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# Differentiable Structures on Spheres and the Kervaire Invariant

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## Summary

First some classical results and constructions from algebraic topology are discussed. Most of these results are only stated, not proved. An exception is the detailed computation of certain homotopy groups of the special orthogonal group $SO_n$. Next, following [KM63], the group $\Theta_n$ of $h$-cobordism classes of homotopy $n$-spheres is defined. In dimensions other than 4 these coincide with the group of smooth structures on $S^n$. There is an important subgroup $bP^{n+1} \subset S^n$ consisting of those homotopy spheres that occur as the boundary of a parallelizable manifold. The techniques of surgery theory are developed and applied with great effect to the study of $bP^{n+1}$. It is shown that $bP^{2k+1} = 0$, and that $bP^{4k}$ is finite cyclic, and its order is computed. The Pontryagin construction induces a monomorphism $p : \Theta_n/bP^{n+1} \to \pi_n(\mathbb{S})/Im(J)$. Using surgery theory it is shown that $p$ also is an epimorphism unless $n \equiv 2 \mod 4$. For $n \equiv 2 \mod 4$ is is shown that the cokernel is $\mathbb{Z}_2$ if and only if there exist a closed framed $n$-manifold with Kervaire invariant one. This is the Kervaire invariant one problem. A proof is given that if such an $n$ dimensional manifold exists $bP^n = 0$ and otherwise $bP^n = \mathbb{Z}_2$. The celebrated solution of the Kervaire invariant one problem for $n \neq 126$ by Hill, Hopkins and Ravenel, [HHR16] is stated, and Kervaire's proof in dimension 10 [Ker60] is given. Finally a manifold $K_n$ of dimension $n = 4k+2$ with boundary $\partial K_n = \Sigma$ a homotopy sphere and Kervaire invariant one is constructed. Thus $\Sigma = S^{4k+1}$ if and only if there exist a closed $n$-manifold with Kervaire invariant one. If $\Sigma \neq S^{4k+1}$, then filling in a disk results in a piecewise linear manifold which is not homeomorphic to any smooth manifold.



## Oppsummering

Først introduseres noen klassiske konstruksjoner og teoremer fra algebraisk topologi. De fleste resultatene formuleres uten bevis. Enkelte homotopi grupper av de spesielle ortogonale gruppene, $SO_n$, beregnes i full detalj. Deretter defineres som i [KM63] gruppen $\Theta_n$ bestående av $h$-kobordisme klasser av homotopi $n$-sfærer. For $n \neq 4$ kan $\Theta_n$ identifiseres med gruppen av glatte strukturer på $S^n$. En viktig undergruppe $bP^{n+1} \subset \Theta_n$ defineres. Kirurgi teori utvikles og anvendes med stort hell til å studere $bP^{n+1}$. Pontryagins konstruksjon induserer en monomorfi $p : \Theta_n/bP^{n+1} \to \pi_n(\mathbb{S})/Im(J)$. Med kirurgi teori vises det at $p$ også er en epimorfi med mindre $n \equiv 2 \mod 4$. Kervaire invarianten defineres, og det vises at for $n \equiv 2 \mod 4$ har $p$ en kokjerne $\mathbb{Z}_2$ hvis det eksisterer en lukket $n$ dimensjonal mangfoldighet med Kervaire invariant en, i motsatt fall vises det at $p$ er en epimorfi. Dette er Kervaire invariant en problemet: Finnes det en lukket, glatt mangfoldighet med Kervaire invariant en? Vi formulerer Hill, Hopkins og Ravenels teorem [HHR16] som løser Kervaire invariant en problemet for alle $n \neq 126$. Kervaires løsning for $n = 10$ fra [Ker60] gis i full detalj. Til slutt konstrueres for hver $n \equiv 2 \mod 4$ en $n$-dimensjonal mangfoldighet $K_n$ begrenset av en homotopi sfære $\partial K_n = \Sigma$ og med Kervaire invariant en. Hvis det finnes en lukket $n$-dimensjonal mangfoldighet med Kervaire invariant vises det at $\Sigma = S^{n-1}$. I motsatt fall bærer $\Sigma$ en eksotisk glatt struktur. Ved å lime en $n$-dimensjonal ball på $\Sigma = \partial K_n$ oppnås en stykkevis lineær mangfoldighet $M_0$ som ikke kan være homeomorf med noen glatt mangfoldighet: En slik mangfoldighet ville hatt Kervaire invariant en.

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# Contents

# 1 Introduction

There are essentially three categories of manifolds, differentiable[1] manifolds, piecewise linear manifolds and topological manifolds. A natural question is to ask how different these categories are. J.H.C. Whitehead, extending work of Cairns, showed in [Whi40] that any $C^1$ differentiable manifold carries an essentially unique piecewise linear structure. We think of this as a forgetful functor $F$ from smooth manifolds to piecewise linear ones. At first it seems intuitive that their could be an inverse construction, for note that $F$ is satisfactory close to being dense: Every continuous map $N \to M$ is homotopic to a smooth map $N \to M$. Next we can ask wether $F$ is full, i.e. can we smooth out any piecewise linear manifold? We could certainly start out with smoothing one corner, and then extend that smooth structure along the edges from that corner. Remarkably we will see that there are manifolds on which this program cannot be carried out to each corner in a compatible way! We shall also see that $F$ is far from being faithful: We follow Kervaire and Milnor [KM63] in investigating the number of smooth structures on $S^n$, a number which is almost always greater than 1. On the other hand Kirby and Siebenmann showed in [KS69] that a topological manifold $M$ of dimension at least 5 can support a piecewise linear structure if $H^4(M;\mathbb{Z}_2) = 0$, and furthermore that it is unique if $H^3(M;\mathbb{Z}_2) = 0$. Thus for $n$ at least 5, $S^n$ has a unique piecewise linear structure.

The thesis is organized as follows: First we will recall some results from algebraic topology which will be used throughout the thesis without further ado. Most results are only stated without proofs. Next we follow Kervaire-Milnor, [KM63], in defining and investigating the group $\Theta_n$, which for $n \neq 4$ can be identified with the group of diffeomorphism classes of differentiable structures on $S^n$. We see how modulo a certain subgroup, $bP^{n+1} \subset \Theta_n$ maps into a quotient of $\pi_n(\mathbb{S})$, the $n$-th stable homtopy group of the spheres. We develop the techniques of framed surgery as in [KM63] and [Mil61], and follow Kervaire-Milnor closely in employing it to the study of $\Theta_n$. However we follow [Lev85] in defining the Kervaire invariant. Finally we discuss the implication of the Kervaire invariant one problem to the structure of $\Theta_n$. Along the way we give Kervaire's construction from [Ker60] of a triangulated $4k+2$ manifold[2] which for certain values of $k$ does not admit any differentiable structures, and give his proof that it in fact does not in dimension $4k+2 = 10$. .

[1]For us differentiable will mean $C^\infty$ unless otherwise specified. The terms "smooth" and differentiable will be used interchangeably.

[2]Kervaire initially studied the manifold in dimension $4k+2 = 10$, but the generalization to arbitrary $k$ is immediate.

# 2 Some theorems and constructions

This section is somewhat chaotic in nature. It includes certain constructions and theorems that will be needed, but are outside of the scope of this thesis. Thus the reader can freely skip this section, and consult it whenever results from it are employed.

## 2.1 The Universal Coefficient Theorem

For an abelian group $A$, let $TA$ denote the torsion subgroup, $TA = \{x \in A; ax = 0 \text{ for some } a \in \mathbb{Z}\}$. We denote by $C_n(X)$ the $n$-th singular chain group of $X$, and $C^n(X) = \mathrm{Hom}_{\mathbb{Z}}(C_n, \mathbb{Z})$. When coefficients are not indicated we use $\mathbb{Z}$ coefficients. For the torsion subgroups of the (co)homology groups we will sometimes use the notation $T_n = TH_n(X)$ and $T^n = TH^n(X)$. One can also consider homology groups with coefficients in other groups than $\mathbb{Z}$.

**Theorem 2.1.1** (Theorem 3.17 in [Ran02])**.** *For any field $\mathbb{F}$ and any $n \geqslant 0$, the evaluation morphism*

$$e : H^n(X; F) \to \mathrm{Hom}_{\mathbb{F}}(H_n(X; \mathbb{F}), \mathbb{F}); f \mapsto (x \mapsto f(x))$$

*is an isomorphism.*

*For any $n \geqslant 0$ the evaluation morphism*

$$e : H^n(X) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z}); f \mapsto (x \mapsto f(x))$$

*is onto, and the morphism*

$$\overline{e} : \ker(e) = TH^n(X) \to \mathrm{Hom}_{\mathbb{Z}}(TH_{n-1}(X), \mathbb{Q}/\mathbb{Z}); f \mapsto \left(x \mapsto \frac{f(y)}{s}\right)$$
$$(f \in C^n(X), x \in C_{n-1}(X), y \in C_n(X), \mathbb{Z} \ni s \neq 0, sx = \partial y)$$

*is an isomorphism so that there is a naturally defined short exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(T_{n-1}, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(X) \xrightarrow{e} \mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z}) \longrightarrow 0\ .$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(T_{n-1}, \mathbb{Q}/\mathbb{Z})$ is a torsion group isomorphic to $T_{n-1}$, and that $\mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z})$ is free and isomorphic to $H_n(X)/T_n$. It follows from the fundamental theorem of finitely generated abelian groups that the above short exact sequence splits, but the splitting is not natural. Nevertheless this yields the abstract (as opposed to canonical or natural) isomorphism:

**Theorem 2.1.2.**

$$H^n(X) \simeq T_{n-1} \oplus H_n(X)/T_n$$

## 2.2 Poincaré Duality

Poincaré duality is a classical result from the theory of manifolds. In this thesis we shall use the term to refer to any one of the below theorems, stated without proof for later reference.

In this section manifolds are not assumed to be smooth. Let $R$ be a ring. By an $R$-orientation of a connected and closed manifold $M$ of dimension $n$ we shall mean simply an assignment of a generator $[M]$ of the $R$-module $H_n(M;R)$. We call $[M]$ the fundamental class of the $R$-oriented manifold $M$. A $\mathbb{Z}$-orientation, also just called an orientation, gives $R$ orientations for all $R$. The significance of allowing general $R$ is that even nonorientable manifolds admits a canonical $\mathbb{Z}_2$ orientation. This notion of orientability coincides with more intuitive notions, see [Hat02, p. 233] and [Lee03]. Let $T_k \subset H_k(M;\mathbb{Z})$ and $T^k \subset H^k(M;\mathbb{Z})$ be the respective torsion subgroups.

**Theorem 2.2.1** (Theorem 3.30 in [Hat02])**.** *Let $M$ be an $R$-oriented closed manifold. For each $k$ there is an isomorphism $PD : H^k(M;R) \to H_{n-k}(M;R)$. Since $H_*(-;R)$ and $H^*(-;R)$ are functors of opposite variance, co- and contra- respectively, it does not make sense to ask if $PD$ is a natural isomorphism. However $PD$ is natural in the sense that the following diagram commutes for each $f : M \to N$.*

$$\begin{tikzcd} H^k(M;R) \arrow[r, "PD"] \arrow[d, phantom] & H_{n-k}(M;R) \arrow[d, "f_*"] \\ H^k(N;R) \arrow[u, "f^*"] \arrow[r, "PD"'] & H_{n-k}(N;R) \end{tikzcd}$$

□

We shall also be using a version of Poincaré duality for manifolds with boundaries. In this case an $R$ orientation assign a generator $[M] \in H_n(M, \partial M; R)$.

**Theorem 2.2.2** (Theorem 3.43 in [Hat02])**.** *Let $M$ be an $R$-oriented manifold with boundary $\partial M = M_1 \sqcup M_2$. For each $k$, there are isomorphisms*

$$H^k(M, M_1; R) \simeq H_{n-k}(M, M_2; R)$$

*which are natural in the same sense as in Theorem 2.2.1.*

This theorem is sometimes calls Lefschetz, or Poincaré-Lefschetz, duality. Note that $M_1$ or $M_2$ can be empty, yielding

$$H_k(M, \partial M; R) \simeq H^{n-k}(M; R)$$

and the corresponding statement

$$H^k(M, \partial M; \mathbb{Z}) \simeq H_{n-k}(M; R)$$

where of course the assumptions of Theorem 2.2.2 are assumed to hold.

## 2.3 The Intersection Pairing

For proper treatments of the intersection pairing, see [ST80] and [Ran02].

The intersection pairing on a compact, $R$-oriented manifold, $M$, is most easily defined as Poincaré dual of the cup product pairing. For each $0 \leqslant k \leqslant n$, we have a pairing

$$H_k(M;R) \otimes H_{n-k}(M;R) \to R$$

defined by

$$\tau \otimes \sigma \mapsto \tau \cdot \sigma := PD^{-1}(\tau) \smile PD^{-1}(\sigma)[M]$$

where $[M] \in H_n(M, \partial M; R)$ is the fundamental class of $M$, i.e. the generator corresponding to the given orientation. In [Hat02, Prop. 3.38] it is shown (the proof is short and simple) that the cup product pairing with $\mathbb{Z}$-coeffients is non-degenerate when torsion is factored out, or if coefficients are taken in a field. Hence the same holds for the intersection pairing. The intersection pairing has a geometric interpretation, which indeed was Poincarés original definition, in the case of smooth manifolds, justifying its name. So assume now that manifolds are smooth. We say that two sub-manifolds $N_1, N_2 \subset M$ intersect transversely if for every $x \in N_1 \cap N_2$ we have $T_xM = T_xN_1 + T_xN_2$. The Intersection of two transverse sub-manifolds is again a submanifold and has codimension the sum of the codimensions. It is a fact that each pair of homology classes $\tau$ and $\sigma$ can be represented by transverse immersed sub-manifolds. Thus if the dimensions of $\tau$ and $\sigma$ are as above, the intersection of such representatives will have dimension $n - (n-k) - k = 0$. The intersection pairing simply counts the points of intersection of transverse representatives algebraically with signs given by the orientation. That this is well defined is not obvious. Nor is it obvious that this really is the same as the above definition using the cup product. Nonetheless, since this geometric definition is often more convenient we shall freely apply it.

## 2.4 The Linking Number

See [ST80] for the classic viewpoint containing what we shall need. Compare also [Ran02] and [Ran81] to see how the ideas have developed over the decades.

Let $M$ be a manifold of dimension $n = r + l + 1$. We define the linking form, giving rise to the isomorphisms $T_l \simeq T_r$ where we again use the notation $T_k \subset H_k(M;\mathbb{Z})$ for the torsion subgroup. The short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

induces a short exact sequence of chain complexes,

$$0 \longrightarrow C(M;\mathbb{Z}) \longrightarrow C(M;\mathbb{Q}) \longrightarrow C(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow 0\ ,$$

and so a long exact sequence of homology groups

$$\cdots \longrightarrow H_{l+1}(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H_l(M;\mathbb{Z}) \xrightarrow{i_*} H_l(M;\mathbb{Q}) \xrightarrow{p_*} H_l(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow \cdots .$$

All elements of finite order, i.e. all of $T_l$, are in $\ker i_* = \mathrm{im}\ \beta$. There is for each $p$ a pairing

$$H_p(M;\mathbb{Q}/\mathbb{Z}) \otimes H_{n-p}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

induced by multiplication of coefficients (any abelian group, such as $\mathbb{Q}/\mathbb{Z}$, is a $\mathbb{Z}$ module). It can be defined by

$$x \otimes y \mapsto x \cdot y := PD(x) \smile PD(y)[M]$$

just like the ordinary intersection pairing. Here $[M]$ denotes a generator of $H_n(M;\mathbb{Q}/\mathbb{Z})$ and $PD$ is the Poincaré duality map. Given $\mu \in T_r(M;\mathbb{Z})$ and $\lambda \in T_l$ we lift $\mu$ back to $\eta \in H_{n-l}(M;\mathbb{Q}/\mathbb{Z})$, i.e. $\beta(\eta) = \mu$, and define the linking number

$$L(\lambda, \mu) = \lambda \cdot \eta \in \mathbb{Q}/\mathbb{Z}.$$

**Lemma 2.4.1.** *The linking number is well defined and satisfies*

$$L(\lambda, \mu) + (-1)^{lr} L(\mu, \lambda) = 0.$$

*When $l = r$, $L$ can be computed as follows: Let $y$ be a cycle representing $\mu \in T_r$. Since $\mu$ is torsion, $s\mu = 0$ for some $s$. Hence $sy$ is a boundary, $sy = \partial w$ say. Then $L(\lambda, \mu) = \lambda \cdot w/s \in \mathbb{Q}/\mathbb{Z}$.*

*Proof.* The first statement follows from the corresponding formula for the intersection pairings. The last statement, regarding computation, follows from the universal coefficient theorem: $L$ is the adjoint of the isomorphism

$$\overline{e} : \ker(e) = T^{l+1} \to \mathrm{Hom}_{\mathbb{Z}}(T_l, \mathbb{Q}/\mathbb{Z}).$$

Explicitly that is, $L(\lambda, \mu) \mapsto \overline{e}(\lambda)(\mu)$, which is non-degenerate since $\overline{e}$ is an isomorphism. □

When $l = r$ we call $L$ the linking form on $H_l(M; \mathbb{Z})$.

## 2.5 Homotopy Theory

We give for the convenience of the reader some definitions and theorems from homotopy theory. For a thorough, but elementary, treatment of homotopy theory, see [Hat02, Chapter 4] where proofs of all the following assertions can be found. First we work in the category of pointed topological spaces. As a set we define $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$, the set of homotopy classes of maps from $(S^n, s_0)$ to $(X, x_0)$. This becomes a group in a natural way as follows: $S^n$ (or more generally, any suspension) comes with a pinching map, $p$: By definition, $S^n$ is the unit sphere in $\mathbb{R}^{n+1}$, and the intersection with $\mathbb{R}^n$ is $S^{n-1}$. Pinching is the map collapsing this "equator", $S^{n-1}$, to a point. Clearly the image of $p$ is homeomorphic to $S^n \vee S^n$, the union of two copies of $S^n$ with only the basepoint in common. We define $+$ in $\pi_n(X, x_0)$ by $[f] + [g] = [f \vee g \circ p]$, i.e. post composing pinching with $f$ on one copy of $S^n$, and $g$ on the other. There is no conflict in the point they have in common since both $f$ and $g$ take $s_0$ to $x_0$. This gives $\pi_n$ the structure of a group which is abelian for $n > 1$. If $X$ is path connected the choice of basepoint does not matter for the isomorphism type of $\pi_n(X, x_0)$, however there is in general no canonical way to identify homotopy groups with different basepoints: it requires a choice of a path, and different paths may induce different isomorphisms. In this way $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$.

Nevertheless we shall sometimes suppress the basepoint from the notation. This is justified at least when $\pi_1(X) = 0$, or more generally when $\pi_1(X)$ acts trivially on $\pi_n(X)$. If $X$ satisfies this latter less restrictive assumption we say that $X$ is $n$-simple. There are also relative homotopy groups, $\pi_n(X, A, x_0)$, for $A$ a subspace of $X$. A map $(S^n, s_0) \to (X, x_0)$ is the same thing as a map $(I^n, \partial I^n) \to (X, x_0)$. The generalization to relative homotopy groups is to allow $I^{n-1} \subset \partial I^n$ to be mapped anywhere in $A$. Again we may write $\pi_n(X, A) = \pi_n(X, A, x_0)$, at least whenever $\pi_1(X, A) = 0$. The set $\pi_n(X, A)$ is a group for $n \geqslant 2$, and abelian for $n \geqslant 3$. As in homology, we get a long exact sequence of homotopy groups for the pair $(X, A)$. (See Theorem 2.5.2 below.)

We say that $p : E \to B$ satisfies the homotopy lifting property (HLP) with respect to $X$ if for any maps making the following square commute, the dotted arrow exists such that the triangles commute.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

If $p$ satisfies the HLP with respect to all spaces we call $p$ a fibration. It is an important fact that the homotopy type of $p^{-1}(x)$ is constant on each path component of $B$ when $p$ is a fibration. We call $F = p^{-1}(b_0)$ the fiber of the fibration, where $b_0$ is the basepoint of $B$. If $p : E \to B$ satisfies HLP with respect to CW-complexes, $X$, then we call $p$ a Serre-fibration. When $E \to B$ is a (Serre-)fibration with fiber $F$ we may say things such as "$F \to E \to B$ is a (Serre-)fibration". This notation is reminiscent of short exact sequences of groups, which is not a coincidence, but we will not elaborate on this. Fibrations are in particular Serre-fibrations. It will be important that for any Serre-fibration $F \to E \to B$ there is a long exact sequence of homotopy groups,

$$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \cdots .$$

We describe the boundary map, $\partial : \pi_{n+1}(B) \to \pi_n(F)$. A map $(S^n, s_0) \to (B, b_0)$ is the same thing as a map $f : (I^n, \partial I^n) \to (B, b_0)$. We provide a lift of $f$ on $I^{n-1} \times \{0\}$ by the constant map at $e_0$, the basepoint of $E$. By the homotopy lifting property, we obtain $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "e_0"] \arrow[d] & E \arrow[d] \\ I^n \arrow[r, "f"'] \arrow[ur, "\tilde{f}"'] & B \end{tikzcd}$$

Now $\partial f : S^{n-1} \to F$ is defined to be $\tilde{f}|_{\partial I^n}$ which takes values in $F$ since $f(\partial(I^n)) = \{b_0\}$.

There is a natural homomorphism $h : \pi_n(X) \to H_n(X;\mathbb{Z}) =: H_n(X)$, called the Hurewicz homomorphism. It is most easily defined by $h([f]) = f_*(\iota)$ where $\iota$ is a generator of $H_n(S^n)$. There is of course an ambiguity in this definition: a generator has not been specified. There is also a relative Hurewicz map, $\pi_n(X, A) \to H_n(X, A)$, which is analogously defined. We say that a pair $(X, A)$ is $n$ connected if $\pi_i(X, A) = 0$ for $i \leqslant n$. (Note $\pi_n(X, x_0) = \pi_n(X, x_0, x_0)$, so we have also defined what it means for a space (as opposed to a pair of spaces) to be $n$-connected.)

**Theorem 2.5.1** (Th.4.32 in [Hat02])**.** *If $(X, A)$ is $(n-1)$-connected with $n \geqslant 2$, $A \neq \emptyset$, and $A$ is 1-connected, then $h : \pi_i(X, A) \to H_i(X, A)$ is an isomorphism for $i \leqslant n$. In other words, $H_i(X, A) = 0$ for $i < n$ and $h : \pi_n(X, A) \simeq H_n(X, A)$.* □

**Theorem 2.5.2.** *If care is taken in choosing the generators of $H_m(S^m)$, the following diagram, "the Hurewicz ladder", commutes. (Without care it might only commute up to sign in the square containing $\partial$.)*

$$\begin{tikzcd} \pi_m(A) \arrow[r] \arrow[d, "h"] & \pi_m(X) \arrow[r] \arrow[d, "h"] & \pi_m(X, A) \arrow[r, "\partial_\pi"] \arrow[d, "h"] & \pi_{m-1}(A) \arrow[d, "h"] \\ H_m(A) \arrow[r] & H_m(X) \arrow[r] & H_m(X, A) \arrow[r, "\partial_H"] & H_{m-1}(A) \end{tikzcd}$$

□

The map $\partial_\pi$ can be described as follows: an element of $\pi_m(X, A)$ is represented by a map of triples, $(I^m, I^{m-1}, J^m) \to (X, A, x_0)$. (Here $J^m = \overline{\partial I^m \setminus I^{m-1}}$, recall only $I^{m-1}$ is allowed to go anywhere in $A$.) We simply restrict the map to $I^{m-1}$ to obtain a map $(I^{m-1}, \partial I^{m-1}) \to (A, x_0)$. This induces $\partial_\pi$. Thus, in particular, applying $\partial_\pi$ to the characteristic map of a $(k+1)$ cell of a (relative) $CW$-complex yields its attaching map.

### 2.6 Suspension and $\pi_n(\mathbb{S})$

Given pointed spaces $(X, x_0)$ and $(Y, y_0)$ we define their wedge $X \vee Y \subset X \times Y$ by $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. Their smash product is defined by $X \wedge Y = X \times Y / X \vee Y$ which is given the basepoint $[X \vee Y]$. The (reduced) suspension of a pointed topological space $(X, x_0)$ is defined by $\Sigma X = S^1 \wedge X$. This is a functor

$$\Sigma : \mathrm{Top}_* \to \mathrm{Top}_*,$$

where $\mathrm{Top}_*$ denotes the category of pointed topological spaces. For if $f : X \to Y$ is a continuous map, then $id \times f$ maps $S^1 \vee X$ to $S^1 \vee Y$, hence induces a map $\Sigma f : \Sigma X \to \Sigma Y$, and this assignment is easily seen to be functorial. (Smashing is also a functor, $\mathrm{Top}_* \times \mathrm{Top}_* \to \mathrm{Top}_*$, and $\Sigma = S^1 \wedge -$.)

We get a homomorphism $E : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ since $\Sigma S^n = S^{n+1}$. ($E$ is short for “Einhängung”, the German word for suspension.) We state the Freudenthal suspension theorem.

**Theorem 2.6.1** (Corollary 4.24 in [Hat02])**.** *If $X$ is a $k$-connected $CW$-complex, then*

$$E : \pi_n(X) \to \pi_{n+1}(\Sigma X)$$

*is an isomorphism for $n < 2k+1$ and an epimorphism for $n = 2k+1$.* $\square$

Hence, for CW-complexes, $\pi_{i+k}\left(\Sigma^k(X)\right)$ is independent of $k$, assuming $k$ is sufficiently large: $\Sigma^k(X)$ is at least $k-1$ connected, hence $E$ is an isomorphism for $n = i+k < 2k-1 \iff k > i+1$. We say the homotopy groups in this range are stable. In particular this applies to spheres. We denote the $n$-th stable homotopy group of the spheres, $\pi_{k+n}(S^k)$, by $\pi_n(\mathbb{S})$.

### 2.7 Fiber Bundles, Orthogonal Groups and Stiefel Manifolds

A fiber bundle, like a fibration, consists of a projection map $p : E \to B$. This projection is however subject to more severe constraints than that of a fibration. We are requiring that each $x \in B$ admits a neighborhood $U$ such that $p^{-1}(U)$ is homeomorphic to $U \times F$ for a fixed space $F$, called the fiber of the fiber bundle. Furthermore these local trivializations are to be such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r] \arrow[d, "p"'] & U \times F \arrow[dl, "proj_U"] \\ U & \end{tikzcd}$$

It is a theorem, see [Hat02], that every fiber bundle is a fibration.

One way fiber bundles can arise is as quotients of Lie-groups. Two important Lie groups are $O_n$ and $SO_n$. $O_n$ consists of the orthogonal matrices and $SO_n \subset O_n$ consists only of the orientation preserving ones, that is, those with determinant one. In fact $SO_n$ is the component of $O_n$

containing the identity. One example of a manifold arising as the quotient of Lie-groups is the Stiefel manifold $V_{n,k}$. As a set it consists of orthonormal $k$-frames in $\mathbb{R}^n$. That is, a point in $V_{n,k}$ is a set of $k$ orthonormal vectors in $\mathbb{R}^n$. We compare this with the Lie-group $O_n$. An orthonormal $k$-frame, spanning a subspace $V$, can always be extended to an orthonormal $n$-frame since $\mathbb{R}^n = V \oplus V^\perp$, but the extension is not unique. The indeterminacy is the framing of $V^\perp \simeq \mathbb{R}^{n-k}$, i.e. an element of $O_{n-k}$. From this analysis it follows that $V_{n,k} = O_n/O_{n-k}$, at least as a set. We can topologize $V_{n,k}$ as this quotient space. We could also topologize $V_{n,k}$ as a subspace of $(\mathbb{R}^n)^k$, and the two topologies do in fact coincide.

**Theorem 2.7.1** (Corollary 14.2, and Proposition 15.5 in [VdB10]).
*Let $G$ be a Lie group, and $H$ a closed subgroup. Then $\pi : G \to G/H$ is a principal $H$ bundle and the coset space $G/H$ (which is a group if and only if $H \subset G$ is normal) has a unique structure of a smooth manifold such that $\pi$ is a smooth submersion. In particular it is a fiber bundle with fiber $H$. Furthermore, the $G$ action on $G/H$ is smooth and transitive, and for any smooth manifold $M$ on which $G$ acts transitively and smoothly, $M \simeq_G G/H_x$, where $H_x$ is the group fixing the point $x$ and $\simeq_G$ denotes a $G$-equivariant diffeomorphism.*

Thus $O_n \to V_{n,k}$ is a fiber bundle with fiber $O_{n-k}$, hence we get a long exact sequence of homotopy groups

$$\pi_{l+1}(V_{n,k}) \to \pi_l(O_{n-k}) \to \pi_l(O_n) \to \pi_l(V_{n,k}).$$

We will later use this sequence to compute $\pi_{n-k}(V_{n,k})$ for $n-k$ odd. (See lemma 2.14.3.)

