)^2} = a\sqrt{\xi^2 + 1/4}$$

Therefore: $$\sin\varphi = \frac{\xi a}{a\sqrt{\xi^2 + 1/4}} = \frac{\xi}{\sqrt{\xi^2 + 1/4}} \quad \text{and} \quad \sin^2\varphi = \frac{\xi^2}{\xi^2 + 1/4}$$

$$\cos\varphi = \frac{a/2}{a\sqrt{\xi^2 + 1/4}} = \frac{1}{2\sqrt{\xi^2 + 1/4}} \quad \text{and} \quad \cos^2\varphi = \frac{1}{4(\xi^2 + 1/4)}$$

The failure moment in skew yield lines is found from Eq. (3.2.11):

$$m_\varphi = m_{dp} \cdot \left(\sin^2\varphi + 0{,}5\cos^2\varphi\right) = m_{dp} \cdot \frac{8\xi^2 + 1}{8\xi^2 + 2}$$

Internal work:

Along BC : $$A_i^{BC} = 2 \cdot m_{dp} \cdot (1-\xi) \cdot 2a \cdot \frac{w}{0,5a} = m_{dp} \cdot 8(1-\xi) \cdot w$$

Along AB : $$A_i^{AB} = m_\varphi \cdot \xi a \cdot \frac{w}{0,5a} + m_\varphi \cdot 0,5a \cdot \frac{w}{\xi a} = m_\varphi \cdot \left(2\xi + \frac{1}{2\xi}\right) \cdot w$$

$$= m_{dp} \cdot \frac{8\xi^2+1}{8\xi^2+2} \cdot \left(2\xi + \frac{1}{2\xi}\right) \cdot w = m_{dp} \cdot \frac{8\xi^2+1}{4\xi} \cdot w$$

Total internal work: $$A_i = A_i^{BC} + 4A_i^{AB} = m_{dp} \cdot \frac{8\xi+1}{\xi} \cdot w$$

External work = Internal work gives: $$q = \frac{m_{dp}}{a^2} \cdot \frac{8\xi+1}{\xi \cdot \left(1 - \xi/3\right)}$$

Minimum value of q found by $\frac{dq}{d\xi} = 0$:

$$\xi^2 + \xi/4 - 3/8 = 0 \quad \text{with root} \quad \xi = 0,5$$

The load-carrying capacity is: $$q_{min} = 12,0 \cdot \frac{m_{dp}}{a^2}$$

Checking the sensibility of the solution for variation of ξ :

For $\xi = 0,4$, 0,6 and 1,0 , the deviations from exact solution are 0,9%, 0,7% and 12,5%, respectively. Again it is clear that small variations of ξ do not influence the solution significantly.

3.2.9 Special yield line patterns

Circular yield line pattern

A concentrated load on a slab may produce a conical-shaped yield line pattern as shown in Figure 3.2.4, by development of n radial yield lines (tension in bottom of slab with failure moment *m*. At a distance r from the concentrated load, a circular yield line develops (tension in top of slab) with failure moment *m'*.

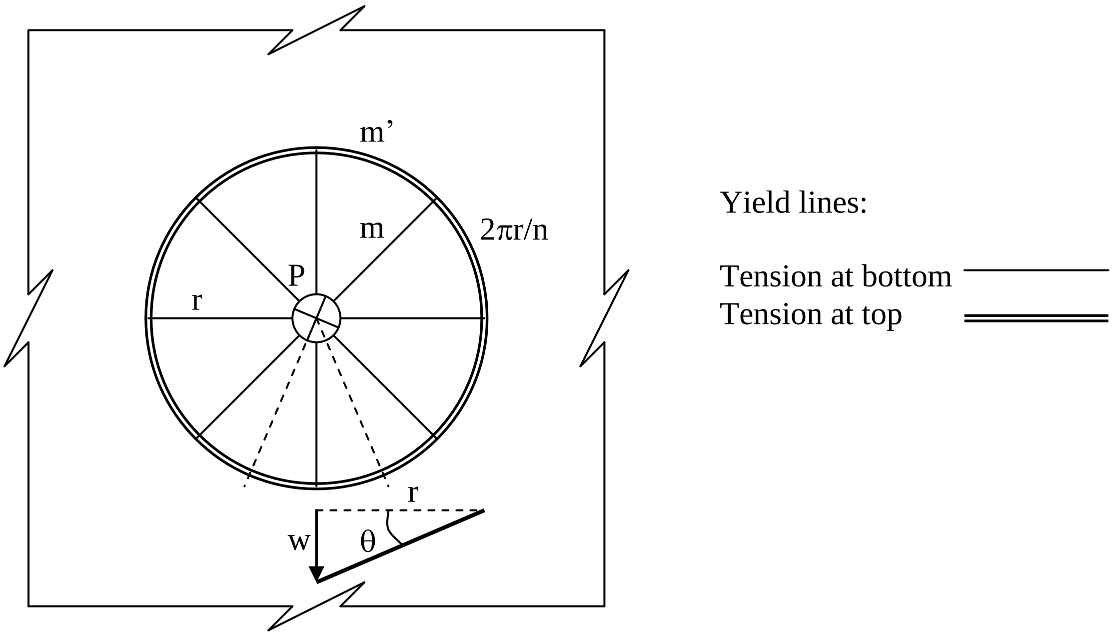


Figure 3.2.4 Circular yield line pattern in slab

With a virtual deflection w at the concentrated load, the external work is:

$$A_y = P \cdot w \tag{3.2.13}$$

Internal work :

Rotation about circular yield line : $\theta = w/r$

Internal work of a slab segment (sector of circle, between dashed lines in figure 2.2.4), a_i, determined by Eq. (3.2.4).

Along radial yield line: $$a_{i1} = m \cdot \frac{2\pi r}{n} \cdot \frac{w}{r} = \frac{2\pi m}{n} \cdot w$$

Along circular yield line: $$a_{i2} = m' \cdot \frac{2\pi r}{n} \cdot \frac{w}{r} = \frac{2\pi m'}{n} \cdot w$$

Hence, for the sector of circle: $a_i = a_{i1} + a_{i2}$

For all n slab segments:

$$A_i = n \cdot a_i = 2\pi \cdot (m + m') \cdot w \tag{3.2.14}$$

External work = Internal work gives the failure load:

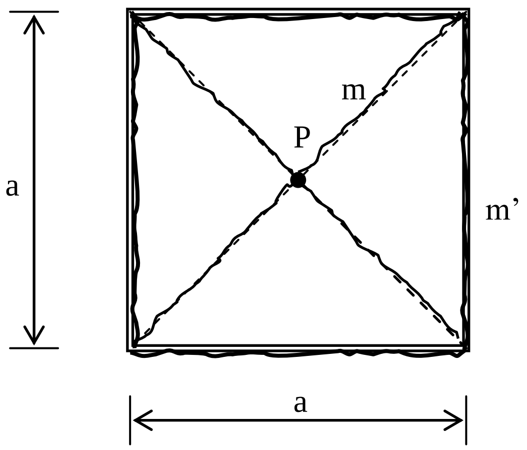
$$P_{failure} = 2\pi \cdot (m + m') \tag{3.2.15}$$

The failure load expression does neither tell how many radial yield lines that form, nor the length of the radius in the yield line pattern.

3.2.10 Quadratic slab with concentrated load

The slab is fixed along all four edges, and subjected to a concentrated load at the midpoint. The figure shows a possible yield line pattern.

Failure moment for bottom tension is *m*, and for top tension *m'*.



With a virtual deflection w at the concentrated load, the external work is :

$$A_y = P \cdot w$$

Internal work is determined by Eq. (3.2.4)

$$\begin{aligned} A_i &= 4 \cdot 2 \cdot m \cdot 0{,}5a \cdot w / 0{,}5a \quad \text{(skew yield lines)} \\ &\quad + 4 \cdot m' \cdot a \cdot w / 0{,}5a \quad \text{(support yield lines)} \\ &= 8mw + 8m'w = 8 \cdot (m + m') \cdot w \end{aligned}$$

External work = Internal work gives :

$$P_{failure} = 8 \cdot (m + m')$$

By comparing this failure load to the failure load for the circular yield line pattern, Eq. (3.2.15), it appears that the circular failure load occurs at a 20% lower load than for this yield line pattern. This patter does therefore not represent the critical, and the slab will fail with a local circular collapse mechanism.

Special corner effects

So far it has been assumed that a yield line forming in the corner of a slab enters directly into the corner. However, it is evident from the elastic theory for slabs that there are strong torsion moments in the corner regions and that if a corner of a simply supported slab is not held down, it will tend to lift off the support.

If the corners of the slab are not anchored in the support, they will lift when the slab is loaded (cf. Chapter 3.1 for elastic two-way slabs). This causes the yield line to split, and form a Y-shaped pattern in the corner, as shown in Figure 3.2.5a.

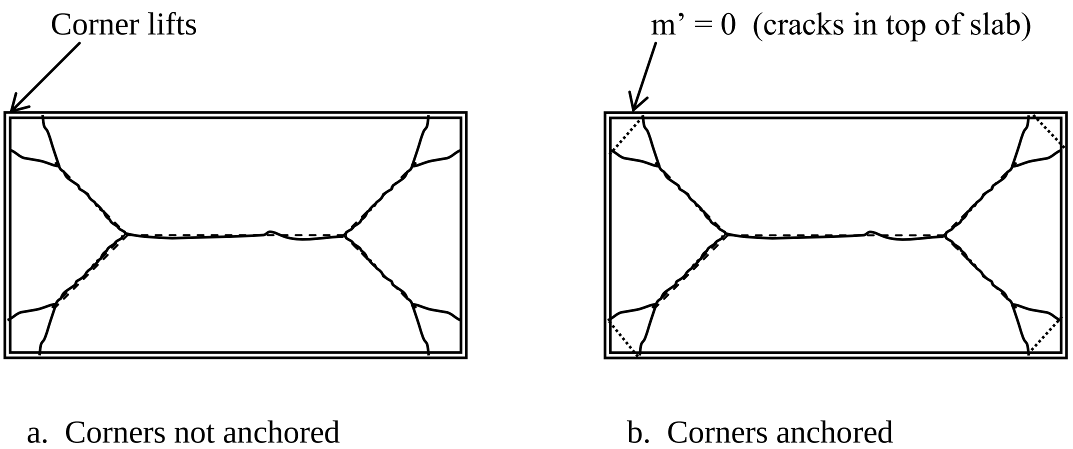


Figure 3.2.5 Y-shaped yield lines at corner of rectangular slab

If the corners are anchored, but without top reinforcement in the slab, similar Y-shaped yield lines may form. In this case cracks will develop as shown in Figure 3.2.5b. Separate reinforcement for the moment which causes tension in top of the slab will change the resistance, and will most likely result in yield lines directly to the corners instead of the Y-shaped yield lines in Figure 3.2.5.

3.2.11 Comparison of methods of analysis

Figure 3.2.6 shows a simply supported quadratic two-way slab with thickness h = 200mm and average effective depth d = 150mm.
The slab is subjected to a uniform load in ULS, q_{Ed} = 15 kN/m^2.

Materials: Concrete B30 → f_{cd} = 17 MPa ; Reinforcement B500NC → f_{yd} = 434 MPa

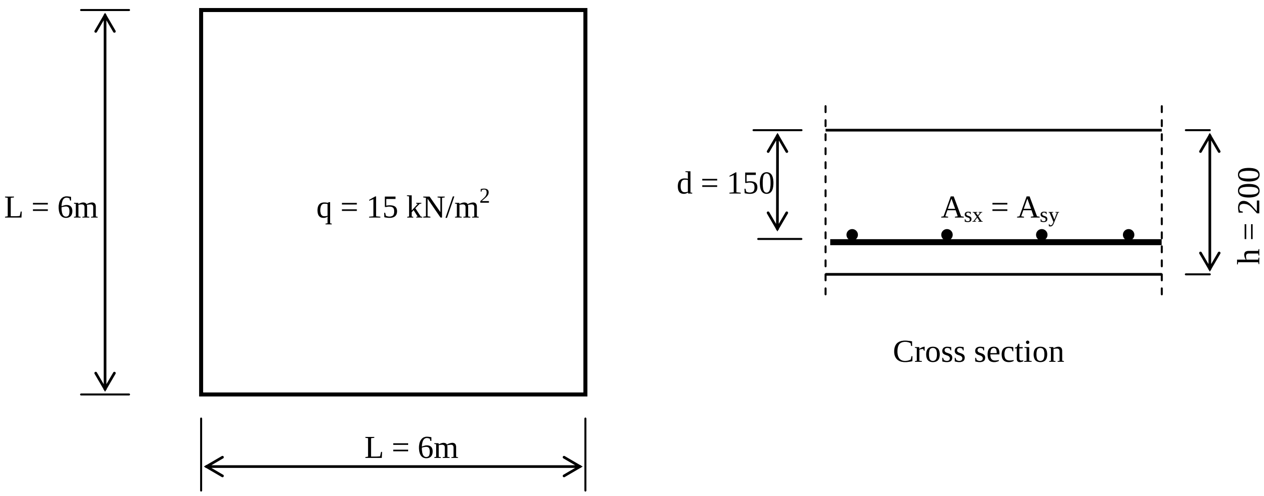


Figure 3.2.6 Simply supported slab with uniform load

Theory of elasticity

Required reinforcement is calculated for moments from Table 3.1.1:
Poisson's ratio for concrete is assumed as ν = 0,2.

From the table (with ν = 0) : $m_x = m_y = \frac{q_{Ed}L^2}{27,2} = \frac{15 \cdot 6^2}{27,2} = 19,85\text{kNm/m}$

The design moments in both directions are:

$$m_{Ed,x} = m_{Ed,y} = 19,85 + 0,2 \cdot 19,85 = 23,8\text{kNm/m}$$

Moment capacity, normal reinforced: $m_{Rd} = 0,275 \cdot 17 \cdot 10^3 \cdot 150^2 \cdot 10^{-6}$ = 150 kNm/m

$m_{Rd} >> m_{Ed}$ → Choose z = 0,95d = 142 mm

Required reinforcement: $A_{sx} = A_{sy} = \frac{m_{Ed}}{zf_{yd}} = \frac{23,8 \cdot 10^6}{142 \cdot 434} = 386\text{mm}^2/\text{m}$

Yield line theory

From the example in chapter 3.2.5: $m_d = \dfrac{qL^2}{24}$

With reinforcement according to elastic theory : m_d = 23,8 kNm/m

Failure load $q_{brudd} = \dfrac{24}{L^2} \cdot m_d = \dfrac{24}{36} \cdot 23{,}8 = 15{,}9 \text{kN/m}$

That is: The load-carrying capacity from yield line theory is approximately 6% higher than the design load, 15 kN/m. This represents an upper limit.

FEM-analysis

The FEM-program DIANA /3.2.2/ is used for a non-linear analysis of the slab.

Material models for concrete and reinforcement are shown in Figure 3.2.7.

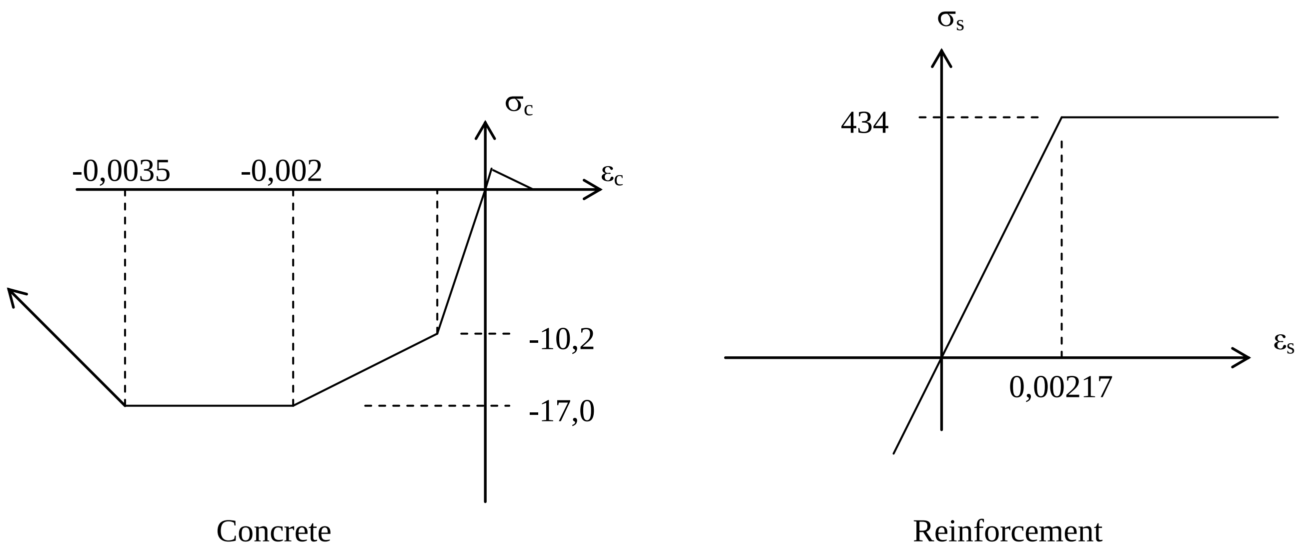


Figure 3.2.7 Material models in the DIANA-analysis

Due to double symmetry, one quart of the slab is modelled by 15x15 elements Q20SH with corner nodes with three translational and two rotational degrees of freedom.

The analysis is carried out for a reference load q = 10 kN/m^2, using arc length control in the incremental/iterative numerical solution process.

Load –deflection of the midpoint in the slab is shown in Figure 3.2.8.

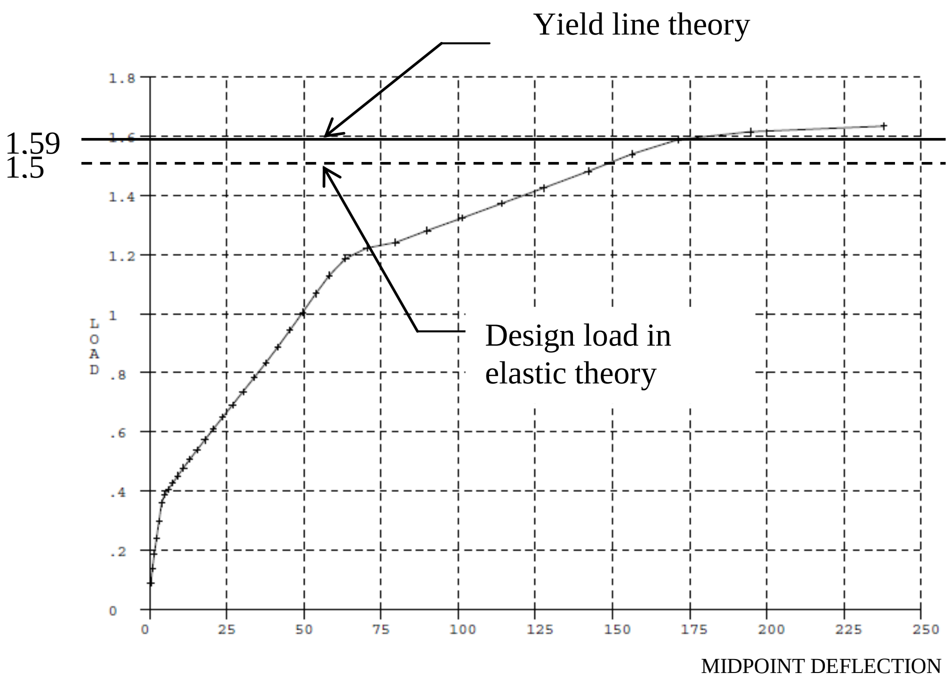


Figure 3.2.8   Load-deflection curve from DIANA-analysis

The non-linear DIANA-analysis which simulates the real behaviour of the slab, shows good correspondence to the yield line theory with a load carrying capacity 16,3 kN/m.

3.2.12 References

/3.2.1/ Johansen, K. W.: ”Brudlinieteorier” Gjellerups Forlag, København, 1943, (in Danish)

/3.2.2/ DIANA FEM Program, TNO Delft

3.3 Strip method for slabs

3.3.1 Basis of the simple strip method

A lower bound design method for reinforced concrete two-way slabs was suggested by Hillerborg i 1956 /3.3.1/.

The equilibrium equation for a plate element was given in Chapter 3.1, Eq. (3.1.1):

$$\frac{\partial^2 m_x}{\partial x^2} + 2 \cdot \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = -q \qquad (3.3.1)$$

Here, m_x og m_y are bending moments in x- and y-direction, respectively, m_{xy} is torsion moment and q is uniformly distributed load.

According to the lower bound theory, any combination of m_x , m_y and m_{xy} that satisfies Eq. (3.3.1) at all points in the slab and the boundary conditions when the ultimate load is applied is a valid design solution provided that reinforcement can be placed to carry these moments.
Thus, the external load q can be apportioned arbitrarily between the terms $\partial^2 m_x / \partial x^2$, $2 \cdot \partial^2 m_{xy} / \partial x \, \partial y$ and $\partial^2 m_y / \partial y^2$.

Hillerborg chooses a solution where $m_{xy} = 0$, and carries the load entirely by the $\partial^2 m_x / \partial x^2$ and $\partial^2 m_y / \partial y^2$ terms. This means that the load is carried entirely by bending in the x- and y-directions, and hence that the slab can be visualized as being composed of two systems of strips running in the x- an y- directions. The method is termed ”Strip method”.

Without the coupling term of Eq. (3.3.1), the equation can be replaced by two equations that represent twistless strip action:

$$\frac{\partial^2 m_x}{\partial x^2} = -\gamma q \qquad (3.3.2)$$

and

$$\frac{\partial^2 m_y}{\partial y^2} = -(1-\gamma) q \qquad (3.3.3)$$

where γ is a factor chosen by the designer, $0 \le \gamma \le 1{,}0$.
The value of γ may vary throughout the slab without affecting its validity. Note that if $\gamma = 1$, all the load is carried by bending of the x-direction strips, and if $\gamma = 0$, all the load is carried by the y-direction strips.

The use of Eqs. (3.3.2) and (3.3.3) to find possible design moment fields will be illustrated for the case of a square, simply supported slab carrying a uniformly distributed ultimate load per unit area, q.

Case 1

Choose γ = 0,5 over the entire area of the slab. The results are shown in Figure 3.3.1.

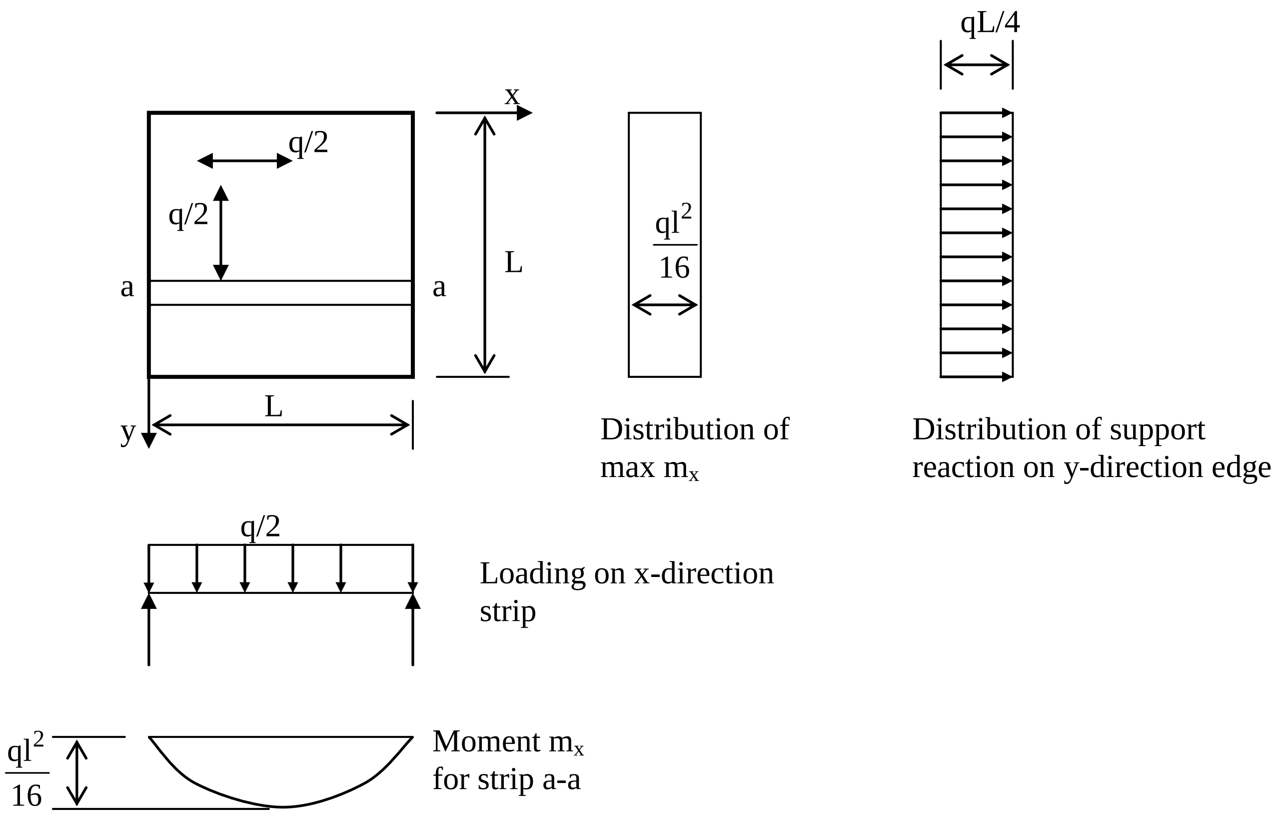


Figure 3.3.1 Quadratic slab – Case 1 , γ = 0,5 over the entire slab

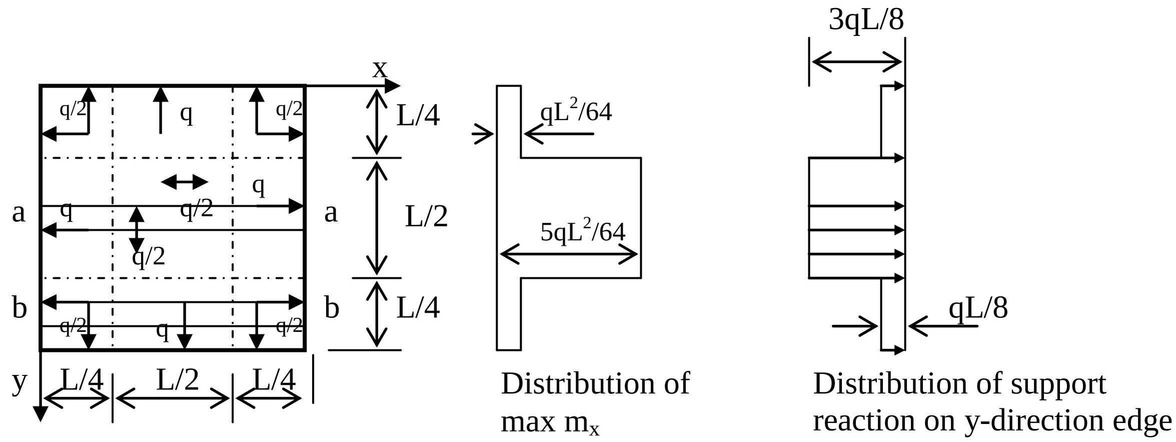
In this case half the load is allocated uniformly to the strips in each direction, as indicated by the dispersion arrows in the figure.

The resulting x-direction moments, obtained by simple statics for uniform load per unit area q/2 on the strips, are shown in the figure. The distribution of y-direction moments is similar to the x-direction moments. Thus, the maximum moment per unit width in each direction is $qL^2/16$ and has a constant value at the midspan sections of the slab.

The distribution of the loading acting on the edge support of the slab is also shown. This loading is simply the end reactions of the strips and acts on the supporting beam or wall.

Case 2

This case, which is shown in Figure 3.3.2, is obtained by giving γ values that depend on the region of the slab. The slab is divided into three regions, corresponding to the slab corners, middle edges and centre region of the slab. The load is allocated to the strips in each direction within the regions in the manner indicated by the load-dispersion arrows. Two basic types of strip loading exist, shown as strips a-a and b-b.



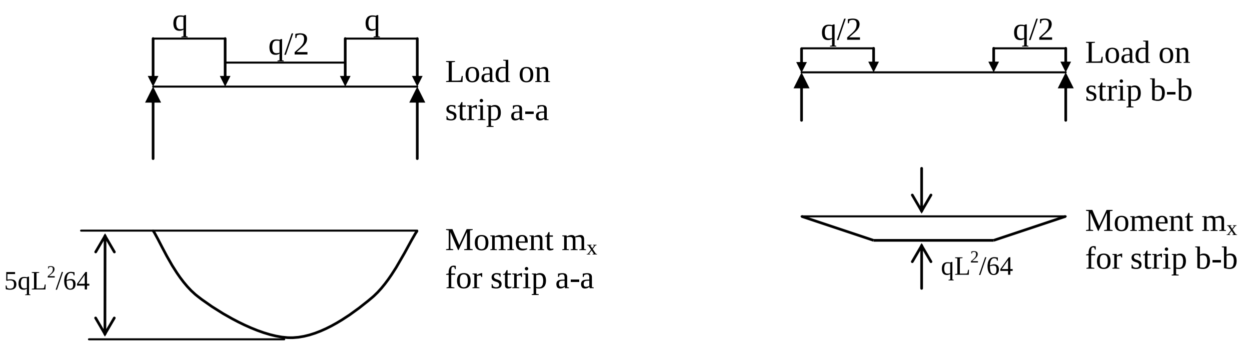


Figure 3.3.2 Quadratic slab - Case 2, γ = 1,0 and 0,5

The resulting x-direction moments can be obtained by simple statics. The maximum x-direction moments for x-direction strips a-a and b-b are different.

The distribution of y-direction moments is similar to the x-direction moments. Thus, the maximum moment per unit width in each direction is $5qL^2/64$, constant across the middle half of the slab, with a moment per unit width of $qL^2/64$ in the edge strips.

The distribution of the loading action on the edge supports of the slab is also shown.

Case 3

This case, which is shown in Figure 3.3.3, is obtained by giving γ values of either 0 or 1, depending on the region of the slab. The slab is divided into triangular regions by diagonal lines, and the load on the triangles is transferred to the nearest support, as indicated by the load-dispersion arrows.
Each strip therefore carries a uniform load per unit area, q , over the end regions.

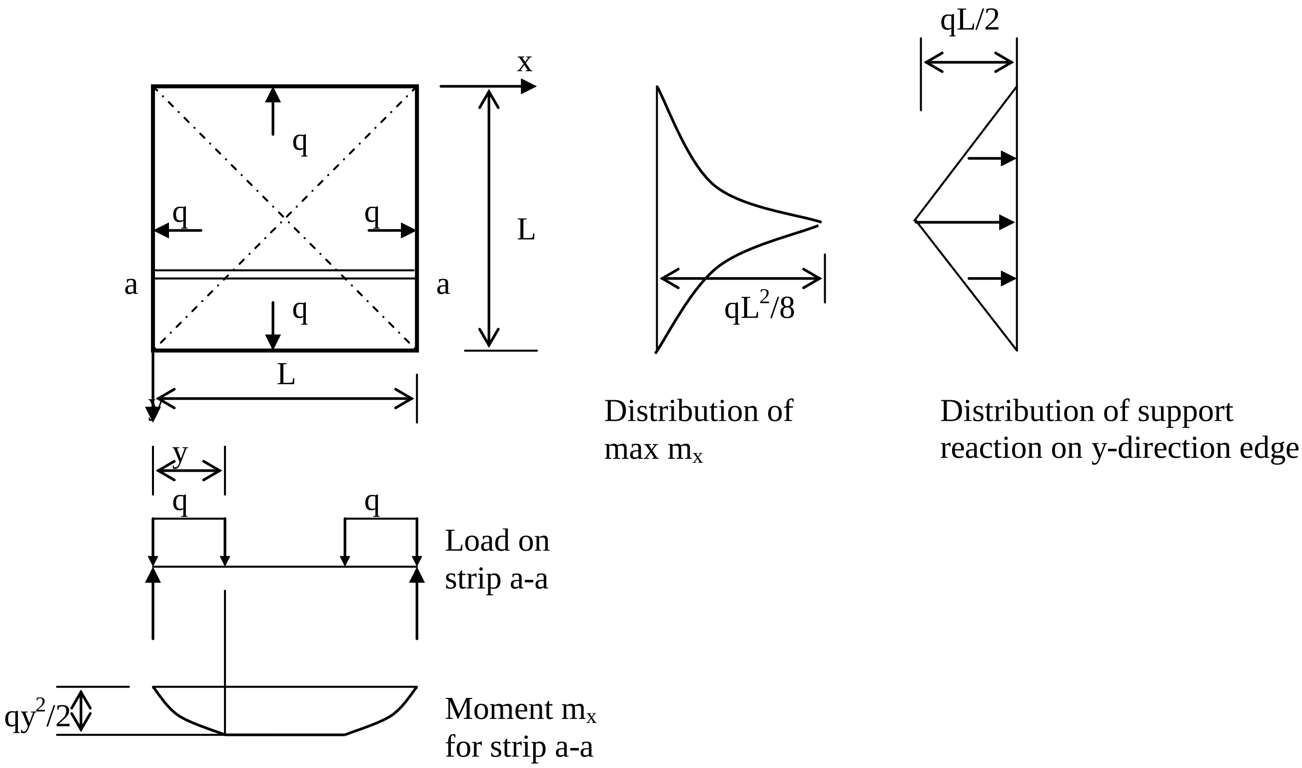


Figure 3.3.3 Quadratic slab – Case 3, γ = 1 or 0

The resulting x-direction moments can be obtained by simple statics. The maximum x-direction moment is a function of y and rises sharply to a peak at the slab centre.

The distribution of y-direction moments is similar to the x-direction moments.
The maximum moment per unit width is $qL^2/8$.

The distribution of the loading acting on the edge support is also shown. The triangular shape of the edge load is similar to that assumed by many designers.

The three cases illustrate two features of the strip method.

The first is the ease with which the moments in the slab and the loads on the supporting system can be obtained by the use of simple statics.
The second is the variety of moment and load distributions possible depending on the assumed manner of load dispersion.

For many years designers have used strip action intuitively to approximate the moments in slabs of awkward shape or boundary conditions. It is of interest to note that such an approach has the full formal backing of lower bound limit design.
It is of interest to look at the relative economy, from the point of view of the reinforcing steel requirements, of the three cases. The area of steel per unit width is proportional to the moment per unit width. Suppose that all the slab bars run the full length L of the slabs, and that all bars have the same effective depth. Then the value of steel in the slab is proportional to the area of the diagram showing the distribution of maximum m_x.

For cases 1, 2 and 3, these areas are in the ratio 1,00 : 0,75 : 0,67, respectively, indicating the relative economies. Case 3 is seen to be the most economical, but note that for case 3 the maximum moment varies continuously over the slab width, which implies that in order to obtain the ideal value of 0,67, the spacing of reinforcement bars has to vary continuously. This is obviously impracticable. Thus, in practice for case 3 the bars would need to be placed in several uniform bands to cope with the distribution of moments, and the ratio will increase towards the value for case 2.

The lines on the slabs that indicate the region of different load dispersion will be referred to as ”discontinuity lines”.
Case 1 uses the simplest possible load dispersion assumption. Case 2 and 3 offer alternatives of discontinuity lines originating from either the slab corners or the slab sides. These two possibilities are discussed further in the following.

3.3.2 Discontinuity lines originating from slab corners

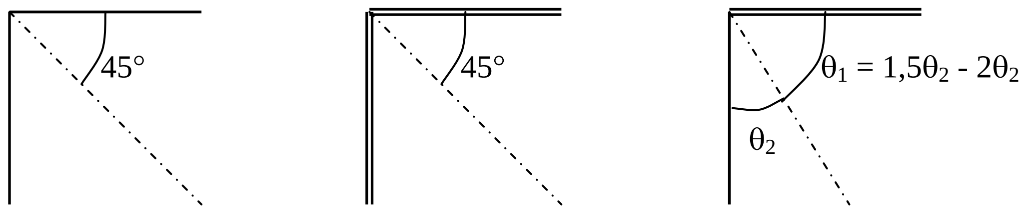
Assuming that the load is carried by the nearest support edge, the discontinuity lines enter the slab corners. Strictly, the discontinuity lines can enter a slab corner at any angle, but angles are best selected on the basis of the moments giving economy of reinforcing steel and reasonable accordance with the elastic moment distribution.

The following rules for right-angle corners were suggested by Hillerborg:

- Where two simply supported or two fixed edges meet, the discontinuity line should make 45° with the edges (bisecting the corner angle).
- Where a simply supported and a fixed edge meet, the discontinuity line should make an angle with the fixed edge about 1,5 – 2 times the angle with the simply supported edge.

These rules can also be used as guidelines for discontinuity lines in skew slabs.

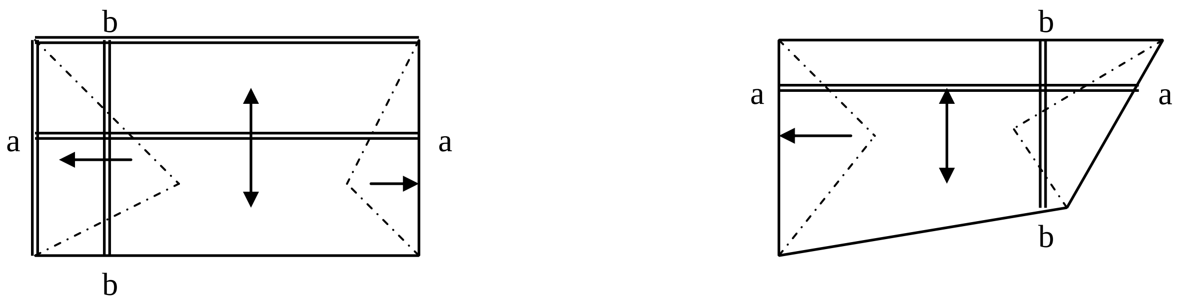
Figure 3.3.4 shows examples of discontinuity lines following these principles.



Two simply supported edges

Two fixed edges

One simply supported and one fixed edge



Rectangular slab with various support conditions

Skew slab

Figure 3.3.4 Discontinuity lines originating from slab corners

A problem that arises when reinforcement is being designed for the bending moments obtained from the strip method with discontinuity lines originating from the corners, is that over a large part of rectangular slabs and throughout non-rectangular slabs, the bending moments can change rapidly and theoretically require continuously variable bar spacing. Reinforcement to follow such a distribution of moments is obviously impracticable.

In such cases the reinforcement can be placed in bands with constant bar spacing, dimensioned for the average maximum moment in the band. Design on the basis of such bands is strictly not in accordance with lower bound theory because at the ultimate load the theoretical moments will exceed the ultimate moments of resistance over a part of each band. However, once yielding occurs, it is reasonable to expect the moments to redistribute themselves. Also, the total available ultimate moment resistance across a band is equal to the required value.

3.3.3 Discontinuity lines originating from slab sides

There is no reason why the discontinuity lines should originate from the corners or be straight. Wood and Armer /3.3.2/ have pointed out that rather than complicating the calculations by using triangular and trapezoidal shapes for the loaded regions of bands, the discontinuity lines could be drawn to cross each band at right angles and thus allow direct determination of the maximum design moment in the bands without any averaging. In addition to simplifying the calculations, the solution is now exact and in accordance with strict lower bound theory. Such a procedure is illustrated in Figure 3.3.5.

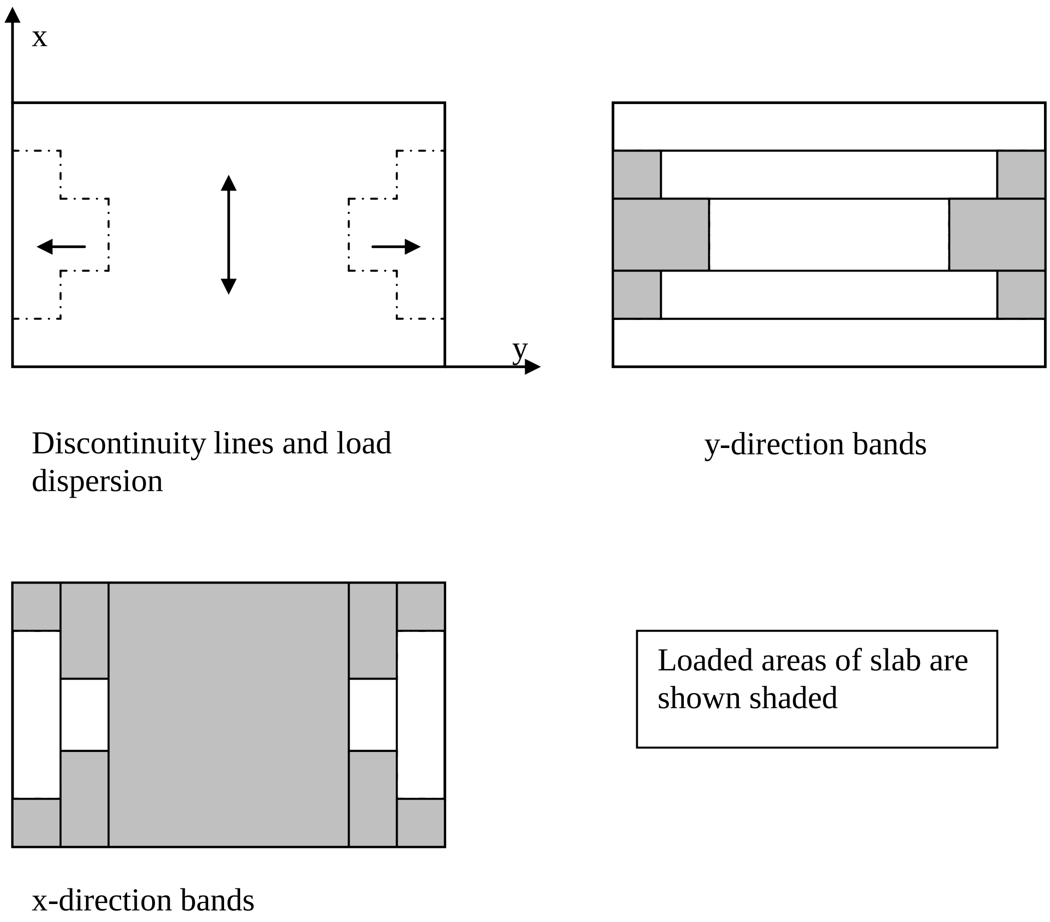


Figure 3.3.5 Discontinuity lines originating from slab sides

All the bands (strips) in Figure 3.3.5 can now be analysed by simple statics as simply supported beams with different loading.

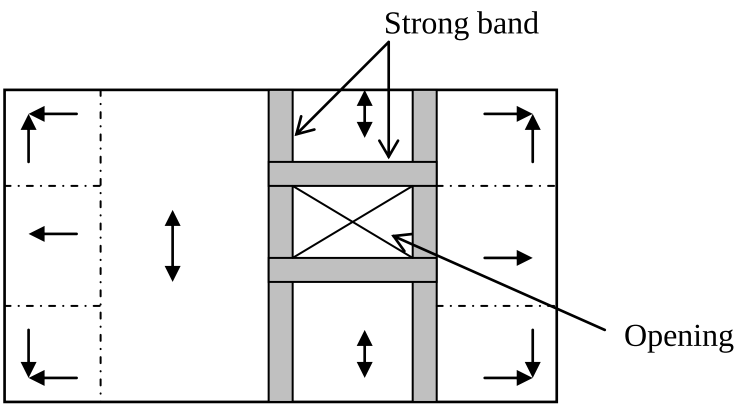
Required reinforcement for maximum moment in each strip makes the design exact according to lower bound theory.

3.3.4 Slabs with free edges and large openings

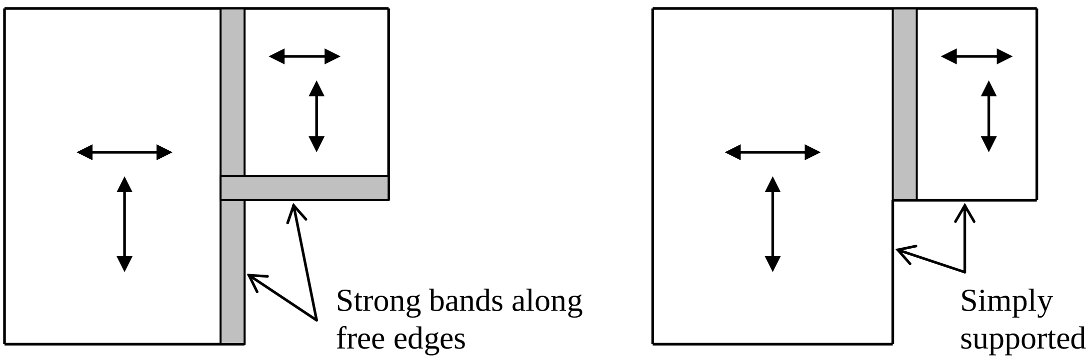
The simple strip method cannot deal with slabs with openings, re-entrant corners, free edges and beamless slabs with column supports.

To cope with this, Wood and Armer /3.3.2/ suggested an approach with so-called “strong bands”. A strong band is a strip of slab of reasonable width that contains a concentration of reinforcement and hence acts as a beam within the slab.

Figure 3.3.6 shows examples of strong bands.



Simply supported slab with opening. Uniform load



Slabs with re-entrant corners – with and without supports

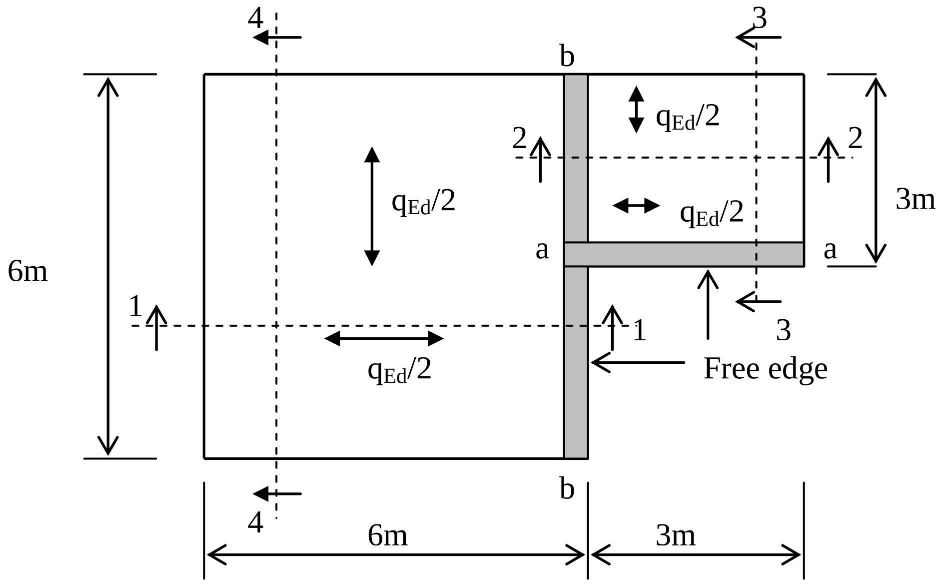
Figure 3.3.6 Slabs with strong bands

The longitudinal strong band reinforcement, acting as “beam reinforcement”, has to be enclosed by minimum shear links according to Eurocode2.

3.3.5 Example - Design of slab with unsupported re-entrant corner

The slab in the figure will be dimensioned in ULS

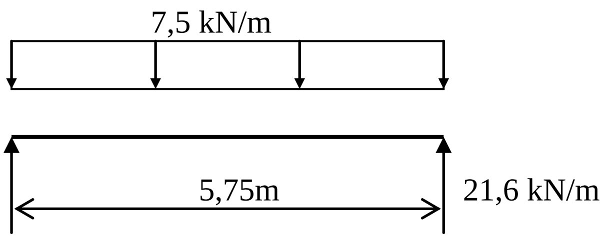
GIVEN DATA:
Load per unit area in ULS (self-weight and live load) : q_{Ed} = 15 kN/m^2
Slab thickness: h = 250mm
Concrete: B35 Reinforcement: B500NC



The figure shows four slab sections and two strong bands that have to be analysed to obtain the design bending moments. The assumed load dispersion is also shown.
The width of the strong bands is chosen as b = 500mm.

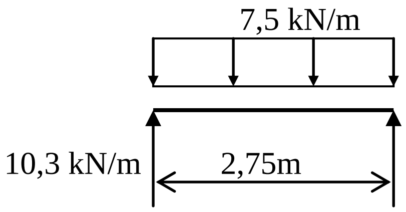
Static analyses :

Slab strip 1-1:



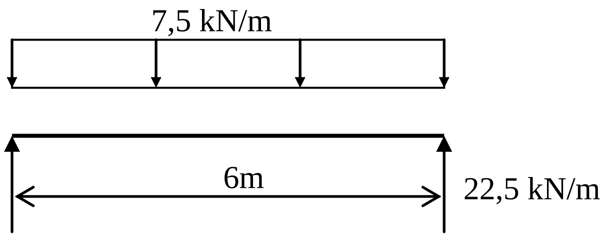
Span is 5,75m to middle of strong band b-b.
Support reaction: 7,5·5,75/2 = 21,6 kN/m
Maximum moment: 7,5·5,75^2/8 = 31,0 kNm/m

Slab strip 2-2 og 3-3:



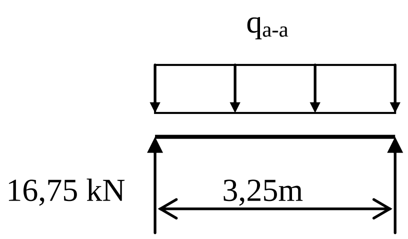
Span is 2,75m to middle of strong bands b-b and a-a.
Support reaction: $7{,}5 \cdot 2{,}75/2 = 10{,}3$ kN/m
Maximum moment: $7{,}5 \cdot 2{,}75^2/8 = 7{,}1$ kNm/m

Slab strip 4-4:



Span 6m.
Support reaction: $7{,}5 \cdot 6/2 = 22{,}5$ kN/m
Maximum moment: $7{,}5 \cdot 6^2/8 = 33{,}75$ kNm/m

Strong band a-a:



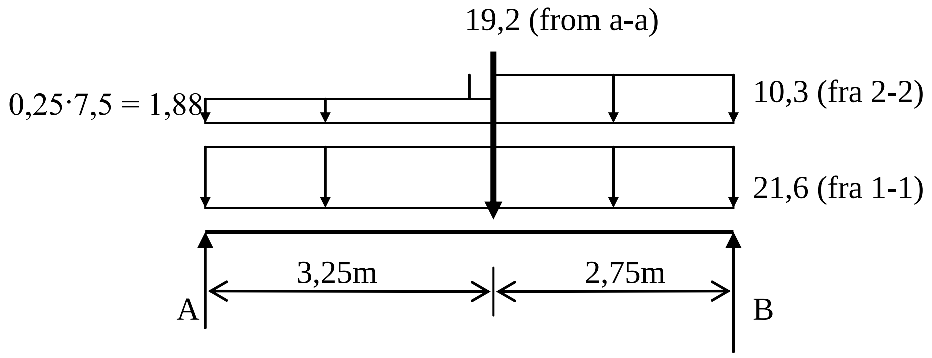
Load: $q_{a\text{-}a} = 10{,}3 + 0{,}25 \cdot 7{,}5 = 12{,}18$ kN/m

(10,3 is support reaction of slab strip 3-3, while $0{,}25 \cdot 7{,}5$ is load outside centre line in a-a)

Support reaction: $12{,}18 \cdot 3{,}25/2 = 19{,}2$ kN

Maximum moment: $M_{Ed,a\text{-}a} = 12{,}18 \cdot 3{,}25^2/8 = 16{,}1$ kNm

Strong band b-b:



Moment equilibrium about left end gives support reaction B = 98,7 kN
Vertical equilibrium gives support reaction A = 84,5 kN

Maximum moment: $M_{Ed,b\text{-}b} = 84{,}5 \cdot 3{,}25 - (21{,}6+1{,}88) \cdot 3{,}25^2/2 = 150{,}6$ kNm

Dimensioning:

Slab strip 4-4

Choose reinforcement diameter 10mm and cover 35mm → $d_{4\text{-}4} = 210$mm

Moment capacity: $M_{Rd,4\text{-}4} = 0{,}275 f_{cd} d_{4\text{-}4}^2 = 0{,}275 \cdot 19{,}8 \cdot 10^3 \cdot 210^2 \cdot 10^{-6} = 240$ kNm/m

$M_{Ed,4\text{-}4} = 33{,}75 << M_{Rd,4\text{-}4}$ → $z_{4\text{-}4} = 0{,}95 d_{4\text{-}4} = 199$mm

Required reinforcement: $A_{s,4-4} = \dfrac{M_{Ed,4-4}}{z_{4-4} \cdot f_{yd}} = \dfrac{33{,}75 \cdot 10^6}{199 \cdot 434} = 391\text{mm}^2/\text{m}$

Slab strip 1-1:

$d_{1\text{-}1} = d_{4\text{-}4} - 10 = 200$mm

$M_{Ed,1\text{-}1} = 31{,}0$ kNm/m $<< M_{Rd,1\text{-}1}$ → $z_{1\text{-}1} = 0{,}95 d_{1\text{-}1} = 190$mm

Required reinforcement: $A_{s,1-1} = \dfrac{M_{Ed,1-1}}{z_{1-1} \cdot f_{yd}} = \dfrac{31{,}0 \cdot 10^6}{190 \cdot 434} = 376\text{mm}^2/\text{m}$

Slab strip 2-2 and 3-3:

$M_{Ed,2\text{-}2} = M_{Ed,3\text{-}3} = 7{,}1$ kNm/m $\ll M_{Rd} \rightarrow$ $z \approx 190$mm

Required reinforcemen : $A_{s,2-2} = A_{s,3-3} = \frac{M_{Ed,2-2}}{z \cdot f_{yd}} = \frac{7{,}1 \cdot 10^6}{190 \cdot 434} = 86\text{mm}^2/\text{m}$

Minimum slab reinforcement according to EC2:

$$A_{s,min} = 0{,}26 \cdot \frac{f_{ctm}}{f_{yk}} \cdot bd = 0{,}26 \cdot \frac{3{,}2}{500} \cdot 10^3 \cdot 210 = 350\text{mm}^2/\text{m}$$

Choose equal slab reinforcement in both directions in entire slab:

ø10s200 $\rightarrow$ $A_s = 392$ mm^2/m

Strong band a-a:

Try reinforcement diameter 12mm because design moment is small.

