

# Mixed integer formulations for a short sea fuel oil distribution problem

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## Abstract

We consider a short sea fuel oil distribution problem occurring in the archipelago at Cape Verde. Here, an oil company is responsible for the routing and scheduling of ships between the islands such that the demand for various fuel oil products is satisfied during the planning horizon. Inventory management considerations are taken into account at the demand side, but not at the supply side. The ports have restricted opening hours each day, so multiple time windows are considered. In contrast to many other studies within ship routing and scheduling, considerable time is spent in the ports compared to at sea. Hence, the time in port is modeled in detail by incorporating both a variable (un)loading time and a set up time for loading different products in the same ports. A mathematical model of the problem is presented and it includes a combined continuous and discrete time horizon due to the multiple time windows and a daily varying consumption rate of the various products in the different ports. We discuss several strategies to improve the proposed model, such as tightening bounds, using extended formulations and including valid inequalities. The computational study shows that the real problem can be solved to optimality within reasonable time by the use of improved formulations based on a combination of such strategies.

*Keywords:* Maritime transportation, Inventory, Routing, Extended formulations, Valid inequalities

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## Introduction

The inter-islands distribution of fuel oil is a real problem of Cape Verde, an archipelago with nine inhabited islands. Fuel oil products are imported and delivered to specific is-

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lands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships. These products are stored in consumption storage tanks. Some ports have both supply and consumption tanks (see Figure 1). For the inter-islands distribution problem we ignore two islands (with a circle in Figure 1) since they are supplied by another ship using a different technology.

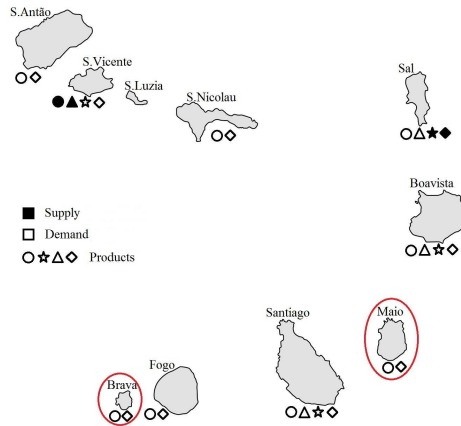


Figure 1: Supply and demand for fuel oil products at several islands in Cape Verde.

The inter-island distribution plan consists of designing routes and schedules for the fleet of ships including determining the (un)loading quantity of each product at each port. This plan must satisfy (i) the demand of each product at each island per period, (ii) time window constraints for the port operations (loading/unloading), and (iii) the capacities of the ships, ports and depots. The total cost of the distribution plan is to be minimized, and includes sailing costs, a fixed cost for each operation and a penalty cost for violation of time windows.

We consider a short-term distribution problem with a planning horizon of twelve days. The input to this problem is the output of a medium-term plan for several weeks (few months). The demands correspond to the quantities to be delivered at each port per day determined in the medium-term plan. Hence, usually the demands at each port follow a pattern where the demands are zero for most periods and relatively large in the rest of the periods. By coordinating the distribution of all products in all ports during the planning horizon, it might be efficient to deliver the demand in periods prior to the specified period by the medium-term plan or in other quantities. This means that we need to keep track of the inventory level at the consumption storage tanks for all products in all ports. Storage capacities in supply and consumption tanks are taken into account in the medium-term planning. In the short-term plan considered here capacities in supply tanks can be ignored, since the global consumption of each product

from all consumption tanks during the time horizon, is much smaller than the capacity of the supply tanks. However, for the consumption tanks the capacity of the tanks for a particular product can be less than the total demand over the planning horizon for that product. When solving our instances, we consider only inventory capacity bounds for the consumption tanks where the corresponding capacity can be lower than the total demand over the entire planning horizon. It is assumed that at most one ship can operate in each port at a given time period. During a port call for a ship, it is possible to load and unload different products. We assume that there is a fixed (un)loading time per unit product (un)loaded. This (un)loading time may vary for different products and different ports. In addition, there exists a considerable set up time between (un)loading different products due to coupling and decoupling of pipes between tanks in the ship and tanks in the port.

Most of the ports are closed during night and some ports have operational restrictions during certain periods of the day. This means that in each period (day), there may exist a time window for (un)loading. These time windows may vary from port to port. A ship cannot start to operate before the beginning of the time window. However, if the operation has begun inside the time window, it can be finished outside that time window, see Figure 2. In that case, an extra man-power cost considered as a penalty cost is incurred.

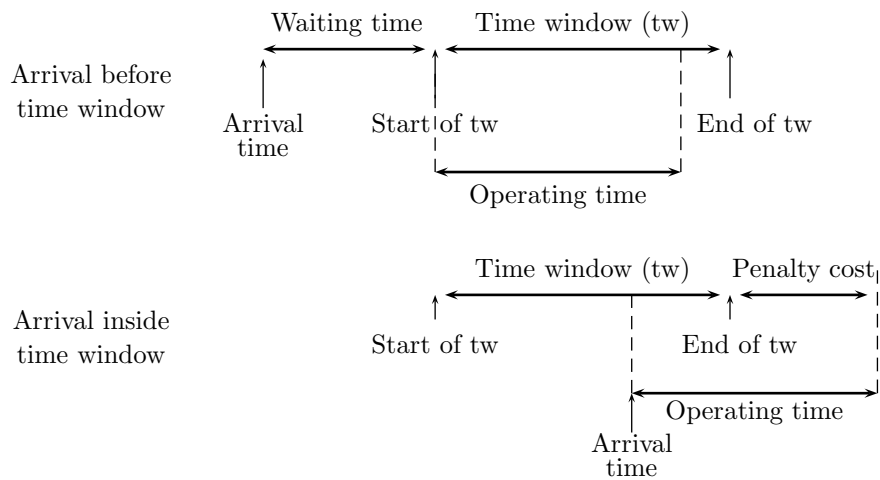


Figure 2: Time windows: operating time inside and outside (with penalty) of the time window, and waiting times.

To transport the fuel products between the islands, the planners control two different ships, but a larger heterogeneous fleet is expected in the future. Each ship has a specified load capacity, fixed speed and cost structure. The setup and (un)loading times are independent of ship. The cargo hold of each ship is separated into several cargo tanks. We consider two scenarios: the case where the allocation of the different fuel products

into different cargo tanks is not considered and the case where there are dedicated tanks for families of products. The later case is the closest to reality (and later denoted as the Real Case) although in some situations it can be considered too restrictive since changes between families of products are possible. However, such changeovers are only allowed under exceptional circumstances. On the other hand, the first case can be regarded as a relaxation of the real situation. We focus on the first case and explain in a later section how to deal with the dedicated tanks case.

We consider a fixed cost associated with each operation at each port and associate a setup time to each operation (loading/unloading) as depicted in Figure 3.

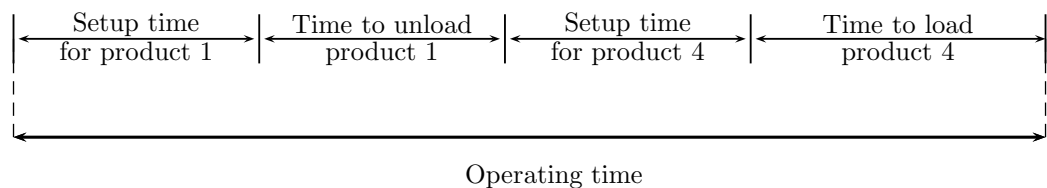


Figure 3: Schedule of operations: the ship unloads product 1 and then loads product 4. A setup time is required for each operation.

In this paper we present mixed integer formulations for the two cases of the short sea fuel oil distribution problem (SSDP) in Cape Verde and provide several strategies to improve the formulation applying techniques such as the use of extended formulations and the inclusion of strong valid inequalities. Based on an extensive computational study we propose improved formulations which can solve the tested instances based on real data to optimality within reasonable time.

The models proposed are based on the underlying real planning problem. However, some simplifications are made and some issues are omitted. Safety stocks are not explicitly considered, but could easily be taken into account by considering net stocks (the stock level minus the safety stock), see Agra et al. [2011]. Fluctuating weather conditions are neglected, and very few contributions in the literature have so far focused on this issue within maritime transportation. In addition, we chose to make some assumptions; such as a maximum number of ships operating in the same port per day, no draft limits in the ports and assuming the described tank allocation policy. However, these simplification and assumptions should not prevent the planners from making valuable short-term decisions based on the SSDP models.

Although the paper is concerned with a real problem our contributions are also of interest for other maritime transportation problems. Besides the application, the contributions comprise (i) new models that include a combined continuous and discrete time horizon. These are fairly general models that deal with multiple time windows and a daily varying consumption rate of the various products. They can also be used for

the constant consumption rates case as this is a particular case of the previous one; (ii) discussion of different strategies to strengthen the proposed models. The computational results also provide some insight into the relevance of each strategy.

We remark that the models presented here can also be used to solve larger instances. For instance, heuristic procedures based on the mathematical formulation such as rolling-horizon heuristics, relax-and-fix heuristics, etc. can be used to derive feasible solutions.

The rest of the paper is organized as follows: In Section 1 we give a brief literature review related to the planning problem and the formulation techniques considered. We have limit ourselves to the maritime transportation literature. Section 2 presents a mixed integer formulation of the SSDP in Cape Verde. Different strategies to improve the initial formulation are discussed in Section 3. These strategies include tightening bounds, including valid inequalities and deriving extended formulations. In Section 4 we discuss the real problem with dedicated tanks. We focus on the main differences between the two cases (with and without dedicated tanks) and explain how to adapt the results from the previous sections to the case with dedicated tanks. Section 5 is devoted to the results of an extensive computational study to compare different ways of combining the improving strategies, and to test our best strategy on the real case and on future scenarios. Finally, the main conclusions of this work follow in Section 6.