## 2.8 Obstruction theory

In this section we will discuss the problem of when a map defined on the $k$-skeleton of a CW-complex can be extended to the $k+1$ skeleton. To avoid referencing basepoints, and having to deal with local coefficients, we assume $Y$ to be connected and $k$-simple for every relevant $k$ throughout this section. Much can be said also if this assumption is dropped. We include some proofs to give a taste of the theory, although what follows is not a self-contained exposition. For a thorough introduction, see [Hu59, Chapter VI]. See also [Hat02].

### 2.8.1 Homotopy Theoretic Obstruction Theory

Let $X$ be a CW-complex with $k$-skeleton $X^k$, let $Y$ be a topological space, and let $f : X^k \to Y$ be a continuous map. Let $E_\phi \subset X^{k+1}$ be a $(k+1)$-cell with attaching map $\phi : S^k \to X^k$. Then $f$ is defined on $\mathrm{Im}(\phi)$. Extending $f$ over the interior of $E_\phi$ is equivalent to providing a null-homotopy of $f \circ \phi : S^k \to Y$ which represents an element of $\pi_k(Y)$. We thus define a function taking $(k+1)$-cells of $X$ to elements of $\pi_k(Y)$ by

$$c^{k+1}(f)(E_\phi) := [f \circ \phi].$$

This is exactly the data of a cellular cochain, to be made precise below. We call $c^{k+1}(f)$ the obstruction cochain of $f$, or the obstruction to extending $f$ over $X^{k+1}$. Denote the $n$-th cellular chain group of $X$ by $\Gamma_n(X) = H_n(X^n, X^{n-1})$. This group is free abelian on the $n$-cells of $X$. There is a boundary map

$$\partial : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

coming from the long exact sequence of homology groups of the triple $(X^n, X^{n-1}, X^{n-2})$. It is a theorem [Hat02, Th.2.27] that the chain complex $\Gamma_\bullet(X)$ is quasi-isomorphic to $C_\bullet(X)$, the singular chain complex of $X$, i.e. there is a chain map inducing isomorphisms on homology. Furthermore the dual complex with $G$ coefficients, $\Gamma^\bullet(X;G) = \mathrm{Hom}(\Gamma_\bullet(X), G)$, is quasi-isomorphic to $C^\bullet(X;G)$.

**Lemma 2.8.1.** *For each continuous map $f : X^k \to Y$, the obstruction cochain $c^{k+1}(f)$ defines an element of $\Gamma^{k+1}(X;\pi_k(Y))$.*

*Proof.* We have defined $c^{k+1}(f)$ on a basis of the $\mathbb{Z}$-module $\Gamma_{k+1}(X;\mathbb{Z})$. It is basic module theory that there is a unique extension to a homomorphism from the whole group. □

Note that the assumption that $Y$ is $k$-simple in the case $k = 1$ reduces to the assumption that $\pi_1(X)$ is abelian, for the action of $\pi_1(X)$ on itself is by conjugation.

**Lemma 2.8.2.** *$c^{k+1}(f)$ is a cocycle.*

Intuitively there should be a simple proof of this lemma exploiting the fact that $c^{k+1}(f)$ resembles a boundary operation, and so ought to take cycles to 0. To make a formal proof we must do some trickery with the Hurewicz homomorphism.

**Lemma 2.8.3.** *$c^{k+1}(f) = f_* \circ \partial \circ h^{-1}$ where $h$ is the Hurewicz map and $\partial$ is the boundary map of the long exact sequence of homotopy groups of pairs*

$$\Gamma_{k+1}(X) = H_{k+1}(X^{k+1}, X^k) \xleftarrow{h} \pi_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial} \pi_k(X^k) \xrightarrow{f_*} \pi_k(Y)\ .$$

*Proof.* First we need to argue that this is well defined. Note that the pair $(X^{k+1}, X^k)$ is $k$-connected so that the Hurewicz map, $h : \pi_{k+1}(X^{k+1}, X^k) \longrightarrow H_{k+1}(X^{k+1}, X^k)$, would be an isomorphism if $k$ is at least 1 by Hurewicz, Theorem 2.5.1, if $X^k$ is simply connected. When $X^k$ is not simply connected it is still true that $h$ is an epimorphism. The kernel is generated by elements of the form $\gamma \cdot [g] - [g]$ for $\gamma \in \pi_1(X^k)$ and $\cdot$ the usual action of $\pi_1(X^k)$ on $\pi_k(X^{k+1}, X^k)$, see [Hat02, Th. 4.37]. The latter of the equalities

$$f_* \circ \partial(\gamma \cdot [g] - [g]) = f_*(\gamma) \cdot f_*(\partial[g]) - f_*(\partial[g]) = 0$$

holds since $Y$ is $k$-simple, and the former by naturality. Hence $f_* \circ \partial \circ h^{-1}$ is a well defined map $\Gamma_{k+1}(X) \to \pi_k(Y)$. Let $E$ be a $k+1$-cell of $X$ with attaching map $g$, and consider $E$ as an element of $\Gamma_{k+1}$. Then by definition $c^{k+1}(f)(E) = [f \circ g]$. On the other hand, if $h(\tau) = E$, then $\partial(\tau) = [g]$ and so $f_* \circ \partial(\tau) = [f \circ g]$. □

*Proof of Lemma 2.8.2.* We consider the following diagram.

$$\begin{tikzcd}
H_{k+2}(X^{k+2}, X^{k+1}) \arrow[d, "\partial"'] & \pi_{k+2}(X^{k+2}, X^{k+1}) \arrow[l, "h"'] \arrow[d, "\partial"] & & \\
H_{k+1}(X^{k+1}) \arrow[d, "i"'] & \pi_{k+1}(X^{k+1}) \arrow[l, "h"'] \arrow[d, "i'"] & & \\
H_{k+1}(X^{k+1}, X^k) & \pi_{k+1}(X^{k+1}, X^k) \arrow[l, "h"'] \arrow[r, "\partial_1"] & \pi_k(X^k) \arrow[r, "f_*"] & \pi_k(Y)
\end{tikzcd}$$

where the $h$'s are Hurewicz maps, and the $\partial$'s and the $i$'s are maps appearing in long exact sequences of homology and homotopy groups of the appropriate pairs. The small squares commute since they both appear in Hurewicz ladders, see Theorem 2.5.2. Note that the composition along the left column coincides with the cellular boundary map $\partial_2 : H_{k+2}(X^{k+2}, X^{k+1}) \to H_{k+1}(X^{k+1}, X^k)$. To prove $\delta c^{k+1}(f) = 0$ it suffices to show $(\delta c^{k+1}(f)) \circ h = 0$ since $h$ is an epimorphism. Using commutativity of the diagram we get

$$\begin{aligned} \delta(c^{k+1}(f)) \circ h &= c^{k+1}(f) \circ \partial_2 \circ h \\ &= f_* \circ \partial_1 \circ i' \circ \partial \end{aligned}$$

where we have used that $c^{k+1} = f_* \circ \delta_1 \circ h^{-1}$. But the composition of $\partial_1$ and $i'$ is 0 since these are consecutive maps in the long exact sequence of homotopy groups of the pair $(X^{k+1}, X^k)$. Hence $c^{k+1}(f)$ is a cocycle. □

Thus the obstruction cochain $c^{k+1}(f)$ is in fact a cocycle and as such represents a cohomology class $\mathfrak{o}^{k+1}(f) \in H^{k+1}(X; \pi_k(Y))$ called the obstruction class. Clearly $\mathfrak{o}^{k+1}(f)$ is 0 on $X^{k-1}$, so we can consider it as a relative class $\mathfrak{o}^{k+1}(f) \in H^{k+1}(X, X^{k-1}; \pi_k(Y))$. We have the following result:

**Theorem 2.8.4.** *A given map $f : X^k \longrightarrow Y$ is homotopic to a map $f' : X^k \longrightarrow Y$ which extendends over $X^{k+1}$ and satisfying $f'|_{X^{k-1}} = f|_{X^{k-1}}$ if and only if $\mathfrak{o}^{k+1}(f) = 0$.*

*Proof.* This is Theorem 5.1 in [Hu59, Chapter IV]. □

### 2.8.2 Sections of Fibrations

We briefly discuss obstructions to extending partially defined sections of fibrations. Let $p : E \to B$ be a fiber bundle with fiber $F$ and $B$ a $CW$-complex. Let $B^k$ denote the $k$-skeleton of $B$. Suppose we have a section, $f : B^k \to E$, i.e, $p \circ f$ is the ordinary inclusion of $B^k$ into $B$. We could as before consider the obstruction $c^{k+1}(f) \in H^{k+1}(B; \pi_k(E))$, but we can in fact do better. Given a $(k+1)$-cell of $B$, $K$, with attaching map $\phi$ and characteristic map $\psi$, consider the following commutative diagram.

$$\begin{tikzcd} & S^k \arrow[r, "f\circ\phi"] \arrow[dl] \arrow[d] & E \arrow[d, "p"] \\ S^k \times I \arrow[r, "\pi"'] \arrow[urr, dotted] & D^{k+1} \arrow[r, "\psi"'] & B \end{tikzcd}$$

We can consider $\psi$ as a null homotopy of $\phi$, $S^k \times I \to B$, by precomposing it with the map collapsing one end, $\pi$. By the homotopy lifting property we get a map $\psi' : S^k \times I \to E$ satisfying $p \circ \psi' = \psi \circ \pi$, i.e. the image of $f \circ \phi$ can be homotoped into $p^{-1}(\psi(0)) = F$. Hence we have defined an element of $C^{k+1}(B; \pi_k(F))$. This element is again a cocycle and defines an obstruction class, $\mathfrak{o}^{k+1}(f) \in H^{k+1}(B; \pi_k(F))$. Strictly speaking we should be using local coefficients here, and keeping track of basepoints. But by assuming that $\pi_1(B) = 0$, or at least that the action on the homotopy groups of the fiber is trivial, we are on safe ground again.

**Theorem 2.8.5.** *A section $f : B^k \to E$ of a fibration $p : E \to B$ with fiber $F$ can be extended over $B^{k+1}$ if and only if $\mathfrak{o}^{k+1}(f) = 0 \in H^{k+1}(B; \pi_k(F))$.*

## 2.9 Vector-Bundles

For a thorough and excellent account including complete proofs of the theory outlined in this section, see [MS74, Chapter 1].

A real vector bundle is a fiber bundle $\pi : E \to B$ where each fiber $\pi^{-1}(b)$ is endowed with the structure of a real vector space. The local triviality assumption is slightly strengthened: Any $b \in B$ must admit a neighborhood $U$ and a homeomorphism $F : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $F|_{\pi^{-1}(b)}$ is a vector space isomorphism for each $b$, where $\{b\} \times \mathbb{R}^n$ is given the obvious vector space structure. The $n$ appearing here is fixed throughout the bundle and called the dimension of the vector bundle. Sometimes it will be indicated in the declaration of the bundle. For example we may say, "let $E$ be a n-plane bundle over $B$".
Let $\pi : E \to B$ and $\pi' : E' \to B'$ be vector bundles. We call a continuous map $F : E \to E'$ a bundle map if it is fiber preserving, i.e. there is a map $B \to B'$ making the following diagram commute

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r] & B' \end{tikzcd}$$

and furthermore $F|_{\pi^{-1}(b)}$ is a vector space isomorphism for each $b$. If a bundle map admits an inverse bundle map it is called a bundle isomorphism. Suppose that we have a map $f : X \to B$. Then we get a bundle over $X$ as in the following pullback diagram:

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

This can be spelled out explicitly as

$$f^*(E) := \{(x, v) \in X \times E | f(x) = \pi(v)\}$$

with projection map $(x, v) \mapsto x$, and bundle map $(x, v) \mapsto v \in E$. It is not difficult to show that the domain of every bundle map is a pullback of the target bundle. Thus one might wonder about the smallest set of bundles $\{E_i\}_{i \in I}$ such that every bundle is a pullback of some $E_i$. It turns out to be the case that only one bundle is needed in each dimension. This bundle is called the universal bundle and denoted $\gamma^n \to G_n$. The construction of this bundle is included because it is beautiful, but the proof that it has the stated properties is tedious, and is referred to [MS74]. $G_n$ is the set of all $n$-dimensional subspaces of $\mathbb{R}^\infty$. We first topologize the set $G_n(\mathbb{R}^{n+k}) := \{V \subset \mathbb{R}^{n+k} | \dim V = k\}$. This is done by identifying a subspace $V \subset \mathbb{R}^{n+k}$ with the orthogonal projection onto it. Thus $G_n(\mathbb{R}^{n+k})$ is a subset of $\mathrm{End}(\mathbb{R}^{n+k})$ and as such inherits a topology. The inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces inclusions $G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k+1})$. The limit of these inclusions is $G_n$ which is topologized as such. That is, $U \subset G_n$ is closed if for each $k$, $U \cap G_n(\mathbb{R}^{n+k})$ is closed. Finally $G_n$ comes with a bundle, $\gamma^n$. A point of $G_n$ is a $n$-space in $\mathbb{R}^\infty$. We can therefore define the bundle $\gamma^n$ to consist of pairs $(x, v)$ where $x$ is a point of $G_n$, that is a space, and $v$ is a vector in that space. Formally that is

$$\gamma^n = \{(x, v) \in G_n \times \mathbb{R}^\infty | v \in x\}.$$

Similarly $G_n(\mathbb{R}^{n+k})$ comes with a bundle $\gamma^n_{n+k}$. We can now state the classification theorem for vector bundles.

**Theorem 2.9.1.** *Let $B$ be a paracompact space. Then the assignment $[f] \mapsto f^*(\gamma^n)$ is a bijection from $[B, G_n]$ to isomorphism classes of $n$-vector bundles over $B$.*

We call the map $f$ the classifying map of the bundle $f^*(\gamma^n)$. Thus any bundle admits a classifying map which is unique up to homotopy.

The space $G_n(\mathbb{R}^{n+k})$ is called a Grassmannian manifold. It is canonically isomorphic to $G_k(\mathbb{R}^{n+k})$ through the map sending a space $V$ to its orthogonal complement. The space $G_1(\mathbb{R}^{n+1})$ is usually denoted $\mathbb{R}P^n$ and called the real projective space of dimension $n$. Since each line is uniquely determined by the two points it intersects the sphere in, $\mathbb{R}P^n$ is a quotient of $S^n$ by the antipodal relation. We can pull the bundle $\gamma^n_{n+1}$ back to $S^n$ by the quotient map. A bundle map to $\gamma^n_{n+1}$ factors through this bundle over $S^n$ if and only if it is orientable. In other words, the bundle over $S^n$ classifies oriented line bundles. The trick with taking orthogonal complements apply also here, so there is also a bundle over $S^n$ that classifies oriented $n$-plane bundles. It can be described as pairs $(x, v)$ where $x$ is a point of $S^n$ and $v$ is a vector orthogonal to $x$. Thus:

**Lemma 2.9.2.** *$S^n$ classifies oriented $n$-vector bundles.*

A trivialization of a vector bundle $p : E \longrightarrow B$ is an isomorphism $E \simeq B \times \mathrm{R}^n$. We think of this as a continuous choice of basis for each fiber. We make this rigorous in two different ways:

1. A trivialization is a set of $n$ continuous sections of $E$ which form a basis at every point.
2. We may form a fiber bundle, $V(E) \to B$ with fiber over $x$ the set of bases of $E_x$. For each $x$, $V(E)_x$ can be identified with $GL(E_x)$ which is homeomorphic to $GL(n, \mathbb{R})$ and has the homotopy type of $SO_n$. The space $V(E)$ is an open subset of $E^n = E \times E \times \cdots \times E$ and therefore inherits a topology and if $E$ is a manifold, a smooth structure. A section $B \to V(E)$ is a trivialization of $E$.

The two notions agree: Given a section $s$ of $V(E)$, post composing with the projection maps $\pi_i : V(E) \subset E^n \to E$ we get sections of $E$ as in 1. $\pi_1 \circ s, \cdots, \pi_n \circ s$. Conversely, $n$ everywhere linearly independent sections of $E$ is in particular a section of $E^n$. That they are everywhere linearly independent shows that the corresponding section of $E^n$ factors through the inclusion of $V(E)$. It is clear that either notion is the same as a bundle isomorphism $E \simeq B \times \mathbb{R}^n$. The advantage of 2. is that it gives us the following lemma:

**Lemma 2.9.3.** *Suppose $E \to B$ is a vector bundle of dimension $n$ over a CW-complex $B$. Given a trivialization $F$ of $E$ over the $k$-skeleton of $B$ the obstruction to extending $F$ is a cohomology class*

$$c^{k+1}(F) \in H^{k+1}(B, \pi_k(SO_n)).$$

*Proof.* This is an application of Lemma 2.8.5. □

## 2.10 Oriented Vector Bundles over Spheres

Suppose we have a map $f : S^{n-1} \to SO_k$. We think of $S^{n-1}$ as the equator of $S^n$. Over each hemisphere, $D_n$ and $D_s$ ($s$ and $n$ being "northern" and "southern", irrespectively) we consider the trivial $k$-plane bundle $D_i \times \mathbb{R}^k$. To form a vector bundle over $S^n$, we need to glue these bundles together over the equator, $S^{n-1}$. We do this using $f$. We identify $(x, v) \in \partial D_s \times \mathbb{R}^k$ with $(x, f(x)v) \in \partial D_n \times \mathbb{R}^k$. Denote the resulting space by $E_f$. The map $E_f \to S^n$ given by $(x, v) \mapsto x$ is a vector bundle and we say that $f$ is a clutching function for $E_f$. Let $\mathrm{Vect}^k_+(S^n)$ denote the set of isomorphism classes of oriented $k$-plane bundles over $S^n$.

**Theorem 2.10.1** (Proposition 1.14 in [Hat09])**.** *The map $\pi_{n-1}(SO_k) \to \mathrm{Vect}^k_+(S^n)$ defined by $[f] \mapsto [E_f]$ is well defined and a bijection. We identify these objects with each other and allow ourselves to write $\xi \in \pi_{n-1}(SO_k)$ for a bundle $\xi$.*

□

We view $SO_k$ as the subset of $SO_{k+1}$ leaving the last coordinate of $\mathbb{R}^{k+1}$ fixed. Let $i$ denote the inclusion.

**Lemma 2.10.2.** $[E_{i\circ f}] = [E_f \oplus \epsilon]$

*Proof.* The clutching function for $E_{i\circ f}$ is $f \times id$ by the above description of $i$. But $f \times id$ is also the clutching function of $E_f \oplus \epsilon$. □

## 2.11 Framed cobordism

We call $TM \oplus \epsilon$ the stable tangent bundle of $TM$. A framing of a manifold $M$ is a trivialization of the stable tangent bundle of $M$, $\phi : M \to V(TM \oplus \epsilon)$. Thus a framed manifold is in particular stably parallelizable, but a stably parallelizable manifold can be given different framings. Therefore requiring a framing as part of the data gives us a richer category. We say that two framed manifolds, $(M, \phi)$ and $(N, \psi)$ are framed cobordant if $M \coprod N$ bounds a manifold $W$ and the following condition holds. There exists a trivialization of $TW$, $F : W \to V(TW)$, such that $F|_M = \phi$ and $F|_N = \psi$ where we interpret $\epsilon$ as the trivial line bundle of outward normal vector fields. It is clear that framed cobordism is an equivalence relation. We denote the set of framed cobordism classes of manifolds of dimension $n$ by $\Omega^{fr}_n$. It is a group under the operation induced by the disjoint union (sometimes called disjoint sum): $[M] + [N] = [M \sqcup N]$.

## 2.12 The Pontryagin Construction

Suppose we have a framed $n$-manifold, $M$, and let $i : M \to \mathbb{R}^{2n+2}$ be an embedding into Euclidean space. We get a trivialization $f$ of the corresponding normal bundle, $NM$. An element of the stable group $\phi(M, f) \in \pi_{2n+2}(S^{n+2})$ is defined as follows. $NM$ is diffeomorphic to a tubular neighborhood $U$ of $M$. Hence $f$ gives rise to a diffeomorphism $f' : U \simeq M \times \mathbb{R}^{n+2}$. We post compose $f'$ with the projection onto $\mathbb{R}^{n+2}$. We extend this map to a map from all of $\mathbb{R}^{2n+2}$ to the sphere $S^{n+2}$ by sending the complement of $U$ to the point at $\infty$. This is continuous since points of $U$ sufficiently close to $\partial U$ are sent to points of $\mathbb{R}^{n+2}$ of arbitrarily large norm. Finally this extends to the required map $S^{2n+2} \to S^{n+2}$ since a neighborhood of $\infty \in S^{2n+2}$ is sent to $\infty \in S^{n+2}$. This construction turns out to be invariant under framed cobordism: A framed cobordism induces a homotopy. It is a deep theorem that $\phi$ induces an isomorphism.

**Theorem 2.12.1** (The Pontryagin construction)**.** *complement The map $p : \Omega_n^{fr} \to \pi_n(\mathbb{S})$ defined by $p([M,F]) = \phi(M,F)$ is a group isomorphism for each $n$.*

The inverse of $\phi$ is defined as follows. Each class $\alpha \in \pi_{2n+2}(S^{n+2})$ contains a smooth map $g$. By Sards theorem the critical values of $g$ have measure 0, so $g$ admits regular values, call one $x_0$. It is a theorem that the inverse image of a regular value under a smooth map is a manifold of codimension equal to the dimension of the codomain. Thus $M := g^{-1}(x_0)$ is a manifold of dimension $2n+2-(n+2) = n$.[3] It is a theorem that if $U$ is a sufficiently small neighborhood of $x_0$, $g^{-1}(U)$ is a tubular neighborhood of $M$. Of course $U$ is parallelizable, and a trivialization of $TU$ induces a diffeomorphism $g^{-1}(TU) \simeq M \times \mathbb{R}^{n+2}$. Since $g^{-1}(TU)$ is diffeomorphic to the normal bundle of the embedding, we have obtained a framing of $M$. The proof that these constructions indeed are well defined and inverse to each other is long and tedious. The interested reader is referred to [Pon55]. We will use the following.

**Corollary 2.12.2.** *If $p(M,F) = 0$, then $M$ bounds a parallelizable manifold such that the trivialization of the tangent bundle restricts to $F$ on $M$.*

*Proof.* $p$ is a group isomorphism, so $(M,F)$ is the 0 element of $\Omega_n^{fr}$. That is, $(M,F)$ is framed cobordant to $\emptyset$, i.e. there exist a framed manifold $V$ such that $\partial V = M \sqcup \emptyset$ inducing the given framing $F$. That $V$ is parallelizable is an application of Lemma 3.3.3. □

## 2.13 Some Theorems of Whitney

Whitney proved a number of theorems concerning the existence of certain kinds of immersions and embeddings. We state the results we need, without proofs which can be found in [Whi44a], [Whi36] and [Whi44b]. See also [Ran02]. The application of most of these theorems follow [Lev85], where similar theorems are stated.

**Theorem 2.13.1.**
*For $2n \leqslant m$ every map $f : N^n \to M^m$ is homotopic to an immersion $f : N \looparrowright M$, and for $2n+1 \leqslant m$ any two homotopic immersions are regularly homotopic.* □

**Theorem 2.13.2.** *For $n \geqslant 3$ and simply connected $M$ every map $f : N^n \to M^{2n}$ is homotopic to an embedding $N \hookrightarrow M$.* □

**Theorem 2.13.3.**
*If $f : (N^{n+1}, \partial N) \to (M^{2n+1}, \partial M)$ is a map with $n \geqslant 2$ and $f|_{\partial N}$ an embedding, then $f$ is homotopic to an immersion through a homotopy leaving the boundary fixed.* □

A regular homotopy between immersions is a homotopy $f_t$ which for each value of $t$ is an immersion. Theorem 2.13.3 has the following corollary:

**Theorem 2.13.4.** *Any two homotopic embeddings $f_0, f_1 : N^n \to M^{2n}$ are regularly homotopic.*

[3] If $g^{-1}(x_0) = \emptyset$ we consider $\emptyset$ as a representative of the trivial cobordism class. This is consistent, $g^{-1}(x_0) = \emptyset$ forces $\alpha = 0$.

*Proof.* Denote the homotopy by $(x,t) \mapsto F_t(x)$. Then we get a map $N \times I \to M \times I$ defined by $(t,x) \mapsto (F_t(x), t)$. Restricted to the boundary this is an embedding, hence homotopic to an immersion: $G : N \times I \to M \times I$. Projecting onto $M$ yields a regular homotopy from $f_0$ to $f_1$. □

We consider the self intersection number of a map $f : N^n \to M^{2n}$. If every self intersection of $f$ is transverse, then the self intersection number can be defined as a signed count of the intersection points with signs induced by the orientations. If $n$ is odd, or $M$ or $N$ is non-orientable, then it turns out that the intersection number is well defined on regular homotopy classes of maps, only up to parity.

**Theorem 2.13.5.** *Let $f : N^n \to M^{2n}$ be an immersion. $f$ has self intersection number* 0 *(defined modulo 2 if $n$ is odd or $N$ or $M$ is non-orientable) if and only if $f$ is regularly homotopic to an embedding.* □

### 2.14 Some Homotopy Groups of $SO_n$

This section differs from the previous ones in that it contains complete proofs of all of its propositions. The choice to include these proofs was made because of how essential these computations are for different aspects of the theory presented in the ensuing sections. Furthermore it is important to not become lost in abstractions and forget about hands on computations which usually is where the real mathematics happen. Quite a few of the results below are similar to results appearing in the "preliminaries" section of [Lev85], but the discussions and lines of proof strives to be more elementary.

$O_{n+1}$ acts on $S^n \subset \mathbb{R}^{n+1}$ by matrix multiplication, and the action is smooth and transitive. Let $e_{n+1}$ be the last basis vector of $\mathbb{R}^{n+1}$. The group fixing $e_{n+1}$ is the subgroup $O_n \subset O_{n+1}$. This is the standard inclusion of $O_n$ into $O_{n+1}$. Forming the colimit of the inclusions $O_1 \subset O_2 \subset \cdots$ we obtain the orthogonal group $O$. Similarly $SO_1 \subset SO_2 \subset \cdots$ has colimit $SO$. We have a fiber bundle $O_n \xrightarrow{i_n} O_{n+1} \xrightarrow{j_n} S^n$ where $j_n$ is defined by $j_n(T) = Te_{n+1}$. This fiber bundle shows inductively that the $k$-skeleton of $SO_n$ depends only on $SO_{k+2} \subset SO_n$. We combine long exact sequences of homotopy groupshowss corresponding to these fiber bundles for various values of $n$ to obtain the following diagram:

$$
\begin{tikzcd}
 & & \vdots \arrow[d, "(j_{k+1})_*"] & & \vdots \arrow[d] & \\
 & & \pi_{k+1}(S^{k+1}) \arrow[d, "\partial"] \arrow[dr, dotted] & & \pi_{k-1}(SO_{k-1}) \arrow[d] & \\
\cdots \arrow[r] & \pi_k(SO_k) \arrow[r, "(i_k)_*"] & \pi_k(SO_{k+1}) \arrow[r, "(j_k)_*"] \arrow[d] & \pi_k(S^k) \arrow[r] \arrow[dr, dotted] & \pi_{k-1}(SO_k) \arrow[r, "i_*"] \arrow[d] & \cdots \\
 & & \pi_k(SO_{k+2}) \arrow[d] & & \pi_{k-1}(S^{k-1}) \arrow[d] & \\
 & & \vdots & & \vdots &
\end{tikzcd}
$$

**Lemma 2.14.1.**

$$(j_k)_* \circ \partial = \begin{cases} 0 & \text{even} \\ 2 & k \text{ odd} \end{cases}$$

*and*

$$(j_k)_*\pi_k(SO_{k+1}) = \begin{cases} 0 & k \text{ even} \\ 2\pi_k(S^k) & k \text{ odd},\ k \neq 1,3,7 \\ \pi_k(S^k) & k = 1,3,7 \end{cases}$$

Before proving this, let us use it to make some computations.