Effective depth: $d_{a\text{-}a} = 209$mm

Moment capacity : $M_{Rd,a\text{-}a} = 0{,}275 f_{cd} b d_{a\text{-}a}^2 = 0{,}275 \cdot 19{,}8 \cdot 500 \cdot 209^2 \cdot 10^{-6} = 120$ kNm

$M_{Ed,a\text{-}a} = 16{,}1$ kNm $\rightarrow$ $z_{a-a} = \left(1 - 0{,}17 \cdot \frac{16{,}1}{120}\right) d_{a-a} = 0{,}98 d_{a-a}$ $\rightarrow$ $z = 0{,}95 d_{a\text{-}a} = 199$mm

Required reinforcement : $A_{s,a-a} = \frac{16{,}1 \cdot 10^6}{199 \cdot 434} = 186\text{mm}^2$ $\rightarrow$ 2ø12 = 226,2mm^2

Strong band b-b:

Try reinforcement diameter 20mm because design moment is large.

Effective depth : $d_{b\text{-}b} = 205$mm

Moment capacity : $M_{Rd,b\text{-}b} = 0{,}275 f_{cd} b d_{b\text{-}b}^2 = 0{,}275 \cdot 19{,}8 \cdot 500 \cdot 205^2 \cdot 10^{-6} = 114$ kNm

$M_{Ed,b\text{-}b} = 150{,}6$ kNm $> M_{Rd,b\text{-}b}$ $\rightarrow$ Compression reinforcement for $\Delta M_{Ed} = 36{,}6$ kNm

Required reinforcement:

$$A_{s1} = \frac{M_{Rd,b-b}}{0,83d_{b-b} \cdot f_{yd}} = \frac{114 \cdot 10^6}{0,83 \cdot 205 \cdot 434} = 1544\text{mm}^2$$

$$A_{s2} = \frac{\Delta M_{Ed}}{h' \cdot f_{yd}} = \frac{36,6 \cdot 10^6}{160 \cdot 434} = 527\text{mm}^2$$

Bottom reinforcement $A_{s,b-b} = A_{s1} + A_{s2} = 2057$ mm^2 → 7ø20 = 2198 mm^2

Top reinforcement : $A_{s,b-b}' = A_{s2} = 527$ mm^2 → 2ø20 = 628 mm^2

The longitudinal reinforcement has to be enclosed by minimum shear links according to EC2, NA.9.2.2(5):

$$\frac{A_{sw}}{s} = \rho_{w,min} \cdot b = 0,1 \cdot \frac{\sqrt{35}}{500} \cdot 500 = 0,592\text{mm}^2/\text{mm}$$

With stirrups ø8 (double section) is $A_{sw} = 100,5$ mm^2

Spacing: $s \leq 100,5/0,592 = 170$ mm

Choose stirrups ø8s170

3.3.6 References

/3.3.1/ Hillerborg, A : ”Jamviktsteori for armerade betongplattor”, Betong, Vol.41, nr.4, 1956. (In Swedish)

/3.3.2/ Wood, R.H and Armer, G.S.T : ”The theory of the Strip method for Design of Slabs”, Proceedings Institute of Civil Engineering, Vol.41, Oct. 1968

/3.3.3/ Hillerborg, A : ”Strimlemetoden for plattor på pelare, vinkelplattor m.m”, Utgiven av Svenska Riksbyggen, 1959. (In Swedish)

3.4 Flat slabs

3.4.1 Definitions

A flat slab is a plate which is supported directly on columns, without beams between the columns. The columns are normally arranged in a rectangular pattern.

Figure 3.4.1 shows a flat slab with a direct connection between the column cross section and the slab. This is often termed “flat plate”.

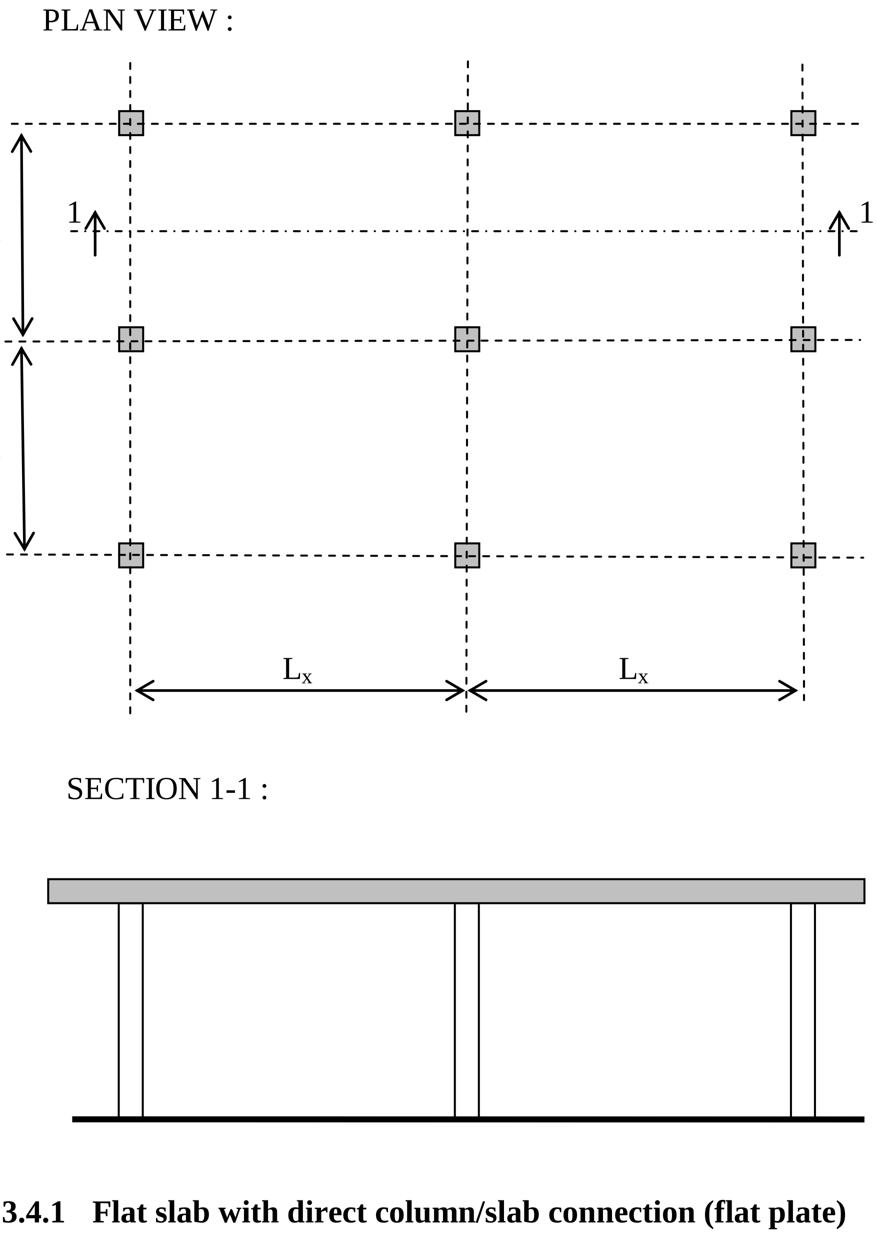


Figure 3.4.1 Flat slab with direct column/slab connection (flat plate)

Local shear at the columns is often critical for flat slabs. If the shear capacity is too low, a local punching failure may happen. This will require shear reinforcement or alternatively increased support area.

This can be done as shown in Figure 3.4.2, with capitals and drop panels.

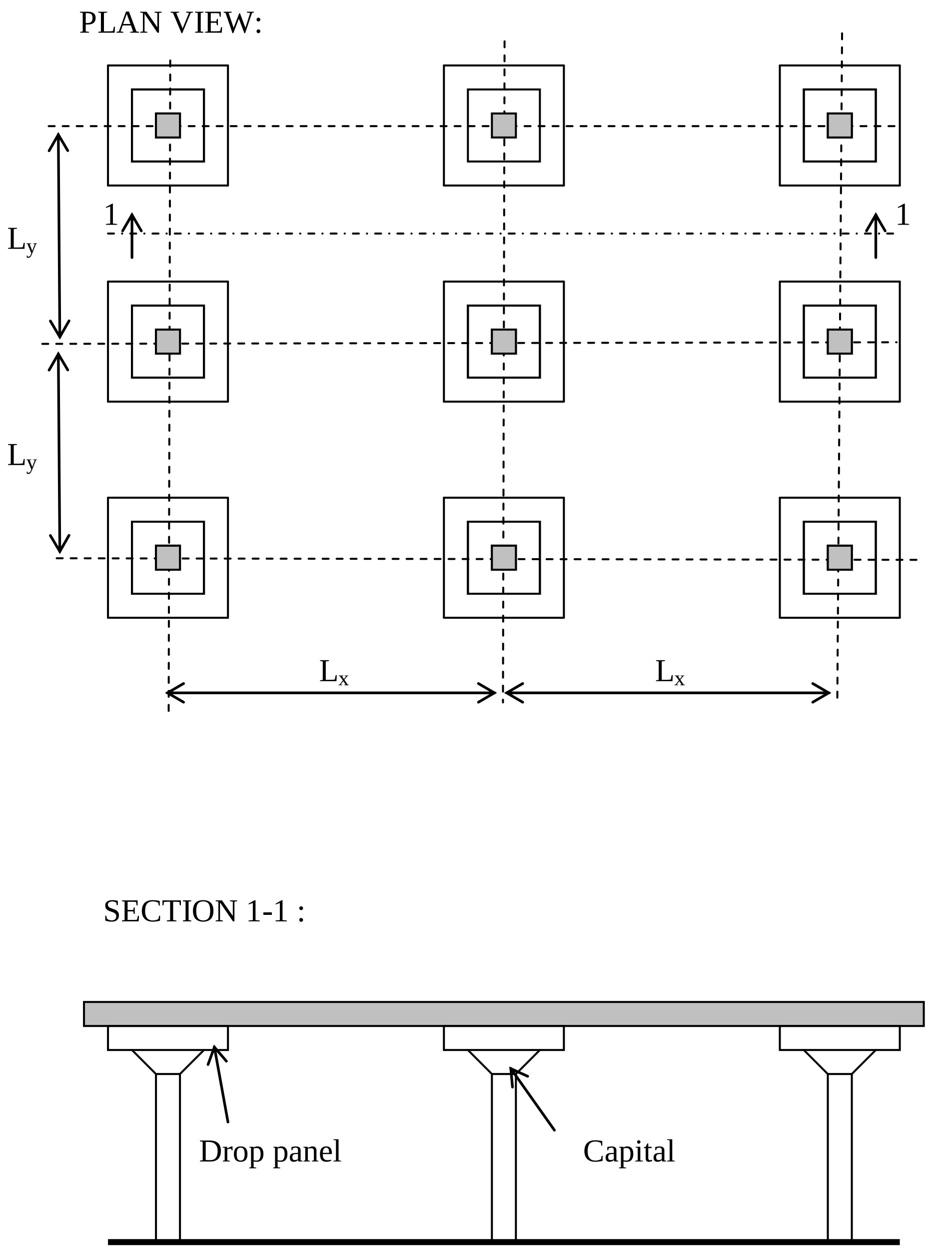
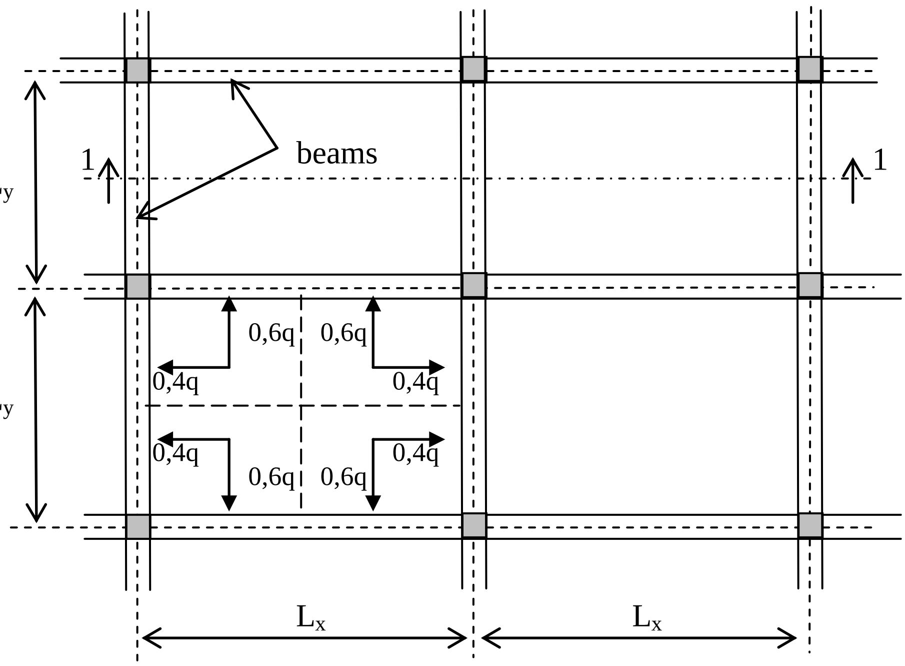


Figure 3.4.2 Flat slab with capitals and drop panels

3.4.2 Static behaviour

Figure 3.4.3 shows a traditional plate/beam – slab, where the plate regions act as two-way slabs, and the load in each region is carried by the four supporting beams.
The figure shows example of load dispersion according to the strip method. Based on the actual span ratio – i.e. the primary direction is the y-direction.

PLAN VIEW :



SECTION 1-1 :

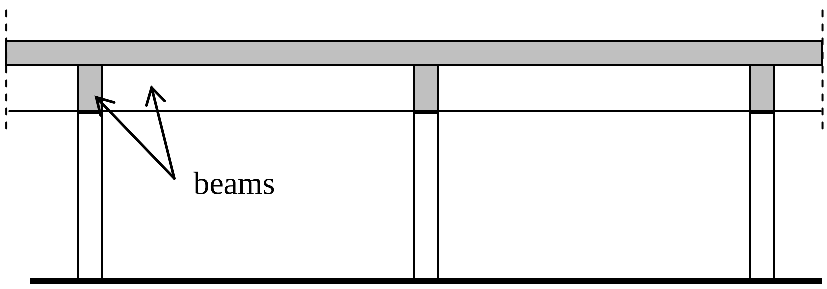


Figure 3.4.3 Plate/beam – slab with load dispersion according to the strip method

The static behaviour of a flat slab can be compared to a plate/beam slab, as shown in Figure 3.4.3.
Imagine that the beam depths are decreased to the thickness of the plate. The result is a flat slab. The slab can still be considered as a system with very wide crossing beams, and where parallel beams are touching each other as shown in Figure 3.4.4.

The width of the strip is the span length in the transverse direction.

The flat slab strip has to carry the entire load on the strip in its span direction (x-direction in Figure 3.4.4). The total load also has to be carried by the y-direction strips.

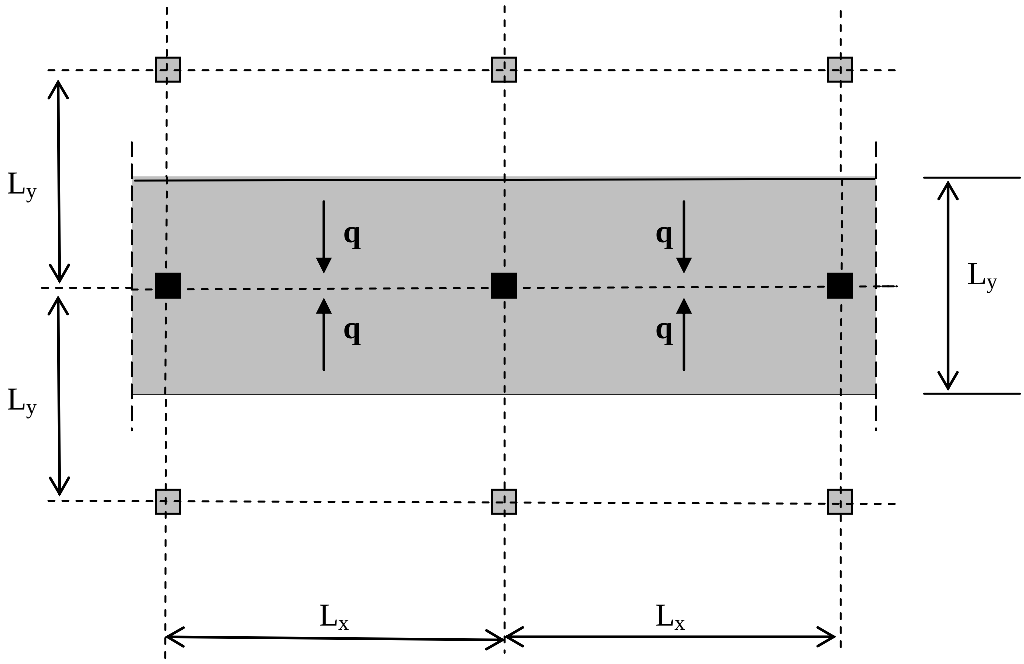


Figure 3.4.4 Load on flat slab strip in one direction

3.4.3 Flat slabs in Eurocode 2

EC2, Annex I gives the following recommendations for analysis of flat slabs:

EC2, I.1.1 General

(1) Flat slabs may be of uniform thickness or they may incorporate drops (capitals or drop panels over columns).

(2) Flat slabs should be analysed using a proven method of analysis, such as grillage, finite element, yield line or equivalent frame. Appropriate geometric and material properties should be employed.

Equivalent frame analysis is the most common method of analysis for determination of load actions in the plate and columns in a flat slab, and will therefore be used here.

Eurocode 2 also gives recommendations for idealizing the three-dimensional column/plate structure as plane frames:

EC2, I.1.2 Equivalent frame analysis

(1) The structure should be divided longitudinally and transversely into frames consisting of columns and sections of slabs contained between the centre lines of adjacent panels (area bounded by four adjacent supports). The stiffness off members may be calculated from their gross cross-sections. For vertical loading the stiffness may be based on the full width of the panels. For horizontal loading 40% of this value should be used to reflect the increased flexibility of the column/slab joints in flat slab structures compared to that of column/beam joints.
Total load on the panel should be used for the analysis in each direction (i.e. as pointed out in chapter 3.4.2).

(2) The total bending moments obtained from analysis should be distributed across the width of the slab. In elastic analysis negative moments (tension in top) tend to concentrate towards the centre lines of the columns.

(3) The panels should be assumed to be divided into column and middle strips (see Figure 3.4.5), and the bending moments should be apportioned as given in Table 3.4.1.

(4) Where the width of the column strip is different from $0,5L_x$, as shown in Figure 3.4.5, the middle strip width should be adjusted accordingly.
(This is different from common practice in Norway, where the same relation between widths of column and middle strips are used in both directions).

(5) Unless there are perimeter beams, which are adequately designed for torsion, moments transferred to edge or corner columns should be limited to the moment of resistance of a rectangular section equal to $0,17f_{ck}b_ed^2$ ($\approx 0,275f_{cd}b_ed^2$).
The width b_e is defined in EC2, 9.4.2.
The positive moment in the end span should be adjusted accordingly.

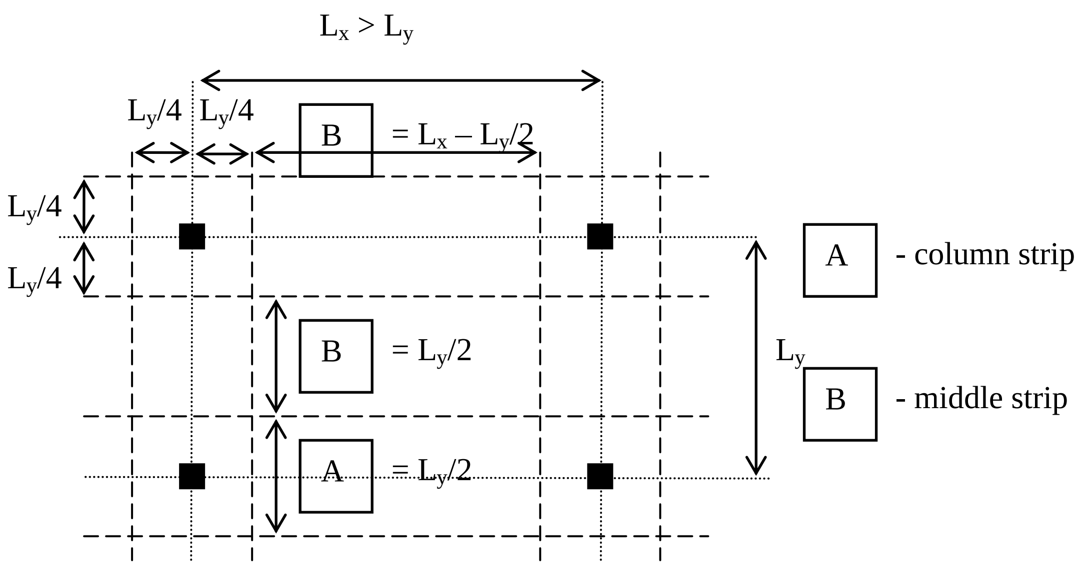


Figure 3.4.5 Division of panels in flat slabs in column and middle strips

Table 3.4.1 Simplified apportionment of bending moments for a flat slab

	Negative moments	**Positive moments**
Column strip	60 – 80 %	50 – 70 %
Middle strip	40 – 20 %	50 – 30 %
NOTE: Total negative and positive moments to be resisted by the column and the middle strip together should always add up to 100 %		

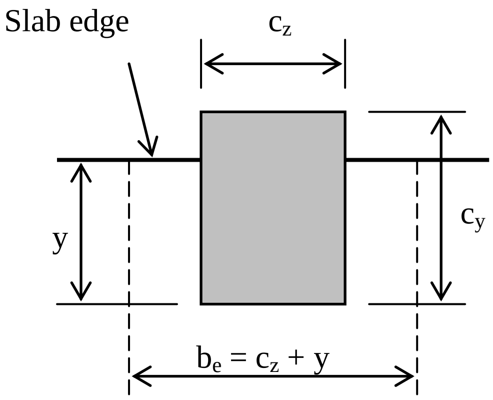
EC2, 9.4 gives structural rules for flat slabs:

EC2, 9.4.1(1):
The arrangement of reinforcement in flat slab construction should reflect the behaviour under working conditions. In general this will result in a concentration of reinforcement over the columns.

EC2, 9.4.1(2):
At internal columns, unless rigorous serviceability calculations are carried out, top reinforcement of area $0{,}5A_t$ should be placed in a width equal to the sum of 0,125 times the panel width on either side of the column. A_t represents the area of reinforcement required to resist the full negative moment from the sum of the two half panels at each side of the column.

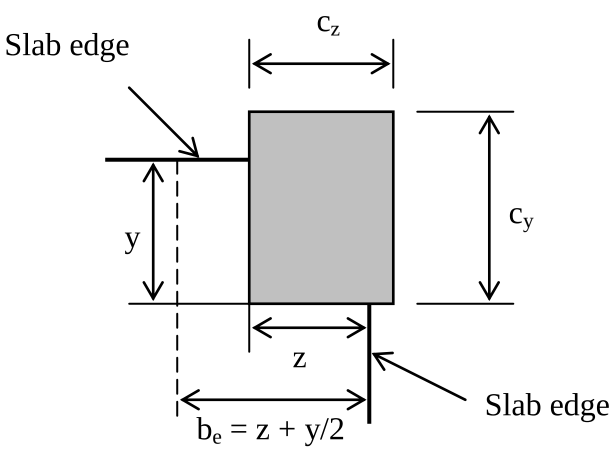
EC2, 9.4.1(3):
Bottom reinforcement (≥ 2 bars) in each orthogonal direction should be provided at internal columns and the reinforcement should pass through the column.

EC2, 9.4.2(1):
Reinforcement perpendicular to a free edge required to transmit bending moments from slab to an edge or corner column should be placed within the effective width b_e shown in Figure 3.4.6.
This is the effective width in the moment capacity formula in EC2, I.1.2 (5).



Note : y can be > c_y

a) Edge column



NOTE : y can be > c_y
and z can be > c_z

b) Corner column

Figure 3.4.6 Effective width, b_e , of a flat slab

3.4.4 Flat slab analysis by equivalent frame method

The flat slab in Figure 3.4.7 is modelled by plane frames in both directions. The slab width is chosen as the transverse span length.

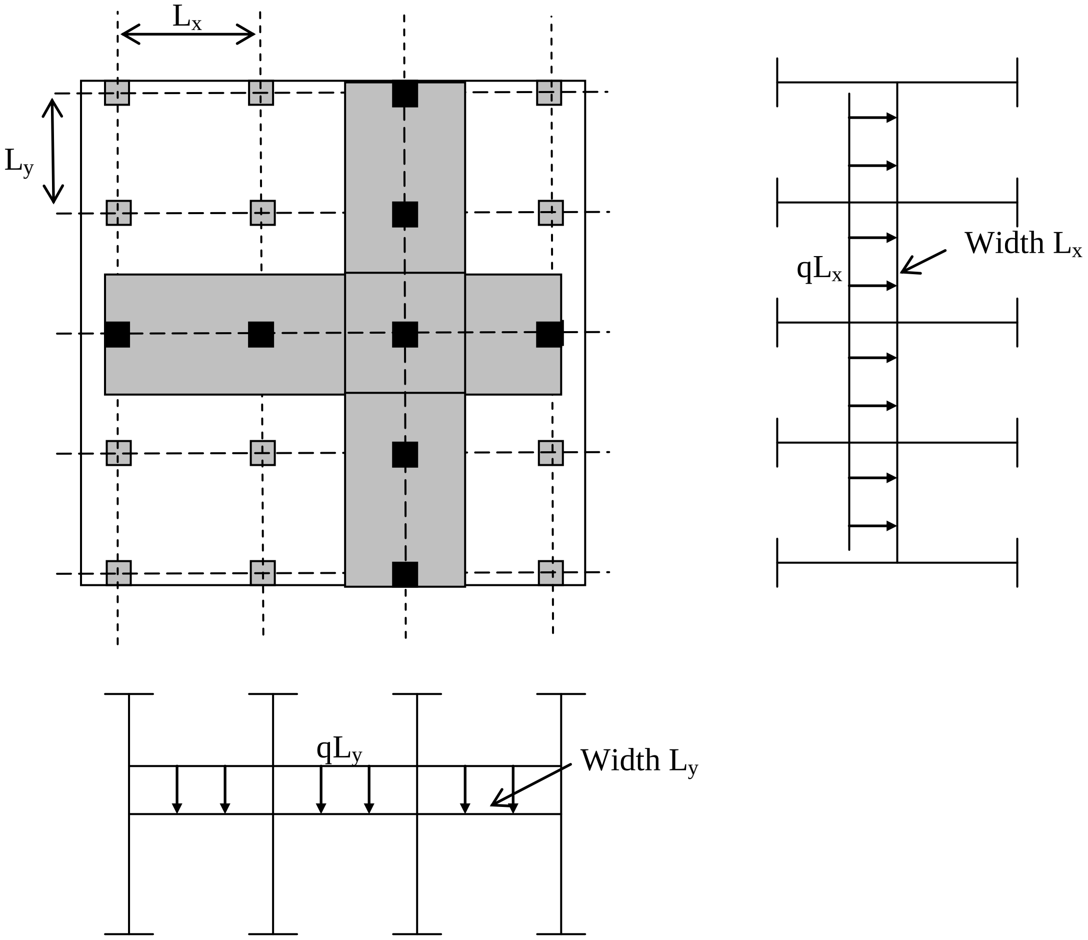


Figure 3.4.7 Equivalent frames

For a multi-storey building it is sufficient to analyse one floor slab for vertical load, as shown in the figure.
For horizontal loads the entire frame should be analysed, unless the horizontal forces are taken by stiffening panels.

In order to determine maximum negative and positive moments in the slab at the columns and mid-span, respectively, the live load has to be placed unfavourably with respect to each load action.

Earlier, this was rather time consuming, and several approximate methods were often used. One alternative was to analyse the slab as a continuous beam (simply supported on columns), with successive approximations for the column moments (beam method).
Another alternative was an approximation of the beam method, based on moment coefficients for outer and inner spans. This method is valid provided certain ratio between neighbouring spans.
With today's availability of computer programs for plane frames, there should be no reason for focusing on the approximation methods.
Therefore, static analyses of the frames should be carried out by using some available computer program for plane frames.

The results from the analyses for the slab moments in an equivalent three-span frame will in principle look like shown in Figure 3.4.8.

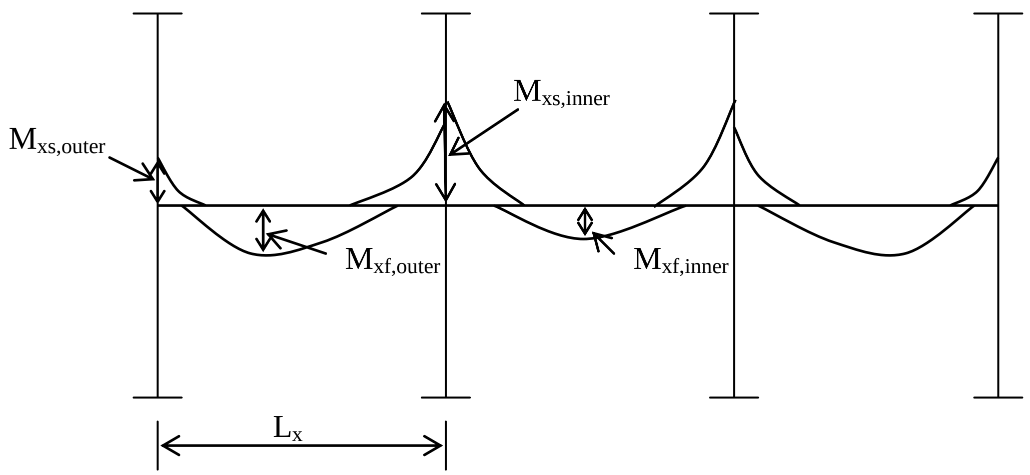


Figure 3.4.8 Design total moments in flat slab strip with width L_y

The slab moments uniformly distributed across the width of the strip is:

Over columns: $m_{xs} = \dfrac{M_{xs}}{L_y}$ In spans : $m_{xf} = \dfrac{M_{xf}}{L_y}$

According to EC2, I.1.2 (2), the negative moments at the columns should be concentrated to the centre lines of the columns.

Further, EC2, 9.4.1(2) says that at internal columns, unless rigorous serviceability calculations are carried out, top reinforcement of area $0{,}5A_t$ should be placed in a width equal to the sum of 0,125 times the panel width on either side of the column. A_t represents the area of reinforcement required to resist the full negative moment from the sum of the two half panels at each side of the column.
This means that 50% of $M_{s,inner}$ should be distributed across a width $0{,}25L_y$.

In Norwegian Concrete Society's Publication no. 33 /3.4.1/, the flat slab strip is divided into inner and outer column strips and middle strips, as shown in Figure 3.4.9.
The figure also shows the assumed transverse moment distribution.

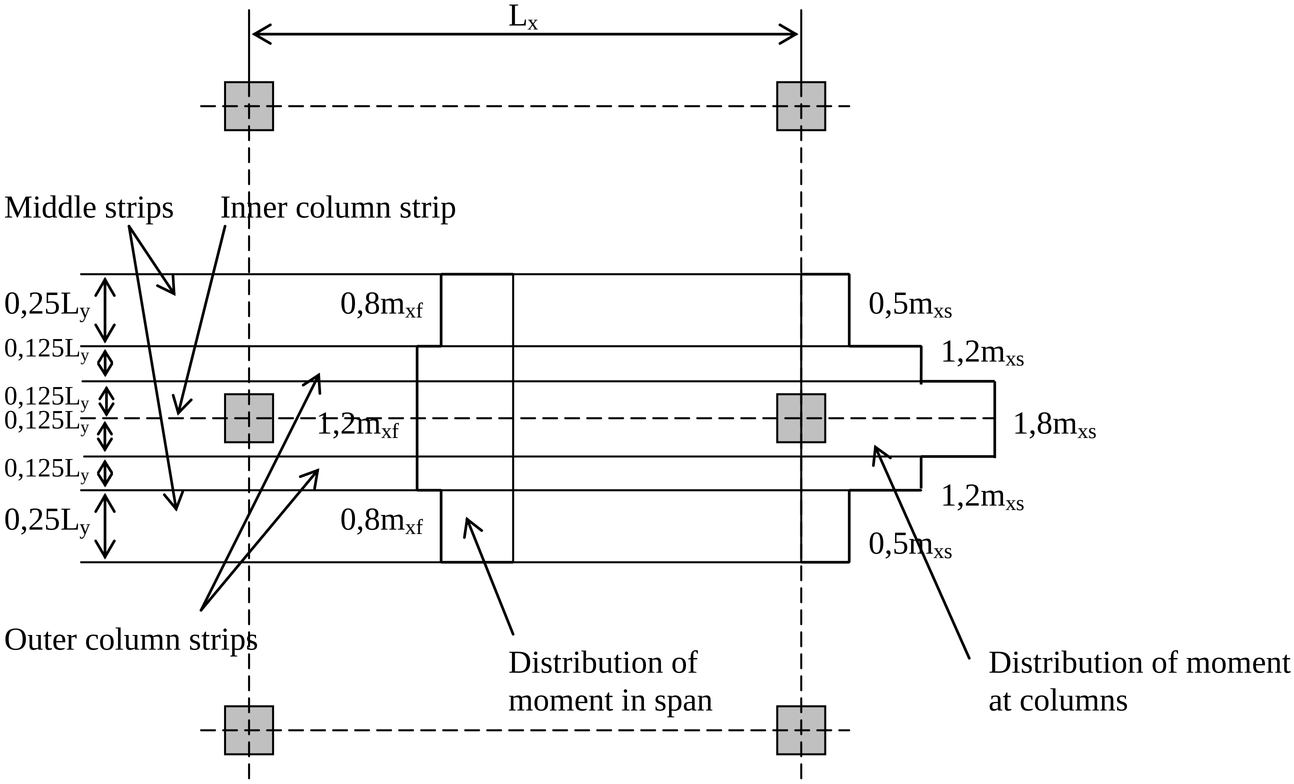


Figure 3.4.9 Transverse distribution of moments according to NB Publikasjon no. 33

This distribution results in the following part of the moment concentrated in $0{,}25L_y$ over the columns :

$$0{,}25L_y \cdot 1{,}8m_{xs} = 0{,}45m_{xs}L_y = 0{,}45M_{xs}$$

This means that $0{,}45A_t$ is concentrated over the columns, instead of $0{,}5A_t$ according to EC2, 9.4.1(2).
In order to satisfy this recommendation in EC2, the factor for the inner column strip has to be 2,0 instead of 1,8 according to NB Publikasjon nr. 33. However, the recommendations in Table 3.4.1 are satisfied.

3.4.5 Comparison to theory of elasticity

Figure 3.4.10 shows a finite element model in DIANA of ¼ of a flat slab with 3·3 panels (16 columns).
The column supports are pinned in each column corner.
The symmetry edge BC is restrained for x-displacement and rotation about the y-axis, while CD is restrained for y-displacement and rotation about the x-axis. The slab thickness is 250mm the span lengths are shown in the figure. The slab is subjected to a uniformly distributed load $q = 10$ kN/m^2.

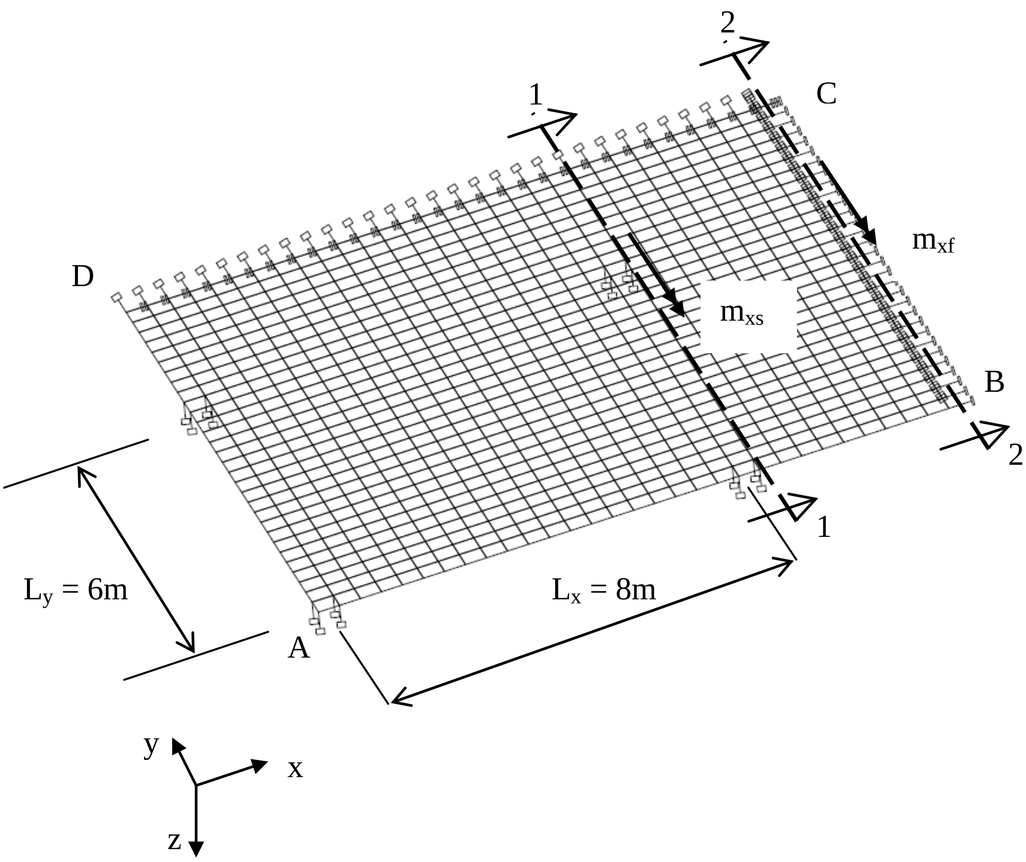


Figure 3.4.10 Finite element model of flat slab

Moment distributions at inner column support, section 1-1, and in the span, section 2-2, are shown in Figure 3.4.11.

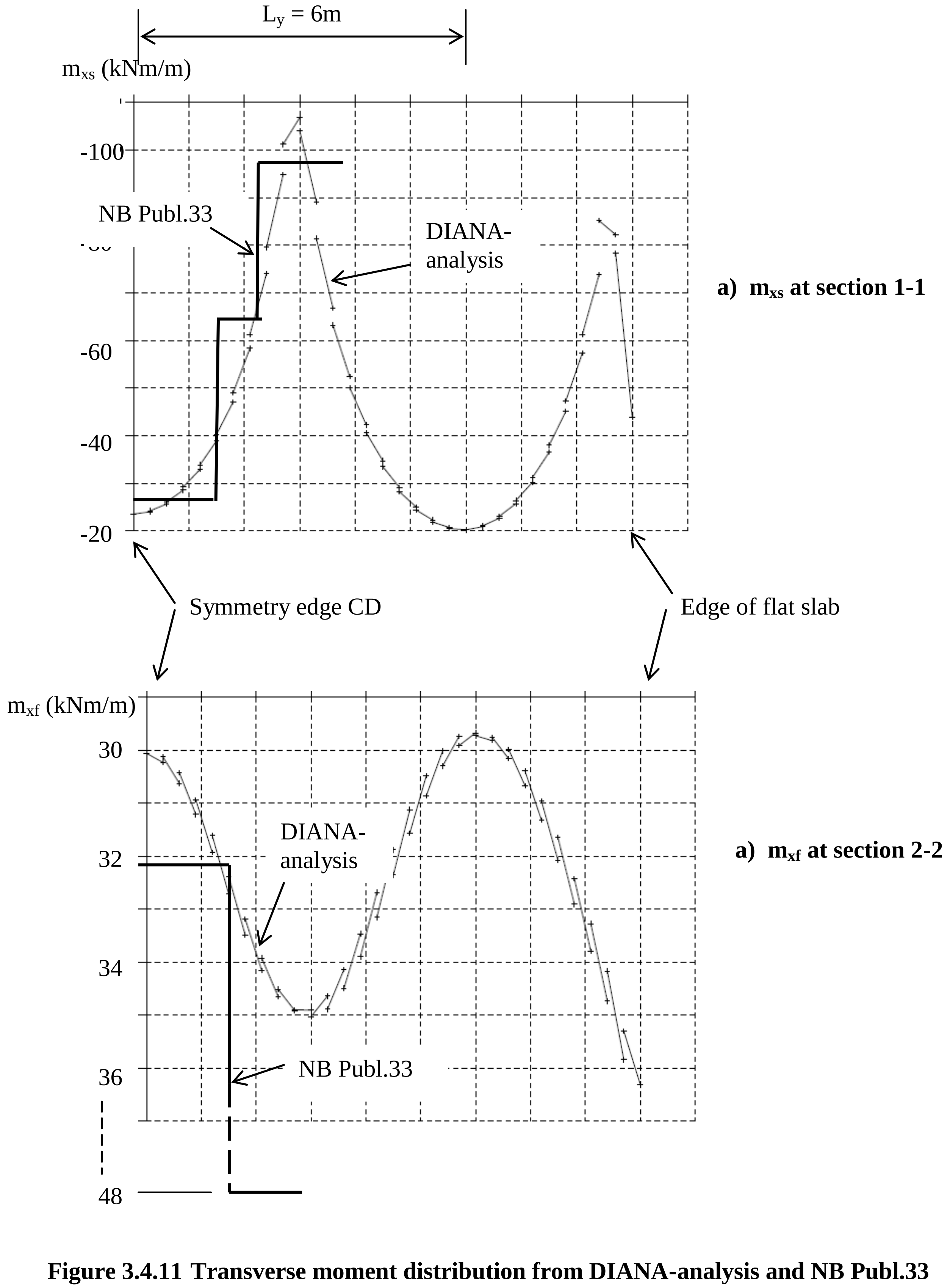


Figure 3.4.11 Transverse moment distribution from DIANA-analysis and NB Publ.33

The transverse moment distribution from NB Publ. 33 in Figure 3.4.11 is based on the following slab moments :
According to moment coefficient methods, the moment at inner column support may be approximated as $qL_x^2/12$ (i.e. assumed fixed), while the moment in the span may be approximated as $qL_x^2/16$.
This results in the following according to NB Publ. 33;

Over the column support :

$$M_{xs} = \frac{qL_yL_x^2}{12} = \frac{10 \cdot 6 \cdot 8^2}{12} = 320\text{kNm/m} \rightarrow m_{xs} = \frac{M_{xs}}{L_y} = \frac{320}{6} = 53{,}3\text{kNm/m}$$

Inner column strip : $1{,}8 \cdot 53{,}3 = 96$ kNm/m
Outer column strips : $1{,}2 \cdot 53{,}3 = 54$ kNm/m
Middle strips : $0{,}5 \cdot 53{,}3 = 26{,}7$ kNm/m

In the span :

$$M_{xf} = \frac{qL_yL_x^2}{16} = \frac{10 \cdot 6 \cdot 8^2}{16} = 240\text{kNm/m} \rightarrow m_{xf} = \frac{M_{xf}}{L_y} = \frac{240}{6} = 40\text{kNm/m}$$

Column strip : $1{,}2 \cdot 40 = 48$ kNm/m
Middle strip : $0{,}8 \cdot 40 = 32$ kNm/m

Figure 3.4.12 shows that the assumed transverse distribution of the column support moments are close to the results obtained by theory of elasticity (finite element solution).

The moments in the span, however, are over-estimated compared to elastic theory.
However, approximate moment coefficients are used for calculation of the total span moment. A complete frame analysis would probably give better correspondence between elastic theory and NB Publ. 33.
Also, the differences are not as large as it seems in Figure 3.4.13, because of the difference in scale of the moment axes at column support and in the span.

In order to satisfy the recommendation in EC2, 9.4.1(2), to concentrate 50% of the total moment to the inner column strip, the factor has to be increased from 1,8 to 2,0.
Further, the factor for the outer column strips has to be reduced from 1,2 to 1,0 to give a total moment M_{xs} . This moment distribution is shown in Figure 3.4.12.

A transverse moment distribution with a $L_y/2$ wide column strip and constant moment $1{,}33m_{xs}$, middle strips with $0{,}67m_{xs}$ should also be mentioned.. This distribution was recommended in the Norwegian design rules NS427A /3.4.2/ from 1963, and replaced by NS3473, 1st edition /3.4.3/ in 1973. The distribution is shown with a dashed line in Figure 3.4.12.
Prior to the release of NB Publikasjon nr. 33 in 2004, most flat slabs in Norway were designed based on this distribution, and as can be seen from Figure 3.4.14, this is a significant under-estimation of the moment concentration at the columns. However, for flat slabs with capitals and drop panels over the columns (commonly used earlier), this distribution is reasonable.

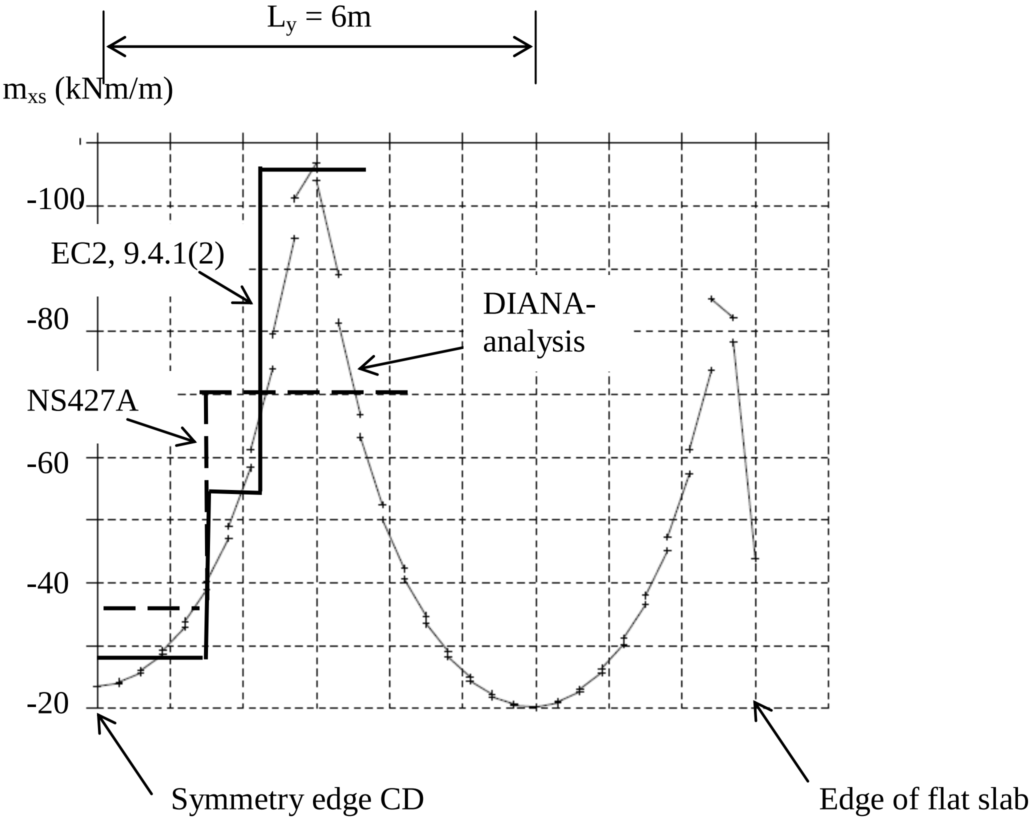


Figure 3.4.12 Transverse moment distribution from DIANA-analysis, modified NB Publ. 33 and NS427A.

3.4.6 Common design of flat slabs

When designing flat slabs, one has to choose if the plate is supported directly on the columns, or if the column top area should be increased by capitals and/or drop panels.
The choice will influence eventual need of local shear reinforcement in the slab at the columns, and slab thickness to satisfy deflection requirement.

The slab edges may include perimeter beams in order to increase the slab stiffness and reduce moments that are transferred to the columns. Including perimeter beams makes the practical reinforcement detailing in joints between slab and columns near the slab edge easier. Appropriate reinforcement detailing is easiest to achieve if the columns are not too close to (or at) the slab edge.
In recent years several flat slabs have been designed with spans up to 7,2m. In some cases the deflections have become too large and have resulted in problems with respect to serviceability. It is therefore urgent to choose slab thickness and reinforcement based on deflection calculations. The slab thickness has the major influence on the deflections.

The former Norwegian design rules required minimum slab thickness h ≥ L/30.   Experience has shown that this minimum thickness requires large reinforcement quantities in order to limit the deflections to acceptable sizes.
It is therefore recommended to choose slab thickness larger than the minimum requirement, e.g. in the order of magnitude L/25, for normal live loads and span lengths ≤ 7,2m. If the live load is close to the self-weight, still larger slab thicknesses should be considered.
For smaler spans, e.g. ≤ 5m, the minimum thickness is probably sufficient.

3.4.7 Deflection calculation in SLS

According to EC2, 7.4.1(4), the slab deflection for quasi-permanent load should not exceed 1/250 of the span. Quasi-permanent load is self-weight plus a permanent part of the live load (often 40% – 50%).

Since transverse distribution of moments in the equivalent frame analysis is based on elastic theory solutions, the same distribution assumptions can be used in SLS.
Further, the actual load case for control according to EC2, 7.4.1(1) is permanent load in all spans, because EC2 does not specify any limit for maximum deflection when short term loads are included.

Figure 3.4.13 shows in principle how the deflection of a slab panel can be calculated.

