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Dimensional reduction of the Two-Higgs Doublet Model with a softly broken Z_2 symmetry at one-loop

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Abstract

In the collaboration between NTNU, the University of Stavanger, and the University of Helsinki we pursue a satisfactory answer to the problem of baryogenesis, i.e. the origin of the asymmetry in the amount of baryons and antibaryons in the universe. Baryogenesis at the electroweak phase transition cannot be explained by the Standard Model as the phase transition is a crossover, not strongly first order as required. In addition, the amount of CP violation in the Standard Model is insufficient. Therefore, we investigate extensions of the Standard Model to find a viable candidate for explaining baryogenesis.

Using the imaginary-time formalism for quantum field theories at finite temperature, we have applied the method of dimensional reduction to the Two-Higgs Doublet Model with a softly broken Z_2 symmetry. An effective three-dimensional Euclidean bosonic theory was constructed by integrating out all non-zero Matsubara modes. The parameters of the effective three-dimensional theory were determined in terms of the parameters of the original four-dimensional theory, by matching the correlators at long distances. The effective potential was used to find the scalar correlators. The discussion was extended to the N-Higgs Doublet Model, where CP violation is only present in the mass-mixing terms. The results obtained here will be used in a numerical simulation of the electroweak phase transition in a future paper.

Samandrag

I samarbeidet mellom NTNU, Universitetet i Stavanger og Universitet i Helsingfors prøver me å finne eit fullnøyande svar på problemet om opphavet til asymmetrien i mengda av materie og antimaterie i universet. I Standardmodellen er den elektrosvake faseovergangen kontinuerleg, ikkje sterkt fyrste ordens som vert kravd for å lage ei netto mengd materie. I tillegg er mengda av brot på CP-symmetrien i Standardmodellen ikkje tilstrekkeleg. Derfor undersøkjer me utvidingar av Standardmodellen for å finne ein modell som kan forklåre dei kosmologiske observasjonane.

Ved hjelp av imaginær-tid formalismen for kvantfeltteoriar ved endeleg temperatur har vi brukt metoden kalla dimensjonsreduksjon på 2-Higgs-dublet-modellen med ein mjukt brota Z_2 symmetri. Ein effektiv 3-dimensjonal Euklidisk bosonsk teori blei konstruert ved å integrere ut alle dei endelege Matsubara-frekvensane. Parameterane til den effektive 3-dimensjonale teorien blei uttrykt som ein funksjon av parameterane til den fulle 4-dimensjonale teorien og temperaturen, ved å setje korrelasjonsfunksjonane på lange avstandar til dei to teoriane lik kvarandre. Det effektive potensialet blei brukt til å finne dei skalare korrelasjonsfunksjonane. Diskusjonen ble utvida til ein N-Higgs-dublet-modell, kor brotet på CP-symmetren berre er til stades i masseparameterane. Resultata me kom fram til her vil bli brukt i ei numerisk simulering av den elektrosvake faseovergangen i ein framtidig artikkel.

Preface

I have applied the method of dimensional reduction to the Two-Higgs Doublet Model (2HDM) with a softly broken Z_2 symmetry, and extended the calculation to the N-Higgs Doublet Model (NHDM). This work is my master thesis in theoretical physics at the Norwegian University of Science and Technology (NTNU) as part of the study program Physics and Mathematics, and was carried out during the spring semester of 2017, at the Department of Physics at NTNU in collaboration with the University of Stavanger (UiS). My supervisor at NTNU has been Professor Jens O. Andersen, and my supervisor at UiS has been Professor Tomas Brauner.

I thank both Professor Andersen and Professor Brauner for their willingness to give excellent advice whenever I was in need of it. Professor Andersen enabled me to be part of the larger collaboration between NTNU, UiS and the University of Helsinki, for which I am grateful. I thank Professor Brauner for his generosity and hospitality when I visited UiS. Lastly, I thank my friends and family for always supporting me.

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Abbreviations

1PI	=	One-particle irreducible
2HDM	=	Two-Higgs Doublet Model
C	=	Charge conjugation
EWPT	=	Electroweak phase tranition
IR	=	Infrared
ITF	=	Imaginary-time formalism
MSSM	=	Minimal Supersymmetric Standard Model
NHDM	=	N-Higgs Doublet Model
NTNU	=	Norwegian University of Science and Technology
P	=	Parity
QCD	=	Quantum chromodynamics
QED	=	Quantum electrodynamics
QGP	=	Quark-gluon plasma
SM	=	Standard Model
SSM	=	Singlet-extended Standard Model
TFT	=	Thermal field theory
UiS	=	University of Stavanger
UV	=	Ultraviolet
VEV	=	Vacuum expectation value

CHAPTER 1

INTRODUCTION

Observations have ruled out the presence of a significant amount of antimatter in the universe on scales ranging from the solar system to clusters of galaxies, and even distances comparable to the scale of the present horizon [1]. The dominance of matter in the universe remains one of the unresolved puzzles in cosmology. Generally, two distinct solutions can be put forward. Either the universe can be assumed to start out in an asymmetric state, with no net generation of matter, or the universe starts out in a symmetric state, and the net amount of matter we observe today has to be generated. It is most natural, both from an aesthetic and intellectual point of view, to rule out the first alternative. It is by no stretch of the imagination a satisfactory explanation, as the question of why the initial conditions of the universe were the way we observe is left unanswered. We will focus on the second alternative, which requires a mechanism for generating a net amount of baryons in the universe, also known as baryogenesis.

In 1967, Sakharov recognised that a mechanism for the generation of an asymmetry in the baryon number in the universe must satisfy certain criteria, the so-called Sakharov criteria [2]:

1. Baryon number violation
2. C and CP violation
3. Deviation from thermal equilibrium

Here C and P stand for the discrete symmetries charge conjugation (C) and parity (P). CP is the combination of the two discrete symmetries charge conjugation and parity.

Firstly, we consider baryon number violation. The baryon number is defined as a third of the difference between the number of quarks and antiquarks. The baryons (antibaryons) are bound states of three quarks (antiquarks), which give rise to the numerical factor. The requirement of baryon number violation follows immediately from the assumption of a symmetric initial state of the universe (vanishing baryon number), and the observations of an asymmetric universe (non-zero baryon number). As both quantum electrodynamics (QED) and quantum chromodynamics (QCD) preserve baryon number, we look to the electroweak sector of the Sandard Model (SM) for help, where baryon number conservation is broken by the chiral anomaly [3; 4]. The rate of the baryon number non-conservating processes is negligible at zero temperature, but at high temperature the rate $\sim \exp[-\frac{\pi M_W(T)}{\alpha_w T}]$, where $M_W(T)$ is the mass of the intermediate vector bosons $W^{\pm}$ [5]. Above the electroweak phase transition the mass of the intermediate vector bosons vanishes together with the vacuum expectation value (VEV) of the Higgs field [6], and the rate of the baryon number violating processes is rapid compared to the rate of expansion of the universe.

The second criterion is that the discrete symmetries C and CP must be violated. If this were not the case, then the rate of processes involving baryons would be equal to the rate of the processes involving antibaryons. No net baryon number would be generated. Baryon number violating processes, called sphaleron processes, must be biased in producing more baryons than antibaryons. For this to be possible, C and CP must be violated.

The last criterion is that the universe must be out of thermal equilibrium, which is connected with the symmetry CPT, i.e. CP together with the discrete symmetry time reversal (T). Any unitary, Lorentz invariant quantum field theory has been shown to be invariant under the CPT symmetry [7]. The transformation properties of the baryon number are such that it is invariant under P and T, while it changes sign under C. Thus, when calculating the thermal average of the baryon number at equilibrium, the average can be shown to vanish. In order to avoid reaching the obviously wrong conclusion of a vanishing baryon number, the universe must have been, at some stage, away from thermal equilibrium, and thereby invalidating the argument above. A possible way of ensuring a deviation from thermal equilibrium is to let the universe undergo a first order phase transition.[1]

The electroweak phase transition (EWPT) has been intensively investigated due to its possible connection to the generation of the asymmetry in the baryon number in the universe [5; 8] (see refs. [9; 10] for reviews). One of the most important features of the phase transition is the requirement to be of first order. This is to ensure deviation from thermal equilibrium. If the phase transition is first order, bubbles of the broken phase will nucleate in a sea of the symmetric phase. Then the bubbles will expand, collide, coalesce, and fill the whole universe [5; 11]. The baryon asymmetry is generated in the vicinity of the expanding bubble walls, away from thermal equilibrium.

For the baryon asymmetry generated during the phase transition to survive until today, sphaleron processes (baryon number violating processes) must be suppressed immediately after the phase transition [8]. The strength of the phase transition is reduced when increasing the mass of the Higgs boson [10], and weakly first order phase transitions will not sufficiently suppress the sphaleron processes, and no net baryon number will survive. This problem arises in the SM: with a Higgs mass of 125 GeV [12; 13], the phase transition is not strongly first order, but a crossover transition, and the SM is unable to account for baryogenesis at the EWPT [14; 15; 16]. In addition, the amount of CP violation is suppressed at high temperatures and it is clear that the amount of CP violation in the SM is insufficient [17; 18; 19; 20; 21; 22; 23; 24]. Since the SM is unable to explain the baryon-antibaryon asymmetry in the universe, we need physics beyond the SM.

As the SM is unable to account for baryogenesis, extensions of the SM have been investigated, as the Minimal Supersymmetric Standard Model (MSSM) [25], scalar-extended Standard Model (SSM) [26] and the Two-Higgs Doublet Model (2HDM) [27]. The theo-

[1] The requirement that the universe undergo a first order phase transition is not sufficient; the phase transition must be a strong first order phase transition in order to ensure the suppression of sphaleron processes in the broken phase.

ries are investigated to find regions of the parameter space where the phase transition is strongly first order, to avoid the wash out of the excess of baryon number by sphaleron processes.

Investigating the EWPT has been made feasible by applying the techniques of dimensional reduction [28] and effective field theory [29]. The idea behind dimensional reduction is that in the imaginary-time formalism, discussed in section 2.1.2, the bosonic and fermionic fields acquire a Matsubara frequency, which acts as a mass term. All but one bosonic mode, the zero mode, have a non-zero thermal mass, and decouple from the bosonic zero mode at high temperature and weak coupling, according to the decoupling theorem by Appelquist and Carazzone [30]. We can integrate out the so-called *superheavy* modes, i.e. the modes with a mass of order T, and are left with an effective three-dimensional theory with only zero modes. All the infrared problems of finite temperature field theory are associated with the zero modes, so the method of dimensional reduction is free of infrared problems. Several momentum scales are present if we consider a non-abelian gauge theory [28]. The temporal component of the gauge field acquires a mass of order gT, while the spatial component of the gauge field provides a momentum scale of order g^2T non-perturbatively. The scalar mass in theories with a single Higgs doublet will generally be of order g^2T close to the phase transition. When extending the number of Higgs doublets, the masses of the additional scalar fields are normally of order gT [25]. It is useful to also integrate out the fields of order gT, i.e. the temporal component of the gauge field and possibly some scalar fields, to construct a second effective field theory [29]. Studying the resulting effective field theory at high temperature has been useful. The effective theory contains severe infrared problems in the symmetric phase, and perturbation theory breaks down. Therefore, the phase transition was studied on the lattice [31; 14; 15; 16].

The recent detection of gravitational waves [32] is relevant for baryogenesis. During a first order phase transition, the colliding bubble walls and the aftermath of the bubble collisions will produce gravitational waves. The possibility of detecting the gravitational waves produced at a first order electroweak phase transition would be a direct probe of the mechanism for baryogenesis [33; 34; 35]. Thus, cosmological observations will give insight into the particle content of the underlying theory. The detection (or absence of) primordial gravitational waves will help verify or falsify possible extensions of the SM, and will give complementary information to the information obtained through collider experiments. The space-based detector eLISA [36], expected to be launched in 2034, may observe gravitational waves originating from the EWPT. Thus, investigating the strength of the EWPT in extensions of the SM is of great interest.

The outline of the thesis is as follows. In chapter 2 background material needed for the rest of the thesis is briefly discussed. In chapter 3 the Two-Higgs doublet model is introduced. In chapter 4 correlators needed for dimensional reduction are presented. β-functions for the gauge couplings are also calculated. Chapter 5 discusses the effective potential, and extracts the counterterms for the scalar sector. β-functions for the scalar couplings are also presented there. The method of dimensional reduction is applied to the 2HDM in Chapter 6, and the connection between the parameters in the effective three-dimensional theory and the full four-dimensional theory is discussed. In Chapter 7 the discussion is generalised to the N-Higgs doublet model. Chapter 8 contains a conclusion and outlook for future work.

CHAPTER 2

PRELIMINARIES

This chapter contains a discussion of some of the fundamental concepts needed to understand the thesis. For a basic introduction to thermal field theory (TFT), see e.g. [37; 38; 39]. The informed reader may skim through or skip altogether this chapter.

2.1 Thermal field theory

Extending the framework of quantum field theory to finite temperature is of interest in many areas of research, from the interior of compact stars to heavy ion collisions and the evolution of the universe. New phenomena arise at finite temperature, as e.g. a new phase in QCD called the quark-gluon plasma [40], relevant for both heavy ion collisions and the early universe [41; 42]. Another phenomenon arising at finite temperature, central to baryogenesis, is the electroweak phase transition [8; 9; 10].

Two main frameworks for describing quantum field theories at finite temperature have been developed, the imaginary-time formalism and the real-time formalism. There are advantages and disadvantages with both frameworks, as we will shortly discuss.

The imaginary-time formalism naturally connects statistical mechanics with the path integral of quantum field theory. Many of the same methods can be employed in evaluating the path integral representation of the partition function, both perturbative methods (Feynman diagrams) and numerical calculations. One shortcoming of the imaginary-time formalism is that it is unable to describe out-of-equilibrium phenomena.

The real-time formalism can account for both equilibrium and non-equilibrium behaviour, but the technical evaluation is more cumbersome than in the imaginary-time formalism. We will use the imaginary-time formalism throughout the thesis.

2.1.1 Statistical field theory

We recall some basic notions from statistical field theory. A field theory in thermal equilibrium can be described by the partition function. In the grand canonical ensemble the partition function is a function of the temperature T, volume V, and the chemical potential μ. The partition function is defined as the trace of the density matrix of the system $\hat{\rho}$,

$$Z = \mathrm{Tr}\,\hat{\rho} = \sum_{\phi} \langle \phi | e^{-H/T} | \phi \rangle \tag{2.1}$$

$$\hat{\rho} = e^{-H/T}, \tag{2.2}$$

where H is the Hamiltonian, and $|\phi\rangle$ is an eigenstate of the field ϕ. The sum is over a basis of eigenstates. We can find macroscopic quantities of the system through the relations

$$P = T\frac{\partial \log Z}{\partial V} \tag{2.3}$$

$$N = T\frac{\partial \log Z}{\partial \mu} \tag{2.4}$$

$$S = \frac{\partial T \log Z}{\partial T} \tag{2.5}$$

$$E = -PV + TS + \mu N, \tag{2.6}$$

where P is the pressure, N is the number of particles, S is the entropy, and E is the internal energy. The density matrix can be used to calculate the thermal average of a physical observable $\langle O \rangle$,

$$\langle O \rangle = \frac{\mathrm{Tr}\, O\hat{\rho}}{Z}. \tag{2.7}$$

More details can be found in any decent textbook on statistical field theory. From now on we set the chemical potential to zero. In the next section, we will connect the partition function to the path integral formalism.

2.1.2 Imaginary-time formalism

We want to connect the statistical mechanics partition function with the path integral of quantum field theory. The main idea of the imaginary-time formalism is to recognise that a four-dimensional theory at finite temperature is equivalent to a 3+1 dimensional theory, with three dimensions of space and a compact dimension of time, with (anti-)periodic boundary conditions.

When deriving the path integral formalism for finite temperature field theory, we follow [43]. Consider the transition amplitude for going from $|\phi_0\rangle$ at $t = 0$ to $|\phi_1\rangle$ at $t = t_1$

$$\langle \phi_1 | e^{-iHt_1} | \phi_0 \rangle = \mathcal{N} \int \mathcal{D}\pi \mathcal{D}\phi \exp\left[i \int_0^{t_1} dt \int d^3x \left[\pi\dot{\phi} - \mathcal{H}(\pi, \phi) \right] \right], \tag{2.8}$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are eigenstates of the field ϕ, $\mathcal{H}$ is the Hamiltonian density, π is the conjugate momentum of the field ϕ, $\dot{\phi} = \partial\phi/\partial t$, and $\mathcal{N}$ is a normalisation factor. The integral $\int \mathcal{D}\phi$ goes over all possible field configurations respecting the initial and final conditions, while the integral $\int \mathcal{D}\pi$ is unconstrained.

In quantum field theory at zero temperature, we often analytically continue our theory from real to imaginary time: $t \to -i\tau$, where τ is real. This means that we have moved from Minkowski to Euclidean space, as the metric takes the form of a Euclidean metric (with a change of sign): $t^2 - \boldsymbol{x}^2 \to -(\tau^2 + \boldsymbol{x}^2)$.

We now rotate our path integral to Euclidean space. Also, we identify $it_1 = 1/T$. The transition amplitude in eq. (2.8) has become

$$\langle \phi_1 | e^{-H/T} | \phi_0 \rangle = \mathcal{N} \int \mathcal{D}\pi \mathcal{D}\phi \exp\left[\int_0^{1/T} d\tau \int d^3x \left[i\pi\dot{\phi} - \mathcal{H}(\pi, \phi) \right] \right], \tag{2.9}$$

where now $\dot{\phi} = \partial\phi/\partial\tau$. We recognise that eq. (2.9) is very similar to eq. (2.1). The main difference is that in eq. (2.1) the initial and final states are identical, and we sum over all such states. To connect the partition function to the path integral we restrict the integral $\int \mathcal{D}\phi$ to go over all periodic paths, i.e. paths where the field configuration is the same at $t = 0$ and $t = 1/T$. We then write

$$\begin{aligned} Z = \mathrm{Tr}\, e^{-H/T} &= \sum_{\phi} \langle \phi | e^{-H/T} | \phi \rangle \\ &= \mathcal{N} \int \mathcal{D}\pi \int_{\text{periodic}} \mathcal{D}\phi \exp\left[\int_0^{1/T} d\tau \int d^3x \left[i\pi\dot{\phi} - \mathcal{H}(\pi, \phi) \right] \right]. \end{aligned} \tag{2.10}$$

The momentum integration is unrestricted, as before. Most Hamiltonians are at most quadratic in the conjugate momentum, and we can perform the integral $\int \mathcal{D}\pi$ by completing the square. We can replace the conjugate momentum in favour of $\dot{\phi}$, as we go from a Hamiltonian to a Lagrangian description of the system. Thus, we have that

$$Z = \mathcal{N}' \int_{\text{periodic}} \mathcal{D}\phi \exp\left[\int_0^{1/T} d\tau \int d^3x \mathcal{L}(\phi, \dot{\phi}) \right], \tag{2.11}$$

where $\mathcal{N}'$ is a new temperature-dependent normalisation constant.

We want to take a closer look at the boundary conditions for the fields, and consider a Euclidean Lagrangian $\mathcal{L}(\Phi, \Psi)$ with bosonic and fermionic fields Φ and Ψ, respectively. Take the bosonic thermal Green function for propagation from the point $(\vec{y}, 0)$ to the point $(\vec{x}, \tau)$,

$$G_B(\vec{x}, \vec{y}; \tau, 0) = \frac{\mathrm{Tr}\left\{ T_\tau \left[\Phi(\vec{x}, \tau) \Phi(\vec{y}, 0) \right] \hat{\rho} \right\}}{Z}, \tag{2.12}$$

where T_τ is the imaginary-time ordering operator,

$$T_\tau \left[\Phi(\tau_1), \Phi(\tau_2) \right] = \Phi(\tau_1)\Phi(\tau_2)\theta(\tau_1 - \tau_2) + \Phi(\tau_2)\Phi(\tau_1)\theta(\tau_2 - \tau_1), \tag{2.13}$$

and $\theta(\tau)$ is the Heaviside step function. We use the Heisenberg time evolution

$$e^{H/T} \Phi(\vec{y}, 0) e^{-H/T} = \Phi(\vec{y}, 1/T), \tag{2.14}$$

and the commutation property of the imaginary-time ordering operator and the Hamiltonian, $\left[T_\tau, e^{-H/T} \right] = 0$, to get

$$G_B(\vec{x}, \vec{y}; \tau, 0) = G_B(\vec{x}, \vec{y}; \tau, 1/T), \tag{2.15}$$

which implies

$$\Phi(\vec{x},0) = \Phi(\vec{x},1/T). \tag{2.16}$$

For the fermions, we make the exact same steps. The only difference is the definition of the imaginary-time ordering operator,

$$T_\tau\left[\Psi(\tau_1),\Psi(\tau_2)\right] = \Psi(\tau_1)\Psi(\tau_2)\theta(\tau_1-\tau_2) - \Psi(\tau_2)\Psi(\tau_1)\theta(\tau_2-\tau_1), \tag{2.17}$$

coming from the statistics for the fermionic fields. This minus sign goes through the calculation, and we end up with

$$\Psi(\vec{x},0) = -\Psi(\vec{x},1/T). \tag{2.18}$$

Hence, the bosonic fields obey periodic boundary conditions, while the fermionic fields obey antiperiodic boundary conditions in the compactified imaginary-time direction.

In summary, to construct a path integral representation of the partition function from a zero-temperature Lagrangian, we perform the following steps:

i Do a Wick rotation from Minkowski to Euclidean space, where $\tau \equiv it$ is the imaginary-time.

ii Let
$$\mathcal{L} = -\mathcal{L}_M(\tau = it) \tag{2.19}$$
where $\mathcal{L}_M$ is the zero-temperature Lagrangian in Minkowski space.

iii Compactify the imaginary-time dimension, e.g. restrict τ to the interval $(0, 1/T)$.

iv Impose (anti-)periodic boundary conditions for the bosonic (fermionic) fields,
$$\Phi(\boldsymbol{x},0) = \Phi(\boldsymbol{x},\tau) \tag{2.20}$$
$$\Psi(\boldsymbol{x},0) = -\Psi(\boldsymbol{x},\tau). \tag{2.21}$$

Because of step (i), the method is known as the *imaginary-time formalism* (ITF).

The Euclidean action takes the form

$$S = \int_0^{1/T} d\tau \int d^3x \mathcal{L}, \tag{2.22}$$

where the imaginary-time integration only goes over the interval $(0, 1/T)$. Because of the boundary condition eqs. (2.20) and (2.21), the bosonic and fermionic fields can be expanded as

$$\Phi(x,\tau) = \sqrt{T}\sum_{n=-\infty}^{\infty}\phi_n(x)e^{i\omega_n^B\tau} \tag{2.23}$$

$$\Psi(x,\tau) = \sqrt{T}\sum_{n=-\infty}^{\infty}\psi_n(x)e^{i\omega_n^F\tau}. \tag{2.24}$$

The values of the so-called Matsubara frequencies are

$$\omega_n^B = 2n\pi T, \qquad \text{for bosons} \tag{2.25}$$
$$\omega_n^F = (2n+1)\pi T, \qquad \text{for fermions} \tag{2.26}$$

where $n \in \mathbb{Z}$. By inserting eqs. (2.23) and (2.24) into the Euclidean action in eq. (2.22), we can trivially perform the integral over imaginary time. The resulting action is a three-dimensional integral over space, and a sum over the Matsubara frequencies. Thus, we can see that a quantum field theory at finite temperature is the same as a three-dimensional Euclidean theory with an infinite number of fields. This feature of the imaginary-time formalism will be useful later when we consider effective theories and dimensional reduction.

Going to momentum space, where we normally perform our calculations, we see that the finite temperature effect amounts to replacing the normal four-momentum integration by an infinite sum and a three-momentum integration times the temperature,

$$\int \frac{d^{d+1}k}{(2\pi)^{d+1}} \to T \sum_n \int \frac{d^d k}{(2\pi)^d}. \tag{2.27}$$

We will use the short-hand notation $\SumInt_K$ and $\SumInt_{\{K\}}$ defined in eq. (A.4).

2.1.3 Thermal mass

To get more acquainted with the ITF, we will start by looking at a scalar theory. We will assume that the temperature is sufficiently high so that any bare mass scales can be neglected. The Euclidean Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g^2}{4!}\phi^4 \tag{2.28}$$

where g^2 is the scalar coupling constant. We use the notation g^2 instead of the conventional λ because it turns out that the perturbation theory is an expansion in g, not $\lambda = g^2$ as at zero temperature. This point will be discussed in more detail in section 2.1.6.

We can divide the Lagrangian in eq. (2.28) into a free and an interacting part,

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\partial_\mu \phi)^2, \qquad \mathcal{L}_{\text{int}} = \frac{g^2}{4!}\phi^4. \tag{2.29}$$

The Feynman rules are derived in a similar fashion to the zero temperature case [43]. The scalar propagators take the form $1/K^2$, and the interaction vertices give a factor of $-g^2$. The first quantum correction to the self-energy comes in the form of

$$I^{(1)} = \frac{1}{2} g^2 \SumInt_K \frac{1}{K^2} = \frac{1}{2} g^2 I_1^{4b}. \tag{2.30}$$

We use dimensional regularisation with $d = 3 - 2\epsilon$ to regularise the ultraviolet (UV) divergences. The UV divergence in eq. (2.30) is set to zero in dimensional regularisation, and we get a finite result. At zero temperature the corresponding integral will simply be zero. The sum-integral is evaluated in appendix C, and gives the value $I_1^{4b} = T^2/12$. Thus, the self-energy at one-loop is

$$I^{(1)} = \frac{g^2 T^2}{24} \equiv m_\beta^2. \tag{2.31}$$

The scalar field has acquired a thermal mass m_β^2 of order gT, arising from interactions with the heat bath.

2.1.4 Phase transitions

In section 2.1.3 we found the thermal mass of a scalar theory to one-loop accuracy to be $m_\beta^2 = \frac{g^2}{24}T^2$. This is an *effective* mass, and arises because the propagation of particles in a heat bath is altered by their continuous interactions with the medium. The mass is also called the Debye mass, from the similar effect in QED plasma [44]. This is one of the major results of thermal field theory. A similar phenomenon is present in QCD at finite temperature [45].

The thermal mass has important consequences for cosmology. Consider a potential with a negative mass-squared term at low temperatures

$$V_{\text{low }T}(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{g^2}{4!}\phi^4. \tag{2.32}$$

Clearly, one extremum of the potential is at a vanishing value of the field, $\phi_{\text{max}} = 0$. Other extrema of the potential are at non-zero values of the field, $\phi_{\text{min}} = \pm\sqrt{6\mu^2/g^2}$. If the mass term is truely negative ($\mu^2 > 0$), then the global minimum is away from the origin. We have a phenomenon called spontaneous symmetry breaking, where a symmetry possessed by the Lagrangian is not shared by the ground state. Here the system has a $\phi \to -\phi$ discrete symmetry, while for the ground state, we must choose either the left or the right minimum. The same happens for continuous symmetries as well, e.g. in the Higgs mechanism [46].

However, interactions with the heat bath induce a *positive* mass-squared term, and at very high temperatures the effective potential takes the form

$$V_{\text{high }T}(\phi) = \frac{g^2}{48}T^2\phi^2 + \frac{g^2}{4!}\phi^4. \tag{2.33}$$

The unique minimum of the potential is at $\phi_{\text{min}} = 0$. Hence, there must be a *phase transition* between the high- and low-temperature regimes. To find the order of the phase transition, subleading terms must be taken into account. Even so, this is an important result. A spontaneously broken symmetry can be restored at sufficiently high temperature [6]. This symmetry restoration is the underlying mechanism for the electroweak phase transition, where the nonzero VEV of the Higgs field we observe today vanishes at high temperatures.

For the universe to move away from thermal equilibrium, the phase transition must be of first order. A barrier between the high-temperature minimum at the origin and the low-temperature minimum away from the origin arises near the critical temperature T_c. The critical temperature is the temperature where the high- and low-temperature minima are both global minima. Figure 2.1 shows a sketch of a first order phase transition, where the minimum at the origin has been shifted to coincide with the horizontal axis. The other possibility is that the phase transition happens continuously, i.e. that no potential barrier is present at the critical temperature. This is called a second order (or continuous) phase

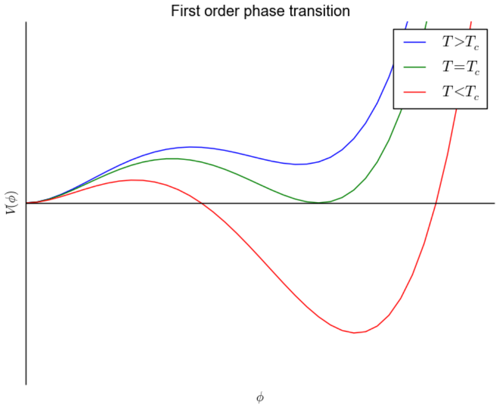


Figure 2.1: A sketch of a first order phase transition. A barrier separates the two minima at the critical temperature. The value of the minimum at the origin is shifted to zero.

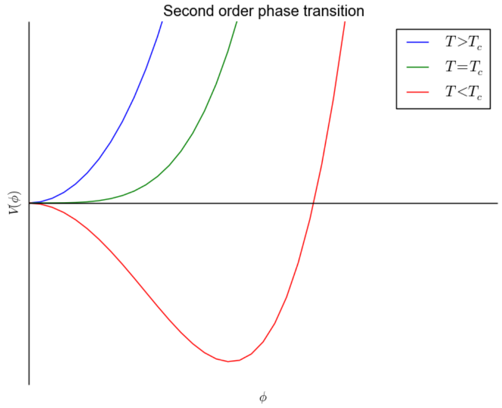


Figure 2.2: A sketch of a second order phase transition. No barrier is present at the critical temperature. The value of the potential at the origin is shifted to zero.

transition, and fig. 2.2 shows a sketch of a continuous phase transition. Alternatively, the transition can be a crossover, where we cannot distinguish between the two phases.

The relevance of the order of the phase transition goes back to the criteria for baryogenesis by Sakharov [2]. One of the criteria for baryogenesis is the deviation from thermal equilibrium. If the electroweak phase transition is a strong first order phase transition, the deviations from equilibrium are large, and a net baryon number can be produced by CP violating scatterings of the bubble walls with the surrounding plasma. The net baryon number will not be washed-out by sphaleron processes, and survive until today. However, if the phase transition is of second order or is a crossover, then no net baryon number will be generated, as the sphaleron processes are not suppressed. Thus, the order of the phase transition determines the generation of any net baryon number through electroweak baryogenesis, and is therefore responsible for all the matter we observe in the universe today, including you and me.