**Lemma 2.14.2.**

$$I \ \operatorname{coker}(\pi_k(SO_k) \longrightarrow \pi_k(SO)) = \begin{cases} 0 & k \neq 1,3,7 \\ \mathbb{Z}_2 & k = 1,3,7 \end{cases}$$

$$II \ \ker(\pi_{k-1}(SO_k) \longrightarrow \pi_{k-1}(SO)) = \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & k = 1,3,7 \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

*Proof.* Since $\pi_k(SO) = \pi_k(SO_{k+2})$ we need only consider $(i_{k+1})_* \circ (i_k)_*$. First of all, since $\pi_k(S^{k+1}) = 0$, $(i_{k+1})_*$ is always onto. If $k$ is even, then $(j_k)_* = 0$, and so $(i_k)_*$ is also onto. Next suppose $k \neq 1,3,7$ is odd. We chase the diagram: Let $(i_{k+1})_*(b) = a \in \pi_k(SO_{k+1})$. Say $(j_k)_*(b) = c$. Then for some $d \in \pi_{k+1}(S^{k+1})$, $c = (j_k)_* \circ \partial(d)$. Now $(i_{k+1})_*(b - \partial(d)) = a$ and $(j_k)_*(b - \partial(d)) = c - c = 0$, hence $b - \partial(d) \in \operatorname{im}\,(i_k)_*$, and we conclude $(i_{k+1})_* \circ (i_k)_*$ is onto as required. If $k = 1,3,7$, the diagram chase does not quite go through: It is possible to find $d \in \pi_{k+1}(S^{k+1})$ such that $(j_k)_* \circ \partial(d) = c$ if and only if $c$ is even.

To prove II, if $k$ is even, then $(j_{k-1})_* \circ \partial = 2$ shows that $\operatorname{im}(\partial) \simeq \mathbb{Z}$. Next, if $k = 1,3,7$, then $(j_k)_*$ is onto, so $(\partial : \pi_k(S^k) \to \pi_{k-1}(SO_k)) = 0$. Finally, if $k \neq 1,3,7$, then $\operatorname{im}(\partial) = \mathbb{Z}_2$. □

We are now in a position to prove the following theorem, initially proved by Eduard Stiefel in [Sti35].

**Theorem 2.14.3.** *If $n-k$ is odd and $k > 1$, then $\pi_{n-k}(V_{n,k}) = \mathbb{Z}_2$.*

*Proof.* We consider the long exact sequence:

$$\pi_{n-k}(SO_{n-k}) \longrightarrow \pi_{n-k}(SO_n) \xrightarrow{p_*} \pi_{n-k}(V_{n,k}) \xrightarrow{\partial} \pi_{n-k-1}(SO_{n-k}) \longrightarrow \pi_{n-k-1}(SO_n)$$

Note that $\pi_{n-k}(SO_n) = \pi_{n-k}(SO)$ if $k > 1$. If $n-k$ is odd, then by Lemma 2.14.2, exactly one of $\partial$ and $p^*$ is 0, and the other one has kernel or cokernel $\mathbb{Z}_2$, respectively. Thus the sequence breaks up as

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_{n-k}(V_{n,k}) \longrightarrow 0$$

or

$$0 \longrightarrow \pi_{n-k}(V_{n,k}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0\ .$$

In either case it immediately follows that $\pi_{n-k}(V_{n,k}) \simeq \mathbb{Z}_2$.[4] □

[4] The same argument shows $\pi_{n-1}(V_{n,n-1}) = \mathbb{Z}$, but we already knew this: $V_{n,n-1} = S^{n-1}$.

*Proof of Lemma 2.14.1.* The following proof is in essence extracted from [MT91, Th.6.13, Ch.4]. The group $O_{k-1}$ acts on the fiber bundle $O_k \to O_{k+1} \to S^k$ in the sense that it acts on each space, as a subgroup on $O_k$ and $O_{k+1}$, and trivially on $S^k$, and the maps are equivariant. Quotienting out this action we obtain a new fiber bundle. We have the following commutative diagram, relating the two fiber bundles.

$$\begin{tikzcd}
 & O_k \arrow[rr, "j_{k-1}"] \arrow[d] & & S^{k-1} \arrow[dl] \\
I^{n-1} \arrow[r, dotted, "e"] \arrow[d, dotted] & O_{k+1} \arrow[r] \arrow[d, "j_k"] & SO_{k+1}/SO_{k-1} \arrow[dl] & \\
I^n \arrow[r, dotted] \arrow[ur, dotted] & S^k & &
\end{tikzcd}$$

The dashed maps, together with commutativity and the definition of the boundary maps (preceding Theorem 2.5.1), proves that the boundary maps $\Delta_n : \pi_n(S^k) \to \pi_{n-1}S^{k-1}$ and $\partial_n : \pi_n(S^k) \to \pi_{n-1}(SO_k)$ are related by $\Delta_n = (j_{k-1})_* \circ \partial_n$. Since

$$\pi_k(S^k) \longrightarrow \pi_{k-1}(S^{k-1}) \longrightarrow \pi_{k-1}(SO_{k+1}/SO_{k-1}) \longrightarrow \pi_{k-1}(S^k) = 0$$

is exact, $\pi_{k-1}(S^{k-1})/\mathrm{Im}(\Delta_k) \simeq \pi_k(SO_{k+1}/SO_{k-1})$ and so the following lemma immediately implies the first statements.

**Lemma 2.14.4.**

$$\pi_k(SO_{k+2}/SO_k) = \begin{cases} 0 & m \text{ even} \\ \mathbb{Z}_2 & m \text{ odd} \end{cases}$$

Before proving Lemma 2.14.4 we finish the proof of Lemma 2.14.1. If $k$ is odd,

$$(j_k)^* \pi_k(SO_{k+1}) \supset \Delta_{k+1}\pi_{k+1}(S^{k+1}) = 2\pi_k(S^k).$$

We consider a representative of the generator of $\pi_k(O_{k+1})$, $f$, and the self map of $S^k$, $j_k \circ f = (x \mapsto f(x, e_{k+1}))$. This composition being homotopic to the identity is equivalent to the assignment $\mu(x, y) = f(x)(y)$ endowing $S^k$ with an $H$-space multiplication with identity (up to homotopy) $e_{k+1}$. By Adams theorem regarding elements of Hopf invariant one, proven in [Ada60], $S^k$ can support such an endowment if and only if $k = 1, 3, 7$. This finishes the proof in the case $k$ odd. If $k$ is even, then $k-1$ is odd, and as we just saw, $\Delta_k = j_{k-1} \circ \partial_k$ is a monomorphism, namely multiplication by $\pm 2$ after choosing generators. Thus $\partial_k$ is a monomorphism, and by exactness of

$$\pi_k(O_{k+1}) \xrightarrow{j_k} \pi_k(S^k) \xrightarrow{\partial_k} \pi_{k-1}(O_k) ,$$

$j_k$ must equal 0. □

*Proof of Lemma 2.14.4.* For each $v \in S^k$, let $r_k(v) \in O_{k+1}$ be defined by $r_k(v)(v') = v' - 2\langle v, v' \rangle v$, the reflection fixing the orthogonal complement of $v'$. Denote the upper and lower hemisphere of $S^k$ by $E_+^k$ and $E_-^k$ respectively, $E_\pm^k = \{(x_1, \cdots, x_{k+1}) \in S^k | \pm x_{k+1} \geqslant 0\}$. We allow ourselves to write $E_+^k \cap E_-^k = S^{k-1} \subset \mathbb{R}^k \subset \mathbb{R}^{k+1}$. We claim that the map $r_\pm : (E_\pm^k, S^{k-1}) \to (S^k, e_{k+1})$ defined by $v \mapsto r_k(v)e_{k+1}$ is a relative homeomorphism. It is clear that for $v \in S^{k-1}$, $r_k(v)e_{k+1} = e_{k+1}$ and that for each $v \in S^k$, $r_k(v) = r_k(-v)$. It suffices to show that the inverse map $S^k \backslash \{e_{k+1}\} \to \dot{E}^k_{\pm}$ is continuous. Let $\alpha$ be the angle between $v$ and $e_{k+1}$, i.e. $\cos(\alpha) = \langle e_{k+1}, v\rangle$. Then the angle between $r_k(v)e_{k+1}$ and $e_{k+1}$ is $\pi + 2\alpha$:

$$\langle r_k(v)(e_{k+1}), e_{k+1}\rangle = 1 - 2\cos^2(\alpha) = -\cos(2\alpha)$$

where the last equality is standard trigonometry. From this description we see that $r_{\pm}^{-1}$ can be expressed as

$$-\sin(\theta)x - \cos(\theta)e_{k+1} \longmapsto \pm(\sin(\theta/2)\, x + \cos(\theta/2)\, e_{k+1}),$$

for $x \in S^{k-1}$, proving the claim. Let $s$ be the composition $s = p \circ r_{m+1}$ as in the following diagram:

$$\begin{tikzcd}
(E^{m+1}_+, S^m) \arrow[r, "(r_{m+1}, r_m)"] \arrow[dr, "r_+"'] & (O_{m+2}, O_{m+1}) \arrow[r, "(p, j_m)"] \arrow[d, "proj'"] & (O_{m+2}/O_m, S^m) \arrow[dl, "proj"] \\
& (S^{m+1}, e_{m+1}) &
\end{tikzcd}$$

Here *proj* and *proj'* are quotienting out the action of $O_{m+1}$. Commutativity is immediate from the definitions. Consider next this diagram:

$$\begin{tikzcd}
H^m(S^m) \arrow[r, "\delta"] \arrow[d, "(j_m \circ r_m)^*"'] & H^{m+1}(O_{m+2}/O_m, S^m) \arrow[r, "(proj^*)^{-1}"] \arrow[d, "s^*"'] & H^{m+1}(S^{m+1}, e_{m+1}) \arrow[dl, "r_+^*"] \\
H^m(S^m) \arrow[r, "\delta'"] & H^{m+1}(E^{m+1}_+, S^m) &
\end{tikzcd}$$

The triangle commutes, and so does the square since maps of pairs yields commutative ladders on cohomology. Since $(proj^*)^{-1}$ and $r_+^*$ are isomorphisms, so is $s^*$. Since $\delta'$ is also an isomorphism, the following statements are equivalent:

$$(j_m \circ r_m)^*(\text{generator}) = \pm(1 + (-1)^{m+1}) \cdot \text{generator} \tag{1}$$

$$\delta(\text{generator}) = \pm(1 + (-1)^{m+1}) \cdot \text{generator} \tag{2}$$

We have almost already proven (1): We have seen that $j_m \circ r_m$ factors through the folding map $S^m \vee S^m \to S^m$ since it takes $S^{m-1}$ to $e_{m+1}$. Furthermore, we have seen that the corresponding induced map $S^m \to S^m \vee S^m$ maps $(E^m_{\pm}, S^{m-1})$ homeomorphicly onto $(S^{m-1}, e_k)$, hence we need only determine if the maps $r_{\pm}$ have the same, or opposite degree. Note that $r_-$ is the composition $(E^m_-, S^{m-1}) \xrightarrow{-1} (E^m_+, S^{m-1}) \xrightarrow{r_+} (S^m, e_m)$ (this is just the formula $r_k(v) = r_k(-v)$ reappearing). Hence we need only determine the degree of the antipodal map, i.e. in which path component of $O(m+1)$ $-1$ lies. This data is revealed by determinant, $\det(-I_{m+1}) = (-1)^{m+1}$. Thus we have proven (1), and (2) follows. We know $H^{m+1}(O_{m+2}/O_m, S^{\cdot}) \simeq H^{m+1}(S^{m+1}, e_{m+1}) \simeq \mathbb{Z}$. This data along with (2) and the exact sequence

$$H^m(S^m) \xrightarrow{\delta} H^{m+1}(O_{m+2}/O_m, S^m) \longrightarrow H^{m+1}(O_{m+2}/O_m) \longrightarrow H^{m+1}(S^m) = 0$$

proves that $H^{m+1}(O_{m+2}/O_m) = \mathbb{Z}_2$ for $m$ odd and 0 for $m$ even. Using the corollary of the universal coefficient theorem that $H^i \simeq H_i/T_i \oplus T_{i-1}$ where $T_i \subset H_i$ is the torsion subgroup, see [Hat02, Cor.3.3], we deduce $H_m(O_{m+2}/O_m) = \mathbb{Z}_2$ for $m$ odd, and 0 for $m$ even. Since $O_{m+2}/O_m$ is $(m-1)$-connected, Hurewicz, Theorem 2.5.1, finishes the proof:

$$\pi_m(O_{m+2}/O_m) = H_m(O_{m+2}/O_m).$$

□

# 3 Homotopy Spheres

This section is closely following [KM63]. From now on we strive to go into full detail in every proof.

A homotopy $n$-sphere is a smooth oriented compact manifold $M$, without boundary and of the same homotopy type as $S^n$. Note that this has nothing to do with the smooth structure on $M$: We are only requiring that the homotopy equivalences, and the corresponding homotopies, be continuous. It follows from the $h$-cobordism theorem for topological manifolds [Sma61, Theorem A] that for $n$ at least 5, any homotopy sphere is in fact homeomorphic to a sphere. Hence for $n$ at least 5 a homotopy sphere is just a sphere with a smooth structure, which may be exotic, that is, not diffeomorphic to the standard smooth structure on $S^n$.

We say that two smooth oriented manifolds $M$ and $N$ are h-cobordant if there exists a smooth manifold $W$ with $\partial W = M \sqcup -N$ where the $-$ indicates orientation reversal. Furthermore both $M$ and $-N$ are required to be deformation retracts of $W$, i.e. the inclusion maps have to be homotopy equivalences. Being h-cobordant is stronger than being homotopy equivalent since $W$ is required to be smooth. The h-cobordism theorem applies also in the category of smooth manifolds [Sma62]: For $n$ at least 5, $n$-manifolds are h-cobordant (in the smooth sense) if and only if they are diffeomorphic. Thus for $n$ at least 5, the h-cobordism classes of homotopy $n$-spheres coincides with the diffeomorphism classes of smooth structures on $S^n$. This is a good reason to care about the following definition:

**Definition 3.0.5.** *$\Theta_n$ is the set of h-cobordism classes of homotopy n-spheres.*

## 3.1 The Connected Sum

There is a natural group structure for $\Theta_n$. The group operation is induced by the connected sum, denoted by #, which is defined for arbitrary connected manifolds as follows. Given connected $n$-dimensional manifolds, $M$ and $N$, choose embeddings

$$i_0 : D^n \to M \qquad\qquad i_1 : D^n \to N$$

which are orientation preserving and orientation reversing respectively. Then form $M\#N$ from

$$M\backslash i_0(0) \bigsqcup N\backslash i_1(0)$$

by for each $0 < t < 1$ and $u \in S^{n-1}$ identifying $i_0(tu)$ with $i_1((1-t)u)$. Note that $i_0(tu) \mapsto i_1((1-t)u)$ is an orientation preserving diffeomorphism from $i_0(\mathrm{int}D^n\backslash 0)$ to $i_1(\mathrm{int}D^n\backslash 0)$, hence $M\#N$ is locally euclidean and get an induced smooth structure. Seccond countability is also immediate. We need to check Hausdorffness. It is clear that points in $M\#N$ corresponding to points in $M$ or $N$ that can be separated from interior$i_s(D^n)$, $s = 1, 2$, can be separated from each other. For points in interior $i_s(D^n)$ there is also nothing to check: we have just identified two copies of the cylinder $S^{n-1} \times (0, 1)$. Thus we are left with checking that points on $i_s(\partial D^n)$ can be separated from other points. Locally, around a point $p \in i_1(\partial D^n) \subset M$, the gluing looks like the identification of $\dot{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 > 0\}$ with its image under the inclusion into $\mathbb{R}^n$. This is clearly Hausdorff. Note that there was something to prove around $i_s(\partial D^n)$: If we had defined an operation similar to the connected sum, but identified $i_0(tu)$ with $i_1(tu)$, $0 < t < 1$

$u \in S^{n-1}$, then around points of $i_s(\partial D^n)$ the identification would look like the identification space of two copies of $\mathbb{R}^n$, by identifying the respective copies of $H^n$. This would not be Hausdorff: Points of the form $(0, x_2, \cdots, x_n)$ could then not be separated from the corresponding point in the other copy.

**Lemma 3.1.1.** *The connected sum of two manifolds is well defined, i.e.i.e. independent of the choice of embeddings $i_0$ and $i_1$.*

To prove this result we state without proof the following.

**Theorem 3.1.2** (Theorem B in [Pal60])**.** *Given any two orientation preserving embeddings $i, j : D^n \to V^n$, there exists a diffeomorfism $\phi : V \to V$ such that $j = \phi \circ i$.* □

*Proof of Lemma 3.1.1.* Suppose we formed the connected sum using the embeddings $j_0 : D^n \to M$ and $j_1 : D^n \to N$ instead of $i_0$ and $i_1$, and denote this version by $M\#'N$, just to keep the two apart. This clumsy notation will never be used again after this proof. By the theorem $j_k = \phi_k \circ i_k$ for orientation preserving diffeomorphisms $\phi_k$, $k = 0, 1$. We consider the map

$$M\backslash i_0(0) \bigsqcup -N\backslash i_1(0) \xrightarrow{\quad F \quad} M\backslash j_0(0) \bigsqcup -N\backslash j_1(0)$$

defined as the restriction of $\phi_0 \sqcup \phi_1$ to the stated domain. It is clear that this is an orientation preserving diffeomorphism. It is also clear that the identification on the left hand side to form $M\#N$ coincides under $F$ with the identification on the right hand side to form $M\#'N$. Hence $F$ induces a diffeomorphism of the two a priori different constructions. □

When we form the connected sum $M\#N$ from $M \sqcup N$ for triangulated $n$-manifolds $M$ and $N$, we are only changing the $n$-skeleton. Hence the homology of $M\#N$ is the same as for $M \sqcup N$, except in dimensions $n$ and $n-1$. Any immersion representing classes of $H_i(M)$ and $H_{n-i}(N)$ gives immersions representing the corresponding classes in $H_i(M\#N)$ and $H_{n-1}(M\#N)$, and they intersect no more in $M\#N$ than they did in $M \sqcup -N$. Hence we conclude:

**Lemma 3.1.3.** *The intersection pairing splits over $\#$. More precicely, for $i < n-1$, we have*

$$\psi_i : H_i(M\#N) \simeq H_i(M) \times H_i(N)$$

*and letting $p_M$ and $p_N$ denote projections onto $H_i(M)$ and $H_i(N)$ respectively,*

$$\alpha \cdot \beta = p_M(\psi_i(\alpha)) \cdot p_M(\psi_i(\beta)) + p_N(\psi_i(\alpha)) \cdot p_N(\psi_i(\beta))$$

*for $1 < i < n-1$.* □

If both $M$ and $N$ have connected boundaries there is another useful construction. Let $H^n \subset D^n$ be the set consisting of points $(x_1, \cdots, x_n)$ such that $x_n \geqslant 0$. We consider $D^{n-1} \subset H^n$ as consisting of those $x = (x_1, \cdots, x_n)$ for which $x_n = 0$. Suppose we are given embeddings

$$i_M : (H^n, D^{n-1}) \to (M, \partial M) \qquad\qquad i_N : (H^n, D^{n-1}) \to (N, \partial N)$$

which are orientation preserving and reversing, respectively. Then we form $W = (M\#N, \partial M\#\partial N)$ from

$$M\backslash i_M(0) \bigsqcup N\backslash i_N(0)$$

by identifying $i_M(tu)$ with $i_N(1-t)u$ for $u \in S^{n-1} \cap H^n$ and $0 < t < 1$. It is entirely analogous to the preceding discussion that $W$ is a differentiable manifold, and clearly it has boundary $\partial W = \partial M \# \partial N$. We call $W$ **the connected sum of $M$ and $N$ along the boundary**. If $M$ or $N$ do not have connected boundaries the diffeomorphism type of $W$ might depend on which boundary components we choose to form the connected sum along. Hence we shall specify that choice whenever it is relevant. We prove one more important and interesting property of the connected sum:

**Lemma 3.1.4.** *There is a cobordism $\Omega_{M,N}$ between $M \sqcup N$ and $M \# N$. If the manifolds are framed, the cobordism can also be framed. In particular, if $M$ is (framed) cobordant to $N$, then $M \# -N$ belongs to the trivial (framed) cobordism class.*

*Proof.* We start out with $I \times M$ and $I \times N$. We glue the two together with a connected sum along the boundaries $\{1\} \times M$ and $\{1\} \times N$ to form the space $\Omega_{M,N}$. Then one component of $\partial \Omega_{M,N}$ is $M \# N$. The other boundary components of $I \times M$ and $I \times N$, namely $\{0\} \times M$ and $\{0\} \times N$, which we identify with $-M$ and $-N$ respectively, have not been altered. Therefore

$$\partial \Omega_{M,N} = M \# N \bigsqcup -(M \sqcup N).$$

For the second statement, if $M$ and $N$ are framed, then we get an induced trivialization of $T(I \times M)$ and $T(I \times N)$ by identifying the trivial bundle $\epsilon$ with the tangential direction along $I$. The diffeomorphism used to glue $I \times M \backslash (1,p)$ together with $I \times N \backslash (1,q)$ gives an isomorphism also of the tangent bundles over those areas, hence patching together the trivializations to a trivialization of $T\Omega_{M,N}$.

The last statement is just composition of cobordisms. For concreteness, connect $M$ and $-N$ with the given cobordism to obtain a manifold with only the boundary component $M \# N$. □

## 3.2 The Group $\Theta_n$

Following [KM63] we shall spend some time on rigorously proving that $\Theta_n$ becomes a group under the operation induced by $\#$. We strive to give every detail of every proof.

**Lemma 3.2.1.** *The sphere $S^n$ serves as identity for $\#$.*

*Proof.* Note that it is clear from Lemma 3.1.4 that forming the connected sum with $\# S^n$ (with the standard framing) at least preserves the (framed) cobordism class since $S^n$ is a boundary.

Let $M^n$ be a smooth manifold, and let $2D^n$ denote the closed disk of radius 2. Let

$$i_0 : 2D^n \to M$$

be an orientation preserving embedding. We consider $S^n$ as the quotient of two copies of $\mathrm{int} D^n$, $\dot{D}^n_1$ and $\dot{D}^n_2$, under the relation that for $0 < t < 1$ and $u \in S^{n-1}$, $tu$ in the first copy is identified with $(1-t)u$ in the second. Let $i_2 : D^n \to S^n$ be multiplication with $1/2$ followed by the inclusion of $\dot{D}^n_2$. To form $M \# S^n$ we cut out $i_2(0)$ from $S^n$. The remaining is $\dot{D}^n_1$, the inclusion of which is denoted $i_1$. Thus $M \# S^n$ is the quotient of $M \backslash i_0(0) \sqcup \dot{D}^n_1$ by the identification $i_0(tu) = i_2((1-t)u) = \frac{1}{2}(1+t)u$, $0 < t < 1$ and $u \in S^{n-1}$. Denote the quotient map by $\pi$. Then $\pi \circ i_1$ is a diffeomorphism onto its image which therefore is diffeomorphic to $i_0(D^n)$. The only thing to worry about is if this diffeomorphism can be patched together with the identity outside of $\pi \circ i_1(D^n)$ to form a global diffeomorphism $M \to M\#S^n$. I.e, is the map $F$

$$i_0(2D^n) \longrightarrow i_0(2D^n\backslash 0)\bigsqcup i_1(D^n)/\sim \subset M\#S^n$$
$$i_0(tu) \longmapsto \begin{cases} \pi(i_0(tu)) & \text{if } t \geqslant 1 \\ \pi(i_1(tu)) & \text{if } 0 < t \leqslant 1 \end{cases}$$

a diffeomorphism? There is a problem around $t = 1$ since $\pi(i_1(tu)) = \pi(i_0(2t-1)u)$ for $\frac{1}{2} < t < 1$. We must redefine our map. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism with $\phi(t) = t$ for $t \geqslant \frac{3}{2}$ and $\phi(t) = 2t - 1$ for $t \leqslant 1$. (Such a map can for example be constructed using a partition of unity.) We redefine $F$ by

$$i_0(tu) \longmapsto \begin{cases} \pi(i_0(\phi(t)u)) & \text{if } t \geqslant \frac{1}{2} \\ \pi(i_1(tu)) & \text{if } 0 < t \leqslant \frac{1}{2} \end{cases}$$

Now the problem at $t = 1$ has been fixed. $F$ can clearly be extended to a diffeomorphism $M \to M\#S^n$ ◻

**Lemma 3.2.2.** *$\#$ is commutative.*

*Proof.* Given manifolds $M$ and $N$ and embeddings $i_0$ and $i_1$ as above, note that the oriented manifolds below have the same underlying manifolds, but different orientation. Hence the identity can be considered an orientation reversing diffeomorphism:

$$F : M\backslash i_0(0)\bigsqcup -N\backslash i_1(0) \to N\backslash i_1(0)\bigsqcup -M\backslash i_0(0)$$

But on the right hand side we find that quotienting to form $N\#M$ will not coincide with our definition since it was the inclusion into the first summand that was supposed to be orientation preserving. Thus we need new maps $i'_k$, $k = 0, 1$ ($i'_k$ having the same target as $i_k$ to avoid confusion). Again invoking theorem 3.1.2 we get orientation reversing diffeomorphisms $\phi_k$ such that $\phi_k \circ i_k = i'_k$, $k = 0, 1$. Now the composition

$$M\backslash i_0(0) \bigsqcup -N\backslash i_1(0) \xrightarrow{F} N\backslash i_1(0) \bigsqcup -M\backslash i_0(0) \xrightarrow{\phi_0 \sqcup \phi_1} N\backslash i'_1(0) \bigsqcup -M\backslash i'_0(0)$$

takes the relation defining $M\#N$ to the relation defining $N\#M$ and therefore induces the desired diffeomorphism $M\#N \to N\#M$. ◻

**Lemma 3.2.3.** *$\#$ is associative.*

*Proof.* We choose two embeddings $i, j : D^n \to M^n$ with disjoint images. We consider $(M\#N)\#N'$ for $n$-manifolds $N$ and $N'$. It is clear that whether we attach $N'$ to $M\#N$ using $j$, or if we attach $N$ to $M\#N'$ using $i$ does not matter. ◻

So far we have considered $\#$ as a binary operation on $n$-manifolds. The next lemma show that it induces an operation on h-cobordism classes.

**Lemma 3.2.4.** *If $N^n$, $N'^n$ and $M^n$ are simply connected closed manifolds, and $N$ is h-cobordant to N', then $M\#N$ is h-cobordant to $M\#N'$.*

*Proof.* If $n = 1$, the statement is vacuously true as there are no simply connected closed 1-manifolds. If $n = 2$, then the classification of surfaces forces the manifolds to be $S^2$, and so the lemma is clearly true since we have seen $M\#S^n \simeq M$. Now we deal with the remaining cases, $n$ at least 3. Let $W$ be an h-cobordism from $N$ to $N'$. We may choose a path $f : [0,1] \to W$ from $f(0) = p \in N$ to $f(1) = p' \in N'$, and we may choose it such that it admits a tubular neighborhood $U$ diffeomorphic to $I \times \mathbb{R}^n$, where $I = [0,1]$, through some diffeomorphism $F$. Let $i : \mathbb{R}^n \to M$ be an embedding. We modify $W$ as follows: Cut out the image of the path $f$. For each $a \in I$, $t \in \mathbb{R}_{>0}$ and $u \in S^{n-1}$, identify $F^{-1}(a, tu)$ with $(a, i(tu)) \in M\backslash i(0) \times I$ and let us denote the resulting space by $W'$. We denote the quotient map by $\pi$. Since we have glued along open sets with a diffeomorphism, $W'$ is locally Euclidean and carries a smooth structure. It is also clear that $W'$ is second countable. $W'$ is Hausdorff by an analysis entirely analogous to the analysis showing that the connected sum gives a Hausdorff space. Recalling the definition of the connected sum it is clear that $\partial W' = M\#N \sqcup -(M\#N')$, so it remains only to show that both boundary components are deformation retracts of $W'$. We cover $W'$ with the two open sets $\pi(W\backslash f(I))$ and $\pi(M\backslash i(0) \times I)$. Note that the intersection of these sets is $\mathbb{R}^n\backslash 0 \times I$ which has the homotopy type of $S^{n-1}$. Note also that $\pi$ restricted to each component of $W\backslash f(I) \sqcup M\backslash i(0) \times I$ is a diffeomorphism. We map to the Mayer-Vietoris sequence of this covering from the Mayer-Vietoris sequence of the covering of $M\#N$ given by $M\backslash i(0)$ and $N\backslash p$, or rather their diffeomorphic images under the identification map. Here the intersection is again homotopy equivalent to $S^{n-1}$. We get the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(S^{n-1}) \arrow[r] \arrow[d, "\simeq"'] & H_k(N\backslash p) \oplus H_k(M\backslash i(0)) \arrow[r] \arrow[d, "j_* \oplus iso"'] & H_k(M\#N) \arrow[r] \arrow[d, "i_*"'] & \cdots \\
\cdots \arrow[r] & H_k(S^{n-1}) \arrow[r] & H_k(W\backslash f(I)) \oplus H_k(M\backslash i(0) \times I) \arrow[r] & H_k(W') \arrow[r] & \cdots
\end{tikzcd}$$

where the map denoted *iso* is the isomorphism induced by inclusion at 0, $M\backslash i(0) \to M\backslash i(0) \times I$. Note that since $n \geq 3$ both $M\backslash i(0)$ and $N\backslash p$ are simply connected, so Van Kampen's theorem yields that $\pi_1(W')$ and $\pi_1(M\#N)$ fits in the following pushout diagram as the group $G$. They must therefore be trivial.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & G
\end{tikzcd}$$

It follows as a corollary of Whitehead's theorem and Hurewhicz's theorem that a map between simply connected CW-complexes which induces isomorphisms on homology is a homotopy equivalence. Thus, by the 5-lemma, it now only remains to show that the inclusion $j : N\backslash p \to W\backslash f(I)$ induces isomorphisms on homology. Consider the commutative diagram induced by the map of pairs $(N, N\backslash p) \to (W', W'\backslash f(I))$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(N\backslash p) \arrow[r] \arrow[d] & H_k(N) \arrow[r] \arrow[d, "\simeq"'] & H_k(N, N\backslash p) \arrow[r] \arrow[d] & H_{k-1}(N\backslash p) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_k(W\backslash f(I)) \arrow[r] & H_k(W) \arrow[r] & H_k(W, W\backslash f(I)) \arrow[r] & H_{k-1}(W\backslash f(I)) \arrow[r] & \cdots
\end{tikzcd}$$

By excision and homotopy invariance of homology, for each $k > 0$,

$$H_k(W, W\backslash f(I)) \simeq H_k(\mathbb{R}^n \times I, \mathbb{R}^n\backslash 0 \times I) \simeq H_k(\mathbb{R}^n, \mathbb{R}^n\backslash 0) \simeq H_k(N, N\backslash p)$$

and the composition from right to left is induced by the inclusion. Thus the 5-lemma implies that $j$ induces isomorphisms on homology, and we can conclude that $M\#N$ is a deformation retract of $W'$. The exact same argument also applies to $M\#N'$, so $W'$ is an h-cobordism. □

To show that $\Theta_n$ is an abelian group, it now only remains to verify that it admits inverses. Given a homotopy sphere $M$ we need to produce a homotopy sphere $-M$ such that $M\#-M$ is h-cobordant to $S^n$. Indeed $-M$ turns out to be just $M$ with the orientation reversed, justifying the notation. We need another lemma:

**Lemma 3.2.5.** *embedded Let $M^n$ be a simply connected manifold. $M$ is h-cobordant to $S^n$ if and only if $M$ bounds a contractible manifold.*

*Proof.* Suppose first that $M$ is h-cobordant to $S^n$, through some manifold $W$. $S^n$ admits a neighborhood $U$, diffeomorphic to $S^n \times [1/2, 1)$, via a diffeomorphism $\phi$. We attach to $U$ a disk by identifying $(u, a) \in \phi(U)$ with $au \in \mathrm{D}^{n+1}$. There is as usual no problem in giving the resulting manifold $W'$ a smooth structure since we have glued along an open set, and even Hausdorffness is routine to verify. Also note that no problem arises in orienting $W'$. Since $W$ deformation retracts onto $S^n$ it is clear that $W'$ is contractible. It is also clear that $\partial W' = M$.