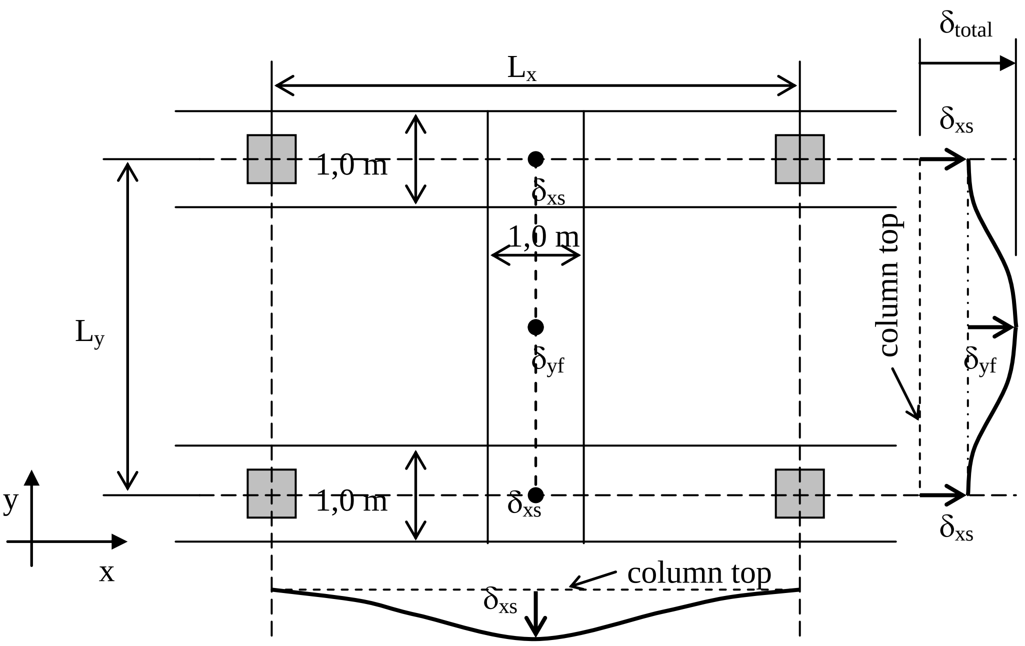
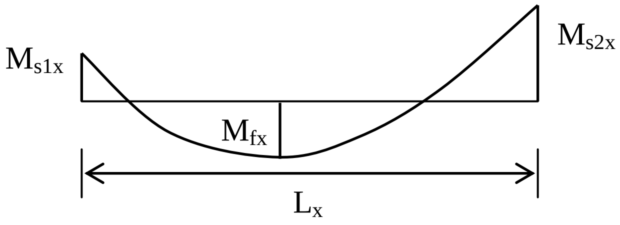


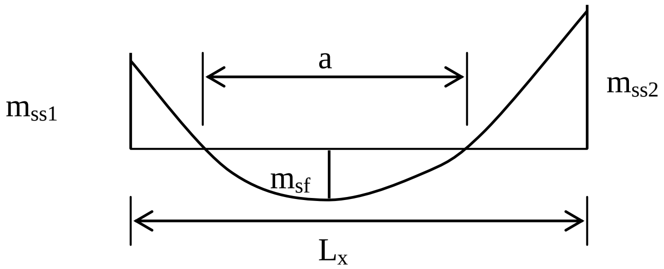
Figure 3.4.13 Principle for deflection calculation in slab panel

Analysis procedure:

1) Total moments in flat slab strip in x-direction with width L_y are determined by frame analysis for permanent load in all spans in SLS. For actual span :

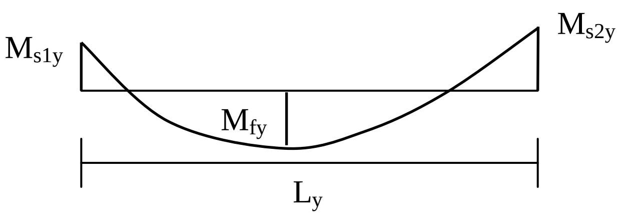


Moments per unit width in column strip according to NB Publikasjon 33:

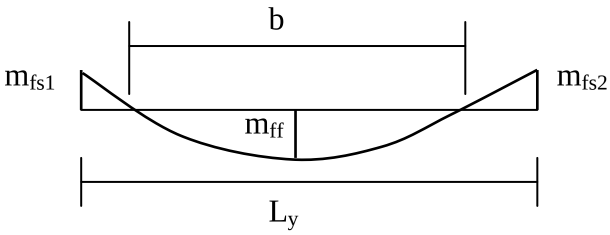


Here : $m_{ss1} = \frac{M_{s1x}}{L_y} \cdot 1{,}8$; $m_{ss2} = \frac{M_{s2x}}{L_y} \cdot 1{,}8$; $m_{sf} = \frac{M_{fx}}{L_y} \cdot 1{,}2$

2) Total moments in flat slab strip in y-direction with width L_x are determined by frame analysis for permanent load in all spans in SLS. For actual span :



Moments per unit width in middle strip according to NB Publikasjon 33:



Her er : $m_{fs1} = \frac{M_{s1y}}{L_x} \cdot 0{,}5$; $m_{fs2} = \frac{M_{s2y}}{L_x} \cdot 0{,}5$; $m_{ff} = \frac{M_{fy}}{L_x} \cdot 0{,}8$

3) The required reinforcement in each section is determined in ULS.

For each section, calculate flexural stiffness for cracked state (State II):

$$\rho = \frac{A_s}{10^3 \cdot d} \quad ; \quad \eta = \frac{E_s}{E_{c,mean}}$$

$E_{c,mean}$ is determined from long term E-moduli for self-weight and permanent part of live load.

$$\alpha = \sqrt{(\eta\rho)^2 + 2\eta\rho} - \eta\rho \; ; \quad I_{ce} = \frac{1}{2}\alpha^2\left(1 - \frac{\alpha}{3}\right)10^3 \cdot d^3 \; ; \qquad EI = E_{c,mean} \cdot I_{ce}$$

4) Calculate a quasi-average flexural stiffness for the regions with tension in top and bottom for the strips in x- and y-direction :

Column strip in x-direction: $\frac{1,2qa^2}{8} = m_{sf} \rightarrow a = \sqrt{\frac{8m_{sf}}{1,2q}} \rightarrow \beta_u = \frac{a}{L_x}$

$$EI_{x,average} = \beta_u EI_{x,bottom} + (1 - \beta_u)EI_{x,top}$$

Middle strip in y-direction: $\frac{0,8qb^2}{8} = m_{ff} \rightarrow b = \sqrt{\frac{8m_{ff}}{0,8q}} \rightarrow \gamma_u = \frac{b}{L_y}$

$$EI_{y,average} = \gamma_u EI_{y,bottom} + (1 - \gamma_u)EI_{y,top}$$

5) Deflections δ_{xs} and δ_{yf} in Figure 3.4.13 are calculated from the moment distributions in 1) and 2) for x- and y-strips based on virtual work method with average flexural stiffness from 4).

Further, the deflection is $\delta_{total} = \delta_1 = \delta_{xs} + \delta_{yf}$

Ideally, deflection determined by analyses of column strip in y-direction and middle strip in x-direction should be the same, i.e. $\delta_{total} = \delta_2 = \delta_{ys} + \delta_{xf}$.

Most probably, $\delta_1 \neq \delta_2$, hence, the deflection can be determined as an average value for the two directions :

$$\delta_{total} = \frac{\delta_1 + \delta_2}{2}$$

Acceptable deflection according to EC2 is L/250, where $L = \min\{ L_x ; L_y \}$

3.4.8 Example - Design and deflection calculation

Figure 3.4.14 shows a 3·3-span flat slab with span lengths L_x = 7,2m og L_y = 6,0m.
The slab thickness is chosen as h = 300mm, which is slightly over L_x/25.
The columns are quadratic with 300·300. The column lengths are 3,0m.

Loads: Self-weight : g = 0,3·25 = 7,5 kN/m^2 ;
Live load : p = 5,0 kN/m^2 , 40% assumed permanent.

Design load in ULS : q_{Ed} = 1,2g + 1,5p = 16,5 kN/m^2

Permanent load in SLS : q = g + 0,4p = 9,5 kN/m^2

Materials: B30 → f_{cd} = 17 N/mm^2 B500NC → f_{yd} = 434 N/mm^2

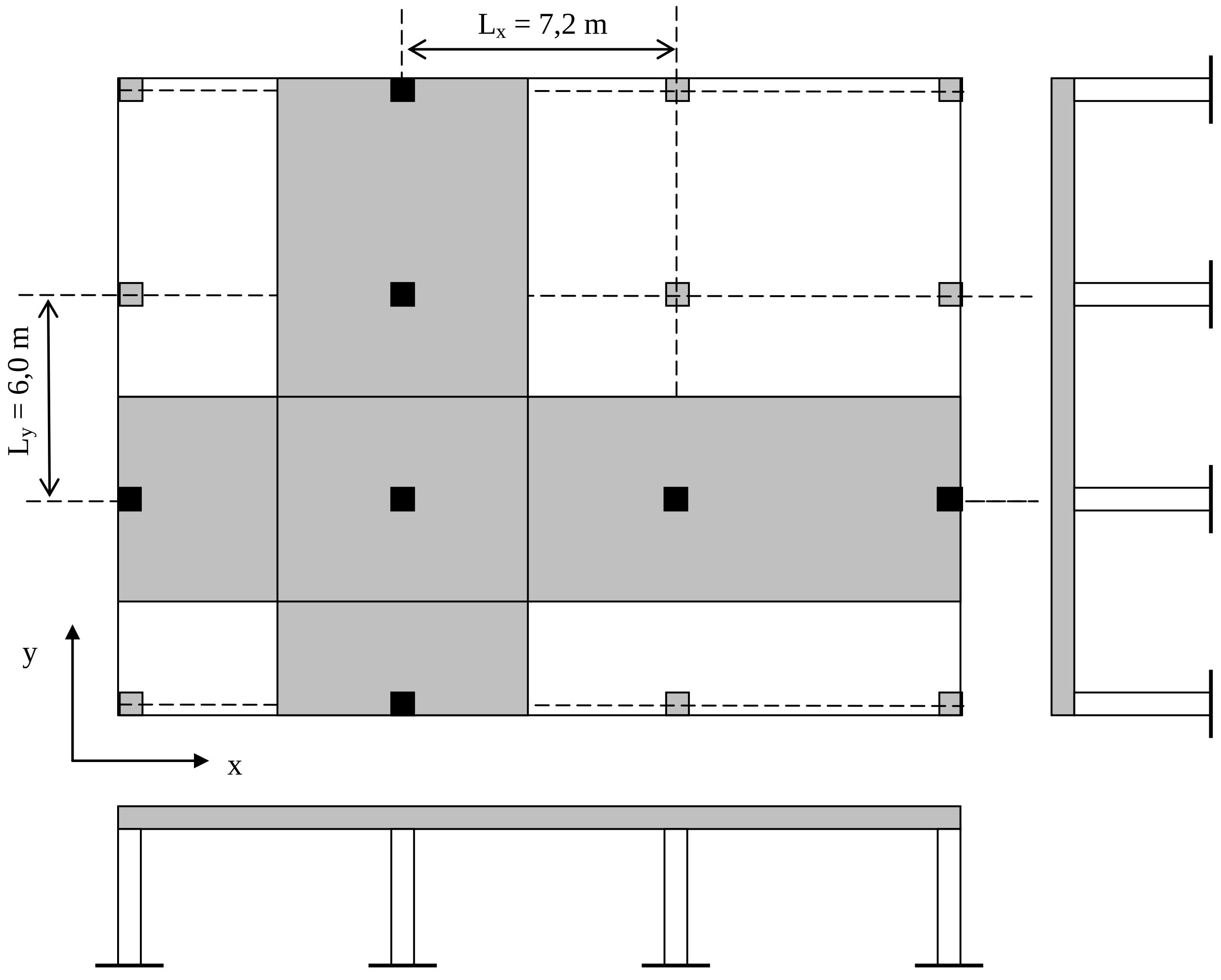


Figure 3.4.14 Flat slab- plan view and frames in both directions

Due to double symmetry, half the frames in x- and y-direction are modelled and analysed by DIANA for actual load cases that give maximum design support and span moments in ULS. Diagrams for total moment for the entire flat slab strip in the x- and y-direction frames, respectively, are shown in Figure 3.4.15 and Figure 3.4.16.

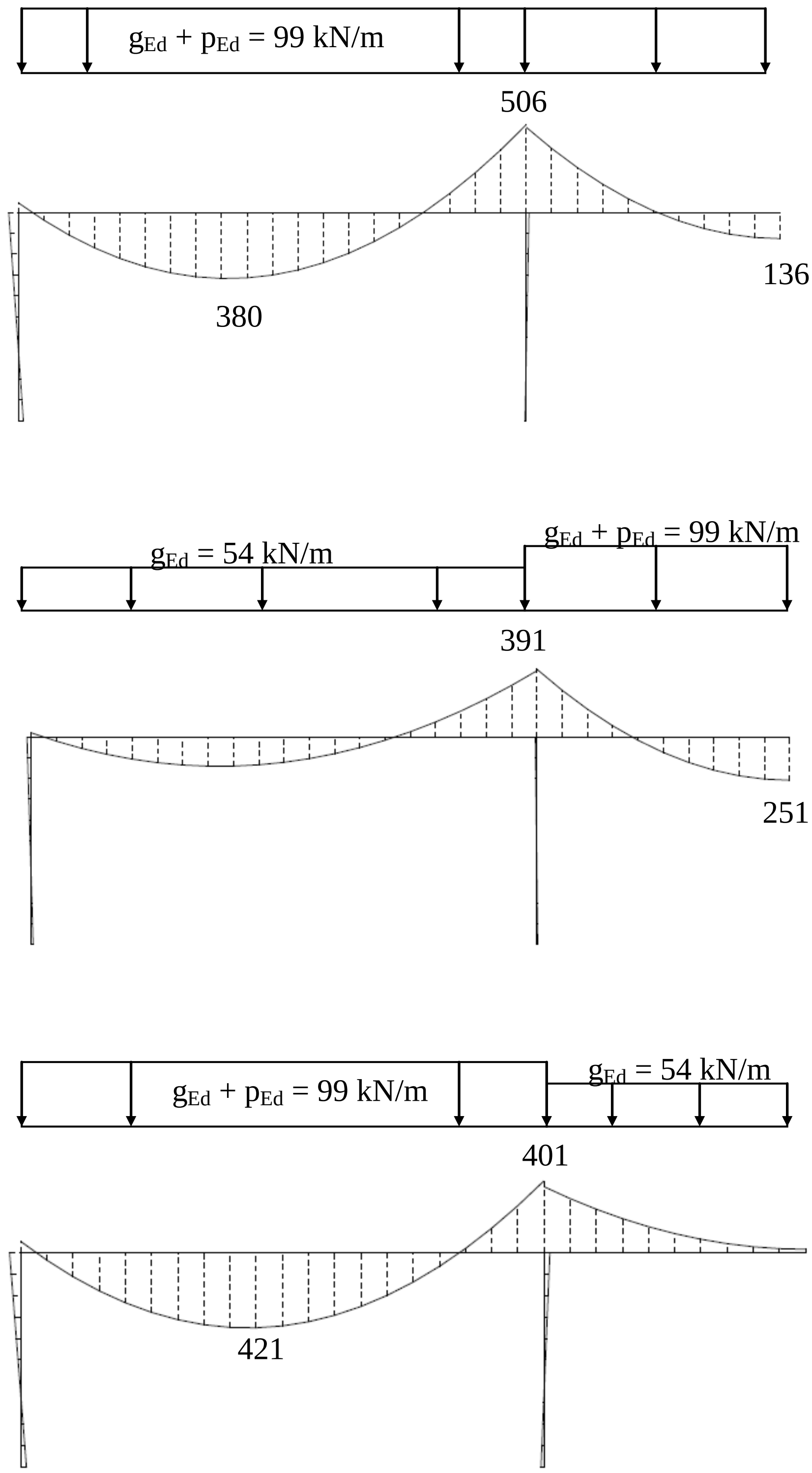


Figure 3.4.15 Moments in x-direction frame [kNm]

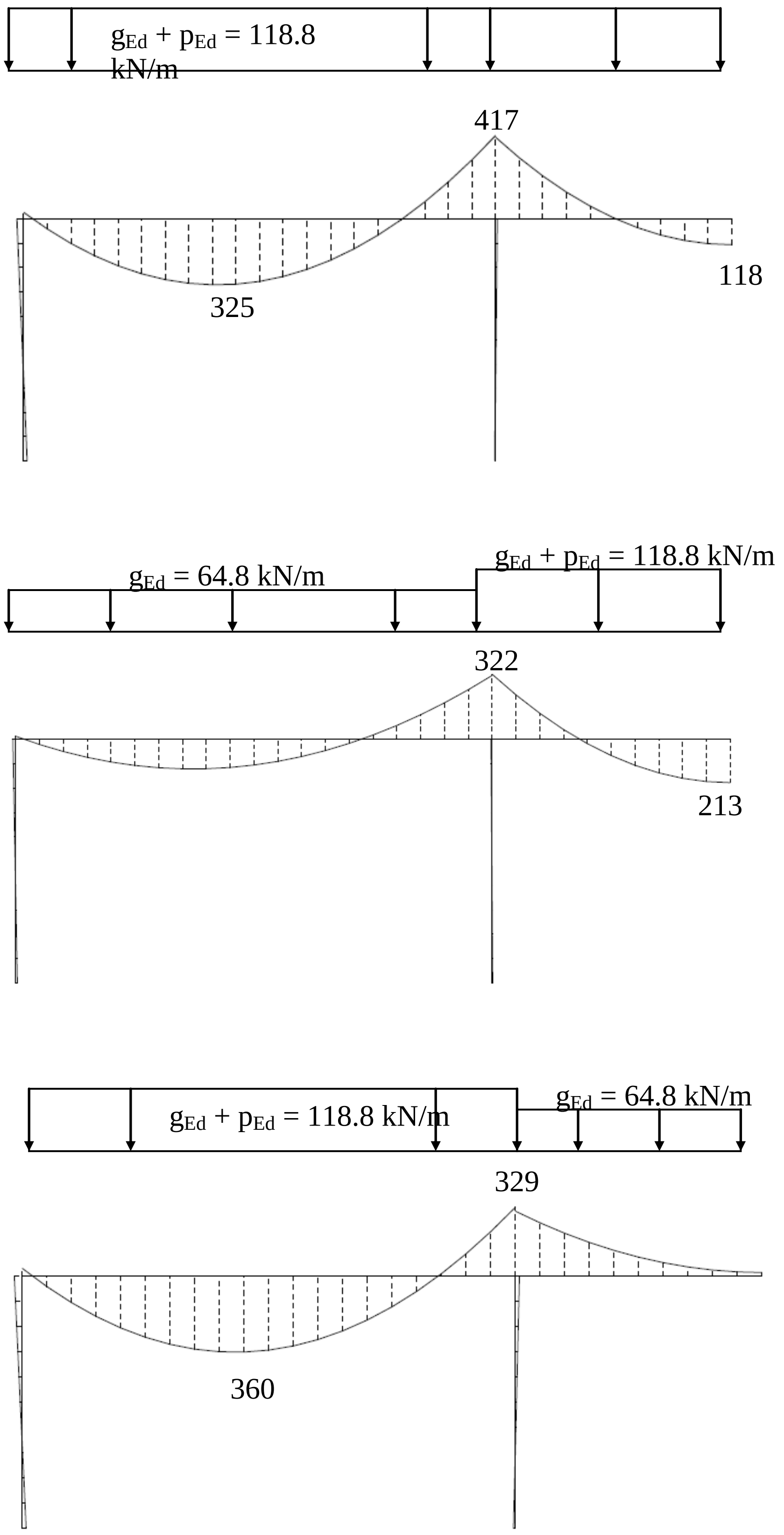


Figure 3.4.16 Moments in y-direction frame [kNm]

Determination of required reinforcement:

Flat slab strip in x-direction

The transverse moment distribution at inner column is shown in Figure 3.4.17.
The uniform moment intensity is: m_{xs} = 506/6 = 84,33 kNm/m

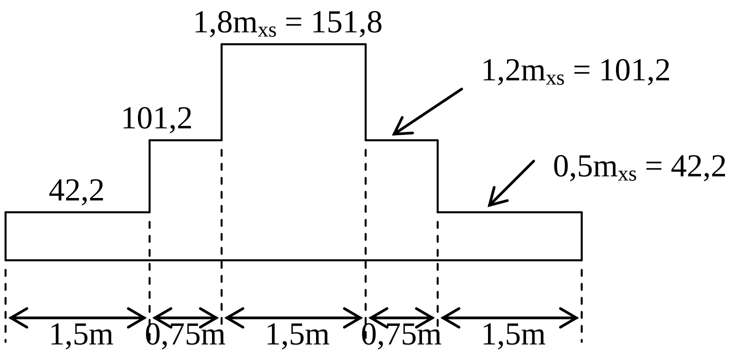


Figure 3.4.17 Moment distribution at inner column in x-direction strip

Assume an average effective depth for all sections: d = 250mm

Moment capacity: $m_{Rd} = 0,275 f_{cd} b d^2 = 0,275 \cdot 17 \cdot 10^3 \cdot 250^2 \cdot 10^{-6} = 304 \text{kNm/m}$

Inner column strip: $z \approx \left(1 - 0,17 \cdot \frac{151,8}{304}\right) d = 0,915 \cdot 250 = 229\text{mm}$

$$A_{sx,s1} = \frac{151,8 \cdot 10^6}{229 \cdot 434} = 1527\text{mm}^2/\text{m} \rightarrow$$ <u>ϕ16s130 gives 1546 mm^2/m</u>

Outer column strip: $z \approx \left(1 - 0,17 \cdot \frac{101,2}{304}\right) d = 0,94 \cdot 250 = 235\text{mm}$

$$A_{sx,s2} = \frac{101,2 \cdot 10^6}{235 \cdot 434} = 992\text{mm}^2/\text{m} \rightarrow$$ <u>ϕ16s200 gives 1005 mm^2/m</u>

Middle strip: $z \approx 0,95d = 237\text{mm}$

$$A_{sx,f} = \frac{42,2 \cdot 10^6}{237 \cdot 434} = 410\text{mm}^2/\text{m} \rightarrow$$ <u>ϕ12s270 gives 419 mm^2/m</u>

This is all top reinforcement !

The transverse moment distribution at inner span is shown in Figure 3.4.18.
The uniform moment intensity is : m_{xf} = 251/6 = 41,83 kNm/m.

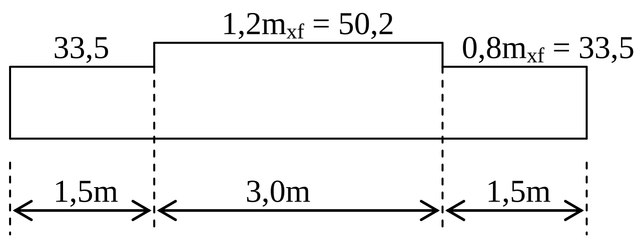


Figure 3.4.18 Moment distribution at inner span in x-direction strip

Column strip : Small moments, hence z = 0,95d = 237 mm

$$A_{sx,fs} = \frac{50,2 \cdot 10^6}{237 \cdot 434} = 488 mm^2 / m \rightarrow \underline{\phi 12s230 \text{ gir } 492 \text{ mm}^2/\text{m}}$$

Middle strip : $$A_{sx,ff} = \frac{33,5 \cdot 10^6}{237 \cdot 434} = 326 mm^2 / m$$

Minimum reinforcement according to EC2 : $A_{s,min} = 0,26 \cdot \frac{2,9}{500} \cdot 10^3 \cdot 250 = 377 mm^2 / m$

Choose <u>ϕ12s300 which gives 377 mm^2/m</u>

<u>Flat slab strip in y-direction</u>

The transverse moment distribution at inner column is shown in Figure 3.4.19.
The uniform moment intensity is : m_{ys} = 417/7,2 = 57,9 kNm/m

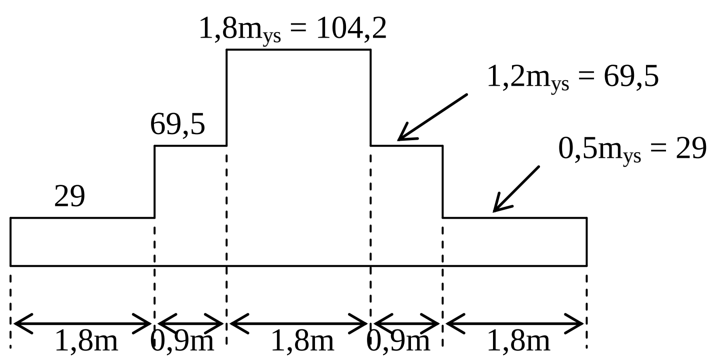


Figure 3.4.19 Moment distribution at inner column in y-direction strip

The transverse moment distribution at inner span is shown in Figure 3.4.20.
The uniform moment intensity is : m_{yf} = 213/7,2 = 29,6 kNm/m.

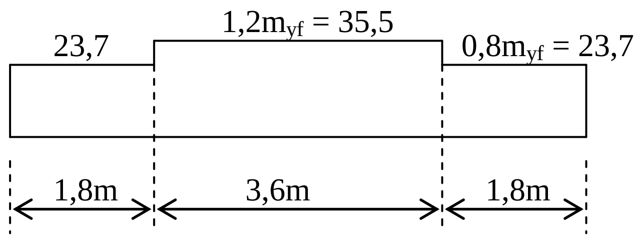


Figure 3.4.20 Moment distribution at inner span in y-direction strip

Reinforcement at all sections can be calculated similarly as for the x-direction strip

Deflection calculation only requires the reinforcement in the middle strip, where the magnitude of the moments results in minimum reinforcement, i.e.:

$A_{sy,f} = A_{sy,ff}$ = 377 mm^2/m → <u>ϕ12s300</u>

Deflection calculation at inner span :

Moments in ULS are scaled to SLS by the factor

$$\frac{q}{q_{Ed}} = \frac{9,5}{16,5} = 0,576$$

Moments in column strip in x-direction are determined from Figure 3.4.17 and Figure 3.4.18, and are shown in Figure 3.4.21.

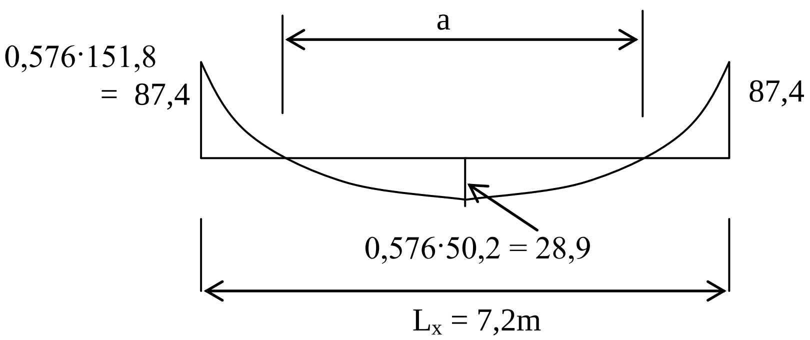


Figure 3.4.21 Moments in inner span in column strip in x-direction

Length with bottom tension, a:

$$\frac{1,2qa^2}{8} = 28,9 \quad \rightarrow \quad a = \sqrt{\frac{28,9 \cdot 8}{1,2 \cdot 9,5}} = 4,5m \quad \rightarrow \quad \beta_u = \frac{a}{L_x} = \frac{4,5}{7,2} = 0,625$$

<u>Flexural stiffness at column:</u>

$$\rho_{sx} = \frac{A_{sx,s1}}{10^3 \cdot d} = \frac{1546}{10^3 \cdot 250} = 0,00618$$

Assume creep coefficient $\varphi = 2,2$, resulting in $E_{c,eff} = 10000$ N/mm^2

$$\eta = \frac{E_s}{E_{c,eff}} = \frac{200000}{10000} = 20 \quad \rightarrow \quad \eta\rho_{sx} = 0,124$$

$$\alpha_{sx} = \sqrt{0,124^2 + 2 \cdot 0,124} - 0,124 = 0,39$$

$$I_{cs\,x} = \frac{1}{2} \cdot 0,39^2 \left(1 - \frac{0,39}{3}\right) \cdot 10^3 \cdot 250^3 = 10,34 \cdot 10^8 \text{ mm}^4 \quad \rightarrow \quad EI_{sx} = 10,34 \cdot 10^{12} \text{ Nmm}^2$$

<u>Flexural stiffness in span:</u>

$$\rho_{fx} = \frac{A_{sx,f}}{10^3 \cdot d} = \frac{492}{10^3 \cdot 250} = 0,00197 \quad \rightarrow \quad \eta\rho_{fx} = 0,0394$$

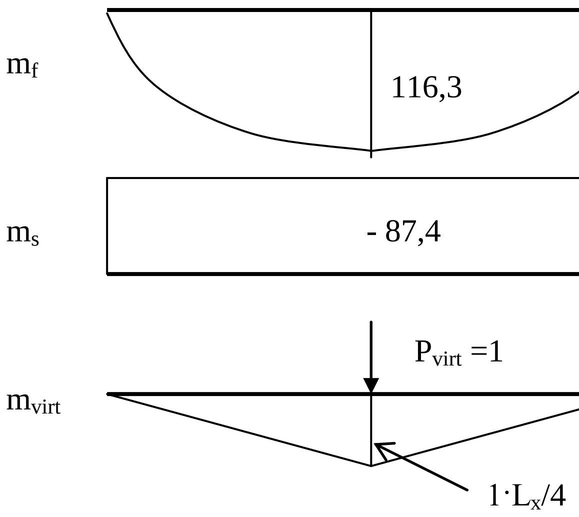
$$\alpha_{fx} = \sqrt{0,0394^2 + 2 \cdot 0,0394} - 0,0394 = 0,244$$

$$I_{cs\,x} = \frac{1}{2} \cdot 0,244^2 \left(1 - \frac{0,244}{3}\right) \cdot 10^3 \cdot 250^3 = 4,27 \cdot 10^8 \text{ mm}^4 \quad \rightarrow \quad EI_{fx} = 4,27 \cdot 10^{12} \text{ Nmm}^2$$

Average flexural stiffness:

$$EI_{xm} = \beta_u EI_{fx} + (1 - \beta_u) EI_{sx} = 0,625 \cdot 4,27 \cdot 10^{12} + 0,375 \cdot 10,34 \cdot 10^{12} = 6,55 \cdot 10^{12} \text{ Nmm}^2$$

Split the moment diagram in Figure 3.4.21, and calculate deflection using principle of virtual work:



$$\delta_{xs} = \left(\frac{5}{12} \cdot m_f \cdot \frac{L_x}{4} \cdot L_x + \frac{1}{2} \cdot m_s \cdot \frac{L_x}{4} \cdot L_x \right) \cdot \frac{1}{EI_{xm}}$$

$$= \left(\frac{5}{12} \cdot 116,3 \cdot 10^6 \cdot \frac{7,2^2 \cdot 10^6}{4} + \frac{1}{2} \cdot (-87,4) \cdot \frac{7,2^2 \cdot 10^6}{4} \right) \cdot \frac{1}{6,55 \cdot 10^{12}} = 9,4\text{mm}$$

Moments in middle strip in y-direction are determined from Figure 3.4.19 and Figure 3.4.20, and are shown in Figure 3.4.22.

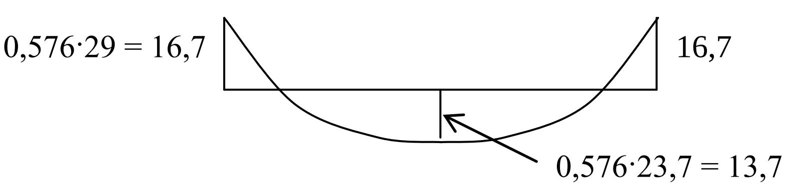


Figure 3.4.22 Moments in inner span for middle strip in y-direction

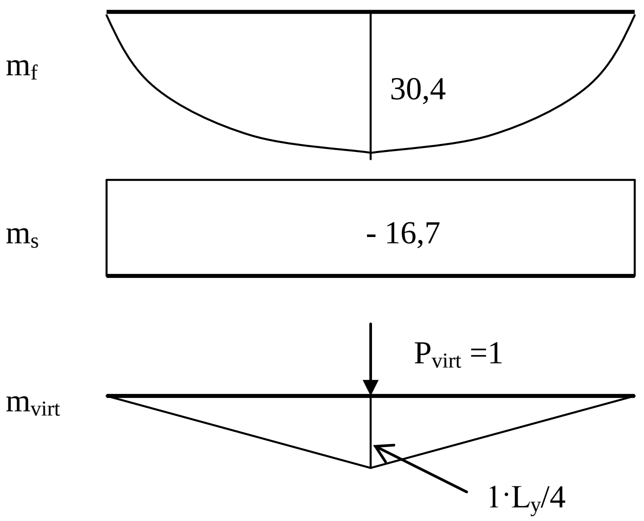
Here it is minimum reinforcement both in top and bottom, hence the flexural stiffness is constant throughout the span.

$$\rho_{fy} = \frac{A_{s,min}}{b \cdot d} = \frac{377}{10^3 \cdot 250} = 0,00151 \;\rightarrow\; \eta\rho_{fy} = 0,03$$

$$\alpha_{fy} = \sqrt{0,03^2 + 2 \cdot 0,03} - 0,03 = 0,217$$

$$I_{cs\,x} = \frac{1}{2} \cdot 0,217^2 \left(1 - \frac{0,217}{3}\right) \cdot 10^3 \cdot 250^3 = 3,41 \cdot 10^8\,\text{mm}^4 \;\rightarrow\; EI_y = 3,41 \cdot 10^{12}\,\text{Nmm}^2$$

Split the moment diagram in Figure 3.4.22, and calculate deflection using principle of virtual work:



$$\delta_{yf} = \left(\frac{5}{12} \cdot 30,4 \cdot 10^6 \cdot \frac{6^2 \cdot 10^6}{4} + \frac{1}{2} \cdot (-16,7) \cdot \frac{6^2 \cdot 10^6}{4}\right) \cdot \frac{1}{3,41 \cdot 10^{12}} = 11,4\text{mm}$$

The total deflection at inner span is : $\delta_{total} = \delta_{xs} + \delta_{yf} = 9,4 + 11,4 = 20,8\text{mm}$

Acceptable deflection according to EC2 : $\dfrac{L_y}{250} = \dfrac{6000}{250} = 24\text{mm}$

Additional deflection due to shrinkage is not accounted for. Tension stiffening, which counteracts the shrinkage is neither included. Therefore, the calculated deflection may be assumed to represent the reality with reasonable degree of accuracy.

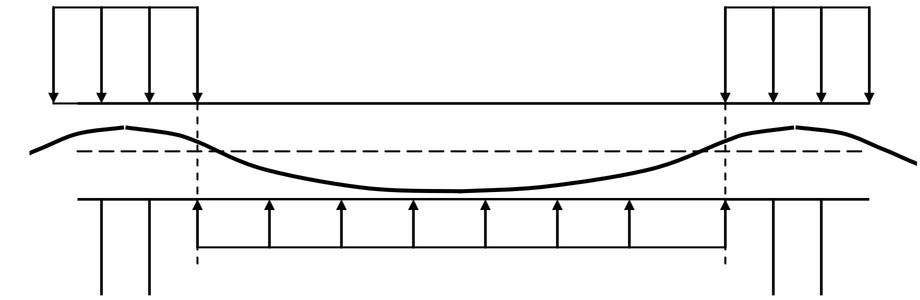
3.4.9 Pre-stressed flat slabs

Post-tensioned, unbonded pre-stressing is an effective way of reducing deflections in flat slabs, and thereby allowing larger spans.

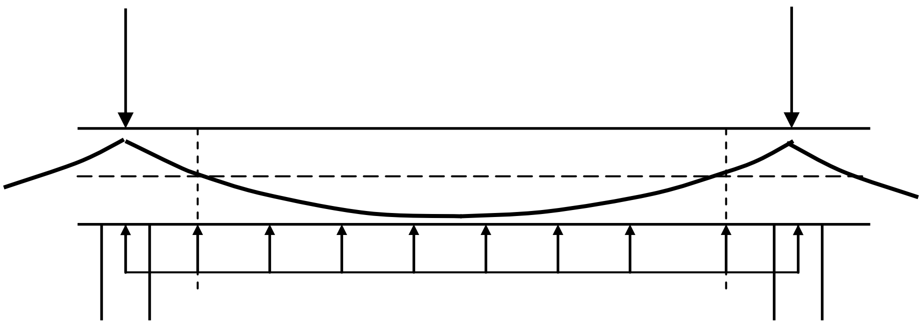
Pre-stressing tendons are placed in greased tendon ducts, which protect against corrosion and reduce friction. Hence, the pre-stressing force can be assumed as constant along the tendon.

Commonly used parabolic shaped tendon profile with positive and negative curvature and resulting equivalent forces is shown in Figure 3.4.23a.

The simplified tendon profile in Figure 3.4.23b is often used as an approximation



a) Real tendon profile

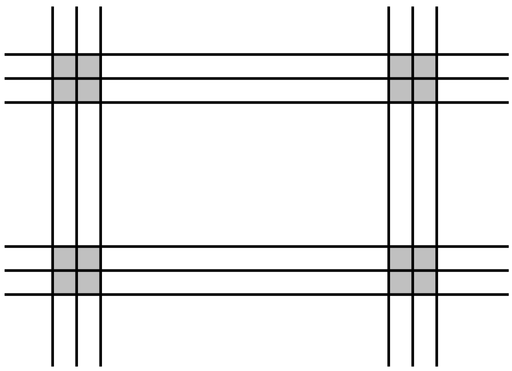


b) Idealised tendon profile

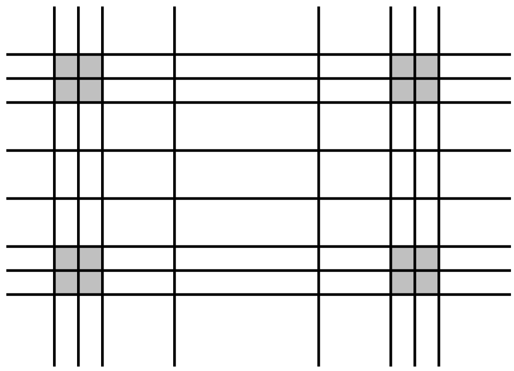
Figure 3.4.23 Pre-stressing tendon profile with equivalent forces

The uniformly distributed equivalent load in the span counteracts self-weight and live load (so-called load balancing), and is included in the equivalent frame analysis. Creep, shrinkage and relaxation will influence the pre-stress force, and hence the equivalent uniform load. Such pre-stress losses are described in ref. /3.4.4/.

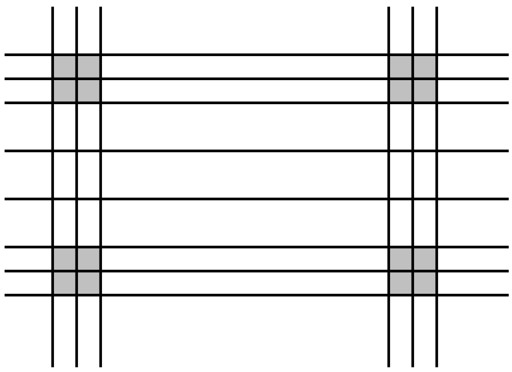
The tendons may be placed in various patterns. Some examples from ref. /3.4.1/ are shown in Figure 3.4.24. Here, a) is the simplest, b) is difficult to produce due to crossing tendons in the span, while c) and d) are the most commonly used.



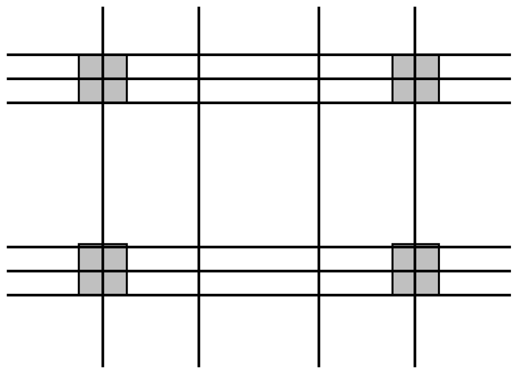
a) Tendons only in column strips



b) Tendons which is most effective related to elasticity theory



c) Tendons in the span only in largest span direction



d) Tendons in column and middle strips in separate directions

Figure 3.4.24 Various tendon patterns

3.4.10 References

/3.4.1/ Norsk Betongforening, Publikasjon nr.33: Flatdekker – Beregning og konstruktiv utforming, 2004 (in Norwegian)

/3.4.2/ NS427A Prosjektering av betongkonstruksjoner, 1963 (in Norwegian)

/3.4.3/ NS3473 Prosjektering av betongkonstruksjoner, 1.utgave, 1973 (in Norwegian)

/3.4.4/ Sørensen, S.I.: Betongkonstruksjoner – Beregning og dimensjonering etter Eurocode 2, Tapir akademisk forlag, 2010. (in Norwegian)

CHAPTER 4

PUNCHING OF CONCRETE SLABS

Jan Arve Øverli

4.1 Introduction

In design of concrete slabs, shear is generally not critical when slabs carry distributed loads and are supported by beams or walls. However, in slabs with concentrated loads the maximum shear force per unit length can be relatively high close to the loaded area, as illustrated in Figure 4.1.1. The effect of concentrated loading on slabs is referred to as punching shear. Punching shear can result from a concentrated load or reaction forces acting on å relatively small area. In civil engineering structures punching shear must typically be considered in:

- Slab-column connections in flat slabs floors
- Column-footing connection in a foundation
- External loads such as wheel loads

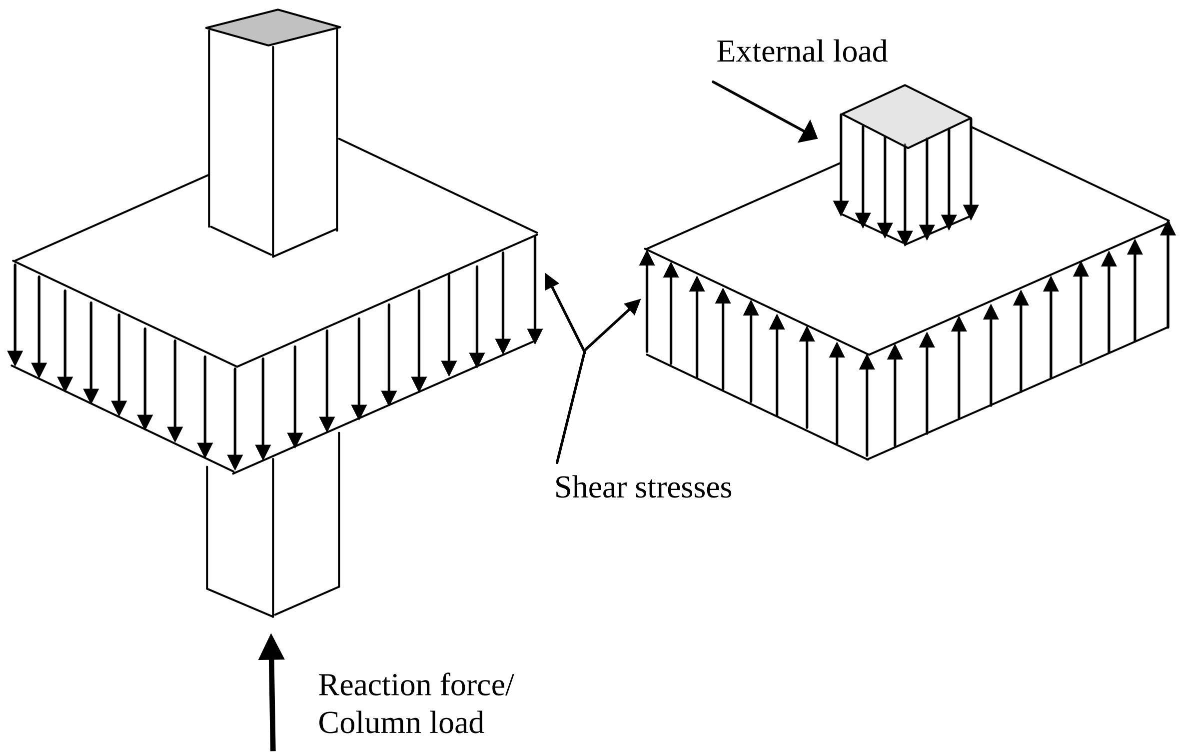


Figure 4.1.1 Large shear stresses close to a column

If punching shear is critical and governs the design, this can influence the geometry of the structure. An easy way of reducing shear stresses is to increase the size of the column or the slab thickness. To maintain storey heights in a flat slab the shear resistance around the column can be increased by employing a drop panel or a column head, as illustrated in Figure 4.1.2. A drop panel is easy to execute with regard to formwork and reinforcement. Of architectural or durability considerations a column head with either a pyramid or a cone shape can be used. Another advantage of using a column head or a drop panel is that the moments and deflections are reduced since the effective span is reduced.

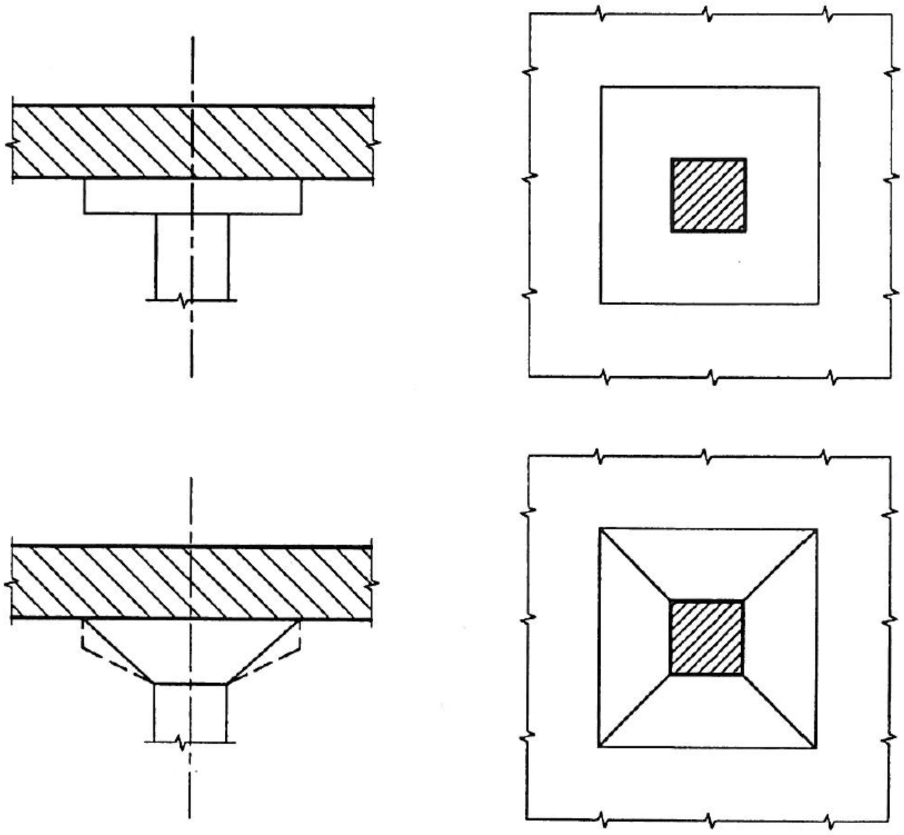


Figure 4.1.2 Flat slab with a) Drop panel; b) Column head, /4.4/

4.2 Punching shear failure

Punching shear is a local shear failure around a concentrated load on a slab. Figure 4.2.1 shows a typical punching shear failure observed in a flat slab. The failure occurs along a truncated cone caused by diagonal tensile cracks, see Figure 4.2.2. The cracks form a failure surface around the loaded area. Hence, punching shear is a three-dimensional problem. In beams there are specified critical sections to do the shear design. For punching shear the critical section is defined as a failure perimeter some distance from the loaded area. In design this perimeter is defined as the basic control perimeter.

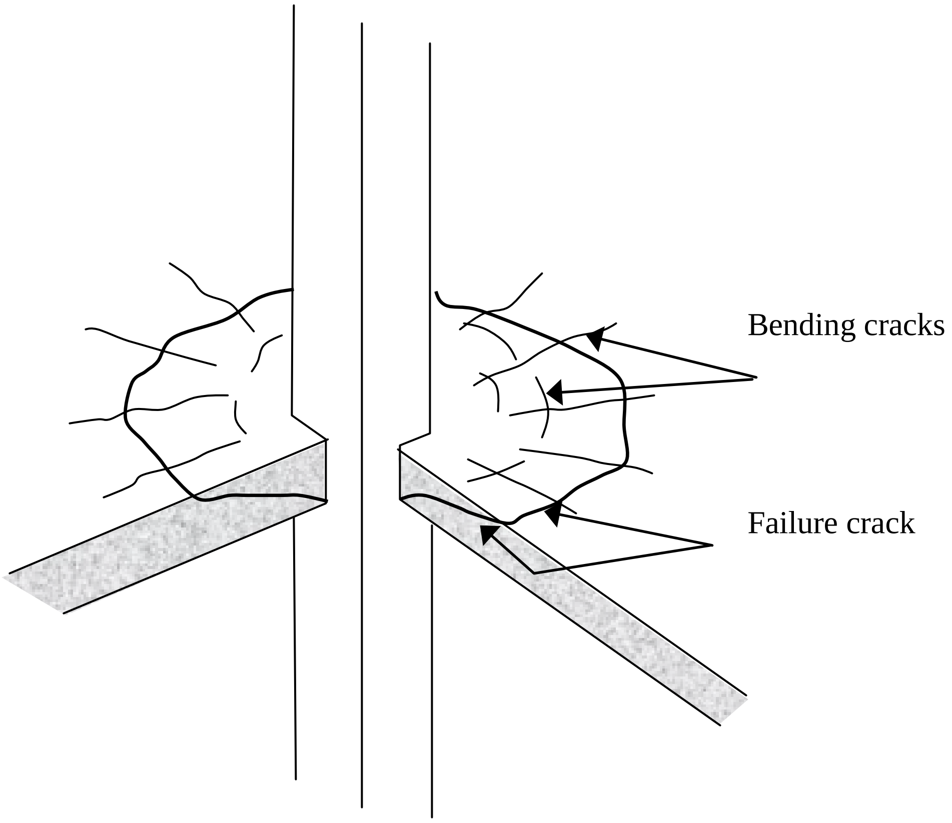


Figure 4.2.1 Punching shear failure in a flat slab

Punching shear is one of the most difficult problems in design of concrete structures. Much experimental and theoretical work has been carried out to develop mechanical models, analyses methods and code regulations to find the shear resistance for punching /4.2/. Among the factors which in general influence shear resistance in concrete are concrete quality, amount of longitudinal reinforcement and size. In addition the size of the loaded area and definition of the control perimeter influence in punching shear.

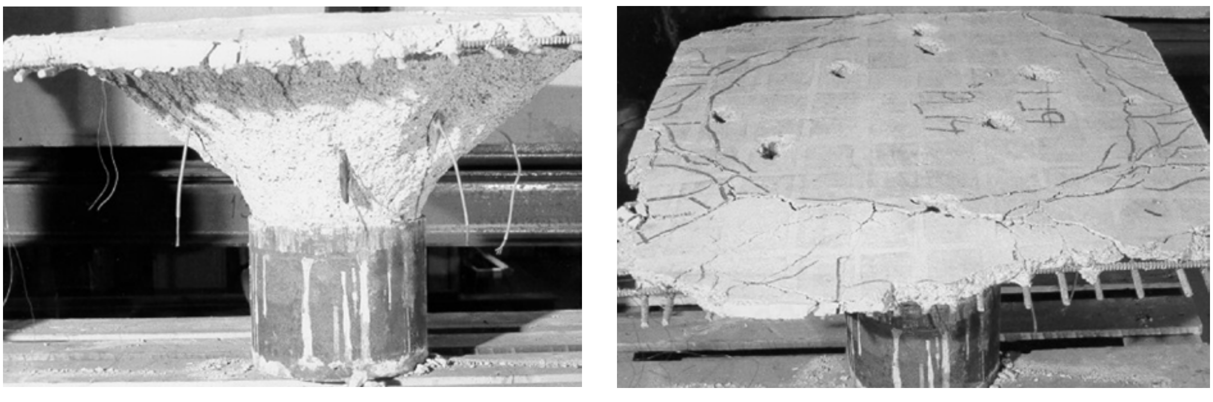


Figure 4.2.2 Typical punching shear failure a) Truncated cone; b) failure cracks /4.5/

4.3 Basic control perimeter

In design for punching shear, the critical perimeter must be defined, both the shape and the distance from the loaded area. Compared to a beam with a critical section, definition of the critical perimeter is more important for punching shear. As the perimeter gets closer to the loader area, the perimeter rapidly gets shorter and the shear force rapidly gets higher.