From the discussion above, we were able to find that there must be a phase transition between the low and high temperature regimes. However, we were not able to see if the phase transition was first or second order. We again have the potential

$$V(\phi, T) = \frac{g^2}{48}\left(T^2 - 24\frac{\mu^2}{g^2}\right)\phi^2 + \frac{g^2}{4!}\phi^4 = D(T^2 - T_0^2)\phi^2 + \frac{g^2}{4!}\phi^4, \tag{2.34}$$

where we have defined $D = g^2/48$ and $T_0^2 = 24\mu^2/g^2$. At $T < T_0$, the global minima are at $\phi_{\min}(T < T_0) = \pm\sqrt{\frac{12D(T_0^2 - T^2)}{g^2}}$, while a local maximum is at $\phi_{\max}(T = 0) = 0$. The Z_2 symmetry $\phi \leftrightarrow -\phi$ of the Lagrangian is spontaneously broken. At $T > T_0$, we have only one global minimum, $\phi_{\min}(T > T_0) = 0$. For $T = T_0$, both the solutions collapse at $\phi_{\min}(T = T_0) = 0$. There is no barrier between the low and high temperature solution for the minimum, so this phase transition is a *second order* phase transition.

However, for baryogenesis we need a *first order* phase transition in order to have deviations from thermal equilibrium. Consider the potential

$$V(\phi, T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{g^2}{4!}\phi^4, \tag{2.35}$$

where we have included a cubic term with the constant coefficient E. This is actually the form of the SM effective potential at one-loop and analysed in refs. [47; 48; 49]. However, perturbation theory is not to be trusted too close to the phase transition. We will treat the potential as an example of a first order phase transition. At temperatures above T_1 the only minimum is at $\phi_{\min}(T > T_1) = 0$, where

$$T_1^2 = \frac{4g^2 D T_0^2}{4g^2 D - 27E^2}. \tag{2.36}$$

T_1 is also the temperature when a local minimum at $\phi(T_1) \neq 0$ appears. This is an inflection point. The value of the inflection point is $\langle \phi(T_1) \rangle = 9ET_1/g^2$. For $T < T_1$ a barrier between the global minimum at the origin and the local minimum away from the origin starts to develop. The local minimum and the maximum of the barrier are at $\phi(T) = \left(9ET \pm \sqrt{81E^2T^2 - 12g^2D(T^2 - T_0^2)}\right)/g^2$, respectively. At the critical temperature T_c, the origin and the other minimum become degenerate, where

$$T_c^2 = \frac{g^2 D T_0^2}{g^2 D - 6E^2}. \tag{2.37}$$

Below the critical temperature, the global minimum will no longer be at the origin, and the local minimum at the origin becomes metastable. Finally, the barrier disappears at $T = T_0$, and the origin becomes a local maximum.

One last comment is in order. The SM phase transition cannot be reliably analysed using only perturbation theory, as perturbation theory fails close to the phase transition. The SM phase transition was analysed using Monte Carlo simulations in the mid 90s, and the conclusion was that the strength of the phase transition depended on the mass of the Higgs boson [14; 15; 16]. The main result was that the phase transition turns into a crossover when the mass of the Higgs is above about 80 GeV. The phase diagram of the electroweak phase transition is similar to the liquid-vapour phase diagram of water, with a first order line ending in a second order point, called the critical point. At higher temperatures and pressures the transition is a crossover, where it is impossible to distinguish between the liquid and the vapour phase.

2.1.5 Infrared problems

It is well known that quantum field theories at finite temperature are plagued with infrared problems [50]. We will use a simple scalar theory to illustrate some of the problems arising from loop diagrams at finite temperature.

Consider again the massless $g^2\phi^4$theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g^2}{4!}\phi^4, \tag{2.38}$$

where g^2 is the scalar coupling. The usual procedure of replacing the four-dimensional momentum integrals with an infinite sum and a three-dimensional momentum integral is applied. As for the scalar propagator, we get at the two-loop level the contribution[1]

$$I^{(2a)} = -\frac{g^4}{4} \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \sum\!\!\!\!\!\!\int_K \frac{1}{(K^2)^2}. \tag{2.39}$$

For the zero mode $n = 0$, we have the integral

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2)^2} \sim \int dk \frac{1}{k^2} \sim \infty \quad \text{as } k \to 0. \tag{2.40}$$

The sum-integral diverges at large distances, or small momenta. We call this kind of divergence an infrared (IR) divergence. There is an infinite set of infrared divergent Feynman diagrams, with increasingly severe divergences. The problem of the infrared behaviour of quantum field theories at finite temperature looks at the offset to be an unmanageable problem. However, much work has been devoted to curing the problem, as we will see shortly.

We will look at one example of resummation, and see that the infrared divergences disappear when an infinite set of Feynman diagrams is summed.

2.1.6 Resummation

We will here present the resummation program initiated by Braaten and Pisarski [51]. A review of the program of resummation is presented in [52].

In the 1980s, a major problem of the apparent gauge dependence of the gluon damping rate γ_{damp} stimulated progress in understanding quantum field theories at finite temperature. As the damping rate is a physical quantity, it cannot be gauge dependent. Pisarski [53] pointed out that the one-loop calculations performed to date were incomplete, and an infinite subset of diagrams was to be included in order to get the correct result. For more details see ref. [54].

We will continue to use the massless scalar theory to get a better look at the IR divergences. In eq. (2.40) we first encountered an IR divergent diagram. Perturbation theory breaks down because of these kinds of IR divergences. However, in eq. (2.31) we calculated the thermal mass, which in practice screens the IR divergences. We include this effect by using the effective propagator

$$\Delta(\omega_n, k) = \frac{1}{K^2 + m_\beta^2}, \qquad \text{with } m_\beta \sim gT \ll T. \tag{2.41}$$

The thermal mass can be omitted if the momentum in the propagator is *hard*, i.e. of order T. However, the thermal mass must be included if the momentum in the propagator is *soft*, i.e. of order gT. The thermal mass provides an IR cutoff of order gT, and the zeroth mode contribution in eq. (2.39) now becomes

$$-\frac{1}{4} g^4 \sum\!\!\!\!\!\!\int_K \frac{1}{K^2} T \int_p \frac{1}{(p^2 + m_\beta^2)^2} = -\frac{1}{4} g^4 \left(\frac{T^2}{12}\right)\left(\frac{T}{8\pi m_\beta}\right) + \mathcal{O}(g^4 mT). \tag{2.42}$$

Since $m_\beta \sim gT$, the contribution to the self-energy is of order g^3T^2, and not the naively expected g^4T^2. Other bubble diagrams also contribute at order g^3T^2, and must be included. It can be shown that the bubble diagrams form so-called daisy diagrams, with

[1] There is also a contribution from a sunset diagram at the two-loop level, which is logarithmic IR divergent.

additional bubbles connected to a central bubble [52]. An infinite subset of these daisy diagrams must be summed, and the reorganisation of the perturbation expansion is called *resummation*.

We have seen that resummation in scalar theories simply amounts to replacing the propagator with an effective propagator, where the thermal mass is included. A thermal mass term must also be included in the interaction part of the Lagrangian. For gauge theories, the situation is more complicated, as the thermal "mass" will depend non-trivially on the external momentum. In addition, the vertices must be replaced by effective vertices, which also depend on the external momentum in a non-trivial way. For more details on resummation in hot field theories, see ref. [52].

2.2 β-function

We will be using dimensional regularisation to regularise the ultraviolet divergences that arise both in zero and finite temperature quantum field theory. One feature of dimensional regularisation is that the regularisation scheme introduces an arbitrary renormalisation scale μ. All physical quantities should be independent of the renormalisation scale. It is convenient, and optimal, to set the renormalisation scale such that the contributions from higher orders are minimised.

The bare coupling constant is independent of the renormalisation scale μ,

$$\mu \frac{d}{d\mu}\lambda_{(b)} = \mu \frac{\partial}{\partial \mu}\lambda_{(b)} + \mu \frac{\partial \lambda}{\partial \mu}\frac{\partial}{\partial \lambda}\lambda_{(b)} = 0 \tag{2.43}$$

where $\lambda_{(b)}$ and λ are the bare and renormalised coupling constants, respectively. The β-function is defined as

$$\beta_\lambda \equiv \mu \frac{\partial \lambda}{\partial \mu} = \frac{\partial \lambda}{\partial \log \mu}. \tag{2.44}$$

The β-function can be determined from the counterterms, as we will see shortly.

2.2.1 Structure of the β-function

We will follow the lecture notes by Kaplunovsky [55]. Our goal for this section is to construct a general expression for the β-function from the relevant counterterms. We will begin our discussion with a theory with only one coupling constant (e.g. $\lambda\phi^4$ theory), and afterwards generalise to a theory with multiple coupling constants (e.g. SM or 2HDM). We will be using dimensional regularisation and the MS renormalisation scheme[2] to regularise and renormalise our theory. An L-loop amplitude can give rise to a pole in ϵ of at most order L. The MS scheme simply says that we absorb the poles into the counterterm, which takes the form

$$\delta Z_L = g^{2L}\left[\frac{A_L}{\epsilon^L} + \frac{A_{L-1}}{\epsilon^{L-1}} + \cdots + \frac{A_1}{\epsilon}\right]. \tag{2.45}$$

The bare coupling constant λ_b can be written as

[2]The final answer will be scheme independent to the two-loop level [55]. We can go to the renormalisation scheme $\overline{\text{MS}}$ by redefining the renormalisation scale μ.

$$\lambda_b = \mu^{2\epsilon} \frac{\lambda(\mu) + \delta\lambda(\mu)}{\Big[1 + \delta Z(\mu)\Big]^2}, \tag{2.46}$$

where μ is the renormalisation scale introduced by dimesional regularisation, and $\delta\lambda$ and δZ are the coupling constant and field renormalisation counterterms, respectively. We can write the counterterms as power series in $1/\epsilon$,

$$\delta\lambda = \sum_{L=1}^{\infty} \lambda^{L+1} \sum_{k=1}^{L} \frac{A_{L,k}}{\epsilon^k} \tag{2.47}$$

$$\delta Z = \sum_{L=1}^{\infty} \lambda^{L} \sum_{k=1}^{L} \frac{B_{L,k}}{\epsilon^k} \tag{2.48}$$

for some constant coefficients $A_{L,k}$ and $B_{L,k}$. Taking the limit $\lambda \to 0$ before the limit $\epsilon \to 0$, we have

$$\frac{\lambda(\mu) + \delta\lambda(\mu)}{\Big[1 + \delta Z(\mu)\Big]^2} = \lambda(\mu) + \sum_{L=1}^{\infty} \lambda^{L+1}(\mu) \sum_{k=1}^{L} \frac{C_{L,k}}{\epsilon^k}, \tag{2.49}$$

where the constant coefficients $C_{L,k}$ are given by polynomials in $A_{L',k'}$ and $B_{L',k'}$, with $L' \leq L$ and $k' \leq k$. The first few coefficients take the form

$$C_{1,1} = A_{1,1} - 2B_{1,1}, \quad C_{2,1} = A_{2,1} - 2B_{2,1}, \quad C_{2,2} = A_{2,2} - A_{1,1}B_{1,1} + 3B_{1,1}^2 - 2B_{2,2}, \tag{2.50}$$

It is convenient to re-express the sum as

$$\lambda_b = \mu^{2\epsilon}\lambda(\mu) + \mu^{2\epsilon} \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}, \quad \text{where} \quad f_k(\lambda) = \sum_{L=k}^{\infty} C_{L,k}\lambda^{L+1}. \tag{2.51}$$

We will be doing calculations at one-loop, and it is sufficient to know $f_1(\lambda)$, which is given as

$$f_1(\lambda) = \text{Residue of simple } \frac{1}{\epsilon} \text{ pole of } \delta\lambda - 2\lambda\delta Z. \tag{2.52}$$

Now, we will use eq. (2.43) to find the β-function. The bare coupling is independent of the renormalisation scale μ, so the left-hand side of eq. (2.51) becomes

$$\mu \frac{d}{d\mu} \lambda_b = 0. \tag{2.53}$$

The right-hand side of eq. (2.51) is more involved,

$$\mu \frac{d}{d\mu} \mu^{2\epsilon} \Big[\lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}\Big] = 2\epsilon \Big[\lambda(\mu) + \sum_{k=1}^{\infty} \frac{f_k(\lambda(\mu))}{\epsilon^k}\Big] + \beta(\lambda) \Big[1 + \sum_{k=1}^{\infty} \frac{f_k'(\lambda(\mu))}{\epsilon^k}\Big], \tag{2.54}$$

where $f_k'(\lambda(\mu)) = \frac{d}{d\lambda(\mu)} f_k(\lambda(\mu))$. By equating the eqs. (2.53) and (2.54) we get

$$0 = 2\epsilon\Big[\lambda(\mu) + \sum_{k=1}^{\infty}\frac{f_k(\lambda(\mu))}{\epsilon^k}\Big] + \beta(\lambda)\Big[1 + \sum_{k=1}^{\infty}\frac{f'_k(\lambda(\mu))}{\epsilon^k}\Big]. \tag{2.55}$$

By making the β-function dependent on the spacetime dimension, $\beta(\lambda) = \beta(\lambda, \epsilon)$, we can express it as a power series in ϵ

$$\beta(\lambda, \epsilon) = \sum_{n=0}^{\infty} \beta_n(\lambda)\epsilon^n \tag{2.56}$$

where we have only included non-negative powers of ϵ as the β-function is not singular in the limit $\epsilon \to 0$. By combining eqs. (2.55) and (2.56), and rearranging the terms we end up with

$$-2\epsilon\lambda - 2\epsilon\sum_{k=1}^{\infty}\frac{f_k(\lambda)}{\epsilon^k} = \Big[\sum_{n=0}^{\infty}\beta_n(\lambda)\epsilon^n\Big]\Big[1 + \sum_{k=1}^{\infty}\frac{f'_k(\lambda)}{\epsilon^k}\Big]. \tag{2.57}$$

As we can vary ϵ, the two sides should match for any power of ϵ. Since the left-hand side is at most linear in ϵ, we conclude that

$$\beta(\lambda, \epsilon) = \beta_0 + \epsilon\beta_1(\lambda). \tag{2.58}$$

This simplifies eq. (2.57) a great deal,

$$-2\epsilon\lambda(\mu) - 2\epsilon\sum_{k=1}^{\infty}\frac{f_k(\lambda)}{\epsilon^k} = \epsilon\beta_1(\lambda) + \beta_0 + \beta_1\sum_{k=1}^{\infty}\frac{f'_k(\lambda)}{\epsilon^{k-1}} + \beta_0\sum_{k=1}^{\infty}\frac{f'_k(\lambda)}{\epsilon^k}. \tag{2.59}$$

By comparing equal powers in ϵ, we find that

$$\beta_1(\lambda) = -2\lambda(\mu) \tag{2.60}$$

$$\beta_0(\lambda) = -2f_1(\lambda) - \beta(\lambda)f'_1(\lambda). \tag{2.61}$$

Recursion relations for the $f_k(\lambda)$'s can also be obtained from eq. (2.59), but are not needed to find the β-function, which takes the form

$$\beta(\lambda) = \beta_0(\lambda) + \beta_1(\lambda)\epsilon = -2\epsilon\lambda + \Big(2\lambda\frac{d}{d\lambda} - 2\Big)f_1(\lambda). \tag{2.62}$$

The quantity $f_1(\lambda)$ is given by the counterterms, e.g. eq. (2.52) in the $\lambda\phi^4$ theory.

For a theory with multiple coupling constants we reach a similar result. Let $g_s(\mu)$ be the coupling constants, with $s = 1, \dots, n$. The bare coupling constants can be expressed as

$$g_{s,\text{bare}} = \mu^{\Delta_s}\frac{g_s(\mu) + \delta g_s(\mu)}{\prod\limits_{\substack{\text{Appropriate}\\ \text{fields } i}}\sqrt{1+\delta Z_i}} = \mu^{\Delta_s}\left(g_s(\mu) + \sum_{k=1}^{\infty}\frac{f_{k,s}(g_1(\mu), \dots, g_n(\mu))}{\epsilon^k}\right), \tag{2.63}$$

where the Δ_s is the dimensionality of the renormalised coupling constant $g_s(\mu)$. It is sufficient to know $f_{1,s}(g_1, \dots, g_n)$ in order to calculate the β-functions. In a similar fashion to eq. (2.52), we find that

$$f_{1,s}(g_1,\ldots,g_n) = \text{Residue of simple } \frac{1}{\epsilon} \text{ pole of } \left[\delta g_s - \frac{g_s}{2} \sum_{\substack{\text{Appropriate} \\ \text{fields } i}} \delta Z_i\right]. \tag{2.64}$$

The β-functions take the form

$$\beta_s(g_1,\ldots,g_n,\epsilon) = -\Delta_s(\epsilon)g_s + \left[\sum_{p=1}^{n} K_p g_p \frac{\partial}{\partial g_p} - K_s\right] f_{1,s}(g_1,\ldots,g_n), \tag{2.65}$$

where K_s is given by $\Delta_s(\epsilon) = \Delta_s(0) + K_s\epsilon$. For marginal coupling constants $\Delta_s(0) = 0$. For scalar coupling constants $K_s = 2$, while for gauge and Yukawa coupling constants $K_s = 1$. In chapters 4 and 5 we use eq. (2.65) to find the β-functions.

2.2.2 Temperature-independence of the counterterms

We have seen in section 2.2.1 that the β-functions can be extracted from the counterterms. We are familiar with how to extract the counterterms from Feynman diagrams at zero temperature. However, one may wonder if the counterterms remain the same, or if finite temperature effects will also contribute to the UV divergences. It turns out that the UV divergences are the same at zero and finite temperature. Thus, the counterterms and the β-functions are the same for zero and finite temperature. We can find the counterterms by calculating the UV divergences at either zero and finite temperature. We will be doing all the calculations at finite temperature in this thesis, as the finite temperature correlators are needed for dimensional reduction (see section 2.4).

2.3 Effective potential

The ground state of a quantum field theory including quantum fluctuations can be determined by the effective potential V_{eff}. The effective potential was used in studying theories with a spontaneously broken symmetry by Goldstone, Salam, S. Weinberg [56] and Jona-Lasinio [57]. The famous one-loop calculation of the effective potential was done by Coleman and E. Weinberg [58], and the extension to higher-loop was performed by Jackiw [59].

2.3.1 Generating functionals

We start by finding the effective action in a theory with a scalar field $\phi(x)$ and action $S[\phi]$. The generating functional can be used to calculate vacuum amplitudes with sources,

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\phi \exp\left[iS[\phi] + i\int d^4x \phi(x)J(x)\right]. \tag{2.66}$$

The functional $W[J] = -i\log Z[J]$ is the generator of all connected diagrams. We define the effective action $\Gamma[\overline{\phi}]$ as a Legendre transform of the functional $W[J]$,

$$\Gamma[\overline{\phi}] = W[J] - \int d^4x J(x)\overline{\phi}(x), \tag{2.67}$$

where

$$\overline{\phi}(x) = \frac{\delta W[J]}{\delta J(x)}. \tag{2.68}$$

Varying Γ with respect to ϕ using eqs. (2.67) and (2.68), we obtain

$$\frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}} = \frac{\delta W[J]}{\delta J}\frac{\delta J}{\delta\overline{\phi}} - J - \overline{\phi}\frac{\delta J}{\delta\overline{\phi}} = -J. \tag{2.69}$$

where we have used the notation $\phi J = \int d^4x\phi(x)J(x)$. In particular, the vacuum in the absence of external sources is defined by

$$\left.\frac{\delta\Gamma[\overline{\phi}]}{\delta\overline{\phi}}\right|_{J=0} = 0. \tag{2.70}$$

We can expand the generating functionals $Z[J]$ and $W[J]$ in powers of the external source J, to obtain a representation in terms of Green functions,

$$Z[J] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n G_{(n)}(x_1,\dots,x_n)J(x_1)\dots J(x_n) \tag{2.71}$$

$$iW[J] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n G^c_{(n)}(x_1,\dots,x_n)J(x_1)\dots J(x_n) \tag{2.72}$$

$$\tag{2.73}$$

where $G_{(n)}$ are the n-point Green functions, and $G^c_{(n)}$ are the n-point connected Green functions. In a similar fashion, the effective action can be expanded in powers of $\overline{\phi}$ as[3]

$$\Gamma[\overline{\phi}] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\dots d^4x_n\Gamma^{(n)}(x_1,\dots,x_n)\overline{\phi}(x_1)\dots\overline{\phi}(x_n), \tag{2.75}$$

where $\Gamma^{(n)}$ are the one-particle irreducible (1PI) Green functions.

We can Fourier transform the 1PI Green functions and the field as,

$$\Gamma^{(n)}(x) = \int\prod_{i=1}^{n}\left[\frac{d^4p_i}{(2\pi)^4}e^{ip_ix_i}\right](2\pi)^4\delta^{(4)}(p_1+\dots+p_n)\Gamma^{(n)}(p) \tag{2.76}$$

$$\tilde{\phi}(p) = \int d^4x e^{-ipx}\overline{\phi}(x), \tag{2.77}$$

so eq. (2.75) becomes

$$\Gamma[\overline{\phi}] = \sum_{n=0}^{\infty}\int\prod_{i=1}^{n}\left[\frac{d^4p_i}{(2\pi)^4}\tilde{\phi}(-p_i)\right](2\pi)^4\delta^{(4)}(p_1+\dots+p_n)\Gamma^{(n)}(p_1,\dots,p_n). \tag{2.78}$$

[3]The effective action can be expanded in an alternative way, in powers of external momentum

$$\Gamma[\overline{\phi}] = \int d^4x\left[-V_{\text{eff}}(\overline{\phi}) + \frac{1}{2}\left(\partial_\mu\overline{\phi}(x)\right)^2 Z(\overline{\phi}) + \dots\right], \tag{2.74}$$

where the expansion point is where all external momenta vanish.

We assume that the classical field is translational invariant, $\overline{\phi}(x) = \phi_c$. We define the effective potential as

$$\Gamma[\phi_c] = -\int d^4x V_{\text{eff}}(\phi_c). \tag{2.79}$$

Using eq. (2.77) and the definition of the δ-function, $\delta^{(4)}(p) = \int \frac{d^4x}{(2\pi)^4} e^{-ipx}$, we find that eq. (2.78) becomes

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_c^n (2\pi)^4 \delta^{(4)}(0) \Gamma^{(n)}(p_i = 0) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_c^n \Gamma^{(n)}(p_i = 0) \int d^4x. \tag{2.80}$$

Comparing with eq. (2.79) we end up with

$$V_{\text{eff}}(\phi_c) = -\sum_{n=0}^{\infty} \frac{1}{n!} \phi_c^n \Gamma^{(n)}(p_i = 0). \tag{2.81}$$

So, the effective potential is a sum of 1PI Green functions at zero external momenta. As we will see, this is exactly what we need when performing dimensional reduction.

2.3.2 Background fields

We will calculate the effective potential using the background field method [60]. We start by shifting the fields $\phi \to \phi + \hat{\phi}$, where $\hat{\phi}$ is an arbitrary non-dynamical field.[4] The shifted action is denoted by $S_b[\hat{\phi}, \phi] \equiv S[\phi + \hat{\phi}]$, with a corresponding shifted effective action $\Gamma_b[\hat{\phi}, \phi]$. The generating functional of the connected diagrams is defined as before

$$\exp\Big(iW_b[\hat{\phi}, J]\Big) = \int \mathcal{D}\phi \exp\Big\{iS_b[\hat{\phi}, \phi] + i\int d^4x J(x)\phi(x)\Big\}, \tag{2.82}$$

with

$$\overline{\phi}_b(x) = \frac{\delta W_b[\hat{\phi}, J]}{\delta J(x)} \tag{2.83}$$

being the analogue of eq. (2.68). By shifting the field $\phi \to \phi - \hat{\phi}$ in eq. (2.82) we find

$$W_b[\hat{\phi}, J] = W[J] - \int d^4x J(x)\hat{\phi}(x), \tag{2.84}$$

which implies that $\overline{\phi}_b = \overline{\phi} - \hat{\phi}$. This is as expected, since it only indicates a shift in the expectation value when we shift the field. By defining the shifted effective action as the Legendre transform of $W_b[\hat{\phi}, J]$, we find that

$$\Gamma_b[\hat{\phi}, \phi] = W_b[\hat{\phi}, J] - \int d^4x J(x)\phi(x) = W[J] - \int d^4x J(x)\Big[\phi(x) + \hat{\phi}(x)\Big] = \Gamma[\phi + \hat{\phi}], \tag{2.85}$$

where we have used eq. (2.84). In particular, we have that $\Gamma[\hat{\phi}] = \Gamma_b[\hat{\phi}, 0]$, which means that we can find the functional form of $\Gamma[\phi]$ by calculating $\Gamma_b[\hat{\phi}, 0]$. Afterwards we simply replace the background field by the original field in $\Gamma[\hat{\phi}]$.

[4] Non-dynamical means that we do not integrate over the field configuration in the path integral.

2.3.3 One-loop effective potential at zero temperature

We will now calculate the effective potential at one-loop order for a scalar theory. We assume that the quantum fluctuations are small, and perform a saddle-point expansion around the classical solution ϕ_0, given by the solution to $\Box\phi_0 + V'(\phi_0) = J(x)$. The field is separated into the classical solution and quantum fluctuations, $\phi = \phi_0 + \tilde{\phi}$. The path integral can be approximated by

$$Z = e^{iW} \approx \exp\Big\{iS[\phi_0] + i\langle J\phi_0\rangle\Big\} \int \mathcal{D}\tilde{\phi} \exp\Big\{i \int d^4x \Big[\frac{1}{2}\Big(\partial_\mu\tilde{\phi}\Big)^2 - V''(\phi_0)\tilde{\phi}^2\Big]\Big\}. \quad (2.86)$$

We have neglected terms of order $\mathcal{O}(\hbar^2)$, which corresponds to contributions from two- and higher-loop orders. The functional integral over $\tilde{\phi}$ is Gaussian, given by $\det(\Box + V'')^{-1/2}$. We find that

$$W = S[\phi_0] + \langle J\phi_0\rangle + \frac{i}{2}\operatorname{Tr}\log(\Box + V''(\phi_0)) + \mathcal{O}(\hbar^2), \quad (2.87)$$

where we have used the identity $\log\det A = \operatorname{Tr}\log A$. The trace is a summation over discrete and integration over continuous quantum numbers. For a scalar particle, we only have continuous quantum numbers, and we have to integrate the matrix element $\langle x|\log(\Box + V'')|x\rangle$ only over space-time. In order to do so, we insert a complete set of plane waves

$$\begin{aligned}\operatorname{Tr}\log(\Box + V'') &= \int d^4x \langle x|\log(\Box + V'')|x\rangle = \int d^4x \frac{d^4k}{(2\pi)^4}\langle x|\log(\Box + V'')|k\rangle\langle k|x\rangle \\ &= \int d^4x \frac{d^4k}{(2\pi)^2}\log(-k^2 + V'')\langle x|k\rangle\langle k|x\rangle = \Omega\int \frac{d^4k}{(2\pi)^4}\log(-k^2 + V'').\end{aligned} \quad (2.88)$$

where Ω is a space-time volume. After a Legendre transform, and using that $S[\phi_0] = -\Omega V(\phi_0)$, we get the effective potential with the first quantum corrections included

$$V_{\text{eff}}(\phi_0) = V(\phi_0) - \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\log(-k^2 + V''(\phi_0)) + \mathcal{O}(\hbar^2). \quad (2.89)$$

After a Wick rotation to Euclidean space the effective potential is given by

$$V_{\text{eff}}(\phi_0) = V(\phi_0) + \frac{1}{2}\int \frac{d^4K}{(2\pi)^4}\log(K^2 + V''(\phi_0)) + \mathcal{O}(\hbar^2). \quad (2.90)$$

We can also write the contribution from fermions and gauge fields. For fermions, the one-loop contribution to the effective potential is

$$V^{(1)}_{\text{eff,f}} = -\frac{1}{2}\lambda\int \frac{d^4K}{(2\pi)^4}\log(K^2 + M_f^2(\phi_0)) \quad (2.91)$$

where $M_f^2(\phi_0)$ is the mass matrix squared for the fermion fields, which is a function quadratic in the scalar field ϕ_0. The prefactor λ counts the number of degrees of freedom of the fermions; $\lambda = 4$ for Dirac fermions and $\lambda = 2$ for Weyl fermions. Notice the sign difference compared to eq. (2.90), which comes from the different statistics between fermions and bosons.

For the gauge bosons, we get the one-loop contribution

$$V^{(1)}_{\text{eff,gb}} = \operatorname{Tr}\left(\mathcal{P}_T(Q)\right)\frac{1}{2}\int\frac{d^4K}{(2\pi)^4}\log(K^2 + (M_{\text{gb}})^2(\phi_0)) \tag{2.92}$$

where $M^2_{\text{gb}}(\phi_0)$ is the ϕ_0-dependent mass squared of the gauge bosons, and $\mathcal{P}_T(K)_{\mu\nu} = \delta_{\mu\nu} - K_\mu K_\nu/K^2$ is a projection operator, as defined in appendix B.

The effective potential is divergent, and we have to introduce counterterms to eliminate the divergent parts. We will determine the counterterms, and thereby the β-functions, for the scalar couplings in chapter 5.