## 1. Related Literature

We have witnessed an increased interest in studying optimization problems within maritime transportation, see the reviews on maritime transportation, Christiansen et al. [2007], and maritime inventory routing problems, Christiansen and Fagerholt [2009]. Combined routing and inventory management within maritime transportation have been present in the literature the last one and a half decades only. Christiansen [1999] considers a supply chain for ammonia consisting of several facilities that either produce or consume ammonia and the transportation network between those facilities. Ammonia is produced and stored in inventories at given loading ports and transported at sea to inventories at unloading ports. Inventory capacities are defined in all ports. Here, the production and consumption rates are given and fixed during the planning horizon in all ports. The planning problem is to find routes and schedules for a fleet of ships that minimize the transportation costs without interrupting production or consumption at the storages. The overall problem is solved by a branch-and-price method in Christiansen and Nygreen [1998a] and Christiansen and Nygreen [1998b] and by a heuristic in Flatberg et al. [2000]. Unlike the problem studied in Christiansen [1999], the short sea fuel oil distribution problem (SSDP) includes several products. Also Hwang [2005] studies a maritime inventory routing problem that allowed for multiple products on board the ship and with dedicated compartments in the ship for various products. Al-Khayyal and Hwang [2007] give a mathematical formulation for such a problem where the products

are assumed to require dedicated compartments in the ship. For this problem there exist inventory limits and production/consumption rates for each product in each port, just as for our SSDP. We include the product-compartment allocation case in Section 5.3. As for the SSDP, Al-Khayyal and Hwang [2007] include a fixed setup time for switching between (un)loading different products in a port. The maritime inventory routing problem described in Ronen [2002] also includes multiple products. The underlying model focuses on the inventory management and not the routing part of the problem, as the model solution suggests shipment sizes that are assumed to be input for a cargo routing problem at a later stage.

Both Christiansen [1999] and Al-Khayyal and Hwang [2007] present continuous time models and introduce an index indicating the visit number to a particular port. For both models it is assumed that the production/consumption rate is fixed and constant during the planning horizon. In Grønhaug et al. [2010] and Ronen [2002] discrete time models are developed to overcome the complicating factors with varying production and consumption rates. In the SSDP, the production inventory side is not considered, but we have varying consumption rates during the planning horizon.

In most studies in the literature, the inventory management is considered both at the production and consumption sites. However, Rakke et al. [2011] consider a liquefied natural gas inventory routing problem with just one large production port and no inventory management aspects considered at the consumption ports.

Most of the maritime transportation planning problems studied in the literature are within the deep sea segment, see Christiansen et al. [2007]. However, we are considering a short sea distribution problem with relatively low-activity ports. Considerable time is spent in port, and some ports are closed during night. This corresponds to a problem with multiple time windows. Within maritime transportation, this is considered in Christiansen and Fagerholt [2002] for a tramp ship scheduling problem without inventory management considerations. In contrast to the SSDP, no loading and unloading are possible after the end of the time window. This means that a ship might stay idle in port during night or in the weekend if it did not finish its service in the port opening hours. In order to avoid such idle times due to unexpected delays, the authors have introduced penalties for finishing the service in port just before the end of time windows. In this way, they expect that more robust schedules are designed. We can also find a few other contributions within maritime transportation where penalties are used in connection with time windows. In Fagerholt [2001], the hard time windows are extended to soft ones. There the penalty costs occur outside the hard time windows. The work of Christiansen [1999] is extended in Christiansen and Nygreen [2005] to reduce the possibility of violating the inventory limits at the storages. Here another pair of soft inventory limits within the hard ones is introduced. This means that those soft inventory limits can be violated at a penalty, but it is not possible to exceed the storage capacity

or be under the lower inventory limits. They show that the soft inventory constraints can be transformed into soft time windows.

Although the study of valid inequalities for mixed-integer sets and the derivation of extended formulations is currently receiving large attention with several applications to other mixed-integer problems, little work has been done in applying these techniques to maritime transportation problems. However, a few contributions already exist. Sherali et al. [1999] include valid inequalities in order to enhance the proposed formulations of an oil products transportation problem, and Persson and Göthe-Lundgren [2005] develop valid inequalities within a column generation approach for a maritime inventory routing problem. Also, Grønhaug et al. [2010], include valid inequalities to improve the path-flow formulation presented for the liquefied natural gas inventory routing problem.

## 2. Mathematical Formulation

In this section, we describe a mathematical model for the SSDP without dedicated tanks.

The nature of the production and consumption rates affects the underlying model. If it is assumed that the production and consumption rates are fixed and constant during the planning horizon, then a mathematical model based on continuous time can be used (e.g. Al-Khayyal and Hwang [2007] and Christiansen [1999]). When the production and/or consumption rate is variable or fixed but varying during the planning horizon a discrete time model is applied (see Grønhaug et al. [2010] and Ronen [2002]). The case of variable production and consumption rates is, of course, the most general one, but often the rates are fixed. In practice, the production and consumption rates are most often varying, although, in some applications, the simplification made by assuming a constant rate is acceptable.

In this paper we consider a combined continuous and discrete time horizon. The discrete time horizon corresponds to the continuous one divided into periods (corresponding to days). The discrete time horizon allows us to easily handle the multiple time windows and non-constant demand requests. The drawback of this approach is the large number of variables involved in the mathematical model.

In this formulation, the decision variables are written in lower case letters and the parameters and sets are written in upper case letters.

### Indices

- $k$  products;
- $i, j$  ports;
- $v$  ships;
- $i_v$  initial port position of ship  $v$ ;
- $m, n$  time periods.

## Sets

$V$  set of ships;

$N$  set of ports;

$K$  set of products;

$M$  set of periods.  $M = \{1, \dots, |M|\}$ , where  $|M|$  is the number of periods considered.

## Parameters

$T_{ijv}$  time required by ship  $v$  to sail from port  $i$  to port  $j$ ;

$T_{im}^A$  start of time window in period  $m$  at port  $i$ ;

$T_{im}^B$  end of time window in period  $m$  at port  $i$ ;

$C_{ik}^W$  fixed cost of operating (loading/unloading) product  $k$  at port  $i$ ;

$T_{ik}^Q$  time required to (un)load one unit of product  $k$  at port  $i$ ;

$D_{imk}$  demand of product  $k$  at port  $i$  in period  $m$ ;

$C_{ijv}$  total transportation cost for ship  $v$  to sail from port  $i$  to port  $j$ ;

$V_v^{CAP}$  total storage capacity of ship  $v$ ;

$U_{ik}$  storage capacity of the depot for product  $k$  at port  $i$ ;

$W_{ik}$  setup time required for operating product  $k$  at port  $i$ ;

$Q_{vk}$  quantity of product  $k$  on board ship  $v$  at the beginning of the planning horizon;

$C_{im}^P$  penalty cost, per hour, for operating outside the time window at port  $i$  in period  $m$ ;

$J_{ik}$  =1 if port  $i$  is a producer of product  $k$ ; =-1 if port  $i$  is a consumer of product  $k$ ; 0 otherwise.

## Continuous Variables

$t_{im}$  start time of operation at port  $i$  in time period  $m$ ,  $i \in N, m \in M$ .

We assume  $t_{i_v,1} = 0, v \in V$ ;

$t_{im}^E$  ending time of the operation that started during period  $m$  in port  $i$ ,  $i \in N, m \in M$  (these variables are not necessary for the model but they are useful to ease the reading);

$p_{im}$  operating time outside the time window of period  $m$  at port  $i$ ,  $i \in N, m \in M$ ;

$q_{imvk}$  amount of product  $k$  loaded onto or unloaded from ship  $v$  at port  $i$  in time period  $m$ ,  $i \in N, m \in M, v \in V, k \in K$ . We assume  $q_{imvk} = 0$  if  $J_{ik} = 0$ ; or  $m = 1, i \neq i_v$ ; or  $m = 1, i = i_v, Q_{vk} = 0, J_{ik} = -1$ ;

$l_{imvk}$  amount of product  $k$  onboard ship  $v$  when leaving port  $i$  after an operation that started in time period  $m$ ,  $i \in N, m \in M, v \in V, k \in K$ . We assume  $l_{i_1vk} = 0$  if  $i \neq i_v$ ;

$s_{imk}$  stock level of product  $k$  at port  $i$  at the end of time period  $m$ ,  $i \in N, m \in M, k \in K$ . ( $s_{i0k}$  is the stock level at the beginning of period 1).

## Binary variables



- $x_{imjnv}$  1 if ship  $v$  starts to operate at port  $i$  in period  $m$  and then sails from port  $i$  to port  $j$  and starts to operate at port  $j$  in period  $n$ , 0 otherwise,  $i, j \in N$ ,  $m, n \in M$ ,  $v \in V$ . We assume  $x_{imjnv} = 0$  if  $m \geq n$ ; or  $i = j$ ; or  $m = 1, i \neq i_v$ ;
- $z_{imv}$  1 if ship  $v$  ends its route at port  $i$  after an operation that started in time period  $m$ , 0 otherwise,  $i \in N, m \in M, v \in V$ ;
- $o_{imvk}$  1 if product  $k$  is loaded onto or unloaded from ship  $v$  at port  $i$  in time period  $m$ ; 0 otherwise,  $i \in N, m \in M, v \in V, k \in K$ . We assume  $o_{imvk} = 0$  if  $J_{ik} = 0$ ; or  $m = 1, i \neq i_v$ ; or  $m = 1, i = i_v, Q_{vk} = 0, J_{ik} = -1$ .

### The MIP model for the SSDP:

$$\min \sum_{i,j \in N} \sum_{n,m \in M} \sum_{v \in V} C_{ijv} x_{imjnv} + \sum_{i \in N} \sum_{m \in M} \sum_{v \in V} \sum_{k \in K} C_{ik}^W o_{imvk} + \sum_{i \in N} \sum_{m \in M} C_{im}^P p_{im}, \quad (1)$$

subject to:

$$\sum_{j \in N} \sum_{n \in M} x_{i_v 1 j n v} + z_{i_v 1 v} = 1, \quad \forall v \in V, \quad (2)$$

$$\sum_{j \in N} \sum_{n \in M} x_{j n i m v} - \sum_{j \in N} \sum_{n \in M} x_{i m j n v} - z_{i m v} = 0, \quad \forall i \in N, m \in M, m > 1, v \in V, \quad (3)$$

$$\sum_{i \in N} \sum_{m \in M} z_{i m v} = 1, \quad \forall v \in V, \quad (4)$$

$$\sum_{j \in N} \sum_{n \in M} \sum_{v \in V} x_{j n i m v} \leq 1, \quad \forall i \in N, m \in M, \quad (5)$$

$$T_{im}^A \leq t_{im} \leq T_{im}^B, \quad \forall i \in N, m \in M, \quad (6)$$

$$(t_{im}^E + T_{ijv} - t_{jn}) x_{imjnv} \leq 0, \quad \forall i, j \in N, m, n \in M, v \in V, \quad (7)$$