Different codes have adopted different perimeters. According to NS-EN 1992-1-1 Eurocode 2 [EC2] /4.1/, the basic control perimeter u_1 may normally be taken to be at a distance 2,0d from the loaded area and should be constructed so as to minimise the length, see Figure 4.3.1. The effective depth of the slab can normally be taken as a mean value:

$$d_{eff} = \frac{d_y + d_z}{2} \tag{4.3.1}$$

where d_y and d_z are the effective depths of the reinforcement in two orthogonal directions. For a circular loaded area with diameter *D*, and a rectangular loaded area with dimension $c_1 \cdot c_2$, the length of the control perimeter becomes respectively:

$$\mathrm{u}_1^{\text{circular}} = \pi \cdot (D + 4d) \tag{4.3.2}$$

$$\mathrm{u}_1^{\text{rectangular}} = 4\pi \cdot d + 2 \cdot c_1 + 2 \cdot c_2 \tag{4.3.3}$$

In reality the choice of shape and u_1 is not too important. The treatment of shear in EC2 is empirical, i.e. it is based on experimental work. Hence, the same strength can be obtained with a short perimeter and large shear strength or with a longer perimeter and lower shear strength. By employing the distance 2,0d from the loaded area, the same formula in EC2 for shear resistance can be used for both punching shear and shear in beams.

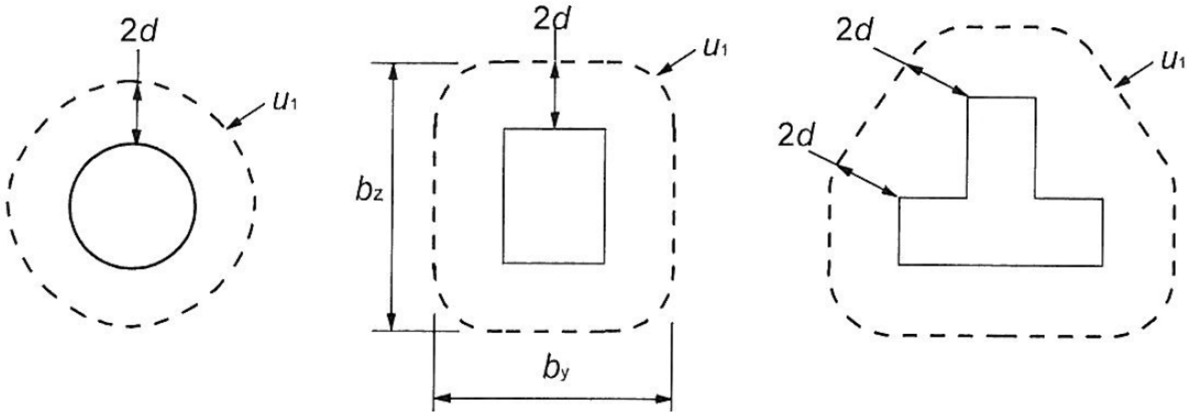


Figure 4.3.1 Typical basic control perimeters around loaded areas /4.1/

The choice of a minimised length to define the shape of the control perimeter in EC2, is realistic and in agreement with experiments, as seen in Figure 4.2.2. However, it is not necessary practical. If shear reinforcement is needed, this must be provided within the perimeter. With a minimised perimeter this is difficult with a rectangular reinforcement grid.

With the control perimeter located 2,0d from the loaded area, the appropriate verification model for checking punching shear failure in ultimate limit state in EC2 is given in Figure 4.3.2.

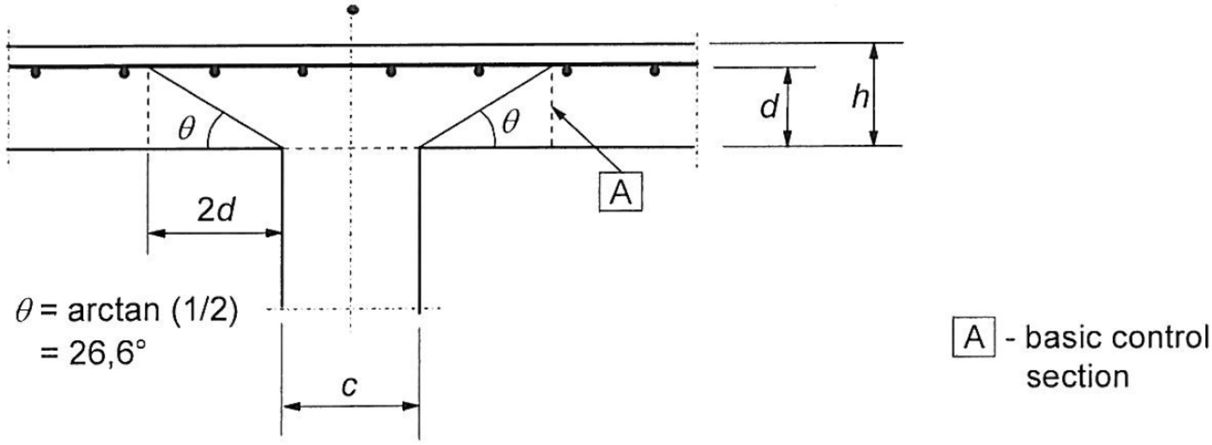


Figure 4.3.2 Model for punching shear at the ultimate limit state /4.1/

For loaded areas situated near an opening, corner or edge, the control perimeter must be reduced compared to Figure 4.3.1. The effect of the free edges must be taken into account. In EC2 the basic control perimeters for loaded areas close to an edge or corner are given as shown in Figure 4.3.3. The perimeter continues to the free edge must be smaller than that obtained from Figure 4.3.1.

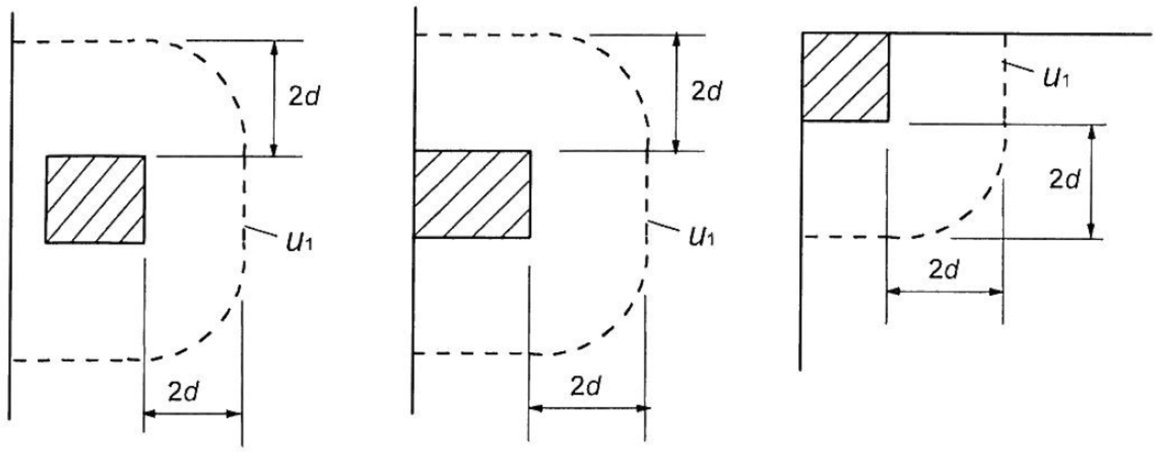


Figure 4.3.3 Basic control perimeters around loaded areas close to or at edge or corner /4.1/

Openings in a slab reduce the punching shear capacity. Depending on the distance between openings and loaded area, the length of the control perimeter must be reduced. When the distance between the perimeter of the loaded area and the edge of opening is less than 6d, EC2 employs a simplified approach. The part of the control perimeter contained between two tangents drawn to the outline of the opening from the centre to the loaded area is considered to be ineffective,

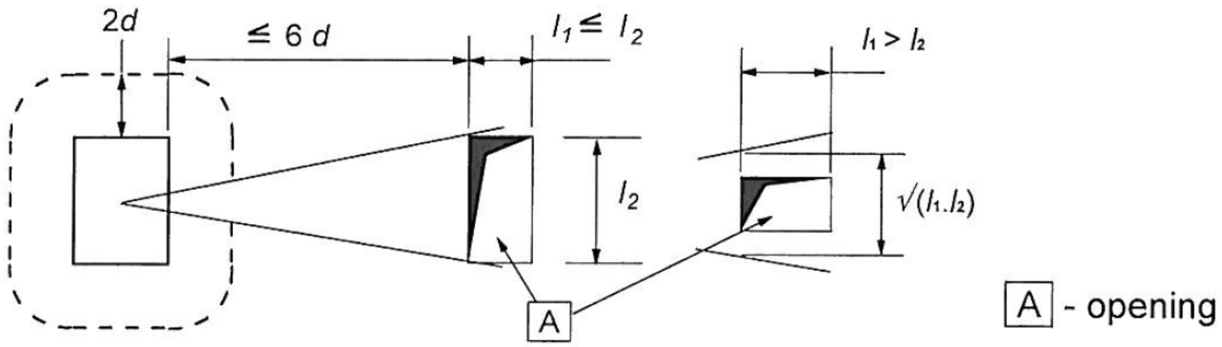


Figure 4.3.4 Control perimeter near an opening /4.1/

In flat slabs where drop panels or column heads are provided, control sections must in general be checked both within the head and in the slab. Detailed specifications are given in EC2. With basis in Figure 4.3.7, distinction must be made between cases where $l_H < 2.0h_H$ and $l_H > 2.0h_H$, where

l_H = distance from the column face to the edge of the column face
h_H = height of head below soffit of slab

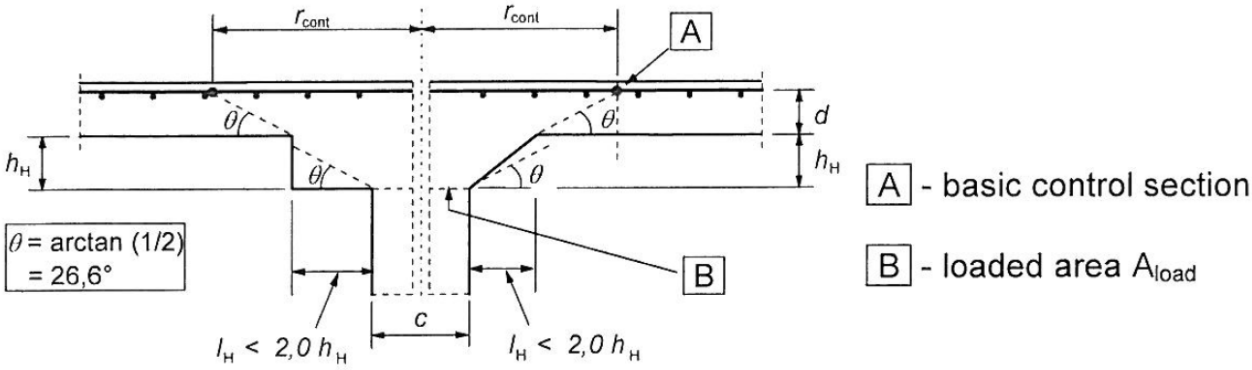


Figure 4.3.5 Location of control perimeter with enlarged column head, $l_H < 2.0h_H$, /4.1/

When $l_H < 2.0h_H$ a circular control perimeter is assumed with a radius r_{cont} outside the enlarged column head. Even for a rectangular column head, EC2 specify a circular perimeter. The radiuses are given in EC2 as:

$$r_{cont} = 2d + l_H + 0{,}5c \quad \text{for circular column head} \tag{4.3.4}$$

$$r_{cont} = \min\begin{cases} 2d + 0{,}56\sqrt{l_1 l_2} \\ 2d + 0{,}69 l_1 \end{cases} \quad \text{for a reactangular column with a reactangular column head} \tag{4.3.5}$$

where l_1 and l_2 are the overall dimensions of the rectangular head, and $l_1 \leq l_2$. To validate the length of the control perimeter by using the radius in Eq. (4.3.5) for a rectangular head, the length is also calculated a distance 2,0d outside the column head employing the minimised length according to Figure 4.3.1. Figure 4.3.6 gives the difference for different l_1/l_2 ratios relative to effective height d in the slab. As seen in the figure, the difference for a quadratic head is small. For an elongated head with l_1=0,5·l_2 the difference is much larger. However, in a flat slab, the drop panel or column head are normally quadratic.

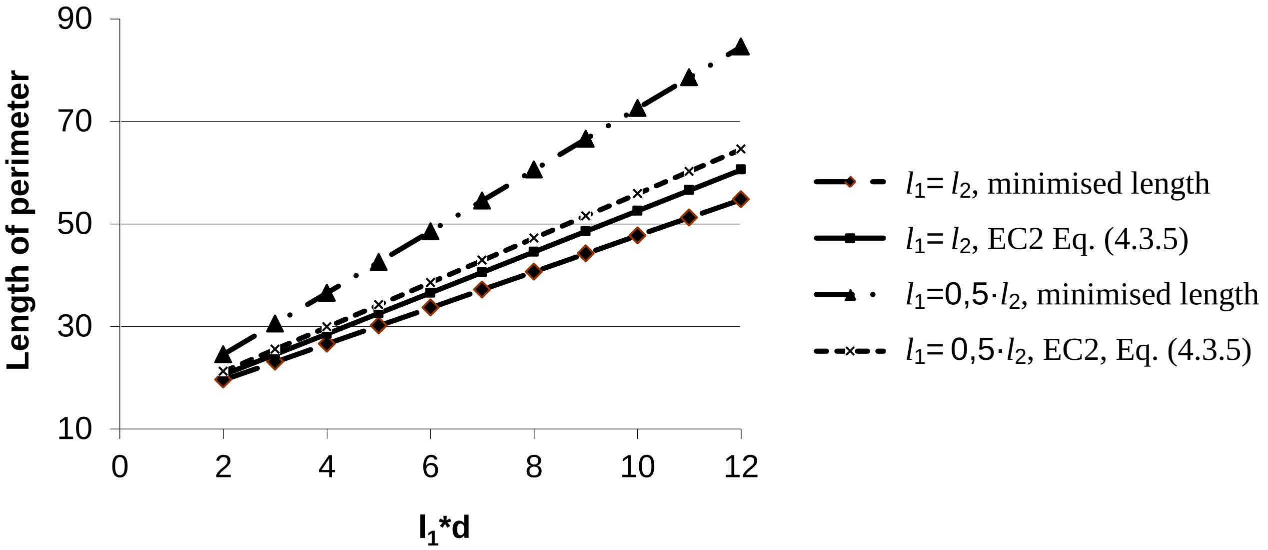


Figure 4.3.6 Difference in perimeter length according to EC2 and minimised length

When $l_H > 2.0h_H$ control sections within and outside the enlarged column heads must be defined, as seen in Figure 4.3.7. For circular columns EC2 specifies the distances from the centroid of the column to the control sections as:

$$r_{cont,ext} = 2d + l_H + 0{,}5c \tag{4.3.6}$$

$$r_{cont,int} = 2(d + h_H) + 0{,}5c \tag{4.3.7}$$

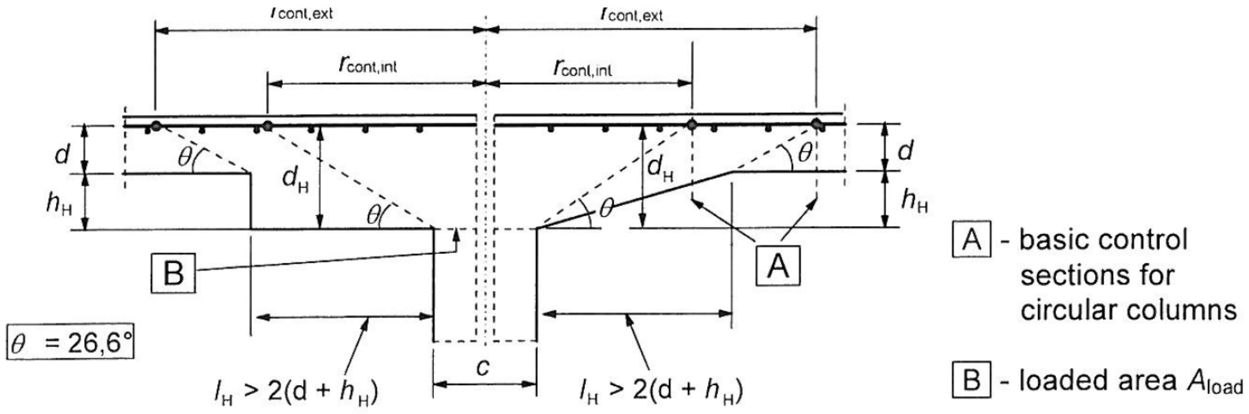


Figure 4.3.7 Location of control perimeter with enlarged column head, $l_H > 2.0h_H$, /4.1/

4.4 Design procedure and design shear force

Design for punching shear is based on verification of applied shear stresses, v_{Ed}, along defined control perimeters around the loaded area. The shear force acts over an area $u \cdot d_{eff}$, where u is the length of the perimeter and d_{eff} is the effective depth of the slab according to Eq. (4.3.1). For an external load or a slab-column connection with no moment transfer between the slab and the column, a uniform shear stress distribution can be assumed, see Figure 4.1.1.

$$v_{Ed} = \frac{V_{Ed}}{u \cdot d_{eff}} \tag{4.4.1}$$

The following design shear stresses (MPa) along the control sections are defined in EC2:

$v_{Rd,c}$ –design value for punching shear stress resistance of a slab without punching shear reinforcement.

$v_{Rd,cs}$ –design value for punching shear stress resistance of a slab with punching shear reinforcement.

$v_{Rd,max}$ –design value for maximum punching shear stress resistance.

The following checks must be carried out to fulfil the design for punching shear in a slab:

a) At the column perimeter, or the perimeter of the loaded area, ensure that the maximum punching shear stress is not exceeded, $v_{Ed} < v_{Rd,max}$. Otherwise, the practical solution is to resize the slab.

b) At the control perimeters considered, determine if punching shear reinforcement is required, $v_{Ed} > v_{Rd,c}$.

c) If $v_{Ed} > v_{Rd,c}$, provide necessary punching shear reinforcement so that $v_{Ed} < v_{Rd,cs}$.

Because of unsymmetrical loading, unequal spans or boundary conditions, moment transfer is practically always present from slab to column in a flat slab. This influence the shear stresses in the slab around the column. Experimental work shows that punching shear strength is reduced when moment transfer occurs. The strength is usual most critical at corner or edge columns, because the critical perimeter for punching does not extend all around the column and, hence, is weaker than at interior columns.

Design codes normally take into account the unbalanced moment from columns by using a multiplier to the uniform shear stress distribution in Eq. (4.4.1). In EC2 the increased stress is given as:

$$v_{Ed} = \beta \frac{V_{Ed}}{u \cdot d} \tag{4.4.2}$$

where β is the multiplication factor.

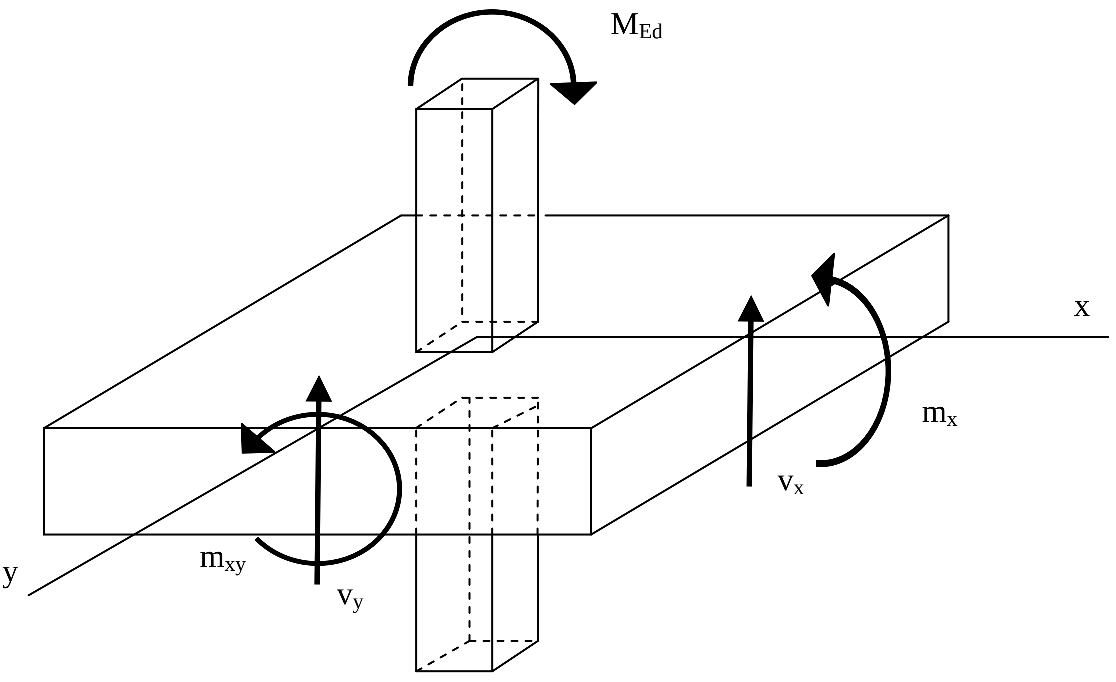


Figure 4.4.1 Transfer of moment from column to slab

Some distance from the column edge, an unbalanced moment M_{Ed} in the column is balanced by a distributed bending moment m_x, torsional moment m_{xy}, and partly by shear forces v_x and v_y around the considered perimeter, see Figure 4.4.4. Thus, the β factor in Eq. (4.4.2) accounts for the increased distributed shear forces due to the unbalanced column moment. The factor is a function of the geometry of the critical perimeter, column size and the moment transferred and EC2 defines it as:

$$\beta = 1 + k \cdot \frac{M_{Ed}}{V_{Ed}} \cdot \frac{u_1}{W_1} \tag{4.4.3}$$

$$W_1 = \int_0^{u_i} |e| dl \tag{4.4.4}$$

where k is a value defining the proportion of the unbalanced moment transmitted by uneven shear, on one hand, and by bending and torsion on the other. Values of k in EC2 are given in Table 4.1, depending on the aspect ratio of the column size, $c_1$and c_2, for rectangular loaded areas. For round columns c_1/c_2=1,0, so k=0,6.

Table 4.1 Values of k for rectangular loaded areas

c_1/c_2	≤0,5	1,0	2,0	≥3,0
k	0,45	0,60	0,70	0,80

W_1 in Eq. (4.4.4) depends on the distribution of shear stresses around the control perimeter due to an unbalanced moment M_{Ed}. On the basis of elastic analyses /4.7/, the distribution of shear stresses in a slab in the vicinity of a column can be calculated, and they approach the distribution given in Figure 4.4.2. In Eq. (4.4.4), *dl* is a length increment of the perimeter and *e* is the distance of dl from the axis of which the moment M_{Ed} acts.

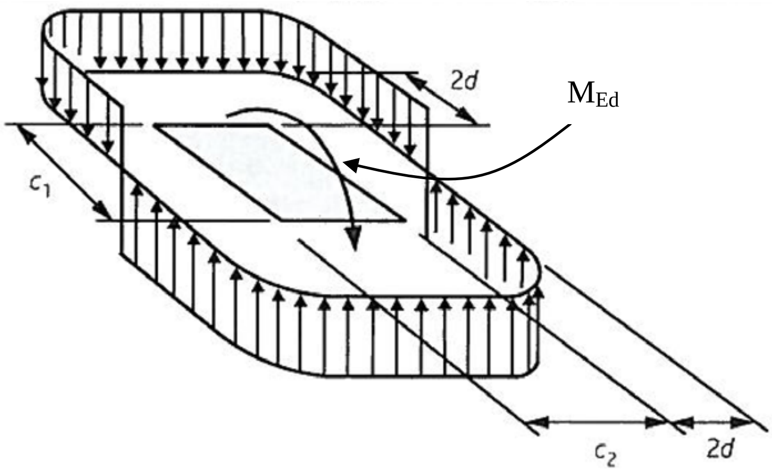


Figure 4.4.2 Shear distribution due to an unbalanced moment at a slab internal column connection /4.8/

EC2 assumes the shear distribution in Figure 4.4.2. The moment transferred between column and slab must equal the moment produced by the shear distributed around the perimeter. This can be written as:

$$M_{Ed} = 4 \cdot v_M \left[\frac{c_1}{2} \cdot \frac{c_1}{4} + \frac{c_2}{2} \cdot (\frac{c_1}{2} + 2d) + 2d \cdot \frac{\pi}{2} (\frac{c_1}{2} + 2d \cdot \frac{2}{\pi}) \right] \tag{4.4.5}$$

$$= v_M \left(\frac{c_1^2}{2} + c_1 \cdot c_2 + 4 \cdot c_2 \cdot d + 16d^2 + 2\pi d c_1 \right) = W_1 \cdot v_M$$

$$\Rightarrow$$

$$W_1 = \frac{c_1^2}{2} + c_1 \cdot c_2 + 4 \cdot c_2 \cdot d + 16d^2 + 2\pi d c_1$$

$$v_M = \frac{M_{Ed}}{W_1}$$

The shear v_M due to an unbalanced column moment and the definition of W_1 in Eq. (4.4.5) for an internal column corresponds to the equations given in EC2. Figure 4.4.3 gives the total assumed shear distribution around a rectangular loaded area.

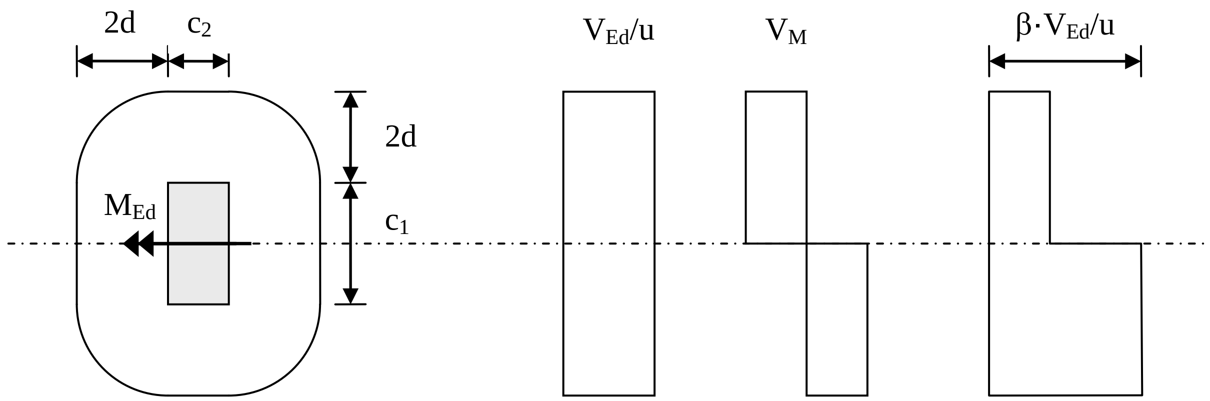


Figure 4.4.3 Total assumed shear distribution around the control perimeter

The punching shear enhancement factor *β*, is also given in EC2 for other configurations of columns, such as corner and edge columns and biaxial unbalanced moments in the column.

Following EC2, the calculation of the β factors can be rather complicated. Therefore EC2 offers simplified values for *β*. For structures where the lateral stability does not depend on the frame action between the slabs and columns, and where the adjacent spans do not differ in length by more than 25%, approximate values given in Figure 4.4.4 can be used.

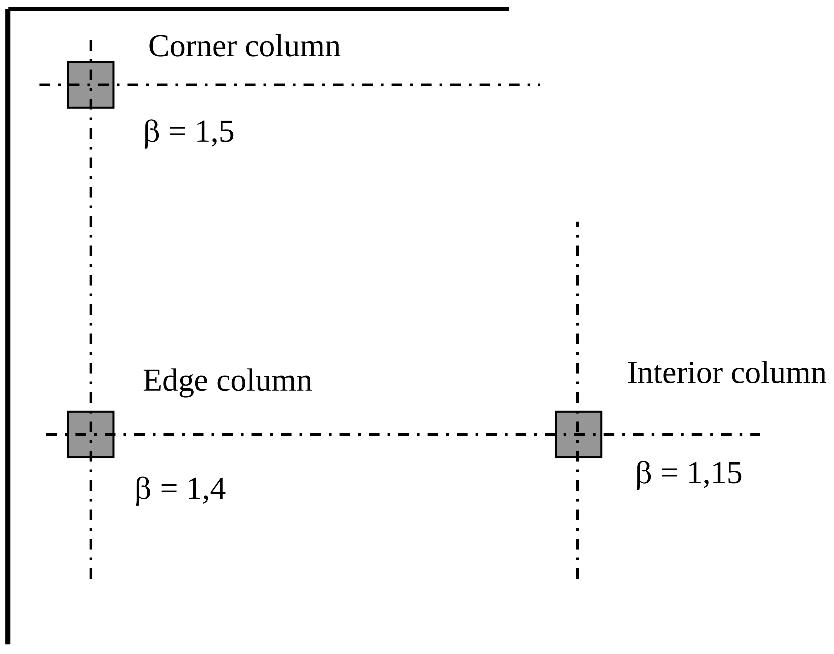


Figure 4.4.4 Recommended values for β

4.5 Punching shear resistance without shear reinforcement

The basic control section u_1 must be checked to determine whether punching shear reinforcement is required. The design shear stress resistance $v_{Rd,c}$ is in EC2 given as:

$$v_{Rd,c} = C_{Rd,c} \cdot k \cdot (100\rho_l f_{ck})^{1/3} + k_1 \cdot \sigma_{cp} \geq v_{\min} \tag{4.5.1}$$

where

$$C_{Rd,c} = 0{,}18 / \gamma_c$$
$$k = 1 + \sqrt{200/d} \leq 2{,}0$$
$$k_1 = 0{,}1 \text{ in compression and - } 0{,}3 \text{ in tension}$$
$$\sigma_{\mathrm{cp}} = \text{normal concrete stresses}$$
$$\rho_{\mathrm{l}} = \sqrt{\rho_{\mathrm{ly}} \cdot \rho_{\mathrm{lz}}} \leq 0{,}02$$
$$\rho_{\mathrm{ly}}, \rho_{\mathrm{lz}} = \text{the mean ratios of reinforcement in each direction over a width equal to the column dimension plus 3d on each side}$$
$$\mathrm{v}_{\min} = 0{,}035 \cdot k^{3/2} \cdot f_{ck}^{1/2}$$

The expression in Eq. (4.5.1) is almost similar to the shear resistance for a beam unreinforced in shear. Compared to a beam, average values of reinforcement ratio ρ_l and normal concrete stresses σ_{cp} must be used in Eq. (4.5.1). σ_{cp} must be calculated based on the longitudinal forces across the full bay for internal columns, and across the control section for edge columns. In addition the factor k_1 applied to the effect of axial stresses, is reduced from 0,15 to 0,1 compared to a beam.

4.6 Design of reinforcement for punching shear

Where the applied shear stress v_{Ed} at the basic control perimeter u_1 exceeds $v_{Rd,c}$, shear reinforcement must be provided to achieve necessary resistance. EC2 uses the following relationship for punching shear resistance with shear reinforcement:

$$v_{Rd,cs} = 0{,}75 \cdot v_{Rd,c} + 1{,}5 \cdot (d/s_r) \cdot A_{sw} \cdot f_{ywd,ef} \cdot (1/u_1 d) \cdot \sin\alpha \tag{4.6.1}$$

where A_{sw} is the area of shear reinforcement in one perimeter around the column, s_r is the radial spacing of perimeters of shear reinforcement, and $f_{ywd,ef}$ is effective design strength of punching shear reinforcement given as:

$$f_{ywd,ef} = (250 + 0{,}25d) \leq f_{ywd} \tag{4.6.2}$$

The summation principle of concrete contribution and punching reinforcement component is employed in Eq. (4.6.1). This in difference to shear reinforcement design of beams which is based on a truss model with no concrete contribution. However, the concrete contribution is reduced to 75% for punching shear. The factor 1,5 for the shear reinforcement contribution would correspond to an angle of cotθ for the strut in a truss model. However, the use of 1.5 does not imply a steeper failure plane, but reflects observations from test results that shear reinforcement at the ends of the shear planes are less effective. Punching shear in EC2 are based on a model with a failure plane cotθ=2,0 (radial distance 2,0d). Always keep in mind that formulas for punching shear resistance given in EC2 are calibrated against test results

The introduction of effective design strength of the shear reinforcement can partly be explained by anchoring efficiency. It is argued that it is hard to find adequately anchored punching shear reinforcement at both sides of a critical crack.

By restructuring Eq. (4.6.1) , the required vertical shear reinforcement per perimeter can be written as:

$$A_{sw} = (v_{Ed} - 0{,}75 \cdot v_{Rd,c}) \cdot s_r \cdot u_1 / (1{,}5 \cdot f_{ywd,ef}) \tag{4.6.3}$$

To maintain the designed amount of reinforcement, A_{sw}, around a circle loaded area, the number of links must be kept constant around each perimeter, see Figure 4.6.1.

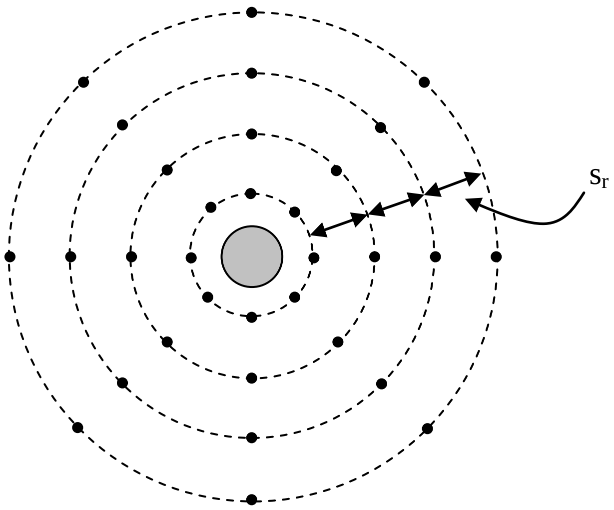


Figure 4.6.1 Shear reinforcement

In practice it can be inconvenient to keep control of the required shear reinforcement per perimeter according to Eq. (4.6.3), especially in a rectangular reinforcement grid. The contribution of shear reinforcement to the total shear resistance in Eq. (4.6.1), is assumed to be take part over a radial distance of 1.5d. To verify the required amount of shear reinforcement it is sometimes easier to find the total area of reinforcement in a radial band around the loaded area, as illustrated in Figure 4.6.2. Within the band with radial distance of 1.5d the required shear reinforcement area is given as:

$$\sum A_{sw} = 1{,}5 \cdot (d / s_r) \cdot A_{sw} \tag{4.6.4}$$

The total punching shear reinforcement in the radial band must be controlled in successive perimeters from the loaded area to ensure that the reinforcement in each zone satisfies Eq. (4.6.4).

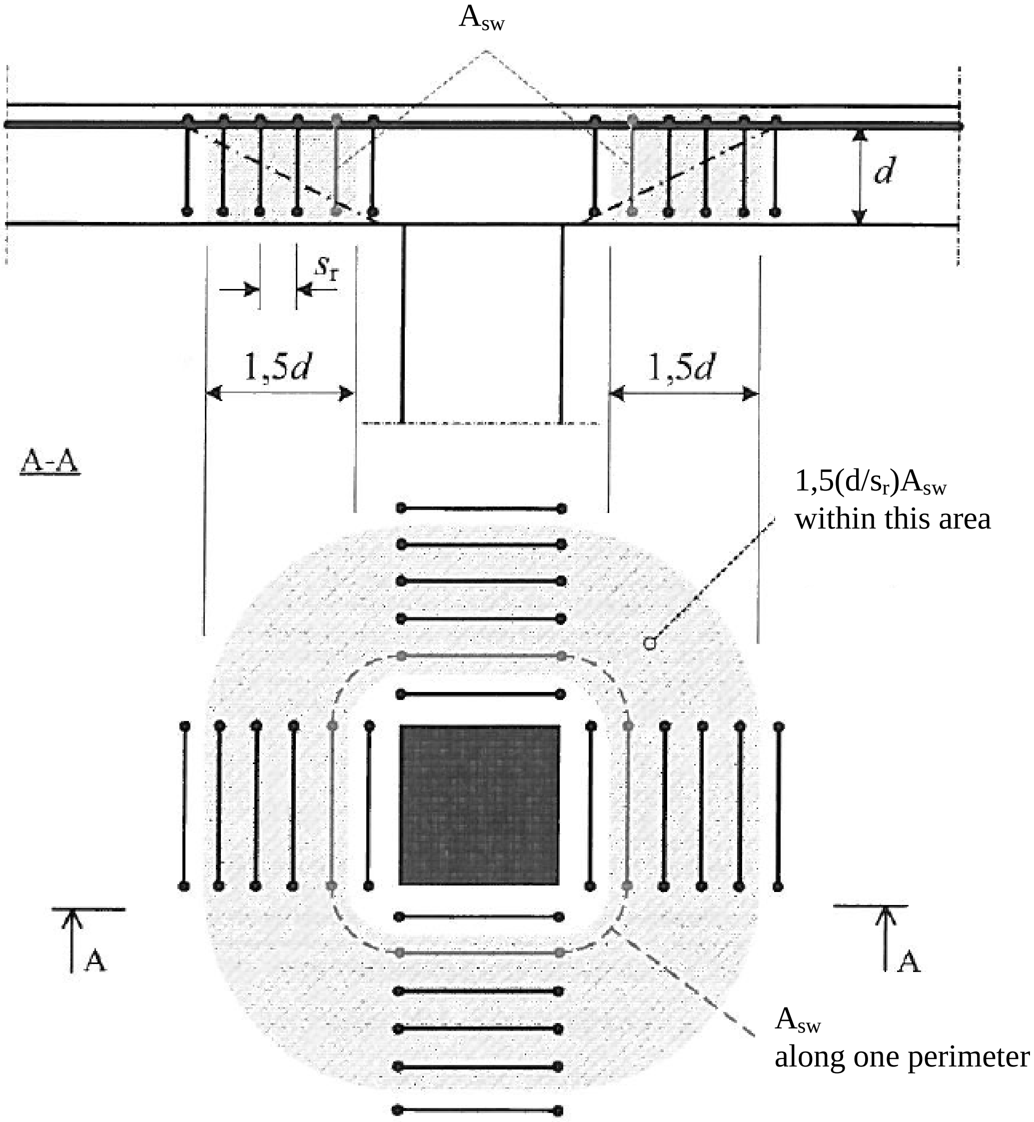


Figure 4.6.2 Contributory zone of shear reinforcement /4.9/

4.7 Punching shear resistance adjacent to loaded area

The punching shear stress must be checked at the loaded area perimeter, u_0, normally the column perimeter, to ensure that

$$v_{Ed} = \beta \cdot \frac{V_{Ed}}{u_0 \cdot d} \le v_{Rd,\max} \qquad (4.7.1)$$

where

$$\begin{aligned} u_0 &= \text{column perimeter for an interior column} \\ &= c_2 + 3d \le c_2 + 2c_1 \text{ for an edge column} \\ &= 3d \le c_2 + 2c_1 \text{ for an corner column} \end{aligned}$$

The check in Eq. (4.7.1) is to avoid a compression failure in the concrete. It is applicable to sections with or without shear reinforcement, but is unlikely to be critical for slabs without shear reinforcement. The value for maximum punching shear stress resistance, $v_{Rd,max}$, is in EC2 given as:

$$v_{Rd,\max} = 0,4 \cdot v \cdot f_{cd} \qquad (4.7.2)$$

where

$$v = 0,6 \cdot \left[1 - (\mathrm{f_{ck}} / 250)\right] \qquad (4.7.3)$$

Even by employing Eq. (4.7.2), experimental work on punching resistance of slabs with shear reinforcement has shown that the addition formula in Eq. (4.6.1) can give too high resistances. Hence, in the National Annex for Norway in EC2, a limitation is defined for maximum punching shear stress resistance, to reduce the influence of high ratios of shear reinforcement. The limit is defined as:

$$v_{Rd,\max} \le \frac{1,6 \cdot v_{Rd,c} \cdot u_1}{\beta \cdot u_0} \qquad (4.7.4)$$

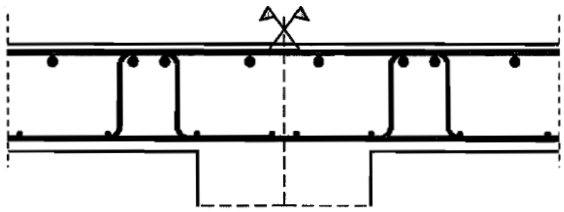
In Eq. (4.7.4), $v_{Rd,c}$ is calculated according to Eq. (4.5.1), but with no contribution from normal stresses (i.e. $k_1 \cdot \sigma_p = 0$). If required punching shear reinforcement is calculated according to Eq. (4.6.1) with no concrete contribution (i.e. $0,75 \cdot v_{Rd,c} = 0$), $v_{Rd,max} = 0,4 \cdot v \cdot f_{cd}$, can be employed with no upper limitation.

4.8 Detailing shear reinforcement

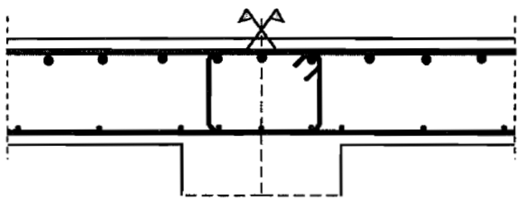
In design of slabs, the main problem is often the punching shear resistance. To increase the resistance it is possible to:

- Increase column diameter → often rejected by the architect.
- Use a drop panel or enlarged column head → can be in conflict with utilisation of the structure
- Larger slab depth → increased dead load and cost of footings and columns.
- More flexural reinforcement → not so effective
- Increase concrete compressive strength→ expensive and not so effective
- Shear reinforcement

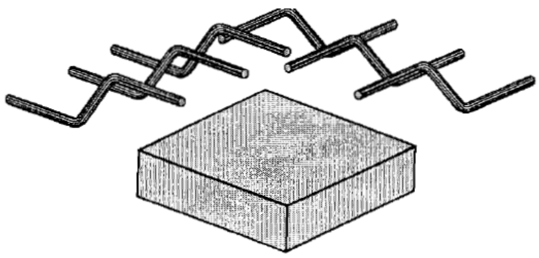
Providing effective shear reinforcement is often the most economic solution to increase the punching resistance in a slab. There are many different shear systems to increase the shear resistance. Figure 4.8.1 shows some of the most popular systems.



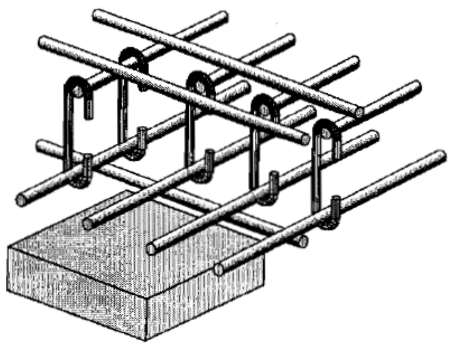
a) Conventional stirrup



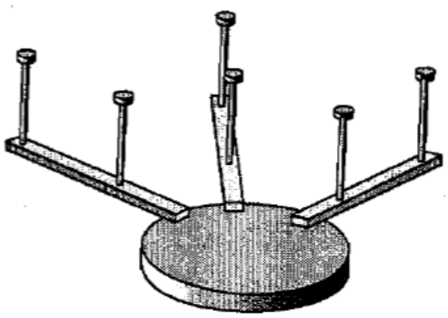
b) Closed links



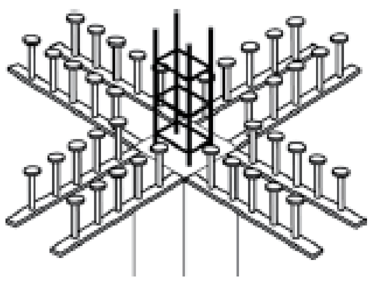
c) Bent up bars



d) Hooks



e) Stud-rails



f) Concentrated stud-rails

Figure 4.8.1 Different systems for punching shear reinforcement /4.2/

Conventional shear links and stirrups are cheap, but they are difficult and time-consuming to set out and fix. Normally they are placed in rectangular grid around the column. Stud-rails are a prefabricated system where the studs are welded to flat base rails. They can be placed radial, in a rectangular grid or concentrated along the column axes. It is also possible to use double headed studs with no rails.

Design of punching shear reinforcement is governed by the shape of control perimeter around the loaded area. The required area of shear reinforcement in EC2, given in Eq. (4.6.3), must be provided in each perimeter with a radial distance of s_r. Orthogonal rectangular reinforcement grids, see Figure 4.8.2, are often employed since they can coincide with horizontal reinforcement arrangements. Such arrangements lead to an increase in area of shear reinforcement per perimeter on successive perimeters away from the loaded area. It can also be difficult to find and fit the exact number of shear links in a rectangular grid around the perimeter, as illustrated in Figure 4.8.2. The difficulty is to estimate the effect of shear reinforcement not lying exact on the minimised perimeter around the rectangular loaded area. Use of shear reinforcement in a radial arrangement, e.g. stud rails, would simplify the shear reinforcement requirements. Another option is to employ the approach in section 4.6, where the total area of shear reinforcement, within an area of 1.5d inside the control perimeter under consideration, is calculated according to Eq. (4.6.4).

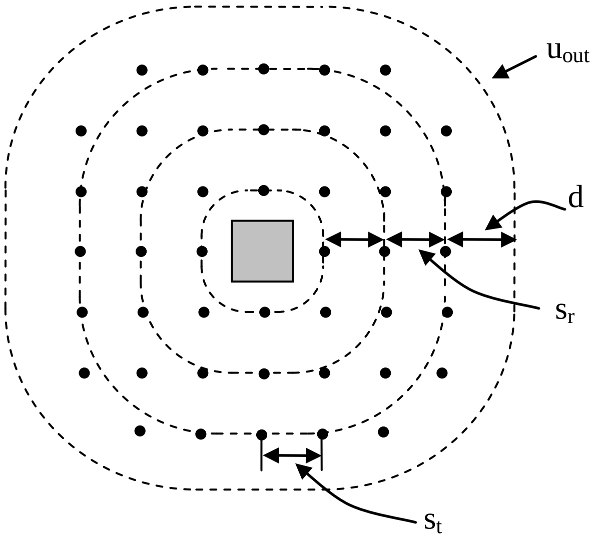


Figure 4.8.2 Shear links in an orthogonal rectangular grid

Having found the required shear reinforcement at the basic control perimeter u_1, and outer perimeter, at which shear reinforcement is not required must be decided. An outer control perimeter can be calculated according to EC2 as:

$$u_{out,ef} = \beta \cdot \frac{V_{Ed}}{v_{Rd,c} \cdot d} \tag{4.8.1}$$

Knowing the perimeter $u_{out,ef}$, the distance from the loaded area can be found. According to EC2, the outermost perimeter of shear reinforcement must be placed at a distance no greater than the effective depth 1,5d (1,0d in national annex in Norway), within the perimeter where reinforcement is not required. This is illustrated in Figure 4.8.2 for a rectangular grid and in Figure 4.8.3 for radial reinforcement and cruciform shape of concentrated reinforcement. The limitation of the distance inside $u_{out,ef}$ is to ensure that an inclined punching shear plane cannot develop within this perimeter without passing through a set of shear reinforcement legs.

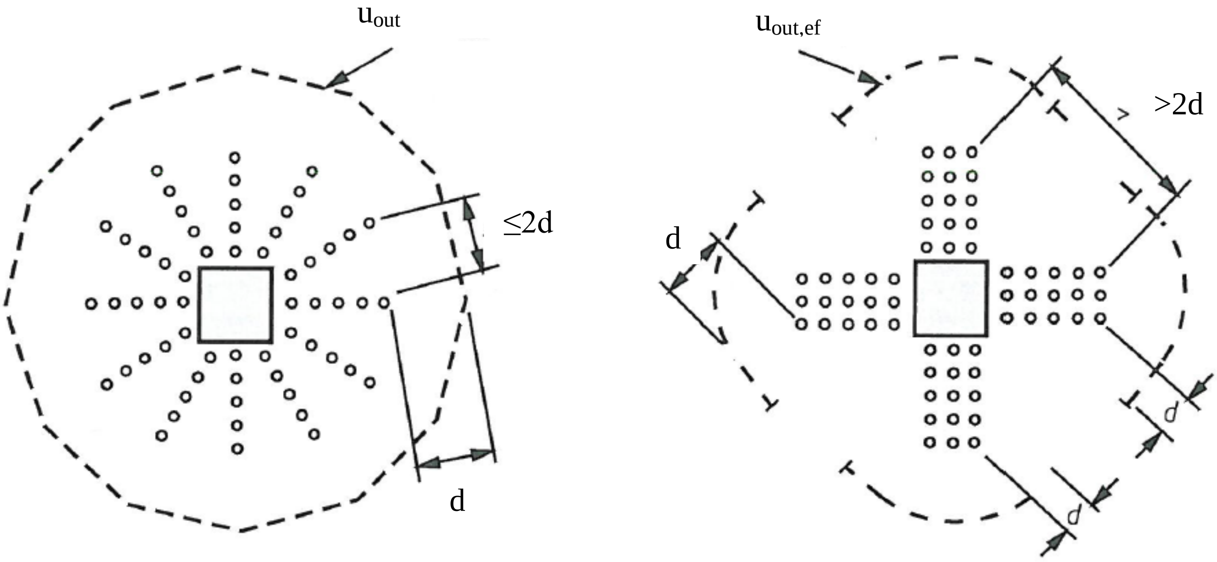


Figure 4.8.3 Outer control perimeter at internal columns

Shear reinforcement too close to the loaded area is unlikely to be effective to resist shear. Thus, the detailing rules in EC2 claim the first perimeter with reinforcement must not be closer than 0,3d from the edge of the loaded area, as seen in Figure 4.8.4. To avoid critical shear cracks to form between perimeters of reinforcement, the radial spacing of reinforcement must not be larger than 0,75d, and the first perimeter with reinforcement not more than 0,5d from the face of the loaded area. Shear reinforcement must be provided in at least two perimeters. The spacing of link legs around a perimeter, s_t, must not exceed 1,5d within the first control perimeter (2d from loaded area) or 2d for perimeters outside the first control perimeter.