2.3.4 One-loop effective potential at finite temperature

We wish to extend the framework of the effective potential to include thermal fluctuations. We can generalise the effective potential found in the previous section using the imaginary-time formalism. The effective potential for a scalar field found in eq. (2.90) becomes

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{1}{2}\sum\!\!\!\!\!\!\int_K \log\left[K^2 + V''(\phi)\right] + \mathcal{O}(\hbar^2), \tag{2.93}$$

where $K = (\omega_n, \boldsymbol{k})$ as usual. By including fermion and gauge fields we adopt the same procedure. From [28] we need the integrals

$$\mathcal{C}_S(m) = -\sum\!\!\!\!\!\!\int_K \log\left(\frac{1}{K^2 + m^2}\right)^{1/2} = J_b(m) \tag{2.94}$$

$$\mathcal{C}_V(M) = -\sum\!\!\!\!\!\!\int_K \log\left(\det\frac{\delta_{\mu\nu} - K_\mu K_\nu/K^2}{K^2 + M^2}\right)^{1/2} = (3 - 2\epsilon)J_b(M) \tag{2.95}$$

$$\mathcal{C}_F(m_f) = -\sum\!\!\!\!\!\!\int_{\{K\}} \log\left(\frac{1}{i\not{K} + m_f}\right)^{1/2} = -4J_f(m_f). \tag{2.96}$$

where m, M, and m_f are the eigenvalues of the mass matrices for the scalars, gauge bosons and fermions, respectively. The prefactor in eq. (2.95) is the trace of the projection operator in the gauge field propagator, eq. (B.1), in $d = 3 - 2\epsilon$ dimensions, while the prefactor in eq. (2.96) indicates that we are working with Dirac fermions. We can see the similarity to eqs. (2.90) to (2.92). The integrals

$$J_b(m) = \frac{1}{2}\sum\!\!\!\!\!\!\int_K \log(K^2 + m^2) \tag{2.97}$$

$$J_f(m) = \frac{1}{2}\sum\!\!\!\!\!\!\int_{\{K\}} \log(K^2 + m^2) \tag{2.98}$$

are evaluated in eqs. (C.15) and (C.16). We should sum up the contribution from all the degrees of freedom in our theory, using eqs. (2.94) to (2.96), to get the full one-loop effective potential.

2.4 Dimensional reduction

We see that in the imaginary-time formalism, a $(d+1)$-dimensional theory at finite temperature can be viewed as a d-dimensional Euclidean theory with an infinite number of fields. The fields are characterised by their Matsubara frequencies, $\omega_n^B = 2n\pi T$ for bosons and $\omega_n^F = (2n+1)\pi T$ for fermions, which act as mass terms. In the high-temperature limit all fermionic and non-static bosonic fields will be very massive, with mass of order πT. We call the modes with mass of order πT for *superheavy*. In an abelian field theory, e.g. QED, the static component of the gauge field ($n = 0$) acquires a thermal mass, also known as a Debye mass, of order gT, similar to what we saw in section 2.1.3. The fields with mass of order gT are called *heavy*. The inverse of the Debye mass is the electric screening length, as it screens the electric forces. In a non-abelian field theory, we have one additional mass scale, g^2T [61], which screens the colour-magnetic (for QCD) forces. The fields with mass of order g^2T or lower are called *light*. The fermionic and non-static bosonic fields are superheavy, the temporal part of the zero-mode of the gauge fields is heavy, while the spatial part of the zero-modes of the gauge fields is light. The zero-mode of scalar fields can be superheavy, heavy, or light, depending on the zero-temperature mass.

If the coupling constant g is small, then we can separate the different scales of the problem, $g^2T \ll gT \ll T$. We integrate out all the superheavy modes in the theory, and are left with an effective three-dimensional theory [62]. This process is called dimensional reduction [63; 64; 65; 66; 67], and was made into a tool for quantitative calculations [68; 69; 70; 71]. The idea is based on the observation that at high temperatures, the non-zero modes become superheavy, and decouple from the bosonic zero modes, according to the decoupling theorem by Appelquist and Carazzone [30]. All fermionic and non-zero bosonic modes are integrated out, and the effective theory consists of only static, i.e. zero-mode, bosonic fields. If the theory contains an additional scale g^2T, we can also integrate out all the heavy modes.

The way we integrate out the heavier modes, is to write down the most general effective, three-dimensional theory with the relevant fields, consistent with the underlying symmetries. This effective theory will have a superrenormalisable Lagrangian, along with higher-order operators. The effective theory should be able to reproduce the same results as the original theory, by including more and more higher-order operators to increase the accuracy. We will restrict ourselves to the superrenormalisable part of the effective theory.

We want to express the parameters of the effective theory in terms of the parameters of the original theory. This is done by matching correlators at zero external momentum in the two theories. We require the two theories to predict the same behaviour at long distances, as it is only the short-distance modes that we have integrated out.

We will illustrate the method of dimensional reduction in the context of quantum chromodynamics (QCD). Later, in the main part of the thesis we perform dimensional reduction on the 2HDM, where we have omitted the contribution from the colour sector.

2.4.1 Dimensional reduction of quantum chromodynamics

The theory describing the strong nuclear force, called quantum chromodynamics (QCD), is given by the four-dimensional Lagrangian

$$\mathcal{L}_{\text{QCD,4d}} = \frac{1}{4}H^n_{\mu\nu}H^n_{\mu\nu} + \sum_i \overline{\Psi}_i \not{D}\Psi_i, \tag{2.99}$$

where $H^n_{\mu\nu} = \partial_\mu C^n_\nu - \partial_\nu C^n_\mu + g_s f^n_{mr} C^m_\mu C^r_\nu$ is the field strength tensor and g_s is the gauge coupling constant. C^n_μ and f^n_{mr} are the gauge fields and the structure constants of the non-abelian gauge group SU(3), respectively, where $n = 1, \ldots, 8$ is the colour index. We have assumed that the temperature is much larger than any zero-temperature mass scales, i.e. the temperature is much larger than the masses of the quarks.

The theory of quarks (the fermion fields Ψ) and gluons (the gauge fields) is characterised by asymptotic freedom and confinement [72; 73]. Asymptotic freedom means that the quarks act as if they were free at very high energy scales, i.e. the strength of the strong interactions goes to zero at very short distances. On the flip side, the strength of the strong interactions increases at large distances, and the quarks are confined to bound states. This is known as quark confinement.

At finite temperature, the quarks and gluons can undergo a phase transition from the confined phase (hadron phase) to the unconfined phase (quark-gluon phase) [40]. The quark-gluon phase, called quark-gluon plasma (QGP), is relevant for both the early universe and heavy-ion collision experiments. Thus, finite temperature (and density) can change the characteristics of a theory dramatically.

The dimensional reduction of QCD has been performed by several authors [65; 66; 74; 75]. We will here briefly outline the procedure.

Firstly, we know that the temporal gauge fields, C^a_0, receive a thermal mass of order gT, while the spatial gauge fields, C^a_i, remain massless. From this we construct the most general three-dimensional Lagrangian containing the (three-dimensional) fields[5]

$$\mathcal{L}_{\text{QCD,3d}} = \frac{1}{4}H^n_{ij}H^n_{ij} + \frac{1}{2}\left(D_i C^a_0\right)^2 + \frac{1}{2}m''^2_D C^a_0 C^a_0 + \frac{1}{4}\lambda_C \left(C^a_0 C^a_0\right)^2 + \delta\mathcal{L} \tag{2.100}$$

where $D_i C^a_0 = \left(\partial_i - i g_{s,3}\frac{\vec{\lambda}}{2}\cdot\vec{C}_i\right) C^a_0$ is the covariant derivative in the adjoint representation, $g_{s,3}$ is the three-dimensional gauge coupling constant, m''^2_D is the three-dimensional mass of the scalar octet C^a_0, and λ_C is the three-dimensional self-coupling of the scalar octet C^a_0. $\delta\mathcal{L}$ contains all higher-order operators and counterterms.

We match the two theories by requiring that the correlators at zero external momentum should be the same, i.e. that the two theories predict the same long distance behaviour. We can use the results by [71; 76; 77], calculated at one-loop accuracy in the $\overline{\text{MS}}$ scheme,

$$g_3^2 = g^2(\mu)T\left[1 + \frac{g^2}{(4\pi)^2}\left(11L_b - \frac{2}{3}n_f L_f + 1\right)\right] \tag{2.101}$$

$$m''^2_D = g^2(\mu)T^2\left(1 + \frac{n_f}{6}\right) \tag{2.102}$$

$$\lambda_A = \frac{6g^4 T}{(4\pi)^2}\left(1 - \frac{n_f}{9}\right) \tag{2.103}$$

where n_f is the number of fermion flavours, $L_b = 2\log\left(\frac{\mu}{4\pi T}\right) + 2\gamma_E$, $L_f = L_b + 4\log 2$, μ is the renormalisation scale coming from dimensional regularisation and γ_E is the Euler-Mascheroni constant.

[5]We have maintained the same notation for the three- and four-dimensional fields for simplicity.

After integrating out the superheavy fields, we end up with eq. (2.100), which contains two mass scales. One is associated with the Debye mass, of order gT, and the other is the three-dimensional gauge coupling constant $g_3^2 = g^2T$, which is not dimensionless. Because of asymptotic freedom, the gauge coupling becomes very small at sufficiently high temperatures. With the assumption of a weakly coupled theory, we can separate the mass scales, $g_3^2 \ll m''_D$. This suggests that we can simplify our theory even more. We can integrate out the *heavy* fields, i.e. fields with a mass of order gT, and be left with a theory containing only *light* fields, i.e. fields with a mass of order g^2T or less. The resulting theory is a pure Yang-Mills theory with the gauge group SU(3),

$$\mathcal{L}_{\mathrm{YM,3d}} = \frac{1}{4} H_{ij}^n H_{ij}^n + \delta\mathcal{L}. \tag{2.104}$$

The new gauge coupling can be determined by requiring the correlators to match at zero external momentum. The result is [76]

$$\overline{g}_{s,3}^2 = g_{s,3}^2 \left(1 - \frac{g_{s,3}^2}{(4\pi)^2 m''_D}\right). \tag{2.105}$$

The final theory contains only one scale $g_{s,3}^2$, and is strongly coupled despite the asymptotic freedom of QCD. Thus, perturbative methods have some limitation. For the free energy, as an example, it is not possible to compute the $\mathcal{O}(g^6T^4)$ correction perturbatively, as an infinite number of diagrams contribute at that level of accuracy [76]. We have to use non-perturbative methods such as Monte Carlo simulations.

CHAPTER 3

TWO-HIGGS DOUBLET MODEL

In this chapter we introduce the Two-Higgs Doublet Model (2HDM). For a detailed review of the 2HDM phenomenology, see refs. [78; 79].

3.1 The full theory

Firstly, we discuss the general Two-Higgs Doublet Model. The full Lagrangian of the 2HDM is

$$\mathcal{L}_{\text{2HDM}} = \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{ghost}} + \delta\mathcal{L} \tag{3.1}$$

where the fermion, gauge, scalar, and ghost sectors of the Lagrangian are defined as

$$\mathcal{L}_{\text{fermion}} = \sum_A \Big(\bar{l}_A \not{D} l_A + \bar{e}_A \not{D} e_A + \bar{q}_A \not{D} q_A + \bar{u}_A \not{D} u_A + \bar{d}_A \not{D} d_A\Big) \tag{3.2}$$

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{1}{4} H^n_{\mu\nu} H^n_{\mu\nu} \tag{3.3}$$

$$\mathcal{L}_{\text{scalar}} = \Big(D_\mu \Phi_1\Big)^\dagger D_\mu \Phi_1 + \Big(D_\mu \Phi_2\Big)^\dagger D_\mu \Phi_2 + V(\Phi_1, \Phi_2) \tag{3.4}$$

$$\mathcal{L}_{\text{ghost}} = \partial_\mu \bar{\xi} \partial_\mu \xi + \partial_\mu \bar{\eta}^a D_\mu \eta^a + \partial_\mu \bar{\zeta}^n D_\mu \zeta^n. \tag{3.5}$$

The scalar potential and the Yukawa sector will be discussed in sections 3.2 and 3.3, respectively. All the fields and couplings are renormalised, and $\delta\mathcal{L}$ contains the counterterms, specified in chapters 4 and 5. The field strength tensors $F_{\mu\nu}$, $G^a_{\mu\nu}$, and $H^n_{\mu\nu}$ contain the gauge fields of the gauge groups $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, and $\mathrm{SU}(3)$, with corresponding ghost fields ξ, η^a, and ζ^n, respectively. The field strength tensors are given as

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{3.6a}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^a_{bc} A^b_\mu A^c_\nu \tag{3.6b}$$

$$H^n_{\mu\nu} = \partial_\mu C^n_\nu - \partial_\nu C^n_\mu + g_s f^n_{mr} C^m_\mu C^r_\nu \tag{3.6c}$$

where B_μ, A^a_μ, and C^n_μ are the $\mathrm{U}(1)_Y, \mathrm{SU}(2)_L$, and SU(3) gauge fields, and ϵ^a_{bc} and f^n_{mr} are the structure constants of the non-abelian gauge groups $\mathrm{SU}(2)_L$ and SU(3), respectively. The fermionic fields form left-handed doublets and right-handed singlets under the $\mathrm{SU}(2)_L$ gauge group,

	Left-handed	Right-handed
Leptons	$l = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$e = e_R$
Quarks	$q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$u = u_R, d = d_R.$

The covariant derivatives of the various fields are

$$D_\mu \Psi = \left(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu - ig'\frac{Y_\Psi}{2} B_\mu - ig_s \frac{\vec{\lambda}}{2} \cdot \vec{C}_\mu \right) \Psi \qquad \text{for } \Psi = q \tag{3.7a}$$

$$D_\mu \Psi = \left(\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu - ig'\frac{Y_\Psi}{2} B_\mu \right) \Psi \qquad \text{for } \Psi = l, \Phi_{1,2} \tag{3.7b}$$

$$D_\mu \Psi = \left(\partial_\mu - ig'\frac{Y_\Psi}{2} B_\mu \right) \Psi \qquad \text{for } \Psi = e \tag{3.7c}$$

$$D_\mu \Psi = \left(\partial_\mu - ig'\frac{Y_\Psi}{2} B_\mu - ig_s \frac{\vec{\lambda}}{2} \cdot \vec{C}_\mu \right) \Psi \qquad \text{for } \Psi = u, d \tag{3.7d}$$

where $\vec{\tau}$ and $\vec{\lambda}$ are the Pauli and Gell-Mann matrices and g', g, and g_s are the coupling constants for the $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, and SU(3) gauge fields, respectively. The hypercharge Y is defined by the Gell-Mann-Nishijima relation [80; 81; 82]

$$Q = I_3 + \frac{1}{2} Y \tag{3.8}$$

where Q is the electric charge and I_3 is the third component of isospin of the various fields. Explicitly, we have that

$$Y_l = -1, \quad Y_e = -2, \quad Y_q = \frac{1}{3}, \quad Y_u = \frac{4}{3}, \quad Y_d = -\frac{2}{3}, \quad Y_{\Phi_{1,2}} = 1. \tag{3.9}$$

Some sums which regularly arise when doing loop calculations are

$$\sum_f Y_f^2 = N_f[2Y_l^2 + Y_e^2 + N_c(2Y_q^2 + Y_u^2 + Y_d^2)] = N_f \left[6 + N_c \frac{22}{9} \right] \tag{3.10a}$$

$$\sum_f Y_f^4 = N_f[2Y_l^4 + Y_e^4 + N_c(2Y_q^4 + Y_u^4 + Y_d^4)] = N_f \left[18 + N_c \frac{274}{81} \right] \tag{3.10b}$$

$$\sum_{\text{left}} = N_f[1 + N_c] \tag{3.10c}$$

where N_f is the number of fermion families, and N_c is the number of colours. The first two sums are taken over all fermions, while the last sum is the sum of all left-handed fermions. We will also need some traces,

$$g^2_{Y,1} = \mathrm{Tr}[h^{(e)}h^{(e)\dagger} + N_c h^{(d)}h^{(d)\dagger} + N_c h^{(u)}h^{(u)\dagger}] \tag{3.11}$$
$$g^2_{Y,2} = \mathrm{Tr}[(Y_l^2 + Y_e^2)h^{(e)}h^{(e)\dagger} + N_c(Y_q^2 + Y_d^2)h^{(d)}h^{(d)\dagger} + N_c(Y_q^2 + Y_u^2)h^{(u)}h^{(u)\dagger}] \tag{3.12}$$
$$g^2_{Y,3} = \mathrm{Tr}[h^{(e)}h^{(e)\dagger} + h^{(d)}h^{(d)\dagger} - h^{(u)}h^{(u)\dagger}] \tag{3.13}$$
$$g^2_{Y,4} = \mathrm{Tr}[-h^{(e)}h^{(e)\dagger} + h^{(d)}h^{(d)\dagger} + h^{(u)}h^{(u)\dagger}] \tag{3.14}$$
$$G^4_{Y,1} = \mathrm{Tr}[h^{(e)}h^{(e)\dagger}h^{(e)}h^{(e)\dagger} + N_c h^{(d)}h^{(d)\dagger}h^{(d)}h^{(d)\dagger} + N_c h^{(u)}h^{(u)\dagger}h^{(u)}h^{(u)\dagger}]. \tag{3.15}$$

We will also use the conventions from [28], and define

$$L_b = 2\log\left(\frac{\mu}{4\pi T}\right) + 2\gamma_E \tag{3.16}$$
$$L_f = L_b + 4\log(2). \tag{3.17}$$

Inserting the specific values of the 2HDM, we get

$$\sum_f Y_f^2 = 40 \tag{3.18a}$$
$$\sum_f Y_f^4 = \frac{760}{9} \tag{3.18b}$$
$$\sum_{\text{left}} = 12 \tag{3.18c}$$
$$N_h = 2 \tag{3.18d}$$
$$g^2_{Y,1} = \mathrm{Tr}[h^{(e)}h^{(e)\dagger} + 3h^{(d)}h^{(d)\dagger} + 3h^{(u)}h^{(u)\dagger}] \tag{3.18e}$$
$$g^2_{Y,2} = \mathrm{Tr}[5h^{(e)}h^{(e)\dagger} + \frac{5}{3}h^{(d)}h^{(d)\dagger} + \frac{17}{3}h^{(u)}h^{(u)\dagger}] \tag{3.18f}$$

where N_h is the number of Higgs doublets.

3.1.1 Renormalisation

The fields in the Lagrangian given in eq. (3.1) are the renormalised fields, with $\delta\mathcal{L}$ containing the counterterms. The bare fields are denoted by a subscript (b), while the renormalised fields have no subscript. The relations between the bare and renormalised fields are

$$\Phi_{n(b)} \equiv (1 + \delta Z_{\Phi_n})^{1/2}\Phi_n \tag{3.19}$$
$$\vec{A}_{\mu(b)} \equiv (1 + \delta Z_A)^{1/2}\vec{A}_\mu \tag{3.20}$$
$$B_{\mu(b)} \equiv (1 + \delta Z_B)^{1/2}B_\mu \tag{3.21}$$

where $n = 1, 2$. The relations between the bare and renormalised couplings are

$$g_{(b)} \equiv g + \delta g \tag{3.22}$$

$$g'_{(b)} \equiv g' + \delta g' \tag{3.23}$$
$$\lambda_{n(b)} \equiv Z_{\Phi_n}^{-2}(\lambda_n + \delta\lambda_n) \tag{3.24}$$
$$\lambda_{i(b)} \equiv Z_{\Phi_1}^{-1} Z_{\Phi_2}^{-1}(\lambda_i + \delta\lambda_i) \tag{3.25}$$
$$m^2_{nm(b)} \equiv Z_{\Phi_n}^{-1/2} Z_{\Phi_m}^{-1/2}(m^2_{nm} + \delta m^2_{nm}) \tag{3.26}$$

where $n, m = 1, 2$ and $i = 3, 4, 5$.

3.2 Scalar potential

The most general form of the 2HDM potential is [78]

$$\begin{aligned} V(\Phi_1, \Phi_2) = &- \frac{1}{2}\Big\{m^2_{11}(\Phi_1^\dagger\Phi_1) + m^2_{22}(\Phi_2^\dagger\Phi_2) + [m^2_{12}(\Phi_1^\dagger\Phi_2) + \text{h.c.}]\Big\} && (3.27)\\ &+ \frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\\ &+ \frac{1}{2}[\lambda_5(\Phi_1^\dagger\Phi_2)^2 + \lambda_5^*(\Phi_2^\dagger\Phi_1)^2] + \Big\{[\lambda_6(\Phi_1^\dagger\Phi_1) + \lambda_7(\Phi_2^\dagger\Phi_2)](\Phi_1^\dagger\Phi_2) + \text{h.c.}\Big\} && (3.28) \end{aligned}$$

where λ_i are coupling constants, m^2_{ij} are the squared masses, and Φ_i are the scalar Higgs fields. h.c. stands for the Hermitian conjugate. The potential is required to be Hermitian, which restricts λ_{1-4}, m^2_{11} and m^2_{22} to be real. In general can λ_{5-7} and m^2_{12} be complex. The potential contains 14 independent parameters.

The most general Lagrangian for the scalar sector violates the Z_2 symmetry

$$\Phi_1 \leftrightarrow \Phi_1, \quad \Phi_2 \leftrightarrow -\Phi_2 \qquad \text{or} \qquad \Phi_1 \leftrightarrow -\Phi_1, \quad \Phi_2 \leftrightarrow \Phi_2. \tag{3.29}$$

Thus, the Lagrangian permits the transformations $\Phi_1 \leftrightarrow \Phi_2$. We impose the Z_2 symmetry on the quartic couplings, eq. (3.29), which dispenses of λ_6 and λ_7. We can make a global phase transformation of one of the Higgs fields to cancel the phase of λ_5, and thereby making λ_5 real. The new scalar potential takes the form

$$\begin{aligned} V(\Phi_1, \Phi_2) = &- \frac{1}{2}\Big\{m^2_{11}(\Phi_1^\dagger\Phi_1) + m^2_{22}(\Phi_2^\dagger\Phi_2)\Big\} - \frac{1}{2}\Big[m^2_{12}(\Phi_1^\dagger\Phi_2) + \text{h.c.}\Big]\\ &+ \frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\\ &+ \frac{\lambda_5}{2}[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2] \end{aligned} \tag{3.30}$$

The complex coupling m^2_{12} does not respect the discrete Z_2 symmetry, and we say that the symmetry is softly broken by the mass-mixing term. The potential now contains 9 independent parameters. We will start by considering the simpler potential where the Z_2 symmetry is respected both by the couplings and the masses. Thus, we drop the mass-mixing term $\mathcal{L}_{\text{mix}} = -1/2(m^2_{12}\Phi_1^\dagger\Phi_2 + \text{h.c.})$. The potential now contains 7 parameters. Later, we will treat the mass-mixing as a perturbation, and include it as our source for additional CP violation. It will be clear when the mass-mixing term is reintroduced.

3.3 Yukawa sector

There is still some freedom in defining the Yukawa sector of the theory. The different possible couplings of the scalar fields with the fermions separate the 2HDM into distinct types. We will briefly discuss the most common types, but will focus on the Type I in this thesis. In all the various types of 2HDM presented here, the Φ_1 Higgs field is taken to couple to the up-type quarks, by convention.[1]

3.3.1 Type I

The Type I 2HDM is also called the fermiophobic 2HDM, since only one of the two Higgs doublets directly couples to the charged fermions. The Yukawa sector of the Lagrangian reads

$$L_{\text{Yukawa}} = \sum_{AB} \left[h^{(e)}_{AB} \bar{l}_A e_B \Phi_1 + h^{(d)}_{AB} \bar{q}_A d_B \Phi_1 + h^{(u)}_{AB} \bar{q}_A u_B \tilde{\Phi}_1 \right] + \text{h.c.} \tag{3.31}$$

where $\tilde{\Phi}_i = i\tau_2 \Phi_i^*$ is the charge conjugated Higgs doublet, and τ_2 is the second Pauli matrix. Thus, the Yukawa sector of Type I 2HDM takes the same form as the corresponding SM Yukawa sector. Some authors use the convention that the up-type quarks always couple to the Φ_2. We will let the fermions only couple to Φ_1.

3.3.2 Type II

In the Type II 2HDM both Higgs doublets couple directly to the charged fermions. Up-type quarks couple to Φ_1, while down-type quarks and the charged leptons couple to Φ_2. The Yukawa sector of the Lagrangian takes the form

$$L_{\text{Yukawa}} = \sum_{AB} \left[h^{(e)}_{AB} \bar{l}_A e_B \Phi_2 + h^{(d)}_{AB} \bar{q}_A d_B \Phi_2 + h^{(u)}_{AB} \bar{q}_A u_B \tilde{\Phi}_1 \right] + \text{h.c.} \tag{3.32}$$

In the Type II 2HDM the Higgs fields couple to the charged fermions in a similar fashion as in the MSSM. The Type II 2HDM has been investigated for explaining electroweak baryogenesis, and has been disfavoured [83].

3.3.3 Other models

Other types of 2HDM are Type III, X, and Y. In the Type III, both Higgs fields couple to all charged fermion fields. Flavor-changing neutral currents at tree level are induced, which makes the theory unattractive. The X type is also called lepton-specific, as the Φ_2 Higgs field only couples to the charged leptons, while Φ_1 couples to the quarks. The Y type is the flipped version of the X type, as the Φ_2 Higgs field only couples to the down-type quarks while Φ_1 couples to the up-type quarks and the charged fermions.

[1] Other authors may use the opposite convention, where the Φ_2 Higgs field couples to the up-type quarks, as in e.g. [78].

3.4 Mass spectrum

We will illustrate the method for finding and diagonalising the mass matrix, also used in [84], in order to find the mass spectrum of the scalar particles. In Chapters 5 and 7 we will use some of these techniques when finding the effective potential, counterterms, and β-functions for the scalar particles.

We start with two complex doublets, resulting in a total of 8 degrees of freedom. We expect 3 of these to be would-be Goldstone bosons, absorbed to give a longitudinal degree of freedom to the 3 massive vector bosons in the electroweak interactions, through the Higgs mechanism [46], similarly to the SM. We are then left with 5 degrees of freedom, and expect to find 5 massive scalars as a result.

We have the scalar potential of the Lagrangian from eq. (3.30),

$$\begin{aligned} V(\Phi_1, \Phi_2) = &-\frac{1}{2}m_{11}^2(\Phi_1^\dagger\Phi_1) - \frac{1}{2}m_{22}^2(\Phi_2^\dagger\Phi_2) + \frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 \\ &+ \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{\lambda_5}{2}[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2] \end{aligned} \tag{3.33}$$

where we have dropped the mass-mixing term. By writing

$$\Phi_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{bmatrix}, \quad \Phi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_5 + i\phi_6 \\ \phi_7 + i\phi_8\end{bmatrix} \tag{3.34}$$

we get that $\Phi_1^\dagger\Phi_1 = \frac{1}{2}\vec{\Phi}_1 \cdot \vec{\Phi}_1$ and $\Phi_2^\dagger\Phi_2 = \frac{1}{2}\vec{\Phi}_2 \cdot \vec{\Phi}_2$, where $\vec{\Phi}_1 = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $\vec{\Phi}_2 = (\phi_5, \phi_6, \phi_7, \phi_8)^T$. We want to find the minimum of the potential, where the vacuum expectation values of the Higgs doublets are

$$\langle\Phi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ v_1\end{bmatrix}, \quad \langle\Phi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ v_2\end{bmatrix}. \tag{3.35}$$

We have imposed that the vacuum expectation values of the Higgs doublets are electrically neutral, and that they do not break the CP symmetry. When setting the vacuum expectation values to be $v_1/\sqrt{2}$ and $v_2/\sqrt{2}$, we have implicitly imposed the extremum condition

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_3=v_1, \phi_7=v_2} = 0. \tag{3.36}$$

The only non-trivial constraints we obtain are

$$v_1\Big(-\frac{1}{2}m_{11}^2 + \frac{1}{2}\lambda_1 v_1^2 + \lambda_+ v_2^2\Big) = 0 \tag{3.37}$$

$$v_2\Big(-\frac{1}{2}m_{22}^2 + \frac{1}{2}\lambda_2 v_2^2 + \lambda_+ v_1^2\Big) = 0 \tag{3.38}$$

where $\lambda_+ = \frac{1}{2}(\lambda_3 + \lambda_4 + \lambda_5)$. The system of equations yields two independent solutions when we require v_1 to be non-zero,

$$\text{Case A:} \qquad v_1^2 = \frac{m_{11}^2\lambda_2 - 2m_{22}^2\lambda_+}{\lambda_1\lambda_2 - 4\lambda_+^2}, \quad v_2^2 = \frac{m_{22}^2\lambda_1 - 2m_{11}^2\lambda_+}{\lambda_1\lambda_2 - 4\lambda_+^2} \tag{3.39}$$

$$\text{Case B:} \qquad v_1^2 = \frac{m_{11}^2}{\lambda_1}, \quad v_2^2 = 0. \tag{3.40}$$

We will take advantage of these conditions shortly. We find the mass matrix by

$$M_{ij}^2 = \left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_{\phi_3=v_1,\phi_7=v_2}. \tag{3.41}$$

The 8×8 mass matrix takes the form

$$M^2 = \begin{bmatrix} M_{11}^2 & 0 & 0 & 0 & M_{15}^2 & 0 & 0 & 0 \\ 0 & M_{22}^2 & 0 & 0 & 0 & M_{26}^2 & 0 & 0 \\ 0 & 0 & M_{33}^2 & 0 & 0 & 0 & M_{37}^2 & 0 \\ 0 & 0 & 0 & M_{44}^2 & 0 & 0 & 0 & M_{48}^2 \\ M_{15}^2 & 0 & 0 & 0 & M_{55}^2 & 0 & 0 & 0 \\ 0 & M_{26}^2 & 0 & 0 & 0 & M_{66}^2 & 0 & 0 \\ 0 & 0 & M_{37}^2 & 0 & 0 & 0 & M_{77}^2 & 0 \\ 0 & 0 & 0 & M_{48}^2 & 0 & 0 & 0 & M_{88}^2 \end{bmatrix} \tag{3.42}$$

where

$$\begin{aligned} M_{11}^2 &= -\frac{1}{2}\tilde{m}_{11}^2 & M_{22}^2 &= -\frac{1}{2}\tilde{m}_{11}^2 \\ M_{33}^2 &= -\frac{1}{2}\tilde{m}_{11}^2 + \lambda_1 v_1^2 + \frac{1}{2}(\lambda_4+\lambda_5)v_2^2 & M_{44}^2 &= -\frac{1}{2}\tilde{m}_{11}^2 + \frac{1}{2}(\lambda_4-\lambda_5)v_2^2 \\ M_{55}^2 &= -\frac{1}{2}\tilde{m}_{22}^2 & M_{66}^2 &= -\frac{1}{2}\tilde{m}_{22}^2 \\ M_{77}^2 &= -\frac{1}{2}\tilde{m}_{22}^2 + \lambda_2 v_2^2 + \frac{1}{2}(\lambda_4+\lambda_5)v_1^2 & M_{88}^2 &= -\frac{1}{2}\tilde{m}_{22}^2 + \frac{1}{2}(\lambda_4-\lambda_5)v_1^2 \\ M_{15}^2 &= \frac{1}{2}(\lambda_4+\lambda_5)v_1 v_2 & M_{26}^2 &= \frac{1}{2}(\lambda_4+\lambda_5)v_1 v_2 \\ M_{37}^2 &= 2\lambda_+ v_1 v_2 & M_{48}^2 &= \lambda_5 v_1 v_2 \end{aligned}$$

where $\tilde{m}_{11}^2 = m_{11}^2 - \lambda_1 v_1^2 - \lambda_3 v_2^2$ and $\tilde{m}_{22}^2 = m_{22}^2 - \lambda_2 v_2^2 - \lambda_3 v_1^2$. We divide the discussion into two parts, one for each choice of the vacuum expectation values.