Conversely, if $M$ bounds a contractible manifold $W'$, then form $W$ by cutting out the interior of an embedded disk. By construction $\partial W = M \sqcup -S^n$, but we need to verify that both $S^n$ and $M$ are deformation retract of $W$. We consider the commutative diagram with exact rows, induced by the map of pairs $(D^{n+1}, S^n) \to (W', W)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(S^n) \arrow[r] \arrow[d] & H_k(D^{n+1}) \arrow[r] \arrow[d] & H_k(D^{n+1}, S^n) \arrow[r] \arrow[d] & H_{k-1}(S^n) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_k(W) \arrow[r] & H_k(W') \arrow[r] & H_k(W', W) \arrow[r] & H_{k-1}(W) \arrow[r] & \cdots
\end{tikzcd}$$

We excise from $(W', W)$ everything but a neighborhood of the embedded disk $D^n$, which deformation retracts onto $D^n$. Then it is clear that $H_k(D^n, S^n) \to H_k(W', W)$ is an isomorphism for each $k$. It follows from the 5-lemma, or from observing that the horizontal maps of the above rightmost square are isomorphisms, that $S^n \to W$ is a homology isomorphism, and hence a homotopy equivalence. That is, $S^n$ is a deformation retract of $W$. To show that $W$ also deformation retracts onto $M$, we apply the Poincaré duality isomorphism $H_k(W, S^n) \simeq H^{n+1-k}(W, M)$. Since $H_k(S^n) \to H_k(W)$ is an isomorphism it follows from the long exact sequence of homology groups of the pair $(W, S^n)$ that $H_k(W, S^n) = 0$. Then the long exact sequence of cohomology groups of the pair $(W, M)$ shows that $M \to W$ induces isomorphisms on cohomology, thus is a homotopy equivalence since $M$ and $W$ are simply connected. □

Using this we can show the following.

**Lemma 3.2.6.** *If $M$ is a homotopy $n$-sphere, then $M\#-M = S^n$.*

We construct a contractible manifold $W$ with boundary $M\#-M$. Denote by $H^2 \subset D^2$ the set of points $(r\sin(\theta), r\cos(\theta))$ where $0 \leqslant r \leqslant 1$ and $0 \leqslant \theta \leqslant \pi$. Let $i : D^n \to M$ be an embedding, and denote by $\frac{1}{2}D^n$ the disk of radius $\frac{1}{2}$. Glue $(M \backslash i(\frac{1}{2}D^n)) \times [0, \pi]$ and $H^2 \times S^{n-1}$ together by identifying $(i(ru), \theta)$ with $(((2r-1)\sin(\theta), (2r-1)\cos(\theta)), u)$ for each $u \in S^{n-1}$, $\frac{1}{2} < r \leqslant 1$ and $0 \leqslant \theta \leqslant \pi$. It is clear that the identification space $W$ is a manifold. We consider the

Mayer Vietoris sequence for $W$ from the two open subsets we patched together to form it. Their intersection is $H^2\backslash 0 \times S^{n-1}$, the inclusion of which into $H^2 \times S^{n-1}$ s a homotopy equivalence. Thus, since $M\backslash\frac{1}{2}D^n$ is contractible, the sequence is

$$\cdots \to H_k(S^{n-1}) \to H_k(S^{n-1}) \oplus 0 \to H_k(W) \to \cdots$$

and we conclude that $H_k(W) = 0$ for each $k$. Now we investigate the boundary of $W$. Note first that the part of $\partial H^2 \times S^{n-1}$ with $r = 1$ is identified with interior points, unless $\theta = 0, \pi$, and therefore do not contribute to the boundary of $W$. Note that $M\backslash i(\frac{1}{2})\times[0,\pi]$ has two diffeomorphic, but oppositely oriented, boundary components corresponding to $\theta = 0, \pi$. A point of the form $(i(ru), 0)$ gets identified with $((0, 2r-1), u)$, and a point of the form $(i(ru), \pi)$ with $((0, 1-2r), u)$. Thus the boundary of $W$ is formed from

$$M\backslash i\left(\frac{1}{2}D^n\right) \times \{0, \pi\} \bigsqcup [-1, 1] \times S^{n-1}$$

by the above identification. We can view this as first attaching the "cylinder" $[-1, 1] \times S^{n-1}$ to $M\backslash i(\frac{1}{2}D^n) \times \{0\}$, a procedure which crucially does not change the diffeomorphism type, and then attach the second copy on the other side of the cylinder. This is the same as just glueing the two copies of $M\backslash i\left(\frac{1}{2}\right)$ together using the relation $(i(ru), 0) \sim \left(i\left(\left(\frac{3}{2} - r\right) u\right), \pi\right)$ □

We have now proved the following theorem.

**Theorem 3.2.7.** *$\Theta_n$ is an abelian group under $\#$. The standard sphere, $S^n$, serves as identity element and orientation reversal as inversion.* □

We end this section with some concrete results:

**Theorem 3.2.8.** *For $n = 1, 2, 3$, the group $\Theta_n$ is trivial.*

*Proof.* It is a classical result that any topological manifold of dimension $n < 4$ admits a unique smooth structure. For $n = 1$ the only connected and closed manifold is $S^1$. For $n = 2$ the classification of surfaces implies that there is only one homeomorphism class of homotopy 2-spheres. For $n = 3$ the statement that any homotopy 3-sphere is homeomorphic to $S^3$ is known as the Poincaré conjecture. Perelman was able to show it was true approximately 100 years after Poincaré started thinking about it. □

## 3.3 Stable Parallelizability

We call a vector bundle $E \to B$ stably trivial if $E \oplus \epsilon^k \simeq \epsilon^{n+k}$ for some $k$. Here $\oplus$ denotes the Whitney sum, defined by taking the direct sum in each fiber. We say that a manifold is stably parallelizable, or s-parallelizable for short, if its tangent bundle is stably trivial. Note that a vector bundle $E_f$, over a sphere is stably trivial if and only if $i \circ f$ is null-homotopic for some $i : SO_n \to SO_{n+k}$ by Lemma 2.10.2.

**Lemma 3.3.1.** *If a vector bundle $E$ of dimension $k$ over an $n$-dimensional CW-complex $B$ with $n < k$ is stably trivial, then $E$ is trivial.*

*Proof.* By assumption $E \oplus \epsilon^r \simeq \epsilon^{r+k}$ for some $r$. There is no harm in assuming $r = 1$. For if it holds in that case, then if $E \oplus \epsilon^r$ is trivial, $E \oplus \epsilon^{r-1}$ must be trivial, and so on until we finally get that $E$ must be trivial. Since both $\epsilon$ and $\epsilon^{k+1}$ are oriented bundles, $E$ also gets an induced orientation. Namely, a basis for a fiber of $E$ is deemed positively oriented if adding a positively oriented basis for $\epsilon$ (as the last basis-vector) yields a positively oriented basis for $\epsilon^{k+1}$. Thus $E$ is classified by a map $B \to S^k$ by Lemma 2.9.2. But $k > n$, so by homotoping the classifying map into a cellular map we see that it must be null-homotopic. Hence $E$ is trivial as claimed. $\square$

Suppose $M^n$ is a stably parallelizable manifold. Then, by the above lemma, $TM \oplus \epsilon^1$ is already trivial. We may imbed $M$ into some high dimensional euclidean space, $\mathbb{R}^{2n+k}$, $k > 0$. Denote the normal bundle of this embedding by $NM$. Then $NM \oplus TM \simeq \epsilon^{2n+k}$. We get

$$NM \oplus \epsilon^{n+1} \simeq NM \oplus (TM \oplus \epsilon^1) \simeq \epsilon^{2n+k+1}.$$

By Lemma 3.3.1 we conclude that $NM$ must be trivial since it has dimension $n + k > n$, and apparently is stably trivial. Note that the above argument runs equally well with the assumption that $NM$ is trivial and conclusion that $TM \oplus \epsilon$ also is. Thus $M^n$ is stably parallelizable if and only if the normal bundle of any embedding into $\mathbb{R}^{2n+1}$ is trivial. We also get the following from the discussion:

**Lemma 3.3.2.** *A manifold $M^n$ is s-parallelizable if and only if the normal bundle of any embedding into $\mathbb{R}^{2n+k}$, $k > 0$ is trivial. Furthermore there is a one-to-one correspondence between trivializations of $TM \oplus \epsilon^k$ and trivializations of the normal bundle of an embedding $M \to \mathbb{R}^{2n+k}$.* $\square$

Note that if we have a stably parallellizable connected $n$-manifold, $W$, with non-vacuous boundary, then as in the proof of Lemma 3.3.1 we get a map $W \to S^n$ classifying its tangent bundle. A theorem of Hopf, [Hu59, Theorem $\mathbf{C^n}$, p.53], states that $[W, S^n]$ is in bijection with $H^n(W)$ for triangulable spaces, $W$, of dimension $\leqslant n$. By Poincaré duality (note that s-parallelizability again gives us an orientation, thus ensuring orientability so that Poincaré duality holds) this group coincides with $H_0(W, \partial W; \mathbb{Z})$. The assumptions that $W$ is connected and that $\partial W \neq \emptyset$ guarantees that $H_0(\partial W) \to H_0(W)$ is onto, hence has cokernel $H_0(W, \partial W) = 0$. Thus there is only one homotopy class of maps $W \to S^n$, and the classifying map of $TM$ must be null-homotopic. Equivalently $TW$ is trivial, which is to say $W$ is parallelizable. We summarize this in a lemma for future reference. We have even proved slightly more:

**Lemma 3.3.3.** *Any stably trivial $n$-vector bundle over a connected, orientable manifold $W^n$ with non-vacuous boundary is trivial. In particular oriented manifolds with non-vacuous boundary are stably parallelizable if and only if they are parallelizable.* $\square$

## 3.4 Homotopy Spheres are Stably Parallelizable

Let $M$ be a homotopy $n$-sphere. We choose a CW-structure for $M$ and attempt to trivialize the stable tangent bundle $E = TM \oplus \epsilon$. The obstructions lie in $H^k(M; \pi_{k-1}(SO_{n+1}))$ by Lemma 2.9.3. A trivialization of course exist over the 0-skeleton of $M$, since that is just a discrete space. The obstructions, $\mathfrak{o}^i$, to extending this section lies by Theorem 2.8.5 in the groups $H^i(B; \pi_{k-1}(SO_{n+1}))$ which are zero unless $k = n$. Note that $\pi_{k-1}(SO_{n+1}) = \pi_{k-1}(SO)$ for $k \leqslant n$. The computation of the groups $\pi_{n-1}(SO)$ was carried out by Bott in [Bot57].

**Theorem 3.4.1** (Bott-periodicity).

| $n-1$ *mod* 8 | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|---|
| $\pi_{n-1}(SO)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | *0* | $\mathbb{Z}$ | 0 | *0* | *0* | $\mathbb{Z}$ |

*Proof.* There are a number of proofs of this theorem. See for example [Mil63], [Ati68] or the original proof in [Bot57]. □

Thus the proof of Theorem 3.4.4 splits into three cases, depending on the structure of the group containing the obstruction.

**Case 1,** $\pi_{n-1}(SO) = 0$**:**
There is no need to say anything.

In the remaining cases we need to understand better what the obstruction $\mathfrak{o}^n$ is. We will state some results from [MK60]. The proofs of these results involve computations with characteristic classes and the stable Hopf-Whitehead homomorphism, $J : \pi_k(SO_m) \to \pi_{r+k}(S^q)$. To give the foundations necessary to understand the proofs are outside the scope of this thesis, although for the reader well acquainted with the theory of characteristic classes and the $J$ homomorphism they are not hard. A manifold $(M^n, F)$ is called almost framed if removing a single point $x_0$, renders a parallelizable manifold, and

$$F : \epsilon \oplus T(M\backslash x_0) \to \mathbb{R}^{n+1} \times (M\backslash x_0)$$

is a bundle isomorphism, trivializing the stable tangent bundle of $M$ except at $x_0$. The trivialization is defined on every cell of $M$ except a single $n$-cell, thus there is a well defined obstruction, $\mathfrak{o}^n(F) \in H^n(M; \pi_{n-1}(SO_{n+1}))$ (Lemma 2.9.3) to extending it to all of $M$. Since there is only a single problematic cell, in fact the obstruction lies in $\pi_{n-1}(SO_{n+1}) = \pi_{n-1}(SO)$.

**Lemma 3.4.2** (Lemma 1 of [MK60]). *Let $\alpha \in \pi_k(SO)$. Then $J\alpha = 0$ if and only there exists an almost framed manifold $(M, F)$ such that $\alpha = \pm\mathfrak{o}^n(F)$.* □

**Lemma 3.4.3** (Lemma 2 of [MK60]). *Let $E \to K$ be a vector bundle of dimension $m$ over a CW-complex $K$ of dimension*[5] *$k < m+1$, and let $f : K^{4k-1} \to V(E)$ be a trivialization of $E$ restricted to the $4k-1$ skeleton of $K$. Then the Pontryagin class $p_k(E)$ is related to $\mathfrak{o}^{4k}(f)$ by $p_k(E) = a_k(2k-1)!\mathfrak{o}^{4k}(f)$, where $a_k$ is 1 if $k$ is even and $k$ if $n$ is odd.* □

**Case 2,** $\pi_{n-1}(SO) = \mathbb{Z}$**:**
This occurs when $n = 4k$. By Lemma 3.4.3 $\mathfrak{o}^n(F)$ must be a nonzero multiple of the top Pontryagin class of the bundle, $p_k$. Since all other Pontryagin classes are 0 ($H^i(M) = 0$ for $i < n$) we get from Hirzebruch's signature theorem, stated later in Theorem 6.1.1, that in fact a nonzero multiple of $p_n$ equals $\sigma(M)$. But $H^{2k}(M) = 0$, so $\sigma(M) = 0$. Therefore $p_k = 0$, and finally $\mathfrak{o}^n = 0$.

**Case 3,** $\pi_{n-1}(SO) = \mathbb{Z}_2$**:**
Lemma 3.4.2 implies that $\mathfrak{o}^n(F)$ is in the kernel of the Hopf-Whitehead homomorphism $J_{n-1}$. Adams has proved in [Ada66] that for $n-1 \equiv 0$ or $1 \mod 8$, $J$ is a monomorphism. Thus $\mathfrak{o}^n(F) = 0$.

Thus we have shown the following theorem.

**Theorem 3.4.4.** *Homotopy spheres are s-parallelizable.* □

[5] This means that $K$ has no cells of dimension higher than $k$.

### 3.5 Connection With Stable Homotopy Theory

Suppose $M$ is a homotopy sphere. Since $M$ is s-parallelizable, it can support a framing, say $\phi$. Then we can apply the isomorphism $p : \Omega_n^{fr} \to \pi_n(\mathbb{S}) = \pi_{n+k}(S^k)$ (for $k$ large) of the Pontryagin construction. Of course $p(M, \phi)$ depends on $\phi$, so we consider the set $p(M) := \{p(M, \phi) | \phi \text{ is a framing of } M\}$. We shall see that this defines a homomorphism to a certain quotient group of $\pi_n(\mathbb{S})$.

**Lemma 3.5.1.** *The set $p(M)$ contains $0$ if and only if $M$ bounds a parallelizable manifold.*

*Proof.* Suppose $M^n = \partial W$ with $W$ parallelizable. Then we may choose an embedding $i : W \to D^{n+k+1}$, $k > n$ such that $i(M) \subset S^{n+k}$. Let $\psi$ be a trivialization of $TW$. Then $\phi = \psi|_M$ is a framing of $M$. Since $\phi$ extends over $W$, $p(M, \phi)$ extends over $D^{n+k+1}$. Equivalent it is null-homotopic. One way to see this is to put $W$ into a tubular neighborhood $F : W \times \mathbb{R}^{k+1} \simeq U \subset D^{n+k+1}$. Then the extension of $p(M, \phi)$ can be defined on $U$ as $F^{-1}$ followed by projection onto $\mathbb{R}^{k+1}$, and constantly $\infty$ outside of $U$.

Conversely, suppose $p(M)$ contains 0, say $0 = p(M, \phi)$. Having a null-homotopy is the same as a map $\psi : D^{n+k+1} \to S^k$ which we can assume to be smooth. Suppose $x_0$ is a regular value such that $M = \psi|_{S^{n+k}}^{-1}(x_0)$ and set $W = \psi^{-1}(x_0)$. Then $\partial W = S^{n+k} \cap W = M$. As in the Pontryagin construction, for a sufficiently small neighborhood $V$ of $x_0$, $\psi^{-1}(V)$ is a tubular neighborhood of $W$, and a trivialization of $TV$ induces a trivialization of $NW$. By Lemma 3.3.2, $W$ is s-parallelizable, and so by Lemma 3.3.3, $W$ is parallelizable. □

**Lemma 3.5.2.** *If $M_0$ and $M_1$ are h-cobordant, then $p(M_0) = p(M_1)$.*

*Proof.* Note that we know that if $M_i$ is given a framing $\phi_i$, $i = 0, 1$, and if $(M_0, \phi_0)$ is framed cobordant to $(M_1, \phi_1)$, then $p(M_0, \phi_0) = p(M_1, \phi_1)$. Hence we only need to show that if $\phi_0$ is a framing of $M_0$, then the h-cobordism $W$ from $M_0$ to $M_1$ can be framed. Consider

$$TW|_{M_0} \simeq TM_0 \oplus \epsilon.$$

This is a trivial bundle since $M_0$ is s-parallelizable, and $\phi_0$ gives us a concrete trivialization. Since $M_0$ is a deformation retraction of $W$, we get a trivialization of $TW \oplus \epsilon$. That is, the framing of $M_0$ extends to $W$. Hence $p(M_0) = p(M_1)$. □

**Lemma 3.5.3.** *If $M$ and $N$ are s-parallelizable, then $p(M_0) + p(M_1) \subset p(M_0 \# M_1)$.*

*Proof.* Given framings $\phi_i$ of $M_i$, $i = 0, 1$, it is clear that $p(M_0, \phi_0) + p(M_1, \phi_1) = p(M_0 \sqcup M_1, \phi_0 \sqcup \phi_1)$. We have seen in Lemma 3.1.4 that $M_0 \# M_1$ is framed cobordant to $M_0 \sqcup M_1$. This finishes the proof. □

Define $bP^{n+1} \subset \Theta_n$ to consist of the homotopy spheres bounding parallelizable manifolds.

**Theorem 3.5.4.** *The set $p(S^n)$ is a subgroup of $\pi_n(\mathbb{S})$, and for any other homotopy sphere $M$, $p(M)$ is a coset of $p(S^n)$. Hence $p$ is a homomorphism $\Theta_n \to \pi_n(\mathbb{S})/p(S^n)$ and it has kernel $bP^{n+1}$.*

*Proof.* We apply Lemma 3.5.3 and obtain:

$$S^n \# S^n = S^n \implies p(S^n) + p(S^n) \subset p(S^n) \tag{3}$$
$$S^n \# M = M \implies p(S^n) + p(M) \subset p(M) \tag{4}$$
$$M \# -M = S^n \implies p(M) + p(-M) \subset p(S^n) \tag{5}$$

(3) shows that $p(S^n)$ is closed under addition. Together with the fact that

$$p(M,\phi) + p(M,-\phi) = 0 \in \pi_n(\mathbb{S})$$

since $(M,\phi) \sqcup (M,-\phi)$ bounds $M \times I$ with the obvious induced framing, this proves that $p(S^n)$ is a subgroup of $\pi_n(\mathbb{S})$. Now (4) shows that $p(M)$ is a union of cosets of $p(S^n)$. Finally (5) proves that $p(M)$ is no more than a single coset of $p(S^n)$. This proves that $p$ is a homomorphism as claimed. The additional observation that the kernel of $p$ is $bP^{n+1}$ follows from Lemma 3.5.1. □

It is known that $p(S^n) = \mathrm{Im}(J)$. Levine gives a short proof of this in [Lev85]. A proof is not included here so that we can avoid working with $J$ explicitly. We proceed to show that $bP^{n+1}$ is finite for each $n$. Since $\pi_n(\mathbb{Z})$ is a finite group, the following theorem will then follow from Theorem 3.5.4.

**Theorem 3.5.5.** *$\Theta_n$ is a finite group for each $n$.*

# 4 Surgery Theory and $bP^{n+1}$

In this section we develop the machinery of surgery theory. The proofs and exposition follows [KM63] and [Mil61] closely. Our main reason for studying surgery theory is to compute the groups $bP^{n+1}$, and also the index of $\Theta_n/bP^{n+1} \subset \pi_n(\mathbb{S})/Im(J)$. Our approach to compute $bP^{n+1}$ is the same as that in [KM63]. Take a framed manifold $(V,F)$ bounded by a homotopy sphere $\Sigma$, and try to replace $V$ by a manifold which is still bounded by $\Sigma$, but has simpler homotopy groups. If we manage to kill all of the homotopy groups, then we have shown that $\Sigma$ bounds a contractible manifold $V'$ and so is $h$-cobordant to $S^n$. Note that it is not essential here that $V$ and $V'$ are framed cobordant: We only care about producing some contractible $V'$ which need have no structure in common with $V$ except $\partial V = \partial V' = \Sigma$. As for computing the index $\Theta_n/bP^{n+1} \subset \pi_n(\mathbb{S})/Im(J)$: If we can manage to show that an arbitrary closed framed manifold $(M,F)$ is framed cobordant to a homotopy sphere, then that is exactly saying that every $\alpha \in \pi_n(\mathbb{S})$ arises through the Pontryagin construction from a framed homotopy sphere. In other words, the index of $\Theta_n/bP^{n+1} \subset \pi_n(\mathbb{S})/Im(J)$ is 1. Since the Pontryagin construction is invariant under framed cobordism, but not in general cobordism, it is important that the arbitrary closed framed manifold $(M,F)$ actually is framed cobordant to a homotopy sphere, not just cobordant.

## 4.1 Spherical Modifications

Let $M$ be a manifold of dimension $n = p+q+1$, and let $i : S^p \times D^{q+1} \to M$ be an embedding. We form a new manifold $\chi(M,i)$ from $M \backslash i(S^p \times \{0\}) \bigsqcup D^{p+1} \times S^q$ by identifying $i(u,tv)$ with $(tu,v)$ for each $(u,v) \in S^p \times S^q$ and $0 < t < 1$. It is easy to see that $\chi(M,i)$ is Hausdorff, hence a smooth manifold since we are gluing open sets together with a diffeomorphism. It is also clear that the boundary of $M$ (vacuous or not) is left unaltered by this procedure. We say that $\chi(M,i)$ is obtained from $M$ by the spherical modification $\chi(i)$, and we call $\chi(i)$ a spherical modification of type $(p,q)$. The application of spherical modifications is called surgery. We may describe surgery neatly as follows: $S^p \times S^q$ bounds $S^p \times D^{q+1}$, but it also bounds $D^{p+1} \times S^q$. Hence, given an embedding of one of these, we may cut out the interior and replace it with the interior of the other. Thus the effect of surgery may be undone by simply cutting the just inserted interior of $S^p \times S^q$ back out and replacing it with the alternative. Thus:

**Lemma 4.1.1.** *If $\chi(i)$ is a surgery on $M$ of type $(p,q)$, then $M$ can be obtained from $\chi(M,i)$ by surgery of type $(q,p)$.* □

**Theorem 4.1.2.** $bP^{2k+1} = 0$

The case $k = 1$ is easy since $\Theta_2 = 0$. The rest of section 4 is dedicated to proving the result for $k > 1$.

First of all we introduce a construction which will show that performing surgery does not alter the cobordism class. Define

$$L \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$$

by the relations

$$-1 \leqslant \|x\| - \|y\| \leqslant 1, \quad \|x\|\|y\| < \sinh(1)\cosh(1), \quad \text{for } x \in \mathbb{R}^{p+1} \text{ and } y \in \mathbb{R}^{q+1}.$$

We consider the level sets, $S_c = \{(x,y) \in L : \ \|x\| - \|y\| = c\}$ for $c \in [-1,1]$. The boundary of $L$ is $S_{-1} \sqcup S_1$. Consider the curve $\gamma_{(x,y)} : \mathbb{R}_{>0} \to \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ given by $t \mapsto (tx, t^{-1}y)$. Observe that the product $\|x\|\|y\|$ is constant on these curves, that $L$ is a bounded set, and that the curves are unbounded in both directions unless $x$ or $y$ is 0. It follows that if $x$ or $y$ equals 0, this curve is a straight line connecting $(x,y)$ with the origin, and otherwise the curve intersect each $S_c$, and in particular both $S_{-1}$ and $S_1$, in exactly one point. We have diffeomorphisms

$$\begin{aligned} &\dot{D}^{p+1} \times S^q \simeq S_{-1} \\ &(ut, v) \longleftrightarrow (u\sinh(t), v\cosh(t)) \end{aligned}$$

and

$$\begin{aligned} &S^p \times \dot{D}^{q+1} \simeq S_1 \\ &(u, tv) \longleftrightarrow (u\cosh(t), v\sinh(t)) \end{aligned}$$

where $(u,v) \in S^p \times S^q$ and $0 < t < 1$ and $\dot{X}$ = interior $X$. Note that if $\cosh(t) \neq 0 \neq \sinh(t)$, then $(u\sinh(t), v\cosh(t)) = (u\cosh(t)\tanh(t), v\cosh(t)\tanh(t)^{-1})$. In other words the curves $\gamma_{(x,y)}$ connect the point corresponding to $(ut,v) \in \dot{D}^{p+q} \times S^q$ to the point corresponding to $(u,tv) \in S^p \times \dot{D}^{q+1}$. It seems reasonable that we can use $L$ to obtain a smooth transition from $M$ to $\chi(M,i)$, i.e. a cobordism, at least if we assume that $M$ has no boundary, or just that $i : S^p \times D^{q+1} \to M$ does not take any values in $\partial M$, so we can avoid dealing with corners. Let $i : S^p \times D^{q+1} \to M$ be an embedding. Then form $\omega(M,i)$ from

$$M \backslash i(S^p \times 0) \times D^1 \sqcup L$$

by identifying $(i(u,tv),c)$ with the point on $S_c$ which lies on $\gamma_{(u\cosh(t), v\sinh(t))}$. It is clear that $\omega = \omega(M,i)$ is a smooth manifold. The boundary of $\omega$ corresponds to $c = \pm 1$. At $c = 1$ we are patching $M\backslash i(S^p \times 0)$ with $S_1 \simeq S^p \times \dot{D}^{q+1}$ by the relation $i(u, tv) \sim (u, tv)$. Hence we are just recovering $M$. At $c = -1$ we are patching with $S_{-1} \simeq \dot{D}^{p+1} \times S^q$ with the relation $i(u, tv) \sim (tu, v)$, i.e. we obtain $\chi(M, i)$. We have proved:

**Theorem 4.1.3.** *The manifold $\omega(M, i)$ is a cobordism between $M$ and $\chi(M, i)$.* □

The converse is almost true as well: Any pair of cobordant closed oriented manifolds can be obtained from each other through a sequence of spherical modifications. Milnor proves this in [Mil59], but we will not use it and do not include the proof here. In general we really do need more than one spherical modification in general.