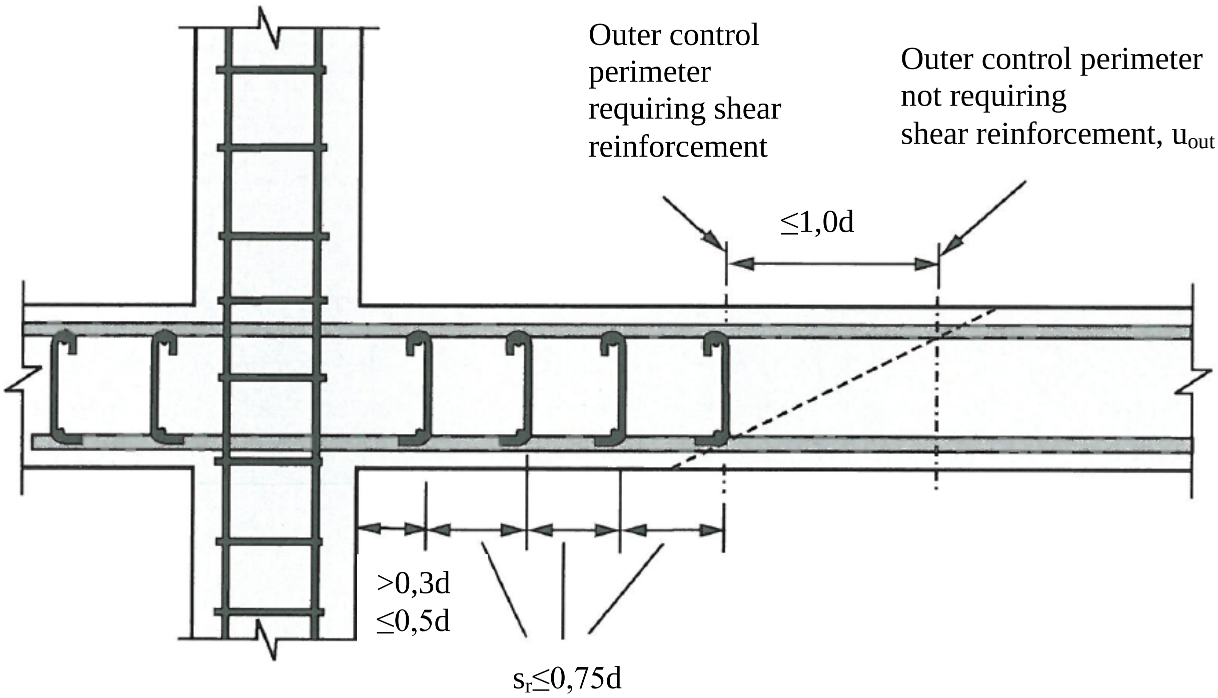


Figure 4.8.4 Spacing of shear reinforcement links /4.8/

Where shear reinforcement is required, the minimum area of a single leg of link is given as:

$$A_{sw,\min} \geq \frac{0{,}08 \cdot \sqrt{f_{ck}} \cdot s_r \cdot s_t}{f_{yk} \cdot (1{,}5 \sin\alpha + \cos\alpha)} \tag{4.8.2}$$

where

α = angle between main reinforcement and shear reinforcement; for vertical reinforcement $\sin\alpha = 1{,}0$

s_r = is the spacing of shear links in the radial direction

s_t = is the spacing of shear links in the tangential direction

Detailing shear reinforcement around a column is sometimes a trial and error procedure. The result is a compromise between costs, work on site and to satisfy requirements for the reinforcement. The minimum total shear reinforcement is achieved by a short radial distance between perimeters, but is time consuming on site. Choice of bar diameter and spacing of links also depends on the layout of bending reinforcement. Using the same arrangement of reinforcement for so many columns as possible can for instance avoid confusion in detailing or on site.

The required amount of shear reinforcement will often be defined by the circumferential and radial spacing rules rather than the required strength. In particular, the radial maximum spacing of 0,75d ensures that in many situations, three perimeters of reinforcement within the failure zone.

4.9 Example: Internal slab-column connection

Design a slab for punching shear around an internal column according to EC2. The column supports a slab 225 mm thick and the column size is 300x300 mm.

Material properties:

Reinforcement B500NC: $f_{yk} = 500$ N/mm^2 2.4.2.4(1)

Concrete B30: $f_{cd} = \alpha_{cc} \cdot f_{ck} / \gamma_c = 0{,}85 \cdot 30 / 1{,}5 = 17{,}0$ N/mm^2 3.1.6(1)

Nominal concrete cover $c_{nom} = 25$ mm 4.4.1.1(2)

Assuming 12 mm diameter bars give the average effective depth

$d_{eff} = (d_y + d_z)/2 = 225 – 25 - 12 = 188$ mm Eq. 6.32

Design shear force (column reaction force) $V_{Ed} = 600$ kN

Bending moment transferred between column and slab, about one axis only,

$M_{Ed} = 40$ kNm

The design for flexure gave tension reinforcement on top surface of the slab of ø12c80 in y-direction and ø12c120 in z-direction

$\rho_{ly} = A_{sy}/(b \cdot d) = 1413/(1000 \cdot 188) = 0{,}0075$

$\rho_{lz} = A_{sz}/(b \cdot d) = 942/(1000 \cdot 188) = 0{,}0050$

Basic control perimeter

6.4.2(1)

$u_1 = 2(c_1+c_2) + 2\pi \cdot 2d = 4 \cdot 300+2\pi \cdot 2 \cdot 188 = 3562$ mm

Shear stress at control perimeter u_1 (2d from face of load)

$v_{Ed} = \beta \cdot V_{Ed}/(u_1 \cdot d)$ Eq. 6.38

$\beta = 1+k \cdot (M_{Ed} /V_{Ed}) \cdot (u_1/W_1)$ Eq. 6.39

$k = 0{,}6$ for $c_1/c_2 =1{,}0$ Tab. 6.1

$W_1 = c_1^2/2+ c_1 \cdot c_2 +4c_2d+16d^2+2\pi \cdot d \cdot c_1$

$= 300^2/2+300^2+4 \cdot 300 \cdot 188+16 \cdot 188^2+2\pi \cdot 188 \cdot 300 = 1{,}28 \cdot 10^6 \text{ mm}^2$ Eq. 6.41

$\beta = 1+0{,}6 \cdot (40 \cdot 10^3 /600) \cdot (3562/1{,}28 \cdot 10^6) = 1{,}11$

$v_{Ed} = 1{,}11 \cdot 600 \cdot 10^3 /(3562 \cdot 188) = \underline{1{,}01 \text{ N/mm}^2}$

A value of β=1,11 is only a small reduction compared to approximate value of 1,15 given in EC2 for an internal column. To avoid shear reinforcement it may still be economic in some situations to do a calculation of β where the shear resistance is close to the design shear force. 6.4.3(6)

Shear stress resistance without shear reinforcement

Eq. 6.47

$v_{Rd,c} =0{,}18/\gamma_c \cdot k \cdot (100\rho_l \cdot f_{ck})^{1/3} \geq v_{min}$

$k = 1+(200/d)^{0,5} = 1+(200/188)^{0,5} =2{,}03 \leq 2{,}0 \rightarrow k = 2{,}0$

$\rho_l = (\rho_{ly} \cdot \rho_{lz})^{0,5} = (0{,}0075 \cdot 0{,}0050)^{0,5} = 0{,}006$

$v_{min} = 0{,}035 \cdot k^{1,5} \cdot (f_{ck})^{0,5} = 0.035 \cdot 2{,}0^{1,5} \cdot 30^{0,5} = 0{,}54 \text{ N/mm}^2$

$v_{Rd,c} =0{,}18/1{,}5 \cdot 2{,}0 \cdot (100 \cdot 0{,}006 \cdot 30)^{1/3} = \underline{0{,}63 \text{ N/mm}^2} \geq v_{min}$

$v_{Ed} > v_{Rd,c}$ meaning punching shear reinforcement is required

Shear stress at perimeter of column u_0 — 6.4.5(3)

$v_{Ed} = \beta \cdot V_{Ed}/(u_0 \cdot d) \leq v_{Rd,max}$ — Eq. 6.53

$u_0 = 2(c_1+c_2) = 4 \cdot 300 = 1200$ mm

$v_{Ed} = 1,11 \cdot 600 \cdot 10^3 /(1200 \cdot 188) = 2,95$ N/mm^2

NA.6.4.5(3)

$v_{Rd,max} = 0,4 \cdot v \cdot f_{cd} < 1,6 \cdot v_{Rd,c} \cdot u_1/(\beta \cdot u_0)$

$v = 0,6(1-f_{ck}/250) = 0,6(1-30/250) = 0,528$ — Eq. NA.6.6N

$v_{Rd,max} = 0,4 \cdot 0,528 \cdot 17,0 < 1,6 \cdot 0,63 \cdot 3562/(1,11 \cdot 1200)$
$= 3,59 < 2,69$

$v_{Ed} = 2,95 > v_{Rd,max}=2,69$ N/mm^2

NA.6.4.5(3)

The maximum punching shear resistance is too low. A possible solution could be to increase the slab thickness or the amount of bending reinforcement, or use a drop panel or enlarged column head. However, in EC2 +NA the lower limit of $v_{Rd,max}$ can be used if the concrete contribution, $v_{Rd,c}$, is not taken account when calculating required shear reinforcement. This will be done in this example since it is already clear that shear reinforcement is required. In practise this should also be compared to the minimum shear reinforcement.

$v_{Ed} = 2,95 < v_{Rd,max}=3,59$ N/mm^2

Perimeter at which shear reinforcement is not required

$u_{out} = \beta \cdot V_{Ed}/(v_{Rd,c} \cdot d) = 1,11 \cdot 600 \cdot 10^3 /(0,63 \cdot 188) = 5623$ mm — Eq. 6.54

Distance from face of column to u_{out}, is $(5623-4 \cdot 300)/2\pi = 704$ mm $= 3,74d$

Perimeters of shear reinforcement may stop a distance 704-d = 704-188 = 516 mm from the face of column. — NA.6.5.4(4)

Required shear reinforcement

Shear reinforcement are placed in a rectangular arrangement of links.

Maximum spacing of reinforcement: — 9.4.3(1)

$s_{r,max} = 0,75 \cdot d = 0,75 \cdot 188 = 141$ mm, say 140 mm

Inside 2d control perimeter, $s_{t,max} = 1,5 \cdot d = 1,5 \cdot 188 = 288$ mm, say 280 mm
Outside 2d control perimeter, $s_{t,max} = 2,0 \cdot d = 2,0 \cdot 188 = 376$ mm, say 370 mm

Required shear reinforcement at perimeter u_1 assuming vertical links: 6.4.5.1(1)

$A_{sw} = (v_{Ed}-0,75 \cdot v_{Rd,c}) \cdot s_r \cdot u_1/(1,5 \cdot f_{ywd,ef})$ Eq. 6.52

$f_{ywd,ef} = (250+0,25d) = 250+0,25 \cdot 188 = 297$ N/mm^2

$0,75 \cdot v_{Rd,c} = 0$, due to definition of $v_{Rd,max}$

$A_{sw} = (1,01-0) \cdot 140 \cdot 3562/(1,5 \cdot 297) =$ 1130 mm^2 pr perimeter

Minimum reinforcement of one link leg: 9.4.3(2)

$A_{sw,min} \geq 0,08 \cdot f_{ck}^{0,5} \cdot s_r \cdot s_t/(1,5 \cdot f_{yk}) = 0,08 \cdot 30^{0,5} \cdot 140 \cdot 370/(1,5 \cdot 500) = 30$ mm^2 Eq. 9.11

→ ø8 legs of links = 50 mm^2

With ø8 links the maximum tangential spacing at perimeter u_1 :

$s_t = 50 \cdot u_1/A_{sw} = 50 \cdot 3562/1130 = 158$ mm, say 140 mm

It is difficult to find the optimal reinforcement arrangement because the required reinforcement is per perimeter with the same shape as the critical perimeter. The first perimeter with reinforcement must be >0,3d but <0,5d from the face of the column. Here 60mm (0,32d) is used. Then a regular square arrangement of links, $s_r = s_t = 140$ mm, is possible, which simplifies the work at the building site. Figure 4.9.1 shows a possible layout using a regular square grid arrangement. Requirements for maximum distance in radial direction makes it difficult to get the outmost perimeter of reinforcement exact on the perimeter where punching reinforcement is no longer required. With 5 perimeters of reinforcement, a total of 136ø8 = 6836 mm^2 is required. As seen in Figure 4.9.1, double links must be employed in the first perimeter and partly in the second perimeter from the column face.

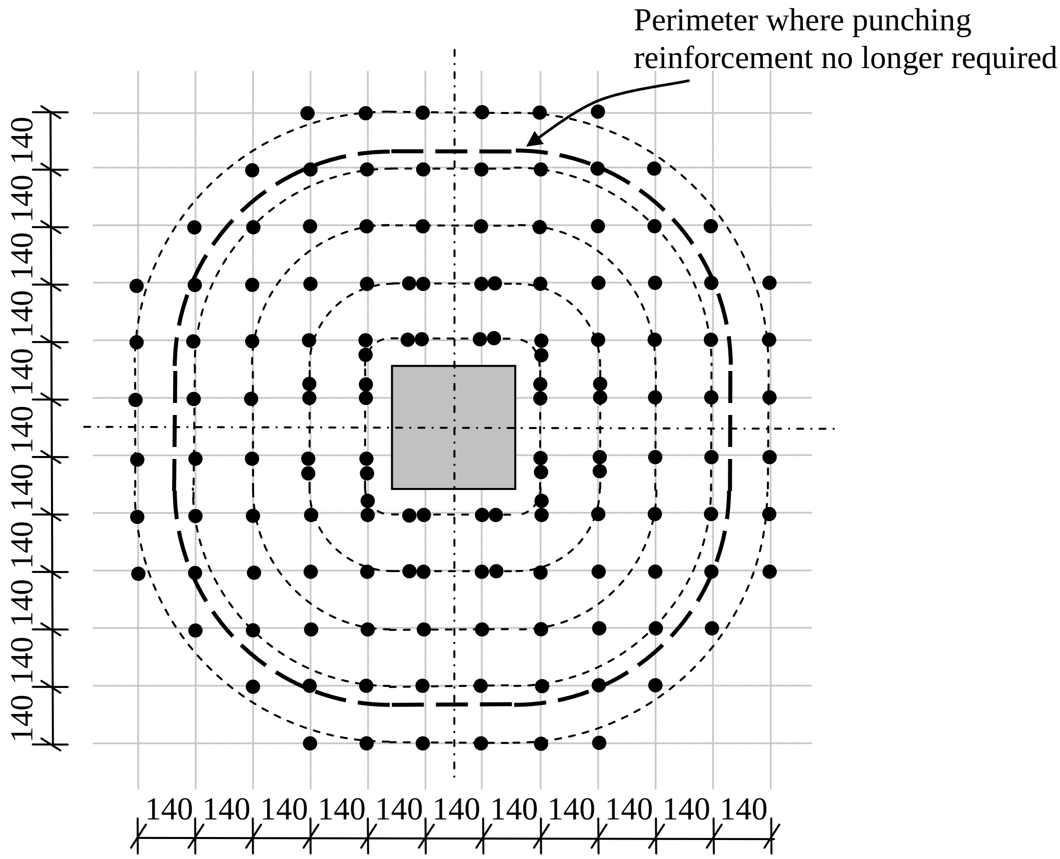


Figure 4.9.1 Regular square arrangement of shear links

To ensure that A_{sw} > 1130 mm^2 pr perimeter, it is necessary to check the number of links in close vicinity to each perimeter of reinforcement. By inspection of Figure 4.9.1, the number of links can be counted and are presented in Table 4.2 . However, by using a square arrangement of links, there will always be a discussion if links not lying exact on a perimeter can be taken into account.

Table 4.2 Area of reinforcement in perimeters

Distance from face of column	Number of links	A_s [mm^2]
0,32d = 60 mm	2x12ø8=24ø8	1200
1,06d = 200mm	8ø8+2x8ø8=24ø8	1200
1,81d = 340 mm	24ø8	1200
2,55d = 480mm	28ø8	1400
3,30d = 620 mm	32ø8	1600

A rectangular grid 140x240 of ø10 would have been possible. However, then the grid need to change orientation around the column, and it is considered better to use a regular square grid arrangement to avoid confusion in detailing or on site.

Using stud rails as shear reinforcement, would simplify the reinforcement requirements. In Figure 4.9.2 a combination of radial and square arrangement of shear links is illustrated. By using 16ø10 links around each perimeter, the total reinforcement required for 5 perimeters is 80ø10 = 6283 mm^2. This is almost a 10% reduction compared to the regular square arrangement of links in Figure 4.9.1. In addition, using stud rails makes it easier to fit the radial spacing of reinforcement, and have a perimeter exact where shear reinforcement no longer is required. The number of perimeters with shear reinforcement can for instance be reduced to 4, giving a total amount of 5026 mm^2.

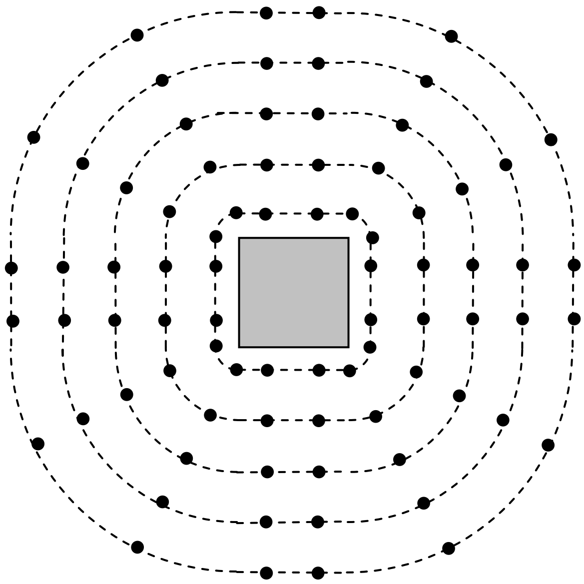


Figure 4.9.2 Optimised arrangement with ø10 shear links

4.10 Punching resistance of column bases

In general in Eurocode 2, the punching shear resistance for a slab must be assessed at the basic control perimeter 2d from the loaded area. However, for column bases the punching resistance must be verified at control perimeters closer than 2d from the face of the column, as seen in Figure 4.10.1. This is because of the punching cone may be steeper in bases due to the favourable reaction from the soil. In pad foundations of variable depth, the effective depth can be taken to be the depth at the perimeter of the loaded area, since the shear plane must pass through the full depth.

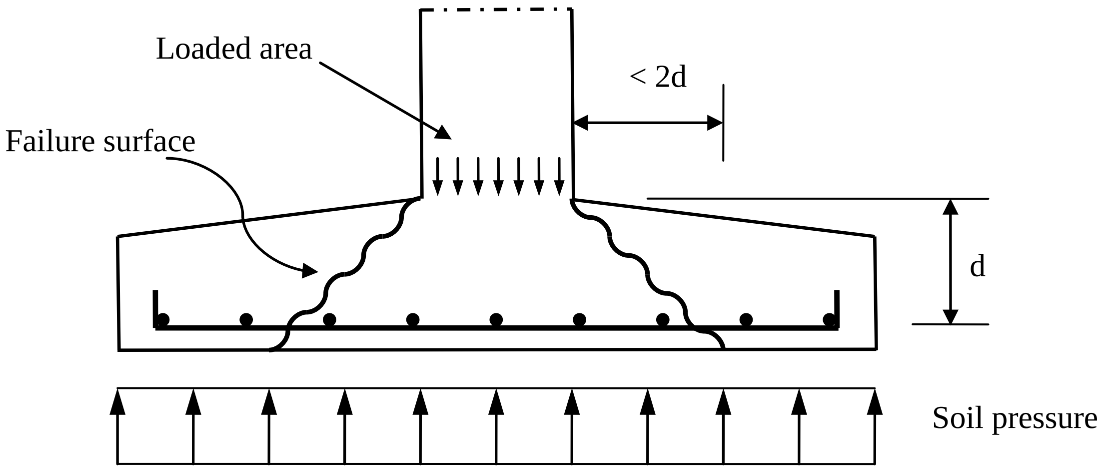


Figure 4.10.1 Model for punching shear of column bases

I practice several perimeters inside 2d from the column face must be checked. The perimeter giving the lowest value of the punching shear resistance must be taken. Due to the reliving soil pressure, a reduced concentrated load can be applied in the design, defined as:

$$V_{Ed,red} = V_{Ed} - \Delta V_{Ed} \tag{4.10.1}$$

where V_{Ed} is the applied shear force (column load) and ΔV_{Ed} is the net upward force within the control perimeter considered, i.e. upward pressure from soil minus self weight of the base. A check of the punching shear resistance at the basic perimeter 2d from the column face, ignoring the reliving pressure from the soil is conservative.

In absence of a moment transfer between the column and the base, the design shear stress is:

$$v_{Ed} = \frac{V_{Ed,red}}{ud} \tag{4.10.2}$$

where *u* relates to the actual perimeter being checked. If the column axial load is accompanied by a bending moment, the design shear stress is given as:

$$v_{Ed} = \frac{V_{Ed,red}}{ud}\left[1 + k \cdot \frac{M_{Ed}}{V_{Ed,red}} \cdot \frac{u}{W}\right] \tag{4.10.3}$$

In Eq. (4.10.3) u and W are calculated for the perimeter under consideration. Compared to the general expression of the design shear stress for punching shear in Eq. (4.4.3), the reduced shear force is introduced. In reality also the unbalanced moment M_{Ed} can be replaced with a reduced value due to the soil pressure. However, Eurocode 2 does not take this into account, which is conservative.

The design shear stress resistance $v_{Rd,c}$ for bases is in EC2 given as:

$$v_{Rd,c} = C_{Rd,c} \cdot k \cdot (100 \rho f_{ck})^{1/3} \cdot 2d / a \geq v_{\min} \cdot {}^{2d}\!/_{a} \tag{4.10.4}$$

where a is the distance from the periphery of the column to the control perimeter considered. In perimeters closer than 2d from the column face, the shear resistance is enhanced. This is in contradiction to shear design of beams, where the shear force itself is reduced.

For bases, required shear reinforcement and control of maximum shear stress, are calculated as for punching shear in slabs, given in sections 4.6 and 4.7 respectively.

4.11 Example: Punching resistance of a pad footing

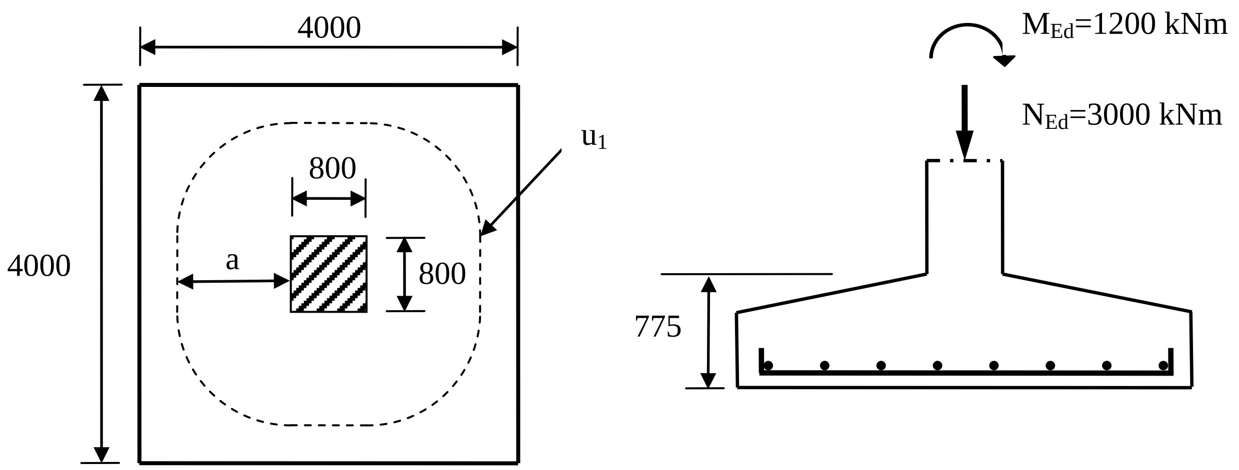


Figure 4.11.1 Reinforced concrete pad footing

The punching resistance of the pad footing in Figure 4.11.1 is investigated /4.10/.

Material properties:

Reinforcement B500NC: $f_{yk} = 500\ N/mm^2$

Concrete B35: $f_{ck} = 35\ N/mm^2$

Nominal concrete cover: $c_{nom} = 50$ mm

Assuming 25 mm diameter bars give the average effective depth:

$d_{eff} = (d_y+d_z)/2 = 775 – 50 - 25 = 700$ mm

The average base pressure excluding the pad self-weight is p = 187,5 kN/m^2.

The punching resistance must be checked at perimeters, u_i, closer than 2d from the edge of the column. The distance from the periphery of the column to the control perimeter considered is denoted *a*. To find the reduced concentric loading, $V_{Ed,red}$, the area A_i within the perimeter u_i must be calculated to take into account the reliving pressure from soil.

$u_i = 2\cdot(c_1+c_2) + 2\pi\cdot a$
$A_i = c_1\cdot c_2 + 2\cdot c_1\cdot a + 2\cdot c_2\cdot a+\pi\cdot a^2$
$V_{Ed,red} = V_{Ed} - \Delta V_{Ed} = V_{Ed} - A_i\cdot p$

Shear stress at control perimeter $\underline{u_i}$

$v_{Ed} = \beta\cdot V_{Ed,red}/(u_i\cdot d)$

$\beta = 1 + k\cdot(M_{Ed}/V_{Ed,red})\cdot(u_i/W_i)$

$k = 0{,}6$ for $c_1/c_2 =1{,}0$

$W_i = c_1^2/2 + c_1\cdot c_2 + 2c_2\cdot a + 4\cdot a^2 + a\cdot\pi\cdot c_1$

Shear stress resistance at critical control perimeter $\underline{u_1}$

$v_{Rd,c} =0{,}18/\gamma_c \cdot k\cdot(100\rho_l \cdot f_{ck})^{1/3} \geq v_{min}$

$k = 1 + (200/d)^{0,5} = 1 + (200/700)^{0,5} =1{,}534 \leq 2{,}0 \rightarrow k = 1{,}534$

$\rho_l = 0{,}35\%$ (assumed)

$v_{min} = 0{,}035\cdot k^{1,5}\cdot(f_{ck})^{0,5} = 0.035\cdot 1{,}534^{1,5}\cdot 35^{0,5} = 0{,}393\ N/mm^2$

$v_{Rd,c} =0{,}18/1{,}5 \cdot 2{,}0\cdot(100\cdot 0{,}0035 \cdot 35)^{1/3} = \underline{0{,}424\ N/mm^2} \geq v_{min}$

Shear stress resistance at control perimeter $\underline{u_i}$

$v_{Rd,c} =0{,}18/\gamma_c \cdot k\cdot(100\rho_l \cdot f_{ck})^{1/3} \cdot 2d/a = 0{,}424\cdot 2\cdot 700/ a = \underline{594\cdot a}$

Table 4.3 presents the results from the calculation of the necessary parameters to find the design shear stress and the shear stress resistance for different perimeters. The utilisation with respect to punching shear is also given as the ratio $v_{Rd,c}/v_{Ed}$. A ratio below 1,0 implies that the punching shear capacity must be increased, e.g. with shear reinforcement, increased depth of the footing or more longitudinal reinforcement.

Table 4.3 Punching shear utilisation for different distances from the column

Distance from column	u_i [mm]	A_i [mm^2]	$V_{Ed,red}$ [kN]	W_i [m^2]	β	v_{Ed} [N/mm^2]	$v_{Rd,c}$ [N/mm^2]	$v_{Rd,c}/ v_{Ed}$
2d	11996	11,28	886	14,6	1,67	0,176	0,424	2,41
1,5d	9797	7,46	1601	9,7	1,45	0,340	0,565	1,66
1,0d	7598	4,42	2171	5,8	1,43	0,586	0,848	1,45
0,5d	5399	2,14	2598	2,9	1,52	1,043	1,696	1,63

In Figure 4.11.2 the variation of the utilisation ratios are plotted for different distances from the column edge. The lowest ratio, 1,44, is obtained for a distance of 0.9d from the column. As seen in the figure it is important to consider several control perimeters inside 2d to find the one giving lowest punching shear capacity.

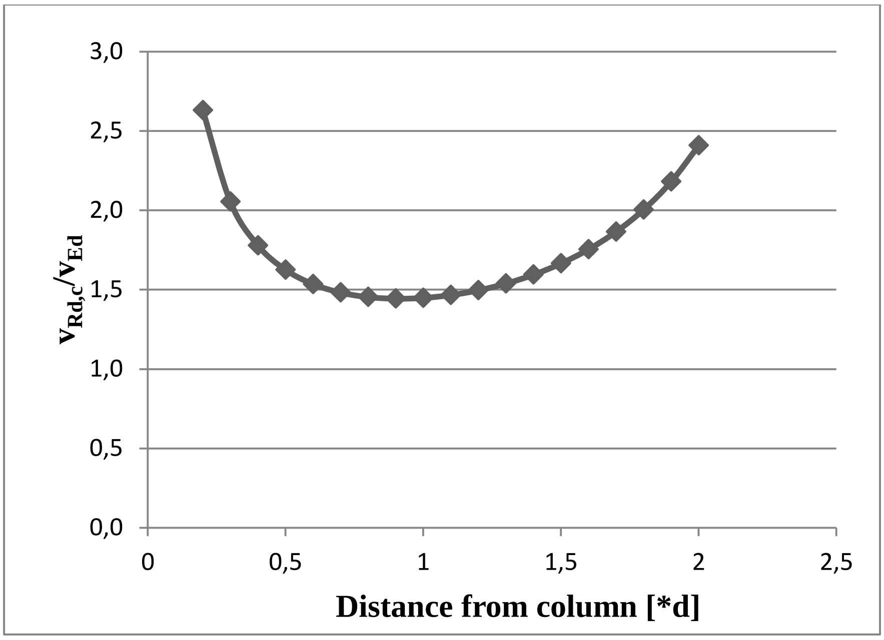


Figure 4.11.2 Relative shear resistance for different control perimeters

A conservative approach is to ignore the reliving soil pressure and control the punching resistance at the critical perimeter u_1, 2d from the loaded area. By employing this approach in the example, the results are:

$v_{Rd,c} = 0,424\ N/mm^2$
$\beta\ \ = 1,20$
$v_{Ed}\ = 0,428\ N/mm^2$

$v_{Rd,c}/\ v_{Ed} = 0,99$

There is a considerable reduction in the punching shear resistance using the conservative approach. In this example shear reinforcement is required since the utilisation ratio is below 1,0.

For small foundations the critical perimeter 2d from the column will sometimes be located outside the foundation area. Hence, for small basements the punching must controlled at perimeter closer than 2d.

4.12 References

/4.1/ Standard Norge, *NS-EN 1992-1-1:2004+NA:2008, Eurocode 2: Design of concrete structures. Part 1-1: General rules and rules for buildings.* 2008.

/4.2/ CEB-FIP, *Punching of structural concrete slabs*, Bulletin 12, 2001.

/4.3/ European Concrete Platform ASBL, *Commentary Eurocode 2*, Brussels, 2008.

/4.4/ Norsk Betongforening, *Flatdekker: Beregning og konstruktiv utforming*, Publikasjon nr. 33, 2004.

/4.5/ M. Hallgren, S. Kinnunen and B. Nylander, *Punching shear tests on column footings*, Nordic Concrete Research, 21 (1) (1998), pp. 1–23.

/4.6/ R. S Narayanan and A. Beeby, *Designers' guide to EN 1992-1-1 and EN 1992-1-2*, Thomas Telford, 2005.

/4.7/ P. E. Mast, *Stresses in flat plates near columns*, ACI Journal, October 1970, pp. 761-768.

/4.8/ R.S. Narayanan and A.H. Goodchild, *Concise Eurocode 2*, The Concrete Centre, 2006.

/4.9/ Svenska Betongförening, *Handbok till Eurokode2, Volym I och II*, Betongrapport nr 15, 2010.

/4.10/ C.R. Hendy and D.A. Smith, *Designers' guide to EN 1992-2.* Thomas Telford, 2007.

CHAPTER 5

DESIGN OF CONCRETE SHELLS

Jan Arve Øverli

5.1 Introduction

Structural elements can be classified according their geometry and loading. Elements subjected to pure membrane action are denoted membranes, like shear walls and webs in beams. Elements subjected to pure bending 1are normally called plates or slabs. By these definitions, membrane and slabs are mainly plane structures, as seen in Figure 5.1.1. Shells are defined as elements subjected to both membrane and bending forces. They can be part of plane or curved structures. Typical curved structures are shell roofs and storage tanks. Examples of plane structures exposed to both membrane and bending actions are prestressed cantilever bridges and post-tensioned flat slabs.

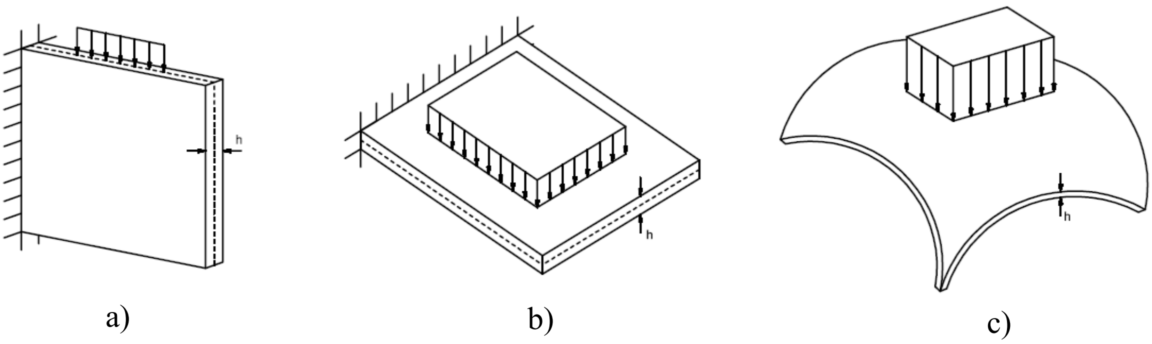


Figure 5.1.1 Structural elements. a) Membrane; b) Plate/slab; c) Shell

Most concrete shells structures are thin shells. A thin shell is a curved slab where the thickness *h* is small compared to its other dimensions and compared to the radius of curvature *R* of the geometry. The surface that bisects the shell is called the middle surface or the middle plane, as illustrated in Figure 5.1.2.

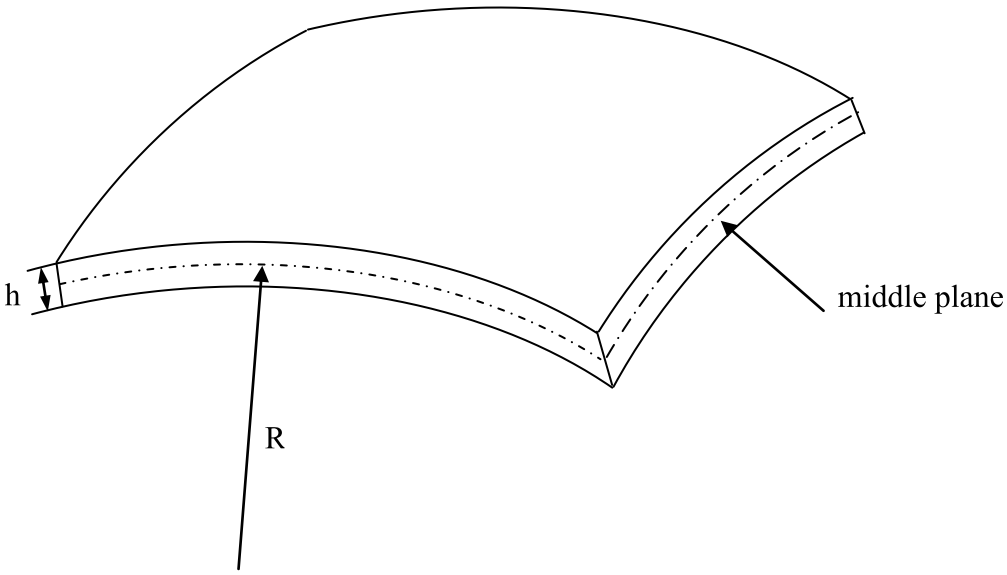


Figure 5.1.2 Middle plane, radius of curvature and thickness of a thin shell

In classical thin shell theory, Love-Kirchoff theory, the following assumptions applies:

- The shell thickness is negligibly small in comparison to the curvature of the shell middle surface.
- Strains and stresses are small
- Straight lines that are normal to the middle surface prior to deformation remain straight during deformation.
- Stresses normal to the shell middle surface is negligible.

5.2 Shell forces

Figure 5.2.1 shows the stress resultants acting along the boundaries of a shell element. The resultants are two bending moments, m_x and m_y, torsional moment m_{xy}, two transverse shear forces, v_x og v_y, and three membrane forces n_x, n_y and n_{xy}. Hence, in design of concrete shell sections, 8 stress resultants must be taken into account. All forces and moments have units per length.

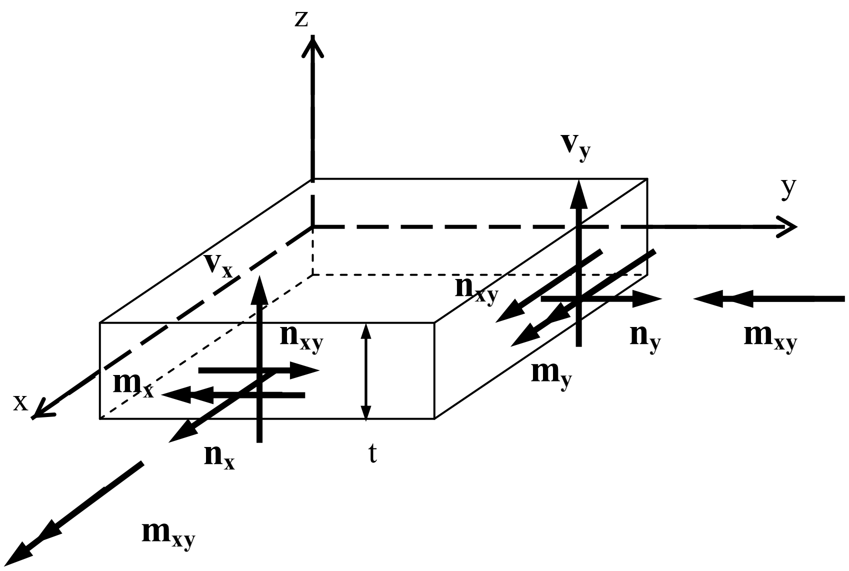


Figure 5.2.1 Stress resultants in a plane shell element

In a general shell the curvature of the geometry will affect the definition of the stress resultants so that $m_{xy} \neq m_{yx}$ and $n_{xy} \neq n_{yx}$. Typical civil engineering concrete shell structures are thin shells. Hence, assuming $m_{xy} = m_{yx}$ and $n_{xy} = n_{yx}$ is a good approximation.

Figure 5.2.2 defines the governing stresses along the shell thickness.

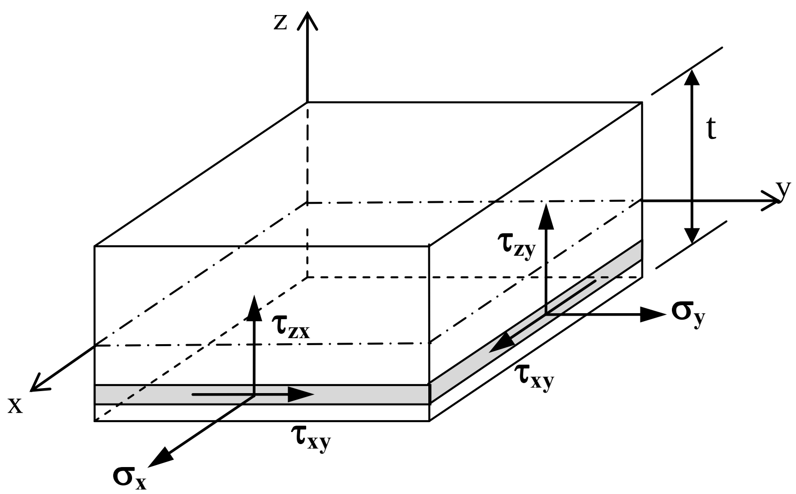


Figure 5.2.2 Stresses in a shell element at level z

The stress resultants are obtained by integrating the stresses along the boundaries of a shell element.

$$n_x = \int_{-t/2}^{t/2} \sigma_x dz \qquad n_y = \int_{-t/2}^{t/2} \sigma_y dz \qquad n_{xy} = n_{yx} = \int_{-t/2}^{t/2} \tau_{xy} dz \tag{5.2.1}$$

$$m_x = \int_{-t/2}^{t/2} \sigma_x z dz \qquad m_y = \int_{-t/2}^{t/2} \sigma_y z dz \qquad m_{xy} = m_{yx} = -\int_{-t/2}^{t/2} \tau_{xy} z dz \tag{5.2.2}$$

$$v_x = \int_{-t/2}^{t/2} \tau_{zx} z dz \qquad v_y = \int_{-t/2}^{t/2} \tau_{zy} z dz \tag{5.2.3}$$

5.3 Analysis of shells with the finite element method

Structural analyses of shell structures are normally carried out by means of the finite element method (FEM). FEM is a powerful numerical method which can analyse almost any type of geometry and loading. The main goal of the analysis is to find the distribution of the stress resultants defined in Figure 5.2.1. In a design process a linear stress analysis is performed as a basis for the design of the reinforcement.

Shell finite elements can in general have a 3-dimensional curved geometry, but they do not have a physical thickness. Figure 5.3.1 shows two examples of different geometrical shaped shell elements. The outputs are directly the stress resultants. Transverse shear forces are important in design of concrete shells. Thus, transverse shear deformation must be included in the finite element formulation, e.g. according to the Mindlin-Reissner theory, which assumes that normals remain straight, but not necessarily normal to the middle plane of the shell.

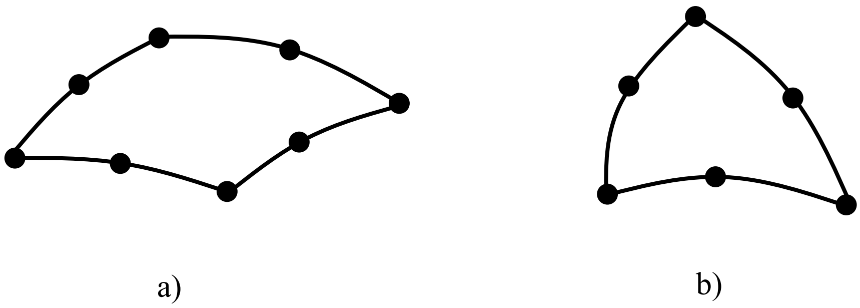


Figure 5.3.1 Shell elements. a) quadrilateral with 8 nodes: b) triangle with 6 nodes

Figure 5.3.2 shows a finite element mesh with shell elements of a cantilever bridge. Any type of structures where the thickness of the cross-section is small compared to dimensions of structure can be modelled with shell elements.

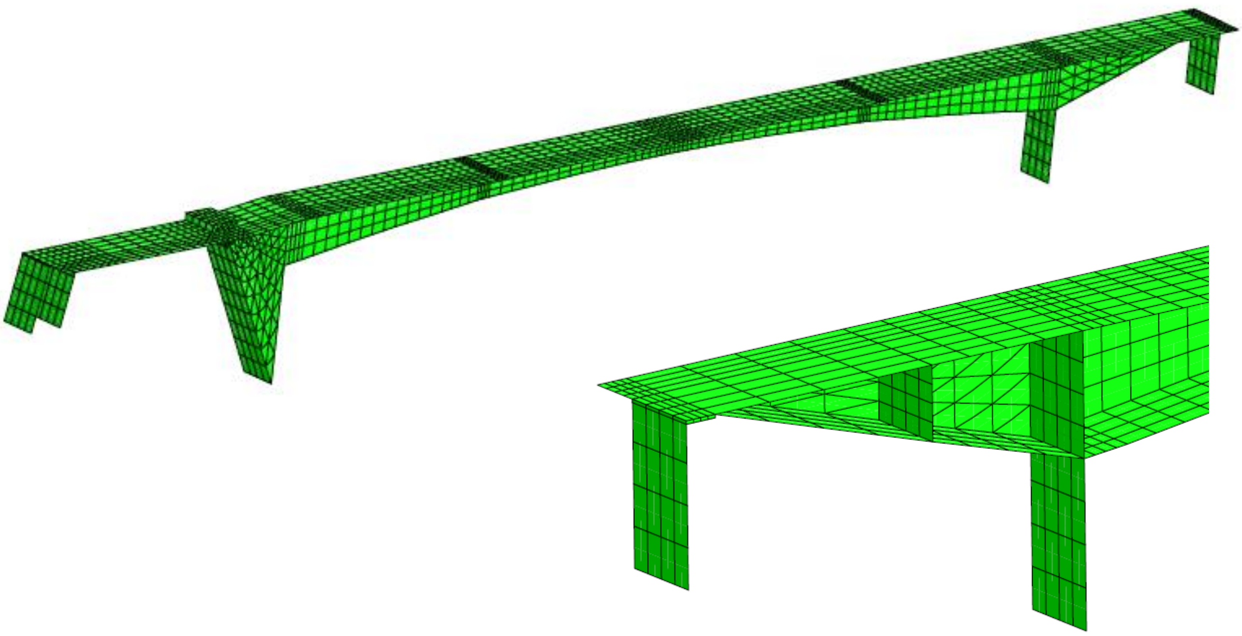


Figure 5.3.2 FEM model of a cantilever bridge with shell elements

It is also possible to model shell structure with solid elements. Solid elements are 3-dimensional elements as seen in Figure 5.3.3. They only have translation degrees of freedom. Hence, the outputs from the analysis are stresses and not stress resultants.

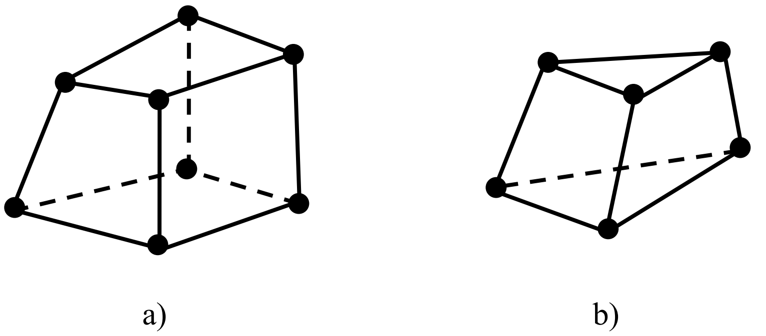


Figure 5.3.3 Solid elements. a) brick with 8 nodes; b) wedge with 6 nodes

Figure 5.3.4 illustrates the difference in the finite element mesh for a simple cylinder modelled with both shell elements and solid elements. Since stresses are the outcome employing solid elements, Eqs. (5.2.1)-(5.2.3) must be used to obtain the stress resultants which are the basis for the design.

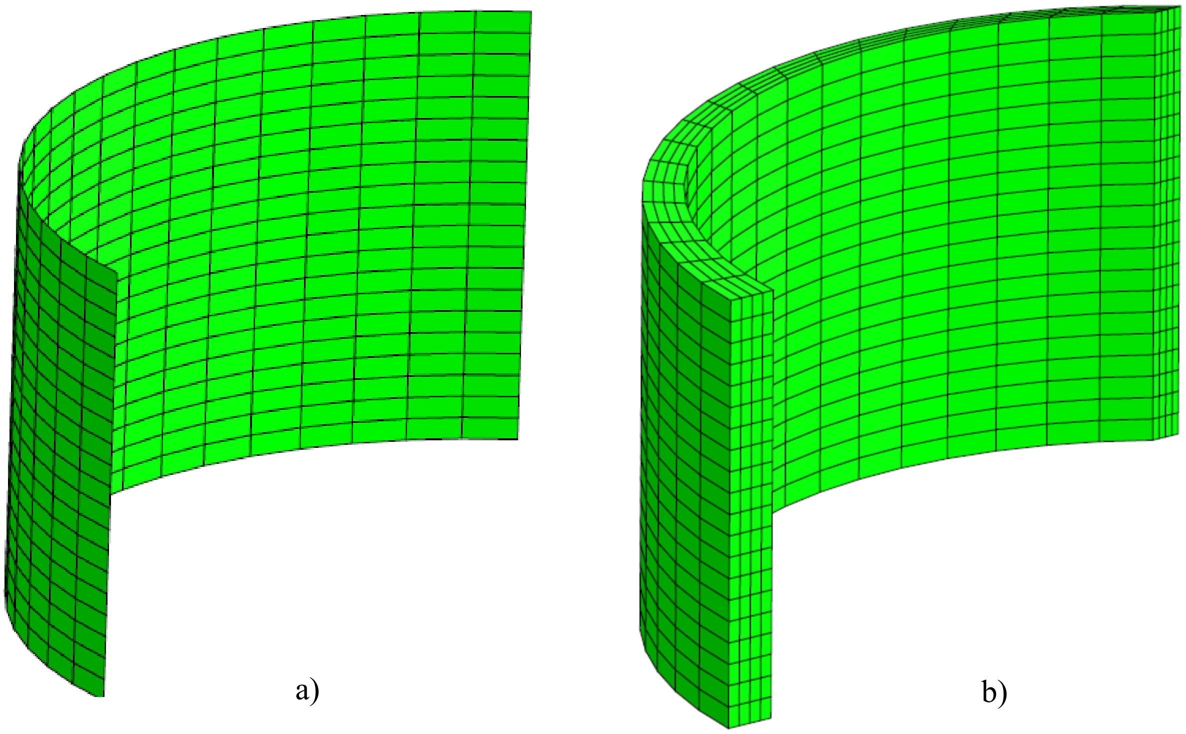


Figure 5.3.4 Finite element mesh of a cylinder with a) shell elements; b) solid elements

The major advantage of solid elements compared to shell elements is the correct modelling of the stiffness in structures. The joint in Figure 5.3.5 is properly described with solid elements. Shell elements, which are defined by middle planes, do not have correct stiffness in the joint due to overlap of cross section areas. Defining design sections are also more realistic using solid elements, because they can be placed at the edge of the shell surfaces and not at the middle planes. Shells with geometrical discontinuities, like the change in shell thickness from t_1 to t_2 in Figure 5.3.5, can easily be modelled with solid elements. Shell elements must apply linear constraints to capture the eccentricity of the middle planes in the shell.

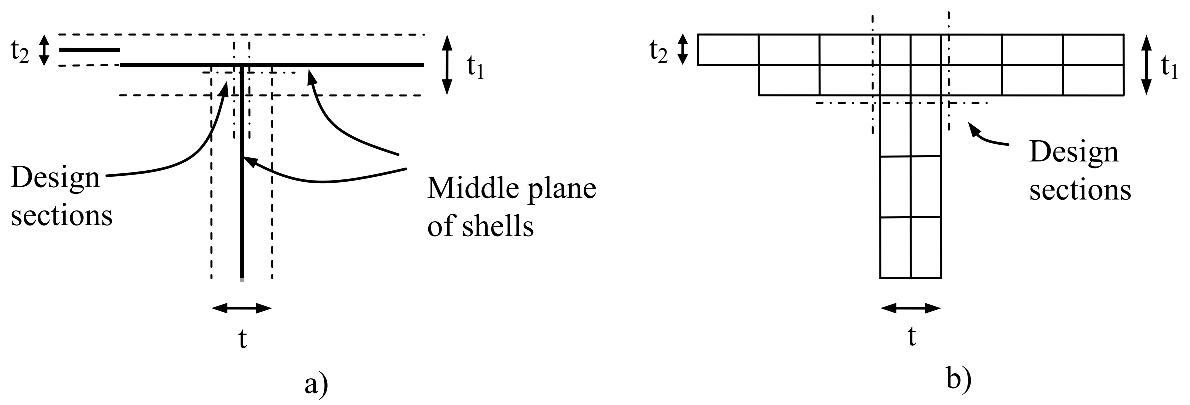


Figure 5.3.5 Finite element mesh of a joint. a) shell elements; b) solid elements

5.4 Cylindrical shells with axisymmetric loads

5.4.1 Introduction

Container structures in reinforced concrete are often a combination of different types of axisymmetric shells, as seen in Figure 5.4.1. Geometrically they can be cylinders, spheres, cones and circular plates. Often also the loading is symmetrical, like gas and water pressures and silo loads.