3.4.1 Case A

We notice that if we rearrange the matrix $(1,2,3,4,5,6,7,8) \to (1,5,2,6,3,7,4,8)$, and notice that $M_{11}^2 = M_{22}^2$, $M_{55}^2 = M_{66}^2$ and $M_{15}^2 = M_{26}^2$, the matrix takes a block-diagonal form,

$$M^2 = \left[\begin{pmatrix} M_{11}^2 & M_{15}^2 \\ M_{15}^2 & M_{55}^2 \end{pmatrix} \otimes \mathbb{I}_{2\times 2}\right] \oplus \begin{pmatrix} M_{33}^2 & M_{37}^2 \\ M_{37}^2 & M_{77}^2 \end{pmatrix} \oplus \begin{pmatrix} M_{44}^2 & M_{48}^2 \\ M_{48}^2 & M_{88}^2 \end{pmatrix} \tag{3.43}$$

where

$$M_{11}^2 = -\frac{1}{2}(\lambda_4+\lambda_5)v_2^2 \qquad M_{15}^2 = \frac{1}{2}(\lambda_4+\lambda_5)v_1 v_2$$

$$M_{55}^2 = -\frac{1}{2}(\lambda_4 + \lambda_5)v_1^2 \qquad M_{33}^2 = \lambda_1 v_1^2$$
$$M_{37}^2 = 2\lambda_+ v_1 v_2 \qquad M_{77}^2 = \lambda_2 v_2^2$$
$$M_{44}^2 = -\lambda_5 v_2^2 \qquad M_{48}^2 = \lambda_5 v_1 v_2$$
$$M_{88}^2 = -\lambda_5 v_1^2$$

where we have used the minimum condition in eq. (3.39). We need to find the eigenvalues and eigenvectors of a 2×2 matrix, on the form

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \tag{3.44}$$

The procedure is familiar, we want to find the solution of the equation $M\vec{u}_i = k_i\vec{u}_i$, for $i = 1, 2$, which is the same as setting the determinant of $M - k_i\mathbb{I}$ equal to zero for non-zero eigenvectors $\vec{u}_i$,

$$\det(M - k_{1,2}\mathbb{I}) = \begin{vmatrix} a - k_{1,2} & c \\ c & b - k_{1,2} \end{vmatrix} = k_{1,2}^2 - (a+b)k_{1,2} + ab - c^2 = k_{1,2}^2 - \operatorname{Tr} M k_{1,2} + \Delta = 0 \tag{3.45}$$

where $\operatorname{Tr} M = a + b$ and $\Delta = \det M = ab - c^2$. The eigenvalues are found by solving the quadratic equation for $k_{1,2}$,

$$k_{1,2} = \frac{1}{2}\Big[a + b \pm \sqrt{(a-b)^2 + 4c^2}\Big] = \frac{1}{2}\Big[\operatorname{Tr} M \pm \sqrt{(\operatorname{Tr} M)^2 - 4\Delta}\Big]. \tag{3.46}$$

The corresponding orthonormal eigenvectors are given by

$$\begin{pmatrix} \cos\delta \\ \sin\delta \end{pmatrix} \equiv \vec{u}_1 \leftrightarrow k_1; \quad \begin{pmatrix} -\sin\delta \\ \cos\delta \end{pmatrix} \equiv \vec{u}_2 \leftrightarrow k_2 \tag{3.47}$$

where

$$\sin 2\delta = \frac{2c}{\sqrt{(a-b)^2 + 4c^2}}; \quad \cos 2\delta = \frac{(a-b)}{\sqrt{(a-b)^2 + 4c^2}}. \tag{3.48}$$

We will now use this generic procedure to diagonalise the $(1, 5)$ and $(2, 6)$ submatrices. The submatrices are identical, and determine the masses of the charged Higgs bosons. The eigenvectors and eigenvalues of the submatrices are

$$\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \leftrightarrow 0 \tag{3.49}$$
$$\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix} \leftrightarrow -\frac{1}{2}(\lambda_4 + \lambda_5)(v_1^2 + v_2^2) \tag{3.50}$$

where

$$\cos\beta = \frac{v_1}{\sqrt{v_1^2 + v_2^2}}, \quad \sin\beta = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}. \tag{3.51}$$

The mass eigenstates are

$$\begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_{1/2} \\ \phi_{5/6} \end{pmatrix}, \tag{3.52}$$

The masses of the charged Higgs bosons are

$$m^2_{G^\pm} = 0, \qquad m^2_{H^\pm} = -\frac{1}{2}(\lambda_4 + \lambda_5)(v_1^2 + v_2^2). \tag{3.53}$$

Notice the two massless would-be Goldstone bosons $G^\pm$, which are absorbed into the longitudinal degree of freedom of the charged vector bosons $W^\pm$. Two massive charged bosons $H^\pm$ are left, as expected.

In a similar fashion, we find the mass eigenstates and masses of the remaining scalar fields. For the $(3, 7)$ submatrix, we get

$$m^2_{H^0,h^0} = \frac{1}{2}[\lambda_1 v_1^2 + \lambda_2 v_2^2 \pm \sqrt{(\lambda_1 v_1^2 - \lambda_2 v_2^2)^2 + 16\lambda_+^2 v_1^2 v_2^2}] \tag{3.54}$$

$$\tan 2\alpha = \frac{4v_1 v_2 \lambda_+}{(\lambda_1 v_1^2 - \lambda_2 v_2^2)} \tag{3.55}$$

$$\begin{pmatrix} H^0 \\ h^0 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_7 \end{pmatrix}. \tag{3.56}$$

Lastly, for the $(4, 8)$ submatrix, we have that

$$m^2_{G^0} = 0, \quad m^2_{A^0} = -\lambda_5(v_1^2 + v_2^2) \tag{3.57}$$

$$\begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_4 \\ \phi_8 \end{pmatrix} \tag{3.58}$$

where β is defined in eq. (3.51). We have found the mass spectrum of the scalar particles. 3 of the 8 degrees of freedom are massless, and are would-be Goldstone bosons, absorbed by the longitudinal degree of freedom of the massive vector bosons $W^\pm$, Z^0. The remaining 5 degrees of freedom are massive scalar particles, two charged scalars $H^\pm$, two neutral scalars H^0 and h^0, and one neutral pseudoscalar A^0.

In summary, the mass spectrum of the scalar fields is

$$m^2_{H^\pm} = -\frac{1}{2}(\lambda_4 + \lambda_5)(v_1^2 + v_2^2) \tag{3.59a}$$

$$m^2_{H^0,h^0} = \frac{1}{2}[\lambda_1 v_1^2 + \lambda_2 v_2^2 \pm \sqrt{(\lambda_1 v_1^2 - \lambda_2 v_2^2)^2 + 16\lambda_+^2 v_1^2 v_2^2}] \tag{3.59b}$$

$$m^2_{A^0} = -\lambda_5(v_1^2 + v_2^2). \tag{3.59c}$$

3.4.2 Case B

When using eq. (3.40), we notice that the mass matrix is diagonal, and we obtain

$$M^2_{11} = 0$$

$$\begin{aligned}
M_{22}^2 &= 0 \\
M_{33}^2 &= \lambda_1 v_1^2 \\
M_{44}^2 &= 0 \\
M_{55}^2 &= -\frac{1}{2}m_{22}^2 + \frac{1}{2}\lambda_3 v_1^2 \\
M_{66}^2 &= -\frac{1}{2}m_{22}^2 + \frac{1}{2}\lambda_3 v_1^2 \\
M_{77}^2 &= -\frac{1}{2}m_{22}^2 + \lambda_+ v_1^2 \\
M_{88}^2 &= -\frac{1}{2}m_{22}^2 + \lambda_- v_1^2
\end{aligned}$$

where $\lambda_- = \frac{1}{2}(\lambda_3 + \lambda_4 - \lambda_5)$. Two charged and one neutral degrees of freedom have been absorbed into the longitudinal component of the (now) massive vector bosons $W^\pm$ and Z^0. We identify the remaining 5 degrees of freedom with two charged Higgs bosons, $\varphi_{5,6} = H^\pm$, one heavy neutral scalar $\varphi_7 = H^0$, one light neutral scalar, $\varphi_3 = h^0$, and one neutral pseudoscalar $\varphi_8 = A^0$. The mass spectrum becomes

$$m_{H^\pm}^2 = -\frac{1}{2}m_{22}^2 + \frac{\lambda_3}{2}v_1^2 \tag{3.60a}$$

$$m_{H^0}^2 = -\frac{1}{2}m_{22}^2 + \frac{1}{2}(\lambda_3 + \lambda_4 + \lambda_5)v_1^2 \tag{3.60b}$$

$$m_{h^0}^2 = \lambda_1 v_1^2 \tag{3.60c}$$

$$m_{A^0}^2 = -\frac{1}{2}m_{22}^2 + \frac{1}{2}(\lambda_3 + \lambda_4 - \lambda_5)v_1^2. \tag{3.60d}$$

CHAPTER 4

CORRELATORS AND β-FUNCTIONS

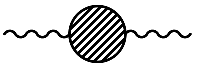
This chapter contains the counterterms needed in order to renormalise the theory, obtained from Feynman diagrams using the renormalisation scheme $\overline{\text{MS}}$. The detailed calculations can be found in appendix D, while the definitions of the renormalised fields and couplings are given in section 3.1.1. From the counterterms we calculate the β-functions for the gauge couplings. Also, additional correlators are calculated, needed in the dimensional reduction step in chapter 6. We have kept the number of Higgs doublet N_h arbitrary as the calculations easily generalise to a fermiophobic N_h-Higgs Doublet Model (NHDM) at one-loop. $N_h = 1$ reduces to the SM result, while $N_h = 2$ is the result for the 2HDM. We can compare our results for the β-functions for the 2HDM with the results in ref. [78].

4.1 Self-energies

The momentum dependent divergences in the two-point functions at one-loop are absorbed into the field renormalisation counterterms of the various fields. Thermal mass terms also arise, but they are not explicitly given here.

$\mathrm{U}(1)_Y$ gauge boson self-energy

The Feynman diagrams for the $\mathrm{U}(1)_Y$ gauge boson self-energy are found in eqs. (D.1) to (D.5). The $\mathrm{U}(1)_Y$ gauge boson self-energy at one-loop is

μ  ν

$$
\begin{aligned}
=& -g'^2(d-1)\Big[N_h + \frac{1}{2}(1-2^{2-d})N_f\Big(6+N_c\frac{22}{9}\Big)\Big]I_1^{4b} \qquad (4.1)\\
&-\frac{1}{6}g'^2\Big[(4-d)N_h + \frac{1}{2}(d-1)(2^{4-d}-1)N_f\Big(6+N_c\frac{22}{9}\Big)\Big]P^2 I_2^{4b}\\
&\text{for } \mu=0,\nu=0,\\
=&\frac{1}{6}g'^2\Big[N_h + (2^{4-d}-1)N_f\Big(6+N_c\frac{22}{9}\Big)\Big](P_iP_j-\delta_{ij}P^2)I_2^{4b} \qquad (4.2)\\
&\text{for } \mu=i,\nu=j,
\end{aligned}
$$

where $d = 3 - 2\epsilon$ is the spatial dimension, N_c is the number of colours, and N_f is the number of fermion families. The sum-integrals I_1^{4b} and I_2^{4b} are given in appendix C. We see that the Lorentz symmetry is broken because the particles interact with the heat bath; the self-energy is divided into a temporal and a spatial part. The resulting divergences are absorbed into the $\mathrm{U}(1)_Y$ field renormalisation counterterm,

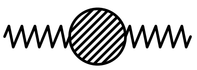
$$\delta Z_B = -\frac{g'^2}{6(4\pi)^2\epsilon}\Big[N_f\Big(6 + N_c\frac{22}{9}\Big) + N_h\Big]. \tag{4.3}$$

With the 2HDM specifics of eq. (3.18), we get

$$\delta Z_B = -\frac{7g'^2}{(4\pi)^2\epsilon}. \tag{4.4}$$

$\mathbf{SU}(2)_L$ gauge boson self-energy

The Feynman diagrams for the $\mathrm{SU}(2)_L$ gauge boson self-energy can be found in eqs. (D.6) to (D.13). The $\mathrm{SU}(2)_L$ gauge boson self-energy is

$a\mu$  $b\nu$

$$\begin{aligned} &= - g^2\delta_{ab}(d-1)\Big[2(d-1) + N_h + (1-2^{2-d})N_f(1+N_c)\Big]I_1^{4b} \\ &\quad + \frac{1}{6}g^2\delta_{ab}\Big[2(d^2-2d+10) - (4-d)N_h \\ &\quad - (d-1)(2^{4-d}-1)N_f(1+N_c)\Big]P^2 I_2^{4b} \end{aligned} \tag{4.5}$$

for $\mu = 0, \nu = 0$,

$$= -\frac{1}{3}g^2\delta_{ab}\Big[(16-d) - \frac{N_h}{2} - (2^{4-d}-1)N_f(1+N_c)\Big](P_iP_j - \delta_{ij}P^2)I_2^{4b} \tag{4.6}$$

for $\mu = i, \nu = j$.

We see that the structure is very similar to the $\mathrm{U}(1)_Y$ gauge boson self-energy, except for the terms not proportional to N_h or N_f. These contributions come from the $\mathrm{SU}(2)_L$ gauge boson self-interactions, which are not present for the $\mathrm{U}(1)_Y$ gauge boson. Also, only the left-handed fermions couple to the $\mathrm{SU}(2)_L$ gauge boson, while all fermions couple to the $\mathrm{U}(1)_Y$ gauge boson. This gives a difference in the numerical factor in the N_f term. The $\mathrm{SU}(2)_L$ field renormalisation counterterm takes the form

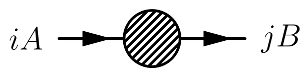
$$\delta Z_A = \frac{g^2}{3(4\pi)^2\epsilon}\Big[13 - \frac{N_h}{2} - N_f(1+N_c)\Big]. \tag{4.7}$$

By inserting the 2HDM specifics of eq. (3.18), we get

$$\delta Z_A = 0. \tag{4.8}$$

Lepton doublet self-energy

When working with a fermiophobic N_h-Higgs doublet model, where only one Higgs doublet directly couples to the fermions, the lepton doublet self-energy takes the same form as for the SM at one-loop. The Feynman diagram for the divergent part of the lepton doublet self-energy is calculated in eq. (D.14). The divergent part of the lepton doublet self-energy is



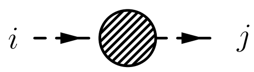
$$= \frac{1}{2} i \left[h^{(e)\dagger} h^{(e)} \right]_{AB} \delta_{ij} \not{P} I_2^{4b}. \tag{4.9}$$

The divergence is absorbed into the lepton doublet field renormalisation counterterm,

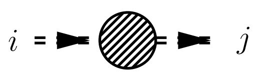
$$\left(\delta Z_l \right)_{AB} = -\frac{1}{2(4\pi)^2 \epsilon} \left[h^{(e)\dagger} h^{(e)} \right]_{AB}. \tag{4.10}$$

Higgs doublet self-energies

The Feynman diagrams for the Higgs doublet self-energies are calculated in eqs. (D.15) to (D.19). We need only the parts of the self-energies which depend on the external momentum to determine the wave function renormalisation. The mass counterterm will be determined by using the effective potential. The P^2 pieces of the Higgs doublet self-energies are



$$= \left(\frac{9}{4} g^2 + \frac{3}{4} g'^2 \right) \delta_{ij} P^2 I_2^{4b} - g_{Y,1}^2 \delta_{ij} P^2 I_2^{4f} \tag{4.11}$$



$$= \left(\frac{9}{4} g^2 + \frac{3}{4} g'^2 \right) \delta_{ij} P^2 I_2^{4b}, \tag{4.12}$$

where $g_{Y,1}^2$ is defined in eq. (3.11). The fermionic sum-integral I_2^{4f} is also evaluated in appendix C. The divergences are absorbed into the Higgs doublet field renormalisation counterterms,

$$\delta Z_{\Phi_1} = \frac{1}{(4\pi)^2 \epsilon} \left[\frac{9}{4} g^2 + \frac{3}{4} g'^2 - g_{Y,1}^2 \right] \tag{4.13}$$

$$\delta Z_{\Phi_2} = \frac{1}{(4\pi)^2 \epsilon} \left[\frac{9}{4} g^2 + \frac{3}{4} g'^2 \right]. \tag{4.14}$$

We now have all the field renormalisation counterterms we need to calculate the β-functions for the gauge fields. All additional field renormalisation counterterms can be found in a similar fashion.

Later, we might wish to reintroduce the mass-mixing term. Then we will need to calculate the $\Phi_1^\dagger \Phi_2$ correlator. The correlator at zero external momentum is

i  $j = -\frac{1}{2}\left[(\lambda_3 + 2\lambda_4)m_{12}^2 + 3\lambda_5 m_{12}^{*2}\right]\delta_{ij} I_2^{4b}.$ (4.15)

The mass-mixing counterterm takes the form

$$\delta m_{12}^2 = \frac{1}{(4\pi)^2\epsilon}\left[(\lambda_3 + 2\lambda_4)m_{12}^2 + 3\lambda_5 m_{12}^{*2}\right]. \tag{4.16}$$

4.2 Lepton doublet interactions

By calculating the interaction vertices with two external lepton doublet legs and a gauge boson leg, we are able to determine the counterterms for the gauge field coupling constants g' and g. We only calculate the vertices needed to extract the gauge coupling counterterms, since our goal is to determine the β-functions.

Lepton doublet - U$(1)_Y$ gauge boson vertex

The lepton doublet - U$(1)_Y$ gauge boson vertex is used to calculate the contribution from the left-handed leptons to the gauge boson coupling constant counterterm $\delta g'$. We do not need the contribution from the other fermionic fields to calculate the β-function for the U$(1)_Y$ gauge boson. Extending the calculation to include all fermions is trivial. The Feynman diagrams for the lepton doublet - U(1) vertex can be found in eqs. (D.22) to (D.25). The divergence is absorbed by the counterterms of the form

$$\delta g' + \frac{g'}{2}\Big(2(\delta Z_l)_{AB} + \delta Z_B\Big) = -\frac{g'}{2(4\pi)^2\epsilon}\left[h^{(e)\dagger}h^{(e)}\right]_{AB}. \tag{4.17}$$

The structure comes from the interaction between the lepton doublet and the gauge field, and from the definition of the gauge boson coupling constant counterterm in eq. (3.23). Using our results for the wave function counterterms, eqs. (4.3) and (4.10), the coupling counterterm becomes

$$\delta g' = \frac{g'^3}{12(4\pi)^2\epsilon}\left[N_h + \frac{40}{3}N_f\right]. \tag{4.18}$$

Lepton doublet - SU$(2)_L$ gauge boson vertex

Similarly, we need only the lepton contribution to the SU$(2)_L$ gauge boson coupling constant counterterm in order to calculate the β-function for the SU$(2)_L$ gauge boson. The Feynman diagrams for the lepton doublet - SU$(2)_L$ vertex can be found in eqs. (D.26) to (D.29). The divergence is absorbed by the counterterms of the form

$$\delta g + \frac{g}{2}\Big(2(\delta Z_l)_{AB} + \delta Z_A\Big) = -\frac{g}{2(4\pi)^2\epsilon}\left[3g^2 + \left[h^{(e)\dagger}h^{(e)}\right]_{AB}\right], \tag{4.19}$$

which is a similar structure to the $\mathrm{U}(1)_Y$ gauge boson case. By using the wave function renormalisation counterterms in eqs. (4.7) and (4.10), we find the coupling counterterm

$$\delta g = -\frac{g^3}{2(4\pi)^2\epsilon}\Big[\frac{44 - N_h}{6} - \frac{4}{3}N_f\Big]. \tag{4.20}$$

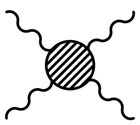
Again, notice the similarities between eqs. (4.18) and (4.20). The main difference is the term not proportional to N_h or N_f, which is only present in eq. (4.20). This comes from the $\mathrm{SU}(2)_L$ gauge boson self-interactions. Also, we get a difference in the numerical factor in the N_f term because only left-handed fermions couple to the $\mathrm{SU}(2)_L$ gauge boson, while all fermions couple to the $\mathrm{U}(1)_Y$ gauge boson.

4.3 4-point correlators

An integral part of the dimensional reduction procedure is to match the effective three-dimensional theory with the original four-dimensional theory, in order to determine the parameters of the effective theory in terms of the original parameters. We match the two theories by requiring the correlators of the two theories to be the same at large distances, or at zero external momentum. We list the correlators in the original four-dimensional theory at zero external momentum needed in chapter 6. Detailed calculations can be found in appendix D.

The B_0^4 correlator

We will need the four-point function with zero external momentum of the temporal component of the $\mathrm{U}(1)_Y$ gauge field when matching the three-dimensional theory to the original theory. The sum of eqs. (D.30) to (D.33) is

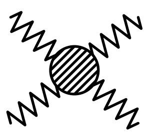


$$= \frac{1}{2}(d-1)(d-3)\Big[N_h - (2^{4-d}-1)\Big(9 + \frac{137}{81}N_c\Big)N_f\Big]g'^4 I_2^{4b}. \tag{4.21}$$

Notice that the factor $(d-3)$ cancels the divergence in I_2^{4b}, and we get a finite result. Our theory is renormalisable, and any divergences should be absorbed by counterterms. As we have no counterterms of this type, no divergence should arise. Making sure that all divergences cancel will serve as a check for consistency when doing the matching.

The $A_0^a A_0^b A_0^c A_0^d$ correlator

Similarly, we need the temporal component of the four-point function of the $\mathrm{SU}(2)_L$ gauge field at zero external momentum. The sum of eqs. (D.34) to (D.41) is

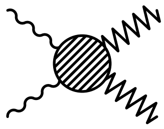


$$\begin{aligned}=&\frac{1}{6}(d-1)(d-3)\Big[8(d-1) + N_h \\ &- (2^{4-d}-1)N_f(1+N_c)\Big]g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b}.\end{aligned} \tag{4.22}$$

Again, the factor $(d-3)$ cancels the divergence in I_2^{4b}. The factor $8(d-1)$ comes from the $\mathrm{SU}(2)_L$ gauge boson and the ghost loops, and is not present in the correlator for the $\mathrm{U}(1)_Y$ gauge boson.

The $A_0^a A_0^b B_0^2$ correlator

The correlator with two B_0 legs and two A_0^a legs can be found by summing eqs. (D.42) to (D.47). It takes the form

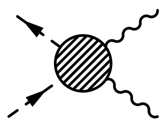


$$= \frac{1}{2}(d-1)(d-3)\Big[N_h - (2^{4-d}-1)N_f(Y_l^2 + N_c Y_q^2)\Big] g^2 g'^2 \delta_{ab} I_2^{4b} \tag{4.23}$$

Notice the similarities between eqs. (4.21) and (4.23). The fermionic part is different because only left-handed fermions interact with the $\mathrm{SU}(2)_L$ gauge boson.

The $\Phi_1^{\dagger i}\Phi_1^j B_\mu B_\nu$ correlator

The correlator with two scalar legs and two gauge field legs is the sum of eqs. (D.57) to (D.64), and takes the form

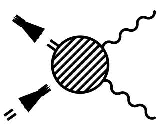


$$\begin{aligned} =& g'^2\Big[(d-3)\Big(\frac{3}{2}\lambda_1 + \lambda_3 + \frac{1}{2}\lambda_4\Big) + \frac{d}{8}g'^2 + \frac{3}{8}dg^2 \\ & - \frac{1}{2}(2^{4-d}-1)\Big(g_{Y,1}^2 - 2\epsilon g_{Y,2}^2\Big)\Big]\delta_{ij} I_2^{4b} \qquad (4.24) \\ & \text{for } \mu = 0, \nu = 0 \\ =& g'^2\Big[\frac{3}{8}g'^2 + \frac{9}{8}g^2 - \frac{1}{2}(2^{4-d}-1)g_{Y,1}^2\Big]\delta_{ij}\delta_{rs} I_2^{4b} \qquad (4.25) \\ & \text{for } \mu = r, \nu = s \end{aligned}$$

where $g_{Y,1}^2$ and $g_{Y,2}^2$ are defined in eqs. (3.11) and (3.12). Now the divergence in I_2^{4b} is not cancelled, neither for the temporal nor the spatial part. The divergence can be absorbed by counterterms. This is also our first encounter with a correlator that does not directly generalise to the NHDM, due to the presence of the scalar couplings in the temporal part of the correlator. This correlator will be re-calculated in chapter 7, to include the effect of additional scalar doublets.

The $\Phi_2^{\dagger i}\Phi_2^j B_\mu B_\nu$ correlator

The coupling with two scalar Φ_2 legs and two gauge field legs takes the form



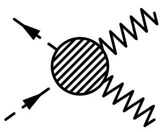
$$= g'^2\Big[(d-3)\Big(\frac{3}{2}\lambda_2 + \lambda_3 + \frac{1}{2}\lambda_4\Big) + \frac{d}{8}g'^2 + \frac{3}{8}dg^2\Big]\delta_{ij} I_2^{4b} \tag{4.26}$$

$$
\begin{aligned}
&\text{for } \mu = 0, \nu = 0 \\
=& g'^2\Big[\frac{3}{8}g'^2 + \frac{9}{8}g^2\Big]\delta_{ij}\delta_{rs}I_2^{4b} \qquad (4.27)\\
&\text{for } \mu = r, \nu = s.
\end{aligned}
$$

The correlator is similar to the $\Phi_1^\dagger\Phi_1 B_\mu B_\nu$ correlator, by replacing $\lambda_1 \to \lambda_2$ and dropping the coupling to the fermions. The generalised correlator for the NHDM is found in eq. (7.3).

The $\Phi_1^{\dagger i}\Phi_1^j A_\mu^a A_\nu^b$ correlator

The diagrams contributing to the correlator with two scalar legs and two gauge field legs can be found in eqs. (D.48) to (D.56). The total contribution to the correlator is

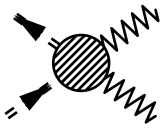


$$
\begin{aligned}
=& g^2\Big[\Big(-\frac{25}{8}+d\Big)dg^2 - (3-d)\Big(\frac{3}{2}\lambda_1 + \lambda_3 + \frac{1}{2}\lambda_4\Big) + \frac{1}{8}dg'^2 \\
&- \frac{1}{2}(2^{4-d}-1)(d-2)g_{Y,1}^2\Big]\delta_{ab}\delta_{ij}I_2^{4b} \qquad (4.28)\\
&\text{for } \mu = 0, \nu = 0 \\
=& \Big[-\frac{3}{8}g^4 + \frac{3}{8}g^2g'^2 - \frac{1}{2}(2^{4-d}-1)g^2g_{Y,1}^2\Big]\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4b} \qquad (4.29)\\
&\text{for } \mu = r, \nu = s.
\end{aligned}
$$

Notice that the divergences do not cancel. Also, the scalar couplings make the expression special to the 2HDM.

The $\Phi_2^{\dagger i}\Phi_2^j A_\mu^a A_\nu^b$ correlator

The coupling with two scalar Φ_2 legs and two gauge field legs takes the form



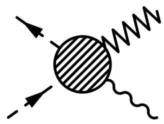
$$
= g^2\Big[\Big(-\frac{25}{8}+d\Big)dg^2 - (3-d)\Big(\frac{3}{2}\lambda_2 + \lambda_3 + \frac{1}{2}\lambda_4\Big) + \frac{1}{8}dg'^2\Big]\delta_{ab}\delta_{ij}I_2^{4b} \qquad (4.30)
$$

$$
\begin{aligned}
&\text{for } \mu = 0, \nu = 0 \\
=& \Big[-\frac{3}{8}g^4 + \frac{3}{8}g^2g'^2\Big]\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4b} \qquad (4.31)\\
&\text{for } \mu = r, \nu = s
\end{aligned}
$$

Notice the similarity to the $\Phi_1^\dagger\Phi_1 A_\mu^a A_\nu^b$ correlator.

The $\Phi_1^{\dagger i}\Phi_1^j A_0^a B_0$ correlator

Lastly, we compute the temporal part of the correlator with two scalar legs and two different gauge field legs. The sum of eqs. (D.65) to (D.72) takes the form

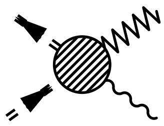


$$= gg'\Big[\frac{1}{2}(d-3)(\lambda_1+\lambda_4)+\frac{1}{8}d(g^2+g'^2) - \frac{1}{2}(2^{4-d}-1)(g_{Y,3}^2-2\epsilon g_{Y,4}^2)\Big](\tau_a)_{ij}I_2^{4b}, \tag{4.32}$$

where $g_{Y,3}^2$ and $g_{Y,4}^2$ are defined in eqs. (3.13) and (3.14). This correlator also diverges, and we must make sure that the divergences are cancelled in the final matching between the three- and four-dimensional theories.

The $\Phi_2^{\dagger i}\Phi_2^j A_0^a B_0$ correlator

The coupling with two scalar Φ_2 legs and two different gauge field legs takes the form



$$= gg'\Big[\frac{1}{2}(d-3)(\lambda_2+\lambda_4)+\frac{1}{8}d(g^2+g'^2)\Big](\tau_a)_{ij}I_2^{4b}, \tag{4.33}$$

which is identical to eq. (4.32), if we discard the coupling to the fermions. The general NHDM correlator with two scalar legs and two different gauge boson legs is given in eq. (7.5).