$$p_{im} \geq t_{im}^E - T_{im}^B, \quad \forall i \in N, m \in M, \quad (8)$$

$$t_{im}^E = t_{im} + \sum_{v \in V} \sum_{k \in K} W_{ik} o_{imvk} + \sum_{v \in V} \sum_{k \in K} T_{ik}^Q q_{imvk}, \quad \forall i \in N, m \in M, \quad (9)$$

$$t_{im} \geq t_{i, m-1}^E, \quad \forall i \in N, m \in M, m > 1 \quad (10)$$

$$x_{imjnv} (l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk}) = 0, \quad \forall i, j \in N, m, n \in M, v \in V, k \in K, \quad (11)$$

$$Q_{vk} + J_{i_v k} q_{i_v 1 vk} - l_{i_v 1 vk} = 0, \quad \forall v \in V, k \in K, \quad (12)$$

$$q_{imvk} \leq V_v^{CAP} o_{imvk}, \quad \forall i \in N, m \in M, v \in V, k \in K : J_{ik} \neq 0, \quad (13)$$

$$\sum_{k \in K} l_{imvk} \leq V_v^{CAP} \sum_{j \in N} \sum_{n \in M} x_{imjnv}, \quad \forall i \in N, m \in M, v \in V, \quad (14)$$

$$s_{i,m-1,k} + \sum_{v \in V} q_{imvk} = D_{imk} + s_{imk}, \quad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \quad (15)$$

$$s_{imk} \leq U_{ik}, \quad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \quad (16)$$

$$x_{imjnv} \in \{0, 1\}, \quad \forall i, j \in N, i \neq j, m, n \in M, m < n, v \in V, \quad (17)$$

$$z_{imv} \in \{0, 1\}, \quad \forall i \in N, m \in M, v \in V, \quad (18)$$

$$o_{imvk} \in \{0, 1\}, \quad \forall i \in N, m \in M, v \in V, k \in K, \quad (19)$$

$$q_{imvk}, l_{imvk} \geq 0, \quad \forall i \in N, m \in M, v \in V, k \in K, \quad (20)$$

$$s_{imk} \geq 0, \quad \forall i \in N, m \in M, k \in K, \quad (21)$$

$$t_{im}, t_{im}^E, p_{im} \geq 0, \quad \forall i \in N, m \in M. \quad (22)$$

The objective function (1) is to minimize the cost (transportation cost, setup cost of operations and penalty cost).

The set of routing constraints (2)-(4) is defined under a network whose set of nodes is  $\{(i, m) \in N \times M\}$ . Hence, each node corresponds to a port-period pair. Constraints (2) ensure that ship  $v$  either departs from the initial port position  $i$  to another port  $j$  or it ends its route in port  $i$  ( $z_{imv} = 1$ ). Constraints (3) are the flow conservation constraints for each port and each time period. That is, if ship  $v$  starts an operation in port  $i$  at period  $m$ , then either it must travel to another port  $j$ , or it finishes its route in port  $i$ . Constraints (4) ensure that ship  $v$  ends its route at some port. Constraints (5) guarantee that at most one ship  $v$  can operate in port  $i$  at a given time period. The time window constraints are given by (6). Constraints (7) ensure that if ship  $v$  sails from port  $i$  (after an operation started in period  $m$ ) to port  $j$  (to initialize an operation in period  $n$ ), then the operation at port  $j$  can only start after the end time of operation at port  $i$  plus the time required to travel from  $i$  to  $j$ . These constraints can be linearized as follows:

$$t_{im}^E + T_{ijv} - t_{jn} \leq B(1 - x_{imjnv}), \quad \forall i, j \in N, m, n \in M, v \in V, \quad (23)$$

where  $B = \max\{0, T_{ik}^Q V_v^{CAP} + T_{im}^B + T_{ijv} - T_{jn}^A\}$ .

Constraints (8) enforce  $p_{im}$  to assume, at least, the value of the duration of operations outside the time windows. Notice that since the cost of  $p_{im}$  is positive,  $p_{im}$  assumes exactly the operation time violating the corresponding time window. Equations (9) define the end time of each operation. Constraints (10) ensure that, for each port and for each period, a ship can only start to operate if the operation of the previous period is already finished. Constraints (11) and (12) relate the quantity onboard to the quantity loaded and/or unloaded. Constraints (11) ensure that if ship  $v$  sails from port  $i$  (after an operation started in period  $m$ ) to port  $j$  (to initialize an operation in period  $n$ ), then the quantity of product  $k$  onboard at the departure from port  $j$  should be equal to the quantity onboard at departure from port  $i$  plus/minus the quantity loaded/unloaded

from port  $j$ . Following Desrosiers et al. [1995], equations (11) can be linearized by replacing them with the following two sets of constraints :

$$\left[ \begin{array}{c} l_{imvk} + J_{jk}q_{jnvk} - l_{jnvk} \\ + V_v^{CAP} x_{imjnv} \end{array} \right] \leq V_v^{CAP}, \quad \forall i, j \in N, m, n \in M, v \in V, k \in K, \quad (24)$$

$$\left[ \begin{array}{c} l_{imvk} + J_{jk}q_{jnvk} - l_{jnvk} \\ - V_v^{CAP} x_{imjnv} \end{array} \right] \geq -V_v^{CAP}, \quad \forall i, j \in N, m, n \in M, v \in V, k \in K. \quad (25)$$

Equations (12) relate the quantity onboard with the quantity loaded/unloaded in the starting port. Constraints (13) ensure that if an operation occurs, that is,  $q_{imvk} > 0$ , then the setup variable  $o_{imvk}$  must be one. They also impose an upper bound on the quantity loaded/unloaded. Constraints (14) impose an upper bound on the quantity onboard. They also ensure that if the quantity onboard is positive, then the ship must travel to some other port. Constraints (15) are the inventory management balance constraints and, together with non-negativity constraints (20) and (21), ensure that the demand for each product at each port in each time period is satisfied. The storage capacity at each port of each product are given by constraints (16). Finally, (17)-(22) are the non-negativity and integrality constraints.

Modeling a real problem implies to make assumptions and, in some cases, simplifications. In order to clarify our modeling options, we next present some observations related to the modeling issues.

(i) Considering the real operation (load/unload) times, a ship can start to operate in one period and finish the operations in that same period or in the next one. For the definition of variables we use the starting period.

(ii) We consider a penalty cost by violation of the time window. That penalty is considered during the operating time outside the time window where the operation started. However, it is in theory possible that a ship finishes the operations during the next time window. In this case we also penalize the operating time occurring in the next time window because this is not a desirable solution.

(iii) We impose that at most one ship can operate in each port per period. Since there is a large uncertainty with docking operations in maritime transportation and since we are considering a small fleet consisting of two ships, it is not desirable to schedule two ships at the same port in the same period.

(iv) The following set of constraints

$$o_{imvk} \leq \sum_{j \in N} \sum_{n \in M} x_{jnimv}, \quad \forall i \in N, m \in M, m > 1, v \in V, k \in K, \quad (26)$$

ensure that if there is an operation involving ship  $v$  at a port  $i$  during a period  $m$ , then port  $i$  must belong to a ship route. These inequalities are not necessary in the model

since the fixed cost associated with the  $x_{jnimv}$  variables are positive. However, we will include these inequalities in order to derive strong formulations for the model.

The “basic” MIP model for the SSDP consists of (1)-(6), (8)-(10), (12)-(26) and will be denoted the B-SSDP.

### 3. Formulation improvements

In this section we explore some directions to derive stronger models, which means models whose linear relaxations are tighter than the original one. Deriving stronger models may lead to better bounds which can be useful to reduce the number of nodes in a branch and bound-based scheme.

We consider different types of improvements. The first one consists of the tightening of bounds. The second one is based on reformulations of the model with the inclusion of additional variables (extended formulations). We propose an arc-load flow formulation and an arc-load multi-commodity formulation. The last improvement is related to the inclusion of valid inequalities. These inequalities are based on inequalities derived for simple mixed-integer sets arising from relaxations from the set of feasible solutions of B-SSDP.

#### 3.1. Tighter bounds

Here we explain how to tighten certain constraints. Basically, for certain constraints, we replace the upper bound given by the capacity of the ship,  $V_v^{CAP}$ , by the total amount of fuel the ship can carry in order to satisfy the remaining demand.

Constraints (13), (14), (24) and (25) can be replaced, respectively, by the following constraints:

$$q_{imvk} \leq A_{imvk} o_{imvk}, \quad \forall i \in N, m \in M, v \in V, k \in K : J_{ik} \neq 0, \quad (27)$$

$$\sum_{k \in K} l_{imvk} \leq A_{imv} \sum_{j \in N} \sum_{n \in M} x_{imjnv}, \quad \forall i \in N, m \in M, v \in V, \quad (28)$$

$$l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} + A_{imjnvk} x_{imjnv} \leq A_{imjnvk}, \\ \forall i, j \in N, m, n \in M, v \in V, k \in K, \quad (29)$$

$$l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} - \bar{A}_{imjnvk} x_{imjnv} \geq -\bar{A}_{imjnvk}, \\ \forall i, j \in N, m, n \in M, v \in V, k \in K, \quad (30)$$

where, for all  $i \in N, m \in M, v \in V, k \in K$ ,

$$A_{imvk} = \begin{cases} \min\{V_v^{CAP}, \sum_{n \in M: n > m} \sum_{u \in N: u \neq i} D_{unk}\}, & \text{if } J_{ik} = 1; \\ \min\{V_v^{CAP}, \sum_{n \in M: n \geq m} D_{ink}\}, & \text{if } J_{ik} = -1, \end{cases}$$

for all  $i, j \in N, m, n \in M, v \in V, k \in K$ ,  $A_{imjnvk} = \min\{V_v^{CAP}, \sum_{t \in M: t > m} \sum_{u \in N: u \neq j} D_{utk}\}$ ,

$$\bar{A}_{imjnvk} = \begin{cases} \min\{V_v^{CAP}, \sum_{t \in M: t > n} \sum_{u \in N, u \neq j} D_{utk}\}, & \text{if } J_{jk} \neq -1; \\ \min\{V_v^{CAP}, \sum_{t \in M: t \geq n} \sum_{u \in N} D_{utk}\}, & \text{if } J_{jk} = -1, \end{cases}$$

and for all  $i \in N, m \in M, v \in V$ ,  $A_{imv} = \min\{V_v^{CAP}, \sum_{u \in N: u \neq i} \sum_{n \in M: n > m} \sum_{k \in K} D_{unk}\}$ .