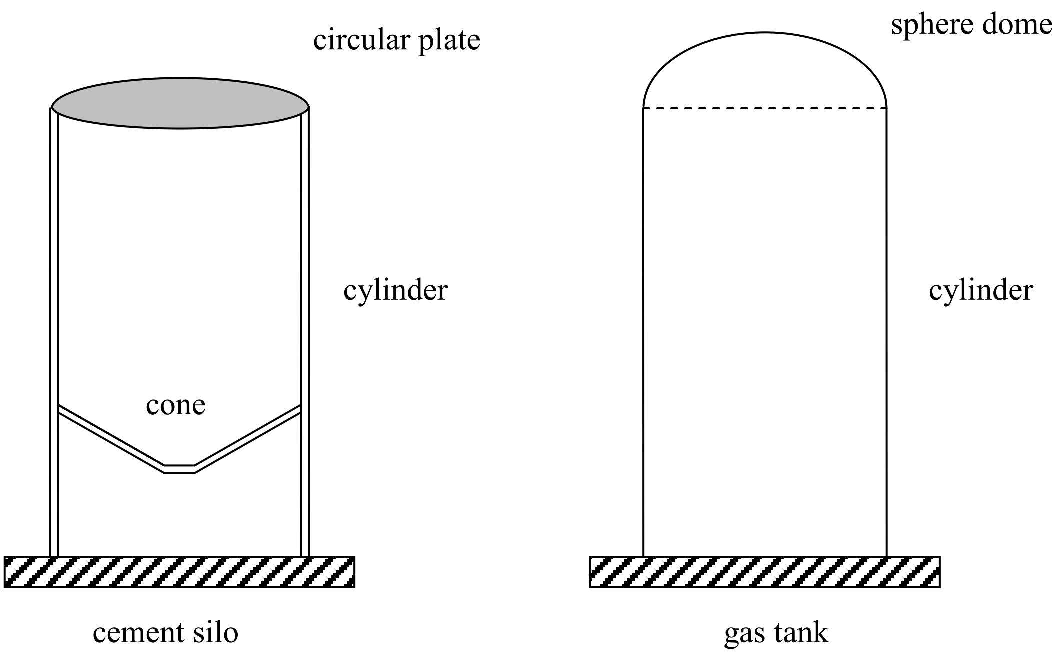


Figure 5.4.1 Shells of revolution

As long as both the geometry and the loads are axisymmetric, it is possible to solve the differential equation for the structure to find the distributions of shell forces. However, for combined structures, e.g. a cylinder with a spherical dome at the top, or non-symmetrical loading like wind are waves, the calculations gets rather complicated. Analyses of concrete shell structures are normally performed with the FEM method. Nevertheless, the results from an FEM analysis need to be verified. Comparison with simplified shells of revolution can be a useful tool for the verification. In this chapter calculation of a cylindrical shell with axisymmetric loading will be presented. For other geometrical shapes and non-symmetrical loading, the shell forces can be found in many textbooks, e.g. /5.5/,/5.6/.

5.4.2 Differential equation

Figure 5.4.2 shows part of a cylindrical shell, with cylindrical coordinates x, r and φ. The shell has constant thickness h. The applied loads are also symmetric with respect to angle φ, but can vary in the x-direction. Hence, the governing forces in the cylindrical shell are the bending moments, m_x and m_φ, membrane forces n_x and n_φ, and the shear force v_x. Due to the axisymmteric condition, the membrane shear force $n_{\varphi x}$, the torsional moment $m_{\varphi x}$, and the shear force v_φ, are zero.

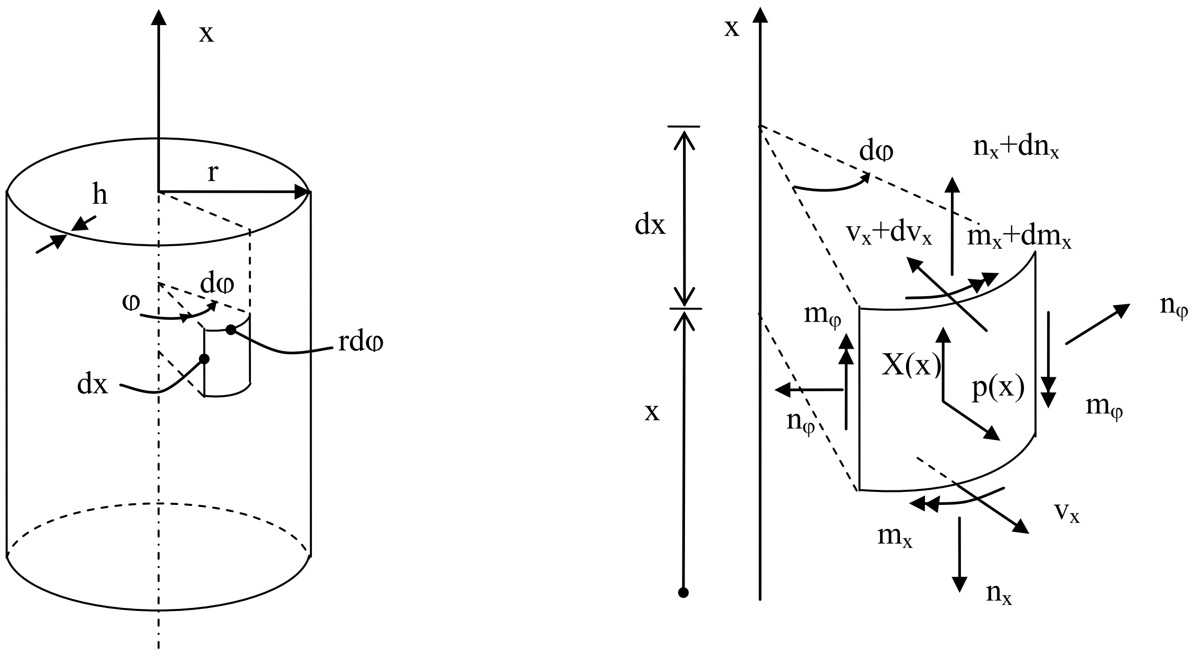


Figure 5.4.2 Forces in a cylindrical shell

By summation of forces and moments, the following equations of equilibrium are obtained:

$$\sum F_{radial} = 0; \qquad \frac{dv_x}{dx} + \frac{1}{r} \cdot n_\varphi = p(x) \tag{5.4.1}$$

$$\sum M_{\text{tangential}} = 0; \qquad \frac{dm_x}{dx} - v_x = 0 \tag{5.4.2}$$

$$\sum F_x = 0; \qquad \frac{dn_x}{dx} + X(x) = 0 \tag{5.4.3}$$

Eq. (5.4.3) is not coupled to Eq. (5.4.1) and (5.4.2). To simplify the derivation of forces in a cylindrical shell, n_x is assumed to be zero. By substituting v_x from Eq. (5.4.2) into Eq. (5.4.1), the differential equation for a cylinder is given by

$$\frac{d^2 m_x}{dx^2} + \frac{1}{r} \cdot n_\varphi = p(x) \tag{5.4.4}$$

M_x and N_φ are unknowns and functions of the radial deflection w, illustrated in Figure 5.4.3 . The relationship between longitudinal strains, ε_x, and circumferential (hoop) strains, ε_φ, in the cylinder, is given by Hooke's law in the plane stress situation. With the assumption of $N_x = 0$ (σ_x=0), the strains becomes

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \upsilon\sigma_\varphi) = -\frac{\upsilon}{E}\sigma_\varphi = -\frac{\upsilon N_\varphi}{Eh} \tag{5.4.5}$$
$$\varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \upsilon\sigma_x) = \frac{1}{E}\sigma_\varphi = \frac{N_\varphi}{Eh}$$

where σ_x and σ_φ are normal stresses in longitudinal and hoop direction of the cylinder respectively, and ν is Poisson' ratio. The hoop strain is obtained from Figure 5.4.3 as

$$\varepsilon_\varphi = \frac{2\pi(r+\Delta r) - 2\pi r}{2\pi r} = \frac{\Delta r}{r} = \frac{w}{r} \tag{5.4.6}$$
$$\Rightarrow N_\varphi = \frac{Eh}{r}w$$

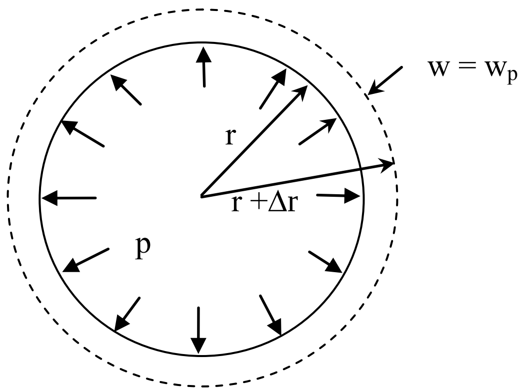


Figure 5.4.3 Radial deflection of cylinder

From the theory for plate structures, the relationship between moment and curvatures is known as

$$\mathrm{M}_x = D(\kappa_x - \upsilon\kappa_\phi) \tag{5.4.7}$$
$$\mathrm{D} = \frac{\mathrm{Eh}^3}{12(1-\upsilon^2)}$$
$$\kappa_x = \frac{d^2w}{dx^2}$$
$$\kappa_\phi = \frac{1}{r+w} - \frac{1}{r} = -\frac{w}{r(r+w)} \approx -\frac{w}{r^2} \approx 0, \quad \text{see Figure 5.4.3}$$
$$\Rightarrow \mathrm{M}_x = D \cdot \frac{d^2w}{dx^2}$$

By combining Eq. (5.4.1), (5.4.3) and (5.4.4), the differential equation for a cylindrical shell as a function of radial displacement can be expressed as

$$\frac{d^4 w}{dx^4} + \frac{Eh}{Dr^2} \cdot w = \frac{p(x)}{D} \tag{5.4.8}$$

In general, solution of a differential equation can be split into a homogenous and a particular part. The particular solution, w_p, represents a membrane action, like the uniform radial displacement in Figure 5.4.3. The homogenous solution, w_h, represents effect of boundary constraints, see Figure 5.4.4. Constraints can be supports or forces. The general solution of Eq. (5.4.8) can be found in many textbooks, /5.5/, /5.6/.

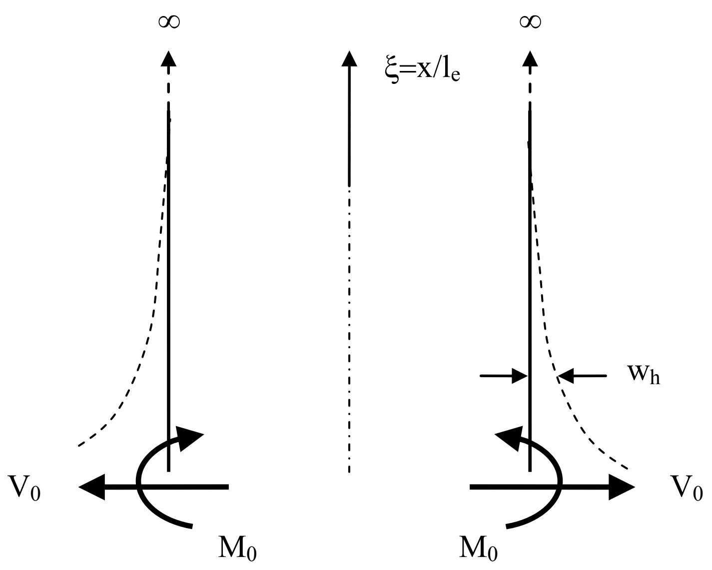


Figure 5.4.4 Infinite long cylinder with shear and moment applied at edge

The solution of the differential equation, $w = w_p + w_h$, for an infinite long cylinder with boundary constraints M_0 and V_0 at one edge, is given as

Homogeneous solution and shell variables

$$\begin{bmatrix} w \cdot \frac{2D}{l_e^2} \\ N_\phi \cdot \frac{l_e^2}{2r} \\ \frac{dw}{dx} \cdot \frac{2D}{l_e} \\ M_x \\ V_x \cdot l_e \end{bmatrix} = \begin{bmatrix} g_4(\xi) & g_1(\xi) \\ g_4(\xi) & g_1(\xi) \\ -2g_1(\xi) & -g_3(\xi) \\ g_3(\xi) & g_2(\xi) \\ -2g_2(\xi) & g_4(\xi) \end{bmatrix} \begin{bmatrix} M_0 \\ V_0 \cdot l_e \end{bmatrix}, \quad \begin{aligned} g_1(\xi) &= e^{-\xi} \cdot \cos\xi \\ g_2(\xi) &= e^{-\xi} \cdot \sin\xi \\ g_3(\xi) &= g_1(\xi) + g_2(\xi) \\ g_4(\xi) &= g_1(\xi) - g_2(\xi) \\ \xi &= x / l_e \end{aligned} \tag{5.4.9}$$

Particular solution

$$w_p = \frac{r^2}{Eh} \cdot p(x) \tag{5.4.10}$$
$$N_{\phi p} = p(x) \cdot r$$

where l_e is named the elastic length and is defined as

$$l_e = \sqrt[4]{\frac{4Dr^2}{Eh}} = \frac{\sqrt{rh}}{\sqrt[4]{3(1-\upsilon^2)}} \tag{5.4.11}$$

5.4.3 Bending of a long cylindrical shell with internal pressure

Cylinders are often used as storage tanks. As an example of how Eqs. (5.4.9) and (5.4.10) can be employed to find shell forces, a tank with constant internal pressure *p* is calculated. The cylinder is assumed to be infinite long and clamped at one edge as seen in Figure 5.4.5.

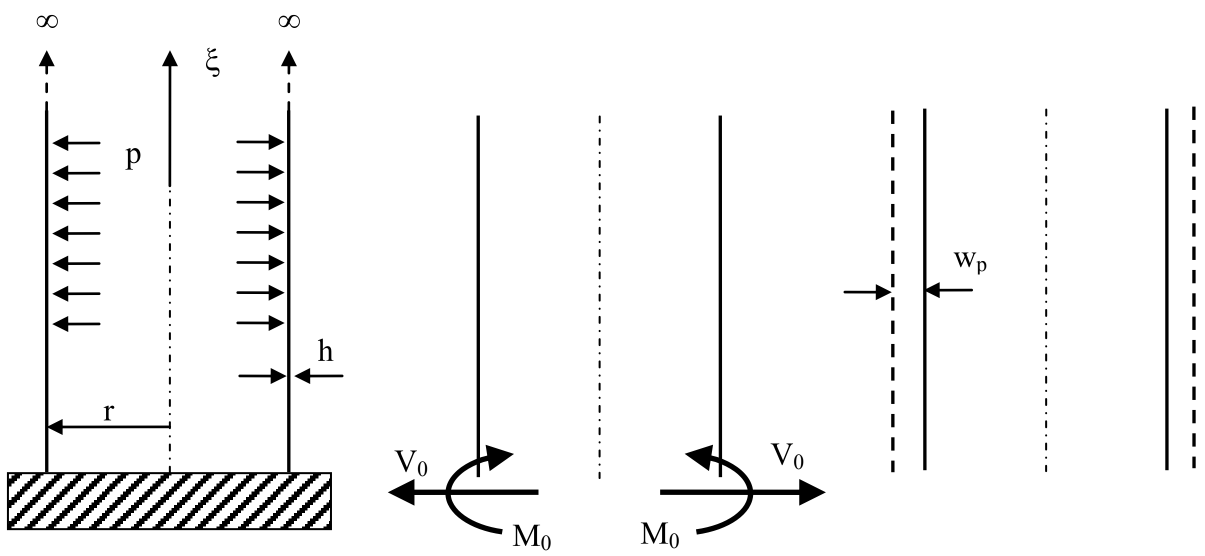


Figure 5.4.5 Infinite long cylinder with internal pressure

The particular solution from Eq. (5.4.10) is a constant radial displacement independent of ξ.

$$w_p = \frac{pr^2}{Eh}$$
$$N_{\varphi p} = p \cdot r$$

The clamped boundary condition introduces the following restraints:

$$\text{Rotation:} \qquad \frac{dw(\xi=0)}{dx}=0$$
$$\text{Displacement:} \quad w(\xi=0)=0$$

Since w_p is constant it does not contribute to the rotation, and the equations to find M_0 and V_0 can be derived.

$$\mathrm{w}=w_p+w_h=\frac{\mathrm{pr}^2}{Eh}+\frac{l_e^2}{2D}(M_0+V_0\cdot l_e)=0$$
$$\frac{dw(0)}{dx}=\frac{dw_p(0)}{dx}+\frac{dw_h(0)}{dx}=\text{-}(2\mathrm{M}_0+\mathrm{V}_0\cdot \mathrm{l}_\mathrm{e})=0$$

$$\Rightarrow M_0=\frac{pl_e^2}{2} \text{ and } V_0\cdot l_e=-pl_e^2$$

Hence, the distribution of shell forces in the longitudinal direction of the cylinder becomes

$$M_x=M_{xh}=\frac{pl_e^2}{2}\cdot g_3(\xi)-pl_e^2\cdot g_2(\xi)=\frac{pl_e^2}{2}\cdot g_4(\xi)$$

$$V_x=V_{xh}=-pl_e\cdot g_2(\xi)--pl_e\cdot g_4(\xi)=-pl_e\cdot g_1(\xi)$$

$$N_\varphi=N_{\varphi p}+N_{\varphi h}=p\cdot r+(pr\cdot g_4(\xi)-2pr\cdot g_1(\xi)=pr(1-g_4(\xi))$$

The distributions of forces are plotted in Figure 5.4.6. The clamped boundary condition at ξ=0, influences the result only close to the boundary. Away from the boundary bending moment M_x and shear force V_x approaches zero, and the hoop force approaches the value for membrane action (particular solution), $N_\varphi = p\cdot r$. This is typical for shell structures.

The distributions of forces in a shell structure are governed by the g-functions in Eq. (5.4.9). As illustrated in Figure 5.4.7, the functions are reduced to low numbers when ξ is in order of 3-4. A damping length L_c is defined corresponding to $\xi = \pi$.

$$\mathrm{L_c}=\pi\cdot l_e \tag{5.4.12}$$

L_c is useful when assessing the influence of boundary constraints. In a distance L_c from the constraint, only the membrane action needs to be taken into account for a cylinder with axisymmetric loading. Storage tanks have a finite length. L_c can then be employed to see if the two edges of the cylinder influence each other.

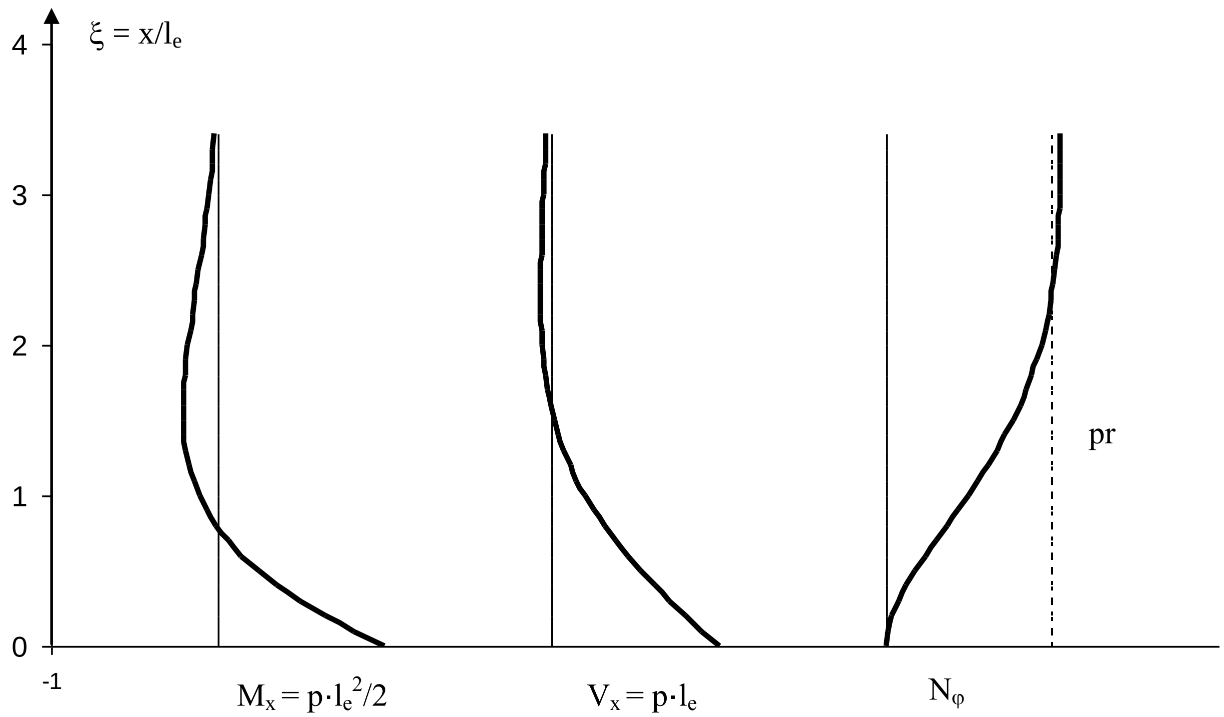


Figure 5.4.6 Distribution of shell forces

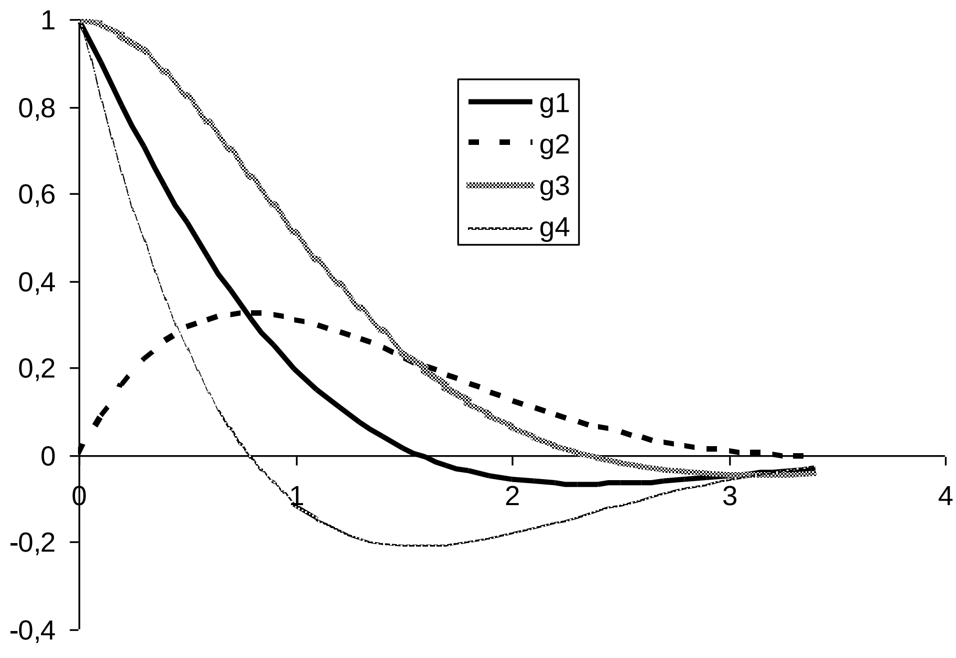


Figure 5.4.7 Shape of g-functions

5.5 Design of concrete shells

The main difficulty designing concrete shells are how to detail the 8 stress resultants into layers of longitudinal reinforcement at the top and bottom surface, and shear reinforcement. The principal moment and principal membrane force directions do not in general coincidence. Hence, establishing interaction diagrams between moments and membrane forces, which are commonly used for columns, is impossible. It is possible to align the longitudinal reinforcement in direction of the principal stresses at the surfaces. However, this requires a lot of complex work on the building site. In addition, with several different load cases the principal stress direction changes, making the method impractical. In concrete shells orthogonal reinforcement is normally provided. Methods for designing concrete shells with orthogonal reinforcement will be presented here.

5.6 Two layered approach - Membrane method

A simple method to design a concrete shell is referred to as the membrane method. In this method the shell element with 8 stress resultants presented in Figure 5.2.1, is replaced by one top and one bottom membrane (layer), as seen in Figure 5.6.1. The intermediate layer between the two membranes is not taken into account in this method. Hence, the method could also be named a two layered approach. Based on the six moments and membrane forces in the shell, both membranes are loaded with equivalent membrane forces.

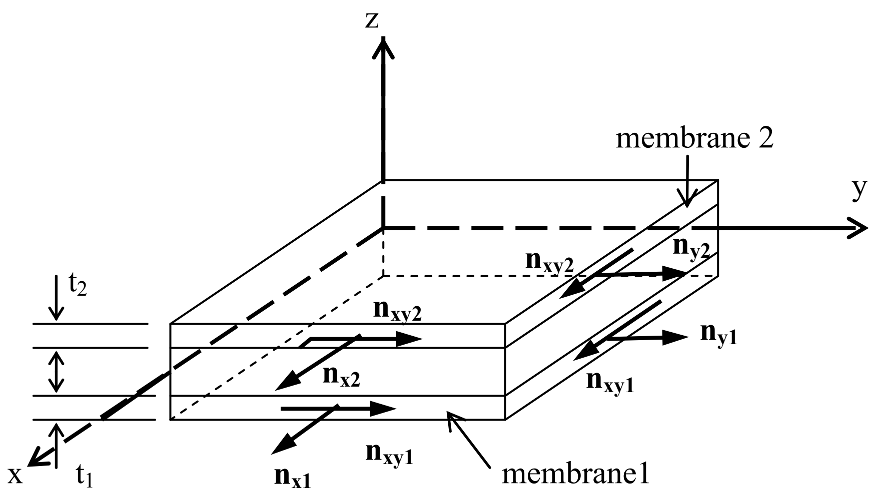


Figure 5.6.1 Equivalent membrane forces

In general the top and bottom membrane have different thickness, t_1 og t_2 respectively. The level arm is given as

$$z = h - 0,5(t_1 + t_2) \tag{5.6.1}$$

where *h* is the shell thickness.

Based on the stress resultants and the level arm, the equivalent membrane forces can be calculated as:

Membrane 1, bottom layer

$$n_{x1} = k_1 n_x + \frac{m_x}{z}; \quad \mathrm{n}_{y1} = k_1 n_y + \frac{m_y}{z}; \quad \mathrm{n}_{xy1} = k_1 n_{xy} + \frac{m_{xy}}{z} \tag{5.6.2}$$

Membrane 2, top layer

$$n_{x2} = k_2 n_x - \frac{m_x}{z}; \quad \mathrm{n}_{y2} = k_2 n_y - \frac{m_y}{z}; \quad \mathrm{n}_{xy2} = k_2 n_{xy} - \frac{m_{xy}}{z} \tag{5.6.3}$$

The factors k_1 og k_2 depends on the membrane thicknesses t_1 and t_2. They are calculated using equilibrium and Figure 5.6.2.

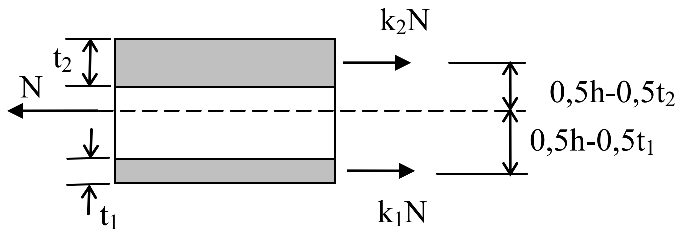


Figure 5.6.2 Equilibrium of membrane forces

$$\sum N = 0 \Rightarrow \quad k_1 \cdot N + k_2 \cdot N = N \Rightarrow \quad k_1 + k_2 = 1 \tag{5.6.4}$$

$$\sum M = 0 \Rightarrow \quad k_1 \cdot N \cdot 0{,}5 \cdot (h - t_1) = k_2 \cdot N \cdot 0{,}5 \cdot (h - t_2) \Rightarrow \quad k_1 = \frac{(1-k_1)(h-t_2)}{h-t_1}$$

The factors k_1 og k_2 is then given as:

$$k_1 = \frac{(h-t_2)}{(2h - t_1 - t_2)}; \qquad k_2 = 1 - k_1 \tag{5.6.5}$$

The thickness of an uncracked membrane is assumed to be t=0,5h. If the membrane is cracked t=2c, where c is the distance from the surface to the reinforcement gravity centre of the two reinforcement directions.

To decide the membrane thicknesses t_1 and t_2, and to calculate the equivalent membrane forces, it is possible to adopt the following step by step procedure:

1) Start with $t_1 = t_2 = h/2 \;\rightarrow\; k_1 = k_2 = 0{,}5$ and z=h/2

2) Calculate membrane forces according to Eqs. (5.6.2) and (5.6.3)

3) Calculate the largest principal membrane force for both membranes assuming a isotropic linear elastic material

$$n_{11}^i = \frac{n_{xi} + n_{yi}}{2} + \sqrt{\left(\frac{n_{xi} - n_{yi}}{2}\right)^2 + n_{xyi}^{\,2}}; \quad \text{i=1,2}$$

If $n^{i}_{11}>0$ the membrane is cracked

4) There are 3 possible configurations of the membrane thicknesses
   - Both membranes are uncracked
     $t_1 = t_2 = 0{,}5h$
   - Both membranes are cracked
     $t_1 = t_2 = 2c \rightarrow k_1 = k_2 = 0{,}5$ and $z = h-2c$
   - One membrane is cracked and one is uncracked
     $t_1 = 2c$ and $t_2 = h/2$ (or vice versa) $\rightarrow$

$$k_1 = \frac{0,25h}{0,75h-c}; \qquad k_2 = 1-k_1 \quad \rightarrow \; z = 0,75\text{-}c$$

5) If membranes are cracked, calculate new equivalent membrane forces according to Eqs. (5.6.2) and (5.6.3)

Knowing the equivalent membrane forces, the two membranes are designed separately using compression field theory.

It must be emphasised that the membrane method is a simplified method with several shortcomings and assumptions. There is no strain compatibility, meaning the membranes have constant strains giving constant stresses. Cracking is checked in the middle plane of the membranes and not at the surfaces. Design for transverse shear force is not part of this method. However, even with the simplifications the method can be a useful tool in a preliminary design. The reinforcement amount found with this method can also be used as input to more advanced methods which will be described later.

The choice of thicknesses for the top and bottom layers, influence the result in the two layered approach. To illustrate the effect the simple beam section in Figure 5.6.3 is designed using material properties from Eurocode 2.

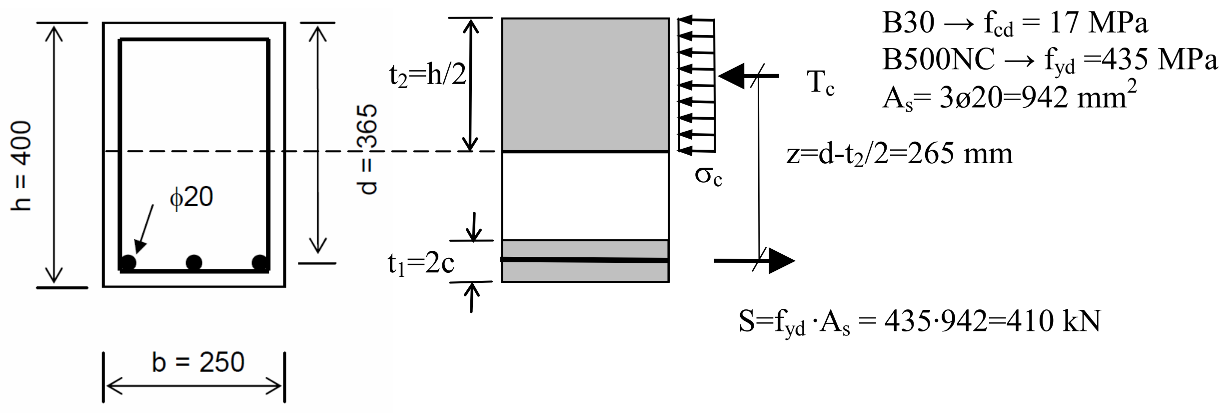


Figure 5.6.3 Beam section to be designed with the membrane method

In a traditional beam design the moment resistance is given by /5.10/:

$$M_{Rd} = 0{,}8 \cdot \alpha \cdot (1 - 0{,}4\alpha) \cdot f_{cd} \cdot d^2$$

$$\alpha = \frac{f_{yd} \cdot A_s}{0{,}8 \cdot f_{cd} \cdot b \cdot d} = \frac{435 \cdot 942}{0{,}8 \cdot 17 \cdot 250 \cdot 365} = 0{,}329$$

$$M_{Rd} = 0{,}8 \cdot 0{,}329 \cdot (1 - 0{,}329) \cdot 17 \cdot 365^2 = 129 \text{ kNm}$$

By using the membrane method and the thicknesses in Figure 5.6.3, the moment resistance and the concrete compressive stresses are given as:

$$M_{Rd} = S \cdot z = 410 \cdot 0{,}265 = 109 \text{ kNm}$$

$$\sigma_c = \frac{T_c}{b \cdot t_2} = \frac{410 \cdot 10^3}{250 \cdot 200} = 8{,}2 \text{ MPa} < \mathrm{f}_{cd}$$

The method underestimates the moment resistance with approximately 20%, and the compressive stresses in the top membrane are low. By decreasing the thickness of the compressive membrane, the resistance will increase. By assuming both membranes having the same thickness 2c, the result is:

$$t_1 = t_2 = 2c = 70 \text{ mm} \quad \rightarrow \quad \mathrm{z} = h - 2c = 320 \text{ mm}$$

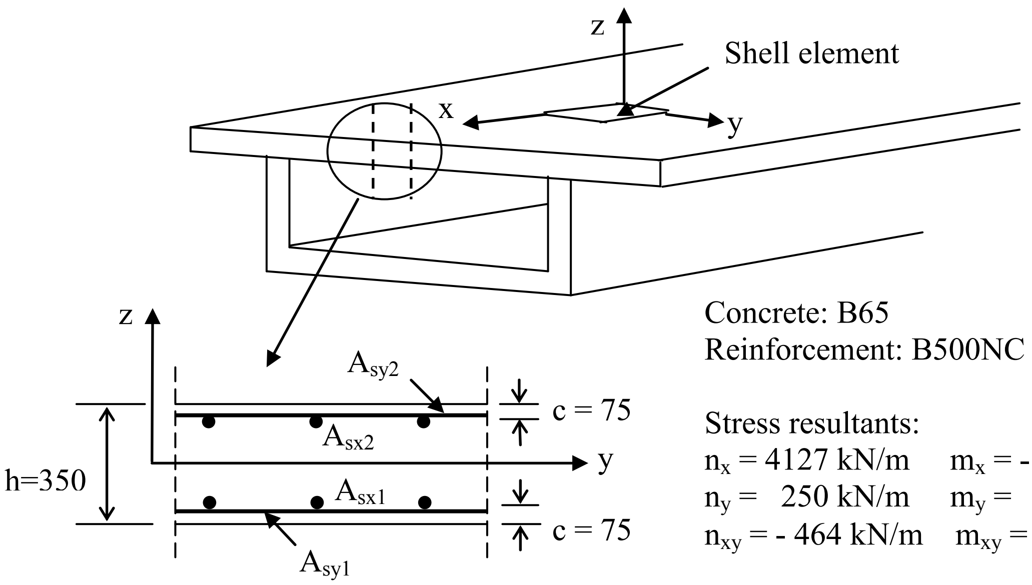
$$M_{Rd} = S \cdot z = 410 \cdot 0{,}32 = 131 \text{ kNm}$$

$$\sigma_c = \frac{T_c}{b \cdot t_2} = \frac{410 \cdot 10^3}{250 \cdot 70} = 23{,}4 \text{ MPa} > \mathrm{f}_{cd}$$

The moment resistance is now slightly overestimated, but the compressive stresses are too high. It is possible to extend the membrane method with an iteration procedure to find the optimum membrane thickness.

5.6.1 Example – Design with the two-layered approach

Figure 5.6.4 shows part of a box girder bridge in reinforced concrete. Material properties and sectional forces taken from a FEM analyse at one point in the top slab is also given in the figure. The design is performed according to EC2.



Concrete: B65
Reinforcement: B500NC

Stress resultants:
n_x = 4127 kN/m m_x = - 38 kNm/m
n_y = 250 kN/m m_y = 70 kNm/m
n_{xy} = - 464 kN/m m_{xy} = 3 kNm/m

Figure 5.6.4 Top slab in a box girder bridge

Assuming membrane thicknesses:

$t_1 = t_2 = h/2 \rightarrow k_1 = k_2 = 0{,}5$ and $z = h/2 = 0{,}175\text{m}$

Eqs. (5.6.2) and (5.6.3):

$n_{x1} = 0{,}5\cdot 4127 + (-38)/0{,}175 = 1846$ kN/m
$n_{y1} = 0{,}5\cdot 250 + 70/0{,}175 = 525$ kN/m
$n_{xy1} = 0{,}5\cdot(-464) + 3/0{,}175 = -215$ kN/m

$n_{x2} = 0{,}5\cdot 4127 - (-38)/0{,}175 = 2280$ kN/m
$n_{y2} = 0{,}5\cdot 250 - 70/0{,}175 = -275$ kN/m
$n_{xy2} = 0{,}5\cdot(-464) - 3/0{,}175 = -249$ kN/m

Largest principal membrane forces:

Membrane 1: $n_{11} = \frac{1846+525}{2} + \sqrt{(\frac{1846-525}{2})^2 + 215^2} = 1881 > 0 \rightarrow t_1 = 2c = 150\text{mm}$

Membrane 2: $n_{11} = \frac{2280-275}{2} + \sqrt{(\frac{2280+275}{2})^2 + 249^2} = 2304 > 0 \rightarrow t_2 = 2c = 150\text{mm}$

$\Rightarrow k_1 = k_2 = 0{,}5$ and $z = h-2c = 0{,}2\text{m}$

Equivalent membrane forces:

$n_{x1} = 0{,}5\cdot 4127 + (-38)/0{,}2 = 1874$ kN/m $\quad n_{x2} = 0{,}5\cdot 4127 - (-38)/0{,}2 = 2254$ kN/m
$n_{y1} = 0{,}5\cdot 250 + 70/0{,}2 = 475$ kN/m $\quad n_{y2} = 0{,}5\cdot 250 - 70/0{,}2 = -225$ kN/m
$n_{xy1} = 0{,}5\cdot(-464) + 3/0{,}2 = -217$ kN/m $\quad n_{xy2} = 0{,}5\cdot(-464) - 3/0{,}2 = -247$ kN/m

The design is based on compression field theory as described in Chapter 1.2.

Crack angle based on yielding of reinforcement:

$$\tan^2\phi + \left(\frac{n_x}{n_{xy}} - \frac{n_y}{n_{xy}}\cdot\frac{A_{sx}}{A_{sy}}\right)\cdot\tan\phi - \frac{A_{sx}}{A_{sy}} = 0$$

Assuming reinforcement ratio in both membranes: $A_{sx}/A_{sy} = 4{,}32$

Membrane 1: $\tan^2\varphi + \left(\frac{1874}{-217} - \frac{475}{-217}\cdot 4{,}32\right)\tan\varphi - 4{,}32 = 0 \rightarrow \tan\varphi = -2{,}529 \rightarrow \varphi = -68{,}4^\circ$

Membrane 2: $\tan^2\varphi + \left(\frac{2254}{-247} - \frac{-225}{-247}\cdot 4{,}32\right)\tan\varphi - 4{,}32 = 0 \rightarrow \tan\varphi = -0{,}323 \rightarrow \varphi = -17{,}9^\circ$

Internal forces in reinforcement and compression field:

$F_c = n_{xy} / \sin\varphi\cos\varphi$
$F_{sx} = n_x + n_{xy}\cdot\tan\varphi$
$F_{sy} = n_y + n_{xy}\cdot\cot\varphi$

Membrane 1: $F_{sx} = 1874 - 217\cdot(-2{,}529) = 2423$ kN/m ;
$F_{sy} = 475 - 217/(-2{,}529) = 561$ kN/m
$F_c = -217/(-0{,}93\cdot 0{,}368) = 634$ kN/m

Membrane 2: $F_{sx} = 2254 -247\cdot(-0{,}323) = 2334$ kN/m
$F_{sy} = - 225 - 247/(-0{,}323) = 540$ kN/m
$F_c = -247/(-0{,}307\cdot 0{,}952) = 845$ kN/m

Required reinforcement: $A_{sx}=F_{sx}/f_{yd}$, $A_{sy}=F_{sy}/f_{yd}$

Membrane 1: $A_{sx} = 2423\cdot 10^3/435 = 5570$ mm^2/m $A_{sy} = 561\cdot 10^3/435 = 1289$ mm^2/m

Membrane 2: $A_{sx} = 2334\cdot 10^3/435 = 5365$ mm^2/m $A_{sy} = 540\cdot 10^3/435 = 1241$ mm^2/m

Control of concrete stresses in compression field:

Concrete grade B65 $\rightarrow$ $f_{cd} = 0{,}85\cdot 65/1{,}5 = 36{,}8$ N/mm^2

$v_{Rd,max}= 0{,}6\cdot v'\cdot f_{cd} = 0{,}6\cdot (1-65/250)\cdot 36{,}8 =16{,}4$ N/mm^2

Membrane 1: $\sigma_c = F_{c1}/t_1 = 634/150 = 4{,}2$ N/mm^2 $< v_{Rd,max} \rightarrow$ OK !
Membrane 2: $\sigma_c = F_{c2}/t_2 = 845/150 = 5{,}6$ N/mm^2 $< v_{Rd,max} \rightarrow$ OK !

5.7 Three layered approach - Sandwich model

The sandwich model is an extension of the membrane method from the previous section. The model was originally presented by Marti /5.11/. Later it has been modified to adapt it to limit state design /5.4/. The model is also a presented in Annex LL of Eurocode 2 part 2: Bridges /5.2/. The sandwich model will be described as outlined in EC2. Therefore the 8 shell forces with notations and indexes from EC2 are defined in Figure 5.7.1.

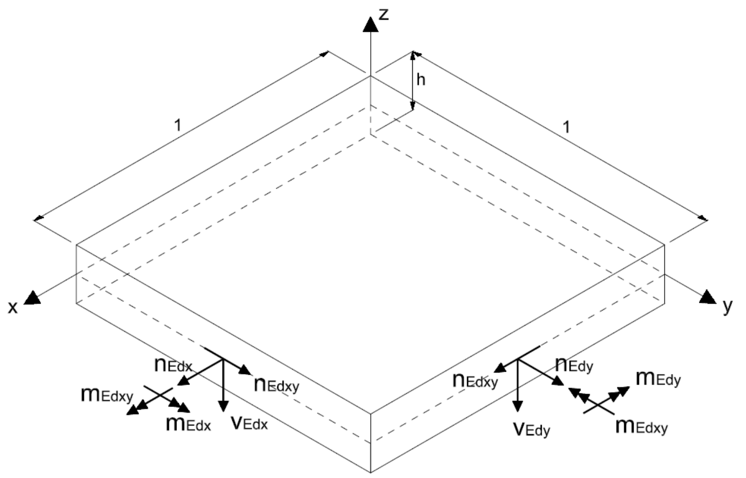


Figure 5.7.1 Shell forces with indexes according to Eurocode 2

The basic concept of the sandwich model is to divide the shell element in three layers. The two outer layers resist membrane actions arising from n_{Edx}, n_{Edy}, n_{Edxy}, m_{Edx}, m_{Edy} and m_{Edxy}. This is similar to the two layered approach. The inner layer carries the transverse shear forces v_{Edx} and v_{Edy}.

Figure 5.7.2 illustrates the forces in the different layers, and definition of thicknesses and level arms. The thicknesses of the upper and lower layer are denoted t_s and t_i, respectively. In general they can be different. The internal level arm z, and the distances y_s and y_i are evaluated to the middle planes of the reinforcement in x and y directions.

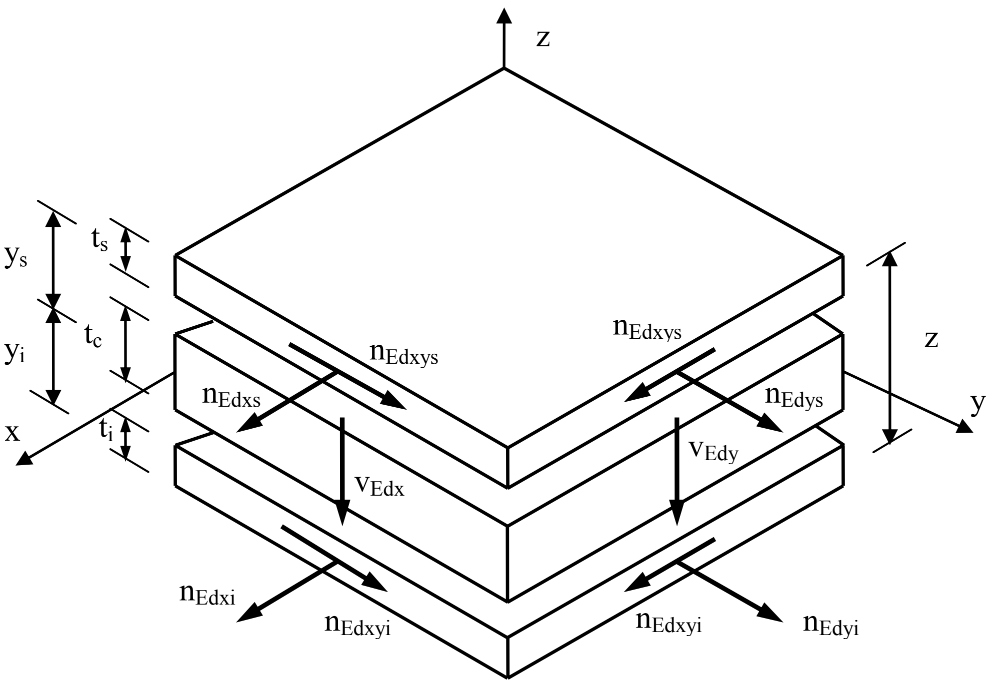


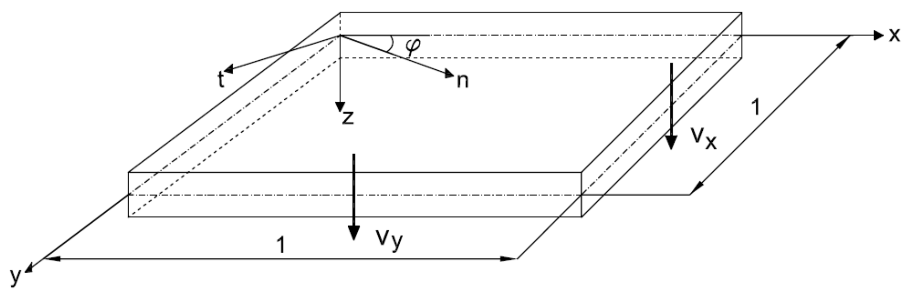
Figure 5.7.2 Definition of forces in different layers

5.7.1 Design of the inner layer

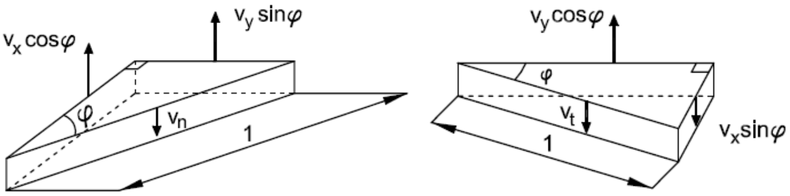
Figure 5.7.3 illustrates the transformation of shear forces from the x-y coordinate system by rotating the z-axis. By imposing the equilibrium equations, the governing transformation equations are given as:

$$v_n = v_{Edx} \cdot \cos\varphi + v_{Edy} \cdot \sin\varphi \tag{5.7.1}$$

$$v_t = -v_{Edx} \cdot \sin\varphi + v_{Edy} \cdot \cos\varphi \tag{5.7.2}$$



a)



b) c)

Figure 5.7.3 Transverse shear components. a) Definitions; b) and c) Free body diagram

The sum of squares in Eqs. (5.7.1) and (5.7.2) is independent with respect to φ, since $\cos^2\varphi + \sin^2\varphi = 1$.

$$v_n^2 + v_t^2 = v_{Edx}^2 + v_{Edy}^2 = v_o^2 \quad \Rightarrow \quad v_o = \sqrt{v_{Edx}^2 + v_{Edy}^2} \tag{5.7.3}$$

where v_0 can be denoted principal transversal shear force. The principal shear direction is defined by the angle φ_0 to the x-axis:

$$\varphi_0 = \tan^{-1}\left(\frac{v_{Edy}}{v_{Edx}}\right) \tag{5.7.4}$$

Perpendicular to the principal shear direction, the shear force is zero ($v_t = 0$). Hence, the shell behaves like a beam running in φ_0 direction. The sandwich core can then be designed as a beam only loaded with the principal shear force v_0. Following the design rules in EC2, it is necessary to distinguish between members requiring shear reinforcement and not. According to EC2 shear reinforcement is not required if:

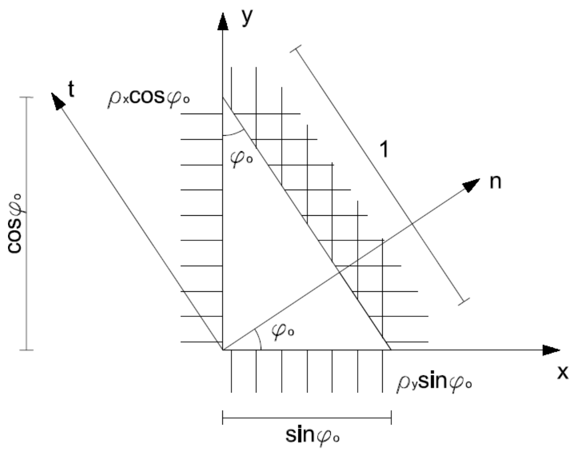
$$v_0 \leq v_{Rd,c} \tag{5.7.5}$$

where $v_{Rd,c}$ is the design shear stress resistance per unit of length and defined as:

$$v_{Rd,c} = C_{Rd,c} \cdot k \cdot (100\rho_l f_{ck})^{1/3} \cdot d \tag{5.7.6}$$

The reinforcement ratio ρ_l must be assessed in the principal shear direction. By inspection of Figure 5.7.4, the effective longitudinal reinforcement can be determined as:

$$\rho_l = \rho_x \cdot \cos^2\varphi_0 + \rho_y \cdot \sin^2\varphi_0 \tag{5.7.7}$$



$$\rho_x = \frac{A_{sx}}{b_x \cdot d}; \qquad \rho_y = \frac{A_{sy}}{b_y \cdot d}$$

Figure 5.7.4 Reinforcement ratio in principal shear direction

If the inner layer requires shear reinforcement, the resisting mechanism is governed by the variable strut inclination method /5.1/. Assuming vertical stirrups are used as shear reinforcement, the following verification equations are necessary:

$$v_0 \le v_{Rd,\max} = v_1 \cdot f_{cd} \cdot z_c \, v_1 \cdot f_{cd} \cdot z_c / (\cot\theta + \tan\theta) \tag{5.7.8}$$

$$v_0 \le v_{Rd,s} = \frac{A_s}{s} \cdot z_c \cdot f_{ywd} \cdot \cot\theta \tag{5.7.9}$$

where $v_{Rd,max}$ is the resistance of the compressive strut, and $v_{Rd,s}$ the tensile force in the shear reinforcement. The inclination angle θ is limited in EC2 to $1 \le \cot\theta \le 2{,}5$.

The resisting mechanism of the variable strut method introduces an additional axial force in the principal shear direction. The compressive strut and the principal shear force are balanced by a longitudinal force of $v_0 \cot\theta$, as seen in Figure 5.7.5.