4.4 Counterterms

The counterterms are defined in section 3.1.1. We have used dimensional regularisation to regularise the UV divergences, and the renormalisation scheme $\overline{\text{MS}}$ to renormalise the theory. The UV divergences are independent of the temperature, so we could calculate the correlators at finite or zero temperature. We have done all calculations at finite temperature. We will here summarise the counterterms found previously, valid in the Landau gauge,

$$\delta Z_A = \frac{g^2}{(4\pi)^2\epsilon}\Big[\frac{26-N_h}{6}-\frac{4}{3}N_f\Big] \tag{4.34}$$

$$\delta Z_B = -\frac{g'^2}{(4\pi)^2\epsilon}\Big[\frac{N_h}{6}+\frac{20}{9}N_f\Big] \tag{4.35}$$

$$(\delta Z_l)_{AB} = -\frac{1}{2(4\pi)^2\epsilon}\Big[h^{(e)\dagger}h^{(e)}\Big]_{AB} \tag{4.36}$$

$$\delta Z_{\Phi_1} = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{4}g^2+\frac{3}{4}g'^2-g_{Y,1}^2\Big] \tag{4.37}$$

$$\delta Z_{\Phi_2} = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{4}g^2+\frac{3}{4}g'^2\Big] \tag{4.38}$$

$$\delta g = -\frac{g^3}{2(4\pi)^2\epsilon}\Big[\frac{44-N_h}{6}-\frac{4}{3}N_f\Big] \tag{4.39}$$

$$\delta g' = \frac{g'^3}{12(4\pi)^2\epsilon}\Big[N_h + \frac{40}{3}N_f\Big]. \tag{4.40}$$

For $N_h = 1$ the counterterms agree with the SM results [26]. The counterterms for the scalar couplings will be calculated using the effective potential in chapter 5.

4.5 β-functions

We find the β-functions by calculating the quantities $f_{g'}$ and f_g. From the definitions of the gauge coupling counterterms, eqs. (3.22) and (3.23), the contributions from the wave function renormalisation to the renormalised gauge couplings have already been taken into account, and the quantities $f_{g'}$ and f_g simply become

$$\begin{aligned} f_{g'} &= \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta g'\Big\} \\ &= \frac{g'^3}{12(4\pi)^2}\Big[N_h + N_f\Big(6 + N_c\frac{22}{9}\Big)\Big] = C'g'^3 \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} f_g &= \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta g\Big\} \\ &= -\frac{g^3}{6(4\pi)^2}\Big[9 + 13 - \frac{N_h}{2} - N_f(1 + N_c)\Big] = Cg^3 \end{aligned} \tag{4.42}$$

In the limit $\epsilon \to 0$, we have that $\beta_{g'} = 2C'g'^3$,

$$\beta_{g'} = \frac{g'^3}{6(4\pi)^2}\Big[N_h + N_f\Big(6 + N_c\frac{22}{9}\Big)\Big]. \tag{4.43}$$

In a similar fashion, we have that

$$\beta_g = -\frac{g^3}{3(4\pi)^2}\Big[22 - \frac{N_h}{2} - N_f(1 + N_c)\Big]. \tag{4.44}$$

Using the 2HDM specifics of eq. (3.18), we get

$$\beta_{g'} = \frac{7g'^3}{(4\pi)^2} \tag{4.45}$$

$$\beta_g = -\frac{3g^3}{(4\pi)^2}. \tag{4.46}$$

The values of the β-functions for the gauge bosons agree with the known results, both for the SM [26] and for the 2HDM [78].

CHAPTER 5

EFFECTIVE POTENTIAL

We want to find the β-functions for the scalar couplings, which are determined from the scalar counterterms in a similar fashion to the discussion in chapter 4. Also, we want to calculate the contributions from the superheavy modes to the two- and four-point scalar correlators at vanishing external momentum, in order to match the four-dimensional scalar correlators with their three-dimensional counterparts. A convenient way to extract the contributions from the superheavy modes to the scalar correlators, and also to find the scalar counterterms, is to evaluate the effective potential $V(\Phi_1, \Phi_2)$. For a preliminary discussion on the effective potential, see section 2.3. The effective potential contains one-particle irreducible Green's functions $G_{n,m}$ at vanishing external momenta of the form $G_{n,m}\Phi_1^n\Phi_2^m$, so the quadratic and quartic terms in Φ_1 and Φ_2 give the two- and four-point correlators. It is enough to calculate the one-loop effective potential to extract the contribution to the coupling, but, in order to be consistent, for the mass parameter the two-loop effective potential is needed for an accuracy of $\mathcal{O}(g^4)$. We will only calculate the one-loop effective potential, and leave the extension to two-loop for the mass parameters as a part of the outlook for this project.

In order to calculate the effective potential $V(\Phi_1, \Phi_2)$, we shift the scalar fields by some arbitrary background fields, $\Phi_i \to \Phi_i + \varphi_i$. The mass matrix is extracted from the parts quadratic in the scalar, gauge or fermion fields. We diagonalise the mass matrix for different choices of the background fields, and extract the contributions to the two- and four-point correlators at zero external momentum. The counterterms are found by absorbing the divergences, in the usual fashion. Lastly, we write down the β-functions for the scalar couplings.

The effective potential can be expanded in terms of the background fields. The parts of the effective potential quadratic or quartic in the background fields take the form

$$\begin{aligned} V_{\text{eff}} =& V_{11}\varphi_1^\dagger\varphi_1 + V_{22}\varphi_2^\dagger\varphi_2 + V_1\Big[\varphi_1^\dagger\varphi_1\Big]^2 + V_2\Big[\varphi_2^\dagger\varphi_2\Big]^2 \\ &+ V_3\Big[\varphi_1^\dagger\varphi_1\Big]\Big[\varphi_2^\dagger\varphi_2\Big] + V_4\Big[\varphi_1^\dagger\varphi_2\Big]\Big[\varphi_2^\dagger\varphi_1\Big] + \frac{V_5}{2}\Big[\Big[\varphi_1^\dagger\varphi_2\Big]^2 + \Big[\varphi_2^\dagger\varphi_1\Big]^2\Big]. \end{aligned} \tag{5.1}$$

Our goal is to find the coefficients V_{ii} and V_n, where $i = 1, 2$ and $n = 1, \ldots, 5$. From this we can both find the scalar counterterms and the correlators needed for dimensional reduction.

Only the fields directly coupled to the Higgs doublets $\Phi_{1,2}$ affect the effective potential, apart from a constant. We divide the discussion into the contribution from the scalars, section 5.1, the gauge bosons, section 5.2, and the fermions, section 5.3. For the gauge bosons, we must be careful when choosing a gauge. For the scalars and fermions, we will directly go to the Landau gauge ($\xi = 0$).

5.1 Scalars

As we will see, we have eliminated the mixing between the gauge and scalar fields in section 5.2. Thus, we can safely go directly to the Landau gauge ($\xi = 0$). After shifting the fields $\Phi_i \to \Phi_i + \varphi_i$ and dropping the linear terms in Φ_i and some constant terms, we get

$$\begin{aligned}
V_{\text{scalar}}[\Phi_1+\varphi_1, \Phi_2+\varphi_2] = &-\frac{1}{2}\tilde{m}_{11}^2\Phi_1^\dagger\Phi_1 - \frac{1}{2}\tilde{m}_{22}^2\Phi_2^\dagger\Phi_2 \\
&+ \frac{1}{2}\lambda_1\Big[\Phi_1^\dagger\varphi_1 + \varphi_1^\dagger\Phi_1\Big]^2 + \frac{1}{2}\lambda_2\Big[\Phi_2^\dagger\varphi_2 + \varphi_2^\dagger\Phi_2\Big]^2 \\
&+ \lambda_3\Big[\Phi_1^\dagger\varphi_1 + \varphi_1^\dagger\Phi_1\Big]\Big[\Phi_2^\dagger\varphi_2 + \varphi_2^\dagger\Phi_2\Big] \\
&+ \lambda_4\Big[(\Phi_1^\dagger\Phi_2)(\varphi_2^\dagger\varphi_1) + (\Phi_2^\dagger\Phi_1)(\varphi_1^\dagger\varphi_2) \\
&+ (\Phi_1^\dagger\varphi_2 + \varphi_1^\dagger\Phi_2)(\Phi_2^\dagger\varphi_1 + \varphi_2^\dagger\Phi_1)\Big] \\
&+ \frac{1}{2}\lambda_5\Big[2(\Phi_1^\dagger\Phi_2)(\varphi_1^\dagger\varphi_2) + 2(\Phi_2^\dagger\Phi_1)(\varphi_2^\dagger\varphi_1) \\
&+ \Big[\Phi_1^\dagger\varphi_2 + \varphi_1^\dagger\Phi_2\Big]^2 + \Big[\Phi_2^\dagger\varphi_1 + \varphi_2^\dagger\Phi_1\Big]^2\Big] + \mathcal{O}(\Phi_i^3) + \mathcal{O}(\Phi_i^4)
\end{aligned}$$

where $\tilde{m}_{11}^2 = m_{11}^2 - 2\lambda_1\varphi_1^\dagger\varphi_1 - 2\lambda_3\varphi_2^\dagger\varphi_2$ and $\tilde{m}_{22}^2 = m_{22}^2 - 2\lambda_2\varphi_2^\dagger\varphi_2 - 2\lambda_3\varphi_1^\dagger\varphi_1$.

Now, we notice that distinguishing the contributions to the $\lambda_3 - \lambda_5$ terms is difficult, and we have to carefully choose the background field accordingly. We need to make three separate choices for the background fields, and from the linear combinations that arise extract the individual contributions to the scalar couplings.

Our choices for the backgrounds fields are

$$\text{Case 1}: \quad \varphi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_1\end{pmatrix}, \quad \varphi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_2\end{pmatrix} \tag{5.2}$$

$$\text{Case 2}: \quad \varphi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_1\end{pmatrix}, \quad \varphi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ iw_0\end{pmatrix} \tag{5.3}$$

$$\text{Case 3}: \quad \varphi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_1\end{pmatrix}, \quad \varphi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}w_+\\ 0\end{pmatrix}. \tag{5.4}$$

For each case we will diagonalise the mass matrix, and evaluate the integral of the form of eq. (2.94) to find the scalar contributions to the scalar correlators at zero external momentum.

5.1.1 Case 1

We use the expansion of the effective potential in powers of the background fields, eq. (5.1), together with eq. (5.2) to get

$$V_{\text{eff}} = \frac{1}{2}V_{11}v_1^2 + \frac{1}{2}V_{22}v_2^2 + \frac{1}{4}V_1 v_1^4 + \frac{1}{4}V_2 v_2^4 + \frac{1}{4}(V_3 + V_4 + V_5)v_1^2 v_2^2 \tag{5.5}$$

We see that we cannot distinguish the contributions to V_3, V_4, or V_5, and are only able to extract the contributions to the sum of them from the expansion of the effective potential. The divergent part of V_1 is absorbed into $\delta\lambda_1$, V_2 into $\delta\lambda_2$, and $(V_3 + V_4 + V_5)$ into $\delta\lambda_3 + \delta\lambda_4 + \delta\lambda_5$. The mass matrix can be found by using eq. (3.41), and it takes the form

$$M^2 = \begin{bmatrix} M_{11}^2 & 0 & 0 & 0 & M_{15}^2 & 0 & 0 & 0 \\ 0 & M_{22}^2 & 0 & 0 & 0 & M_{26}^2 & 0 & 0 \\ 0 & 0 & M_{33}^2 & 0 & 0 & 0 & M_{37}^2 & 0 \\ 0 & 0 & 0 & M_{44}^2 & 0 & 0 & 0 & M_{48}^2 \\ M_{15}^2 & 0 & 0 & 0 & M_{55}^2 & 0 & 0 & 0 \\ 0 & M_{26}^2 & 0 & 0 & 0 & M_{66}^2 & 0 & 0 \\ 0 & 0 & M_{37}^2 & 0 & 0 & 0 & M_{77}^2 & 0 \\ 0 & 0 & 0 & M_{48}^2 & 0 & 0 & 0 & M_{88}^2 \end{bmatrix} \tag{5.6}$$

where

$$\begin{aligned}
&M_{11}^2 = -\frac{1}{2}\tilde{m}_{11}^2, \quad M_{22}^2 = -\frac{1}{2}\tilde{m}_{11}^2, \quad M_{33}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \lambda_1 v_1^2 + \frac{1}{2}(\lambda_4 + \lambda_5)v_2^2, \\
&M_{44}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \frac{1}{2}(\lambda_4 - \lambda_5)v_2^2, \quad M_{55}^2 = -\frac{1}{2}\tilde{m}_{22}^2, \quad M_{66}^2 = -\frac{1}{2}\tilde{m}_{22}^2, \\
&M_{77}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \lambda_2 v_2^2 + \frac{1}{2}(\lambda_4 + \lambda_5)v_1^2, \quad M_{88}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \frac{1}{2}(\lambda_4 - \lambda_5)v_1^2, \\
&M_{15}^2 = \frac{1}{2}(\lambda_4 + \lambda_5)v_1 v_2, \quad M_{26}^2 = \frac{1}{2}(\lambda_4 + \lambda_5)v_1 v_2, \quad M_{37}^2 = 2\lambda_+ v_1 v_2, \quad M_{48}^2 = \lambda_5 v_1 v_2.
\end{aligned}$$

We use the same method as in section 3.4 to diagonalise the mass matrix. After diagonalising the mass matrix we find the mass terms

$$m_1^2 = m_2^2 = -\frac{1}{4}\left[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 + \sqrt{(\tilde{m}_{11}^2 - \tilde{m}_{22}^2)^2 + 4(\lambda_4 + \lambda_5)^2 v_1^2 v_2^2}\right] \tag{5.7a}$$

$$m_3^2 = m_4^2 = -\frac{1}{4}\left[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - \sqrt{(\tilde{m}_{11}^2 - \tilde{m}_{22}^2)^2 + 4(\lambda_4 + \lambda_5)^2 v_1^2 v_2^2}\right] \tag{5.7b}$$

$$\begin{aligned} m_5^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 v_2^2) - (\lambda_4 + \lambda_5)(v_1^2 + v_2^2) \\ &+ \sqrt{\left(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 v_2^2 - \lambda_1 v_1^2) + (\lambda_4 + \lambda_5)(v_1^2 - v_2^2)\right)^2 + 16(\lambda_3 + \lambda_4 + \lambda_5)^2 v_1^2 v_2^2}\Big] \end{aligned} \tag{5.7c}$$

$$\begin{aligned} m_6^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 v_2^2) - (\lambda_4 + \lambda_5)(v_1^2 + v_2^2) \\ &- \sqrt{\left(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 v_2^2 - \lambda_1 v_1^2) + (\lambda_4 + \lambda_5)(v_1^2 - v_2^2)\right)^2 + 16(\lambda_3 + \lambda_4 + \lambda_5)^2 v_1^2 v_2^2}\Big] \end{aligned} \tag{5.7d}$$

$$\begin{aligned} m_7^2 =& -\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4 - \lambda_5)(v_1^2 + v_2^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4 - \lambda_5)(v_1^2 - v_2^2)\Big)^2 + 16\lambda_5^2 v_1^2 v_2^2}\Big] \end{aligned} \tag{5.7e}$$

$$\begin{aligned} m_8^2 =& -\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4 - \lambda_5)(v_1^2 + v_2^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4 - \lambda_5)(v_1^2 - v_2^2)\Big)^2 + 16\lambda_5^2 v_1^2 v_2^2}\Big]. \end{aligned} \tag{5.7f}$$

From eq. (2.94) the one-loop contribution to the effective potential from the scalar sector is

$$\sum_{i=1}^{8} \mathcal{C}_S(m_i) = \sum_{i=1}^{8} \frac{1}{2} \SumInt_K \log(K^2 + m_i^2). \tag{5.8}$$

Equation (5.8) is evaluated using eqs. (5.7) and (C.15). The term $\mathcal{O}(m^3 T)$ in eq. (C.15) is omitted as it comes from the zero mode and thus not from a superheavy mode. This term will be present in the three-dimensional theory as well, and will cancel when doing the matching of the three- and four-dimensional theories. By identifying the different terms in the series expansion eq. (5.5), we find the contributions to the effective potential to be

$$V_{11} = \frac{T^2}{12}\Big[3\lambda_1 + 2\lambda_3 + \lambda_4\Big] \tag{5.9a}$$

$$V_{22} = \frac{T^2}{12}\Big[3\lambda_2 + 2\lambda_3 + \lambda_4\Big] \tag{5.9b}$$

$$V_1 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.9c}$$

$$V_2 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.9d}$$

$$V_3 + V_4 + V_5 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_1 + \lambda_2)(\lambda_3 + 3\lambda_+ + \lambda_-) + 8\lambda_+^2 + (\lambda_4 + \lambda_5)^2 + 2\lambda_5^2\Big] \tag{5.9e}$$

where $\lambda_\pm = \frac{1}{2}(\lambda_3 + \lambda_4 \pm \lambda_5)$ and L_b is defined in eq. (3.16). We leave the determination of the counterterms to section 5.4.

5.1.2 Case 2

We see that the procedure for finding the contributions from the scalar sector to the effective potential is quite mechanical. We will follow the exact same steps as in section 5.1.1, with a slightly different choice for the background fields, eq. (5.3). From the expansion of the effective potential, eq. (5.1), we have that

$$V_{\text{eff}} = \frac{1}{2}V_{11}v_1^2 + \frac{1}{2}V_{22}w_0^2 + \frac{1}{4}V_1 v_1^4 + \frac{1}{4}V_2 w_0^4 + \frac{1}{4}(V_3 + V_4 - V_5)v_1^2 w_0^2. \tag{5.10}$$

Now we are able to find the contributions from the scalars to the linear combination $V_3 + V_4 - V_5$. We absorb the divergent part of V_1 into $\delta\lambda_1$, V_2 into $\delta\lambda_2$, and $V_3 + V_4 - V_5$ into $\delta\lambda_3 + \delta\lambda_4 - \delta\lambda_5$.

Using eq. (3.41), we find the mass matrix to be

$$M^2 = \begin{bmatrix} M_{11}^2 & 0 & 0 & 0 & 0 & M_{16}^2 & 0 & 0 \\ 0 & M_{22}^2 & 0 & 0 & M_{25}^2 & 0 & 0 & 0 \\ 0 & 0 & M_{33}^2 & 0 & 0 & 0 & 0 & M_{38}^2 \\ 0 & 0 & 0 & M_{44}^2 & 0 & 0 & M_{47}^2 & 0 \\ 0 & M_{25}^2 & 0 & 0 & M_{55}^2 & 0 & 0 & 0 \\ M_{16}^2 & 0 & 0 & 0 & 0 & M_{66}^2 & 0 & 0 \\ 0 & 0 & 0 & M_{47}^2 & 0 & 0 & M_{77}^2 & 0 \\ 0 & 0 & M_{38}^2 & 0 & 0 & 0 & 0 & M_{88}^2 \end{bmatrix} \tag{5.11}$$

where

$$\begin{aligned} &M_{11}^2 = -\frac{1}{2}\tilde{m}_{11}^2, \quad M_{22}^2 = -\frac{1}{2}\tilde{m}_{11}^2, \quad M_{33}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \lambda_1 v_1^2 + \frac{1}{2}(\lambda_4 - \lambda_5) w_0^2, \\ &M_{44}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \frac{1}{2}(\lambda_4 + \lambda_5) w_0^2, \quad M_{55}^2 = -\frac{1}{2}\tilde{m}_{22}^2, \quad M_{66}^2 = -\frac{1}{2}\tilde{m}_{22}^2, \\ &M_{77}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \frac{1}{2}(\lambda_4 + \lambda_5) v_1^2, \quad M_{88}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \lambda_2 w_0^2 + \frac{1}{2}(\lambda_4 - \lambda_5) v_1^2, \\ &M_{16}^2 = \frac{1}{2}(\lambda_4 - \lambda_5) v_1 w_0, \quad M_{25}^2 = -\frac{1}{2}(\lambda_4 - \lambda_5) v_1 w_0, \quad M_{38}^2 = 2\lambda_- v_1 w_0, \quad M_{47}^2 = \lambda_5 v_1 w_0. \end{aligned}$$

After diagonalising the mass matrix, using the method from section 3.4, we find the mass terms

$$m_1^2 = m_2^2 = -\frac{1}{4}\left[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 + \sqrt{(\tilde{m}_{11}^2 - \tilde{m}_{22}^2)^2 + 4(\lambda_4 - \lambda_5)^2 v_1^2 w_0^2}\right] \tag{5.12a}$$

$$m_3^2 = m_4^2 = -\frac{1}{4}\left[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - \sqrt{(\tilde{m}_{11}^2 - \tilde{m}_{22}^2)^2 + 4(\lambda_4 - \lambda_5)^2 v_1^2 w_0^2}\right] \tag{5.12b}$$

$$\begin{aligned} m_5^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 w_0^2) - (\lambda_4 - \lambda_5)(v_1^2 + w_0^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 w_0^2 - \lambda_1 v_1^2) + (\lambda_4 - \lambda_5)(v_1^2 - w_0^2)\Big)^2 + 16(\lambda_3 + \lambda_4 - \lambda_5)^2 v_1^2 w_0^2}\Big] \end{aligned} \tag{5.12c}$$

$$\begin{aligned} m_6^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 w_0^2) - (\lambda_4 - \lambda_5)(v_1^2 + w_0^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 w_0^2 - \lambda_1 v_1^2) + (\lambda_4 - \lambda_5)(v_1^2 - w_0^2)\Big)^2 + 16(\lambda_3 + \lambda_4 - \lambda_5)^2 v_1^2 w_0^2}\Big] \end{aligned} \tag{5.12d}$$

$$\begin{aligned} m_7^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4 + \lambda_5)(v_1^2 + w_0^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4 + \lambda_5)(v_1^2 - w_0^2)\Big)^2 + 16\lambda_5^2 v_1^2 w_0^2}\Big] \end{aligned} \tag{5.12e}$$

$$\begin{aligned} m_8^2 = -\frac{1}{4}\Big[&\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4 + \lambda_5)(v_1^2 + w_0^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4 + \lambda_5)(v_1^2 - w_0^2)\Big)^2 + 16\lambda_5^2 v_1^2 w_0^2}\Big]. \end{aligned} \tag{5.12f}$$

The contributions to the effective potential from the scalars are

$$V_{11} = \frac{T^2}{12}\Big[3\lambda_1 + 2\lambda_3 + \lambda_4\Big] \tag{5.13a}$$

$$V_{22} = \frac{T^2}{12}\Big[3\lambda_2 + 2\lambda_3 + \lambda_4\Big] \tag{5.13b}$$

$$V_1 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.13c}$$

$$V_2 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.13d}$$

$$V_3 + V_4 - V_5 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_+ + 3\lambda_-) + 8\lambda_-^2 + (\lambda_4 - \lambda_5)^2 + 2\lambda_5^2\Big]. \tag{5.13e}$$

Clearly, the terms V_{11}, V_{22}, V_1, and V_2 should be the same as in section 5.1.1, and we can see from eq. (5.9) that they match.

5.1.3 Case 3

With the choice of eq. (5.4), the expansion of the effective potential becomes

$$V_{\text{eff}} = \frac{1}{2}V_{11}v_1^2 + \frac{1}{2}V_{22}w_+^2 + \frac{1}{4}V_1 v_1^4 + \frac{1}{4}V_2 w_+^4 + \frac{1}{4}V_3 v_1^2 w_+^2. \tag{5.14}$$

We absorb the divergent part of V_1 into $\delta\lambda_1$, V_2 into $\delta\lambda_2$, and V_3 into $\delta\lambda_3$. The mass matrix becomes, using eq. (3.41),

$$M^2 = \begin{bmatrix} M_{11}^2 & 0 & 0 & 0 & 0 & 0 & M_{17}^2 & 0 \\ 0 & M_{22}^2 & 0 & 0 & 0 & 0 & 0 & M_{28}^2 \\ 0 & 0 & M_{33}^2 & 0 & M_{35}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{35}^2 & 0 & M_{55}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{66}^2 & 0 & 0 \\ M_{17}^2 & 0 & 0 & 0 & 0 & 0 & M_{77}^2 & 0 \\ 0 & M_{28}^2 & 0 & 0 & 0 & 0 & 0 & M_{88}^2 \end{bmatrix} \tag{5.15}$$

where

$$\begin{aligned} &M_{11}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \frac{1}{2}(\lambda_4 + \lambda_5)w_+^2, \quad M_{22}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \frac{1}{2}(\lambda_4 - \lambda_5)w_+^2, \\ &M_{33}^2 = -\frac{1}{2}\tilde{m}_{11}^2 + \lambda_1 v_1^2, \quad M_{44}^2 = -\frac{1}{2}\tilde{m}_{11}^2, \quad M_{55}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \lambda_2 w_+^2, \quad M_{66}^2 = -\frac{1}{2}\tilde{m}_{22}^2, \\ &M_{77}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \frac{1}{2}(\lambda_4 + \lambda_5)v_1^2, \quad M_{88}^2 = -\frac{1}{2}\tilde{m}_{22}^2 + \frac{1}{2}(\lambda_4 - \lambda_5)v_1^2, \\ &M_{17}^2 = \frac{1}{2}(\lambda_4 + \lambda_5)v_1 w_+, \quad M_{28}^2 = -\frac{1}{2}(\lambda_4 - \lambda_5)v_1 w_+, \quad M_{35}^2 = \lambda_3 v_1 w_+. \end{aligned}$$

The mass terms we find after diagonalising the mass matrix are

$$m_1^2 = -\frac{1}{2}\tilde{m}_{11}^2 \tag{5.16a}$$

$$m_2^2 = -\frac{1}{2}\tilde{m}_{22}^2 \tag{5.16b}$$

$$\begin{aligned} m_3^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4+\lambda_5)(v_1^2+w_+^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4+\lambda_5)(v_1^2 - w_+^2)\Big)^2 + 4(\lambda_4+\lambda_5)^2 v_1^2 w_+^2}\Big] \end{aligned} \tag{5.16c}$$

$$\begin{aligned} m_4^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4+\lambda_5)(v_1^2+w_+^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4+\lambda_5)(v_1^2 - w_+^2)\Big)^2 + 4(\lambda_4+\lambda_5)^2 v_1^2 w_+^2}\Big] \end{aligned} \tag{5.16d}$$

$$\begin{aligned} m_5^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4-\lambda_5)(v_1^2+w_+^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4-\lambda_5)(v_1^2 - w_+^2)\Big)^2 + 4(\lambda_4-\lambda_5)^2 v_1^2 w_+^2}\Big] \end{aligned} \tag{5.16e}$$

$$\begin{aligned} m_6^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - (\lambda_4-\lambda_5)(v_1^2+w_+^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + (\lambda_4-\lambda_5)(v_1^2 - w_+^2)\Big)^2 + 4(\lambda_4-\lambda_5)^2 v_1^2 w_+^2}\Big] \end{aligned} \tag{5.16f}$$

$$\begin{aligned} m_7^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 w_+^2) \\ &+ \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 w_+^2 - \lambda_1 v_1^2)\Big)^2 + 16\lambda_3^2 v_1^2 w_+^2}\Big] \end{aligned} \tag{5.16g}$$

$$\begin{aligned} m_8^2 = &-\frac{1}{4}\Big[\tilde{m}_{11}^2 + \tilde{m}_{22}^2 - 2(\lambda_1 v_1^2 + \lambda_2 w_+^2) \\ &- \sqrt{\Big(\tilde{m}_{11}^2 - \tilde{m}_{22}^2 + 2(\lambda_2 w_+^2 - \lambda_1 v_1^2)\Big)^2 + 16\lambda_3^2 v_1^2 w_+^2}\Big]. \end{aligned} \tag{5.16h}$$

The contributions from the scalars to the effective potential are

$$V_{11} = \frac{T^2}{12}\Big[3\lambda_1 + 2\lambda_3 + \lambda_4\Big] \tag{5.17a}$$

$$V_{22} = \frac{T^2}{12}\Big[3\lambda_2 + 2\lambda_3 + \lambda_4\Big] \tag{5.17b}$$

$$V_1 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\Big[3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.17c}$$

$$V_2 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\Big[3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.17d}$$

$$V_3 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\Big[(\lambda_1+\lambda_2)(3\lambda_3+\lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2\Big]. \tag{5.17e}$$

The first 4 coefficients are the same as in eqs. (5.9) and (5.13), as they should be.

5.1.4 Contribution to the effective potential

Clearly, the contribution to V_{11}, V_{22}, V_1, and V_2 from sections 5.1.1 to 5.1.3 should match. This requirement serves as a consistency check for our calculations. In order to find the contributions to V_3, V_4, and V_5, we have to solve the system of three equations and three unknowns, eqs. (5.9e), (5.13e) and (5.17e). To summarise, the contributions to the effective potential from the scalars are

$$V_{11} = \frac{T^2}{12}(3\lambda_1 + 2\lambda_3 + \lambda_4) \tag{5.18a}$$

$$V_{22} = \frac{T^2}{12}(3\lambda_2 + 2\lambda_3 + \lambda_4) \tag{5.18b}$$

$$V_1 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\left[3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\right] \tag{5.18c}$$

$$V_2 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\left[3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\right] \tag{5.18d}$$

$$V_3 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\left[(\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right] \tag{5.18e}$$

$$V_4 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\left[(\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4)\lambda_4 + 4\lambda_5^2\right] \tag{5.18f}$$

$$V_5 = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_b\right)\left[\lambda_5(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4)\right]. \tag{5.18g}$$

Notice the term $4\lambda_5^2$ in eq. (5.18f) and the overall factor in eq. (5.18g). The term $4\lambda_5^2$ will be present in both the β-function β_{λ_4} in eq. (5.39d) and in the three-dimensional scalar coupling Λ_4 in eq. (6.57). The overall factor in eq. (5.18g) will go through to the β-function β_{λ_5} in eq. (5.39e) and in the three-dimensional scalar coupling Λ_5 in eq. (6.58). We can compare our results for the β-functions with the results from the review article on the 2HDM [78], and conclude that our results agree. We can also compare our results for the three-dimensional couplings Λ_4 and Λ_5 with the results from a paper by Losada on dimensional reduction of the Minimal Supersymmetric Standard Model (MSSM) and other multiscalar models, therein the 2HDM [25]. Our results are very similar, except for the term $4\lambda_5^2$ and the overall factor in eq. (5.18g). The paper by Losada disagrees with our results by a factor of 2 in both cases. As our results agree with the review paper [78], and the factor trivially goes through for the calculation of the three-dimensional couplings, we conclude that the results found by Losada are incorrect. The paper on dimensional reduction of the 2HDM by Andersen [27] quotes the results found by Losada, and thus have the same incorrect expressions.