### 3.2. Extended formulations

In this section we propose two extended formulations. The new set of variables introduced in each formulation provides additional information about the solution. That information is essential to derive tighter inequalities. In the first extended formulation the new variables indicate the amount of each product carried along each arc, that is, the new variables associate a flow, for each product, to each arc in the ship path. The second formulation is a classical multi-commodity reformulation of the first extended formulation where the flow on each arc is disaggregated accordingly to its destination.

#### 3.2.1. Arc-load flow reformulation

One of the weaknesses of the B-SSDP model is the set of constraints (14) since even if  $\sum_{k \in K} l_{imvk} = V_v^{CAP}$  there can occur solutions with several fractional values of  $x_{imjnv}$ . In order to strengthen the model we introduce new variables, denoted by arc-load flow variables, and use these variables to decompose variables  $l_{imvk}$ . Instead of considering the amount of fuel of each product onboard the ship when it is leaving a port, the new variables indicate also the next port for where that fuel is being transported to. With these new variables we will replace (14) by stronger inequalities (inequalities that imply (14), and once they are included in the model the corresponding linear relaxation feasible set becomes tighter). Let us define the non-negative arc-load flow variables  $f_{imjnvk}$ ,  $i, j \in N, n, m \in M$  as the amount of product  $k$  that ship  $v$  transports from port  $i$ , after an operation that started in period  $m$ , to port  $j$  in order to start an operation in period  $n$ . We assume  $f_{imjnvk} = 0$  whenever  $x_{imjnv} = 0$ .

The two sets of variables  $l_{imvk}$  and  $f_{imjnvk}$  can be related using the following equations

$$l_{imvk} = \sum_{j \neq i} \sum_{n > m} f_{imjnvk}, \quad \forall i \in N, m \in M, v \in V, k \in K, \quad (31)$$

Using the arc-load flow variables we can replace constraints (11), (12) and (14) by

$$\sum_{i \neq j} \sum_{m < n} f_{imjnvk} + J_{jk} q_{jnvk} = \sum_{i \neq j} \sum_{m > n} f_{jnimvk},$$

$$\forall j \in N, n \in M : n > 1, v \in V, k \in K, \quad (32)$$

$$Q_{vk} + J_{i_v k} q_{i_v 1vk} - \sum_{j \neq i_v} \sum_{n > 1} f_{i_v 1jnvk} = 0, \quad \forall v \in V, k \in K, \quad (33)$$

$$\sum_{k \in K} f_{imjnvk} \leq V_v^{CAP} x_{imjnv}, \quad \forall i, j \in N, m, n \in M, v \in V, \quad (34)$$

respectively. Adding constraints (34) for  $j$  and  $n$  we obtain

$$\sum_{j \neq i} \sum_{n > m} \sum_{k \in K} f_{imjnvk} \leq V_v^{CAP} \sum_{j \neq i} \sum_{n > m} x_{imjnv}.$$

Replacing  $\sum_{j \neq i} \sum_{n > m} f_{imjnvk}$  by  $l_{imvk}$  we obtain (14). Hence constraints (14) can be obtained by aggregating constraints (34). Thus, the model using the arc-load flow variables should provide better bounds based on the linear relaxation. The drawback of this model is the huge number of continuous variables. For instances with higher dimension than those we tested, this reformulation can be of no use.

As in the previous subsection, the constant  $V_v^{CAP}$  can in some cases be replaced by a tighter bound.

Notice that with the inclusion of variables  $f_{imjnvk}$ , variables  $q_{jnvk}$  can be eliminated from the model using equations (32) and (33), that is, setting

$$q_{jnvk} = J_{jk} \left( \sum_{i \neq j} \sum_{m > n} f_{jnimvk} - \sum_{i \neq j} \sum_{m < n} f_{imjnvk} \right), \quad \forall j \in N, n \in M : n > 1, v \in V, k \in K, \quad (35)$$

and

$$q_{i_v 1vk} = J_{i_v k} \left( \sum_{j \neq i_v} \sum_{n > 1} f_{i_v 1jnvk} - Q_{vk} \right), \quad \forall v \in V, k \in K. \quad (36)$$

We denote the arc-load flow model by F-SSDP. The F-SSDP includes constraints (1)-(6), (8)-(10), (13), (15)-(22), (23)-(26), (32)-(34).

### 3.2.2. Multi-commodity reformulation

A classical way to derive tighter models for flow formulations, as the arc-load flow formulation presented in the previous section, is to use multi-commodity formulations. The idea is to disaggregate the flow on each arc into different flows, one for each possible

destination. Here, by destination we mean a port-period pair. With this reformulation it is possible to derive tighter models. From the practical point of view however the number of variables can be prohibitive when solving real problems.

Let us introduce the non-negative multi-commodity arc-load flow variables  $\gamma_{imjnvk}^{ut}$ ,  $i, j, u \in N, m, n, t \in M, k \in K$  as the amount of product  $k$  that ship  $v$  transports from port  $i$ , after an operation that started in period  $m$ , to port  $j$  in an operation starting in period  $n$  to be delivered at port  $u$  in period  $t$ . We assume  $\gamma_{imjnvk}^{ut} = 0$  if  $x_{imjnv} = 0$ . These variables can be related with the arc-load flow variables throughout the following equations

$$f_{imjnvk} = \sum_{u \neq i} \sum_{t \geq n} \gamma_{imjnvk}^{ut}, \quad \forall i, j, u \in N, m, n, t \in M, v \in V, k \in K. \quad (37)$$

The tightening of the F-SSDP model can be obtained by replacing constraints (34) by

$$\sum_{k \in K} \gamma_{imjnvk}^{ut} \leq \min\{V_v^{CAP}, \sum_{l \in M, l \geq t} \sum_{k \in K} D_{ulk}\} x_{imjnv}, \quad \forall i, j, u \in N, m, n, t \in M, v \in V. \quad (38)$$

The multi-commodity flow model obtained from F-SSDP by replacing (34) with (38) and including (37) will be denoted by MF-SSDP. Of course the arc-flow variables  $f_{imjnvk}$  can be eliminated from that model using (37).

We note here that different multi-commodity arc-flow formulations could be derived. For instance, instead of considering the amount of product  $k$  delivered at port  $u$  in period  $t$ , one could consider the amount of product  $k$  to be consumed at port  $u$  in period  $t$ .

### 3.3. Valid inequalities

One approach to derive a stronger model is to include valid inequalities for the set of feasible solutions  $X$ . In order to derive valid inequalities we consider simpler substructures that result from relaxations of our formulation. Valid inequalities for the set of feasible solutions of these relaxations are also valid for  $X$ . We focus on deriving only those inequalities with great impact on the integrality gap reduction. For each family of inequalities we consider the separation problem and tune the separation algorithms.

In Section 3.3.1 we develop two types of inequalities based on the inventory constraints, while inequalities based on fixed charge flow sets are developed in Section 3.3.2. Finally, some strong inequalities for the F-SSDP are defined in Section 3.3.3.

#### 3.3.1. Inequalities based on the inventory constraints

Here we consider valid inequalities for  $X$  derived from well-known valid inequalities for inventory lot-sizing sets obtained when considering constraints (5), (13), (15), (19), (20), (21), (26).

First we introduce valid inequalities for the set of feasible solutions based on the well-known  $(\ell, S)$  inequalities derived for lot-sizing problems (see Pochet and Wolsey [2005]). In order to do that we first consider the following set obtained from constraints (5), (13), (15), (19), (20), (21), (26):

$$s_{i,m-1,k} + \sum_{v \in V} q_{imvk} = D_{imk} + s_{imk}, \quad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \quad (39)$$

$$q_{imvk} \leq V_v^{CAP} o_{imvk}, \quad \forall i \in N, m \in M, v \in V, k \in K, \quad (40)$$

$$\sum_{v \in V} o_{imvk} \leq 1, \quad \forall i \in N, m \in M, k \in K, \quad (41)$$

$$s_{imk}, q_{imvk} \geq 0, \quad \forall i \in N, m \in M, v \in V, k \in K, \quad (42)$$

$$o_{imvk} \in \{0, 1\}, \quad \forall i \in N, m \in M, v \in V, k \in K. \quad (43)$$

Constraints (41) are implied by (5) and (26).

The set of solutions satisfying constraints (39)-(43) can be separated for each port  $i$  and each product  $k$ . By fixing a port  $i$  and a product  $k$  (and removing the corresponding indices, for simplicity), we obtain:

$$s_{m-1} + \sum_{v \in V} q_{mv} = D_m + s_m, \quad \forall m \in M, \quad (44)$$

$$q_{mv} \leq V_v^{CAP} o_{mv}, \quad \forall m \in M, v \in V, \quad (45)$$

$$\sum_{v \in V} o_{mv} \leq 1, \quad \forall m \in M, \quad (46)$$

$$s_m, q_{mv} \geq 0, \quad \forall m \in M, v \in V, \quad (47)$$

$$o_{mv} \in \{0, 1\}, \quad \forall m \in M, v \in V. \quad (48)$$

The set of solutions satisfying (44)-(48), denoted by  $X^{LS}$ , is closely related to the feasible set of capacitated lot-sizing problems (see Pochet and Wolsey [2005]). The polyhedral structure of related sets has been intensively study in the past. In Pochet and Wolsey [2005] it is given a very complete and insightful survey of these studies.

Consider  $y_m = \sum_{v \in V} o_{mv}$  and  $x_m = \sum_{v \in V} q_{mv}$ . From (46) and (48) it follows that  $y_m \in \{0, 1\}$ . Let  $C = \max\{V_v^{CAP} : v \in V\}$ . Hence the following set, denoted by  $X^{CLS}$  is a relaxation of  $X^{LS}$  :

$$s_{m-1} + x_m = D_m + s_m, \quad \forall m \in M, \quad (49)$$

$$x_m \leq C y_m, \quad \forall m \in M, \quad (50)$$

$$s_m, x_m \geq 0, \quad \forall m \in M, \quad (51)$$

$$y_m \in \{0, 1\}, \quad \forall m \in M. \quad (52)$$



Set  $X^{CLS}$  is the feasible set of the well-known single-item constant capacitated lot-sizing problem (see Pochet and Wolsey [2005]). For the instances based on real data that we consider in this paper, in general, the demand of each product at each island over the time period does not exceed the capacity of the smallest ship. Hence, for these instances, constants  $V_v^{CAP}$  in (45) can be seen as a large positive constant and therefore  $X^{CLS}$  can be regarded as the incapacitated single-item lot-sizing problem. In this case the set of well-known  $(\ell, S)$  inequalities defined for all  $\ell \in M$ ,  $S \subseteq \{1, \dots, \ell\}$ ,

$$s_{r-1} + \sum_{j \in \{r, \dots, \ell\} \setminus S} x_j + \sum_{j \in S} \left( \sum_{i=j}^{\ell} D_i \right) y_j \geq \sum_{i=r}^{\ell} D_i, \quad (53)$$

where  $r = \min\{i \in S\}$ , suffice to describe the convex hull of  $X^{CLS}$ .