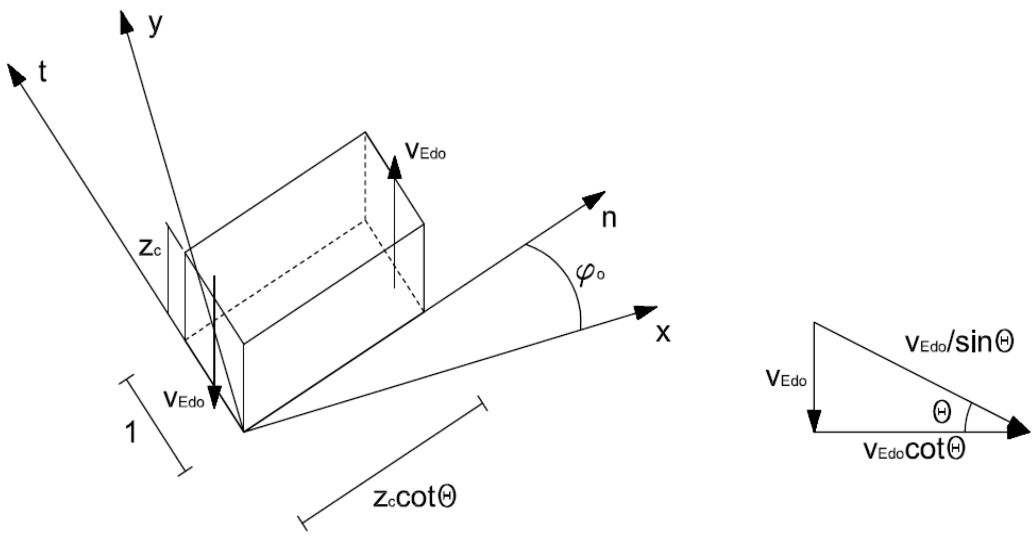


Figure 5.7.5 Principal shear force in inner layer /5.7/

In design of the sandwich covers, the additional force must be projected to the reference directions x and y of the outer layers. By considering the prism in
Figure 5.7.6, and ensure equilibrium, the additional membrane forces in y-direction, n_{Edyc} and n_{Edxyc} , can be found as:

$$n_{Edyc} = v_{Edo} \cdot \cot\theta \sin^2\varphi_o = v_{Edo} \cdot \cot\theta \cdot \frac{\tan^2\varphi_o}{1+\tan^2\varphi_o} \tag{5.7.10}$$

$$= v_{Edo} \cdot \cot\theta \cdot \frac{\dfrac{v_{Edy}^2}{v_{Edx}^2}}{1+\dfrac{v_{Edy}^2}{v_{Edx}^2}} = v_{Edo} \cdot \cot\theta \cdot \frac{v_{Edy}^2}{v_{Edx}^2 + v_{Edy}^2}$$

$$n_{Edxyc} = v_{Edo} \cdot \cot\theta \sin\varphi_o \cos\varphi_o = v_{Edo} \cdot \cot\theta \cdot \frac{\tan\varphi_o}{1+\tan^2\varphi_o} \tag{5.7.11}$$

$$= v_{Edo} \cdot \cot\theta \cdot \frac{\dfrac{v_{Edy}}{v_{Edx}}}{1+\dfrac{v_{Edy}^2}{v_{Edx}^2}} = v_{Edo} \cdot \cot\theta \cdot \frac{v_{Edx} \cdot v_{Edy}}{v_{Edx}^2 + v_{Edy}^2}$$

By employing $v_o^2 = v_x^2 + v_y^2$ for the principal shear force, the expressions are reduced to:

$$n_{Edyc} = \frac{v_{Edy}^2}{v_{Edo}} \cdot \cot\theta \tag{5.7.12}$$

$$n_{Edxyc} = \frac{v_{Edx} \cdot v_{Edy}}{v_{Edo}} \cdot \cot\theta \tag{5.7.13}$$

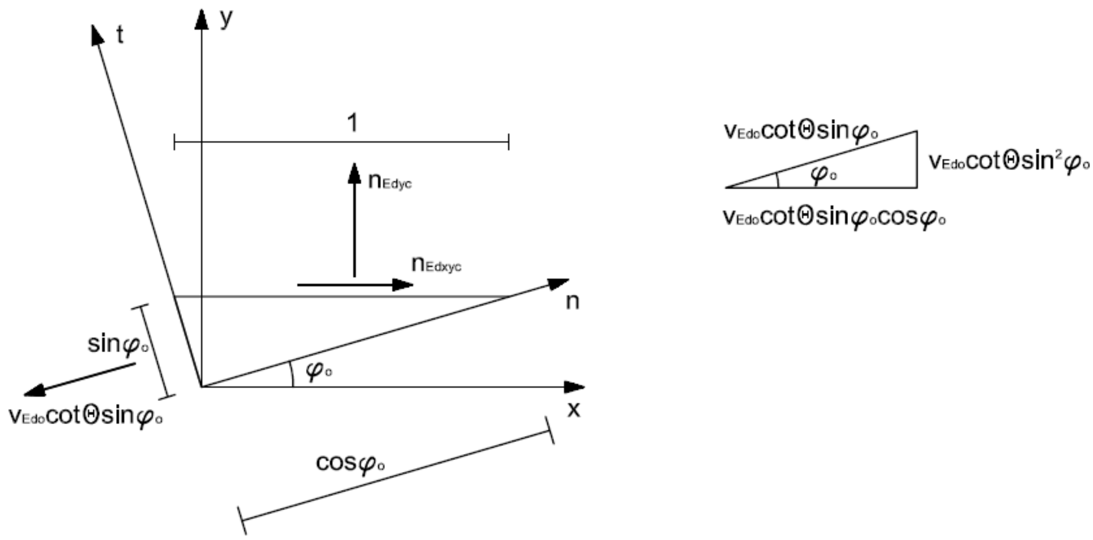


Figure 5.7.6 Axial forces in y-direction due to principal shear /5.7/

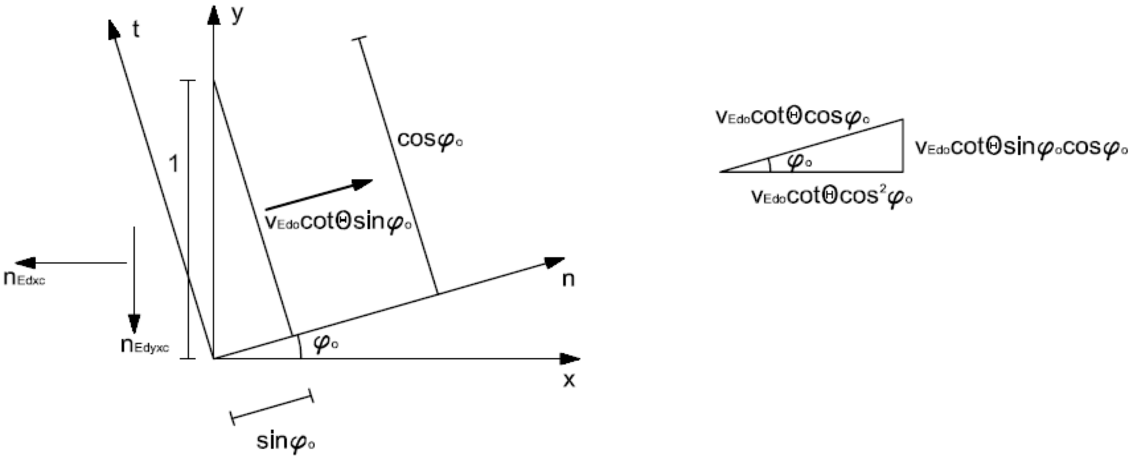


Figure 5.7.7 Axial forces in x-direction due to principal shear /5.7/

By imposing the equilibrium condition on the prism in Figure 5.7.7, the additional membrane forces in x-direction, n_{Edxc} and n_{Edyxc} , can be found as:

$$n_{Edxc} = \frac{v_{Edx}^2}{v_{Edo}} \cdot \cot\theta \tag{5.7.14}$$

$$n_{Edyxc} = n_{Edxyc} = \frac{v_{Edx} \cdot v_{Edy}}{v_{Edo}} \cdot \cot\theta \tag{5.7.15}$$

The additional forces n_{Edxc} , n_{Edyc} and n_{Edxyc} are global contributions and must be distributed to the upper and lower layer. Normally they are equally distributed to the outer layers.

5.7.2 Design of the outer layers

The outer layers are designed as membrane elements according to compression field theory.

The distribution of the membrane forces n_{Edx} , n_{Edy} and n_{Edxy}, to the outer layers depends on their relative thickness. As long as the upper and lower layer thicknesses are equal, the distribution factor $(z-y_{s(i)})/z = 0.5$. The upper and lower layers are subjected to the following membrane forces:

Upper layer:

$$n_{Edxs} = n_{Edx} \cdot \frac{z - y_s}{z} + \frac{m_{Edx}}{z} + \left(\frac{1}{2} \cdot \frac{{v_{Edx}}^2}{v_{Edo}} \cdot \cot\theta \right) \tag{5.7.16}$$

$$n_{Edys} = n_{Edy} \cdot \frac{z - y_s}{z} + \frac{m_{Edy}}{z} + \left(\frac{1}{2} \cdot \frac{{v_{Edy}}^2}{v_{Edo}} \cdot \cot\theta \right)$$

$$n_{Edxys} = n_{Edxy} \cdot \frac{z - y_s}{z} - \frac{m_{Edxy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx} v_{Edy}}{v_{Edo}} \cdot \cot\theta \right)$$

Bottom layer:

$$n_{Edxi} = n_{Edx} \cdot \frac{z - y_i}{z} - \frac{m_{Edx}}{z} + \left(\frac{1}{2} \cdot \frac{{v_{Edx}}^2}{v_{Edo}} \cdot \cot\theta \right) \tag{5.7.17}$$

$$n_{Edyi} = n_{Edy} \cdot \frac{z - y_i}{z} - \frac{m_{Edy}}{z} + \left(\frac{1}{2} \cdot \frac{{v_{Edy}}^2}{v_{Edo}} \cdot \cot\theta \right)$$

$$n_{Edxyi} = n_{Edxy} \cdot \frac{z - y_i}{z} + \frac{m_{Edxy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx} v_{Edy}}{v_{Edo}} \cdot \cot\theta \right)$$

In Eqs. (5.7.16) and (5.7.17) the terms in brackets must be summed if shear reinforcement is required, that is if Eq. (5.7.5) is not satisfied.

The thickness of the layers is an important parameter in the sandwich model. They influence both the distribution of shell forces to each layer and the detail design of each layer. The thickness of the different layers is established by means of an iterative procedure. The initial thickness of the outer layers should not be less than twice the concrete cover evaluated at the centroid of the reinforcement /5.4/. Provision of shear reinforcement is time consuming. It is advisable to try to choose thicknesses such that no shear reinforcement is required. However, in areas close to support or zones with concentrated loading this is not always possible.

If the concrete compressive strength requirement is not satisfied, the thickness of the outer layer should be increased. An alternative solution is to take into account the longitudinal reinforcement in the compression resistance. When increasing the thickness there are two options:

- Increase the concrete cover and move the reinforcement position so it still is in the middle plane of the layer, see Figure 5.7.8b. This will reduce the internal level arm z, and increase the required longitudinal reinforcement. However, in practice it is not economic to maintain large covers in a complete shell structure, and confusing to work with different covers in parts of the structure.

- Increase the concrete cover and leave the reinforcement position unchanged, see Figure 5.7.8c. The reinforcement is then eccentric in to the layer. To restore equilibrium, the amount of reinforcement must change. This influences the internal forces in the entire sandwich since internal bending moments arises. The model in Figure 5.7.9 can be used to assess the new forces in the outer layers.

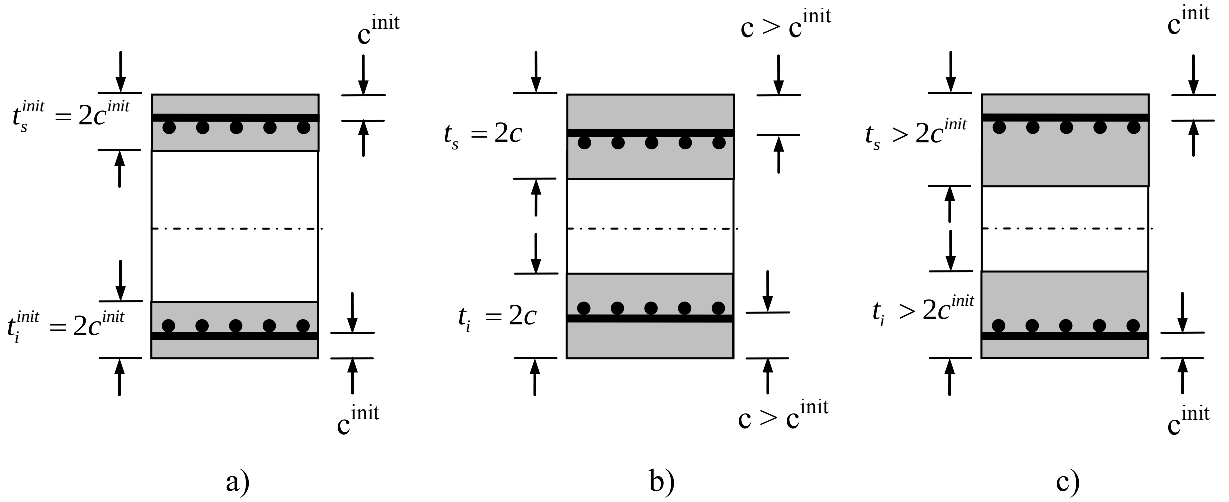


Figure 5.7.8 Thickness of outer layers. a) Initial thickness; b) Reinforcement position unchanged; c) Eccentric reinforcement

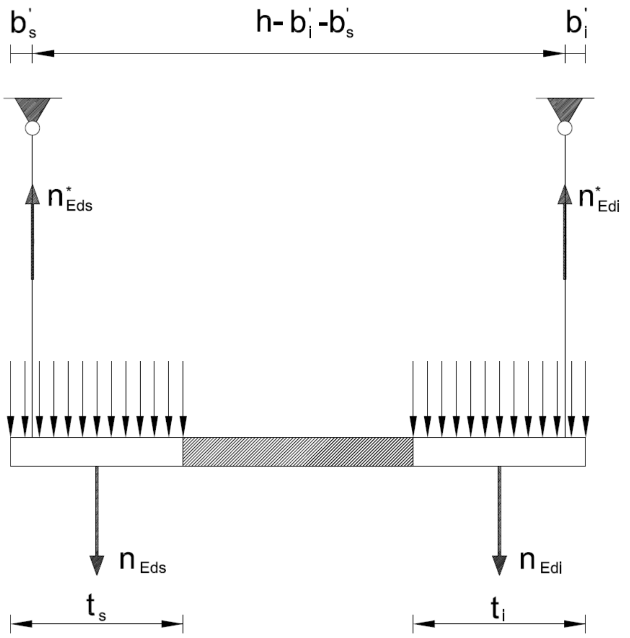


Figure 5.7.9 New forces in reinforcement due to eccentric position of reinforcement

By imposing equilibrium, the new internal forces in the reinforcement become:

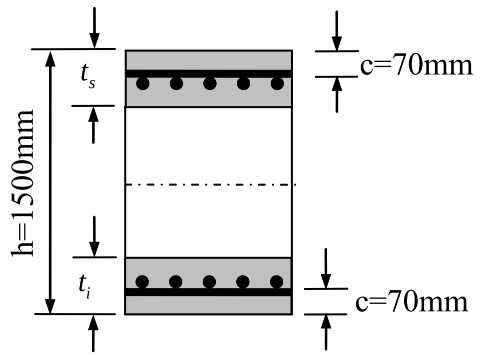
$$n^*_{Eds} = \frac{n_{Eds} \cdot \left(h - \frac{t_s}{2} - b'_i \right) + n_{Edi} \cdot \left(\frac{t_i}{2} - b'_i \right)}{h - b'_i - b'_s} \tag{5.7.18}$$

$$n^{*}_{Edi} = n_{Eds} + n_{Edi} - n^{*}_{Eds} \tag{5.7.19}$$

where $b`_{i,s}$ is the distance from the external surface of the layer to the axis of the reinforcement within the layer.

5.7.3 Example – Design with the sandwich model

Figure 5.7.10 shows a cross section in a reinforced concrete shell. Material properties and sectional forces taken from a FEM analyse at one point in the structure are also given in the figure. The sign definitions of the stress resultants are according to Figure 5.7.1.



B35 $\rightarrow f_{cd} =$ 19,8 MPa

B500NC $\rightarrow f_{yd} = 435$ MPa

$A_{sx}^{min} = A_{sy}^{min} = 2260 \text{ mm}^2/\text{m}$

Stress resultants:

$n_{Edx} = -7748$ kN/m $\quad m_{Edx} = 4274$ kNm/m

$n_{Edy} = -3865$ kN/m $\quad m_{Edy} = 1950$ kNm/m

$n_{Edxy} = -54$ kN/m $\quad m_{Edxy} = -41$ kNm/m

$v_{Edx} = 1095$ kN/m $\quad v_{Edy} = 1124$ kN/m

Figure 5.7.10 Shell section to be designed with the sandwich model

Assuming layer thicknesses:

$t_i = t_s = 2c = 140$ mm $\rightarrow$ $y_i = y_s = 680$ mm and $z = 1360$ mm

<u>Design of inner layer:</u>

$$v_{Edo} = \sqrt{v_{Edx}^2 + v_{Edy}^2} = \sqrt{1095^2 + 1124^2} = 1569 \text{ kN/m}$$

$$\varphi_0 = \tan^{-1}\left(\frac{v_{Edy}}{v_{Edx}}\right) = 45{,}7^{\circ}$$

Assume minimum reinforcement:

$$\rho_x = \rho_y = \frac{A_s}{b \cdot d} = \frac{2260}{1430 \cdot 1000} = 0{,}158\%$$

$$\rho_l = \rho_x \cdot \cos^2 \varphi_0 + \rho_y \cdot \sin^2 \varphi_0 = 0{,}00158$$

Shear resistance:

$$v_{Rd,c} = C_{Rd,c} \cdot k \cdot (100\rho_l f_{ck})^{1/3} \cdot d$$
$$= \frac{0,18}{1,5} \cdot \left(1 + \sqrt{\frac{200}{1500}}\right) \cdot \left(100 \cdot 0,00158 \cdot 35\right)^{1/3} \cdot 1430 = \underline{417 \text{ kN/m}}$$

$v_{Ed0} > v_{Rd,c} \Rightarrow$ need shear reinforcement

$$\rho_z = \frac{v_{Edo} \cdot \tan\theta}{z_c \cdot f_{ywd}}$$, assume $\cot\theta$=2

$$\rho_z = \frac{1569 \cdot 0,5}{1360 \cdot 435} = 1,3\% = \underline{\underline{13260 \text{ mm}^2/\text{m}^2}}$$

<u>Design of outer layers:</u>

Top layer:

$$n_{Edxs} = n_{Edx} \cdot \frac{z - y_s}{z} + \frac{m_{Edx}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx}{}^2}{v_{Edo}} \cdot \cot\theta\right)$$
$$= -7748 \cdot 0,5 + \frac{4274}{1,36} + \frac{1}{2} \cdot \frac{1095^2}{1569} \cdot 2,0 = 33 \text{ kN/m}$$
$$n_{Edys} = n_{Edy} \cdot \frac{z - y_s}{z} + \frac{m_{Edy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edy}{}^2}{v_{Edo}} \cdot \cot\theta\right)$$
$$= -3865 \cdot 0,5 + \frac{1950}{1,36} + \frac{1}{2} \cdot \frac{1124^2}{1569} \cdot 2,0 = 307 \text{ kN/m}$$
$$n_{Edxys} = n_{Edxy} \cdot \frac{z - y_s}{z} - \frac{m_{Edxy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx} \cdot v_{Edy}}{v_{Edo}} \cdot \cot\theta\right)$$
$$= -54 \cdot 0,5 + \frac{41}{1,36} + \frac{1}{2} \cdot \frac{1095 \cdot 1124}{1569} \cdot 2,0 = 788 \text{ kN/m}$$

Check the compression capacity by employing compression field theory, to control the assumed layer thicknesses.

$$\sigma_c = \frac{n_{Edxys}}{t_s \cdot \sin\varphi \cdot \cos\varphi} < \sigma_{Rd,max} = 0,6 \cdot v' \cdot f_{cd}$$

$$\sigma_{Rd,max} = 0,6 \cdot \left(1 - \frac{35}{250}\right) \cdot 19,8 = 10,2 \text{ MPa}$$

Assume crack angle φ = 45°.

$$\sigma_c = \frac{788}{140 \cdot \sin 45 \cdot \cos 45} = 10{,}1 \text{ MPa}$$

$$\sigma_c < \sigma_{Rd,max} \quad \rightarrow \text{ok}$$

Before calculating required reinforcement in the top layer, check the compression capacity in the bottom layer.

$$n_{Edxi} = n_{Edx} \cdot \frac{z - y_s}{z} - \frac{m_{Edx}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx}{}^2}{v_{Edo}} \cdot \cot \theta \right)$$

$$= -7748 \cdot 0{,}5 - \frac{4274}{1{,}36} + \frac{1}{2} \cdot \frac{1095^2}{1569} \cdot 2{,}0 = -6253 \text{ kN/m}$$

$$n_{Edyi} = n_{Edy} \cdot \frac{z - y_s}{z} - \frac{m_{Edy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edy}{}^2}{v_{Edo}} \cdot \cot \theta \right)$$

$$= -3865 \cdot 0{,}5 + \frac{1950}{1{,}36} + \frac{1}{2} \cdot \frac{1124^2}{1569} \cdot 2{,}0 = -2561 \text{ kN/m}$$

$$n_{Edxyi} = n_{Edxy} \cdot \frac{z - y_s}{z} + \frac{m_{Edxy}}{z} + \left(\frac{1}{2} \cdot \frac{v_{Edx} \cdot v_{Edy}}{v_{Edo}} \cdot \cot \theta \right)$$

$$= -54 \cdot 0{,}5 - \frac{41}{1{,}36} + \frac{1}{2} \cdot \frac{1095 \cdot 1124}{1569} \cdot 2{,}0 = 727 \text{ kN/m}$$

Forces in the reinforcement:

$$F_{sx} = n_{Rdx} = n_{Edxi} + n_{Edxyi} \cdot \tan \varphi = -6253 + 727 = -5526 \text{ kN/m}$$
$$F_{sy} = n_{Rdy} = n_{Edyi} + n_{Edxyi} \cdot \cot \varphi = -2561 + 727 = -1834 \text{ kN/m}$$

There is no tension in the reinforcement. Hence, the compression capacity needs to be verified in the principal compression direction.

$$\sigma_c \cdot t_i = \frac{n_{Edxi} + n_{Edyi}}{2} + \sqrt{\left(\frac{n_{Edxi} - n_{Edyi}}{2}\right)^2 + n_{Edxyi}{}^2}$$

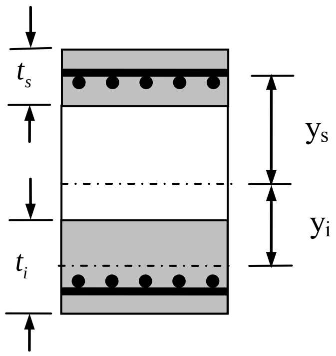
$$= \frac{6253 + 2561}{2} + \sqrt{\left(\frac{6253 - 2561}{2}\right)^2 + 727^2} = 6391 \text{ kN/m}$$

$$\Rightarrow \sigma_c = \frac{6391}{140} = 45{,}6 \text{ MPa}$$

$$\Rightarrow \sigma_c > f_{cd} = 19{,}8 \text{ MPa}$$

$\Rightarrow$ need to increase the bottom layer thickness

A new thickness of 435 mm is assumed for the bottom layer, see Figure 5.7.11. Consequently the membrane forces in the outer layers must be recalculated due to new level arms.



t_s = 140 mm
t_i = 430 mm
y_s = 680 mm
y_i = 535 mm
z = $y_s + y_i$ =1215 mm

Figure 5.7.11 New layer thickness and level arms

Top layer:

$$n_{Edxs} = -7748 \cdot \frac{1215-680}{1215} + \frac{4274}{1,215} + \frac{1}{2} \cdot \frac{1095^2}{1569} \cdot 2,0 = 870 \text{ kN/m}$$

$$n_{Edys} = -3865 \cdot \frac{1215-680}{1215} + \frac{1950}{1,215} + \frac{1}{2} \cdot \frac{1124^2}{1569} \cdot 2,0 = 709 \text{ kN/m}$$

$$n_{Edxys} = -54 \cdot \frac{1215-680}{1215} + \frac{41}{1,215} + \frac{1}{2} \cdot \frac{1095 \cdot 1124}{1569} \cdot 2,0 = 795 \text{ kN/m}$$

Bottom layer:

$$n_{Edxi} = -7748 \cdot \frac{1215-535}{1215} - \frac{4274}{1,215} + \frac{1}{2} \cdot \frac{1095^2}{1569} \cdot 2,0 = -7090 \text{ kN/m}$$

$$n_{Edyi} = -3865 \cdot \frac{1215-535}{1215} - \frac{1950}{1,215} + \frac{1}{2} \cdot \frac{1124^2}{1569} \cdot 2,0 = -2963 \text{ kN/m}$$

$$n_{Edxyi} = -54 \cdot \frac{1215-535}{1215} - \frac{41}{1,215} + \frac{1}{2} \cdot \frac{1095 \cdot 1124}{1569} \cdot 2,0 = 720 \text{ kN/m}$$

Verify bottom layer for compression failure:

$$\sigma_c \cdot t_i = \frac{n_{Edxi} + n_{Edyi}}{2} + \sqrt{\left(\frac{n_{Edxi} - n_{Edyi}}{2}\right)^2 + n_{Edxyi}{}^2}$$

$$= \frac{7090 + 2963}{2} + \sqrt{\left(\frac{7090 - 2963}{2}\right)^2 + 720^2} = 7212 \text{ kN/m}$$

$$\Rightarrow \sigma_c = \frac{7212}{430} = 16,8 \text{ MPa}$$

$$\Rightarrow \sigma_c < f_{cd} = 19,8 \text{ MPa} \Rightarrow \text{ok}$$

Reinforcement in top layer:

$$F_{sx} = n_{Rdx} = n_{Edxi} + n_{Edxyi} \cdot \tan\varphi = 870 + 795 = 1665 \text{ kN/m}$$

$$\mathrm{A}_{sx}^{top} = \frac{F_{sx}}{f_{yd}} = \frac{1665}{435} = \underline{\underline{3827 \ \mathrm{mm}^2/\mathrm{m}}}$$

$$F_{sy} = n_{Rdy} = n_{Edyi} + n_{Edxyi} \cdot \cot\varphi = 709 + 795 = 1504 \text{ kN/m}$$

$$\mathrm{A}_{sy}^{top} = \frac{F_{sy}}{f_{yd}} = \frac{1504}{435} = \underline{\underline{3457 \ \mathrm{mm}^2/\mathrm{m}}}$$

5.8 Resistance of concrete shells – iteration method

5.8.1 Introduction

In its basic form, the iteration method is employed to control the capacity for thin shell structures, where geometry and amount of reinforcement is given. The results are utilisation ratios for the concrete and the reinforcement. On basis of stress resultants from a FEM analysis, the method finds the strain distribution which ensures equilibrium between external and internal forces. With appropriate non-linear material models, equilibrium is achieved by an iteration process. The method as presented here is in accordance with /5.8/ and /5.9/. Figure 5.8.1 shows the volume of a shell section with the stress resultants per length. Out of plane shear forces, V_x and V_y, are not considered in the equilibrium iterations. Design for shear forces are handled separately and will be described later.

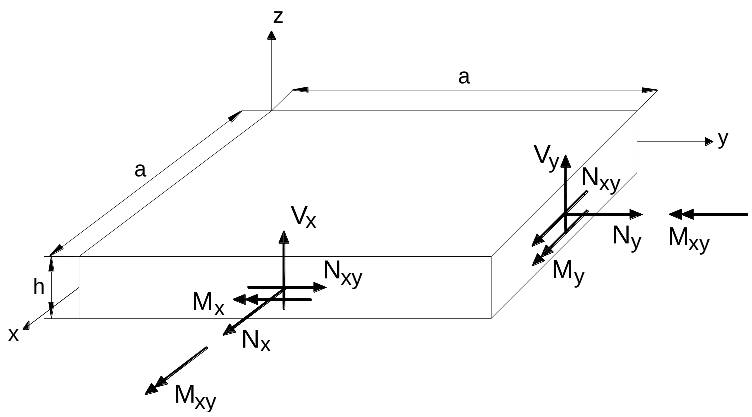


Figure 5.8.1 Shell section with volume a·a·h

In brief, the iteration method employs a displacement formulation to establish the relationship between the external and internal forces. Orthotropic material models in directions of the principal stresses are used to find the internal forces. Iterations by changing the strain distribution are necessary to ensure equilibrium. Within each iteration the material stiffness matrix is updated. The iteration stops when the deviation between external and internal forces is within an acceptable value. Details of the method will be described hereafter.

It is possible to use the iteration method as a design tool for shells, i.e. automatically calculate the required amount of longitudinal reinforcement. However, this is not straightforward and will not be considered here. It is difficult to find the total optimal reinforcement amount, because it is not obvious in which direction reinforcement should be added to increase the capacity of the shell section. A more practical way is first to find required reinforcement with a layered approach as described in section 5.5. Secondly, the iteration method can be employed to check the capacity of shell with the given reinforcement amounts.

5.8.2 Design for in-plane stresses

The design for in-plane stresses is based on Kirchoff's hypothesis of linear strain distribution over the thickness of the shell. Equal to classical beam and plate theory, out of plane normal stresses are assumed to be zero. Hence, the shell can be analysed as a 2-dimensional problem. This assumption and limitation implies that the iteration method for shells cannot be used to analyse complex joints and disturbed regions.

5.8.3 Displacement formulation

The fundamental approach in the iteration method is to decide the state of strain distribution which satisfies equilibrium between external and internal forces:

$$\boldsymbol{R} = \boldsymbol{S}(\boldsymbol{\varepsilon}_{t,r}) \tag{5.8.1}$$

where **R** is the external load vector, **S** is the internal stress resultant vector and $\boldsymbol{\varepsilon}_t$ is the generalised strain vector. They are defined as:

$$\boldsymbol{R} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} \tag{5.8.2}$$

$$\boldsymbol{\varepsilon}_t = \begin{bmatrix} \boldsymbol{\varepsilon}_m \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xm} \\ \varepsilon_{ym} \\ \gamma_{xym} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{5.8.3}$$

where $\boldsymbol{\varepsilon}_m$ and $\boldsymbol{\kappa}$ are the strains and curvatures of the middle plane of the shell element respectively. The assumption of linear strain distribution over the thickness, see Figure 5.8.2, gives the following in-plane strains in a distance z from the middle plane of the shell:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \boldsymbol{\varepsilon}_m - z \cdot \boldsymbol{\kappa} = \boldsymbol{A} \cdot \boldsymbol{\varepsilon}_t = \begin{bmatrix} 1 & 0 & 0 & -z & 0 & 0 \\ 0 & 1 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & -z \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xm} \\ \varepsilon_{ym} \\ \gamma_{xym} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{5.8.4}$$

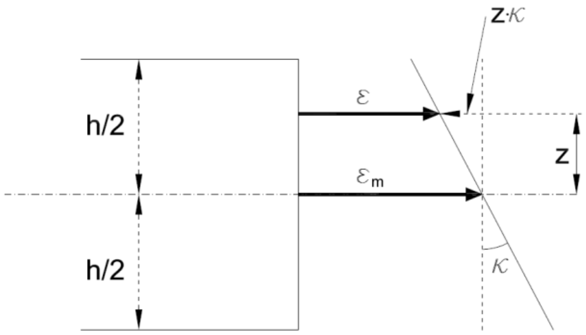


Figure 5.8.2 Strain distribution in a shell section

Eq. (5.8.1) is a nonlinear problem. In order to take into account the inelastic material properties of concrete and reinforcement, including cracking and yielding, a displacement formulation of the equation is defined as

$$\boldsymbol{R} = \boldsymbol{K}(\boldsymbol{\varepsilon}_{t,r}) \cdot \boldsymbol{\varepsilon}_{t,r+1} \tag{5.8.5}$$

where $\mathbf{K}(\boldsymbol{\varepsilon}_{t,r})$ is the secant stiffness matrix for concrete and reinforcement in iteration number *r*. The nonlinear relationship is illustrated in Figure 5.8.3.

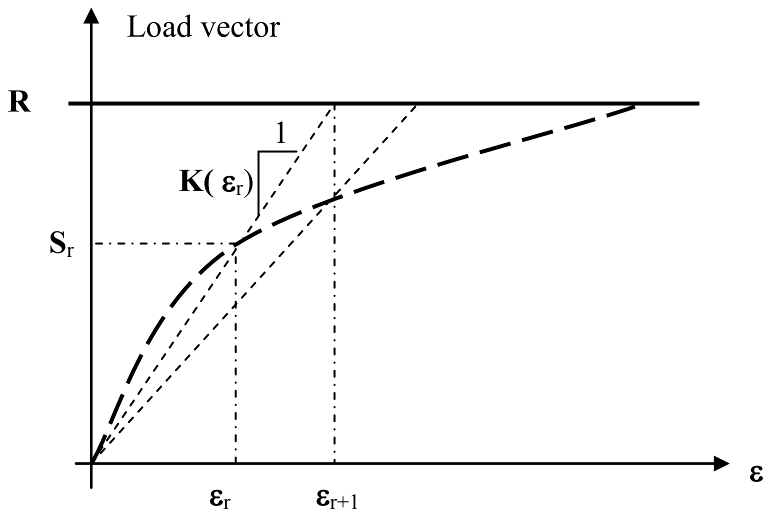


Figure 5.8.3 The nonlinear stiffness relationship

5.8.4 Stiffness matrix and virtual work

The material stiffness matrix **K** can be established by means of the principle of virtual work. The generalised displacement and rotations can be expressed by the displacement vector **r**:

$$\boldsymbol{r} = a \cdot \begin{bmatrix} \boldsymbol{\varepsilon}_m \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_{xy} \\ \theta_x \\ \theta_y \\ \theta_{xy} \end{bmatrix} \tag{5.8.6}$$

where a is the dimension of the shell element, see Figure 5.8.1.

Virtual displacement vector:

$$\delta\boldsymbol{r} = a \cdot \delta\boldsymbol{\varepsilon}_t \tag{5.8.7}$$

External virtual work:

$$A_y = \delta\boldsymbol{r}^T \cdot a \cdot \boldsymbol{R} \tag{5.8.8}$$

Internal virtual work:

$$A_i = \int_V \delta\boldsymbol{\varepsilon}^T \cdot \boldsymbol{\sigma} \cdot dV \tag{5.8.9}$$

External work = Internal work:

$$\delta\boldsymbol{r}^T \cdot a \cdot \boldsymbol{R} = \int_V \delta\boldsymbol{\varepsilon}^T \cdot \boldsymbol{\sigma} \cdot dV \tag{5.8.10}$$

where V is the volume of the shell element, and a·**R** the total external load. At the moment, the material model is defined in a general form, which for the in–plane stresses is given as:

$$\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon} = \boldsymbol{C}_\varepsilon \cdot \boldsymbol{\varepsilon} \; ; \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \; ; \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{5.8.11}$$

where C is the material matrix including contributions from both concrete and reinforcement.

$$A_i = \int_V \delta\boldsymbol{\varepsilon}^T \cdot \boldsymbol{\sigma} dV = \int_V \delta\boldsymbol{\varepsilon}^T \cdot \boldsymbol{C} \cdot \boldsymbol{\varepsilon} \cdot dV = \int_V \delta\boldsymbol{\varepsilon}_t{}^T \cdot \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{A} \cdot \boldsymbol{\varepsilon}_t \cdot dV$$

$$A_y = A_i \rightarrow a^2 \cdot \delta\boldsymbol{\varepsilon}_t{}^T \cdot \boldsymbol{R} = a^2 \cdot \delta\boldsymbol{\varepsilon}_t{}^T \cdot \int_{-h/2}^{h/2} \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{A} \cdot dz \cdot \boldsymbol{\varepsilon}_t$$

Hence, the governing equilibrium equation for the shell element is:

$$\boldsymbol{R} = \int_{-h/2}^{h/2} \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{A} \cdot dz \cdot \boldsymbol{\varepsilon}_t = K \cdot \boldsymbol{\varepsilon}_t \tag{5.8.12}$$

where the stiffness matrix **K** can be expressed by the integral:

$$\boldsymbol{K} = \int_{-h/2}^{h/2} \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{A} \cdot dz \tag{5.8.13}$$

By a congruence multiplication of the integrand, the stiffness matrix becomes:

$$\boldsymbol{K} = \int_{-h/2}^{h/2} \begin{bmatrix} \boldsymbol{C} & -z\boldsymbol{C} \\ -z\boldsymbol{C} & z^2\boldsymbol{C} \end{bmatrix} \cdot dz \tag{5.8.14}$$

With an appropriate material matrix **C**, the strains and curvatures at the middle plane of the shell can be found from the equilibrium equation.

$$\boldsymbol{\varepsilon}_t = \boldsymbol{K}^{-1} \cdot \boldsymbol{R} \tag{5.8.15}$$

The stiffness matrix is a general function of the distance *z* from the middle plane of the shell. The integral is solved numerically by dividing the cross section into layers. With *n* layers, each layer has thickness Δh = h/n, where *h* is the shell thickness, as illustrated in Figure 5.8.4. The layers of reinforcement are defined with separate layers, with given distances z from the middle plane. The material matrix **C** is constant within each layer.

$$\int_{-h/2}^{h/2} dz = \sum_{i=1}^{n} \Delta h$$

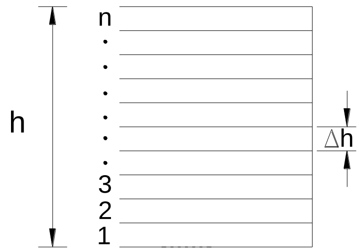


Figure 5.8.4 Shell element divided in concrete layers

The total stiffness matrix is a summation of concrete stiffness $\mathbf{K}_c$ and reinforcement stiffness $\mathbf{K}_s$. Assuming reinforcement only in x- and y-direction, the stiffness's becomes:

$$\text{Concrete:} \quad \boldsymbol{K}_c = \sum_{i=1}^{n} \Delta h \cdot \boldsymbol{A}_i^T \cdot \boldsymbol{C}_i \cdot \boldsymbol{A}_i = \Delta h \sum_{i=1}^{n} \begin{bmatrix} \boldsymbol{C}_i & -z_i \boldsymbol{C}_i \\ -z_i \boldsymbol{C}_i & z_i^2 \boldsymbol{C}_i \end{bmatrix} \tag{5.8.16}$$

$$\text{Reinforcement:} \quad \boldsymbol{K}_s = \sum_{j=1}^{m} (A_{sxj} \cdot \begin{bmatrix} \boldsymbol{C}_{sxj} & -z_j \boldsymbol{C}_{sxj} \\ -z_j \boldsymbol{C}_{0sxj} & z_j^2 \boldsymbol{C}_{sxj} \end{bmatrix} + A_{syj} \cdot \begin{bmatrix} \boldsymbol{C}_{syj} & -z_j \boldsymbol{C}_{syj} \\ -z_j \boldsymbol{C}_{syj} & z_j^2 \boldsymbol{C}_{syj} \end{bmatrix}) \tag{5.8.17}$$

$$\boldsymbol{K} = \boldsymbol{K}_c + \boldsymbol{K}_s \tag{5.8.18}$$

5.8.5 Internal stress resultants

The internal stress resultants can be collected in a stress vector **S**:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_N \\ \boldsymbol{S}_M \end{bmatrix} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} \tag{5.8.19}$$

Both concrete and reinforcement contribute to the stress vector. The internal stress resultants can be expressed as the integrals:

$$\boldsymbol{S}_N = \int_{-h/2}^{h/2} \boldsymbol{\sigma} \cdot dz \tag{5.8.20}$$

$$\boldsymbol{S}_M = \int_{-h/2}^{h/2} -z \cdot \boldsymbol{\sigma} \cdot dz \tag{5.8.21}$$

As for the stiffness matrix, the integral are solved numerically, giving the following equations as a summation of concrete and reinforcement layers:

$$\boldsymbol{S}_N = \sum_{i=1}^{n} \Delta h \cdot \boldsymbol{\sigma}_{ci} + \sum_{j=1}^{m} \begin{bmatrix} A_{sxj} \cdot \sigma_{sxj} \\ A_{syj} \cdot \sigma_{syj} \\ 0 \end{bmatrix} \tag{5.8.22}$$

$$\boldsymbol{S}_M = \sum_{i=1}^{n} \Delta h \cdot (-z) \cdot \boldsymbol{\sigma}_{ci} + \sum_{j=1}^{m} \begin{bmatrix} -z \cdot A_{sxj} \cdot \sigma_{sxj} \\ -z \cdot A_{syj} \cdot \sigma_{syj} \\ 0 \end{bmatrix} \tag{5.8.23}$$

where σ_{ci} are concrete stresses in layer *i*, and σ_{sxj} and σ_{syj} are stresses in reinforcement layer *j*.

5.8.6 Material model

For an isotropic linear elastic material, Hooks law in plane stress situations is given as:

$$\boldsymbol{\sigma} = \boldsymbol{C} \cdot \boldsymbol{\varepsilon} \; ; \qquad \boldsymbol{C} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{5.8.24}$$

where ν is Poisson ratio.

For non-linear materials like concrete, Hooks law is not valid. As seen in Figure 5.8.5, anisotropic behaviour is induced from stresses.

$$\sigma_x \neq \sigma_y \rightarrow E_x \neq E_y$$

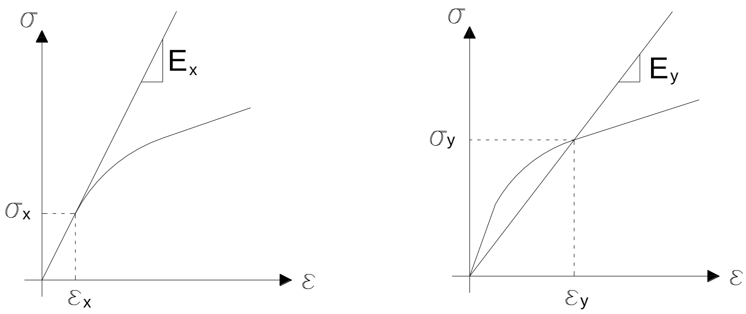


Figure 5.8.5 Anisotropic behaviour

<u>Concrete</u>

Cracking of concrete in tension and non-linear behaviour in compression can be taken into account by using an orthotropic material model in directions of the principal stress.

$$\boldsymbol{\sigma}_p = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \boldsymbol{C}_p \cdot \boldsymbol{\varepsilon}_p = \frac{1}{1-\nu^2} \cdot \begin{bmatrix} E_{11} & \nu E_{12} & 0 \\ \nu E_{12} & E_{22} & 0 \\ 0 & 0 & \dfrac{(1-\nu)E_{12}}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \tag{5.8.25}$$

where $\boldsymbol{\sigma}_p$ are stresses in principal directions and

$$E_{12} = \frac{E_{11} + E_{22}}{2} \tag{5.8.26}$$

The modulus's of elasticity E_{11} og E_{22} are secant modulus's in the principal directions. Hence, it is a linearization of the problem. For non-linear materials they are taken from a defined stress-strain relationship and given as:

$$E_{ii} = \frac{\sigma_i}{\varepsilon_i} \quad \text{for i=1,2} \tag{5.8.27}$$

The orthotropic material model presented in Eq. (5.8.25) is a simple model. Assuming Poisson's ratio ν=0 in ultimate limit state, the two principal stress directions are uncoupled. Hence, the failure criterion is as shown in Figure 5.8.6, assuming the tensile strength is zero.

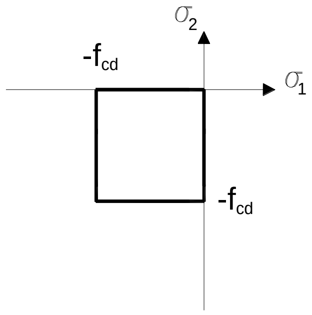


Figure 5.8.6 Biaxial failure criterion for concrete

It is possible to employ more sophisticated material models which takes into account increased compressive strength and ductility for biaxial compression and reduced strength in a compression-tension state of stress.

Using the iteration method, strains, stresses and stiffness matrixes must be transformed between the global xy-axes and the principal stress/strain directions. The global xy-axes are in this context coincident with the definition of axes for external stress resultants in Figure 5.8.1. Transformations of strains from global axes to principal directions are defined by:

$$\boldsymbol{\varepsilon}_p = \boldsymbol{T}(\theta) \cdot \boldsymbol{\varepsilon} \tag{5.8.28}$$

where $\boldsymbol{\varepsilon}_p$ are principal strains, θ the angle for principal strain direction and $\mathbf{T}(\theta)$ the transformation matrix.

$$\boldsymbol{T}(\theta) = \begin{bmatrix} cos^2\theta & sin^2\theta & sin\theta cos\theta \\ sin^2\theta & cos^2\theta & -sin\theta cos\theta \\ -2sin\theta cos\theta & 2sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \tag{5.8.29}$$

$$\theta = \frac{1}{2} \cdot \tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \right) \tag{5.8.30}$$

Assuming coaxility between principal strains and principal stresses, the principal stresses and the material stiffness matrix are transformed to global xy-axes as:

$$\boldsymbol{\sigma}_c = \boldsymbol{T}^T(\theta)\cdot\boldsymbol{\sigma}_p = \boldsymbol{T}^T(\theta)\cdot\boldsymbol{C}_p\cdot\boldsymbol{\varepsilon}_p = \boldsymbol{T}^T(\theta)\cdot\boldsymbol{C}_p\cdot\boldsymbol{T}(\theta)\cdot\boldsymbol{\varepsilon} \tag{5.8.31}$$

$$\boldsymbol{C} = \boldsymbol{T}^T\cdot\boldsymbol{C}_p\cdot\boldsymbol{T} \tag{5.8.32}$$

<u>Reinforcement</u>

Assuming the reinforcement directions are in the global x- and y-directions, the stress-strain relationship for one layer of reinforcement is defined as:

$$\boldsymbol{\sigma}_s = \boldsymbol{C}_s\cdot\boldsymbol{\varepsilon} \tag{5.8.33}$$

$$\boldsymbol{\sigma}_s = \begin{bmatrix}\sigma_{sx}\\ \sigma_{sy}\\ \tau_{sxy}\end{bmatrix} = \begin{bmatrix}E_{sx} & 0 & 0\\ 0 & E_{sy} & 0\\ 0 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{5.8.34}$$

where E_{sx} and E_{sy} are secant modulus's for the reinforcement in the global directions. Eq. (5.8.34) represents two one-dimensional stress-strain relationships.

If the longitudinal reinforcement is not in direction of the global axes, the material matrix for reinforcement must be transformed to the xy-direction.

$$\boldsymbol{C}_s^{xy} = \boldsymbol{T}^T(\alpha)\cdot\boldsymbol{C}_s\cdot\boldsymbol{T}(\alpha) \tag{5.8.35}$$

where α is the angle between the reinforcement and the global x-direction.

5.8.7 Solving the equilibrium equation with iterations

The iteration method is numerical solution of the equilibrium equation between external (**R**) and internal (**S**) load resultants, **R**=**S**. The equation is solved by iterations on the strain distribution. Since it is a numerical solution, a convergence criterion must be defined when acceptable equilibrium is reached. A simple and robust criterion is to use the relative difference between each of the six external and internal stress resultant components. The iteration is stopped when the deviation is within an acceptable value β.

$$\text{Converegence if } \left|\frac{R_k - S_{0k}}{R_k}\right| < \beta\ ;\ \ \text{k=1,2,\ldots,6} \tag{5.8.36}$$

β is typically in order of magnitude 0,01. As long as the reinforcement amount is not automatically increased during the iteration procedure, divergence in the solution must be prevented by a limitation on the number of iteration steps.

The following step by step procedure can describe the iteration method:

1. Decide the external load vector **R**, normally from a FEM analysis, and the reinforcement amount from a preliminary design.

2. Calculate initial stiffness matrix $\mathbf{K}_0$, assuming isotropic linear elastic behaviour for concrete and linear elastic behaviour for reinforcement:

   Concrete:

$$\boldsymbol{K}_{c0} = \sum_{i=1}^{n} \Delta h \cdot \boldsymbol{A}_i^T \cdot \boldsymbol{C}_{0i} \cdot \boldsymbol{A}_i = \Delta h \sum_{i=1}^{n} \begin{bmatrix} \boldsymbol{C}_{0i} & -z_i \boldsymbol{C}_{0i} \\ -z_i \boldsymbol{C}_{0i} & z_i^2 \boldsymbol{C}_{0i} \end{bmatrix}$$

   Reinforcement:

$$\boldsymbol{K}_{s0} = \sum_{j=1}^{m} (A_{sxj} \cdot \begin{bmatrix} \boldsymbol{C}_{0sxj} & -z_j \boldsymbol{C}_{0sxj} \\ -z_j \boldsymbol{C}_{0sxj} & z_j^2 \boldsymbol{C}_{0sxj} \end{bmatrix} + A_{syj} \cdot \begin{bmatrix} \boldsymbol{C}_{0syj} & -z_j \boldsymbol{C}_{0syj} \\ -z_j \boldsymbol{C}_{0syj} & z_j^2 \boldsymbol{C}_{0syj} \end{bmatrix})$$

$$\boldsymbol{K}_0 = \boldsymbol{K}_{c0} + \boldsymbol{K}_{s0}$$

3. Find strains and curvatures in the middle plane of the shell:

$$\boldsymbol{\varepsilon}_{t0} = \boldsymbol{K}_0^{-1} \cdot \boldsymbol{R}$$

4. Find in-plane strains in each concrete and reinforcement layer:

$$\boldsymbol{\varepsilon}_{0i} = \boldsymbol{A}_i \cdot \boldsymbol{\varepsilon}_{t0}$$

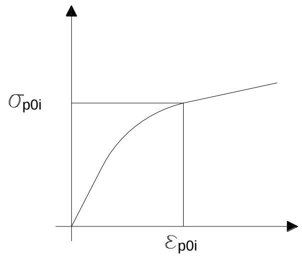
5. Find principal strains and principal directions in the concrete layers:

$$\boldsymbol{\varepsilon}_{p0i} = \boldsymbol{T}_{\varepsilon i}(\theta_i) \cdot \boldsymbol{\varepsilon}_{0i}$$

   where $\mathbf{T}_{\varepsilon i}$ is the transformation matrix from global axes to the local principal direction. Angle θ_i is found from:

$$\theta_i = \frac{1}{2} \cdot \tan^{-1}\left(\frac{\gamma_{xy}^i}{\varepsilon_x^i - \varepsilon_y^i} \right)$$

6. Calculate principal concrete stresses in each concrete layer. The principal stresses $\boldsymbol{\sigma}_{p0i}$ depends on the stress-strain relationship. This is where the non-linear effects in concrete are taken into account.



7. Transform principal stresses in each concrete layer to the global xy-coordinate system:

$$\boldsymbol{\sigma}_{c0i} = \boldsymbol{T}_{\varepsilon i}^{T}(\theta_i) \cdot \boldsymbol{\sigma}_{p0i}$$

$T_{\varepsilon}^{T} = T_{\sigma}^{-1}$ due to orthogonally.

8. Find reinforcement stresses in each reinforcement layer:

$$\boldsymbol{\sigma}_{s0j} = \boldsymbol{C}_{s0j} \cdot \boldsymbol{\varepsilon}_{0j}$$

9. Calculate internal force vector as a summation of concrete and reinforcement contribution:

$$\boldsymbol{S}_0 = \Delta h \cdot \sum_{i=1}^{n} \begin{bmatrix} \boldsymbol{\sigma}_{c0i} \\ -z_i \cdot \boldsymbol{\sigma}_{c0i} \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} A_{sxj} \cdot \sigma_{sx0} \\ A_{syj} \cdot \sigma_{sy0} \\ 0 \\ -z_j A_{sxj} \cdot \sigma_{sx0} \\ -z_j A_{syj} \cdot \sigma_{sy0} \\ 0 \end{bmatrix}$$

10. Find the maximum relative deviation between components of external and internal stress resultants:

$$\text{Maxdiff} = \left| \frac{R_k - S_{0k}}{R_k} \right|; \text{ k=1,2,…,6}$$

11. Check convergence according to a chosen tolerance, for example β = 0,01.

- If Maxdiff ≤ β, equilibrium is achieved and the calculation can be terminated.
- If Maxdiff > β, there is no convergence, and the calculation must proceed with a new

12. Find new secant modulus's for all concrete and reinforcement layers

13. Calculate new material matrix for concrete based on the secant modulus's: $\boldsymbol{C}_{p1i}$;i=1,….,n

14. Transform local material matrixes to global xy-axes:

$$C_{1i} = T_{\varepsilon i}^T \cdot C_{p1i} \cdot T_{\varepsilon i}$$

Repeat step 2-12 with the new material matrix until the convergence criteria is reached.

Figure 5.8.7 visualises the iteration process.