Given an embedding $i : S^p \times D^{q+1}$, we denote the homotopy class of $i$, considered as an element of $\pi_p(M)$ by contracting $D^{q+1}$, by $\lambda(i)$. We say that $i$ represent $\lambda(i)$.

**Lemma 4.1.4.** *Assuming $p < q$, the effect of the spherical modification $\chi(i)$ on the homotopy groups is*

$$\begin{aligned} &\pi_j(\chi(M, i)) \simeq \pi_j(M) \quad \text{for } j < p \\ &\pi_p(\chi(M, i)) \simeq \pi_p(M)/\Lambda \end{aligned}$$

*where $\Lambda$ is a group containing $\lambda = \lambda(i)$.*

*Proof.* We consider the manifold $\omega = \omega(M, i)$. We will not need the smooth structure of $\omega$ for this proof. Note that

$$\big(M\backslash i(S^p \times \{0\})\big) \times I$$

deformation retracts onto

$$M\backslash i(S^p \times \{0\}) \bigcup i(S^p \times (D^{q+1}\backslash\{0\})) \times I$$

and hence $\omega$ has the homotopy type of $M \cup L$ where $L$ is attached to $i(S^p \times D^{q+1}\backslash\{0\}) \times I$ as before. The effect of attaching $L$ is just to fill in the removed "cylinder" $i(S^p \times \{0\}) \times I$. Now, $L$ is contractible and attached to $M$ with $i$, hence $M \cup L$ deformation retracts onto $M \cup D^{p+1}$ where $D^{p+1}$ is attached using $i|_{S^p \times 0}$. Hence the inclusion of $M$ into $\omega$ induces $\pi_j(\omega) \simeq \pi_j(M)$ for $j < p$, and is onto for $j = p$. Clearly $\lambda$ is 0 in $\pi_p(\omega)$. Recalling that $M$ is obtained from $\chi(M, i)$ by surgery of type $(q, p)$ we have actually just shown that $\pi_j(\omega) \simeq \pi_j(\chi(M, i))$ for $j < q$. Since $p < q$ we are done. □

The nice thing about this is that we now have a weapon to kill all $\lambda(i) \in \pi_p(M)$ for $j < n/2$. So we ask which elements $\lambda \in \pi_p(M)$ occurs as $\lambda(i)$. Given $\lambda \in \pi_p(M)$ Theorem 2.13.2 shows that there is an embedding $i : S^p \to M$ provided $p \leqslant n/2$. It extends to $S^p \times D^{q+1}$ if and only if $i$ has trivial normal bundle.

**Lemma 4.1.5.** *If $M$ is an s-parallelizable manifold of dimension $n$, then any $\lambda \in \pi_p(M)$ can be represented by an embedding $i : S^p \times D^{n-p} \to M$ for $p \leqslant n/2$.*

*Proof.* The assumption that $M$ is s-parallelizable ensures that $\nu(i)$, the normal bundle of the embedding $i : S^p \to M$ representing $\lambda$, is stably trivial:

$$\epsilon^{n+k}_{S^p} \simeq i^*(\epsilon^k_M \oplus TM) \simeq \epsilon^k_{S^p} \oplus TS^p \oplus \nu(i) \simeq \epsilon^{k+p}_{S^p} \oplus \nu(i)$$

Since $p < n/2 - 1$, the normal bundle must itself be trivial by Lemma 3.3.1. Thus the embedding extends to all of $S^p \times D^{n-p}$. $\square$

We have thus proved that any element $\lambda$ of the first $n/2 - 1$ homotopy groups of a stably parallelizable manifold can be removed surgically. We have however not yet proved that the surgery leaves s-parallelizability invariant which is necessary if we wish to perform successive spherical modifications: For if $\chi(M, i)$ is not s-parallelizable, Lemma 4.1.5 does not apply to $\chi(M, i)$. We shall prove in Lemma 4.3.2 that $i$ representing $\lambda$ can be chosen so that the resulting manifold also is stably parallelizable. Thus through performing successive surgeries we obtain:

**Theorem 4.1.6.** *Through surgery we can obtain from any stably parallelizable manifold $M^n$ a $p$-connected manifold for any $p \leqslant n/2 - 1$.*

*Proof.* We first kill $\pi_1(M)$, which can be done in finitely many steps since $\pi_1(M)$ is a finitely generated group. Next we kill $\pi_2(M) = H_2(M; \mathbb{Z})$ by surging away the generators. Since all the groups we are killing, $\pi_1(M), \cdots, \pi_p(M)$ are finitely generated, and since killing the next does not revive any of the former by Lemma 4.1.4 we after finitely many steps obtain a $p$-connected manifold. Note that the condition $p \leqslant n/2 - 1$ is equivalent with the condition $p < n - p - 1 = q$ of Lemma 4.1.4. $\square$

Hence any s-parallelizable $2k + 1$ manifold can be made $k - 1$ connected through surgery. We need to get rid of the homotopy group $\pi_k(V)$. Then Poincaré duality along with the Hurewicz theorem easily yields that $V$ has trivial homology if it is bounded closed or bounded by $S^{2k}$. Hence is contratible, if it is closed or bounded by a homotopy sphere by Whiteheads theorem. Killing this middle homotopy group requires more cunning than the lower dimensional groups since Lemma 4.1.4 does not apply. Note that at least $k < \frac{2k+1}{2}$, so Lemma 4.1.5 still applies. By Hurewicz theorem we may study the effect of the surgery on homology instead of homotopy groups. As noted earlier, if $V'$ is obtained from $V$ by a spherical modification $\chi(i)$ of type $(p, q)$, then $V$ is obtained from $V'$ by a spherical modification $\chi(i')$ of type $(q, p)$. In particular recall the embedding $i' : D^{q+1} \times S^p \to V'$. Let $\lambda$ denote the homology class of $i$ in the sense that we identify $H_k(M)$ with $\pi_k(M)$ under the Hurewicz isomorphism. Let similarly $\lambda'$ denote the homology class of $i'$. In our case, $p = q = k$. The apparent symmetry is further expressed in the following lemma:

**Lemma 4.1.7.**

$$H_k(V)/\Lambda \simeq H_k(V')/\Lambda'$$

*where $\Lambda$ and $\Lambda'$ are the groups generated by $\lambda$ and $\lambda'$ respectively.*

The notation introduced in the proof will be used also after the proof.

*Proof of Lemma 4.1.7.* Let $V_0$ be the manifold $V \backslash \text{int } i(S^p \times D^{q+1})$

Note that $V_0 = V' \backslash \text{int } i'(D^{p+1} \times S^q)$. By excision we have

$$H_j(V, V_0) = H_k(S^k \times D^{k+1}, S^k \times S^k) = \delta_{j,k+1}\mathbb{Z},$$

the integers for $j = k + 1$, and 0 otherwise. This follows from the long exact sequence of the pair $(S^k \times D^{k+1}, S^k \times S^k)$: The inclusion is an isomorphism in every dimension, except $H_k(S^k \times S^k) \to H_k(S^k \times D^{k+1})$ has a kernel generated by $h(x_0 \times \iota_k) = x_0 \times S^k$ (where we interpret $S^k$ as the simplex $\Delta^k \to \Delta^k/\partial\Delta^k \simeq S^k$). Using these observations we write a portion of the long exact homology sequence for $(V, V_0)$ as

$$H_{k+1}(V) \longrightarrow \mathbb{Z} \xrightarrow{\epsilon'} H_k(V_0) \longrightarrow H_k(V) \longrightarrow 0$$

where by the preceding discussion $\epsilon' := \epsilon'(1) = i_*(x_0 \times S^k)$. The same analysis applies to $V'$, and combining the two sequences we get the following diagram:

$$\begin{tikzcd}
 & & H_{k+1}(V') \arrow[d] & & \\
 & & \mathbb{Z} \arrow[d, "\epsilon"'] \arrow[dr, "\lambda"] & & \\
H_{k+1}(V) \arrow[r] & \mathbb{Z} \arrow[r, "\epsilon'"] \arrow[dr, "\lambda'"'] & H_k(V_0) \arrow[r] \arrow[d] & H_k(V) \arrow[r] & 0 \\
 & & H_k(V') \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\lambda$ and $\lambda'$ are the maps $1 \mapsto \lambda$ and $1 \mapsto \lambda'$ respectively. It is clear that the diagram commutes. We have $H_k(V) \simeq H_k(V_0)/\epsilon'\mathbb{Z}$ and similarly for $V'$, and we get

$$H_k(V)/\Lambda \simeq H_k(V_0)/(\epsilon'\mathbb{Z} + \epsilon\mathbb{Z}) \simeq H_k(V')/\Lambda'.$$

□

Hence if we can make $\Lambda' = 0$, then $\chi(i)$ kills $\lambda$ without adding anything. The inverse image of $x_0 \times S^k$ under the connecting homomorphism is the relative homology class of $x_0 \times id : \Delta^{k+1} \to S^k \times D^{k+1}$ which we denote by $x_0 \times D^{k+1}$. $i_*(x_0 \times D^{k+1})$ intersects $i_*(S^k \times 0)$, which represents $\lambda$, in a single point. In other words there is a commutative triangle

$$\begin{tikzcd}
H_{k+1}(V) \arrow[r, "\cdot\lambda"] \arrow[dr] & \mathbb{Z} \arrow[d, "\simeq"] \\
 & H_{k+1}(V, V_0)
\end{tikzcd}$$

where $H_{k+1}(V) \to H_{k+1}(V, V_0)$ comes from the long exact sequence for the pair $(V, V_0)$. We may therefore interpret the map $H_{k+1}(V) \to \mathbb{Z}$ in the above diagram as $\cdot\lambda$, and similarly we interpret $H_{k+1}(V') \to \mathbb{Z}$ as $\cdot\lambda'$. The point of giving these interpretations is that if we can ensure that there exists a class $\mu \in H_{k+1}(V)$ with $\mu \cdot \lambda = 1$, then it follows that $\epsilon' = 0$, and so $\lambda' = 0$, i.e. $\Lambda' = 0$. The existence of such a $\mu$ is dependent on $\lambda$. Using Poincaré duality we enjoy some hope that the sought $\mu$ will exist if $\lambda$ generates a free direct summand of $H_k(V)$. For in this case non-degeneracy of the modulo torsion intersection pairing gives an element $\mu \in H_{k+1}(V, M)$ such that $\mu \cdot \lambda = 1$. There is an exact sequence

$$H_{k+1}(V) \to H_{k+1}(V, M) \to H_k(M).$$

Hence we can certainly lift $\mu \in H_{k+1}(V, M)$ back to $H_{k+1}(V)$ if $H_k(M) = 0$. We summarize the following immediate consequence of our discussion.

**Lemma 4.1.8.** *Suppose $V$ is an s-parallelizable $2k+1$-manifold either without boundary, or bounded by a homotopy sphere. If $H_k(V) \simeq F \oplus T$ where $T$ is the torsion subgroup, then through a series of spherical modifications of $V$ we can obtain an s-parallelizable manifold $V'$ with $H_k(V') \simeq T$.* □

Getting rid of the torsion group is even more difficult. The proof splits up into two cases, $k$ odd and $k$ even. The easier case is $k$ even, and so we deal with this first.

## 4.2 Computing $bP^{4k+1} = 0$

This is simpler than the case $4k+3$ because we have at our disposal the following lemma:

**Lemma 4.2.1.** *If $k$ is even, then $\chi(i)$ changes the $k$-th Betti number, i.e. the rank of $H_k(V)$.*

Before proving this lemma, let us enjoy seeing our lemmas coming together in beautiful harmony:

*Proof of Theorem 4.1.2 in the case $k$ even:* Let $M$ be a homotopy sphere of dimension $2k$ bounding a stably parallelizable manifold $V$. Then by Theorem 4.1.6 we may assume $V$ to be $(k-1)-$*connected.* By 4.1.8 we may assume $H_k(V)$ to be a torsion group, $T$. Let $\lambda \in H_k(V)$. Then let $i$ represent $\lambda$ as in Lemma 4.1.5. By Lemma 4.1.7, $T/\Lambda = H_k(\chi(V, i))/\Lambda'$ where $\Lambda$ is generated by $\lambda$ and $\Lambda'$ is generated by the class $\lambda'$ from the canonical embedding $i' \to \chi(V, i)$. The punshline is that since $k$ is even, Lemma 4.2.1 implies that $\Lambda'$ is a free group. Hence we can kill off $\Lambda'$ by Lemma 4.1.8. Since $T$ is finite, repeatedly intervening surgically in this manner will after finitely many repetitions kill $T$ completely. Hence we can assume that $V$ is $k$-connected. We have $H_j(V) = H^{2k+1-j}(V, M) = 0$ for $j > k$ since both $M$ and $V$ have vanishing cohomology in this range. Hence all homology groups of $V$ vanish. Since $V$ is $k > 1$ connected Hurewicz theorem implies that $V$ is weakly contractible. Finally Whitehead's theorem implies that $V$ is contractible, and so $M$ is $h$-cobordant to $S^{2k}$ as claimed. □

We also have:

**Lemma 4.2.2.** *If $M$ is a closed manifold of dimension $2k+1$ with $k$ even, then through surgical intervention we can obtain from $M$ a homotopy sphere $M'$.*

*Proof.* Like in the proof of Theorem 4.1.2 for $k$ even, we can obtain a $k$-connected manifold $M'$ from $M$ through a sequence of spherical modifications. Since $M'$ is closed Poincaré duality implies that it has the same homology groups as $S^{2k+1}$. Note that there are maps inducing this isomorphism: We can collapse all of $M'$ except a single $(2k+1)$-cell. Alternatively we can obtain the map by an application of the following theorem due to Hopf [Hu59, Theorem $C^n$, page 53]: Let $X$ be a $CW$-complex and let $\iota \in H^n(S^n)$ be a generator. Then $f \mapsto f^*(\iota)$ induces a one-to-one correspondence between homotopy classes of maps $X \to S^n$ and $H^n(X)$. □

To prove Lemma 4.2.1 we shall make use of a handy formula. We define the semi-characteristic of a $2r-1$-manifold $M$ (without boundary), $e^*(M,\mathbb{F}) \in \mathbb{Z}_2$ for a field $\mathbb{F}$, to be

$$e^*(M,\mathbb{F}) = \sum_{i=0}^{i=r-1} \dim H_i(M;\mathbb{F}) \mod 2$$

**Lemma 4.2.3.** *The rank of the intersection pairing*

$$H_r(M;\mathbb{F}) \otimes H_r(M;\mathbb{F}) \to \mathbb{F}$$

*is modulo 2 congruent to $e^*(\partial M,\mathbb{F}) + e(M)$ where $e(M)$ is the Euler characteristic of $M$ and $\partial M$ is the boundary of $M$.*

*Proof.* Suppose we have an exact sequence of $\mathbb{F}$ vector spaces,

$$A_{n+1} \xrightarrow{f_n} A_n \xrightarrow{f_{n-1}} \cdots \longrightarrow A_0 \longrightarrow 0\ .$$

Then by induction

$$\text{rank } f_n = \sum_{i=0}^{n} (-1)^i \dim A_{n-i}.$$

The base case $n=0$ is trivial, and the induction step follows from the rank nullity formula, $\dim A_n - \text{rank } f_{n-1} = \text{rank } f_n$. Of particular interest to us is the long exact sequence of the pair $(M,\partial M)$ with coefficients in $\mathbb{F}$,

$$H_r(M) \xrightarrow{h} H_r(M,\partial M) \longrightarrow H_{r-1}(\partial M) \longrightarrow \cdots \longrightarrow H_0(M,\partial M).$$

Note that the rank of $h$ coincides with the rank of the intersection pairing since the intersection pairing by definition is adjoint to the composition

$$H_r(M) \xrightarrow{h} H_r(M,\partial M) \simeq H^r(M) \simeq \text{Hom}(H_r(M),\mathbb{F})\ .$$

Thus we compute the rank of $h$:

$$\text{rank } h = \sum_{i=0}^{r} (-1)^{r-i} \left(\dim H_i(M) + \dim H_i(M,\partial M) + \dim H_i(\partial M)\right)$$

We can forget about the sign since it is of no consequence modulo 2. Since $H_i(M,\partial M) \simeq H^{2r-i}(M)$ by Poincaré duality, and $\dim H^{2r-i}(M) = \dim H_{2r-i}(M)$ by the universal coefficient theorem we obtain the statement of the lemma. ☐

*Proof of Lemma 4.2.1.* We first suppose $\partial V = \emptyset$ and again considering the manifold $\omega = \omega(V,i)$. As we have seen before, $\omega$ deformation retracts onto $V$ with a $k+1$-cell attached. Since all homology groups of $V$ are torsion except $H_0(V) = H_{2k+1}(V) = \mathbb{Z}$, $e(M) = 1-1 = 0$. Thus $e(\omega) = e(V) + (-1)^{k+1} = -1$ since $k$ is even and $e(V) = 0$ . Also since $k$ is even, the intersection pairing with rational coefficients

$$H_{k+1}(\omega,\mathbb{Q}) \otimes H_{k+1}(\omega,\mathbb{Q}) \to \mathbb{Q}$$

is alternating, i.e. $H_{k+1}(\omega,\mathbb{Q})$ is a non-degenerate symplectic vector space over $\mathbb{Q}$, hence has even rank. Thus Lemma 4.2.3 yields

$$e^*(\partial\omega,\mathbb{Q}) \equiv 1 \mod 2.$$

Since $\partial\omega = V \sqcup V'$ we deduce that $e^*(\partial\omega, \mathbb{Q}) = e^*(V, \mathbb{Q}) + e^*(V', \mathbb{Q}) \equiv 1 \mod 2$. But $V$ and $V'$ are $(k-1)$-connected, hence $e^*(V, \mathbb{Q}) = \text{rank } H_k(V, \mathbb{Q})$, and similarly for $V'$. This finishes the proof in the case that $V$ is closed. If $\partial V$ is a homotopy sphere, we cone it off to obtain a closed manifold $V_1$. Since $\chi(i)$ does not change $\partial V$, it does not matter if we cone of the boundary before or after surgery. We have shown that

$$\dim H_k(V_1, \mathbb{Q}) \neq \dim H_k(\chi(V_1, i), \mathbb{Q}).$$

Attaching a $(2k+1)$-cell only interferes with the $2k$ and $2k+1$ homology, hence we are done. □

Note that the only reason we needed $k$ to be even in order to prove Lemma 4.2.1 was to ensure that the intersection form on $\omega(M, i)$ with rational coefficients has even rank. If we for some reason happen to know that all mod 2 self intersection numbers are 0, then it follows from non-degeneracy of the intersection form with field-coefficients that $H_k(\omega, \mathbb{Z}_2)$ must have even rank. Hence in this case the above argument proves the following lemma:

**Lemma 4.2.4.** *Suppose $\omega(M, i)$ is such that every $\xi \in H_k(V; \mathbb{Z}_2)$ have self intersection number $\xi \cdot \xi = 0$. Then a consequence of applying the spherical modification $\chi(i)$ is*

$$\dim H_k(M; \mathbb{Z}_2) \neq \dim H_k(\chi(M, i); \mathbb{Z}_2).$$

□

Dealing with the case $k$ odd requires some different techniques to set up the context of Lemma 4.2.4.

## 4.3 Framed Surgery

From now on we assume that $(M, f)$ is framed. Let $\chi(i)$ be a spherical modification on $M$. If there is a trivialization $F$ of $T\omega$ such that $F|_M = f$, where we have identified $TM \oplus \epsilon$ with $T\omega|_M$ by identifying $\epsilon$ with outward normal vectors, then we call $(\chi(i), F)$ a framed spherical modification and denote it $\chi(i, F)$. If $\chi(i, F)$ is a framed spherical modification, then $\chi(M, i)$ also gets a framing $F|_{\chi(M,i)}$ where we again interpret $\epsilon$ as outward pointing vectors, normal to $T\chi(M, i)$.

We would in general like to frame the spherical modifications $\chi(i)$ we are using since Lemma 4.1.5 then will apply to $\chi(M, i)$. We try to extend the given trivialization $f$ over the cells of $(\omega, M)$. The obstruction to extending over the $r$-skeleton lies in the group $H^{r+1} = H^{r+1}(\omega, M : \pi_r(SO_{n+1}))$. We have seen that $(\omega, M)$ has the homotopy type of a relative CW-complex with only one cell, in dimension $p+1$. Thus we get a nonzero group only if $r+1 = p+1$, so the only obstruction is some well defined class

$$\mathfrak{o}(i) \in H^{p+1}(\omega, M; \pi_p(SO))$$

where we have identified $\pi_p(SO_{n+1})$ with $\pi_p(SO)$. To make sure that the obstruction to framing $\chi(i)$ is 0 we will, if necessary, represent $\lambda(i)$ by a different map than $i$. The new spherical modification will of course also have to kill $\lambda(i)$. Given a smooth map $\alpha : S^p \to (SO_{q+1})$ we consider

$$i_\alpha : S^p \times D^{q+1} \to M$$

defined by

$$i_\alpha(x, y) = i(x, \alpha(x)y)$$

where $SO_{q+1}$ acts on $D^{q+1}$ in the usual way. This is again an embedding of $S^p \times D^{q+1}$ since we are merely precomposing $i$ with a diffeomorphism. Note that $i_\alpha|_{S^p\times 0} = i|_{S^p\times 0}$ so $\lambda(i_\alpha) = \lambda(i) \in \pi_p(M)$. We shall rely heavily on the following lemma. Denote the inclusion $SO_{q+1} \to SO$ by $s$.

**Lemma 4.3.1.** $\mathfrak{o}(i_\alpha) = \mathfrak{o}(i) + s_*(\alpha)$

We will shortly embark on the proof, but let us first note that this addresses our concerns.

**Lemma 4.3.2.** *If $p \leqslant q$, then $\alpha$ can be chosen so that $\mathfrak{o}(i_\alpha) = 0$. I.e. $\chi(i_\alpha)$ can be framed, and in particular $\chi(M, i_\alpha)$ is s-parallelizable. If $p = q-1 \neq 1, 3, 7$, then $\alpha$ can be chosen so that $\chi(i_\alpha)$ can be framed. If $p = q-1 = 1, 3, 7$ the best we can say is that $\alpha$ can be chosen so that $\xi(i_\alpha)$ can be framed if and only if $\mathfrak{o}(i) \in \mathrm{im}(s_*)$.*

*Proof.* Clearly Lemma 4.3.1 implies that $\alpha$ can be chosen so that $\chi(i_\alpha)$ can be framed if and only if $\mathfrak{o}(i) \in \mathrm{im}(s_*)$. For $p \leqslant q$ the map $s_* : \pi_p(SO_{q+1}) \to \pi_p(SO_n)$ is onto. This is elementary since $SO_{q+1}$ and $SO_n$ has the same $p$-skeleton ($n = p+q+1$ still). The second statement follows from the less elementary fact that $s_*\pi_p(SO_p) \to \pi_p(SO)$ is onto for $p \neq 1, 3, 7$, as we proved in Lemma 2.14.2. □

*Proof of Lemma 4.3.1.* Recall that $\omega_\alpha = \omega(M, i_\alpha)$ deformation retracts onto $M \times I$ with a cell $D^{p+1} \times D^{q+1}$ attached using $i_\alpha : S^p \times D^{q+1} \to M \times \{1\}$. Extending a given trivialization $f : M \to V(TM \oplus \epsilon)$ to $M \times I$ is trivial, and extending further to $\omega_\alpha$ will be possible if $f$ can be extended over $D^{p+1} \times D^{q+1}$. So far $f$ is only defined on $i_\alpha(S^p \times D^{q+1})$. Making this extension at one point of $D^{p+1}$ is no more difficult than doing it on all of them at once, since $D^{p+1}$ is contractible. Let $e^l$ be the standard framing of $D^l$, $l$ any integer. Applying the tangent map we obtain a framing $i'_\alpha(e^{n+1})$ of $T\omega_\alpha$ restricted to $i_\alpha(D^{p+1} \times D^{q+1})$. The given framing $f$ can be extended over $D^{p+1} \times 0$ if and only if the map $g_\alpha : i_\alpha(S^p) \to SO_{n+1}$ given by $u \mapsto \langle f(u), i'_\alpha|_{(u,0)}(e^{n+1})\rangle$ is null-homotopic (extending over a disk is the same as providing a null-homotopy), in other words, $\mathfrak{o}(i_\alpha) = [g_\alpha]$.

We elaborate on this point: The trivialization $f : M \to V(TM \oplus \epsilon)$ gives us $n+1$ vector fields, $X_1, \cdots, X_{n+1}$. The assignment $u \mapsto i'_\alpha|_{(u,0)}(e^{n+1})$ also gives us $n+1$ vector fields, $Y_1, \cdots, Y_{n+1}$ over $i_\alpha(S^p)$. These are related by $Y_i = \sum_j a_{ij} X_j$ for continuous functions $a_{ij} : S^p \to \mathbb{R}$. Then $g_\alpha$ is assigning to a point $p$ the matrix $\{a_{ij}(p)\}$, which is invertible at every point since both $\{X_i\}$ and $\{Y_i\}$ are bases. Since $i$ is orientation preserving, $g_\alpha$ has positive determinant everywhere. Hence we can deform $g_\alpha$ into $SO_{n+1}$. Clearly the two bases represent the same element of $\pi_p(SO_{n+1})$ if and only if $g_\alpha$ is null-homotopic, meaning that one basis continuously can be deformed into the other simultaneously at every point of $S^k$. Clearly a trivialization of $\epsilon^N \oplus TS^k$ is homotopic to the standard framing if and only if it extends over $D^k$. Hence $\gamma(i_\alpha) = [g_\alpha]$.