By writing these inequalities in the original variables we obtain the following proposition.

**Proposition 3.1.** *For each  $i \in N$ ,  $\ell \in M$ ,  $S \subseteq \{1, \dots, \ell\}$ ,  $k \in K$  the inequality  $(\ell, S)$*

$$s_{i,r-1,k} + \sum_{n \in \{r, \dots, \ell\} \setminus S} \sum_{v \in V} q_{invk} + \sum_{m \in S} \left( \sum_{n=m}^{\ell} D_{ink} \right) \sum_{v \in V} o_{imvk} \geq \sum_{n=r}^{\ell} D_{ink}, \quad (54)$$

where  $r = \min\{i \in S\}$ , is valid for  $X$ .

**Example 3.1.** Let  $N = \{1, 2, \dots, 7\}$ ,  $M = \{1, 2, \dots, 12\}$ ,  $V = \{1, 2\}$ ,  $K = \{1, 2, 3, 4\}$ . Fix port  $i = 5$ , product  $k = 3$  (supposing  $J_{53} = -1$ ) and consider the demand  $(0, 300, 0, 0, 0, 0, 0, 0, 900, 0, 0, 700)$  for twelve periods at that island for that product. Letting  $\ell = 12$  and  $S = \{8, 10\}$ , then the following inequality is valid for  $X$  :

$$s_{573} + \sum_{n \in \{9, 11, 12\}} \sum_{v=1}^2 q_{5nv3} + 1600(o_{5813} + o_{5823}) + 700(o_{5,10,1,3} + o_{5,10,2,3}) \geq 1600.$$

This inequality states that the demand during periods 8 to 12 must be satisfied either from an unloading operation in period 8,  $1600(o_{5813} + o_{5823})$ , or from a combination of an unloading operation in period 10,  $700(o_{5,10,1,3} + o_{5,10,2,3})$ , the stock from period 7,  $s_{573}$ , and the unload quantity in periods  $\{9, 11, 12\}$ ,  $\sum_{n \in \{9, 11, 12\}} \sum_{v=1}^2 q_{5nv3}$ .

For the few cases (a product-port pair) where the total demand of a product in a given port is greater than the capacity of a ship, inequalities are still valid, however they may no longer define the convex hull of  $X^{CLS}$ .

The family of inequalities  $(\ell, S)$  includes an exponential number of inequalities. As we describe in Section 5 we only use a small number of these inequalities.

Mixed-integer rounding (MIR) is a very powerful technique to derive strong valid inequalities for mixed integer sets, see Marchand and Wolsey [2001]. The well-known MIR-inequalities (see Nemhauser and Wolsey [1988]) can be stated as follows.

**Proposition 3.2.** *Let  $Y = \{(s, y) \in \mathbb{R}_+ \times \mathbb{Z} : s + ay \geq d\}$ . The inequality  $s \geq r(\lceil d/a \rceil - y)$  is valid for  $Y$ , where  $r = d - (\lceil d/a \rceil - 1)a$ .*

Next we apply this proposition to derive valid inequalities for each model, B-SSDP, F-SSDP and MF-SSDP. In order to do that we must define mixed-integer sets of the form of  $Y$  that result from relaxation of the set of feasible solutions of each formulation.

First we consider equations (15). For each port  $i$  and product  $k$  such that  $J_{ik} = -1$ , aggregating equations (15) for all periods in  $M$ , and using nonnegativity of  $s_{i,|M|,k}$  we obtain:

$$s_{i0k} + \sum_{m \in M} \sum_{v \in V} q_{imvk} \geq \sum_{m \in M} D_{imk}. \quad (55)$$

For each  $S \subseteq M$  and  $v' \in V$  (55) can be written as

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V: v \neq v'} q_{imvk} + \sum_{m \in S} o_{imv'k} \geq \sum_{m \in M} D_{imk}. \quad (56)$$

Using (40), it follows that

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V: v \neq v'} q_{imvk} + V_{v'}^{CAP} \sum_{m \in S} o_{imv'k} \geq \sum_{m \in M} D_{imk}. \quad (57)$$

Let  $s = s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V: v \neq v'} q_{imvk}$ ,  $y = \sum_{m \in S} o_{imv'k}$ ,  $a = V_{v'}^{CAP}$ ,  $d = \sum_{m \in M} D_{imk}$ . Applying Proposition 3.2, we obtain the following result.

**Proposition 3.3.** *For each  $i \in N, S \subseteq M, v' \in V, k \in K$  such that  $J_{ik} = -1$ , the following MIR inequality*

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V: v \neq v'} q_{imvk} + r \sum_{m \in S} o_{imv'k} \geq r \left\lceil \frac{\sum_{m \in M} D_{imk}}{V_{v'}^{CAP}} \right\rceil, \quad (58)$$

where  $r = \sum_{m \in M} D_{imk} - \left( \left\lceil \frac{\sum_{m \in M} D_{imk}}{V_{v'}^{CAP}} \right\rceil - 1 \right) \cdot V_{v'}^{CAP}$ , is valid for  $X$ .

**Example 3.2.** Continuing Example 3.1, let  $V_1^{CAP} = 1500$ ,  $V_2^{CAP} = 2000$ . Considering  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $v' = 1$ , then the following inequality is valid for  $X$ :

$$s_{503} + \sum_{m=7}^{12} \sum_{v=1}^2 q_{5mv3} + \sum_{m=1}^6 q_{5m23} + 400 \sum_{m=1}^6 o_{5m13} \geq 800.$$

This inequality states that either the number of unload operations of ship 1 (at port 5 for product 3) during the first six periods is at least two (the minimum number of unload operations from this ship necessary to satisfy all the demand) or else, if there is only one unload operation from ship 1 during the first six periods, then the unloaded quantity from the other ship (ship 2) and from ship 1 during period 7 to period 12 must be at least  $r = 400$ , that is, the total demand (1900) minus the capacity of ship 1, 1500.

### 3.3.2. Inequalities based on fixed charge flow sets

Here we introduce valid inequalities based on the number of ship visits to a set of ports during a given time horizon. We develop and present the family of inequalities for F-SSDP, but similar results can be derived for B-SSDP and MF-SSDP.

Let  $\overline{D}(S, L, Q)$  denote the total demand for the subset of products  $Q \subseteq K$ , in ports  $S \subseteq N$ , such that  $J_{ik} \neq 1$ , for all  $i \in S, k \in Q$ , ( $S$  does not contain any supply port of products in  $Q$ ), during the time horizon  $L = \{1, \dots, \ell\} \subseteq M$ , with  $\ell \geq 2$ . Hence,  $\overline{D}(S, L, Q) = \sum_{i \in S} \sum_{n \in L} \sum_{k \in Q} D_{ink}$ . Let  $D(S, L, Q)$  denote the amount of demand in  $\overline{D}(S, L, Q)$  that must be transported from ports in  $N \setminus S$ , that is,  $D(S, L, Q) = \overline{D}(S, L, Q) - \sum_{v \in V: i_v \in S} \sum_{k \in Q} Q_{vk} - \sum_{i \in S} \sum_{k \in Q} s_{i0k}$ .

For each  $Q \subseteq K, S \subseteq N, L = \{1, \dots, \ell\} \subseteq M$ , such that  $S$  does not contain any supply port of products in  $Q$ , define the following subset  $X(S, L, Q)$ :

$$\sum_{v \in V} \sum_{i \in N \setminus S} \sum_{j \in S} \sum_{m \in L} \sum_{n \in L, n > m} \sum_{k \in Q} f_{imjnvk} \geq D(S, L, Q), \quad (59)$$

$$\sum_{k \in Q} f_{imjnvk} \leq V_v^{CAP} x_{imjnv}, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V, \quad (60)$$

$$f_{imjnvk} \geq 0, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V, k \in Q, \quad (61)$$

$$x_{imjnv} \in \{0, 1\}, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V. \quad (62)$$

In order to verify that  $X(S, L, Q)$  can be obtained as relaxation of  $X$ , we consider the aggregation of constraints (15) over the sets  $S, L, Q$ :

$$\sum_{j \in S} \sum_{k \in Q: J_{jk} = -1} s_{j0k} + \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} q_{jnvk} = \overline{D}(S, L, Q) + \sum_{j \in S} \sum_{k \in Q: J_{jk} = -1} s_{j\ell k}.$$

Since variables  $s_{j\ell k}$  are nonnegative, it follows that

$$\sum_{j \in S} \sum_{k \in Q: J_{jk} = -1} s_{j0k} + \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} q_{jnvk} \geq \overline{D}(S, L, Q). \quad (63)$$

Using (35) and (36), then we obtain

$$\begin{aligned} & \sum_{j \in S} \sum_{k \in Q: J_{jk} = -1} s_{j0k} + \sum_{v \in V: j_v \in S} \sum_{k \in Q: J_{jk} = -1} \left( Q_{vk} - \sum_{i \neq j_v} \sum_{n > 1} f_{j_v 1 i n v k} \right) \\ & + \sum_{v \in V} \sum_{j \in S} \sum_{n \in L, n > 1} \sum_{k \in Q: J_{jk} = -1} \left( \sum_{i \neq j} \sum_{m < n} f_{i m j n v k} - \sum_{i \neq j} \sum_{m > n} f_{j n i m v k} \right) \geq \bar{D}(S, L, Q) \end{aligned} \quad (64)$$

$$\begin{aligned} & \Leftrightarrow \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} \sum_{m < n} \left( \sum_{i \in N \setminus S} f_{i m j n v k} + \sum_{i \in S \setminus \{j\}} f_{i m j n v k} \right) \\ & \geq D(S, L, Q) + \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} \sum_{m > n} \left( \sum_{i \in N \setminus S} f_{j n i m v k} + \sum_{i \in S \setminus \{j\}} f_{j n i m v k} \right). \end{aligned} \quad (65)$$

Constraints (59) are implied by (65) since

$$\sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} \sum_{m \in L: m < n} \sum_{i \in S \setminus \{j\}} f_{i m j n v k} = \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q: J_{jk} = -1} \sum_{m \in L: m > n} \sum_{i \in S \setminus \{j\}} f_{j n i m v k},$$

and using nonnegativity of  $f_{j n i m v k}^k$ .