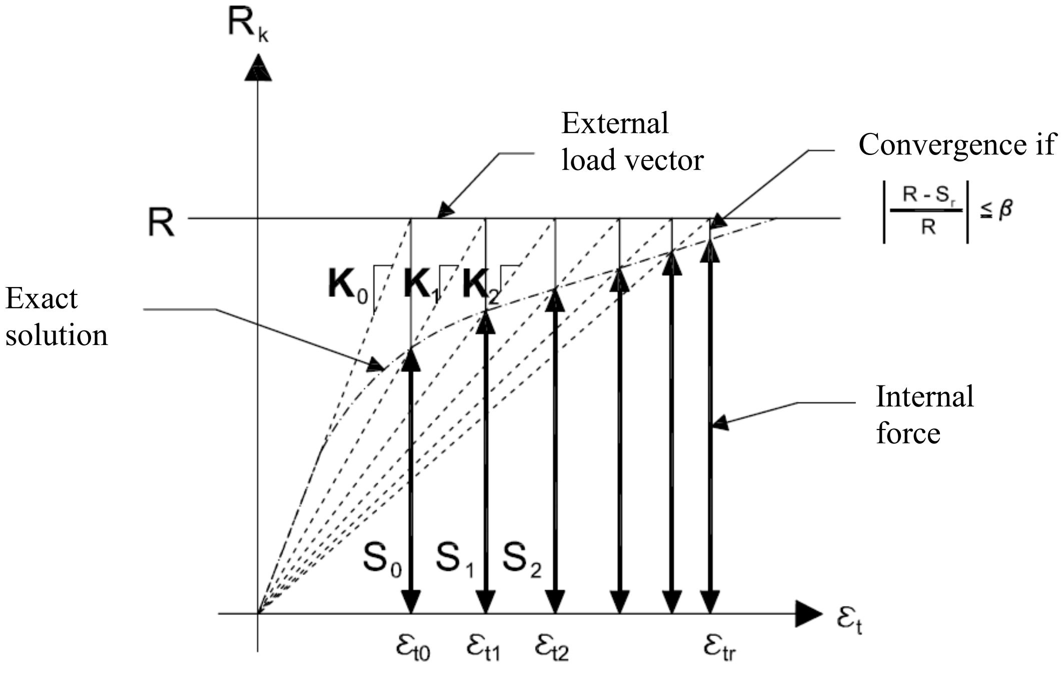


Figure 5.8.7 Visualisation of the iteration process

5.8.8 Definition of utilisation ratios

From the converged solution in the iteration method, the obtained strain distribution can be used to assess the utilisation of the shell section, and to verify that the design loads do not exceed the design resistance. The utilisation of a structural member is expressed by utilisation ratios (UR). In ultimate limit state the ratios are related to strain limits in the concrete and reinforcement. 100% utilisation indicates that the limit is reached. For concrete the utilisation ratio is defined as:

$$UR_c = \frac{\varepsilon_c}{\varepsilon_{cu}} \cdot 100\% \qquad (5.8.37)$$

where ε_c is the maximum compressive principal strain and ε_{cu} the ultimate compressive strain. The utilisation ratio for the reinforcement is defined as:

$$UR_s = \frac{\varepsilon_s}{\varepsilon_{ud}} \cdot 100\% \qquad (5.8.38)$$

where ε_s is the strain in the reinforcement, and ε_{ud} is the strain limit for the reinforcement.

5.8.9 Design for transverse shear

When establishing equilibrium for in-plane forces in section 5.8.2, the transverse shear forces V_x and V_y shown in Figure 5.4.6, were ignored. Consequently a separate control of the shear resistance is necessary. A simplified approach is to consider the shell element as an equivalent beam strip in a general direction α € [0º, 180º], as seen in Figure 5.8.8.

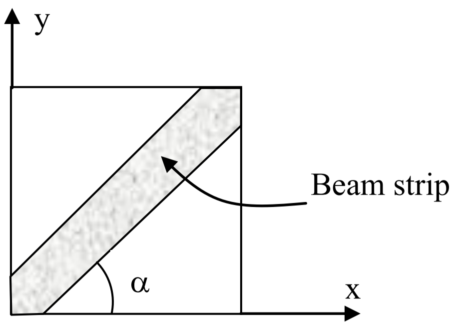


Figure 5.8.8 Equivalent beam strip

The shear design is performed as for a beam with sectional forces V_α, N_α and M_α, see Figure 5.8.9. Forces perpendicular to the beam direction is not taken into consideration.

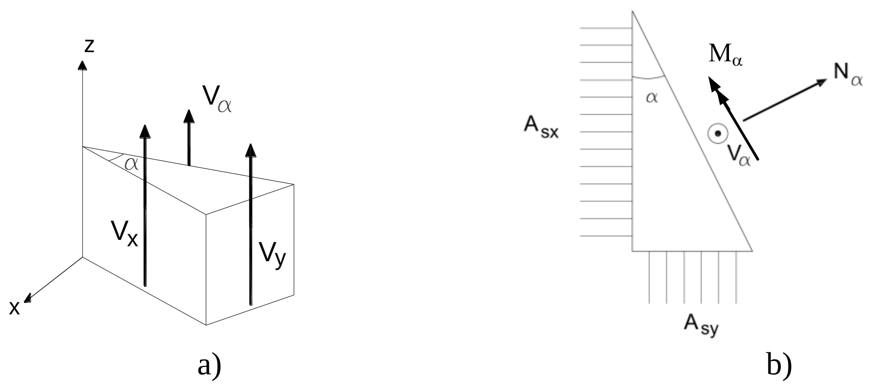


Figure 5.8.9 Design forces in α -direction

Transformation of shell forces to the equivalent beam direction α:

$$V_\alpha = V_x \cdot \cos\alpha + V_y \cdot \sin\alpha \tag{5.8.39}$$

$$N_\alpha = N_x \cdot \cos^2\alpha + N_y \cdot \sin^2\alpha + 2N_{xy} \cdot \sin\alpha\cos\alpha \tag{5.8.40}$$

$$M_\alpha = M_x \cdot \cos^2\alpha + M_y \cdot \sin^2\alpha + 2M_{xy} \cdot \sin\alpha\cos\alpha \tag{5.8.41}$$

$$A_{s\alpha} = A_{sx} \cdot \cos^2\alpha + A_{sy} \cdot \sin^2\alpha \tag{5.8.42}$$

where $A_{s\alpha}$ is the effective cross area of the longitudinal reinforcement in α-direction.

In order to determine the shear resistance and the maximum required amount of shear reinforcement, the equivalent beam must be designed in directions by varying α from 0° to 180° in steps. Steps of 5° are normally sufficient. The shear resistance is defined in the specified design code.

The utilisation ratios for transverse shear are based on the relationship between the design shear force and the shear resistance. Using Eurocode 2 as the design code, the utilisation for members not requiring shear reinforcement is given by:

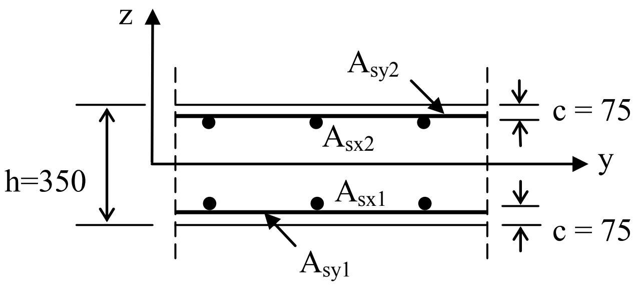
$$UR_c = \frac{V_\alpha}{V_{Rd,c}} \cdot 100\% \tag{5.8.43}$$

For members requiring shear reinforcement the utilisation ratio is defined as:

$$UR_s = \frac{V_\alpha}{V_{Rd,s}} \cdot 100\% \tag{5.8.44}$$

5.8.10 Example – Capacity control with iteration method

Figure 5.8.10 shows the same example as calculated in section 5.6.1. The iteration method requires the longitudinal reinforcement as input. Hence, the amount of reinforcement according to the two-layered approach is also given in the figure.



Stress resultants:
$n_x = 4127$ kN/m $m_x = -38$ kNm/m
$n_y = 250$ kN/m $m_y = 70$ kNm/m
$n_{xy} = -464$ kN/m $m_{xy} = 3$ kNm/m

Concrete: B65
Reinforcement: B500NC

$A_{sx1} = 5570$ mm^2/m
$A_{sy1} = 1289$ mm^2/m
$A_{sx2} = 5365$ mm^2/m
$A_{sy2} = 1241$ mm^2/m

Figure 5.8.10 Top slab in a box girder bridge

The iteration procedure described in section 5.8.7 is implemented in a computer program /5.12/. The uniaxial stress-strain relationships from EC2 are used for concrete and reinforcement in the program. Multiaxial effects on the uniaxial stress-strain relationship are not taken into account.

Table 5.8.1 gives the result from the analysis. Equilibrium was reached after 365 iterations, employing a convergence criterion β=0,001 according to Eq.(5.8.36). When investigating the response of a shell section, the reinforcement strains are often used to evaluate the utilisation of the cross section. Compared to the yield strain in EC2, ε_{yd}=2,17 ‰, the reinforcement in two directions yield. The maximum principal compressive stress and strain are σ_1= 12 MPa and ε_1=0,4 ‰, respectively. Thus, the utilisation in compression is low, which is not surprising since all the reinforcement at the top and the bottom in both directions are in tension.

Table 5.8.1 Stress and strain in reinforcement with initial amount of reinforcement

	Reinforcement [mm^2/m]	Stress [MPa]	Strain [‰]
A_{sx1}	5570	401	2,0
A_{sy1}	1289	435	3,1
A_{sx2}	5365	435	4,1
A_{sy2}	1241	262	1,3

Based on the results of the analysis, the cross section cannot be considered fully utilized. It should be possible to reduce the amount of reinforcement. However, to optimize the total reinforcement is not straightforward in the iteration method. In general, reduction of the reinforcement area in one layer in one direction also influences the response in the unchanged reinforcement bars and the concrete. In practical design of shell structure is even more complicated. First, different load combinations results in different utilisation in different directions. Second, requirements in both ultimate and serviceability limit state must be satisfied, which not necessarily are for the same load combination.

As an example the bottom reinforcement in y-direction is reduced to 500 mm^2/m. The results in Still the maximum principal compressive stress and strain are relatively low, σ_1= 18 MPa and ε_1=0,7 ‰, respectively.

Table 5.8.2 are obtained after 349 equilibrium iterations. As expected, the utilisation of the reinforcement is then higher. However, the reinforcement strains are also much higher. Especially the top reinforcement in x direction where the strain is 9,2 ‰, indicates there will be problems fulfilling requirements in serviceability limit state for crack widths. Still the maximum principal compressive stress and strain are relatively low, σ_1= 18 MPa and ε_1=0,7 ‰, respectively.

Table 5.8.2 Stress and strain in reinforcement with reduced amount of reinforcement

	Reinforcement [mm^2/m]	Stress [MPa]	Strain [‰]
A_{sx1}	5570	423	2,1
A_{sy1}	1289	435	2,2
A_{sx2}	5365	435	9,2
A_{sy2}	500	435	3,7

5.9 References

/5.1/ Standard Norge, *NS-EN 1992-1-1:2004+NA:2008, Eurocode 2: Design of concrete structures. Part 1-1: General rules and rules for buildings.* 2008.

/5.2/ Standard Norge, *NS-EN 1992-2:2005+NA:2010, Eurocode 2: Design of concrete structures, Part2:Concrete bridges*, 2010.

/5.3/ CEB-FIP, *Practitioners' guide to finite element modelling of reinforced concrete structures*, Bulletin 45, 2008.

/5.4/ CEB, *Ultimate Limit State Design Models: Ultimate Limit State Design of Structural Concrete Shell Elements*, Bulletin 223, 1995.

/5.5/ S. I. Sørensen, *Aksesymmetriske skall*, Fag 37047 Beregningsgrunnlaget for betongkonstruksjoner, NTNU, 1998.

/5.6/ M.J. Jawad, *Design of plate and shell structures*, ASME Press, New York, 2004.

/5.7/ I. Betten and S.-M. Bach Johansen, *Design of concrete shell structures*, Master thesis, Department of structural engineering, NTNU, 2010.

/5.8/ E. Åldstedt, *Dimensjonering ved kombinert skive- og platevirkning.* NIF-kurs, NTH Trondheim, 1990.

/5.9/ S. I. Sørensen, *Dimensjonering basert på FEM-analyser*. In TKT4222: Betongkonstruksjoner 3, høst 2009, kompendiesamling. Trondheim, NTNU, 2009.

/5.10/ S. I. Sørensen, *Betongkonstruksjoner: Beregning og dimensjonering etter Eurocode 2,* Tapir, 2010.

/5.11/ P. Marti, *Design of concrete slabs for transverse shear,* ACI Structural Journal, 87(2), 1990.

/5.12/ E. Raknes Brekke and A. Sørvik Hanssen, *Design of concrete shell structures,* Master thesis, Department of structural engineering, NTNU, 2011.

CHAPTER 6

DESIGN OF FOUNDATIONS

Svein I Sørensen

6.1 FOUNDATION TYPES

The structural system and the actual ground properties are decisive for the choice of foundation type.
In ”direct foundation”, the loads are transferred from the structure directly to the ground as a distributed pressure from the foundation.
Figure 6.1.1 shows actual types of direct foundations.

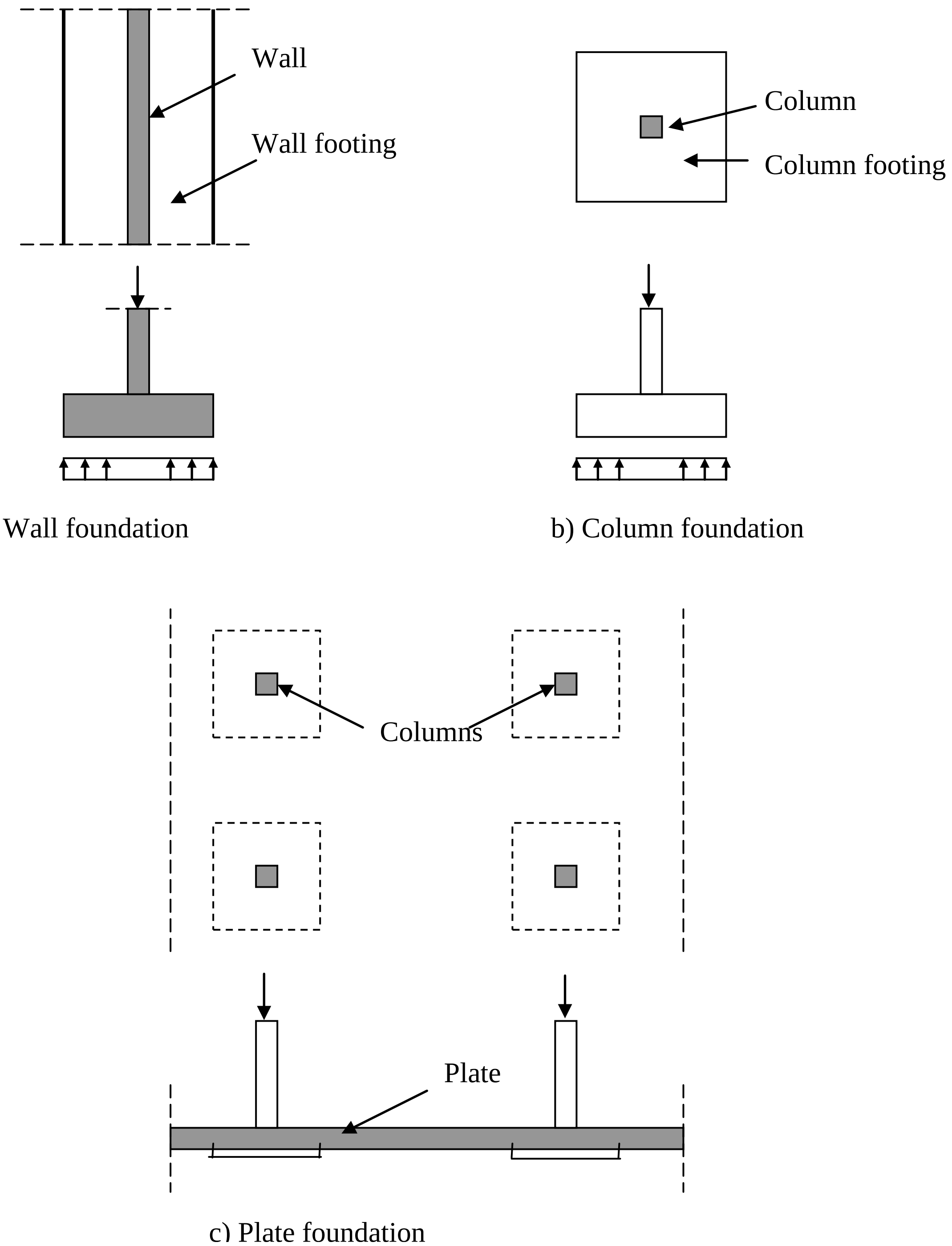


Figure 6.1.1 Direct foundation types

Wall and column foundations as shown in Figure 6.1.1a,b are the most commonly used foundation types when the ground properties are satisfactory.

Plate foundations are used in cases with bad ground properties and/or large loads.
Pile foundations are used in cases with very bad ground properties and/or very large loads.

Here, only direct foundations as wall and column foundations will be considered.

6.2 GROUND PRESSURE – REQUIRED SIZE OF FOOTING

The carrying capacity of the ground is expressed by a acceptable design ground pressure, σ_{gd} , in ULS.

The design ground pressure, σ_{gd} , depends on several factors, like soil type, depth to footing and size of footing. In general this has to be determined in each case by a geotechnical consultant.

The order of magnitude of σ_{gd} for soils may be approximated as follows (kN/m^2):

Gravel, stone……………………………….400
Coarse compressed sand……………………300
Fine compressed sand………………………200
Fine uncompressed sand……………………100
Wet gravel, wet coarse/fine sand………100 – 200
Dry firm clay…………………………...200 – 300
Less firm clay……………….………...50 – 200
Wet clay, sand/clay mix…….………..20 – 100

6.2.1 Wall foundation – required width of footing

<u>Centrically loaded foundation</u>

Figure 6.2.1 shows a section through a wall foundation which is subjected to a centric load from the structure, N_{Ed} (kN/m). N_{Ed} is load in ULS.

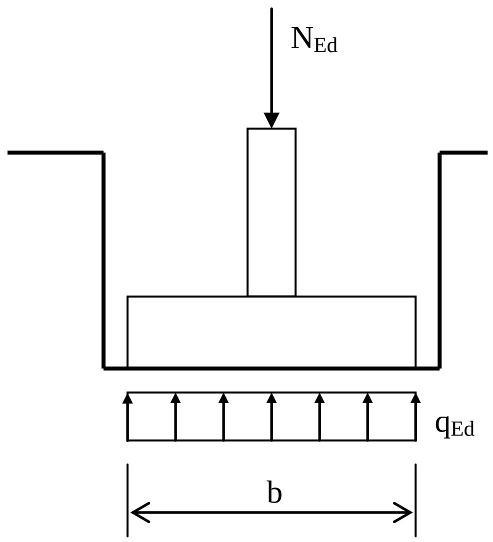


Figure 6.2.1 Centrically loaded wall foundation

Assuming that the footing is stiff, the ground pressure may be considered as uniformly distributed with magnitude:

$$q_{Ed} = \frac{N_{Ed}}{b}$$

The criterion for required width of footing is simply $q_{Ed} \leq \sigma_{gd}$, hence

$$b \geq \frac{N_{Ed}}{\sigma_{gd}} \tag{6.2.1}$$

Eccentrically loaded foundation

Often the foundation has to transfer moment from the wall, M_{Ed}, to the ground, in addition to the vertical load, see Figure 6.2.2a.

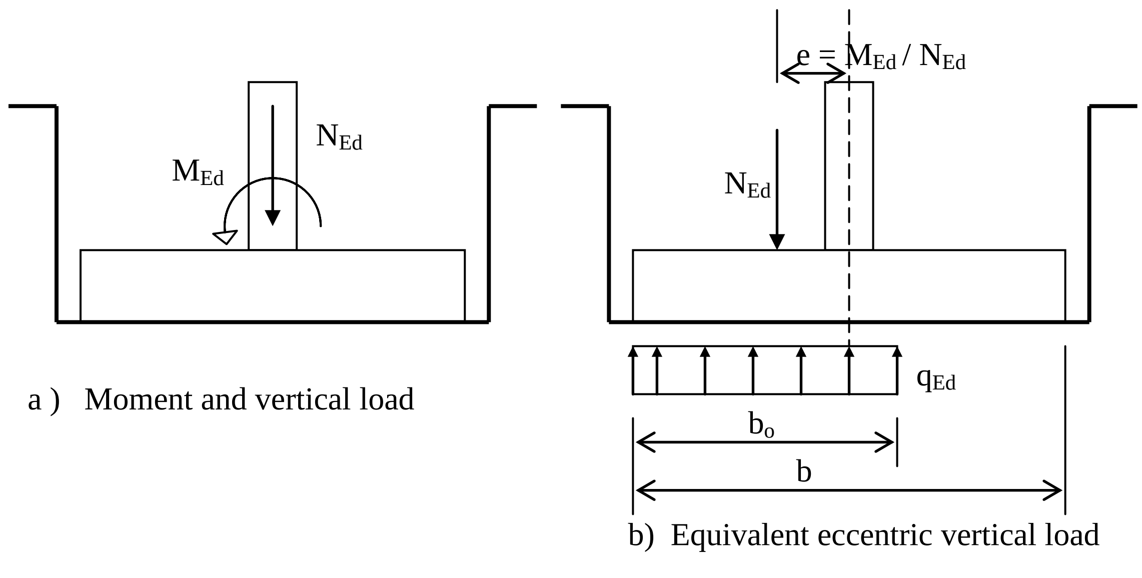


Figure 6.2.2 Eccentrically loaded wall foundation (symmetric)

Combined moment and vertical load can be considered as an equivalent eccentric vertical load, see Figure 6.2.2b.

An ”effective width” can be determined according to Eq. (6.2.1) by assuming uniform ground pressure:

$$b_o = \frac{N_{Ed}}{\sigma_{gd}} \tag{6.2.2}$$

For <u>symmetric wall foundation</u> the total width is found from:

$$\frac{b}{2} = \frac{b_o}{2} + e$$

Hence

$$b = b_o + 2e \tag{6.2.3}$$

For moment acting in one direction the obvious economic choice is an <u>asymmetric wall foundation.</u>

The required width is determined for a moment variation:

$$0 \leq M_{Ed} \leq M_{Ed,\max}$$

This is shown in Figure 6.2.3.

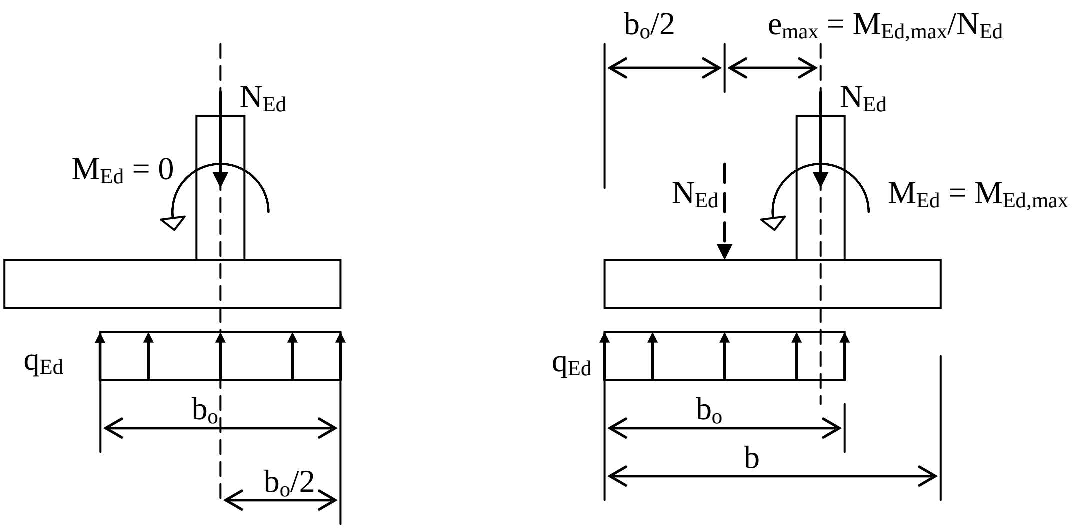


Figure 6.2.3 Eccentrically loaded wall foundation (asymmetric)

Combining the two load cases in Figure 6.2.3 gives the required width:

$$b = \frac{b_o}{2} + \frac{b_o}{2} + e_{\max} = b_o + \frac{M_{Ed,\max}}{N_{Ed}} \tag{6.2.4}$$

6.2.2 Column foundation – required size of footing

Foundations for single columns may be centrically or eccentrically loaded similar to wall foundations.

Eccentrically loaded column foundations may be subjected to moments in one or two directions.

<u>Centrically loaded column foundation</u>

The required size of the footing is determined from
Figure 6.2.**4**.

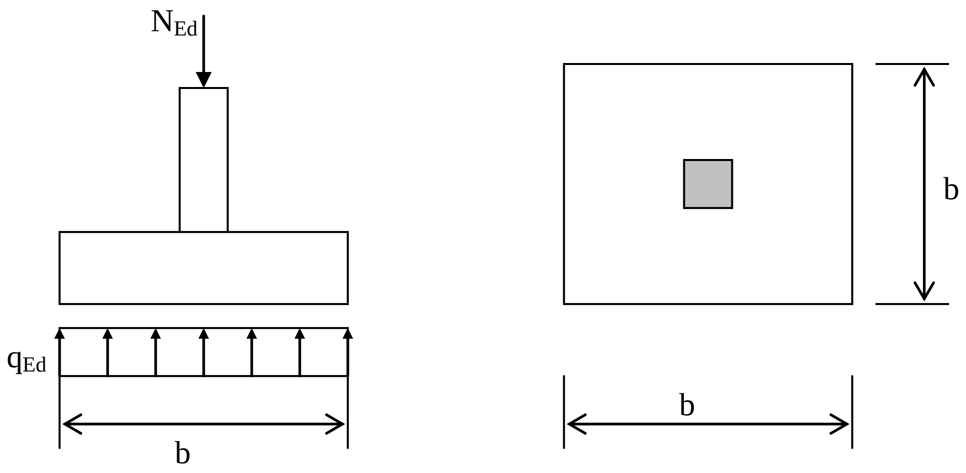


Figure 6.2.4 Centrically loaded column foundation

The required width (quadratic footing) is found from:

$$b^2 \geq \frac{N_{Ed}}{\sigma_{gd}} \Rightarrow \quad b \geq \sqrt{\frac{N_{Ed}}{\sigma_{gd}}} \tag{6.2.5}$$

Eccentrically loaded column foundation

Figure 6.2.5 shows a column foundation transferring moment in one direction to the ground:

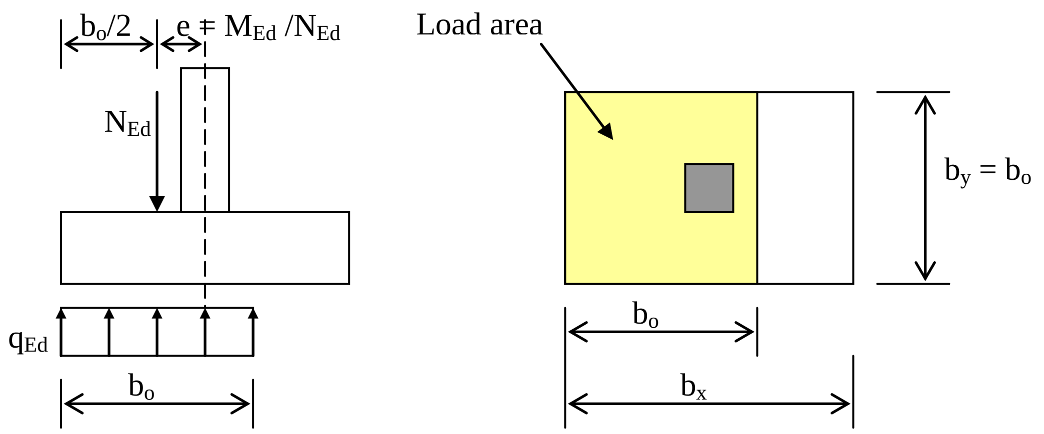


Figure 6.2.5 Eccentrically loaded column foundation – moment in one direction

If a symmetric foundation is chosen, the required area of the footing is $b_x \cdot b_y$, with

$$b_x = b_o + 2e = \sqrt{\frac{N_{Ed}}{\sigma_{gd}}} + 2 \cdot \frac{M_{Ed}}{N_{Ed}} \tag{6.2.6}$$

$$b_y = b_o = \sqrt{\frac{N_{Ed}}{\sigma_{gd}}} \tag{6.2.7}$$

If a symmetric foundation is chosen, b_y is still found from eq. (5.2.7), while b_x is determined similar to eq.(5.2.4) as

$$b_x = b_o + \frac{M_{Ed,\max}}{N_{Ed}} \tag{6.2.8}$$

The most general case, moment in two directions and asymmetric foundation is shown in Figure 6.2.6.

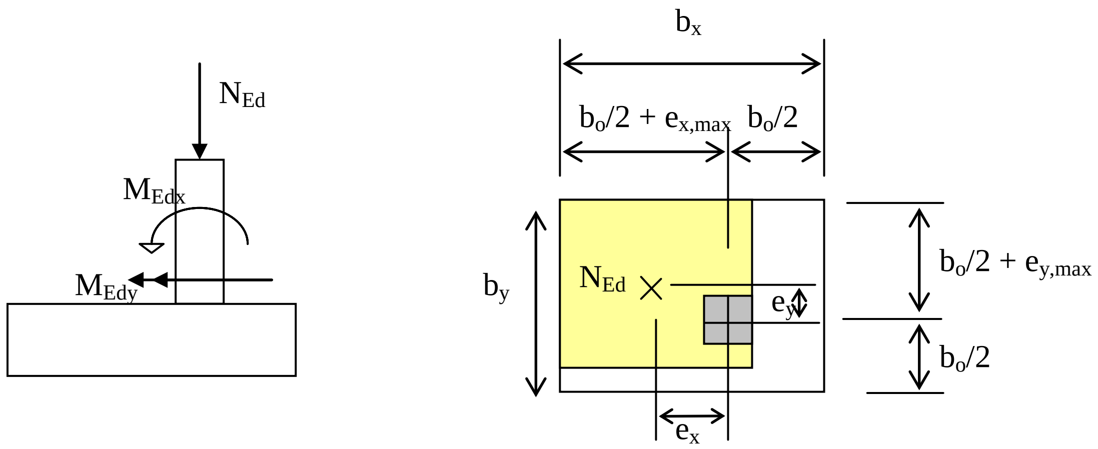


Figure 6.2.6 Asymmetric column foundation with moment in two directions

Similarly as previously for moment in one direction $b_o = \sqrt{\frac{N_{Ed}}{\sigma_{gd}}}$,

$$e_{x,\max} = \frac{M_{Edx,\max}}{N_{Ed}} \qquad e_{y,\max} = \frac{M_{Edy,\max}}{N_{Ed}} \tag{6.2.9}$$

6.3 DESIGN OF SIMPLE FOUNDATIONS

Determination of required footing size in chapter 6.2 ensures that the carrying capacity of the ground is not exceeded.

Furthermore, the footing itself has to be dimensioned, i.e. required depth and reinforcement have to be determined.

In principle, this is a simple dimensioning of a cantilever plate, however, with some special requirements for foundations.

6.3.1 Plain concrete wall foundation

According to EC2, 12.9.3, a wall foundation (strip footing) may be designed and constructed as plain concrete provided that:

$$\frac{h_F}{a} \geq \frac{1}{0{,}85} \cdot \sqrt{\frac{3\sigma_{gd}}{f_{ctd,pl}}} \tag{6.3.1}$$

or as a simplification if

$$\frac{h_F}{a} \geq 2 \tag{6.3.2}$$

where

h_F = the foundation depth
a = the width from the column face to edge of the footing
σ_{gd} = the design ground pressure
$f_{ctd,pl}$ = the design concrete tensile strength (for plain concrete)

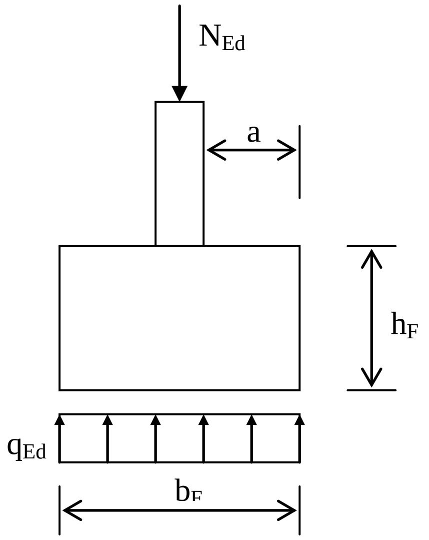


Figure 6.3.1 Plain concrete wall foundation

EC2, 12.3.1 specifies a design tensile strength for capacity calculations of plain concrete:

$$f_{ctd,pl} = 0{,}8 \cdot f_{ctk,0.05} / 1{,}5 = 0{,}533 \cdot f_{ctk,0.05} \tag{6.3.3}$$

Moment capacity, M_{Rd}

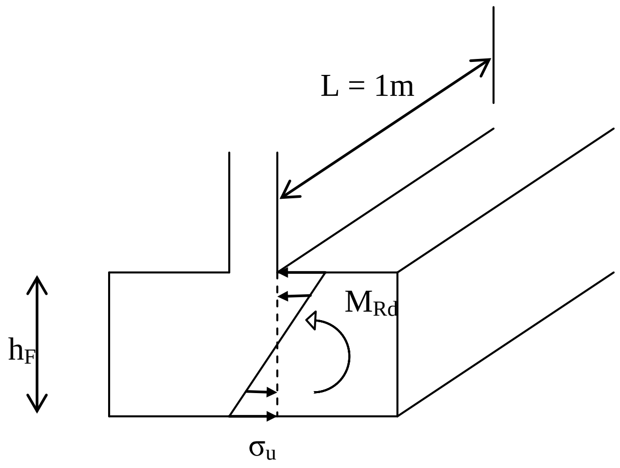


Figure 6.3.2 Linear stress distribution

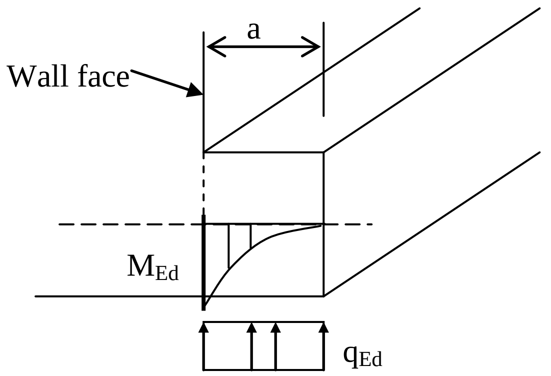
The moment capacity is found when the bottom tensile stress, σ_u , equals the tensile strength according to eq.(5.3.2), i.e.:

$$\sigma_u = \frac{M_{Rd}}{w_u} = \frac{M_{Rd}}{Lh_F^2/6} = f_{ctd,pl} \tag{6.3.4}$$

The moment capacity per meter along the strip footing (L = 1 meter) :

$$M_{Rd} = \frac{h_F^2}{6} \cdot f_{ctd,pl} \tag{6.3.5}$$

External design moment , M_{Ed} :



$$M_{Ed} = \frac{q_{Ed} \cdot a^2}{2}$$

Check that:

$$M_{Rd} \geq M_{Ed}$$

Figure 6.3.3 External design moment at wall face

6.3.2 Reinforced wall foundation

If the requirement in eq. (5.3.1) is not satisfied, the foundation must be reinforced.

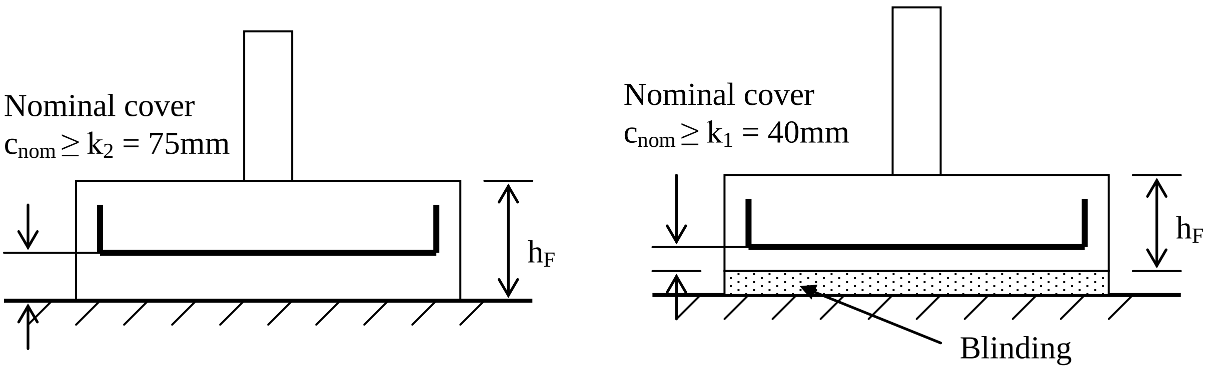
Requirements for reinforced foundations are found several places in Eurocode 2:

- ***EC2, 2.6*** *gives so-called "supplementary requirements for foundations":*
  ***2.6(2)**, **Note 2**: Simple methods ignoring the effects of ground deformation are normally appropriate for the majority of structural designs.*
  This means that soil/structure interaction normally can be ignored.
  ***2.6(3)**, Concrete foundations should be sized in accordance with*
  NS-EN 1997-1 (Geotechnical design). This can be considered satisfied by design according to chapter 5.2.

- ***EC2, Table 4.1*** *categorizes foundations in Exposure class XC2 (Wet, rarely dry).*
  Tabell NA.4.4N *requires minimum cover, $c_{min,dur}$, 25mm or 35mm for 50 or 100 years life time, respectively.*

- ***EC2, NA.4.4.1.3(1)P*** *To calculate nominal cover c_{nom}, a deviation Δc_{dev} = 10mm shall be added to the minimum cover c_{min}, hence, $c_{nom} = c_{min} + \Delta c_{dev}$*

  NA.4.4.1.3(4) *is valid for foundations:*
  For concrete cast against uneven surfaces, the nominal cover should generally be increased by allowing larger deviations in design. The increase should comply with the difference caused by the unevenness, but the nominal cover should be at least k_1=40mm for concrete cast against prepared ground (including blinding) and k_2=75mm for concrete cast directly against soil.
  This is shown in Figure 6.3.4.



a) Cast directly against ground b) Cast against prepared ground

Figure 6.3.4 Cover requirements

DESIGN / CAPACITY CONTROL

EC2, 9.8.2.1(1) *says that the main reinforcement should be anchored according to requirements of 8.4 and 8.5.*
NA.9.8.2.1(1) *requires a minimum bar diameter ϕ_{min} = 8mm.*

Figure 6.3.5 shows a section through a wall foundation with moment reinforcement.

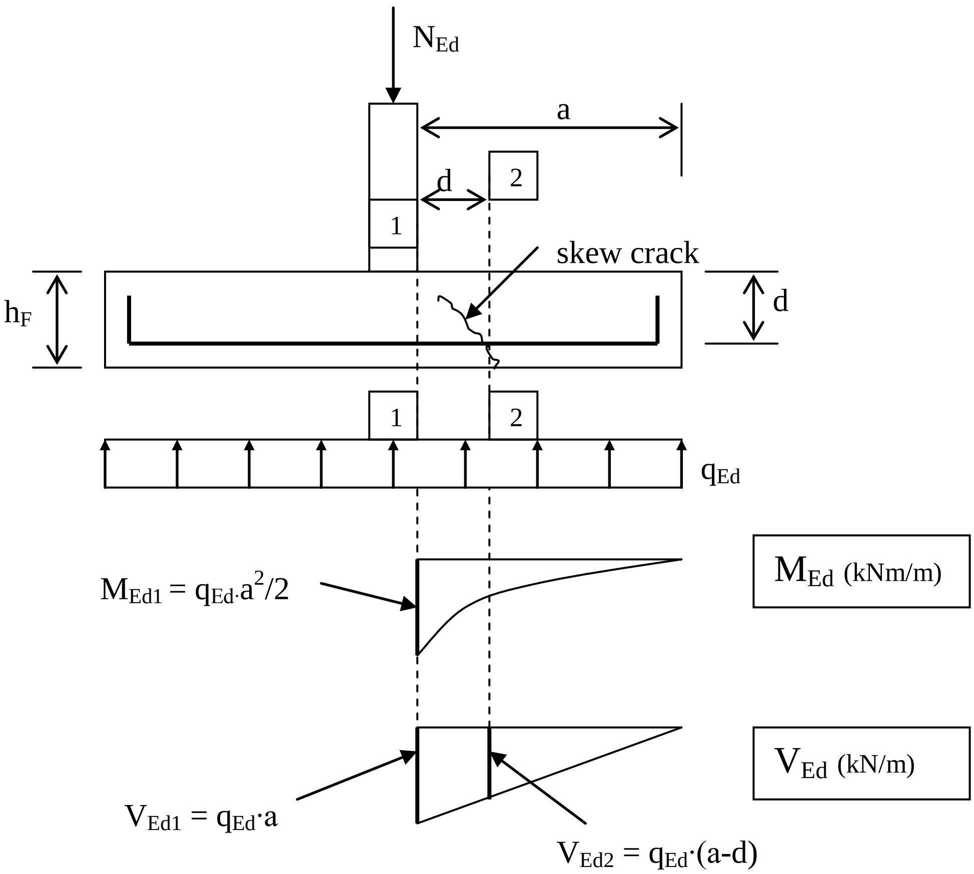


Figure 6.3.5 Reinforced wall foundation with design load actions

<u>Design for moment, Section 1 (wall face)</u>

The concrete compression zone's moment capacity for normal reinforced cross section:

$$M_{Rd} = 0{,}275 \cdot f_{cd} \cdot bd^2 \tag{6.3.6}$$

With b = 1 meter, the dimension of M_{Rd} equals that of M_{Ed} (kNm/m).

Check that $M_{Rd} \geq M_{Ed1}$

This is normally satisfied with good margin, and the compression zone is partly utilized. Hence, the internal lever arm can be approximated as

$$z \approx (1 - 0{,}17 \cdot \frac{M_{Ed1}}{M_{Rd}})d \tag{6.3.7}$$

The required bottom reinforcement in the foundation becomes

$$A_s = \frac{M_{Ed1}}{z \cdot f_{yd}} \tag{6.3.8}$$

<u>Design for shear force</u>

- Shear force at a distance d from wall face, V_{Ed2} , is checked for shear tension failure according to EC2, 6.2.1(8) , 6.2.2 and 6.2.3

- Shear force at wall face , V_{Ed1} , is checked for shear compression failure according to EC2, 6.2.3

According to EC2, 9.3.2(1), slabs with shear reinforcement should at least be 200mm thick. Normally, wall foundations are chosen with sufficient thickness to avoid shear reinforcement.

If shear reinforcement is chosen, it is most practical (and common) with skew reinforcement as shown in Figure 6.3.6.

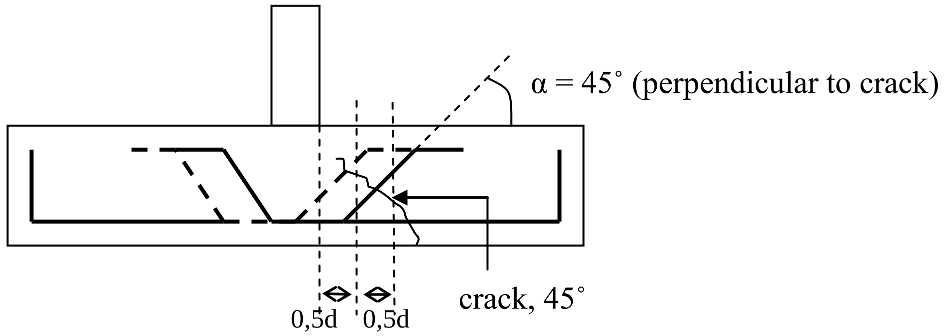


Figure 6.3.6 Shear reinforcement in foundation

Anchoring of longitudinal reinforcement bars

EC2, 9.8.2.2 shows a model that can be used to check if straight reinforcement bars are sufficiently anchored.

EC2, 9.8.2.2(1) – The tensile force in the reinforcement is determined from equilibrium conditions, taking into account the effect of inclined cracks, see Figure 6.3.7. The tensile force at a location x should be anchored in the concrete within the same distance x from the edge of the footing, i.e. $L_b < x$.

EC2, 9.8.2.2(2) – The tensile force F_s to be anchored is given by:

$$F_s = \frac{R \cdot z_e}{z_i} \tag{6.3.9}$$

where
- R is the resultant of ground pressure within distance x
- $N_{Ed,AB}$ is the vertical force corresponding to total ground pressure between sections A and B
- z_e is the external lever arm, i.e. the distance between R and the vertical force $N_{Ed,AB}$
- z_i is the internal lever arm, i.e. distance between the reinforcement and the horizontal force F_c
- F_c is the compressive force corresponding to maximum tensile force $F_{s,max}$

EC2, 9.8.2.2(3) – z_e og z_i can be determined with regard to the necessary compression zones for $N_{Ed,AB}$ and F_c , respectively
As simplifications, z_e may be determined assuming e = 0,15b and
z_i may be taken as 0,9d

EC2, 9.8.2.2(4) – If available anchoring length for straight bars L_b is not sufficient to anchor F_s within x, bars may either be bent up to increase the available length or be provided with end anchorage devices.

EC2, 9.8.2.2(5) – For straight bars without end anchorage, the minimum value of x is the most critical. As a simplification $x_{min} = h_F/2$ may be assumed.

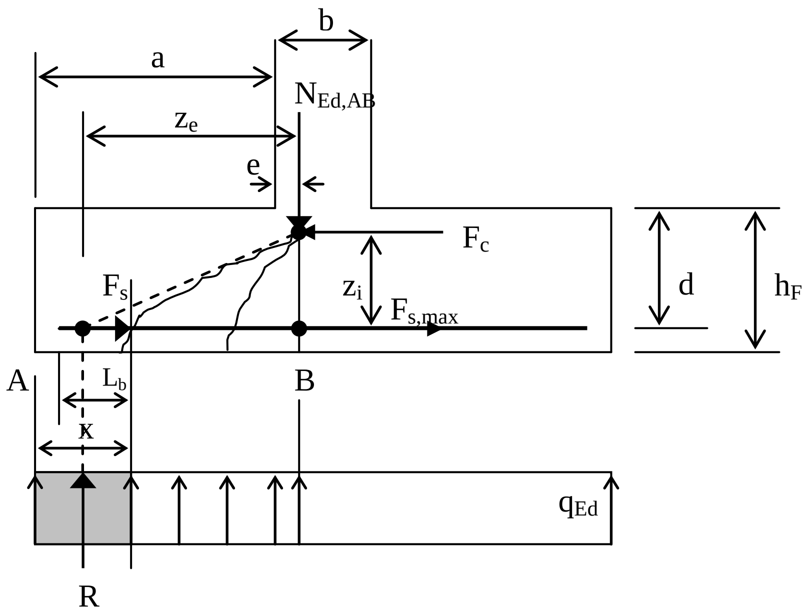


Figure 6.3.7 Model for tensile force with regard to inclined cracks

EC2, 8.4.3(2) defines the "basic anchoring length" as

$$L_{b,rqd} = 0,25 \cdot \phi \cdot \sigma_{sd} / f_{bd} \tag{6.3.10}$$

where ϕ is reinforcement diameter

$\sigma_{sd} = F_s/A_s$ is design bar stress at position from where the anchorage is measured from

f_{bd} is design bond strength according to EC2, 8.4.2(2) (normally equal to $2,25 f_{ctd}$)

The design anchoring length is given in EC2, 8.4.4(1):

$$L_{bd} = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot \alpha_5 \cdot L_{b,rqd} \tag{6.3.11}$$

For straight bars with sufficient cover, transverse reinforcement or transverse pressure is

$$L_{bd} = L_{b,rqd} \tag{6.3.12}$$

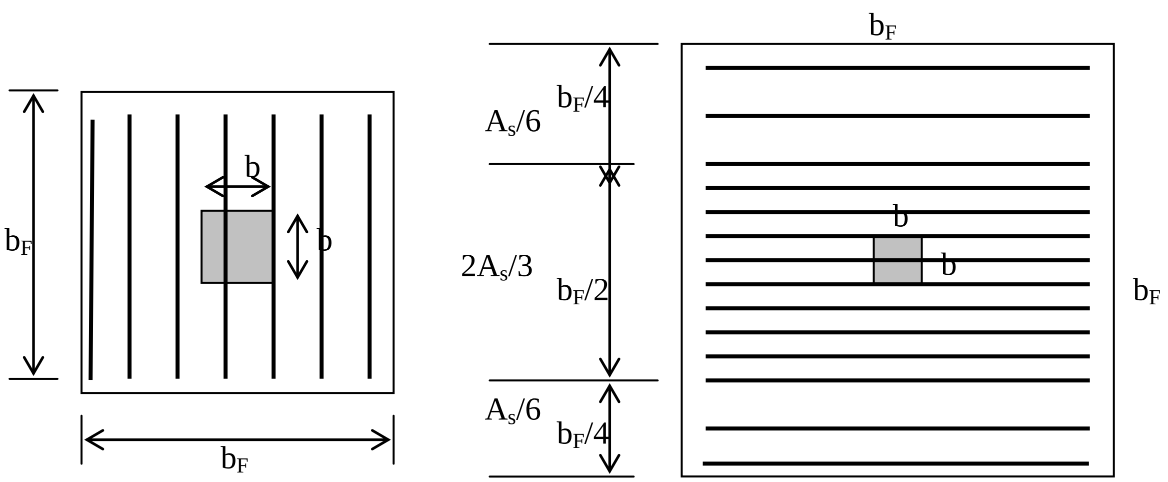
6.3.3 Column foundations

PLAIN CONCRETE COLUMN FOUNDATION

The same design method as for plain concrete wall foundations, see chapter 6.3.1.

REINFORCED COLUMN FOUNDATION

Requirements for cover and depth are as for wall foundations.
The former Norwegian design rules gave recommendations for distribution of moment reinforcement in a column foundation as shown in Figure 6.3.8. It is proposed to satisfy these recommendations also when designing based on EC2.



a) $b_F \le 5b \rightarrow$ uniform spacing b) $b_F > 5b \rightarrow 2/3$ of A_s within middle $b_F/2$

Figure 6.3.8 Recommended distribution of bottom reinforcement in column foundation

<u>Design for moment :</u>

Section at column face – similar as for wall foundation

<u>Design for bond :</u>

Section at distance x from edge of footing – similar as for wall foundation

<u>Design for shear :</u>

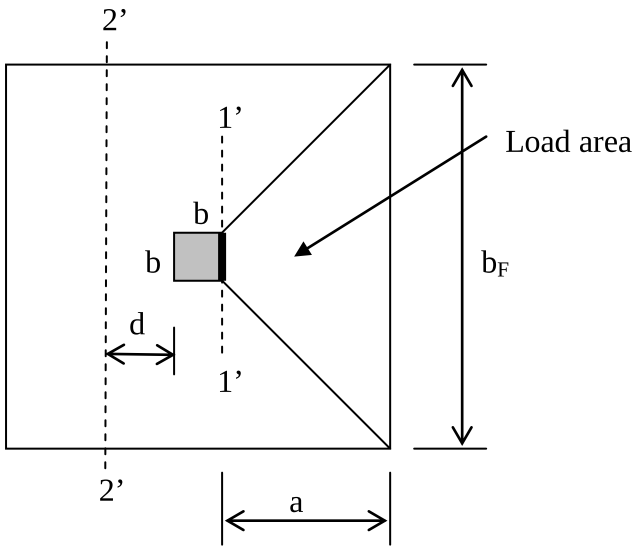


Figure 6.3.9 Critical sections for shear design

<u>Shear compression failure</u>

Design shear force at section 1'-1' in Figure 6.3.9:

$$V_{Ed1}^{*} = q_{Ed} \cdot \frac{b_F + b}{2} \cdot a \tag{6.3.13}$$

The design shear force is checked against shear compression capacity according to EC2.
This may be decisive because the critical section for shear compression is the column width.

<u>Shear tension failure</u>

The design section for bending shear is section 2'-2' in Figure 6.3.9:

<u>Design for punching :</u>

See chapter 4 on "Punching of concrete slabs".