Now we can make some computations. First of all, $i_\alpha|_{S^p\times 0} = i|_{S^p\times 0}$, so using $e^{n+1} = e^{p+1} \times e^{q+1}$ we may write

$$i'_\alpha(e^{n+1}) = i'(e^{p+1}) \times i'_\alpha(e^{q+1}). \tag{6}$$

Clearly it follows from the definitions that at the point $(u, 0)$,

$$i'_\alpha(e^{q+1}) = i'(e^{q+1}) \cdot \alpha(u).$$

Hence $g_\alpha = g \cdot s(\alpha)$, where $g = g_0$. Recalling that the group structure in $\pi_k(X)$ can be defined using the product of $X$ if $X$ is a topological group[6] the lemma follows. This is essentially an application of the Eckmann-Hilton argument, see [Whi12, Theorem 5.21] for an explicit elementary proof. $\square$

If $k = p = q-1 = 1, 3, 7$, then $\mathfrak{o}(i)$ may fail to be in the image of $s_*$. This is a good point to record an observation about $\mathfrak{o}(i)$ from the above proof. Denote by $\phi_0(i_\alpha)$ the homotopy class of the map $S^k \to V_{n+1,k}$ given by $u \mapsto \langle f(u), i'_\alpha|_{(u,0)} e^{k+1}\rangle$, or alternatively $p \mapsto \{a_{ij}(p)\} : \ i \leqslant k+1, \ j \leqslant 2k+1$. Our important observation is this: Equation (6) shows that $\phi_0(i)$ can be identified with the residue class of $\mathfrak{o}(i)$ modulo $s_*\pi_k(SO_k)$. We record for future reference:

**Lemma 4.3.3.** *For $k = 1, 3, 7$ there exist $\alpha$ such that the surgery $\chi(i_\alpha)$ can be framed if and only if $\phi_0(i) = 0$. In other words, if $\mathfrak{o}(i)$ is the obstruction to frame $\chi(i)$, and $p_*$ is the map induced by the projection $SO_{2k+1} \to V_{2k+1,k}$, then $\phi_0(i) = p_*(\mathfrak{o}(i))$.* $\square$

## 4.4 Computing $bP^{4k+3} = 0$

We need to examine in more detail the effect of surgery on the middle dimension. Suppose $M$ is a $(k-1)$-connected manifold of dimension $2k+1$, $k$ odd, and let $i : S^k \times D^{k+1}$ be an embedding such that $\chi(i)$ can be framed. We need to prove that by choosing $\alpha \in \ker s_*$, $s : SO_k \to SO_n$ appropriately we can guarantee that $M'_\alpha := \chi(M, i_\alpha)$ will be "homologically simpler" than $M$. We consider the manifold $M_0 = M \backslash i(\text{int } S^k \times D^{k+1})$. Of course the image of $i$ and $i_\alpha$ is the same for every $\alpha$. Also the class

$$\epsilon' := (i_\alpha)_*(x_0 \times S^k) = i_*(x_0 \times S^k) \in H_k(M_0, \mathbb{Z})$$

is independent of $\alpha$ since $\alpha(x_0) = id$, and so for each $y \in S^k$ we have

$$i_\alpha(x_0, y) = i(x_0, \alpha(x_0)y) = i(x_0, y).$$

The same does not hold for $\epsilon_\alpha = (i_\alpha)_*(S^k \times x_0)$. For $y \in S^k$ we have

$$i_\alpha(y, x_0) = i(y, \alpha(y)x_0)$$

and so we consider the class

$$\beta = [y \mapsto (y, \alpha(y)x_0)] \in \pi_k(S^k \times S^k) = H_k(S^k \times S^k).$$

This group is free abelian on the two generators $S^k \times x_0$ and $x_0 \times S^k$. Clearly $\beta = S^k \times x_0 + c(x_0 \times S^k)$ for some coefficient $c \in \mathbb{Z}$. Introducing the notation $j : SO_{k+1} \to S^k$ for the canonical map $j(\rho) = \rho(x_0)$ we can write $c = j_*(\alpha) \in \pi_k(S^k) = H_k(S^k) = \mathbb{Z}$. We have now shown that

$$\epsilon_\alpha = (i_\alpha)_*(S^k \times x_0) = \epsilon + j_*(\alpha)\epsilon' \in H_k(M_0; \mathbb{Z}).$$

Thus we ask which values $(j_k)_*$ attains on the elements $\alpha$ of $\ker s_*$. We have made this computation in Lemma 2.14.1, $j_*$ attains exactly the even numbers on $\ker s^*$. To summarize our work we recall the diagram of Lemma 4.1.7:

[6] The analogous statement if $X$ is merely an $H$-monoid is no more difficult to prove

$$\begin{tikzcd}
 & & H_{k+1}(V'_\alpha) \arrow[d, "\cdot\lambda"] & & \\
 & & \mathbb{Z} \arrow[d, "\epsilon_\alpha"] \arrow[dr, "\lambda"] & & \\
H_{k+1}(V) \arrow[r, "\cdot\lambda'"] & \mathbb{Z} \arrow[r, "\epsilon'"] \arrow[dr, "\lambda'_\alpha"'] & H_k(V_0) \arrow[r, "in_*"'] \arrow[d, "in'_*"] & H_k(V) \arrow[r] & 0 \\
 & & H_k(V'_\alpha) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

**Lemma 4.4.1.** *The above diagram is commutative and has exact row and column and we have the formula*

$$\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'$$

*where* $\epsilon = \epsilon_{id}$.

□

Let $l$ denote the order of $\lambda \in H_k(V)$. Then $l\epsilon \in \ker in_*$, hence is a multiple of $\epsilon'$, say $l'\epsilon' + l\epsilon = 0$. We insert $\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'$ and get

$$(l' - lj(\alpha))\epsilon' + l\epsilon_\alpha = 0.$$

Hence $(l' - lj(\alpha))\lambda'_\alpha = 0$. Thus by ensuring $0 < |l' - lj(\alpha)| < l$ we ensure that $H_k(V'_\alpha)$ is smaller than $H_k(V)$. This can clearly be achieved through an appropriate choice of the even number $j(\alpha)$ unless $l'$ is a multiple of $l$. We again summarize:

**Lemma 4.4.2.** *Let $M$ be a framed $k-1$ connected manifold of dimension $2k+1$ with $k$ odd such that $H_k(M)$ is a finite group. Suppose $\chi(i)$ is a spherical modification, replacing $\lambda$ of order $l$ with $\lambda'$ of order $l'$. Then, unless $l' \equiv 0 \mod l$, there exist a spherical modification $\chi(i_\alpha)$, which can be framed, such that $|H_k(M'_\alpha)| < |H_k(M)|$.* □

To deal with the problem of $l' \equiv 0 \mod l$ we apply the notion of linking number, see Lemma 2.4.1.

**Lemma 4.4.3.** *With notation as above, $L(\lambda, \lambda) = \pm l'/l \in \mathbb{Q}/\mathbb{Z}$*

*Proof.* Recall that by definition of $l'$ we have $l\epsilon + l'\epsilon' = 0$ in $H_k(V_0)$, hence $l\epsilon + l'\epsilon' = \partial c$ for some $c \in C_{k+1}(V_0)$. Since $in_*$ is onto, of course the same relations hold in $V$. Letting $c_1$ denote the chain $i(x_0 \times D^{k+1}) \subset V$, we see that $\partial c_1 = \epsilon'$. To represent $\lambda$, we use the chain $c_2 = i(S_k \times x_0)$. We have $l\epsilon = \partial(c - l'c_1)$, hence by Lemma 2.4.1 (note $in_*\epsilon = \lambda$)

$$L(\lambda, \lambda) = c_2 \cdot (c - l'c_1)/l.$$

Clearly $c_2 \cdot c = 0$ since $c$ is a chain completely disjoint from im $i$. $c_2$ intersect $c'$ in the point $i(x_0, x_0)$. It is evident that the intersection is transverse, so $c_2 \cdot c_1 = \pm 1$. This finishes the proof. □

Hence saying that for all spherical modifications, $l' \equiv 0 \mod l$, is equivalent to saying that all self intersection numbers vanish, i.e. for all $\lambda$, $L(\lambda, \lambda) = 0$. Since $k$ is odd, and $\lambda \in H_k(V)$, we get from Lemma 2.4.1 that $L(\lambda, \mu) = L(\mu, \lambda)$. Together with the vanishing of self intersection numbers we thus obtain

$$0 = L(\lambda + \mu, \lambda + \mu) = L(\lambda, \lambda) + L(\lambda, \mu) + L(\mu, \lambda) + L(\mu, \mu) = 2L(\lambda, \mu) = L(2\lambda, \mu).$$

By non-degeneracy of the intersection pairing, which follows from Poincaré duality, we deduce $2\lambda = 0$. We have shown:

**Lemma 4.4.4.** *If $M$ is a $2k+1$ dimensional manifold, $k$ odd, for which $H_k(M)$ is a torsion group with vanishing linking form, then $H_k(M)$ is a direct sum of $\mathbb{Z}_2$'s.*

$$H_k(M) = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 =: s\mathbb{Z}_2$$

$\square$

Clearly $H_k(M) = H_k(M; \mathbb{Z}_2)$ where $M$ is as in the above lemma. We again consider a spherical modification, $\chi(i)$, killing $\lambda \in H_k(M)$. Consider$V = \omega(M, i)$, the cobordism between $M$ and $\chi(M, i)$. Since $M$ and $\chi(M, i)$ have vanishing homology in degrees $1 \leqslant i \leqslant k-1$, the groups $H^{k+1+i}(V, \partial V; \mathbb{Z}) = 0$ for those values of $i$. In particular $H^{2k+1}(V; \mathbb{Z}_2)) = 0$. Using the Adem relation,[7] see [Ade52], $Sq^1 Sq^k = Sq^{k+1}$ it follows that $(Sq^{k+1} : H^{k+1}(V; \mathbb{Z}_2) \to H^{2k+2}(V; \mathbb{Z}_2)) = 0$, and so all elements of $H_{k+1}(V)$ have self intersection 0. Hence Lemma 4.2.4 applies, and $\dim H_k(M'; \mathbb{Z}_2) \neq \dim H_k(M'; \mathbb{Z}_2)$ where $M' = \chi(M, i)$. The effect of $\chi(i)$ is to replace $\lambda$, which has order 2, with $\lambda'$ of order $l'$. We can assume $-2 < l' \leqslant 2$, which in combination with the fact that $l'$ is even implies $l' = 0$ or $l' = 2$. We have an exact sequence, where if $l' = 0$, $\mathbb{Z}_0 = \mathbb{Z}$.

$$\mathbb{Z}_{l'} \xrightarrow{\lambda'} H_k(M') \longrightarrow H_k(M)/\Lambda$$

If the sequence splits, the rank of $H_k(M'; \mathbb{Z}_2)$ would also be $s$, this is impossible by Lemma 4.2.4. Hence $H_k(M')$ is a non-trivial extension. There are only the possibilities:

$$H_k(M') = (s-2)\mathbb{Z}_2 \oplus \mathbb{Z}, \qquad H_k(M') = (s-2)\mathbb{Z}_2 \oplus \mathbb{Z}_4$$

In the first case, we can kill a generator of the free direct summand $\mathbb{Z}$ by Lemma 4.1.8. In the second case, since $H_k(M')$ is not a direct sum of copies of $\mathbb{Z}_2$, it follows from Lemma 4.4.4 that not every element of $H_k(M')$ has self-linking number 0. A framed surgery killing a $\lambda$ for which $L(\lambda, \lambda) \neq 0$ is by Lemma 4.4.3 replacing $\lambda$ of order $l$ by $\lambda'$ of order $l'$ such that $l' \not\equiv 0 \mod l$, hence Lemma 4.4.2 allowes us to conduct framed surgery to obtain $M''$ with $|H_k(M'')| < |H_k(M')| = |H_k(M)|$. Since $|H_k(M)| \leqslant \infty$, proceeding in this manner we can make $M$ $k$-connected. We conclude:

**Theorem 4.4.5.** *A framed manifold $M$ of dimension $4k+3$ can through framed surgery be reduced to $k$-connected manifold. Hence, if $M$ is closed, a homotopy sphere is obtained. If $M$ is bounded by a homotopy sphere, a contractible manifold is obtained.*

[7] The Steenrod squares generates the algebra $\mathscr{A}$ of stable cohomology operations with $\mathbb{Z}_2$ coefficients. The only relations are the Adem relations. What we are using here is the important property that for $x \in H^i(X; \mathbb{Z}_2)$ we have $Sq^i(x) = x \smile x$.

*Proof.* The finishing details are the same as for the corresponding results for $k$ even, see Lemma 4.2.2, and the proof immediately above it. $\square$

This concludes the proof of Theorem 4.1.2.

# 5 The Tools to Compute $bP^{2k}$

So far we have managed to show that $bP^{2k+1} = 0$. The proofs also showed that any framed closed manifold of dimension $2k+1$ can be reduced through framed surgery to a homotopy sphere. Next we study the same questions in even dimensions, $n = 2k$. There are certain analogs between the cases $k$ even and $k$ odd. In either case the relevant notion is the intersection form: The intersection product induces a non-degenerate bilinear form on $H_k(M;\mathbb{F})$ for any field $\mathbb{F}$. The obvious difference is that the intersection form is symmetric for $k$ even, and antisymmetric for $k$ odd. Hence we cannot expect a completely uniform treatment of the two. However there are staggering similarities. The ensuing expositions aim at emphasizing those similarities, as well as highlight the differences.

In these dimensions Lemma 4.1.7 is not available anymore. The following lemma takes its place as the main computational tool.

**Lemma 5.1.2.** *Let $M$ be a $(k-1)$-connected manifold of dimension $2k$, $k \geqslant 3$ and suppose $H_k(M)$ is free abelian on $\{\lambda_i, \mu_i\}_{i=1}^r$ satisfying*

$$\lambda_i \cdot \lambda_j = 0, \qquad \text{and} \qquad \lambda_i \cdot \mu_j = \delta_{ij},$$

*for all $1 \leqslant i, j \leqslant r$. If in addition any embedding representing an element of the group generated by $\{\lambda_i\}$ has a trivial normal bundle, then $M$ can through spherical modifications be reduced to $M'$ with $H_k(M') = 0$.*

Note that we are not assuming that the spherical modifications are framed. Of course, unless $k = 3, 7$, we can frame the spherical modifications by Lemma 4.3.2, and so deduce that any framed manifold of dimension $2k \neq 6, 14$ admitting a basis as in Lemma 5.1.2 is framed cobordant to a contractible manifold or a homotopy sphere. However in the cases $k = 3, 7$ it is necessary to further assume that the surgeries removing $\lambda_i$ can be framed to make the cobordism framed.

*Proof of Lemma 5.1.2.* Since[8] $k \geqslant 3$, any homology class of $H_k(M)$ can be represented by an embedding, $S^k \to M$ by Lemma 2.13.2. Choose an embedding $\varphi_0$ representing $\lambda_r$. Since the normal bundle of $\varphi_0$ is trivial, $\phi_0$ extends to an embedding $\phi : S^k \times D^k \to M$. Let $M' = \chi(M, \phi)$. Note that a priori $\phi' : D^{k+1} \times S^{k-1} \to M'$ might represent a non-trivial element of $H_{k-1}(M')$. Let $M_0 = M \setminus \operatorname{int} \varphi(S^k \times D^k) = M' \setminus \operatorname{int} \varphi'(D^{k+1} \times S^{k-1})$. We will consider the long exact sequences of homology groups of the pairs $(M, M_0)$ and $(M', M_0)$. By excision

$$H_i(M, M_0) = H_i(S^k \times D^k, S^k \times S^{k-1}) = \begin{cases} \mathbb{Z} & i = k \\ 0 & \text{otherwise} \end{cases}$$

and the generator for $i = k$ is the relative homology class of $\varphi(x_0 \times D^k) \subset M$ which clearly has intersection number 1 with the chain $\varphi(S^k \times 0)$ representing $\lambda$. Hence the exact sequence of the

[8]For $k = 2$ this fails: Not every homology class of $H_2(M^4)$ can be represented by an embedding $S^2 \to M$

pair $(M, M_0)$ is equivalent with:

$$0 \longrightarrow H_k(M_0) \longrightarrow H_k(M) \xrightarrow{\cdot\lambda_r} \mathbb{Z} \longrightarrow H_{k-1}(M_0) \longrightarrow 0$$

proving that $H_{k-1}(M_0) = 0$ by the existence of $\mu_r$. Similarly $H_{k+1}(M', M_0) = \mathbb{Z}$ is the only non-zero homology group of the pair $(M', M_0)$, and it is generated by $\varphi'(D^{k+1}, x_0)$ which maps to $\phi'(S^k, x_0)$ under the connecting homomorphism. It follows that the following diagram commute:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& \mathbb{Z} \arrow[d] \arrow[dr, "\lambda_r"] & & & \\
0 \arrow[r] & H_k(M_0) \arrow[r] \arrow[d] & H_k(M) \arrow[r, "\cdot\lambda_r"] & \mathbb{Z} \arrow[r] & 0 \\
& H_k(M') \arrow[d] & & & \\
& 0 & & &
\end{tikzcd}$$

The kernel of $\cdot\lambda_r$ is a direct summand of $H_k(M)$ containing $\operatorname{span}\{\lambda_1, \cdots, \lambda_r\}$, say

$$\ker \cdot\lambda_r = \operatorname{span}\{\lambda_1, \cdots, \lambda_r, \mu'_1, \cdots, \mu'_{r-1}\} \simeq H_k(M_0).$$

To obtain $H_k(M')$ we are quotienting out the class which maps to $\lambda_r$ under this isomorphism, hence $\{\lambda''_1, \cdots, \lambda''_{r-1}, \mu''_1, \cdots, \mu''_{r-1}\}$ is a basis for $H_k(M')$ where an isomorphism (which respects the intersection pairing) makes the identifications $\lambda''_i = \lambda_i + \lambda\mathbb{Z}$ and $\mu''_i = \mu_i + \lambda_r\mathbb{Z}$. Each $\lambda''_i$ and $\mu''_j$ is represented by any representative of $\lambda_i$ and $\mu'_j$, respectively, in $H_k(M_0)$. From this observation it is clear that $M'$ satisfies the conditions of the lemma. Hence, iterating $r$ times, we obtain the conclusion of the lemma. $\square$

As in [KM63] we first deal with the case $k = 1, 3, 7$. In these cases $\pi_{k-1}(SO_k) = 0$. [This is a consequence of the $H$-space structure on $S^k$. To reference that essential component of the proof elsewhere, the stable groups $\pi_2(SO)$ and $\pi_6(SO)$ are 0 by Bott periodicity. Lemma 2.14.1 implies that $i_* : \pi_{k-1}(SO_k) \to \pi_k(SO)$ is injetive for $k = 3, 7$.] Hence any embedding $S^k \to M^{2k}$ has trivial normal bundle: There simply are no nontrivial clutching functions. Since $k$ is odd the intersection form is skew-symmetric. Any abelian group with a non-degenerate alternating bilinear form admits a symplectic basis Lemma 5.1.5. Being symplectic is stronger than the assumption of Lemma 5.1.2 which therefore immediately yields:

**Lemma 5.1.3.** $bP^2 = bP^6 = bP^{14} = 0.$ $\square$

Before continuing with serious mathematics, let us note why the proof of Lemma 5.1.3 fail to generalize to $bP^{2k}$. If $k$ is odd, but not $1, 3, 7$, then there is still a symplectic basis, but the embedded spheres representing it may not have trivial normal bundles, and so Lemma 5.1.2 may not apply. This seem to suggests that if we can find a symplectic basis, $\{x_i, y_i\}$, such that any embedded sphere representing $x_i$ has a trivial normal bundle for every $i$, then $H_k(M)$ can be surgically removed. This, and the converse statement, indeed holds true. A more detailed account of this remarkable story is given in section 7.

If $k$ is even there might not be a basis for $H_k(M)$ satisfying the assumptions of Lemma 5.1.2. This would for example be the case if the intersection form could be represented by the identity matrix. In this case the intersection pairing is positive definite, hence $H_k(M)$ is an inner product space. (At least if we have coefficients in $\mathbb{R}$.) Certainly no inner product space can have a basis where half of the basis elements have length 0: $\|\lambda_i\|^2 = \lambda_i \cdot \lambda_i = 0$. Hence definiteness of the intersection form could represent an impass. However it turns out that if the basis can be found, the normal bundles will all be trivial. Thus the assumptions of Lemma 5.1.2 that possibly can fail is very much dependent on the parity of $k$.

Before specializing to $k$ even, we prove some lemmas that are needed both for $k$ even, and odd.

**Lemma 5.1.4.** *If $V$ is a finite dimensional vector space over a field $\mathbb{F}$, $B$ is a symmetric bilinear non-degenerate form on $V$, and $W \subset V$ is a subspace of half the dimension of $V$ on which $B$ restricts to 0: Then there is a basis $\mathscr{B} = \{x_i, y_i\}_{i=1}^r$ for $V$ with $x_i \in W$, such that the matrix of $B$ with respect to $\mathscr{B}$ is $\operatorname{diag}(H, \cdots, H)$ where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

*Proof.* Let $x_1 \in W$. By non-degeneracy of $B$ we can find $y_1'$ such that $B(x_1, y_1') = 1$. Then define $y_1 = y_1' - B(y_1', y_1')x_1$: Now $B(y_1, y_1) = 0$, and $B(x_1, y_1) = 1$. Let the orthogonal complement of $\operatorname{span}(x_1, y_1)$ be denoted $V_1$, and $W_1 := W \cap V_1$. Then $B$ restricts to a non-degenerate form on $V_1$ which restricts to 0 on $W_1$. The second claim is immediate since $W_1 \subset W$. For the first claim, if $y \in V_1$, then there exist $x \in V$ with $B(x, y) = 1$. Note that $x$ is not in $\operatorname{span}(x_1, y_1)$, for $B(x_1, y) = B(y_1, y) = 0$. Hence $V_1 \ni x' = x - B(x, x_1)y_1 - B(x, y_1)x_1 \neq 0$, and $B(x', y) = 1$. Hence $B|_{V_1}$ is non-degenerate. Finally note that $\dim V_1 = \dim V - 2$ and that $\dim W_1 = \dim W - 1$. The first of these statements is completely trivial. For the second one, note that $x_1 \in W$, so $\dim W_1 \leqslant \dim W - 1$. Conversely, $y_1 \notin W$, so $\dim W_1 \geqslant \dim W - 1$. Clearly $H$ represents $B$ with respect to the basis $\{x_1, y_1\}$ for $\operatorname{span}(x_1, y_1)$. We conclude that after $r$ repetitions we have obtained $y_1, \cdots, y_r$ such that the matrix representing $B$ with respect to $\{x_1, y_1, \cdots, x_r, y_r\}$ is block diagonal with $r$ copies of the matrix $H$. □

**Lemma 5.1.5.** *If $A$ is a free abelian group and $B$ is a non-degenerate alternating bilinear form on $A$, then $A$ admits a basis $\mathscr{B} = \{x_i, y_i\}_{i=1}^r$ such that $\operatorname{diag}(H', \cdots, H')$ represents $B$ with respect to $\mathscr{B}$ where $H' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $H \subset A$ is a subgroup of half rank on which $B$ restricts to 0, then we can assume $x_i \in H$ for each $i$.*

*Proof.* The proof is entirely analogous to that of Lemma 5.1.4. □

**Lemma 5.1.6.** *Let $M$ be a manifold of dimension $2k$, ($k$ even or odd) bounding a manifold $V$, $M = \partial V$. Let $i : M \to V$ be the inclusion map. The following is true of the induced map $i_* : H_k(M; \mathbb{F}) \to H_k(V; \mathbb{F})$ for any field $\mathbb{F}$.*

1. $(\ker i_*)^\perp = \ker i_*$
2. $2 \dim \ker i_* = \dim H_k(M; \mathbb{F})$

*Proof.* We consider a the commutative Poincaré duality ladder of the pair $(V, M)$. The rows are exact, and the columns isomorphisms.

$$\begin{tikzcd} H^k(N) \arrow[r, "i^*"] \arrow[d, "PD"] & H^k(M) \arrow[r, "\delta"] \arrow[d, "PD"] & H^{k+1}(N, M) \arrow[d, "PD"] \\ H_{k+1}(N, M) \arrow[r, "\partial"] & H_k(M) \arrow[r, "i_*"] & H_k(N) \end{tikzcd}$$

Coefficients are in $\mathbb{F}$. The reason we choose to have coefficients in a field is to have available the natural isomorphism $\psi : H^i(X; \mathbb{F}) \simeq \mathrm{Hom}(H_i(X; \mathrm{F}), \mathrm{F})$, i.e. the Kronecker pairing, $\psi(x)(\tau) = \langle x, \tau \rangle$ is non-degenerate. We observe that $\ker i_*$ corresponds to im $i^*$ under Poincaré duality. Suppose $PD(x), \tau \in \ker i_*$. Then we have $i^*(z) = x$ for some $z \in H^k(N)$, and

$$\langle i^*(z), \tau \rangle = \langle z, i_*(\tau) \rangle = \langle z, 0 \rangle = 0$$

which implies $\ker i_* \subset (\ker i_*)^\perp$. Suppose $PD(x) \in (\ker i_*)^\perp$. Then for every $y \in H_{2k+1}(N, M; \mathbb{Z})$ we get

$$\langle \delta x, y \rangle = \langle x, \partial(y) \rangle = 0.$$

By non degeneracy of the Kronecker pairing (which holds since $\mathbb{F}$ is a field) we conclude $\delta x = 0$ and by exactness $x \in$ im $i^*$.

The second claim we prove by induction. The relevant data is that the long exact sequence for the pair $(N, M)$, up to isomorphism, has the form

$$0 \longrightarrow A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} A_r \xrightarrow{f_r} H_k(M) \xrightarrow{g_r} A_r \longrightarrow \cdots \xrightarrow{g_1} A_1 \longrightarrow 0\ .$$

The result follows directly from the rank-nullity theorem in the case $r = 1$. For the induction step, note that $\ker f_2 =$ im $f_1 \simeq A_1 \simeq$ coker $g_2$. Hence, replacing $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ by im $f_2 \longrightarrow A_3$ and similarly in the other end, we have obtained a shorter exact sequence, not altering the maps immediately around $H_k(M)$. Thus by induction $\dim H_k(M) = 2 \dim \ker i_*$. $\square$

# 6 The Signature of a Manifold and $bP^{4k}$

When $n = 2k$, $k$ even, the intersection form is symmetric: We have the formula $x \smile y = (-1)^{i+j} y \smile x$ where $i$ and $j$ are the degrees of $x$ and $y$ [Hat02, Theorem 3.11]. It is standard linear algebra that with real coefficients we may choose a basis for $H_k(M; \mathbb{R})$ such that the matrix $D$ satisfying $[x]^t D[y] = x \cdot y$ is diagonal. Here $[x]$ is the vector representing $x$ with respect to the given basis. The signature of $M$, $\sigma(M)$, is defined to be the signature of the intersection form on $H_k(M; \mathbb{R})$. That is, if the intersection pairing is represented by

$$\begin{pmatrix} D_p & 0 \\ 0 & -D_n \end{pmatrix},$$

where $D_p$ and $D_n$ are positive definite diagonal matrices of size $p \times p$ and $n \times n$, then $\sigma(M) = p - n$. If $\dim M \not\equiv 0 \mod 4$ we say that $M$ has signature $\sigma(M) = 0$. Note that if $M$ is $k-1$ connected, and $H_k(\partial M; \mathbb{Z}) = 0 = H_{k-1}(\partial M; \mathbb{Z})$, then $H_k(M; \mathbb{Z})$ is free: The homology assumption on $\partial M$ assures $H_k(M; \mathbb{Z}) \simeq H_k(M, \partial M; \mathbb{Z})$ and the latter group is isomorphic to

$H^k(M;\mathbb{Z})$ by Poincaré duality. Now it follows from the universal coefficient theorem, stating $H^k(M;\mathbb{Z}) \simeq \mathrm{Hom}(H_k(M;\mathbb{Z}),\mathbb{Z}) \oplus T_{k-1}$ where $T_{k-1} \subset H_{k-1}(M;\mathbb{Z}) = 0$, that $H_k(M;\mathbb{Z})$ is free. We state the Hizebruch signature theorem:

**Theorem 6.1.1** (Hirzebruch Signature Theorem)**.** *Let $M$ be a manifold of dimension $n = 4l$. Then $\sigma(M)$ is a polynomial evaluated at the Pontryagin numbers of $M$. The coefficient of the top Pontyagin number $p_l[M]$ is $2^{2l}(2^{2l-1}-1)B_l$ where $B_l$ is the $l$-th Bernoulli number.*

*Proof.* There is a proof in [MS74]. See also [Hir13]. □

The other coefficients are also computable, but we wish to get on with the main story and so state the signature theorem in this simplified form.

**Lemma 6.1.2.** *$\sigma$ distributes over connected sums. That is, for manifolds $M$ and $M'$ of dimensions $2k$, $0 < k$ even, the following equality holds:*

$$\sigma(M\#M') = \sigma(M) + \sigma(M')$$

*Proof.* We have seen in Lemma 3.1.3 that the intersection pairing splits over the connected sum. In particular, so does the intersection form. □

**Theorem 6.1.3.** *The signature is invariant under cobordisms.*

*Proof.* This proof is essentially taken from lectures notes by Dan Freed. It suffices to show that if $V$ is a manifold with connected boundary $\partial V = M$, then $\sigma(M) = 0$. For if that is so, and $M$ is cobordant to $N$, i.e. $M\#-N$ is a boundary (Lemma 3.1.4), then $0 = \sigma(M\#-N) = \sigma(M)-\sigma(N)$ and so $\sigma(M) = \sigma(N)$. By Lemma 5.1.6 $\ker i_* \subset H_k(M;\mathbb{Q})$ satisfies the assumption of Lemma 5.1.4. Note that the matrix $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has characteristic polynomial $x^2-1$, and so eigenvalues $\pm 1$, hence signature $1-1 = 0$. It is elementary that $\mathrm{sign}(\mathrm{diag}(A_1, \cdots, A_n)) = \sum_i \mathrm{sign}(A_i)$, hence $\sigma(M) = r \cdot 0 = 0$. □

**Lemma 6.1.4** (Lemma 7 of [Mil61])**.** *Let $\beta \in H_k(M)$ be represented by an embedding $f : S^k \to M$. The normal bundle of $f$, $\nu(f)$, is trivial if and only if $\beta \cdot \beta = 0$.*

*Proof.* We think of $\nu(f)$ as a subset of $M$, which is justified since $\nu(f)$ is diffeomorphic with a tubular neighborhood of $f(S^k)$. One direction is trivial: If $\nu(f)$ is trivial, then $f$ extends to an embedding $F : S^k \times D^k \to M$, where $f = F_{S^k\times\{0\}}$. Then $F_{S^k\times\{v\}}$, $v \neq 0$, clearly is homologous with $f$ and does not intersect $f$ so $\beta \cdot \beta = 0$. Conversely, suppose $\beta \cdot \beta = 0$ and let $g$ be another embedding representing $\beta$ such that the images of $f$ and $g$ are disjoint. We may assume $\mathrm{Im}(g) \subset \nu(f)$. Hence $g$ is a non-zero section of $\nu(f)$. By [MS74, Proposition 9.7] we deduce that the Euler class vanishes, $\chi(\nu(f)) = 0$. Recall that $\chi(TS^k)[S^k]$ is the Euler characteristic of $S^k$, which is 2 for $k$ even and 0 for $k$ odd. Note that $[f] \in \ker i_*$ since $M$ being stably paralellizable implies that $\nu(f)$ is stably trivial. All we need is to show that $\chi$ is injective on $\ker i_*$. Knowing $\chi(TS^k) \neq 0$, $\ker i_* = \partial\pi_{k+1}(S^{k+1}) \simeq \mathbb{Z}$, ($\partial$ is injective since $j_k \circ \partial = 2\mathbb{Z}$) and that $\chi$ takes values in the group $H^k(S^k) \simeq \mathbb{Z}$, it suffices to show $TS^k \in \mathrm{Im}(\partial)$. But this is of course the case since $TS^k$ is stably trivial. □

Note that in the process of proving Lemma 6.1.4 we proved a sharpened version of the hairy ball theorem: The only stably trivial vector $k$-vector bundle over $S^k$, $k$ even, which admit a nowhere vanishing section is the trivial bundle. We shall not use this, but it is a pleasant result. Note also that in order to apply Lemma 5.1.2 we need now only show the existence of the basis for $H_k(M;\mathbb{Z})$.