Sets  $X(S, L, Q)$  have been intensively studied in the past (e.g. Padberg et al. [1985]). Although some computational tests have been conducted using valid inequalities derived from these sets we focus here only on valid inequalities for a set obtained from aggregation of these  $X(S, L, Q)$  sets. There are two main reasons for this choice: (i) the commercial software used is able to include flow cover inequalities, which are known to be important to strengthening the gap for sets of type  $X(S, L, Q)$ ; (ii) preliminary computational results showed that the inequalities we introduce below provided larger reduction of the integrality gap.

We aggregate the arc-load flow variables corresponding to each ship, that is

$$Y_v = \sum_{i \in N \setminus S} \sum_{j \in S} \sum_{m \in L} \sum_{n \in L} \sum_{k \in Q} f_{i m j n v k}, \forall v \in V,$$

and aggregate the corresponding arc variables:

$$X_v = \sum_{i \in N \setminus S} \sum_{j \in S} \sum_{m \in L} \sum_{n \in L} \sum_{k \in Q} x_{i m j n v}, \forall v \in V.$$

Variables  $Y_v$  indicate the load transported from ports in  $N \setminus S$  to ports in  $S$  during the time horizon  $L$  by ship  $v$ , while  $X_v$  denotes the number of times ship  $v$  visits a port in  $S$  coming from a port not in  $S$  during the time horizon  $L$  (see Figure 4).

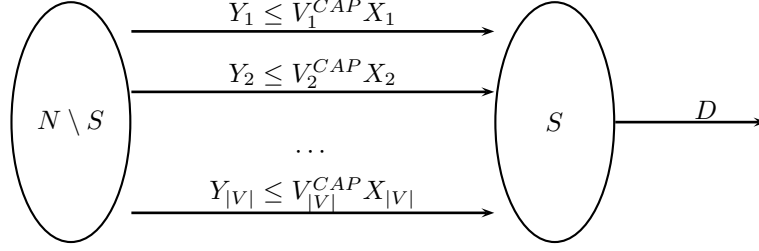


Figure 4: A fixed charge flow set.

Let  $D$  denote the total demand for the subset of products  $Q$ , in ports  $S$ , during the time horizon  $L$ , that must be transported from ports in  $N \setminus S$ , that is,  $D = D(S, L, Q)$ . Hence, the following mixed integer set is a relaxation of the set of the feasible solutions:

$$\left\{ (Y, X) \in \mathbb{R}_+^{|V|} \times \mathbb{Z}_+^{|V|} : \sum_{v \in V} Y_v \geq D, Y_v \leq V_v^{CAP} X_v, v \in V \right\}.$$

In our case there are only two ships, that is  $|V| = 2$ . Then we obtain the following aggregated model, denoted by  $XY^2$ , with two continuous and two integer variables:

$$\{(Y_1, Y_2, X_1, X_2) \in \mathbb{R}_+^2 \times \mathbb{Z}_+^2 : Y_1 + Y_2 \geq D, Y_1 \leq V_1^{CAP} X_1, Y_2 \leq V_2^{CAP} X_2\}.$$

For each valid inequality for  $XY^2$ ,

$$\alpha_1 X_1 + \alpha_2 X_2 + \beta_1 Y_1 + \beta_2 Y_2 \geq \alpha,$$

we obtain a valid inequality,

$$\begin{aligned} & \alpha_1 \sum_{i \in N \setminus S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} x_{imjn1} + \alpha_2 \sum_{i \in N \setminus S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} x_{imjn2} \\ & + \beta_1 \sum_{i \in N \setminus S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q} f_{imjn2k} + \beta_2 \sum_{i \in N \setminus S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q} f_{imjn2k} \geq \alpha, \end{aligned} \quad (66)$$

for  $X$ . This model  $XY^2$  is closely related to the models studies in Agra and Constantino [2006]. The purpose of this paper is not to provide full polyhedral description for  $XY^2$ , but only to identify those valid inequalities with large impact on the gap.

It is easy to verify that facet-defining inequalities for the convex hull of

$$X^2 = \{(X_1, X_2) \in \mathbb{Z}_+^2 : V_1^{CAP} X_1 + V_2^{CAP} X_2 \geq D\}$$

are also facet-defining inequalities for the convex hull of  $XY^2$ .

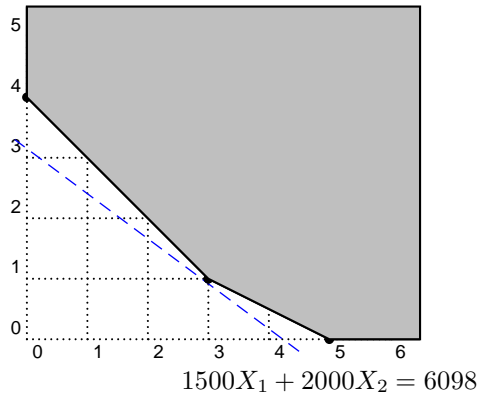


Figure 5: Facet defining inequalities for  $\text{conv}(\{(X_1, X_2) \in \mathbb{Z}_+^2 : 1500X_1 + 2000X_2 \geq 6098\})$ .

In general the convex hull of  $X^2$ , ( $\text{conv}(X^2)$ ) contains non-trivial facet-defining inequalities, that is, facets that are not defined by  $X_1 \geq 0, X_2 \geq 0, V_1^{CAP}X_1 + V_2^{CAP}X_2 \geq D$ . For a polyhedral description of  $\text{conv}(X^2)$  see Agra and Constantino [2006]. Such facet-defining inequalities were already used in Ziarati et al. [1999] for a locomotive assignment problem. Since  $V_1^{CAP}$  and  $V_2^{CAP}$  are large and  $D$  is at most 3 or 4 times the smallest coefficient, it is easy to see that  $V_1^{CAP}X_1 + V_2^{CAP}X_2 \geq D$  does not define a facet of  $\text{conv}(X^2)$  and that  $\text{conv}(X^2)$  has one or two facet-defining inequalities (this is not the general case of integer sets with two variables).

**Example 3.3.** Consider the following set with  $V_1^{CAP} = 1500, V_2^{CAP} = 2000$ , and a demand of  $D = 6098$ . From figure 5 we can see that  $\text{conv}(X^2)$  has two non-trivial facet defining inequalities:  $X_1 + X_2 \geq 4$  and  $X_1 + 2X_2 \geq 5$ .

This family of inequalities on the  $x_{jnimv}$  variables proved to be very important on solving our instances. We call these inequalities Nvisits-inequalities since they are written on the aggregation of  $x_{jnimv}$  variables, thus, on the number of visits to a subset of ports.

Families of inequalities similar to the Nvisits-inequalities have been used in other maritime transportation problems, see for example Persson and Göthe-Lundgren [2005].

Another important family of inequalities that define facets for  $XY^2$  is the MIR inequalities

$$Y_v \geq r \left( \left\lceil \frac{D}{V_v^{CAP}} \right\rceil - X_v \right), \forall v \in V,$$

where  $r = D - \left( \left\lceil \frac{D}{V_v^{CAP}} \right\rceil - 1 \right) V_v^{CAP}$ . Some preliminary tests have shown that these MIR inequalities were ineffective in reducing the integrality gap and, therefore, we ignore them in the computational results.

### 3.3.3. Strong inequalities

The fourth family of inequalities, called strong inequalities (see Chouman et al. [2009] for the use of these inequalities on a related problem), are considered only for the F-SSDP and MF-SSDP. The strong inequalities for the F-SSDP are defined as follows:

$$f_{imjnvk} \leq \min \left\{ V_v^{CAP}, \sum_{u \in N} \sum_{t \in M, t \geq n} D_{utk} \right\} x_{imjnv}, \quad \forall i, j \in N, n, m \in M, v \in V, k \in K. \quad (67)$$

For the MF-SSDP, the strong inequalities are defined as follows:

$$\gamma_{imjnvk}^{ut} \leq \min \left\{ V_v^{CAP}, \sum_{l \in M, l \geq t} D_{ulk} \right\} x_{imjnv}, \quad \forall i, j, u \in N, n, m, t \in M, v \in V, k \in K. \quad (68)$$

Since  $\min \left\{ V_v^{CAP}, \sum_{u \in N} \sum_{t \in M, t \geq n} D_{utk} \right\} \geq \min \left\{ V_v^{CAP}, \sum_{l \in M, l \geq t} D_{ulk} \right\}$ , the strong inequalities are tighter for the MF-SSDP.

The huge number of inequalities in these families makes the use of a separation algorithm necessary in order to choose only a small number of cuts to be included.

## 4. Dedicated tanks case

In this section we consider the tank allocation policy followed by the company. Changing from a dirty product to a cleaner one imposes a major cleaning operation that is time consuming and expensive. In order to avoid such major changeovers the company dedicates tanks to families of products in such a way that the changeover time and cost between products of the same family can be neglected. The families of products do not depend on the ship. Unpredictable situations, such as bad weather conditions, may force the company to change this policy in order to satisfy the demands.

The three models discussed for the non dedicated tanks problem B-SSDP, F-SSDP and MF-SSDP can be adapted to handle the real case with dedicated tanks. Next we give the changes in F-SSDP, since this was the model that provided best result for the case without dedicated tanks. The new model will be denoted by F-SSDP-DC.

We denote by  $S_v^C$  the set of compartments of ship  $v$ . For each compartment  $c \in S_v^C$ , we define its capacity as  $V_v^{CAP^c}$  and define the set of products that  $c$  can transport as  $K_v^c$ . Parameter  $Q_{vk}^c$  denotes the quantity of product  $k$  in compartment  $c$  of ship  $v$  at the beginning of the planning horizon.

When a family has more than one product, we need to specify the compartment where the product is transported for each continuous variable  $f_{imjnvk}$  and  $q_{imvk}$ . In order to do that we define the new continuous nonnegative variables  $f_{imjnvk}^c$  as the amount of flow  $f_{imjnvk}$  transported from compartment  $c$ , and  $q_{imvk}^c$  as the amount of

product  $k$  loaded onto or unloaded in compartment  $c$  of ship  $v$  at port  $i$  in time period  $m$ .

In order to prevent the transportation of more than one product of the same family in the same tank, we define the new binary variables  $\chi_{imjnvk}^c$  indicating whether ship  $v$  transports product  $k$  in compartment  $c$  when sailing from port  $i$ , after an operation that started in period  $m$ , to port  $j$  and starts to operate at port  $j$  in period  $n$ .