5.2 Gauge boson sector

The scalar fields are coupled to the gauge fields through the covariant derivative, $D_\mu\Phi_i^\dagger D_\mu\Phi_i$, where $i = 1, 2$. We proceed to shift the scalar field by some constant background fields, $\Phi_i \to \Phi_i + \varphi_i$, and drop terms linear in the fields A_μ^a, B_μ and Φ_i, due to the equations of motion for the background fields,

$$\begin{aligned} D_\mu\Phi_1^\dagger D_\mu\Phi_1 + D_\mu\Phi_2^\dagger D_\mu\Phi_2 \to& D_\mu\Phi_1^\dagger D_\mu\Phi_1 + D_\mu\Phi_2^\dagger D_\mu\Phi_2 \\ &+ \frac{ig}{2}\vec{A}_\mu\left[\varphi_1^\dagger\vec{\tau}\partial_\mu\Phi_1 - \partial_\mu\Phi_1^\dagger\vec{\tau}\varphi_1 + \varphi_2^\dagger\vec{\tau}\partial_\mu\Phi_2 - \partial_\mu\Phi_2^\dagger\vec{\tau}\varphi_2\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{ig'}{2}B_\mu\Big[\varphi_1^\dagger\partial_\mu\Phi_1-\partial_\mu\Phi_1^\dagger\varphi_1+\varphi_2^\dagger\partial_\mu\Phi_2-\partial_\mu\Phi_2^\dagger\varphi_2\Big]\\
&+\frac{1}{4}(\varphi_1^\dagger\varphi_1+\varphi_2^\dagger\varphi_2)\Big[g^2\vec{A}_\mu\vec{A}_\mu+g'^2B_\mu B_\mu\Big]\\
&+\frac{1}{2}gg'B_\mu\vec{A}_\mu\Big[\varphi_1^\dagger\vec{\tau}\varphi_1+\varphi_2^\dagger\vec{\tau}\varphi_2\Big].
\end{aligned}
$$

Notice the coupling between the gauge bosons and the scalars; a gauge boson can spontaneously turn into a scalar and vice versa. This is not desirable, and we remove it by introducing a gauge fixing term, using the Faddeev-Popov gauge-fixing procedure. We then choose the class of gauges called the R_ξ gauge. We must also include ghost fields, with new ghost interactions and ghost mass terms, all proportional to ξ. When we go to the Landau gauge, $\xi = 0$, the ghost interactions and masses vanish. As there is no bilinear mixing between the ghosts and the gauge bosons, we can safely go to the Landau gauge in the ghost sector. When calculating the sum-integral for the effective potential, eq. (2.95), we drop an infinite constant when using the Landau gauge.

We will focus on the part contributing to the mass of the gauge bosons,

$$
\frac{1}{4}(\varphi_1^\dagger\varphi_1+\varphi_2^\dagger\varphi_2)\Big[g^2\vec{A}_\mu\vec{A}_\mu+g'^2B_\mu B_\mu\Big]+\frac{1}{2}gg'B_\mu\vec{A}_\mu\Big[\varphi_1^\dagger\vec{\tau}\varphi_1+\varphi_2^\dagger\vec{\tau}\varphi_2\Big]. \tag{5.19}
$$

We will use the same three choices for the background fields, as in the preceding section.

5.2.1 Case 1

We will illustrate the method of finding the masses of the gauge bosons here in detail. Recall the first choice of the background fields, eq. (5.2). Equation (5.19) simplifies to

$$
\frac{1}{8}(v_1^2+v_2^2)\Big[g^2\vec{A}_\mu\vec{A}_\mu+g'^2B_\mu B_\mu\Big]-\frac{1}{4}(v_1^2+v_2^2)gg'B_\mu A_{3,\mu}. \tag{5.20}
$$

We see that $A_{1/2,\mu}$ have a mass of $M_W^2 = \frac{1}{4}g^2(v_1^2+v_2^2)$, while $A_{3,\mu}$ and B_μ are mixing. After finding the eigenvalues of the corresponding 2×2 matrix, we find one massless gauge boson (the photon), and one with mass $M_Z^2 = \frac{1}{4}(g^2+g'^2)(v_1^2+v_2^2)$. By the labelling of the masses we have preluded to identifying the gauge bosons with the massive $W^\pm$, Z^0 gauge bosons, and the photon γ. The gauge bosons contributing to the effective potential are $W^\pm$ and Z^0, with masses

$$
M_W^2 = \frac{1}{4}g^2(v_1^2+v_2^2) \tag{5.21}
$$

$$
M_Z^2 = \frac{1}{4}(g^2+g'^2)(v_1^2+v_2^2). \tag{5.22}
$$

The photon does not contribute to the effective potential, since it is massless. From eq. (2.95) the total contribution to the effective potential from the gauge sector is

$$
2\mathcal{C}_V(M_W)+\mathcal{C}_V(M_Z) = 2\mathcal{C}_V^{3d}(M_W)+\mathcal{C}_V^{3d}(M_Z)
$$

$$+\left[\frac{2M_W^2+M_Z^2}{8}T^2-\frac{2M_W^4+M_Z^4}{4(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)\right]+\mathcal{O}(M^6/T^2). \tag{5.23}$$

The contributions labelled by $\mathcal{C}_V^{3d}$ are the terms $\mathcal{O}(m^3T)$, and come from the zero modes, which are also present in the three-dimensional theory. As this contribution will cancel in the matching of the three- and four-dimensional theory in dimensional reduction, we omit it.

The expansion of the effective potential in terms of the background fields is the same as eq. (5.5). By identifying the different terms in the expansion, we find the contributions to the effective potential to be

$$V_{11}=\frac{1}{16}(3g^2+g'^2)T^2 \tag{5.24a}$$
$$V_{22}=\frac{1}{16}(3g^2+g'^2)T^2 \tag{5.24b}$$
$$V_1=-\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)(3g^4+g'^4+2g^2g'^2) \tag{5.24c}$$
$$V_2=-\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)(3g^4+g'^4+2g^2g'^2) \tag{5.24d}$$
$$V_3+V_4+V_5=-\frac{1}{8(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)(3g^4+g'^4+2g^2g'^2). \tag{5.24e}$$

The divergences will be absorbed by the counterterms, similar to section 5.1.1.

5.2.2 Case 2

Following the same procedure as in section 5.2.1, but with a different choice of background field, eq. (5.3), we arrive at the same result. We got two massive, charged gauge bosons, one massive, neutral gauge boson, and one massless, neutral gauge boson. The mass of the $W^\pm$ and Z^0 are

$$M_W^2=\frac{1}{4}g^2(v_1^2+w_0^2) \tag{5.25}$$
$$M_Z^2=\frac{1}{4}(g^2+g'^2)(v_1^2+w_0^2), \tag{5.26}$$

respectively. The contributions to the effective potential from the gauge sector are exactly the same as in section 5.2.1,

$$V_{11}=\frac{1}{16}(3g^2+g'^2)T^2 \tag{5.27a}$$
$$V_{22}=\frac{1}{16}(3g^2+g'^2)T^2 \tag{5.27b}$$
$$V_1=-\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)(3g^4+g'^4+2g^2g'^2) \tag{5.27c}$$
$$V_2=-\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon}+3L_b-2\Big)(3g^4+g'^4+2g^2g'^2) \tag{5.27d}$$

$$V_3 + V_4 - V_5 = -\frac{1}{8(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big)(3g^4 + g'^4 + 2g^2g'^2). \tag{5.27e}$$

As the eqs. (5.24) and (5.27) are exactly the same, we see that the contribution from the gauge bosons to V_5 vanishes.

5.2.3 Case 3

Lastly, with the choice of eq. (5.4) we get four massive gauge bosons. The mass spectrum is now

$$M_W^2 = \frac{1}{4}g^2(v_1^2 + w_+^2) \tag{5.28}$$

$$M_\pm^2 = \frac{1}{8}\Big[(g^2 + g'^2)(v_1^2 + w_+^2) \pm \sqrt{(g^2 - g'^2)^2(v_1^2 + w_+^2)^2 + 4g^2g'^2(v_1^2 - w_+^2)^2}\Big], \tag{5.29}$$

with the two usual charged gauge bosons $W^\pm$ having the same mass as in sections 5.2.1 and 5.2.2, while the masses of the two neutral gauge bosons mix. The total contribution to the effective potential from the gauge sector is

$$2\mathcal{C}_V(M_W) + \mathcal{C}_V(M_+) + \mathcal{C}_V(M_-) = 2\mathcal{C}_V^{3d}(M_W) + \mathcal{C}_V^{3d}(M_+) + \mathcal{C}_V^{3d}(M_-)$$
$$+ \Big[\frac{2M_W^2 + M_+^2 + M_-^2}{8}T^2 - \frac{2M_W^4 + M_+^4 + M_-^4}{4(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big)\Big] + \mathcal{O}(M^6/T^2). \tag{5.30}$$

Again, omitting the zero mode, we find the contributions to the expansion in eq. (5.14) to be

$$V_{11} = \frac{1}{16}(3g^2 + g'^2)T^2 \tag{5.31a}$$

$$V_{22} = \frac{1}{16}(3g^2 + g'^2)T^2 \tag{5.31b}$$

$$V_1 = -\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big)(3g^4 + g'^4 + 2g^2g'^2) \tag{5.31c}$$

$$V_2 = -\frac{1}{16(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big)(3g^4 + g'^4 + 2g^2g'^2) \tag{5.31d}$$

$$V_3 = -\frac{1}{8(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big)(3g^4 + g'^4 - 2g^2g'^2). \tag{5.31e}$$

We can find V_4 by using eqs. (5.24e), (5.27e) and (5.31e).

5.2.4 Contribution to the effective potential

We summarise the results from sections 5.2.1 to 5.2.3. Again, V_{11}, V_{22}, V_1, and V_2 should match for consistency. The different contributions to the effective potential from the gauge sector are

$$V_{11} = \frac{1}{16}(3g^2 + g'^2)T^2 \tag{5.32a}$$

$$V_{22} = \frac{1}{16}(3g^2 + g'^2)T^2 \tag{5.32b}$$

$$V_1 = -\frac{1}{16(4\pi)^2}\left[\frac{3}{\epsilon} + 3L_b - 2\right]\left(3g^4 + g'^4 + 2g^2g'^2\right) \tag{5.32c}$$

$$V_2 = -\frac{1}{16(4\pi)^2}\left[\frac{3}{\epsilon} + 3L_b - 2\right]\left(3g^4 + g'^4 + 2g^2g'^2\right) \tag{5.32d}$$

$$V_3 = -\frac{1}{8(4\pi)^2}\left[\frac{3}{\epsilon} + 3L_b - 2\right]\left(3g^4 + g'^4 - 2g^2g'^2\right) \tag{5.32e}$$

$$V_4 = -\frac{g^2g'^2}{2(4\pi)^2}\left[\frac{3}{\epsilon} + 3L_b - 2\right] \tag{5.32f}$$

$$V_5 = 0. \tag{5.32g}$$

The counterterms are discussed in section 5.4.

5.3 Fermions

The fermion fields are coupled to Φ_1 in the fermiophobic 2HDM through their mass term, $\mathcal{L}_{\text{Yukawa}}$, and are not affected by the choice of gauge. As the fermion fields only couple to the Φ_1 Higgs field, the parts affected by the fermion fields are V_{11} and V_1. Again, we shift the scalar fields by a background field. We do not need to distinguish the contributions to V_{3-5} as they are zero, and we simply use eq. (5.2). Using eq. (2.96), the contributions to the effective potential from the fermion sector are

$$V_{11} = \frac{T^2}{12}g_{Y,1}^2 \tag{5.33}$$

$$V_1 = \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + L_f\right)G_{Y,1}^4 \tag{5.34}$$

with the rest being zero. The definitions of $g_{Y,1}^2$, $G_{Y,1}^4$, and L_f are given in eqs. (3.11), (3.15) and (3.17).

5.4 Effective potential

We sum up the contributions to the effective potential from the scalars, gauge bosons, and fermions. The parts of the effective potential quadratic and quartic in the background fields take the form

$$\begin{aligned} V_{\text{eff}} =& V_{11}\varphi_1^\dagger\varphi_1 + V_{22}\varphi_2^\dagger\varphi_2 + V_1\left[\varphi_1^\dagger\varphi_1\right]^2 + V_2\left[\varphi_2^\dagger\varphi_2\right]^2 \\ &+ V_3\left[\varphi_1^\dagger\varphi_1\right]\left[\varphi_2^\dagger\varphi_2\right] + V_4\left[\varphi_1^\dagger\varphi_2\right]\left[\varphi_2^\dagger\varphi_1\right] + \frac{V_5}{2}\left[\left[\varphi_1^\dagger\varphi_2\right]^2 + \left[\varphi_2^\dagger\varphi_1\right]^2\right] \end{aligned} \tag{5.35}$$

where

$$V_{11} = \frac{T^2}{12}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_1 + 2\lambda_3 + \lambda_4 + g_{Y,1}^2\right] \tag{5.36a}$$

$$V_{22} = \frac{T^2}{12}\Big[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_2 + 2\lambda_3 + \lambda_4\Big] \tag{5.36b}$$

$$V_1 = -\frac{1}{16(4\pi)^2}\Big[\frac{3}{\epsilon} + 3L_b - 2\Big]\Big(3g^4 + g'^4 + 2g^2g'^2\Big) \\ -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] + \frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_f\Big)G_{Y,1}^4 \tag{5.36c}$$

$$V_2 = -\frac{1}{16(4\pi)^2}\Big[\frac{3}{\epsilon} + 3L_b - 2\Big]\Big(3g^4 + g'^4 + 2g^2g'^2\Big) \\ -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2\Big] \tag{5.36d}$$

$$V_3 = -\frac{1}{8(4\pi)^2}\Big[\frac{3}{\epsilon} + 3L_b - 2\Big]\Big(3g^4 + g'^4 - 2g^2g'^2\Big) \\ -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2\Big] \tag{5.36e}$$

$$V_4 = -\frac{g^2g'^2}{2(4\pi)^2}\Big(\frac{3}{\epsilon} + 3L_b - 2\Big) - \frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4)\lambda_4 + 4\lambda_5^2\Big] \tag{5.36f}$$

$$V_5 = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[\lambda_5(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4)\Big]. \tag{5.36g}$$

We can read off the counterterms for the scalar sector directly from the effective potential,

$$\delta\lambda_1 = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2g'^2 + 6\lambda_1^2 + \lambda_3^2 + 2\lambda_+^2 + 2\lambda_-^2 - 2G_{Y,1}^4\Big] \tag{5.37a}$$

$$\delta\lambda_2 = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2g'^2 + 6\lambda_2^2 + \lambda_3^2 + 2\lambda_+^2 + 2\lambda_-^2\Big] \tag{5.37b}$$

$$\delta\lambda_3 = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 - \frac{3}{4}g^2g'^2 + (\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2\Big] \tag{5.37c}$$

$$\delta\lambda_4 = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{3}{2}g^2g'^2 + (\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4)\lambda_4 + 4\lambda_5^2\Big] \tag{5.37d}$$

$$\delta\lambda_5 = \frac{1}{(4\pi)^2\epsilon}\Big[\lambda_5(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4)\Big]. \tag{5.37e}$$

The definitions of the counterterms are given in section 3.1.1.

5.5 β-functions

We are now in a position to calculate the β-functions for the scalar couplings. We follow the procedure of section 2.2. First we calculate the quantities f_{λ_i},

$$f_{\lambda_1} = \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta\lambda_1 - \frac{\lambda_1}{2}\Big[4\delta Z_{\phi_1}\Big]\Big\} \\ = \frac{1}{(4\pi)^2}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2g'^2 + 6\lambda_1^2 + \lambda_3^2 + 2\lambda_+^2 + 2\lambda_-^2$$

$$
\begin{aligned}
&- 2G_{Y,1}^4 - \frac{3}{2}\lambda_1(3g^2 + g'^2) + 2\lambda_1 g_{Y,1}^2\Big] && (5.38a)\\
f_{\lambda_2} =& \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta\lambda_2 - \frac{\lambda_2}{2}\Big[4\delta Z_{\phi_2}\Big]\Big\}\\
=& \frac{1}{(4\pi)^2}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2 g'^2 + 6\lambda_2^2 + \lambda_3^2 + 2\lambda_+^2 + 2\lambda_-^2 - \frac{3}{2}\lambda_2(3g^2 + g'^2)\Big] && (5.38b)\\
f_{\lambda_3} =& \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta\lambda_3 - \frac{\lambda_3}{2}\Big[2\delta Z_{\phi_1} + 2\delta Z_{\phi_2}\Big]\Big\}\\
=& \frac{1}{(4\pi)^2}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 - \frac{3}{4}g^2 g'^2 + (\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2\\
&- \frac{3}{2}\lambda_3(3g^2 + g'^2) + \lambda_3 g_{Y,1}^2\Big] && (5.38c)\\
f_{\lambda_4} =& \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta\lambda_4 - \frac{\lambda_4}{2}\Big[2\delta Z_{\phi_1} + 2\delta Z_{\phi_2}\Big]\Big\}\\
=& \frac{1}{(4\pi)^2}\Big[\frac{3}{2}g^2 g'^2 + \lambda_4(\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4) + 4\lambda_5^2 - \frac{3}{2}\lambda_4(3g^2 + g'^2) + \lambda_4 g_{Y,1}^2\Big] && (5.38d)\\
f_{\lambda_5} =& \text{Residue of simple } \frac{1}{\epsilon} \text{ of } \Big\{\delta\lambda_5 - \frac{\lambda_5}{2}\Big[2\delta Z_{\phi_1} + 2\delta Z_{\phi_2}\Big]\Big\}\\
=& \frac{1}{(4\pi)^2}\Big[\lambda_5(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4) - \frac{3}{2}\lambda_5(3g^2 + g'^2) + \lambda_5 g_{Y,1}^2\Big]. && (5.38e)
\end{aligned}
$$

In the limit $d \to 3$, we have that

$$
\begin{aligned}
\beta_{\lambda_1} =& \frac{1}{(4\pi)^2}\Big[\frac{9}{4}g^4 + \frac{3}{4}g'^4 + \frac{3}{2}g^2 g'^2 + 12\lambda_1^2 + 2\lambda_3^2 + 4\lambda_+^2 + 4\lambda_-^2\\
&- 4G_{Y,1}^4 - 3\lambda_1(3g^2 + g'^2) + 4\lambda_1 g_{Y,1}^2\Big] && (5.39a)\\
\beta_{\lambda_2} =& \frac{1}{(4\pi)^2}\Big[\frac{9}{4}g^4 + \frac{3}{4}g'^4 + \frac{3}{2}g^2 g'^2 + 12\lambda_2^2 + 2\lambda_3^2 + 4\lambda_+^2 + 4\lambda_-^2 - 3\lambda_2(3g^2 + g'^2)\Big] && (5.39b)\\
\beta_{\lambda_3} =& \frac{1}{(4\pi)^2}\Big[\frac{9}{4}g^4 + \frac{3}{4}g'^4 - \frac{3}{2}g^2 g'^2 + 2(\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2(2\lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_3 g_{Y,1}^2)\\
&- 3\lambda_3(3g^2 + g'^2)\Big] && (5.39c)\\
\beta_{\lambda_4} =& \frac{1}{(4\pi)^2}\Big[3g^2 g'^2 + 2\lambda_4(\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4) + 8\lambda_5^2 - 3\lambda_4(3g^2 + g'^2) + 2\lambda_4 g_{Y,1}^2\Big]\\
& && (5.39d)\\
\beta_{\lambda_5} =& \frac{1}{(4\pi)^2}\Big[2\lambda_5(\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4) - 3\lambda_5(3g^2 + g'^2) + 2\lambda_5 g_{Y,1}^2\Big]. && (5.39e)
\end{aligned}
$$

In the review article on the 2HDM [78], the β-functions for the scalar couplings are listed, in agreement with our results.

CHAPTER 6

DIMENSIONAL REDUCTION

In the high temperature limit, a four-dimensional theory can be described by an effective three-dimensional theory, with all non-static modes integrated out. This method is called dimensional reduction [28]. We will follow the program of finding the effective three-dimensional theories of simple extensions of the SM, outlined in ref. [26].

We recall our discussion of a theory at finite temperature, where the action takes the form

$$S = \int_0^\beta d\tau \int d^3x \mathcal{L}, \tag{6.1}$$

where $\mathcal{L}$ is the Lagrangian and $\beta = 1/T$ is the inverse temperature. In the imaginary-time formalism (see section 2.1.2) the bosonic (fermionic) fields obey periodic (antiperiodic) boundary conditions in the Euclidean time direction, which leads to the field expansions (eqs. (2.23) and (2.24))

$$\Phi(x,\tau) = \sqrt{T} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{i\omega_n^B \tau} \tag{6.2}$$

$$\Psi(x,\tau) = \sqrt{T} \sum_{n=-\infty}^{\infty} \psi_n(x) e^{i\omega_n^F \tau}, \tag{6.3}$$

where Φ and Ψ are the bosonic and fermionic fields, respectively. The respective Matsubara frequencies are $\omega_n^B = 2n\pi T$ and $\omega_n^F = (2n+1)\pi T$, where $n \in \mathbb{Z}$. All non-static modes, i.e. all ψ_n and ϕ_n except ϕ_0 receive a thermal mass of order πT, and, in the high-temperature limit, may be integrated out. We are left with an effective three-dimensional theory with static bosonic modes. We can perform the dimensional reduction in two steps, integrating first out the superheavy modes, then integrating out the heavy modes. We will only do the first step, of integrating out the superheavy modes, and let the second step be a part of the outlook.

6.1 Integrating out the superheavy modes

We will focus on a model where both scalar Higgs doublets are either heavy or light. All the fermionic and non-zero bosonic modes are integrated out first.

We write down the most general three-dimensional theory, which respects the underlying symmetries. The Lagrangian takes the form

$$\mathcal{L}^{(3)} = \mathcal{L}^{(3)}_{\text{scalar}} + \mathcal{L}^{(3)}_{\text{spatial}} + \mathcal{L}^{(3)}_{\text{temporal}} + \mathcal{L}^{(3)}_{\text{ghost}} + \delta\mathcal{L}^{(3)}. \tag{6.4}$$

We include only the $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$ gauge fields, i.e. we ignore the SU(3) gauge fields as they decouple from the second Higgs doublet, Φ_2, and thus behave exactly the same way as in the SM at one-loop. Thus, the gauge sector of the Lagrangian reads

$$\mathcal{L}^{(3)}_{\text{spatial}} = \frac{1}{4}G^a_{rs}G^a_{rs} + \frac{1}{4}F_{rs}F_{rs}, \qquad r,s = 1,\ldots,3. \tag{6.5}$$

The couplings are denoted by g'_3 and g_3 for the $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$, respectively. The Lorentz symmetry has been broken by the heat bath, but the smaller O(3) symmetry of the spatial gauge fields is preserved. We will not explicitly consider $\mathcal{L}^{(3)}_{\text{ghost}}$ or $\delta\mathcal{L}^{(3)}$, as they are irrelevant for the discussion below.

The scalar part preserves the structure of the full four-dimensional theory, with some minor modifications,

$$\begin{aligned}\mathcal{L}^{(3)}_{\text{scalar}} = {} & D_r\Phi_1^\dagger D_r\Phi_1 + D_r\Phi_2^\dagger D_r\Phi_2 - \frac{1}{2}\mu_1^2\Phi_1^\dagger\Phi_1 - \frac{1}{2}\mu_2^2\Phi_2^\dagger\Phi_2 + \frac{\Lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2 + \frac{\Lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2 \\ & + \Lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \Lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{\Lambda_5}{2}\Big[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\Big].\end{aligned} \tag{6.6}$$

Notice that the covariant derivative has only a spatial index. As for the temporal part of the gauge fields, we have that

$$\begin{aligned}\mathcal{L}^{(3)}_{\text{temporal}} = {} & \frac{1}{2}(D_rA_0^a)^2 + \frac{1}{2}(\partial_rB_0)^2 + \frac{1}{2}m_D^2(A_0^a)^2 + \frac{1}{2}m_D'^2B_0^2 \\ & + \frac{1}{4}\kappa_1(A_0^a)^4 + \frac{1}{4}\kappa_2B_0^4 + \frac{1}{4}\kappa_3(A_0^a)^2B_0^2 + h_1\Phi_1^\dagger\Phi_1(A_0^a)^2 + h_2\Phi_2^\dagger\Phi_2(A_0^a)^2 \\ & + h_3\Phi_1^\dagger\Phi_1B_0^2 + h_4\Phi_2^\dagger\Phi_2B_0^2 + h_5B_0\Phi_1^\dagger\vec{A}_0\cdot\vec{\tau}\Phi_1 + h_6B_0\Phi_2^\dagger\vec{A}_0\cdot\vec{\tau}\Phi_2,\end{aligned} \tag{6.7}$$

where the covariant derivative of the adjoint temporal gauge field is $D_rA_0^a = \partial_rA_0^a + g\epsilon^{abc}A_r^bA_0^c$. We have maintained the same notation for the fields for simplicity. Our goal is to determine the parameters of the effective three-dimensional theory in terms of the parameters of the original four-dimensional theory.

6.1.1 Thermal masses

Firstly, we determine the thermal masses of the temporal component of the gauge bosons. The mass parameters of the temporal gauge fields at leading order are simply the static limit of the two-point correlators, found in eqs. (4.1) and (4.5),

$$
\begin{aligned}
m_D'^2 &= g'^2(d-1)\Big[N_h + \frac{1}{2}(1-2^{2-d})\sum_f Y_f^2\Big] I_1^{4b} \\
&= g'^2\Big[\frac{N_h}{6} + \frac{N_f}{4}\Big(1 + N_c\frac{11}{27}\Big)\Big]T^2 = 2g'^2T^2 \qquad (6.8)\\
m_D^2 &= g^2(d-1)\Big[2(d-1) + N_h + (1-2^{2-d})\sum_{\text{left}}\Big] I_1^{4b} \\
&= g^2\Big[\frac{2}{3} + \frac{N_h}{6} + \frac{1}{12}N_f(1+N_c)\Big]T^2 = 2g^2T^2, \qquad (6.9)
\end{aligned}
$$

where N_f is the number of lepton families, N_h is the number of Higgs doublets, and N_c is the number of colours. In the last step in eqs. (6.8) and (6.9) we used the 2HDM specifics in eq. (3.18) and let $d \to 3$. For the three-dimensional scalar fields, the squared masses receive a thermal contribution

$$
\mu_1^2 = m_{11}^2 - \frac{T^2}{6}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_1 + 2\lambda_3 + \lambda_4 + g_{Y,1}^2\right] \qquad (6.10)
$$

$$
\mu_2^2 = m_{22}^2 - \frac{T^2}{6}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_2 + 2\lambda_3 + \lambda_4\right], \qquad (6.11)
$$

where we used the effective potential in eqs. (5.36a) and (5.36b). Note that both the temporal gauge fields and the scalars receive a thermal mass of order $\mathcal{O}(gT)$, in agreement with our discussion on thermal masses from section 2.1.3.

6.1.2 Field renormalisation

Secondly, we relate the three-dimensional fields to their four-dimensional counterparts, denoted by the lower indices 3d and 4d, respectively. We have the general relation [28]

$$
\Psi_{3\text{d}}^2 = \frac{\Psi_{4\text{d}}^2}{T}[1 + \Pi'_\Psi(0) + \delta Z_\Psi], \qquad (6.12)
$$

where $\Pi_\Psi(K)$ is the self-energy of the field Ψ, and the prime denotes a derivative with respect to K^2. δZ_Ψ is the field renormalisation counterterm. For the $\text{U}(1)_Y$ gauge field we use eqs. (4.1), (4.3) and (6.12)

$$
B_{3\text{d},0}^2 = \frac{B_{4\text{d},0}^2}{T}\left\{1 + \frac{g'^2}{(4\pi)^2}\left[\frac{1}{6}N_h(L_b+2) + \frac{20}{9}N_f(L_f-1)\right]\right\} \qquad (6.13)
$$

$$
B_{3\text{d},i}^2 = \frac{B_{4\text{d},i}^2}{T}\left[1 + \frac{g'^2}{(4\pi)^2}\left(\frac{1}{6}N_hL_b + \frac{20}{9}N_fL_f\right)\right], \qquad (6.14)
$$

where L_b and L_f are defined in eqs. (3.16) and (3.17), respectively. Note that the divergences from the self-energy and the field renormalisation counterterm must cancel. This is a non-trivial check of the correctness of our calculations.

As for the $\text{SU}(2)_L$ gauge field, we use eqs. (4.5), (4.7) and (6.12)

$$
A_{3\text{d},0}^2 = \frac{A_{4\text{d},0}^2}{T}\left\{1 + \frac{g^2}{(4\pi)^2}\left[-\frac{26-N_h}{6}L_b + \frac{8+N_h}{3} + \frac{4}{3}N_f(L_f-1)\right]\right\} \qquad (6.15)
$$

$$A_{3\text{d},i}^2 = \frac{A_{4\text{d},i}^2}{T}\Big[1 + \frac{g^2}{(4\pi)^2}\Big(- \frac{26 - N_h}{6}L_b - \frac{2}{3} + \frac{4}{3}N_f L_f\Big)\Big]. \tag{6.16}$$

The Higgs fields can be found in a similar fashion, using eqs. (4.11) to (4.14) and (6.12),

$$(\Phi_1^\dagger\Phi_1)_{3\text{d}} = \frac{(\Phi_1^\dagger\Phi_1)_{4\text{d}}}{T}\Big[1 - \frac{1}{(4\pi)^2}\Big(\frac{9}{4}g^2 + \frac{3}{4}g'^2\Big)L_b + \frac{g_{Y,1}^2}{(4\pi)^2}L_f\Big] \tag{6.17}$$

$$(\Phi_2^\dagger\Phi_2)_{3\text{d}} = \frac{(\Phi_2^\dagger\Phi_2)_{4\text{d}}}{T}\Big[1 - \frac{1}{(4\pi)^2}\Big(\frac{9}{4}g^2 + \frac{3}{4}g'^2\Big)L_b\Big], \tag{6.18}$$

where $g_{Y,1}^2$ is defined in eq. (3.11).

6.1.3 Couplings

The matching prescription is that the correlators of the three- and four-dimensional theories should be the same at zero external momentum.