**Theorem 6.1.5.** *Let $M$ be a framed manifold of dimension $2k$, $k$ even, with $\partial M$ a homotopy sphere (respectively $\emptyset$). Then $M$ can through surgery be reduced to a contractible manifold (respectively a homotopy sphere) if and only if $\sigma(M) = 0$.*

*Proof.* If through surgery $M$ can be reduced to a manifold $M'$ for which $H_k(M') = 0$, then by cobordism invariance of $\sigma$, $\sigma(M) = \sigma(M') = 0$. The converse is harder to prove.

We can assume $M$ to be $k-1$ connected by Lemma 4.1.6. Recall that $H_k(M;\mathbb{Z})$ is free abelian, and $\sigma(M) = 0$ implies even rank, say $2r$. We wish to apply Lemma 5.1.2, and therefore we seek a basis $\{\lambda_1, \mu_1 \cdots, \lambda_r, \mu_r\}$ for $H_k(M;\mathbb{Z})$ satisfying $\lambda_i \cdot \lambda_j = 0$ and $\lambda_i \cdot \mu_j = \delta_{ij}$. We apply lemmas 5.1.6 and 5.1.4 to deduce that the sought basis exist over the rational numbers. Adjusting it by multiplying with integers, we obtain indivisible $\lambda_i \in H_k(M;\mathbb{Z})$ satisfying $\lambda_i \cdot \lambda_j = 0$ for all $i$ and $j$. By non-degeneracy we obtain for each $0 < i \leqslant r$ the required $\mu_i$ satisfying $\lambda_i \cdot \mu_i = 1$. By Lemma 6.1.4 any embedded sphere representing an element of $\mathrm{span}(\lambda_1, \cdots, \lambda_r)$ has trivial normal bundle. We conclude that $H_k(M)$ can be killed by applying Lemma 5.1.2. Since $k \neq 1, 3, 7$ the surgeries can be framed by Lemma 4.3.2. The manifold thus obtained is contractible if $\partial M$ is a homotopy sphere, and a homotopy sphere if $M$ is closed. □

**Corollary 6.1.6.** *If $M$ is a framed manifold of dimension $4k$ with $\partial M = \emptyset$, then $M$ is framed cobordant to a homotopy sphere.*

*Proof.* The Pontryagin classes of a vector bundle $E$ are the same as those of $E \oplus \epsilon$, see [MS74]. Hence $M$ being stably paralelliable implies that all the Pontryagin classes $p_i$ of $TM$ vanish. Therefore the Pontryagin numbers, being products of the $p_i$s evaluated at $[M]$, vanish too. The Hirzebruch signature theorem, Theorem 6.1.1, states that $\sigma(M)$ is a polynomial in those numbers, hence $\sigma(M) = 0$. We conclude the proof by reference to Theorem 6.1.5. □

We recall the two lemmas we stated earlier, Lemma 3.4.2 and Lemma 3.4.3. Note that for $n = 2k = 4l$ $J$ is a homomorphism $\mathbb{Z} \simeq \pi_n(SO) \to \pi_n(\mathbb{S})$ by Bott periodicity, Theorem 3.4.1. It is well known that the target group is finite, hence $J$ has a nontrivial kernel. Let $j_n = |Im(J)|$ denote the order of the image of $J$, and let $B_n$ be the $n$-th Bernoulli number. As in Lemma 3.4.2, let $a_l$ be 1 for $l$ even and 2 for $l$ odd. (One way to remember this convention is that $a_l$ is "evenizing" $l$.)

**Theorem 6.1.7.** *There exists a framed manifold $M_0^{4l}$ with boundary $\partial M_0 = S^{4l-1}$ and signature*

$$\sigma_0 = \sigma(M_0) = B_l j_l a_l 2^{2l}(2^{2l-1} - 1)/l,$$

*and the signature of any other framed manifold bounded by $S^{n-1}$ is a multiple of $\sigma_0$.*

*Proof.* Denote a generator of $\ker J$ by $\alpha_0$. By lemma 3.4.2 there exists an almost framed manifold $(M_0'^n, F)$ such that $\alpha_0 = \mathfrak{o}^n(F)$. Lemma 3.4.3 gives us knowledge about the Pontryagin numbers of $M_0'$: First of all, $p_l[M_0'] = a_l(2l-1)!c^n(F)$. Seccondly $p_i(M_0') = 0$ for $i < l$, for we could trivialize $TM_0'$ over the smaller skeleta of $M_0'$. Now the Hirzebruch signature theorem implies the first of the following equalities:

$$\begin{aligned}\sigma(M_0') &= 2^{2l}(2^{2l-1}-1)B_l p_n[M_0']/2l! \\ &= 2^{2l-1}(2^{2l-1}-1)B_l a_l c^n(F).\end{aligned}$$

The second equality is just plugging in the value of $p_n[M_0']$. Finally, since $\alpha_0 = c^n(F)$ is the generator of $\ker J$, it is $\pm j_n$, upon fixing an isomorphism $\pi_{4l-1}(SO) \simeq \mathbb{Z}$. To obtain $M_0$ punch out the interior of an embedded disk containing the unframed point. This does not change the signature since it does not change the intersection form. Since the smooth structure of $M_0$ extends over the punched out disk, $\partial M_0 = S^{n-1}$. Conversely we can fill in a disk $D^n$ into any other framed manifold $M$ bounded by $S^{n-1}$ to obtain an almost framed smooth closed manifold. Hence $\sigma(M)$ is a multiple of $\sigma_0$ by the fact that $\alpha_0$ is a generator. $\square$

The converse also holds: If a homotopy $(2k-1)$-sphere bounds a manifold $M$ with $\sigma(M) = m\sigma_0$, then $\partial M = S^{2k-1}$. We shall shortly state and prove a sharper result. First note that if a parallelizable manifold is bounded by an exotic sphere, we cannot fill in a disk to obtain an almost parallelizable manifold. Therefore almost framed manifolds corresponds to framed manifolds bounded by spheres with the standard smooth structure. Some of the numbers $\mathbb{N}\backslash\sigma_0\mathbb{N}$ will make an appearance as the signature of manifolds $M$ bounded by homotopy spheres. Thus constructing such manifolds $M$ is one way of giving examples of exotic smooth structures. In [Mil59] Milnor refines this idea and obtains a better invariant.

**Lemma 6.1.8.** *If homotopy spheres $\Sigma_1$ and $\Sigma_2$ of dimension $2k-1$ bounds parallelizable manifolds $M_1$ and $M_2$ respectively, then $\Sigma_1$ is h-cobordant to $\Sigma_2$ if and only if $\sigma(M_1) \equiv \sigma(M_2) \mod \sigma_0$.*

*Proof.* Suppose $\sigma(M_1) = \sigma(M_2) + m\sigma_0$. Note that $m\sigma_0$ is the signature of the connected sum of $m$ copies of $M_0$, $mM_0 := M_0\#\cdots\#M_0$, the connected sums being along the boundary so that $\partial mM_0 = S^{2k-1}$. We form the following connected sum along the boundaries:

$$(M, \partial M) = (-M_1, -\Sigma_1)\#(M_2, \Sigma_2)\#(mM_0, S^{2k-1})$$

Then

$$\partial M = (-\Sigma_1)\#\Sigma_2\#S^{2k-1} = (-\Sigma_1)\#\Sigma_2$$

and

$$\sigma(M) = -\sigma(M_1) + \sigma(M_2) + m\sigma_0 = 0.$$

Hence, by Theorem 6.1.5, $-\Sigma_1\#\Sigma_2$ is $h$-cobordant to $S^{2k-1}$. Conversely, if $\Sigma_1$ is h-cobordant to $\Sigma_2$, then $(-\Sigma_1)\#\Sigma_2 = S^{2k-1}$, hence $(M, \partial M) = (-M_1, -\Sigma_1)\#(M_2, \Sigma_2)$ is bounded by $S^{2k-1}$ and has signature $\sigma(M) = -\sigma(M_1)+\sigma(M_2)$. Filling in a disk we obtain a smooth, almost parallelizable manifold with the same signature: Changing the $2k$ skeleton has no impact on the intersection form since $2k-k > 1$. Hence $-\sigma(M_1) + \sigma(M_2)$ is a multiple of $\sigma_0$. $\square$

**Corollary 6.1.9.** *$bP^{4k}$ is isomorphic to a subgroup of $\mathbb{Z}_{\sigma_0}$. In particular we deduce that $bP^{4k}$ is finite cyclic.*

In fact the order of $bP^{4k}$ is $\sigma_0/8$. A proof of this can be extracted from [Lev85].

In [Ada66, Theorem 1.5] Adams prove that $j_l$ is the denominator of $B_l/4l$, when expressed in lowest terms. Hence we obtain the following result.

**Theorem 6.1.10.** *$bP^{4k}$ is cyclic of order*

$$\sigma_0 = a_k 2^{2k-2}(2^{2k-1} - 1) \cdot \text{numerator}(B_k/4k)$$

*Proof.* We have stated that $|bP^{4k}| = \sigma_0/8$. Using this, Adams' result on $j_k$ and $\sigma_0 = 2^{2k-1}(2^{2k-1} - 1)B_k j_k a_k/k$, this is a straight forward calculation. $\square$

# 7 The Kervaire Invariant

## 7.1 Defining $\Phi$

In this section we will consider framed manifolds, $(M, F)$, which are compact, $(k-1)$-connected, and of dimension $n = 2k$, $k$ odd. We follow [Lev85] in using a very geometric definition of the Kervaire invariant. Non the less it is the same invariant that Kervaire introduced in [Ker60], and Kervaire and Milnor used in [KM63] to prove that $bP^{4k+3} = 0$ or $\mathbb{Z}_2$.

Suppose we are given an immersion $f : S^k \to M$. The trivialization $F : M \to V(TM \oplus \epsilon^N)$ of $TM \oplus \epsilon^N$ yields a trivialization $f^*F = F \circ f$ of $\epsilon^N \oplus TS^k \oplus \nu$ where $\nu = \nu(f)$ is the normal bundle of the immersion. We think of $f^*F$ as $N + 2k$ everywhere linearly independent vector fields, $\{X_1, \cdots, X_{N+2k}\}$. We can define a framing, $F_0 = \{Y_1, \cdots, Y_{N+k}\}$, of $\epsilon^N \oplus TS^k$ by restricting the standard framing of $\epsilon^{N-1} \oplus TD^{k+1}$ to $S^k$ and identifying the extra trivial bundle with the normal bundle of $S^k$ in $D^{k+1}$. Combining these two framings we are to each point of $S^k$ assigning a $N+k$ frame in $\mathbb{R}^{N+2k}$. We can express this by saying that there are functions $g^f_{ij} : S^k \to \mathbb{R}$ such that for each $i = 1, ..., N + k$ we have

$$Y_i = \sum_{j=1}^{N+2k} g^f_{ij} X_j$$

Hence we have obtained a map $f' : S^k \to V_{N+2k,N+k}$ given by $p \mapsto \{g^f_{ij}(p)\}$. Define

$$\phi_0(f) = \left[\left\{g^f_{ij}\right\}\right] \in \pi_k(V_{N+2k,N+k}) = \mathbb{Z}_2$$

where the last equality holds by Lemma 2.14.3 since $k$ is odd (and $k > 1$). Clearly the assignment $f \mapsto \{g^f_{ij}\}$ is continuous.

**Lemma 7.1.1.** *$\phi_0$ is invariant under regular homotopy. That is, if immersions $f_1, f_2 : S^k \to M$ are regularly homotopic, then $\phi_0(f_1) = \phi_0(f_2)$. Thus if $f : S^k \to M$ is an immersion which is regularly homotopic to an embedding, then $\phi_0(f) = \phi_0(\alpha)$ where $\alpha \in H_k(M)$ is the homology class represented by $f$.*

*Proof.* Let $f_0$ and $f_1$ be regularly homotopic immersions through $f_t : S^k \to M$. That is, for each $t$, $f_t$ is an immersion, and furthermore $I \times S^k \to M$, $(t, x) \mapsto f_t(x)$, is continuous. Then $t \mapsto \left\{g^{f_t}_{ij}\right\}$ is a homotopy between $\left\{g^{f_0}_{ij}\right\}$ and $\left\{g^{f_1}_{ij}\right\}$, which therefore represent the same element of $\pi_k(V_{N+2k,N+k})$. $\square$

By Theorem 2.13.4 homotopic embeddings $S^k \to M^{2k}$, are regularly homotopic so we get a well defined map

$$\phi_0 : H_k(M) \to \mathbb{Z}_2$$

given by $\phi_0(\alpha) = \phi_0(f)$ where $f$ is any embedding representing $\alpha$.

What is actually $\phi_0(f)$?

**Lemma 7.1.2.** *Unless $k = 3, 7$, $\phi_0(f) = 0$ if and only if $\nu(f)$ is a trivial bundle. In the cases $k = 3, 7$, $\phi_0(f) = 0$ if and only if the surgery via $f$ can be framed.*

*Proof.* We consider the long exact sequence of homotopy groups of the fiber bundle $p : SO_{N+2k} \to V_{N+2k,N+k}$. Recall the proof of Lemma 2.14.3.

$$\pi_k(SO_k) \xrightarrow{i_*} \pi_k(SO_{N+2k}) \xrightarrow{p_*} \pi_k(V_{N+2k,N+k}) \xrightarrow{\partial} \pi_{k-1}(SO_k) \longrightarrow \pi_{k-1}(SO_{N+2k})$$

When $k = 3, 7$ this is Lemma 4.3.3. Suppose $k \neq 3, 7$. Then $\partial$ is a a monomorphism, hence $\phi_0(f) = 0 \iff \partial(\phi_0(f)) = 0$. Upon recalling the definition of $\phi_0$ and $\partial$ it becomes evident that $\partial\phi_0(f) = [\nu(f)]$, the clutching function of the normal bundle, proving the lemma. Less work is required in the following approach: Since $p_* = 0$, $\phi_0(f)$ is trivial if and only if the given trivialization $F_0$ of $\epsilon^N \oplus TS^k$ extends to all of $f^*(\epsilon^N \oplus TM)$. That is, if $F_0$ gives sections $Y_1, \cdots, Y_{N+k}$, then $\phi_0(f)$ is trivial if and only if there are sections $Z_1, \cdots, Z_k$ such that $\{Y_1, \cdots, Y_{N+k}, Z_1, \cdots, Z_k\}$ forms a trivialization of $f^*(\epsilon^N \oplus TM)$. But the same can be said about $\nu(f)$ being zero, for such sections $Z_i$ would trivialize $\nu(f)$, and conversely a trivialization of $\nu(f)$ would yield sections $Z_i$. ☐

**Lemma 7.1.3.** *For $k = 1, 3, 7$ there exist a framing $F$ of $S^k \times S^k$ such that $\Phi(S^k \times S^k, F) = 1$.*

*Proof.* See [Lev85, Prop. 4.11] for a full proof. The argument goes as follows: $H_k(S^k \times S^k)$ has a basis $\alpha, \beta$. We can twist the framing so that $\phi_0(\alpha)$ can become any value. The key is that $p_*$ (in the exact sequence above) is onto for $k = 1, 3, 7$, and that $\pi_{2k-1}(SO) = 0$ for $k = 3, 7$. ☐

We proceed to show that $\phi_0$ is a mod 2 quadratic refinement of the intersection form. We will employ the following construction: Let $f_1, f_2 : S^k \to M$ be immersions representing $\alpha, \beta \in \pi_k(M) = H_k(M)$ (we are not assuming basepoints.). Then choose a nullhomotopic map $\psi : S^k \to M$ which intersects $f_1$ and $f_2$ in disjoint disks. We punch out these disks and assemble the remaining pieces of $f_1$, $f_2$ and $\psi$ to obtain a map $f_1\#f_2 : S^k \to M$. We can assume that $\psi$ is such that $f_1\#f_2$ is an immersion. Clearly $f_1\#f_2$ represents $\alpha + \beta$: Homotoping the remainders of $\psi$ to a point (and stretching the rest of $f_1\#f_2$ to stay continuous) we obtain the map$(f_1 \vee f_2) \circ \mathit{pinch} = f_1 + f_2$. Now we consider $\phi_0(f_1\#f_2) : S^k \to V_{N+2k,k}$. For $k \neq 3, 7$, we can describe $\phi_0(f_1\#f_2)$ by describing $\partial(\phi_0(f_1\#f_2)) = [\nu(f_1\#f_2)]$. Addition in $\pi_{k-1}(S^k)$ corresponds under the clutching function construction to forming the connected sum of vector bundles, defined as follows. First choose trivializations over neighborhoods of the embedded disks. Then, when we cut out the disks, we glue the trivializations together (with the identity as clutching function.) It is clear that $[\nu(f_1\#f_2)] \simeq [\nu(f_1)]\#[\nu(f_2)] \simeq [\nu(f_1] + [\nu(f_2)]$. In other words $\phi_0(f_1\#f_2) = \phi_0(f_1) + \phi_0(f_2)$. This equality also holds for $k = 3, 7$.

**Lemma 7.1.4.** *For $\alpha, \beta \in H_k(M)$,*

$$\phi_0(\alpha + \beta) \equiv \phi_0(\alpha) + \phi_0(\beta) + \alpha \cdot \beta \mod 2$$

*Proof.* Having the equality $\phi_0(\alpha)+\phi_0(\beta)=\phi_0(f\#g)$, the question remains as to wether $f\#g$ is regularly homotopic to an embedding, which is necessary to ensure $\phi_0(f\#g)=\phi_0(\alpha+\beta)$. By Theorem 2.13.5 this happens if and only if $f\#g$ has self intersection number 0 (modulo 2 since $k$ is odd). Since $f$ and $g$ are embeddings they have self intersection number 0, hence the self intersection number of $f\#g$ is the intersection number $\alpha\cdot\beta$. If $\alpha\cdot\beta=0$, then $f\#g$ is regularly homotopic to an embedding, and therefore $\phi_0(\alpha+\beta)=\phi_0(f\#g)=\phi_0(f)+\phi_0(g)$ and we are done. If $\alpha\cdot\beta=1$ we apply a neat trick: There exists an immersion $h_0:S^k\to\mathbb{R}^{2k}$ with self intersection 1. (This follows from the construction in 2. in [Whi44a]). Thus we can choose $h:S^k\to M$ to not intersect $f\#g$ and have self intersection number 1. Now $f\#g\#h$ has self intersection $1+1\equiv 0$ mod 2 by construction, and therefore $f\#g\#h$ is regularly homotopic to an embedding, and so represents $\alpha+\beta+[h]$. Since $h$ is nullhomotopic, $[h]=0$. Thus we obtain

$$\phi_0(\alpha+\beta)=\phi_0(f\#g\#h)=\phi_0(f)+\phi_0(g)+\phi_0(h).$$

and it remains only to show $\phi_0(h)=1$. Clearly $\phi_0(h)=\phi_0(h_0)$, so we only need to produce a single example in which $\phi_0(h)=1$, and it will hold in all. We consider $S^k\times S^k$. $H_k(S^k\times S^k)$ is $\mathbb{Z}\oplus\mathbb{Z}$ generated by $\alpha=S^k\times s_0$ and $\beta=s_0\times S^k$. Clearly $\alpha\cdot\beta=1$. With the framing coming from $D^{k+1}\times D^{k+1}$ it is clear that $\phi_0(\alpha)=\phi_0(\beta)=0$, hence $\phi_0(\alpha+\beta)=\phi_0(h)$. We can represent $\alpha+\beta$ with the diagonal embedding, $d:S^k\to S^k\times S^k$ which has normal bundle $\nu(d)\simeq TS^k$. [It is in general true that the diagonal embedding $\Delta:M\to M\times M$ has normal bundle $TM$. This is because $\Delta^*(T(M\times M))\simeq TM\oplus TM$. Concretely the map $v\mapsto(v,-v)$ is an isomorphism $TM\simeq\nu(\Delta)$.] Hence, by theorem 7.1.2, $\phi(d)=1$ unless $k=3,7$. For $k=3,7$ there is in fact a framing of $S^k\times S^k$ such that no framed surgery can make it a homotopy sphere by Lemma 7.1.3. Consider the basis for $H_k(S^k\times S^k)$ given by $\{\alpha+\beta,\alpha\}$. This basis satisfy the assumption of Lemma 5.1.2: $(\alpha+\beta)\cdot(\alpha+\beta)=\alpha\cdot\beta-\alpha\cdot\beta=0$, $\alpha\cdot\alpha=0$. Thus surgery via $d$ cannot be framed as surgery via $d$ results in a homotopy sphere by Lemma 5.1.2. □

It follows that $\phi_0$ induces a map $\phi:H_k(M;\mathbb{Z}_2)\to\mathbb{Z}_2$. To see this we must show $\phi(2x)=0$. We have

$$\phi_0(x+x)=\phi_0(x)+\phi_0(x)+x\cdot x=x\cdot x=-x\cdot x=0,$$

where the second to last equality holds since $k$ is odd, and the last follows since $x\cdot x$ is an integer. Let $(M,F)$ be a $2k$ dimensional framed manifold, $k$ odd. Then the Kervaire invariant of $(M,F)$ is defined to be the Arf invariant of $\phi$, [9] that is

$$\Phi(M,F)=\sum_{i=1}^{n}\phi(x_i)\phi(y_i)$$

where $\{x_i,y_i\}$ is any symplectic basis for $H_k(M;\mathbb{Z}_2)$. By Lemma 7.1.2, $\phi$, and hence $\Phi$, is independent of the framing $F$ for $k\neq 3,7$. Hence we define the Kervaire invariant of a stably paralellizable manifold $M$ of dimension $2k$, $k\neq 3,7$, $k$ odd, to be $\Phi(M)=\Phi(M,F)$ where $F$ is any framing.

**Theorem 7.1.5.** *A framed manifold $(M,F)$ of dimension $2k$, $k$ odd with $\partial M=\emptyset$ (respectively a homotopy sphere) can through framed surgery be reduced to a homotopy sphere (respectively a contractible manifold) if and only if $\Phi(M,F)=0$.*

[9] See [Dye78] for an elementary exposition of the Arf invariant.

*Proof.* First suppose $\Phi(M, F) = 0$, and let $\{x_i, y_i\}$ be any symplectic basis for $H_k(M)$. Note that we are working with a symplectic basis over the integers and only reducing modulo 2 to evaluate the Kervaire invariant. This is done in order to ensure that we at the appropriate moment can reference Lemma 5.1.2 to kill $H_k(M, F)$. Since $\Phi(M, F) = 0$ there are an even number of pairs $(x_i, y_i)$ for which $\phi_0(x_i)\phi_0(y_i) = 1$. For each pair of such pairs, $x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}$, we make the following substitution:

$$\begin{aligned} x'_{i_1} &= x_{i_1} + x_{i_2} & \qquad y'_{i_1} &= y_{i_1} \\ x'_{i_2} &= y_{i_2} - y_{i_1} & \qquad y'_{i_2} &= x_{i_2} \end{aligned}$$

Then

$$\phi(x'_{i_1}) = \phi(x_{i_1}) + \phi(x_{i_2}) + x_{i_1} \cdot x_{i_2} = 1 + 1 = 0$$

and similarly $\phi(x'_{i_2}) = 0$. Making this substitution result in a new symplectic basis. We may thus assume that $H_k$ has a symplectic basis $\{x_i, y_i\}$ for which $\phi(x_i)\phi(y_i) = 0$ for all $i$. By further substituting $x'_i = y_i$ and $y'_i = -x_i$ if necessary, we can assume that $\phi(x_i) = 0$. If $k \neq 3, 7$, then by Lemma 7.1.2, each $x_i$ can be represented by embeddings with trivial normal bundles, so Lemma 5.1.2 applies, yielding the conclusion. The surgeries in Lemma 5.1.2 can be assumed to be framed by Lemma 4.3.2 since $k \neq 3, 7$. For $k = 3, 7$ it is always the case that the normal bundles are trivial since $\pi_{k-1}(SO_k) = 0$, so we can in this case always kill $H_k(M)$. What is added now is that by Lemma 7.1.2, $\phi_0(x_i) = 0 \implies x_i$ can be surged away in a framed manner. Thus the surgeries of 5.1.2 can be framed, and we are done with "$\Leftarrow$". The converse implication follows from the fact that $\Phi$ is invariant under framed cobordism (which is Lemma 7.1.6 below) since $\Phi(\Sigma) = 0$ for any homotopy $2k$-sphere $\Sigma$. After all $H_k(\Sigma) = 0$. In the proof of Lemma 7.1.6 we will use that we already have showed that if $\partial M$ is a homotopy sphere and $\Phi(M, F) = 0$, then $(M, F)$ is framed cobordant to a contractible manifold. □

**Lemma 7.1.6.** *$\Phi$ is invariant under framed cobordism.*

Hence there is a unique cobordism invariant extension of $\Phi$ to all framed $2k$-manifolds, $k$ odd. (So far we have only defined $\Phi$ on $(k-1)$-connected manifolds.)

**Corollary 7.1.7.** *$\Phi$ induces a map $\Phi : \pi_n(\mathbb{S}) \simeq \Omega_n^{fr} \to \mathbb{Z}_2$.* □

*Proof of Lemma 7.1.6.* By Lemma 3.1.3 the intersection form splits over connected sums, hence $\Phi(M\#N) = \Phi(M) + \Phi(N)$. If $M$ and $N$ are framed cobordant, then $M\# - N$ is a framed boundary by Lemma 3.1.4. We have showed in the proof of Theorem 7.1.5 that if $\Phi(M, F) = 0$, then $(M, F)$ belongs to the trivial framed cobordism class. We will show that if $(M, F)$ is a boundary, then $\Phi(M, F) = 0$. It will then follow that $0 = \Phi(M\# - M) = \Phi(M) + \Phi(-M)$ since $M \sqcup -M$ bounds $I \times M$. Hence if $M$ and $N$ are framed cobordant, then $M\# - N$ bounds, and so $0 = \Phi(M\# - N) = \Phi(M) + \Phi(-N) = \Phi(M) - \Phi(N)$.

We first recall Lemma 5.1.6, which shows that $\ker i_* = (\ker i_*)^\perp \subset H_k(M; \mathbb{Q})$ and $2 \dim \ker i_* = \dim H_k(M; \mathbb{Q})$. Since $H_k(M; \mathbb{Z})$ is free, the same holds with $\mathbb{Z}$ coefficients. We apply Lemma 5.1.5 to obtain a symplectic basis, $\{x_i, y_i\}_{i=1}^r$, for $H_k(M; \mathbb{Z})$ with $x_i \in \ker i_*$. Therefore it is sufficient to show $\phi(\alpha) = 0$ whenever $\alpha \in \ker i_*$. We can assume $V$ to be $(k-1)$-connected by Lemma 4.1.6. Let $f : S^k \to M$ be an embedding representing $\alpha$. The assumption that $i_*(\alpha)$ is homologous to 0 is equivalent to $i \circ f$ being nullhomotopic by the Hurewicz theorem. Thus we obtain $f' : D^{k+1} \to V$ extending $i \circ f$, and by Theorem 2.13.3 we can assume that $f'$ is an immersion. Just as we defined $\phi(f) \in \pi_k(V_{N+2k,N+k})$, $f'$ defines a map $D^{k+1} \to V_{N+2k,N+k}$, a null-homotopy of $\phi(f)$ which therefore equals 0. □

## 7.2 Computation of $bP^{4k+2}$

An analysis equivalent to the one we will carry out in this section and the next can be found in [Lev85]. Certain points are not particularly clear in [KM63], although must have been known to Kervaire and Milnor. What follows that is not contained in [KM63] was probably intend to appear in the promised "Groups of Homotopy Spheres II" which was never published.