The F-SSDP-DC model is obtained from the model F-SSDP by replacing the constraints (13), (32), (33) and (15) with

$$q_{imvk}^c \leq V_v^{CAPc} o_{imvk}, \quad \forall i \in N, m \in M, v \in V, c \in S_v^C, k \in K_v^c : J_{ik} \neq 0, \quad (69)$$

$$\sum_{i \neq j} \sum_{m < n} f_{imjnvk}^c + J_{jk} q_{jnvk}^c = \sum_{i \neq j} \sum_{m > n} f_{jnimvk}^c, \quad \forall j \in N, n \in M : n > 1, v \in V, c \in S_v^C, k \in K_v^c, \quad (70)$$

$$Q_{vk}^c = \sum_{j \neq i_v} \sum_{n > 1} f_{i_v 1 j n v k}^c - J_{i_v k} q_{i_v 1 v k}^c, \quad \forall v \in V, c \in S_v^C, k \in K_v^c, \quad (71)$$

$$s_{i, m-1, k} + \sum_{v \in V} \sum_{c \in S_v^C} q_{imvk}^c = D_{imk} + s_{imk}, \quad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \quad (72)$$

and replacing (34) with

$$f_{imjnvk}^c \leq V_v^{CAPc} \chi_{imjnvk}^c, \quad \forall i, j \in N, m, n \in M, v \in V, c \in S_v^C, k \in K_v^c, \quad (73)$$

$$\sum_{k \in K_v^c} \chi_{imjnvk}^c \leq x_{imjnv}, \quad \forall i, j \in N, n, m \in M, v \in V, c \in S_v^C. \quad (74)$$

By aggregation of variables  $f_{imjnvk}^c$  and  $q_{imvk}^c$ , a feasible solution of F-SSDP from every feasible solution of F-SSDP-DC can be constructed. The converse is not true, since F-SSDP-DC can be feasible when F-SSDP is infeasible. From the computational point of view, this might lead to larger branch and bound trees when using F-SSDP-DC for those instances where finding a feasible solution is difficult. Also, F-SSDP-DC is larger than F-SSDP.

On the other side, the lower bounds based on the linear relaxation of F-SSDP-DC are in general better than those obtained with F-SSDP. This comes from the fact that the coefficients on the linking constraints are smaller since the capacity of the ships is replaced by the capacity of the tanks.

## 5. Computational results

In this section we present some computational results using 20 instances based on real data from Cape Verde. We tested different models resulting from different ways



of combining the improving strategies. All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with processor Intel Core 2 Duo, CPU 2.2GHz, with 4GB of RAM. We consider the present real sized problem consisting of 4 products, 7 islands and 2 ships and a planning horizon of 12 periods (days).

In Section 5.1 we provide a computational comparison of the different models tightened with valid inequalities. Then, in Section 5.2, we test the best model with larger size instances in order to evaluate its performance on hypothetical future scenarios with increase of demand requirements and number of ships. Finally, in Section 5.3, the model F-SSDP-DC is tested on the real case where tanks are dedicated to families of products, and the results are compared with results obtained from the real plans established by the company.

### 5.1. Model comparison

For the B-SSDP model we tested the inclusion of three families of inequalities:  $(\ell, S)$ , MIR and Nvisits inequalities. We tested separately the inclusion of cuts from each of the families. Table 1 reports the results of these tests. For each case, identified in the first line of the table, we present the integrality gap (Gap), the number of cuts (Cuts) added, and the time (Time) in seconds to solve the problem. Here, the gap is defined as  $GAP = \frac{Opt.Value - LowerBound}{Opt.Value} \times 100$ . We also tested the inclusion of all cuts (last three columns in Table 1). In each case the cuts were introduced only at the root node. First we solve the linear relaxation, then we add cuts and finally we execute the branch and bound using the default options. The value LowerBound used to compute the GAP is the lower bound at the root node after the inclusion of these cuts. All the models with exception of the original one, without cuts, include the tightening of bounds. For each instance we consider a time limit of 3 hours. The asterisk in Instance 5 In Table 1 means that this instance was not solved within the time limit using the B-SSDP, the B-SSDP with  $(\ell, S)$  and the B-SSDP with MIR inequalities. In the last line of the table we include the corresponding average value of all the 20 instances.

For the  $(\ell, S)$  inequalities we use the separation procedure described in Pochet and Wolsey [2005]. For the MIR inequalities we include all cuts (inequalities violated by the linear relaxation solution) from this family while the improvement in the bound is greater than 1%. For the Nvisits, the separation algorithm includes all cuts from those inequalities where either  $S$  or  $N \setminus S$  is a singleton.

Since the transportation cost is the most relevant cost for the optimal value, the Nvisit-inequalities are the ones that provide highest reduction of the gap as these cuts consider explicitly the routing variables. The number of cuts introduced from this family is usually very small. Hence as we can see from Table 1 this family alone is the one that provides best results. The gap is half of the original gap. Only the approach that includes cuts from all families provides better results in terms of gap but not on the

Table 1: Computational tests for the B-SSDP.

Int.	B-SSDP		B-SSDP+( $\ell$ , $S$ )			B-SSDP +MIR			B-SSDP+NVISITS			B-SSDP+All		
	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	72,9	903	63,2	26	121	57,2	77	49	36,2	11	78	28,6	114	35
2	86,9	1701	77,9	37	2393	68,1	135	2356	53,2	10	337	42,9	182	526
3	88,1	382	64,1	15	389	78,1	200	237	56,9	11	169	40,5	226	214
4	77,5	33	67,7	3	8	59,1	33	15	15,1	8	7	13,8	44	6
5*	85,5	10800	71,9	4	10801	71,9	30	10801	34,8	10	568	34,1	44	752
6	79,2	184	49,1	9	64	57,9	238	64	36,4	6	33	30,3	253	22
7	75,7	9	35,1	18	6	32,5	107	3	25,2	6	3	11,5	131	4
8	84,8	134	77,3	30	58	73,6	43	55	46,4	11	48	45,1	84	74
9	90,9	19	63,0	6	15	72,3	79	9	61,2	9	11	58,5	94	12
10	84,4	35	66,6	53	34	67,6	160	34	31,1	10	8	27,3	223	8
11	77,2	10	41,1	5	5	32,4	97	5	23,4	3	4	18,4	105	7
12	86,4	64	69,1	13	39	81,7	176	29	52,1	13	22	48,7	202	25
13	88,6	6841	75,5	28	4403	74,5	184	5326	56,4	5	1717	55,7	217	2494
14	83,1	6	62,6	4	3	60,7	34	2	31,3	8	1	29,0	46	1
15	91,2	71	62,9	27	43	79,6	219	44	69,0	10	73	53,1	256	42
16	90,1	52	70,3	22	29	77,4	184	23	54,6	4	25	52,5	210	20
17	84,1	71	72,9	20	40	75,1	80	51	40,0	6	28	37,0	106	31
18	89,3	13	53,6	3	4	64,2	54	3	19,2	5	2	18,3	62	2
19	89,3	28	76,1	19	21	78,2	124	15	63,7	11	13	62,1	154	27
20	84,0	35	66,0	28	30	67,8	238	30	31,4	11	15	28,4	277	12
Av.	<b>84,5</b>	<b>1069,55</b>	<b>64,3</b>	<b>18,5</b>	<b>925,3</b>	<b>66,5</b>	<b>124,6</b>	<b>957,6</b>	<b>41,9</b>	<b>8,4</b>	<b>158,1</b>	<b>36,8</b>	<b>151,5</b>	<b>215,7</b>

time. In this last approach the set of Nvisit cuts are introduced first, then the  $(\ell, S)$  are introduced and, finally, the MIR cuts are introduced. With this approach the average gap is reduced from 84.5% to 36.8%. These gaps do not include the cuts from Xpress.

For the F-SSDP the results are presented in Table 2. For this model we also present the results for the family of strong inequalities (denoted by SI). In this case only the cuts corresponding to the greatest violation are introduced. The approach including all types of cuts follow the sequence of cuts: Nvisits,  $(\ell, S)$ , Strong inequalities and MIR. This approach proved to be the best one by reducing the average gap to 16.2% and by reducing the computational time to less than one minute on average, and always below 10 minutes. When the B-SSDP was used, one instance was not solved within 3 hours. Again, notice that the gaps reported do not include the cuts introduced by Xpress using the default options.

For the MF-SSDP the results are presented in Table 3. As expected, the best bounds were obtained with MF-SSDP, the tightest formulation. However, the size of the model leads to poor running times.

In Table 4 we provide an overview of the average results obtained with the three models. The line Nodes gives the average number of nodes of the branch and bound algorithm.

### 5.2. Future scenarios: larger size instances

To test the model that performed better, F-SSDP, on larger instances we created two artificial future scenarios where the demands as well as the number of ships are increased. One scenario with three ships and where demands are 1.5 times the current demands, and another scenario with four ships and where the demands are doubled. The results are given in Table 5. For each scenario, identified by the number of ships ( $|V| = 3$  and  $|V| = 4$ ), we provide the integrality gap (Gap-I), the gap given by Xpress at the end of the running time, limited to three hours (Gap-E), and the running time (Time). In the four ships case some instances become infeasible because of the port

Table 2: Computational tests for the F-SSDP.