The gauge couplings

We will determine the gauge couplings g_3 and g_3' by matching the correlators computed in the four-dimensional and three-dimensional theory. We will use the correlators of the bare four-dimensional fields calculated in section 4.2 and the wave function renormalisation and coupling counterterms from section 4.4.

The $SU(2)_L$ coupling g_3 will be found by using the correlator with two scalar legs and two spatial gauge field legs. The three-dimensional and four-dimensional theories are matched by equating

$$\begin{aligned}\Phi_{3\text{d},1}^{\dagger,i}\Phi_{3\text{d},1}^{j}A_{3\text{d},r}^{a}A_{3\text{d},s}^{b}\Big(- \frac{1}{2}g_3^2\delta_{ij}\delta_{ab}\delta_{rs}\Big)&\\ =\frac{1}{T}\Phi_{4\text{d},1(b)}^{\dagger,i}\Phi_{4\text{d},1(b)}^{j}A_{4\text{d},r(b)}^{a}A_{4\text{d},s(b)}^{b}\delta_{ij}\delta_{ab}\delta_{rs}\Big[- \frac{1}{2}(g^2 + \delta g^2)&\\ + \Big(- \frac{3}{8}g^4 + \frac{3}{8}g^2g'^2 - \frac{1}{2}(2^{4-d} - 1)g^2g_{Y,1}^2\Big)I_2^{4b}\Big]&,\end{aligned} \tag{6.19}$$

where $\delta g^2 = 2g\delta g$. After some algebra we find the three-dimensional coupling to be

$$g_3^2 = g^2(\mu)T\Big[1 + \frac{g^2}{(4\pi)^2}\Big(\frac{44 - N_h}{6}L_b + \frac{2}{3} - \frac{4N_f}{3}L_f\Big)\Big]. \tag{6.20}$$

We have indicated that the four-dimensional coupling, but not the three-dimensional coupling, is dependent on the renormalisation scale μ. We can use the renormalisation group equation, eq. (2.43), to verify that the three-dimensional coupling g_3 is independent of μ. All other three-dimensional couplings discussed below follow the same pattern. We can use the criterion that the three-dimensional couplings should be independent of the renormalisation scale as another non-trivial check of the correctness of our calculations.

To find the three-dimensional coupling g_3', we use a similar procedure, by using the correlator with two scalar legs and two spatial gauge field legs. We match the two theories by equating

$$\Phi^{\dagger,i}_{3\mathrm{d},2}\Phi^{j}_{3\mathrm{d},2}B_{3\mathrm{d},r}B_{3\mathrm{d},s}\Big(-\frac{1}{2}g'^2_3\delta_{ij}\delta_{rs}\Big)$$
$$=\frac{1}{T}\Phi^{\dagger,i}_{4\mathrm{d},2(b)}\Phi^{j}_{4\mathrm{d},2(b)}B_{4\mathrm{d},r(b)}B_{4\mathrm{d},s(b)}\delta_{ij}\delta_{rs}\Big[-\frac{1}{2}(g'^2+\delta g'^2)+\Big(\frac{3}{8}g'^4+\frac{9}{8}g^2g'^2\Big)I^{4b}_2\Big]. \quad (6.21)$$

It does not matter if we use the correlator with Φ_1 or Φ_2 as the external scalar legs, the result should be the same. This is a check of consistency of the results. The final result for the coupling g'_3 is

$$g'^2_3=g'^2(\mu)T\Big[1-\frac{g'^2}{(4\pi)^2}\Big(\frac{1}{6}N_hL_b+\frac{20}{9}N_fL_f\Big)\Big]. \quad (6.22)$$

The couplings g_3 and g'_3 could also have been found using the four-point correlators of the gauge fields, yielding the same result. However, the calculation is sightly more involved.

Matching of the temporal gauge field self-couplings

The coupling constants for the temporal part of the gauge fields are found by matching the four-point functions of the four-dimensional and three-dimensional theories at zero external momentum. As the self-coupling of the temporal gauge fields is prohibited in the four-dimensional theory at tree-level, the corrections will come at order $\mathcal{O}(g^4)$, and therefore the wave function renormalisation does not contribute. For the $\mathrm{U}(1)_Y$ gauge field we have that

$$-6\kappa_2(B_{3\mathrm{d},0})^4=\frac{1}{T}B_0^4\frac{1}{2}(d-1)(d-3)\Big[N_h+\frac{1}{2}(1-2^{4-d})\sum_f Y_f^4\Big]g'^4I^{4b}_2. \quad (6.23)$$

Thus, we find that

$$\kappa_2=T\frac{g'^4}{(4\pi)^2}\Big[\frac{N_h}{3}-\frac{1}{6}\sum_f Y_f^4\Big]=T\frac{g'^4}{(4\pi)^2}\Big[\frac{N_h}{3}-\frac{380}{81}N_f\Big]. \quad (6.24)$$

Similarly, we find that

$$\kappa_1=T\frac{g^4}{(4\pi)^2}\Big[\frac{16+N_h-4N_f}{3}\Big] \quad (6.25)$$
$$\kappa_3=T\frac{g^2g'^2}{(4\pi)^2}\Big[2N_h-2N_f(Y_l^2+N_cY_q^2)\Big]. \quad (6.26)$$

We see that the couplings κ_1, κ_2 and κ_3 are automatically independent of the renormalisation scale up to the order of accuracy of our calculations.

Matching the Higgs-gauge field couplings

We use the correlators with two scalar legs and two temporal gauge field legs to determine the the Higgs - gauge field couplings,

$$
\begin{aligned}
\Phi^{\dagger,i}_{3\text{d},1}\Phi^{j}_{3\text{d},1}A^{a}_{3\text{d},0}A^{b}_{3\text{d},0}\Big(-2h_1\delta_{ij}\delta_{ab}\Big)& \\
=\frac{1}{T}\Phi^{\dagger,i}_{4\text{d},1(b)}\Phi^{j}_{4\text{d},1(b)}A^{a}_{4\text{d},0(b)}A^{b}_{4\text{d},0(b)}\delta_{ij}\delta_{ab}\Big[&-\frac{1}{2}(g^2+\delta g^2) \\
&+g^2\Big(\Big(-\frac{25}{8}+d\Big)dg^2-(3-d)\Big(\frac{3}{2}\lambda_1+\lambda_3+\frac{1}{2}\lambda_4\Big)+\frac{1}{8}dg'^2 \\
&-\frac{1}{2}(2^{4-d}-1)(d-2)g^2_{Y,1}\Big)I^{4b}_2\Big],
\end{aligned} \tag{6.27}
$$

where $g^2_{Y,1}$ is defined in eq. (3.11). The three-dimensional coupling becomes

$$
\begin{aligned}
h_1=T\Big\{\frac{g^2(\mu)}{4}+\frac{g^2}{4(4\pi)^2}\Big[\Big(\frac{44-N_h}{6}L_b+\frac{53-2N_h}{6}-\frac{4}{3}N_f(L_f-1)\Big)g^2+\frac{1}{2}g'^2-2g^2_{Y,1} \\
+6\lambda_1+4\lambda_3+2\lambda_4\Big]\Big\}.
\end{aligned} \tag{6.28}
$$

The correlator with two scalar Φ_2 legs and two $\mathrm{SU}(2)_L$ legs takes a very similar form,

$$
\begin{aligned}
h_2=T\Big\{\frac{g^2(\mu)}{4}+\frac{g^2}{4(4\pi)^2}\Big[\Big(\frac{44-N_h}{6}L_b+\frac{53-2N_h}{6}-\frac{4}{3}N_f(L_f-1)\Big)g^2+\frac{1}{2}g'^2 \\
+6\lambda_2+4\lambda_3+2\lambda_4\Big]\Big\}.
\end{aligned} \tag{6.29}
$$

Similarly, the couplings h_3 and h_4 are found by using the correlator with two $\mathrm{U}(1)_Y$ gauge boson field legs and two scalar legs, Φ_1 and Φ_2, respectively,

$$
\begin{aligned}
h_3=&T\Big\{\frac{g'^2(\mu)}{4}-\frac{g'^2}{4(4\pi)^2}\Big[\Big(\frac{1}{6}N_hL_b+\frac{2N_h-3}{6}+\frac{20}{9}N_f(L_f-1)\Big)g'^2-\frac{3}{2}g^2+2g^2_{Y,2} \\
&-6\lambda_1-4\lambda_3-2\lambda_4\Big]\Big\}
\end{aligned} \tag{6.30}
$$

$$
\begin{aligned}
h_4=&T\Big\{\frac{g'^2(\mu)}{4}-\frac{g'^2}{4(4\pi)^2}\Big[\Big(\frac{1}{6}N_hL_b+\frac{2N_h-3}{6}+\frac{20}{9}N_f(L_f-1)\Big)g'^2-\frac{3}{2}g^2 \\
&-6\lambda_2-4\lambda_3-2\lambda_4\Big]\Big\}
\end{aligned} \tag{6.31}
$$

where we have defined $g^2_{Y,2}$ in eq. (3.12). The last scalar-gauge field couplings are h_5 and h_6, which can be extracted from the correlator with two scalar legs, Φ_1 and Φ_2, respectively, and two different gauge field legs,

$$
\begin{aligned}
h_5=&T\Big\{\frac{g(\mu)g'(\mu)}{2}+\frac{gg'}{2(4\pi)^2}\Big[2(\lambda_1+\lambda_4)+g^2\Big(\frac{44-N_h}{12}L_b-\frac{5+N_h}{6}-\frac{2}{3}N_f(L_f-1)\Big) \\
&+g'^2\Big(-\frac{1}{12}N_hL_b+\frac{3-N_h}{6}-\frac{10}{9}N_f(L_f-1)\Big)+2g^2_{Y,3}\Big]\Big\}
\end{aligned} \tag{6.32}
$$

$$h_6 = T\Big\{\frac{g(\mu)g'(\mu)}{2} + \frac{gg'}{2(4\pi)^2}\Big[2(\lambda_2+\lambda_4) + g^2\Big(\frac{44-N_h}{12}L_b - \frac{5+N_h}{6} - \frac{2}{3}N_f(L_f-1)\Big) + g'^2\Big(-\frac{1}{12}N_hL_b + \frac{3-N_h}{6} - \frac{10}{9}N_f(L_f-1)\Big)\Big]\Big\} \tag{6.33}$$

where $g^2_{Y,3}$ is defined in eq. (3.13). All the three-dimensional couplings should be independent of the renormalisation scale.

Matching of the scalar couplings

In a similar fashion, we find the scalar coupling constants in the three-dimensional theory by using the scalar correlators at zero external momentum found from the effective potential. The three-dimensional couplings take the form

$$\Lambda_1 = T\Big[\lambda_1(\mu) + \frac{1}{4(4\pi)^2}\Big(3g^4 + g'^4 + 2g^2g'^2 + 8L_f(G^4_{Y,1} - \lambda_1 g^2_{Y,1})\Big) - \frac{2L_b}{(4\pi)^2}\Big(\frac{9}{16}g^4 + \frac{3}{16}g'^4 + \frac{3}{8}g^2g'^2 + 3\lambda_1^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2 - \frac{3}{4}\lambda_1(3g^2+g'^2)\Big)\Big] \tag{6.34a}$$

$$\Lambda_2 = T\Big[\lambda_2(\mu) + \frac{1}{4(4\pi)^2}\Big(3g^4 + g'^4 + 2g^2g'^2\Big) - \frac{2L_b}{(4\pi)^2}\Big(\frac{9}{16}g^4 + \frac{3}{16}g'^4 + \frac{3}{8}g^2g'^2 + 3\lambda_2^2 + \frac{1}{2}\lambda_3^2 + \lambda_+^2 + \lambda_-^2 - \frac{3}{4}\lambda_2(3g^2+g'^2)\Big)\Big] \tag{6.34b}$$

$$\Lambda_3 = T\Big[\lambda_3(\mu) + \frac{1}{4(4\pi)^2}\Big(3g^4 + g'^4 - 2g^2g'^2 - 4L_f\lambda_3 g^2_{Y,1}\Big) - \frac{L_b}{(4\pi)^2}\Big(\frac{9}{8}g^4 + \frac{3}{8}g'^4 - \frac{3}{4}g^2g'^2 + (\lambda_1+\lambda_2)(3\lambda_3+\lambda_4) + 2\lambda_3^2 + \lambda_4^2 + \lambda_5^2 - \frac{3}{2}\lambda_3(3g^2+g'^2)\Big)\Big] \tag{6.34c}$$

$$\Lambda_4 = T\Big[\lambda_4(\mu) + \frac{1}{(4\pi)^2}\Big(g^2g'^2 - L_f\lambda_4 g^2_{Y,1} - L_b\Big(\frac{3}{2}g^2g'^2 + (\lambda_1+\lambda_2+4\lambda_3+2\lambda_4)\lambda_4 + 4\lambda_5^2 - \frac{3}{2}\lambda_4(3g^2+g'^2)\Big)\Big)\Big] \tag{6.34d}$$

$$\Lambda_5 = T\Big[\lambda_5(\mu) - \frac{1}{(4\pi)^2}\Big(L_f\lambda_5 g^2_{Y,1} + L_b\Big(\lambda_5(\lambda_1+\lambda_2+4\lambda_3+6\lambda_4) - \frac{3}{2}L_b\lambda_5(3g^2+g'^2)\Big)\Big)\Big] \tag{6.34e}$$

where we used eqs. (5.36c) to (5.36g) and the field renormalisations eqs. (4.13) and (4.14). We have checked that the three-dimensional couplings are independent of the renormalisation scale.

Mass-mixing

We now include the mass-mixing term in the Lagrangian which softly breaks the Z_2 symmetry,

$$\mathcal{L}_{\rm mix} = -\frac{1}{2}(m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}), \tag{6.35}$$

and treat it as a perturbation. All correlators should be re-evaluated to include contributions from the mass-mixing term. However, at one-loop only the $\Phi_1^\dagger\Phi_2$ and its Hermitian conjugate correlator are affected by the mass-mixing term. All other correlators are affected by the mass-mixing terms at higher orders.

We match the three- and four-dimensional two-point correlators at vanishing external momentum,

$$\left(-\frac{1}{2}\mu_{12}^2\right)(\Phi_1^\dagger\Phi_2)_{3d} = \frac{1}{T}\left(-\frac{1}{2}m_{12}^2 - \frac{1}{2}\delta m_{12}^2 - V_{12}\right)(\Phi_{1(b)}^\dagger\Phi_{2(b)}), \tag{6.36}$$

where V_{12} is the the correlator calculated in eq. (4.15). We use the mass-mixing counterterm from eq. (4.16), the correlator from eq. (4.15) and field renormalisation counterterms from eqs. (4.13) and (4.14), and we end up with

$$\mu_{12}^2 = m_{12}^2 - \left[(\lambda_3+2\lambda_4)m_{12}^2 + 3\lambda_5 m_{12}^{*2} - \left(\frac{9}{4}g^2+\frac{3}{4}g'^2\right)m_{12}^2\right]\frac{L_b}{(4\pi)^2} - \frac{g_{Y,1}^2 m_{12}^2}{2(4\pi)^2}L_f. \tag{6.37}$$

The three-dimensional mass-mixing term should be independent of the renormalisation scale. This requirement serves as an independent check for the correctness of our calculations.

6.1.4 Summary of one-loop matching relations

We here summarise the one-loop matching relations previously obtained, for the reader's convenience. The number of fermion families N_f and the number of Higgs doublets N_h are kept unspecified wherever applicable.

$$m_D^2 = g^2\left[\frac{2}{3} + \frac{1}{6}N_h + \frac{1}{3}N_f\right]T^2 \tag{6.38}$$

$$m_D'^2 = g'^2\left[\frac{1}{6}N_h + \frac{5}{9}N_f\right]T^2 \tag{6.39}$$

$$\mu_1^2 = m_{11}^2 - \frac{T^2}{6}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_1 + 2\lambda_3 + \lambda_4 + g_{Y,1}^2\right] \tag{6.40}$$

$$\mu_2^2 = m_{22}^2 - \frac{T^2}{6}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_2 + 2\lambda_3 + \lambda_4\right] \tag{6.41}$$

$$\mu_{12}^2 = m_{12}^2 - \left[(\lambda_3+2\lambda_4)m_{12}^2 + 3\lambda_5 m_{12}^{*2} - \left(\frac{9}{4}g^2+\frac{3}{4}g'^2\right)m_{12}^2\right]\frac{L_b}{(4\pi)^2} - \frac{g_{Y,1}^2 m_{12}^2}{2(4\pi)^2}L_f \tag{6.42}$$

$$g_3^2 = T\left[g^2(\mu) + \frac{g^4}{(4\pi)^2}\left(\frac{44-N_h}{6}L_b + \frac{2}{3} - \frac{4N_f}{3}L_f\right)\right] \tag{6.43}$$

$$g_3'^2 = T\left[g'^2(\mu) - \frac{g'^4}{(4\pi)^2}\left(\frac{1}{6}N_h L_b + \frac{20}{9}N_f L_f\right)\right] \tag{6.44}$$

$$\kappa_1 =T\frac{g^4}{(4\pi)^2}\Big[\frac{16}{3}+\frac{1}{3}N_h-\frac{4}{3}N_f\Big] \tag{6.45}$$

$$\kappa_2 =T\frac{g'^4}{(4\pi)^2}\Big[\frac{1}{3}N_h-\frac{380}{81}N_f\Big] \tag{6.46}$$

$$\kappa_3 =T\frac{g^2g'^2}{(4\pi)^2}\Big[2N_h-\frac{8}{3}N_f\Big] \tag{6.47}$$

$$h_1 =T\Big\{\frac{g^2(\mu)}{4}+\frac{g^2}{4(4\pi)^2}\Big[\Big(\frac{44-N_h}{6}L_b+\frac{53-2N_h}{6}-\frac{4}{3}N_f(L_f-1)\Big)g^2+\frac{1}{2}g'^2-2g_{Y,1}^2 + 6\lambda_1+4\lambda_3+2\lambda_4\Big]\Big\} \tag{6.48}$$

$$h_2 =T\Big\{\frac{g^2(\mu)}{4}+\frac{g^2}{4(4\pi)^2}\Big[\Big(\frac{44-N_h}{6}L_b+\frac{53-2N_h}{6}-\frac{4}{3}N_f(L_f-1)\Big)g^2+\frac{1}{2}g'^2 + 6\lambda_2+4\lambda_3+2\lambda_4\Big]\Big\} \tag{6.49}$$

$$h_3 =T\Big\{\frac{g'^2(\mu)}{4}-\frac{g'^2}{4(4\pi)^2}\Big[\Big(\frac{1}{6}N_hL_b+\frac{2N_h-3}{6}+\frac{20}{9}N_f(L_f-1)\Big)g'^2-\frac{3}{2}g^2+2g_{Y,2}^2 - 6\lambda_1-4\lambda_3-2\lambda_4\Big]\Big\} \tag{6.50}$$

$$h_4 =T\Big\{\frac{g'^2(\mu)}{4}-\frac{g'^2}{4(4\pi)^2}\Big[\Big(\frac{1}{6}N_hL_b+\frac{2N_h-3}{6}+\frac{20}{9}N_f(L_f-1)\Big)g'^2-\frac{3}{2}g^2 - 6\lambda_2-4\lambda_3-2\lambda_4\Big]\Big\} \tag{6.51}$$

$$h_5 =T\Big\{\frac{g(\mu)g'(\mu)}{2}+\frac{gg'}{2(4\pi)^2}\Big[2(\lambda_1+\lambda_4)+g^2\Big(\frac{44-N_h}{12}L_b-\frac{5+N_h}{6}-\frac{2}{3}N_f(L_f-1)\Big) + g'^2\Big(-\frac{1}{12}N_hL_b+\frac{3-N_h}{6}-\frac{10}{9}N_f(L_f-1)\Big)+2g_{Y,3}^2\Big]\Big\} \tag{6.52}$$

$$h_6 =T\Big\{\frac{g(\mu)g'(\mu)}{2}+\frac{gg'}{2(4\pi)^2}\Big[2(\lambda_2+\lambda_4)+g^2\Big(\frac{44-N_h}{12}L_b-\frac{5+N_h}{6}-\frac{2}{3}N_f(L_f-1)\Big) + g'^2\Big(-\frac{1}{12}N_hL_b+\frac{3-N_h}{6}-\frac{10}{9}N_f(L_f-1)\Big)\Big]\Big\} \tag{6.53}$$

$$\Lambda_1 =T\Big[\lambda_1(\mu)+\frac{1}{4(4\pi)^2}\Big(3g^4+g'^4+2g^2g'^2+8L_f(G_{Y,1}^4-\lambda_1 g_{Y,1}^2)\Big) - \frac{2L_b}{(4\pi)^2}\Big(\frac{9}{16}g^4+\frac{3}{16}g'^4+\frac{3}{8}g^2g'^2+3\lambda_1^2+\frac{1}{2}\lambda_3^2+\lambda_+^2+\lambda_-^2-\frac{3}{4}\lambda_1(3g^2+g'^2)\Big)\Big] \tag{6.54}$$

$$\Lambda_2 =T\Big[\lambda_2(\mu)+\frac{1}{4(4\pi)^2}\Big(3g^4+g'^4+2g^2g'^2\Big) - \frac{2L_b}{(4\pi)^2}\Big(\frac{9}{16}g^4+\frac{3}{16}g'^4+\frac{3}{8}g^2g'^2+3\lambda_2^2+\frac{1}{2}\lambda_3^2+\lambda_+^2+\lambda_-^2-\frac{3}{4}\lambda_2(3g^2+g'^2)\Big)\Big] \tag{6.55}$$

$$\Lambda_3 =T\Big[\lambda_3(\mu)+\frac{1}{4(4\pi)^2}\Big(3g^4+g'^4-2g^2g'^2-4L_f\lambda_3 g_{Y,1}^2\Big) - \frac{L_b}{(4\pi)^2}\Big(\frac{9}{8}g^4+\frac{3}{8}g'^4-\frac{3}{4}g^2g'^2+(\lambda_1+\lambda_2)(3\lambda_3+\lambda_4)+2\lambda_3^2+\lambda_4^2+\lambda_5^2$$

$$
\begin{aligned}
&\quad - \frac{3}{2}\lambda_3(3g^2 + g'^2)\Big)\Big] && (6.56)\\
\Lambda_4 =&T\Big[\lambda_4(\mu) + \frac{1}{(4\pi)^2}\Big(g^2g'^2 - L_f\lambda_4 g_{Y,1}^2 \\
&- L_b\Big(\frac{3}{2}g^2g'^2 + (\lambda_1+\lambda_2+4\lambda_3+2\lambda_4)\lambda_4 + 4\lambda_5^2 - \frac{3}{2}\lambda_4(3g^2+g'^2)\Big)\Big)\Big] && (6.57)\\
\Lambda_5 =&T\Big[\lambda_5(\mu) - \frac{1}{(4\pi)^2}\Big(L_f\lambda_5 g_{Y,1}^2 + L_b\Big(\lambda_5(\lambda_1+\lambda_2+4\lambda_3+6\lambda_4) - \frac{3}{2}L_b\lambda_5(3g^2+g'^2)\Big)\Big)\Big]. && (6.58)
\end{aligned}
$$

The dimensional reduction of the 2HDM has been previously calculated by Losada [25]. In the paper by Losada, neither the hypercharge gauge boson nor the fermions were included. Our calculations have extended the dimensional reduced theory to include contributions from both fermions and the hypercharge gauge boson. When comparing our results with the results found by Losada, we find almost complete agreement. However, there are a couple of discrepancies. For the three-dimensional coupling Λ_4 we get a term $4\lambda_5^2$, while Losada gets twice this value. Also, for the three-dimensional coupling Λ_5 we get a factor $\lambda_5(\lambda_1 + \dots)$, while Losada again gets a factor of 2 extra. Our calculations rely on using the effective potential for the scalar couplings, and we also found the scalar β-functions using the same results. Our results for the β-functions agree with the results from the review article on the 2HDM [78]. As the factors in Λ_4 and Λ_5 follow trivially from the effective potential, which gives the correct values for the β-functions, we conclude that the results found by Losada are incorrect. Also, the article on dimensional reduction the 2HDM by Andersen [27], which extends the calculation to two-loop for the mass terms, quotes the results found by Losada. Therefore, the values for Λ_4 and Λ_5 in that paper are also incorrect.

6.2 Integrating out the heavy modes

Now, it is natural to integrate out the heavy modes, i.e. the modes which have masses of order gT. That includes the temporal parts of the gauge fields, A_0 and B_0. In addition, we can divide the discussion into two parts; we can have one heavy and one light Higgs doublet, or we can have two light Higgs doublets. The first case is the generic one [25], while the other requires the first thermal contribution to the masses to be cancelled by the mass parameters m_{11}^2 and m_{22}^2. Depending on the choice of the scalar mass parameters, additional scalars might be integrated out along with the temporal parts of the gauge fields. The effective theory we end up with is identical to the effective theory for the SM when we have one heavy and one light scalar.

We will not perform this second step of dimensional reduction, but the procedure is identical to the one already outlined, and should not pose any major difficulties. The massive sum-integrals in eqs. (C.17) and (C.18) will be needed. The second step of dimensional reduction is performed in ref. [25], which would not differ much from our case.

CHAPTER 7

N-HIGGS DOUBLET MODEL

We will in this chapter extend the discussion to the N-Higgs Doublet Model (NHDM), find the β-functions for the generalised couplings and perform dimensional reduction where we integrate out the superheavy modes. We let N_h denote the number of Higgs doublets. As we will see, it is easy to generalise to N_h Higgs doublets, and we will rely heavily on the calculations performed earlier.

7.1 The model

We have an extended model of the SM with N_h scalar Higgs doublets, where only one Higgs doublet couples directly to the fermions. This is the fermiophobic NHDM. The interesting part of the Lagrangian is the scalar sector, which takes the form

$$\begin{aligned}\mathcal{L}_{\text{scalar}} = &\sum_{n=1}^{N_h}\left[D_\mu\Phi_n^\dagger D_\mu\Phi_n - \frac{\mu_n^2}{2}\Phi_n^\dagger\Phi_n + \frac{\lambda_{1,n}}{2}(\Phi_n^\dagger\Phi_n)^2\right]\\ &+ \sum_{n=1}^{N_h-1}\sum_{m=n+1}^{N_h}\Big[\lambda_{3,nm}(\Phi_n^\dagger\Phi_n)(\Phi_m^\dagger\Phi_m) + \lambda_{4,nm}(\Phi_n^\dagger\Phi_m)(\Phi_m^\dagger\Phi_n)\\ &+ \frac{\lambda_{5,nm}}{2}\Big((\Phi_n^\dagger\Phi_m)^2 + (\Phi_m^\dagger\Phi_n)^2\Big)\Big]\end{aligned} \tag{7.1}$$

where μ_n^2, $\lambda_{1,n}$, $\lambda_{3,nm}$, $\lambda_{4,nm}$, and $\lambda_{5,nm}$ are real. We also have that $\lambda_{3,nm} = \lambda_{3,mn}$, and similarly for $\lambda_{4,nm}$ and $\lambda_{5,nm}$. The limits of the last two sums are chosen such that we avoid double counting. We have already imposed $(N_h - 1)$ Z_2 symmetries, of the form

$$\Phi_n \to -\Phi_n, \quad \Phi_m \to \Phi_m, \quad m \neq n, \quad n = 1, \ldots, N_h - 1. \tag{7.2}$$

This means that we have discarded couplings with an odd number of the Higgs field Φ_n. We have also made $\lambda_{5,nm}$ real by a field redefinition, as before. A completely general NDHM could also include couplings with an odd number of the Higgs field Φ_n, as e.g. $(\Phi_n^\dagger\Phi_m)(\Phi_m^\dagger\Phi_k)$. The theory considered here will serve as the minimal NHDM, where additional operators could be included. Any general NDHM should reduce to the NHDM considered here.

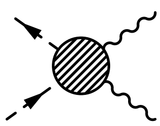
7.2 Correlators

In Appendix D most of the correlators needed have already been calculated with the number of Higgs doublets being arbitrary. We will summarise these correlators here, as well as perform the additional calculations for the new correlators.

The B_0^4, $A_0^a A_0^b A_0^c A_0^d$ and $A_0^a A_0^b B_0^2$ correlators are given in eqs. (4.21) to (4.23), respectively. They remain unchanged, as the number of Higgs doublets is already kept arbitrary there, and no scalar coupling is present.

The $\Phi_n^\dagger \Phi_n B_\mu B_\nu$ correlator

From eqs. (D.57) to (D.64) we find the types of diagrams contributing to the $\Phi_n^\dagger \Phi_n B_\mu B_\nu$ correlator. The only difference between the 2HDM and the NHDM is that we get more diagrams of the form of eqs. (D.58) and (D.62). Thus, with an extra sum over the scalar couplings, the correlator takes the form

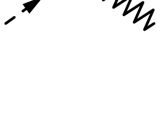


$$
\begin{aligned}
=&g'^2\Big[\frac{1}{2}(d-3)\Big(3\lambda_n + \sum_{m\neq n}(2\lambda_{3,nm}+\lambda_{4,nm})\Big) + \frac{d}{8}g'^2 + \frac{3}{8}dg^2 \\
&- \frac{1}{2}\delta_{n1}(2^{4-d}-1)(g_{Y,1}^2 - 2\epsilon g_{Y,2}^2)\Big]\delta_{ij} I_2^{4b} \qquad (7.3a) \\
&\text{for } \mu = 0, \nu = 0 \\
=&g'^2\Big[\frac{3}{8}g'^2 + \frac{9}{8}g^2 - \frac{1}{2}\delta_{n1}(2^{4-d}-1)g_{Y,1}^2\Big]\delta_{ij}\delta_{rs} I_2^{4b} \qquad (7.3b) \\
&\text{for } \mu = r, \nu = s.
\end{aligned}
$$

Notice the Kronecker delta δ_{n1} that makes sure that only Φ_1 couples directly to the fermions.