What does Theorem 7.1.5 imply about the structure of $bP^{4k+2}$? If $\Sigma$ is a homotopy $n = (4k+2)$-sphere bounding a framed manifold $(V, F)$ we have shown that if $\Phi(V, F) = 0$, then $\Sigma = S^k$. The interesting question is what happens if $\Phi(V, F) = 1$. Any two homotopy spheres $\Sigma$ and $\Sigma'$ bounding such $(V, F)$ and $(V', F')$ are h-cobordant: Form the connected sum along the boundary, $V \# V'$. It has Kervaire invariant $1 + 1 = 0$, thus through a sequence of framed spherical modifications can be made contractible. It is bounded by $\Sigma \# \Sigma'$ which therefore is h-cobordant to $S^n$. So any two such $\Sigma$ represent mutually inverse elements of $\Theta_n$. In particular they are also their own inverses, hence they must represent the same element of $\Theta_n$.

**Lemma 7.2.1.** *If homotopy $(2k-1)$-spheres ($k$ odd) $\Sigma$ and $\Sigma'$ bounds framed manifolds $(V, F)$ and $(V', F')$, then $\Sigma$ and $\Sigma'$ are h-cobordant if and only if $\Phi(V, F) = \Phi(V', F')$.* $\square$

Two questions arise:

1. Does there exist homotopy $(2k-1)$-spheres which bounds manifolds of Kervaire invariant one? If yes, denote the h-cobordism class by $b_{2k}$.
2. If $b_{2k}$ exists, is it non-zero? I.e, if $b_{2k}$ exists, is it represented by $S^{2k-1}$?

**Lemma 7.2.2.** *The class $b_{2k}$ always exists.*

*Proof.* In section 7.4 we construct a concrete manifold $K_{2k}$ which has non-zero Kervaire invariant, at least for $k \neq 1, 3, 7$, and $\partial K_{2k} = S^{2k-1}$. For $k = 1, 3, 7$ the ensuing discussion indicates how to produce $b_{2k}$. $\square$

What makes the analysis of $bP^{4k+2}$ different from the analysis of $bP^{4k}$, with $\sigma$ replaced by $\Phi$, is the absence of an analogue of Corollary 6.1.6. In fact, there might very well be a closed manifold with Kervaire invariant 1, as we saw in Lemma 7.1.3. If there exists a closed framed manifold $(M, F)$ of dimension $2k$ with $\Phi(M, F) = 1$, then punching out a disk we obtain a framed manifold $(V, F|_V)$ with $\partial V = S^k$. Hence in this case $b_{2k} = 0$. Conversely, if $b_{2k} = 0$, then we can fill in a disk over $\partial V = S^k$ to obtain a closed manifold with Kervaire invariant one. Here $V$ is any manifold of Kervaire invariant one, bounded by $S^k$. Thus we have proved:

**Theorem 7.2.3.** *$bP^{4k+2} = 0$ if and only if there exists a closed framed manifold $(V, F)$ of dimension $4k+2$ with $\Phi(V, F) = 1$. Otherwise $bP^{4k+2} = \mathbb{Z}_2$.* $\square$

In [KM63] the corresponding result only states that $bP^{4k+2} = 0$ or $\mathbb{Z}$. However it is impossible to credit anyone other than Kervaire and Milnor with the above condition. The construction of $K_{2k}$ is due to Kervaire, [Ker60], and essentially all theory in this thesis appeared in [KM63], hence Theorem 7.2.3 must have been known to them.

Theorem 7.2.3 compels us to ask the question known as the Kervaire invariant one problem: Does there exist a manifold in dimension $4k+2$ with Kervaire invariant one? Equivalently, is $(\Phi_n : \pi_n(\mathbb{S}) \to \mathbb{Z}_2) = 0$? It is an extremely difficult problem. If the answer is affirmative in dimension $n$, then $K_n$ (see section 7.4) is bounded by $S^{n-1}$ with the usual smooth structure. Thus, filling in the disk $D^n$ we obtain a concrete example of a manifold with Kervaire invariant one. Sadly reposing the question as "is $\partial K_n = S^{n-1}$?" makes it no easier to answer. If the answer is in the negative, then $K_n$ with a disk glued onto it cannot be smoothed since it in fact would have Kervaire invariant one. This was the first application of the invariant:[10] Kervaire gave an example of a PL manifold, $K_{10}$, admitting no differentiable structure. We shall go over his short and elegant proof that $(\Phi_{10} : \pi_{10}(\mathbb{S}) \to \mathbb{Z}_2) = 0$ in Theorem 7.2.5 below. But first we state the celebrated result of Hill, Hopkins and Ravenel in [HHR16].

**Theorem 7.2.4.** *Manifolds with Kervaire invariant* $1$ *exist in dimensions* $n = 2, 6, 14, 30$ *and* $62$. *It is unknown for* $n = 126$. *For any other value of* $n$ *no such manifold exist.*

*The long road leading to 7.2.4.* Kervaire's proof of Theorem 7.2.5 below was given in the article [Ker60] where he introduced $\Phi$. His proof also works in dimension 18. Next Brown and Peterson showed in [BP66] that except for $n = 2$, $\Phi_n = 0$ unless possibly if $n = 8k + 6$. In [Bro69], Browder vastly improved on this result and showed that $\Phi_n = 0$ unless possibly if $n$ is of the form $n = 2^j - 2$. For many years this was the sharpest available result about non-existence. The next progress was that Barratt, Mahowald and Tangora in their 1970 papers [MT67] and [BMT70] were able to compute, using computations about $\pi_n(\mathbb{S})$, that $\Phi_{30} \neq 0$. Jones reproved this in [Jon78] by constructing a concrete manifold, other than $K_{30}$, with Kervaire invariant one. Much later Barrat, Jones and Mahowald showed in [BJM84] that $\Phi_{62} \neq 0$. It must have started to seemed likely that $\Phi_{2^j-2} \neq 0$ for all $j$. Recently Hill, Hopkins and Ravenel has proved in [HHR16] that this is not the case for $j \geqslant 8$. It is a rather lengthy article of more than 200 pages. For an interesting account of the problem and an accessible explanation of their solution, see [HHR11] which is also where this "proof" is taken from. The case $n = 126$ is thus the only one remaining open. □

**Theorem 7.2.5.** *On every smooth, closed* $10$*-manifold,* $\Phi$ *is* $0$. *Equivalently* $\Phi : \pi_{10}(\mathbb{S}) \to \mathbb{Z}_2$ *is identically* $0$.

*Proof.* It is immediate that $\Phi$ vanishes on elements of $\pi_n(\mathbb{S})$ of odd order. Let $\alpha \in \pi_{10}(\mathbb{S})$ satisfy $2\alpha = 0$. We need to show $\Phi(\alpha) = 0$. The reader acquainted with stable homotopy theory is used to seeing diagrams such as in Figure 1. The dot above 10 is connected to the dot above 9, indicating that every $\gamma \in \pi_{10}(\mathbb{S})$ of order 2 is of the form $\gamma = \beta \circ \eta$ for some $\beta \in \pi_9(\mathbb{S})$, where $\eta$ is the stable class of the Hopf map $S^3 \to S^2$. To prove the theorem we will show that each $\alpha$ of order 2 can be represented by a framed homotopy sphere, $(\Sigma, F)$, under the Pontryagin construction. It follows from Lemma 4.2.2 that $\beta$ is obtainable through the Pontryagin construction from a homotopy sphere, $M$, with some framing. We can choose a representative of $\eta$, $f : S^{10+n} \to S^{9+n}$, and a point $x_0 \in S^{9+n}$ such that $f^{-1}(S^{9+n}\backslash x_0) = S^1 \times (S^{9+n}\backslash x_0)$. This is true because it is an $n$-fold suspension of a fiber bundle over $S^2$ with fiber $S^1$. For if $p : E \to B$ satisfies $p^{-1}(B\backslash b_0) = F \times B\backslash b_0$, then $id \times p : S^1 \times E \to S^1 \times B$ satisfies

$$\begin{aligned} id \times p^{-1}(S^1 \times B\backslash S^1 \vee B) &= S^1 \times E\backslash(S^1 \times p^{-1}(b_0) \cup s_0 \times E) \\ &= (S^1\backslash s_0) \times F \times (B\backslash b_0) \\ &\simeq F \times (S^1 \times B\backslash S^1 \vee B). \end{aligned}$$

[10] Arguably Pontryagin used $\Phi$ to compute $\pi_2(\mathbb{S}) = \mathbb{Z}_2$: He proved that the class corresponding to the framing of $S^1 \times S^1$ giving nontrivial Kervaire invariant was non-trivial.

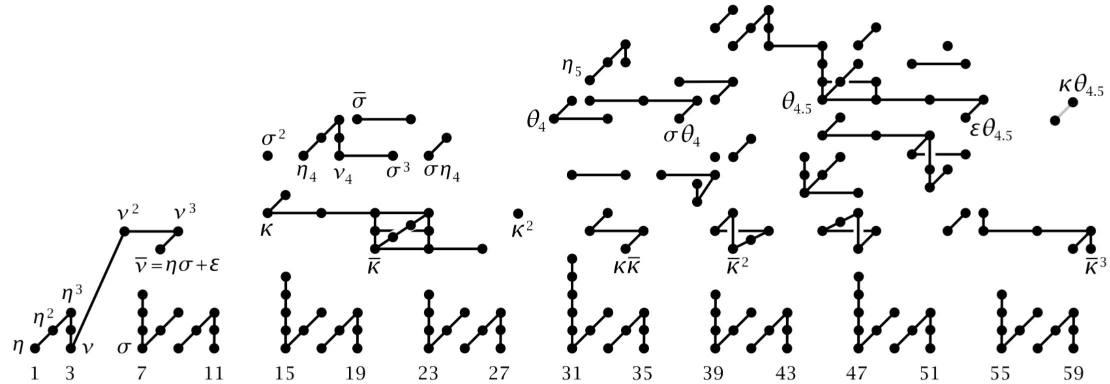


Figure 1: This diagram is taken from [Hat02, page 385]. Each dot above the number $n$ corresponds to a composition factor $\mathbb{Z}_2$ of $\pi_n(\mathbb{S})$. The vertical lines indicate nontrivial extensions. The diagonal and horizontal lines indicate composition with one of the Hopf invariant one maps.

In other words, letting $x_0$ denote the collapsed wedge, the induced map $id \wedge p = \Sigma p : \Sigma E \to \Sigma B$ satisfies

$$id \wedge p^{-1}(\Sigma B \backslash x_0) \simeq F \times (\Sigma B \backslash x_0),$$

and so by induction the claim follows. We may assume $x_0 \notin M$. We get $f^{-1}(M) = S^1 \times M$. Hence there is a framing of $M \times S^1$ giving rise to $\alpha$. Performing framed surgery we can easily kill $\pi_1(M \times S^1)$, obtaining the required homotopy 10-sphere $\Sigma$. Instead of referring to theorems from surgery theory, let us not fear dirt and perform this surgery with our hands, only using the theorems to know that we obtain a framed closed manifold. The class we need to kill can be thought of as the inclusion, $S^1 = S^1 \times \{x_0\} \subset S^1 \times M$. Cutting out a small neighborhood of $S^1 \times \{x_0\}$ we are left with $S^1 \times D^9$. Note that $\partial S^1 \times D^9 = S^1 \times S^8$ which also bound $D^2 \times S^8$, which we attach to it obtaining a manifold $\Sigma$ (see the beginning of section 4.1). We can triangulate $\Sigma$ with one cell in dimensions 0 through 2 and then no more cells until dimension 8. The 2 cell is attached using a generator of $\pi_1(S^1)$, so the 2-skeleton is a disc $D^2$. Since the 5-skeleton coincides with the 2-skeleton, which is contractible, it follows from Poincaré duality (crucially we know that $\Sigma$ is an oriented smooth closed manifold from results on surgery theory) that $\Sigma$ is a simply connected homology sphere, hence a homotopy sphere. (We have seen this argument before: The homology isomorphism is induced by a map. This can be established by referring to Hopf's theorem [Hu59, Theorem $C^n$, page 53] stating that the correspondence $f \mapsto f^*(\iota)$ induced a bijection between $H^n(X)$ and homotopy classes of maps $X \to S^n$ where $\iota$ is the generator of $H^n(S^n)$ and $X$ is any $CW$-complex of dimension at most $n$. Now Hurewicz's and Whitehead's theorem finishes the proof that $\Sigma$ is a homotopy sphere.) $\square$

### 7.3 The Index $[\pi_n(\mathbb{S})/\mathrm{Im}(J) : \Theta_n/bP^{n+1}]$

Suppose we are given an element $\alpha \in \pi_n(\mathbb{S}) \simeq \Omega_n^{fr}$. We represent $\alpha$ by a framed manifold, $(M, F)$. If $M$ is framed cobordant to a homotopy sphere, then that homotopy sphere also represents $\alpha$. Thus in this case, the class of $\alpha$ is in the image of $p : \theta_n \to \pi_n(\mathbb{S})/\mathrm{Im}(J)$. Theorem 4.1.2 together with Corollary 6.1.6 thus implies that $p$ is onto for $n \not\equiv 2 \mod 4$. For $n = 4k + 2$ the situation is more complicated: Theorem 7.1.5 implies that $p$ is onto only if the Kervaire invariant one problem is false in dimension $n$. We have thus proved:

**Theorem 7.3.1.** *$p : \Theta_n \to \pi_n(\mathbb{S})$ is onto unless $n = 2, 6, 14, 30$ or $62$, and possibly also for $n = 126$. When $p$ is not onto,* $\operatorname{coker} p = \mathbb{Z}_2$. □

## 7.4 Kervaire Manifolds

We first introduce a construction due to Milnor. In [Mil59] he used it to construct examples of exotic structures on $S^{4k-1}$.

Suppose we are given a diffeomorphism $F : S^n \times S^m \to S^n \times S^m$. We can form a manifold $M(F)$ from $S^n \times D^{m+1} \sqcup D^{n+1} \times S^m$ by identifying the boundaries under the diffeomorphism $F$ by the rule $(x, y) \sim F(x, y)$. Alternatively, to make $M(F)$ smooth, we may form it from the disjoint union of $S^n \times \mathbb{R}^{m+1}$ and $\mathbb{R}^{n+1} \times S^m$ as follows: Given $(x, y) \in S^n \times S^m$ and $t > 0$ we identify $(x, ty)$ with $(t'x', y')$ where $(x', y') = F(x, y)$ and $t' = \frac{1}{t}$.

We denote by $h$ the last of the $n+1$ coordinate functions $\mathbb{R}^{n+1} \to \mathbb{R}$ restricted to the sphere $S^n$. That is, $h(x_0 \cdots, x_n) = x_n$. Then $h$ has only the two critical values $x_\pm = (0, \cdots, 0, \pm 1)$.

**Lemma 7.4.1.** *Let $F$ be a diffeomorphism:*

$$\begin{aligned} F : S^n \times S^m &\to S^n \times S^m \\ (x, y) &\mapsto (x', y') \end{aligned}$$

*Then if $h(x) = h(x')$ for all $(x, y)$, $M(F)$ is homeomorphic to $S^{n+m+1}$.*

*Proof.* This is Lemma 1 in [Mil59]. □

The way we will construct diffeomorphisms $F : S^n \times S^m \to S^n \times S^m$ will be through maps into rotation groups. Namely, given $f_1 : S^n \to SO_{m+1}$ and $f_2 : S^m \to SO_{n+1}$, we define $F$ by $F(x, y) = (x', y')$ where $y' = f_1(x)y$ and $x' = (f_2(y'))^{-1}x$. We set $M(f_1, f_2) := M(F)$.

**Lemma 7.4.2.** *If $f_1$ factors through the inclusion $SO_m \to SO_{m+1}$, then $M(f_1, f_2)$ is homeomorphic to $S^{n+m+1}$.*

*Proof.* $SO_m$ is the subgroup of $SO_{m+1}$ which leaves the last coordinate fixed. That is to say, if $f_1 \in SO_m$, then

$$h(y') = h(f_1(x)y) = h(y),$$

and Lemma 7.4.1 yields the conclusion. □

It turns out that any $M(f_1, f_2)$ bounds a manifold $W = W(f_1, f_2)$. The construction of $W$ is also found in [Mil59] and goes as follows: Take three copies of $D^{n+1} \times D^{m+1}$, $E_i$ for $i = 1, 2, 3$. We patch together $E_1$ and $E_2$ by identifying $(x_1, y_1) \in S^n \times D^{m+1} \subset E_1$ with $(x_2, y_2) \in E_2$ where $x_2 = x_1$ and $y_2 = f_1(x_1)y_1$. Similarly we identify $(x_2, y_2) \in D^{n+1} \times S^m \subset E_2$ with $(x_3, y_3) \in E_3$ where $y_3 = y_2$ and $x_2 = f_2(y_3)x_3$. Respecting these identifications, the union $W = E_1 \cup E_2 \cup E_3$ is after smoothing out the corners a smooth manifold. $\partial W$ is a quotient of the three copies of $S^n \times S^m$. We note that $\partial E_2$ represents interior points of $W$. The relation forming $\partial W$ from $(S^n \times S^m)_1 \cup (S^n \times S^m)_3$ is $(x_1, y_1) \sim (x_3, y_3)$ where $y_3 = y_2 = f_1(x_1)y_1$ and $x_1 = x_2 = f_2(y_3)x_3$.

Observing that these relations are identical to the defining relations of $M(f_1, f_2)$ we conclude that $\partial W = M(f_1, f_2)$.

Note that in the above construction $E_1 \cup E_2$ and $E_2 \cup E_3$ are disc bundles over $S^{n+1}$ and $S^{m+1}$ respectively. The clutching functions for these disk bundles are $f_1$ and $f_2$. Note that the 0 sections "run" through $E_2$ in opposite directions: Restricted to $E_2$ they are $S^{n+1} \supset D^{n+1} \ni v \mapsto (v, 0)$ and $S^{m+1} \supset D^{m+1} \ni w \mapsto (0, w)$. We know that homotopy classes of clutching functions, i.e. elements of $\pi_{p-1}(SO_q)$, are in one to one correspondence with $Vect^q_+(S^p)$, which obviously is in one to one correspondence with isomorphism classes of disk bundles over $S^p$. Letting $E_f$ be the disk bundle corresponding to the vector bundle with clutching function $f$ we have:

**Lemma 7.4.3.** *If $[f_1] \in \pi_n(SO_{m+1})$ and $[f_2] \in \pi_m(SOn+1)$, then there is a smooth manifold $W(f_1, f_2)$ with boundary $\partial W(f_1, f_2) = M(f_1, f_2)$, and there are embeddings*

$$i_s : E_{f_s} \to W(f_1, f_2), \qquad s = 1, 2$$

*whose images has union $W(f_1, f_2)$ and contractible intersection. Furthermore the images of the zero-sections in $E_{f_1}$ and $E_{f_2}$ intersect transversely in a single point.* □

Let $n = 2k = 4l+2$, and let $[f] = [f_1] = [f_2] \in \pi_{k-1}(SO_k)$ be the clutching function of $TS^k$. Then $K_n$ is defined to be the manifold $W(f_1, f_2)$ of Lemma 7.4.3. From Lemma 2.14.1 $(j_{k-1})_* = 0$, hence by exactness of

$$\pi_{k-1}(SO_{k-1}) \xrightarrow{i_*} \pi_{k-1}(SO_k) \xrightarrow{(j_{k-1})_*} \pi_{k-1}(S^{k-1})$$

it is clear that $i_*$ is epi, so $f$ factors through the inclusion $SO_{k-1} \to SO_k$. Hence Lemma 7.4.2 shows that $M(f_1, f_2)$ is homeomorphic to a sphere. Since $E_f = TS^k$ contracts onto $S^k$, and since $E_{f_1} \cap E_{f_2}$ is contractible, it follows that $K_n$ has the homotopy type of $S^k \vee S^k$. We observe that $i_1$ and $i_2$ represents generators of $H_k(K_n)$, $\alpha$ and $\beta$ respectively. Note that $\{\alpha, \beta\}$ is a symplectic basis: It is clear that $\alpha \cdot \beta = \pm 1$. Furthermore since $k$ is odd, $TS^k$ admits a section which is never 0, hence $\alpha \cdot \alpha = 0 = \beta \cdot \beta$ (both the zero section and the nonzero section represent $\alpha$ and $\beta$ in the respective copies of $TS^k$). Thus by definition $\Phi(K_n) = \phi_0(\alpha)\phi_0(\beta) \mod 2$. Note that a tubular neighborhood of the embedding $i_s$, $s = 1, 2$, is diffeomorphic with the normal bundle of the 0-section $S^k \to TS^k$. Clearly this is $TS^k$. The bundle $TS^k$ is trivial if and only if $k = 3, 7$. Hence, by Lemma 7.1.2, we conclude:

**Theorem 7.4.4.** *For $k \neq 3, 7$, $\Phi(K_{2k}) = 1$.* □

This is a generalization of Kervaire's construction in [Ker60]. However the generalization is so immediate that only Kervaire deserves credit. In particular we have shown that $M(f, f)$ is a homotopy sphere: It represents the non-trivial element of $\partial bP^{10}$. We conclude with Kervaire's astonishing result from [Ker60]. Let the manifold $M_0$ obtained from $K_{10}$ by filling in a disk over the homotopy sphere $\Sigma$. If $M_0$ could be given a smooth structure, then it would be a smooth manifold with Kervaire invariant 1. Hence by Theorem 7.2.5, we deduce that $M_0$ cannot be homeomorphic with any smooth manifold. However clearly $M_0$ can be given a piecewise linear structure.

# References


[Ada60] J. F. Adams, *On the Non-Existence of Elements of Hopf Invariant One*, Annals of Mathematics **72** (1960), no. 1, 20–104.

[Ada66] J. F. Adams, *On the Groups $J(X)$-IV*, Topology **5** (1966), no. 1, 21–71.

[Ade52] J. Adem, *The Iteration of the Steenrod Squares in Algebraic Topology*, Proceedings of the National Academy of Sciences **38** (1952), no. 8, 720–726.

[Ati68] M. F. Atiyah, *Bott Periodicity and the Index of Elliptic Operators*, The Quarterly Journal of Mathematics **19** (1968), no. 1, 113–140.

[BJM84] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *Relations Amongst Toda Brackets and the Kervaire Invariant in Dimension 62*, Journal of the London Mathematical Society (2) **30** (1984), no. 3, 533–550.

[BMT70] M. G. Barratt, M. E. Mahowald, and M. C. Tangora, *Some Differentials in the Adams Spectral Sequence-*II, Topology **9** (1970), no. 4, 309–316.

[Bot57] R. Bott, *The Stable Homotopy of the Classical Groups*, Proceedings of the National Academy of Sciences **43** (1957), no. 10, 933–935.

[BP66] E. H. Brown and F. P. Peterson, *The Kervaire Invariant of $8k + 2$-Manifolds*, American Journal of Mathematics **88** (1966), no. 4, 815–826.

[Bro69] W. Browder, *The Kervaire Invariant of Framed Manifolds and its Generalization*, Annals of Mathematics **90** (1969), no. 1, 157–186.

[Dye78] R. H. Dye, *On the Arf Invariant*, Journal of Algebra **53** (1978), no. 1, 36–39.

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[Hat09] A. Hatcher, *Vector Bundles and $K$-Theory, v 2.1* (2009). Available at: `https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf`. Accessed May 2017.

[HHR11] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *The Arf-Kervaire Problem in Algebraic Topology: Sketch of the Proof*, Current Developments in Mathematics **2010** (2011), 1–44.

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the Non-Existence of Elements of Kervaire Invariant One*, Annals of Mathematics **184** (2016), no. 1, 1–262.

[Hir13] F. Hirzebruch, *Neue Topologische Methoden in der Algebraischen Geometrie*, 2nd ed., Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 9, Springer-Verlag, 2013.

[Hu59] S.-T. Hu, *Homotopy Theory*, 1st ed., Pure and Applied Mathematics, vol. 8, Academic Press, 1959.

[Jon78] J. D. S. Jones, *The Kervaire Invariant of Extended Power Manifolds*, Topology **17** (1978), no. 3, 249–266.

[Ker60] M. A. Kervaire, *A Manifold Which Does Not Admit Any Differentiable Structure*, Commentarii Mathematici Helvetici **34** (1960), no. 1, 257–270.

[KM63] M. A. Kervaire and J. W. Milnor, *Groups of Homotopy Spheres:* I, Annals of Mathematics **77** (1963), no. 3, 504–537.

[KS69] R. C. Kirby and L. C. Siebenmann, *On the Triangulation of Manifolds and the Hauptvermutung*, Bulletin of the American Mathematical Society **75** (1969), no. 4, 742–749.

[Lee03] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2003.

[Lev85] J. P Levine, *Lectures on Groups of Homotopy Spheres*, In: Ranicki, A., Levitt, N., Quinn F. (eds.) Algebraic and Geometric Topology. Lecture Notes in Mathematics, vol. 1126, Springer, 1985, pp. 62–95.

[Mil59] J. Milnor, *Differentiable Structures on Spheres*, American Journal of Mathematics **81** (1959), no. 4, 962–972.

[Mil61] J. Milnor, *A Procedure for Killing Homotopy Groups of Differentiable Manifolds*, Proceedings Symposia in Pure Mathematics, vol. 3, American Mathematical Society, 1961, pp. 39–55. Available at: `http://www.maths.ed.ac.uk/~aar/papers/milnorsurg.pdf`. Accessed May 2017.

[Mil63] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, vol. 51, Princeton University Press, 1963.

[MK60] J. W. Milnor and M. A. Kervaire, *Bernoulli Numbers, Homotopy Groups, and a Theorem of Rohlin*, Proceedings of the International Congress of Mathematicians vol. 1958, The Syndics of the Cambridge University Press, 1960, pp. 454–458.

[MS74] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, 1974.

[MT67] M. Mahowald and M. Tangora, *Some Differentials in the Adams Spectral Sequence*, Topology **6** (1967), no. 3, 349–369.

[MT91] M. Mimura and H. Toda, *Topology of Lie Groups,* I *and* II, Translations of Mathematical Monographs, vol. 91, American Mathematical Society, 1991.

[Pal60] R. S. Palais, *Extending Diffeomorphisms*, Proceedings of the American Mathematical Society **11** (1960), no. 2, 274–277.

[Pon55] L. S. Pontryagin, *Smooth Manifolds and Their Applications in Homotopy Theory*, Trudy Matematicheskogo Instituta imeni VA Steklova **45** (1955), 3–139.

[Ran02] A. Ranicki, *Algebraic and Geometric Surgery*, Oxford University Press, 2002.

[Ran81] A. Ranicki, *Exact Sequences in the Algebraic Theory of Surgery*, Princeton Mathematical Notes, vol. 26, Princeton Univiversity Press, 1981.

[Sma61] S. Smale, *Generalized Poincaré's Conjecture in Dimensions Greater Than Four*, Annals of Mathematics **74** (1961), no. 2, 391–406.

[Sma62] S. Smale, *On the Structure of Manifolds*, American Journal of Mathematics **84** (1962), no. 3, 387–399.

[ST80] H. Seifert and W. Threlfall, *Seifert and Threlfall, A Textbook of Topology*, 1st ed. (J. S. Birman and J. Eisner, eds.), Pure and Applied Mathematics, vol. 89, Academic Press, 1980.

[Sti35] E. Stiefel, *Richtungsfelder und Fernparallelismus in $n$-Dimensionalen Mannigfaltigkeiten*, Commentarii Mathematici Helvetici **8** (1935), no. 1, 305–353.

[VdB10] E. P. Van den Ban, *Lie Groups*, Lecture Notes in Mathematics, MRI, University of Utrecht, 2010. Available at: `https://www.staff.science.uu.nl/~ban00101/lecnotes/lie2010.pdf`. Accessed May 2017.

[Whi12] G. W. Whitehead, *Elements of Homotopy Theory* (J. H. Ewing, F. W. Gehring, and P. R. E. Halmos, eds.), Graduate Texts in Mathematics, vol. 61, Springer-Verlag, 2012.

[Whi36] H. Whitney, *Differentiable Manifolds*, Annals of Mathematics **37** (1936), no. 3, 645–680.

[Whi40] J. H. C. Whitehead, *On $C1$-Complexes*, Annals of Mathematics **41** (1940), no. 4, 809–824.

[Whi44a] H. Whitney, *The Self-Intersections of a Smooth $n$-Manifold in $2n$-Space*, Annals of Mathematics **45** (1944), no. 2, 220–246.

[Whi44b] H. Whitney, *The Singularities of a Smooth $n$-Manifold in $(2n-1)$-Space*, Annals of Mathematics **45** (1944), no. 2, 247–293.