Inst.	F-SSDP		F-SSDP+(I,S)			F-SSDP +MIR			SSDP+NVISITS			SSDP+SI			SSDP+All		
	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	33.9	64	24.1	9	82	23.5	70	33	20.3	8	44	29.5	1	65	16.3	78	28
2	45.5	688	29.5	23	284	33.7	137	336	32.9	9	300	38.0	7	446	21.3	161	150
3	56.5	120	42.5	19	51	30.0	160	81	40.2	9	53	52.8	1	74	7.1	308	46
4	25.0	14	14.4	2	11	14.4	28	8	13.4	5	10	19.4	1	15	7.3	52	5
5	40.0	1736	32.4	4	1726	31.8	42	797	18.3	7	93	33.3	1	1494	15.9	82	180
6	45.4	57	29.6	5	36	27.1	92	25	28.4	6	41	38.6	1	26	18.2	37	25
7	44.5	8	7.7	14	7	16.0	126	3	16.4	6	4	15.4	1	12	3.1	90	4
8	62.2	119	43.2	30	56	52.1	59	46	41.3	10	53	54.1	1	63	35.4	83	37
9	68.3	36	46.4	6	19	36.2	104	12	46.1	9	18	50.8	1	20	28.1	128	9
10	28.7	16	19.3	23	11	21.2	200	8	15.5	7	6	28.1	1	15	10.7	165	6
11	53.0	21	20.8	9	14	20.9	156	6	20.5	4	9	28.9	1	15	5.0	116	7
12	46.7	37	28.3	20	24	38.5	211	21	31.3	10	13	46.5	1	29	22.0	199	14
13	52.8	1592	39.5	4	825	43.6	36	855	39.7	5	750	44.8	1	1147	32.5	49	549
14	59.7	8	30.5	2	5	30.5	17	2	18.7	5	2	44.5	1	9	7.6	61	2
15	64.7	34	35.3	31	32	30.6	394	22	53.7	10	29	60.3	1	35	16.5	243	24
16	50.2	71	30.3	27	40	38.7	197	41	31.0	5	53	35.1	6	63	19.1	129	46
17	37.9	80	28.4	8	73	29.2	206	43	20.2	6	33	31.7	1	35	17.1	115	35
18	64.9	13	29.0	2	7	20.6	34	4	11.1	6	3	49.6	1	13	6.2	53	3
19	47.0	40	39.3	11	20	36.0	119	18	40.0	7	18	33.3	1	35	25.9	206	11
20	39.9	24	18.7	23	19	27.9	317	14	19.6	12	10	26.3	1	21	8.8	229	13
Av.	<b>48.3</b>	238.9	29.5	13.6	167.1	30.1	135.3	118.8	<b>27.9</b>	7.3	77.1	38.0	1.6	181.6	<b>16.2</b>	129.2	<b>59.7</b>

27

Table 3: Computational tests for the MF-SSDP.

Inst.	MF-SSDP		MF-SSDP+(I,S)			MF-SSDP +MIR			MF-SSDP +NV			MF-SSDP +SI			MF-SSDP +All		
	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	22.7	100	21.1	17	115	22.2	17	142	17.9	7	62	22.4	1	86	17.3	14	100
2	29.3	4613	26.6	21	4095	26.6	65	4555	27.0	5	5755	25.9	11	2836	19.8	47	1445
3	15.6	328	15.6	17	213	15.6	6	274	10.8	6	176	15.6	5	268	6.8	14	168
4	16.5	16	14.4	16	12	14.4	26	14	8.7	6	11	16.5	1	12	7.3	93	9
5	29.2	4315	29.2	0	4448	29.2	10	4166	18.4	4	481	29.1	2	3846	13.5	30	294
6	27.6	9	27.6	0	7	27.6	0	11	21.4	4	11	27.6	1	11	21.4	26	6
7	8.3	5	8.3	3	5	8.3	21	7	8.2	1	11	8.3	2	12	7.8	5	6
8	34.1	135	34.1	16	117	34.1	8	155	33.0	5	180	30.7	3	190	27.0	25	129
9	25.6	19	25.6	0	18	25.6	7	23	25.6	4	31	24.5	3	31	19.5	39	17
10	17.4	11	15.6	26	11	15.3	59	12	13.1	5	30	17.4	1	22	11.0	42	11
11	17.4	8	8.2	44	10	8.2	26	12	17.4	1	11	15.3	5	17	4.9	72	11
12	26.2	39	26.2	16	44	26.2	42	31	23.6	6	25	25.8	2	47	16.8	21	19
13	37.8	1173	37.8	9	1280	37.8	14	1250	37.8	2	2347	37.8	2	2023	36.0	60	1384
14	28.4	2	28.4	25	2	28.4	39	3	7.6	5	2	25.9	3	3	6.8	7	2
15	12.8	53	12.8	10	62	12.8	35	63	12.8	2	54	12.7	1	58	12.4	93	82
16	28.1	56	28.1	0	53	28.1	20	65	26.5	1	66	26.9	2	130	22.8	87	70
17	29.6	129	27.3	37	87	24.7	32	118	21.5	3	126	29.2	2	103	19.2	145	104
18	20.1	2	20.1	0	3	20.1	0	3	5.0	5	6	20.1	1	3	3.6	10	2
19	34.9	26	34.5	24	22	34.4	40	31	34.8	3	30	34.6	1	34	26.7	90	29
20	13.0	23	12.8	10	15	12.8	60	30	11.4	6	24	12.9	2	25	8.3	14	11
Av.	23.7	553.1	22.7	14.6	531.0	22.6	26.4	548.3	19.1	4.1	472.0	23.0	2.6	487.85	15.5	46.7	195.0

Table 4: Summary of general information from the three models.

	B-SSDP	B-SSDP+ALL	F-SSDP	F-SSDP+ALL	MF-SSDP	MF-SSDP+ALL
Gap (%)	84.5	36.6	48.3	16.2	23.7	15.5
Nodes	50562.4	16622.6	4301.7	853.9	2307.7	665.9
Time (sec.)	1069.6	215.7	238.9	59.7	472	195

activity restrictions that impose a maximum of one ship operating at each port per time period.

Table 5: Computational results for the F-SSDP with 3 and 4 ships

Inst.	V  = 3			V  = 4		
	Gap-I	Gap-E	Time	Gap-I	Gap-E	Time
1	22.9	0.0	309	24.6	1.1	10800
2	36.4	17.0	10800	34.9	21.4	10800
3	36.7	7.4	10800	30.2	12.9	10800
4	17.0	0.0	72	17.1	0.0	2724
5	23.1	0.0	1411	19.0	6.1	10800
6	19.8	0.0	11	35.5	0.0	686
7	35.8	0.0	119	-	-	-
8	37.1	0.0	195	25.5	0.0	863
9	40.3	0.0	103	32.3	0.9	1037
10	14.7	0.0	454	-	-	-
11	31.0	0.0	651	44.2	7.7	10800
12	14.9	0.0	89	-	-	-
13	22.9	0.0	1697	18.3	1.0	5027
14	20.0	0.0	7	22.7	0.0	21
15	36.4	0.0	5351	32.8	9.5	10800
16	30.1	0.0	2408	25.8	6.5	10800
17	33.7	6.1	10800	29.5	18.3	10800
18	21.3	0.0	11	20.5	1.0	208
19	33.1	0.0	893	31.4	8.6	10800
20	26.2	0.0	2790	-	-	-
Av.	27.7	1.5	2449.6	27.0	4.9	5552.15

In order to derive  $N$ visits inequalities for the three and four ships cases we first generate a  $N$ visits inequality for each subset of  $X^2$  of two variables obtained considering two ships. Then the  $N$ visits inequality is lifted using the subadditive lifting function  $\omega_3$  given in Agra and Constantino [2007].

To improve the running times we also adapted the branching strategy presented in Ziarati et al. [1999] (see also Agra et al. [2011]). We establish high priority for branching on the variables representing the number of ship visits to each port. This strategy proved to be very effective.

We can see from Table 5 that 17 instances were solved to optimality for the three ships case, and only 4 were solved for the four ships case, within the limit of three hours.

### 5.3. Real case: tanks are dedicated to families of products

The F-SSDP-DC is tested on instances based on the real case of dedicated tanks. Three families of products were considered. Two of them with one product only: the fuel

(the dirtiest product) and jet (the cleanest product), and one family with two products, gasoline and diesel.

Since the linear relaxation of F-SSDP-DC provides better lower bounds for this case than those provided by F-SSDP for the non dedicated tanks case, the impact of the inclusion of valid inequalities in F-SSDP-DC is lower than the impact of the inclusion of valid inequalities in F-SSDP. Hence we give the results only for the most relevant valid inequalities, the Nvisits inequalities. The computational results are reported in Table 6.

Table 6: Computational results for the dedicated tanks case.

Instance	F-SSDP-DC			F-SSDP-DC+ NVISITS			
	Gap	Nodes	Time	Gap	Nodes	Cuts	Time
1	26.0	82273	3840	25.7	26653	7	1444
2	29.4	1585	42	8.8	418	7	24
3	26.5	27957	1881	22.4	25680	5	1651
4	25.4	5752	2534	12.7	1746	11	979
5	31.2	35918	1602	10.2	27283	9	1334
6	32.3	23530	2257	4.7	21436	12	1775
7	32.2	59081	6272	20.7	28580	8	2716
8	24.6	22775	2387	23.0	21106	4	2282
9	26.8	1969	97	3.7	1775	6	74
10	22.3	20800	2651	13.4	8077	8	1856
11	30.0	21345	2687	25.9	19257	6	1873
12	35.2	13057	584	21.6	8634	5	351
Av.	28.5	26336.8	2236.2	16.1	16993.9	7.3	1454.0

In this case we tested 12 instances, compared them with the real plans followed by the company and verified an average gain in the cost (not reported in the table) of approximately 15%.

We can observe from Table 6 that, as expected, the average integrality gap is slightly lower in this case of dedicated tanks. Conversely, the running times are larger. In average the running time is less than 25 minutes.

Finally, Table 7 provides a general overview of the average size of the models tested.

Table 7: Average size of the tested models.

Model	V	Binary var.	Continuous var.	Total var.	Constraints
B-SSDP	2	6344	1920	8264	53163
F-SSDP	2	6344	23712	30056	22503
MF-SSDP	2	6344	577824	584168	43003
F-SSDP	3	9516	30076	39592	33397
F-SSDP	4	12688	39896	52584	44291
F-SSDP-DC	2	28184	37704	65888	123268

## 6. Conclusions

We developed a mixed integer model, B-SSDP, for the short sea fuel oil distribution problem occurring in Cape Verde. The model applies a combined discrete and continuous

time horizon in order to take the varying demands and multiple time windows into account.

Both cases with and without dedicated ship tanks for families of products were considered. In order to efficiently solve the instances considered, we tested different approaches to improve the B-SSDP. In particular, we compared the B-SSDP with two extended formulations, an arc-load flow formulation F-SSDP, using additional variables indicating the amount of each type of fuel oil products each ship transports between each pair of ports and a multi-commodity formulation MF-SSDP. We also tightened the constraints and tested the inclusion of cuts from different families of inequalities. Separation algorithms were used such that we could include few inequalities from those inequality families with high impact on the integrality gap reduction.

The extended formulation, F-SSDP, with tighter bounds, combined with the approach of using a small subset of inequalities from each family proved to be the best option. It allowed us to solve all tested instances within reasonable time.

The models introduced are new and can also be used in other maritime transportation problems. Several of the types of cuts presented here have not been developed for maritime transportation problems previously in the literature, and they can easily be used when solving other real maritime inventory routing problems. We have shown how we can transform existing valid inequalities in the literature to maritime inventory routing problems.

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