The $\Phi_n^\dagger \Phi_n A_\mu^a A_\nu^b$ correlator

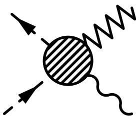
Similarly, the $\Phi_n^\dagger \Phi_n A_\mu^a A_\nu^b$ correlator can be extracted from diagrams of the form of eqs. (D.48) to (D.56). Again, more diagrams of the form of eqs. (D.50) and (D.55) must be included, which results in an extra sum over the scalar couplings. The correlator takes the form



$$
\begin{aligned}
=&g^2\Big[\Big(-\frac{25}{8}+d\Big)dg^2 - \frac{1}{2}(3-d)\Big(3\lambda_n + \sum_{m\neq n}(2\lambda_{3,nm}+\lambda_{4,nm})\Big) + \frac{1}{8}dg'^2 \\
&- \frac{1}{2}\delta_{n1}(2^{4-d}-1)(d-2)g_{Y,1}^2\Big]\delta_{ab}\delta_{ij} I_2^{4b} \qquad (7.4a) \\
&\text{for } \mu = 0, \nu = 0 \\
=&\Big[-\frac{3}{8}g^4 + \frac{3}{8}g^2 g'^2 - \frac{1}{2}\delta_{n1}(2^{4-d}-1)g^2 g_{Y,1}^2\Big]\delta_{ab}\delta_{ij}\delta_{rs} I_2^{4b} \qquad (7.4b) \\
&\text{for } \mu = r, \nu = s.
\end{aligned}
$$

The $\Phi_n^\dagger \Phi_n A_0^a B_0$ correlator

The correlator with two scalar legs and two different gauge field legs can be found from eqs. (D.65) to (D.72). The additional sum over the scalar couplings comes from eqs. (D.66) and (D.70). The correlator becomes



$$
\begin{aligned}
&= gg'\Big[\frac{1}{2}(d-3)(\lambda_n + \sum_{m\neq n}\lambda_{4,nm}) + \frac{1}{8}d(g^2+g'^2) \\
&\quad - \frac{1}{2}\delta_{n1}(2^{4-d}-1)(g_{Y,3}^2 - 2\epsilon g_{Y,4}^2)\Big](\tau_a)_{ij} I_2^{4b}. \qquad (7.5)
\end{aligned}
$$

The correlators will be used when matching the effective three-dimensional theory with the original four-dimensional theory.

7.3 Effective potential

We use the effective potential to extract the counterterms for the scalar couplings, as before. Instead of diagonalising an 8×8 matrix, we will now need to diagonalise a $4N_h \times 4N_h$ matrix. At the outset this seems to be a too great challenge, but we will see that by choosing our background fields in a clever way we will be able to find the mass spectrum.

We shift the scalar fields by a background field, $\Phi_n \to \Phi_n + \varphi_n$. The mass matrix can be found from the terms quadratic in the fields.

7.3.1 Case 1

We set the background fields to be

$$
\varphi_n = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_n\end{pmatrix}, \quad \varphi_m = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_m\end{pmatrix}, \quad \varphi_k = 0, \quad k\neq n,m, \quad n\neq m, \quad k=1,\ldots,N_h. \qquad (7.6)
$$

The mass matrix is similar to the 2HDM mass matrix, with the addition of $4(N_h-2)$ terms on the diagonal. We can rearrange the columns and rows to obtain

$$
M^2 = \begin{bmatrix}
M^2_{nn,1} & 0 & 0 & 0 & M^2_{nm,15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & M^2_{nn,2} & 0 & 0 & 0 & M^2_{nm,26} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & M^2_{nn,3} & 0 & 0 & 0 & M^2_{nm,37} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & M^2_{nn,4} & 0 & 0 & 0 & M^2_{nm,48} & 0 & 0 & 0 & 0 & \cdots \\
M^2_{nm,15} & 0 & 0 & 0 & M^2_{mm,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & M^2_{nm,26} & 0 & 0 & 0 & M^2_{mm,2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & M^2_{nm,37} & 0 & 0 & 0 & M^2_{mm,3} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & M^2_{nm,48} & 0 & 0 & 0 & M^2_{mm,4} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix} \qquad (7.7)
$$

where

$$M^2_{nn,1} = -\frac{1}{2}\tilde{\mu}^2_n \tag{7.8a}$$
$$M^2_{nn,2} = -\frac{1}{2}\tilde{\mu}^2_n \tag{7.8b}$$
$$M^2_{nn,3} = -\frac{1}{2}\tilde{\mu}^2_n + \lambda_{1,n}v^2_n + \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v^2_m \tag{7.8c}$$
$$M^2_{nn,4} = -\frac{1}{2}\tilde{\mu}^2_n + \frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v^2_m \tag{7.8d}$$
$$M^2_{mm,1} = -\frac{1}{2}\tilde{\mu}^2_m \tag{7.8e}$$
$$M^2_{mm,2} = -\frac{1}{2}\tilde{\mu}^2_m \tag{7.8f}$$
$$M^2_{mm,3} = -\frac{1}{2}\tilde{\mu}^2_m + \lambda_{1,m}v^2_m + \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v^2_n \tag{7.8g}$$
$$M^2_{mm,4} = -\frac{1}{2}\tilde{\mu}^2_m + \frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v^2_n \tag{7.8h}$$
$$M^2_{nm,15} = \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v_n v_m \tag{7.8i}$$
$$M^2_{nm,26} = \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v_n v_m \tag{7.8j}$$
$$M^2_{nm,37} = (\lambda_{3,nm} + \lambda_{4,nm} + \lambda_{5,nm})v_n v_m \tag{7.8k}$$
$$M^2_{nm,48} = \lambda_{5,nm}v_n v_m \tag{7.8l}$$
$$M^2_{kk,1} = -\frac{1}{2}\tilde{\mu}^2_k \tag{7.8m}$$
$$M^2_{kk,2} = -\frac{1}{2}\tilde{\mu}^2_k \tag{7.8n}$$
$$M^2_{kk,3} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} + \lambda_{5,kn})v^2_n + \frac{1}{2}(\lambda_{4,km} + \lambda_{5,km})v^2_m \tag{7.8o}$$
$$M^2_{kk,4} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} - \lambda_{5,kn})v^2_n + \frac{1}{2}(\lambda_{4,km} - \lambda_{5,km})v^2_m. \tag{7.8p}$$

The shifted masses are

$$\tilde{\mu}^2_n = \mu^2_n - \lambda_{1,n}v^2_n - \lambda_{3,nm}v^2_m \tag{7.9}$$
$$\tilde{\mu}^2_m = \mu^2_m - \lambda_{1,m}v^2_m - \lambda_{3,nm}v^2_n \tag{7.10}$$
$$\tilde{\mu}^2_k = \mu^2_k - \lambda_{3,kn}v^2_n - \lambda_{3,km}v^2_m. \tag{7.11}$$

The effective potential takes the form

$$V_{\text{eff}} = \frac{1}{2}V_{n,11}v^2_n + \frac{1}{2}V_{m,11}v^2_m + \frac{1}{4}V_{n,1}v^4_n + \frac{1}{4}V_{m,1}v^4_m + \frac{1}{4}(V_{nm,3} + V_{nm,4} + V_{nm,5})v^2_n v^2_m \tag{7.12}$$

where

$$V_{n,11} = \frac{T^2}{12}\Big(3\lambda_{1,n} + \sum_{k\neq n}(2\lambda_{3,kn} + \lambda_{4,kn})\Big) \tag{7.13}$$

$$
\begin{aligned}
V_{n,1} =& -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[3\lambda_{1,n}^2 \\
&+\frac{1}{4}\sum_{k\neq n}\left(2\lambda_{3,kn}^2+(\lambda_{3,kn}+\lambda_{4,kn}+\lambda_{5,kn})^2+(\lambda_{3,kn}+\lambda_{4,kn}-\lambda_{5,kn})^2\right)\Big]
\end{aligned}
\tag{7.14}
$$

$$
\begin{aligned}
V_{nm,3}+V_{nm,4}+V_{nm,5} =& -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[(\lambda_{1,n}+\lambda_{1,m})(3\lambda_{3,nm}+2\lambda_{4,nm}+\lambda_{5,nm}) \\
&+(\lambda_{4,nm}+\lambda_{5,nm})^2+2\lambda_{5,nm}^2+2(\lambda_{3,nm}+\lambda_{4,nm}+\lambda_{5,nm})^2 \\
&+\sum_{k\neq n,m}\left(\lambda_{3,kn}\lambda_{3,km}+(\lambda_{3,kn}+\lambda_{4,kn})(\lambda_{3,km}+\lambda_{4,km})+\lambda_{5,kn}\lambda_{5,km}\right)\Big].
\end{aligned}
\tag{7.15}
$$

The divergence of $V_{n,1}$ is absorbed into $\delta\lambda_{n,1}$ and $V_{nm,3}+V_{nm,4}+V_{nm,5}$ into $\delta\lambda_{nm,3}+\delta\lambda_{nm,4}+\delta\lambda_{nm,5}$.

7.3.2 Case 2

We now set the background fields to be

$$
\varphi_n=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_n\end{pmatrix},\quad \varphi_m=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ iw_m^0\end{pmatrix},\quad \varphi_k=0,\quad k\neq n,m,\quad n\neq m,\quad k=1,\dots,N_h.
\tag{7.16}
$$

The mass matrix is similar to the 2HDM mass matrix, with the addition of $4(N_h-2)$ terms on the diagonal. We can rearrange the columns and rows to obtain

$$
M^2=\begin{bmatrix}
M^2_{nn,1} & 0 & 0 & 0 & 0 & M^2_{nm,16} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & M^2_{nn,2} & 0 & 0 & M^2_{nm,25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & M^2_{nn,3} & 0 & 0 & 0 & 0 & M^2_{nm,38} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & M^2_{nn,4} & 0 & 0 & M^2_{nm,47} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & M^2_{nm,25} & 0 & 0 & M^2_{mm,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
M^2_{nm,16} & 0 & 0 & 0 & 0 & M^2_{mm,2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & M^2_{nm,47} & 0 & 0 & M^2_{mm,3} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & M^2_{nm,38} & 0 & 0 & 0 & 0 & M^2_{mm,4} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\tag{7.17}
$$

where

$$M^2_{nn,1}=-\frac{1}{2}\tilde{\mu}_n^2 \tag{7.18a}$$

$$M^2_{nn,2}=-\frac{1}{2}\tilde{\mu}_n^2 \tag{7.18b}$$

$$M^2_{nn,3}=-\frac{1}{2}\tilde{\mu}_n^2+\lambda_{1,n}v_n^2+\frac{1}{2}(\lambda_{4,nm}-\lambda_{5,nm})(w_m^0)^2 \tag{7.18c}$$

$$M^2_{nn,4} = -\frac{1}{2}\tilde{\mu}^2_n + \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})(w^0_m)^2 \tag{7.18d}$$
$$M^2_{mm,1} = -\frac{1}{2}\tilde{\mu}^2_m \tag{7.18e}$$
$$M^2_{mm,2} = -\frac{1}{2}\tilde{\mu}^2_m \tag{7.18f}$$
$$M^2_{mm,3} = -\frac{1}{2}\tilde{\mu}^2_m + \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v^2_n \tag{7.18g}$$
$$M^2_{mm,4} = -\frac{1}{2}\tilde{\mu}^2_m + \lambda_{1,m}(w^0_m)^2 + \frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v^2_n \tag{7.18h}$$
$$M^2_{nm,16} = \frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v_n w^0_m \tag{7.18i}$$
$$M^2_{nm,25} = -\frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v_n w^0_m \tag{7.18j}$$
$$M^2_{nm,38} = (\lambda_{3,nm} + \lambda_{4,nm} - \lambda_{5,nm})v_n w^0_m \tag{7.18k}$$
$$M^2_{nm,47} = \lambda_{5,nm} v_n w^0_m \tag{7.18l}$$
$$M^2_{kk,1} = -\frac{1}{2}\tilde{\mu}^2_k \tag{7.18m}$$
$$M^2_{kk,2} = -\frac{1}{2}\tilde{\mu}^2_k \tag{7.18n}$$
$$M^2_{kk,3} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} + \lambda_{5,kn})v^2_n + \frac{1}{2}(\lambda_{4,km} - \lambda_{5,km})(w^0_m)^2 \tag{7.18o}$$
$$M^2_{kk,4} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} - \lambda_{5,kn})v^2_n + \frac{1}{2}(\lambda_{4,km} + \lambda_{5,km})(w^0_m)^2. \tag{7.18p}$$

The shifted masses are

$$\tilde{\mu}^2_n = \mu^2_n - \lambda_{1,n}v^2_n - \lambda_{3,nm}(w^0_m)^2 \tag{7.19}$$
$$\tilde{\mu}^2_m = \mu^2_m - \lambda_{1,m}(w^0_m)^2 - \lambda_{3,nm}v^2_n \tag{7.20}$$
$$\tilde{\mu}^2_k = \mu^2_k - \lambda_{3,kn}v^2_n - \lambda_{3,km}(w^0_m)^2. \tag{7.21}$$

The effective potential takes the form

$$\begin{aligned} V_{\text{eff}} =& \frac{1}{2}V_{n,11}v^2_n + \frac{1}{2}V_{m,11}(w^0_m)^2 + \frac{1}{4}V_{n,1}v^4_n + \frac{1}{4}V_{m,1}(w^0_m)^4 \\ &+ \frac{1}{4}(V_{nm,3} + V_{nm,4} - V_{nm,5})v^2_n(w^0_m)^2 \end{aligned} \tag{7.22}$$

where

$$V_{n,11} = \frac{T^2}{12}\Big(3\lambda_{1,n} + \sum_{k\neq n}(2\lambda_{3,kn} + \lambda_{4,kn})\Big) \tag{7.23}$$
$$\begin{aligned} V_{n,1} =& -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda^2_{1,n} \\ &+ \frac{1}{4}\sum_{k\neq n}\Big(2\lambda^2_{3,kn} + (\lambda_{3,kn} + \lambda_{4,kn} + \lambda_{5,kn})^2 + (\lambda_{3,kn} + \lambda_{4,kn} - \lambda_{5,kn})^2\Big)\Big] \end{aligned} \tag{7.24}$$

$$
\begin{aligned}
V_{nm,3}+V_{nm,4}-V_{nm,5} = -\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[&(\lambda_{1,n}+\lambda_{1,m})(3\lambda_{3,nm}+2\lambda_{4,nm}-\lambda_{5,nm}) \\
&+(\lambda_{4,nm}-\lambda_{5,nm})^2+2\lambda_{5,nm}^2+2(\lambda_{3,nm}+\lambda_{4,nm}-\lambda_{5,nm})^2 \\
&+\sum_{k\neq n,m}\Big(\lambda_{3,kn}\lambda_{3,km}+(\lambda_{3,kn}+\lambda_{4,kn})(\lambda_{3,km}+\lambda_{4,km})-\lambda_{5,kn}\lambda_{5,km}\Big)\Big].
\end{aligned}
\tag{7.25}
$$

Now, the divergence of $V_{1,n}$ is absorbed into $\delta\lambda_{1,n}$ and $V_{nm,3}+V_{nm,4}-V_{nm,5}$ into $\delta\lambda_{nm,3}+\delta\lambda_{nm,4}-\delta\lambda_{nm,5}$. The coefficients $V_{n,11}$ and $V_{n,1}$ should be the same as in section 7.3.1.

7.3.3 Case 3

We set the background fields to be

$$
\varphi_n=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_n\end{pmatrix},\quad \varphi_m=\frac{1}{\sqrt{2}}\begin{pmatrix}w_m^+\\ 0\end{pmatrix},\quad \varphi_k=0,\quad k\neq n,m,\quad n\neq m,\quad k=1,\dots,N_h. \tag{7.26}
$$

The mass matrix is similar to the 2HDM mass matrix, with the addition of $4(N_h-2)$ terms on the diagonal. We can rearrange the columns and rows to obtain

$$
M^2=\begin{bmatrix}
M^2_{nn,1} & 0 & 0 & 0 & 0 & 0 & M^2_{nm,17} & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & M^2_{nn,2} & 0 & 0 & 0 & 0 & 0 & M^2_{nm,28} & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & M^2_{nn,3} & 0 & M^2_{nm,35} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & M^2_{nn,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & M^2_{nm,35} & 0 & M^2_{mm,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & M^2_{mm,2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
M^2_{nm,17} & 0 & 0 & 0 & 0 & 0 & M^2_{mm,3} & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & M^2_{nm,28} & 0 & 0 & 0 & 0 & 0 & M^2_{mm,4} & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M^2_{kk,1} & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix} \tag{7.27}
$$

where

$$M^2_{nn,1}=-\frac{1}{2}\tilde{\mu}_n^2+\frac{1}{2}(\lambda_{4,nm}+\lambda_{5,nm})(w_m^+)^2 \tag{7.28a}$$

$$M^2_{nn,2}=-\frac{1}{2}\tilde{\mu}_n^2+\frac{1}{2}(\lambda_{4,nm}-\lambda_{5,nm})(w_m^+)^2 \tag{7.28b}$$

$$M^2_{nn,3}=-\frac{1}{2}\tilde{\mu}_n^2+\lambda_{1,n}v_n^2 \tag{7.28c}$$

$$M^2_{nn,4}=-\frac{1}{2}\tilde{\mu}_n^2 \tag{7.28d}$$

$$M^2_{mm,1}=-\frac{1}{2}\tilde{\mu}_m^2+\lambda_{1,m}(w_m^+)^2 \tag{7.28e}$$

$$M^2_{mm,2}=-\frac{1}{2}\tilde{\mu}_m^2 \tag{7.28f}$$

$$M^2_{mm,3} = -\frac{1}{2}\tilde{\mu}^2_m + \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v_n^2 \tag{7.28g}$$

$$M^2_{mm,4} = -\frac{1}{2}\tilde{\mu}^2_m + \frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v_n^2 \tag{7.28h}$$

$$M^2_{nm,17} = \frac{1}{2}(\lambda_{4,nm} + \lambda_{5,nm})v_n w_m^+ \tag{7.28i}$$

$$M^2_{nm,28} = -\frac{1}{2}(\lambda_{4,nm} - \lambda_{5,nm})v_n w_m^+ \tag{7.28j}$$

$$M^2_{nm,35} = \lambda_{3,nm} v_n w_m^+ \tag{7.28k}$$

$$M^2_{kk,1} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,km} + \lambda_{5,km})(w_m^+)^2 \tag{7.28l}$$

$$M^2_{kk,2} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,km} - \lambda_{5,km})(w_m^+)^2 \tag{7.28m}$$

$$M^2_{kk,3} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} + \lambda_{5,kn})v_n^2 \tag{7.28n}$$

$$M^2_{kk,4} = -\frac{1}{2}\tilde{\mu}^2_k + \frac{1}{2}(\lambda_{4,kn} - \lambda_{5,kn})v_n^2. \tag{7.28o}$$

The shifted masses are

$$\tilde{\mu}^2_n = \mu^2_n - \lambda_{1,n}v_n^2 - \lambda_{3,nm}(w_m^+)^2 \tag{7.29}$$

$$\tilde{\mu}^2_m = \mu^2_m - \lambda_{1,m}(w_m^+)^2 - \lambda_{3,nm}v_n^2 \tag{7.30}$$

$$\tilde{\mu}^2_k = \mu^2_k - \lambda_{3,kn}v_n^2 - \lambda_{3,km}(w_m^+)^2. \tag{7.31}$$

The effective potential takes the form

$$V_{\text{eff}} = \frac{1}{2}V_{n,11}v_n^2 + \frac{1}{2}V_{m,11}(w_m^+)^2 + \frac{1}{4}V_{n,1}v_n^4 + \frac{1}{4}V_{m,1}(w_m^+)^4 + \frac{1}{4}V_{nm,3}v_n^2(w_m^+)^2 \tag{7.32}$$

where

$$V_{n,11} = \frac{T^2}{12}\Big(3\lambda_{1,n} + \sum_{k\neq n}(2\lambda_{3,kn} + \lambda_{4,kn})\Big) \tag{7.33}$$

$$\begin{aligned} V_{n,1} = &-\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[3\lambda^2_{1,n} \\ &+ \frac{1}{4}\sum_{k\neq n}\Big(2\lambda^2_{3,kn} + (\lambda_{3,kn} + \lambda_{4,kn} + \lambda_{5,kn})^2 + (\lambda_{3,kn} + \lambda_{4,kn} - \lambda_{5,kn})^2\Big)\Big] \end{aligned} \tag{7.34}$$

$$\begin{aligned} V_{nm,3} = &-\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_{1,n} + \lambda_{1,m})(3\lambda_{3,nm} + \lambda_{4,nm}) + 2\lambda^2_{3,nm} + \lambda^2_{4,nm} + \lambda^2_{5,nm} \\ &+ \sum_{k\neq n,m}\Big(\lambda_{3,kn}\lambda_{3,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}\Big)\Big]. \end{aligned} \tag{7.35}$$

We can find the individual contributions to the coefficients in the expansion of the effective potential by solving the system of equations from sections 7.3.1 to 7.3.3.

7.3.4 Gauge sector

The gauge bosons couple identically to all the scalar doublets through the covariant derivative. Thus, the gauge boson contributions decouple from each other, and we can write down the contribution directly, using the results obtained previously;

$$
\begin{aligned}
V_{n,11} =& \frac{T^2}{12}\left[\frac{9}{4}g^2 + \frac{3}{4}g'^2\right] && (7.36)\\
V_{n,1} =& -\frac{1}{16(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right)\left[3g^4+g'^4+2g^2g'^2\right] && (7.37)\\
V_{nm,3} =& -\frac{1}{8(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right)\left[3g^4+g'^4-2g^2g'^2\right] && (7.38)\\
V_{nm,4} =& -\frac{g^2g'^2}{2(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right) && (7.39)\\
V_{nm,5} =& 0. && (7.40)
\end{aligned}
$$

7.3.5 Fermion sector

The fermion sector takes the same form as for the 2HDM, since only Φ_1 couples directly to the fermions. The contributions to the effective potential are

$$
\begin{aligned}
V_{1,11} =& \frac{T^2}{12}g_{Y,1}^2 && (7.41)\\
V_{1,1} =& \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_f\right)G_{Y,1}^4 && (7.42)
\end{aligned}
$$

with the rest being zero.

7.3.6 Total contribution to the effective potential

We summarise the total contribution to the effective potential,

$$
\begin{aligned}
V_{n,11} =& \frac{T^2}{12}\left[\frac{9}{4}g^2+\frac{3}{4}g'^2+3\lambda_{1,n}+\sum_{k\neq n}(2\lambda_{3,kn}+\lambda_{4,kn})+\delta_{n1}g_{Y,1}^2\right] && (7.43)\\
V_{n,1} =& -\frac{1}{16(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right)\left[3g^4+g'^4+2g^2g'^2\right]+\frac{\delta_{n1}}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_f\right)G_{Y,1}^4\\
&-\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[3\lambda_{1,n}\\
&+\frac{1}{4}\sum_{k\neq n}\left(2\lambda_{3,kn}^2+(\lambda_{3,kn}+\lambda_{4,kn}+\lambda_{5,kn})^2+(\lambda_{3,kn}+\lambda_{4,kn}-\lambda_{5,kn})^2\right)\Big] && (7.44)\\
V_{nm,3} =& -\frac{1}{8(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right)\left[3g^4+g'^4-2g^2g'^2\right]\\
&-\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[(\lambda_{1,n}+\lambda_{1,m})(3\lambda_{3,nm}+\lambda_{4,nm})+2\lambda_{3,nm}^2+\lambda_{4,nm}^2+\lambda_{5,nm}^2\\
&+\sum_{k\neq n,m}\left(\lambda_{3,kn}\lambda_{3,km}+\frac{1}{2}\lambda_{3,kn}\lambda_{4,km}+\frac{1}{2}\lambda_{3,km}\lambda_{4,kn}\right)\Big] && (7.45)\\
V_{nm,4} =& -\frac{g^2g'^2}{2(4\pi)^2}\left(\frac{3}{\epsilon}+3L_b-2\right)-\frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+L_b\right)\Big[(\lambda_{1,n}+\lambda_{1,m}+4\lambda_{3,nm}+2\lambda_{4,nm})\lambda_{4,nm}
\end{aligned}
$$

$$+ 4\lambda_{5,nm}^2 + \sum_{k\neq n,m}\Big(\lambda_{3,kn}\lambda_{3,km} + \lambda_{4,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}\Big)\Big] \quad (7.46)$$

$$V_{nm,5} = -\frac{1}{(4\pi)^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big[(\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 6\lambda_{4,nm})\lambda_{5,nm} + \sum_{k\neq n,m}\lambda_{5,kn}\lambda_{5,km}\Big]. \quad (7.47)$$

We see that the effective potential reduces to the 2HDM effective potential when $n, m = 1, 2$, as it should. We can directly write down the counterterms

$$\begin{aligned}
\delta\lambda_{1,n} =& \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 + \frac{3}{4}g^2g'^2 - 2\delta_{n1}G_{Y,1}^4 + 6\lambda_{1,n}^2 \\
&+ \frac{1}{2}\sum_{k\neq n}\Big(2\lambda_{3,kn}^2 + (\lambda_{3,kn} + \lambda_{4,kn} + \lambda_{5,kn})^2 + (\lambda_{3,kn} + \lambda_{4,kn} - \lambda_{5,kn})^2\Big)\Big] \quad (7.48)\\
\delta\lambda_{3,nm} =& \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{8}g^4 + \frac{3}{8}g'^4 - \frac{3}{4}g^2g'^2 + (\lambda_{1,n} + \lambda_{1,m})(3\lambda_{3,nm} + \lambda_{4,nm}) + 2\lambda_{3,nm}^2 \\
&+ \lambda_{4,nm}^2 + \lambda_{5,nm}^2 + \sum_{k\neq n,m}\Big(\lambda_{3,kn}\lambda_{3,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}\Big)\Big] \quad (7.49)\\
\delta\lambda_{4,nm} =& \frac{1}{(4\pi)^2\epsilon}\Big[\frac{3}{2}g^2g'^2 + (\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 2\lambda_{4,nm})\lambda_{4,nm} + 4\lambda_{5,nm}^2 \\
&+ \sum_{k\neq n,m}\Big(\lambda_{3,kn}\lambda_{3,km} + \lambda_{4,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}\Big)\Big] \quad (7.50)\\
\delta\lambda_{5,nm} =& \frac{1}{(4\pi)^2\epsilon}\Big[(\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 6\lambda_{4,nm})\lambda_{5,nm} + \sum_{k\neq n,m}\lambda_{5,kn}\lambda_{5,km}\Big]. \quad (7.51)
\end{aligned}$$

To find the β-functions, we need the wave function renormalisation counterterms. From eqs. (4.13) and (4.14) we have that

$$\delta Z_{\Phi_n} = \frac{1}{(4\pi)^2\epsilon}\Big[\frac{9}{4}g^2 + \frac{3}{4}g'^2 - \delta_{n1}g_{Y,1}^2\Big], \quad (7.52)$$

similarly to the 2HDM. Thus, the β-functions can be found by using the same procedure as previously,

$$\begin{aligned}
\beta_{\lambda_{1,n}} =& \frac{1}{(4\pi)^2}\Big[\frac{9}{4}g^4 + \frac{3}{4}g'^4 + \frac{3}{4}g^2g'^2 + 12\lambda_{1,n}^2 \\
&+ \sum_{k\neq n}\Big(2\lambda_{3,kn}^2 + (\lambda_{3,kn} + \lambda_{4,kn} + \lambda_{5,kn})^2 + (\lambda_{3,kn} + \lambda_{4,kn} - \lambda_{5,kn})^2\Big) \\
&- 3\lambda_{1,n}(3g^2 + g'^2) - 4\delta_{n1}(G_{Y,1}^4 - \lambda_{1,n}g_{Y,1}^2)\Big] \quad (7.53)\\
\beta_{\lambda_{3,nm}} =& \frac{1}{(4\pi)^2}\Big[\frac{9}{4}g^4 + \frac{3}{4}g'^4 - \frac{3}{4}g^2g'^2 + 2(\lambda_{1,n} + \lambda_{1,m})(3\lambda_{3,nm} + \lambda_{4,nm}) + 4\lambda_{3,nm}^2 \\
&+ 2\lambda_{4,nm}^2 + 2\lambda_{5,nm}^2 + \sum_{k\neq n,m}\Big(2\lambda_{3,kn}\lambda_{3,km} + \lambda_{3,kn}\lambda_{4,km} + \lambda_{3,km}\lambda_{4,kn}\Big) \\
&- 3\lambda_{3,nm}(3g^2 + g'^2) + 2(\delta_{1n} + \delta_{1m})\lambda_{3,nm}g_{Y,1}^2\Big] \quad (7.54)
\end{aligned}$$

$$\beta_{\lambda_{4,nm}} = \frac{1}{(4\pi)^2}\Big[3g^2g'^2 + 2(\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 2\lambda_{4,nm})\lambda_{4,nm} + 8\lambda_{5,nm}^2 + \sum_{k\neq n,m}\Big(2\lambda_{3,kn}\lambda_{3,km} + 2\lambda_{4,kn}\lambda_{4,km} + \lambda_{3,kn}\lambda_{4,km} + \lambda_{3,km}\lambda_{4,kn}\Big) - 3\lambda_{4,nm}(3g^2 + g'^2) + 2(\delta_{1n} + \delta_{1m})\lambda_{4,nm}g_{Y,1}^2\Big] \tag{7.55}$$

$$\beta_{\lambda_{5,nm}} = \frac{1}{(4\pi)^2}\Big[2(\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 6\lambda_{4,nm})\lambda_{5,nm} + \sum_{k\neq n,m} 2\lambda_{5,kn}\lambda_{5,km} \tag{7.56}$$

$$- 3\lambda_{5,nm}(3g^2 + g'^2) + 2(\delta_{1n} + \delta_{1m})\lambda_{5,nm}g_{Y,1}^2\Big]. \tag{7.57}$$

Most of the terms in the β-functions are just generalisations of the terms in the 2HDM β-functions, while the sums over $k \neq n, m$ are novel, as they arise from interactions between additional Higgs doublets not present in the 2HDM.

7.4 Dimensional reduction

We are now in a position to integrate out the superheavy modes and get a three-dimensional effective theory. The effective theory is similar to eq. (6.4), except for

$$\mathcal{L}^{(3)}_{\text{scalar}} = \sum_n\Big[D_r\Phi_n^\dagger D_r\Phi_n - \mu_{3,n}^2\Phi_n^\dagger\Phi_n + \frac{\Lambda_{1,n}}{2}(\Phi_n^\dagger\Phi_n)^2\Big] + \sum_{m>n}\Big[\Lambda_{3,nm}(\Phi_n^\dagger\Phi_n)(\Phi_m^\dagger\Phi_m) + \Lambda_{4,nm}(\Phi_n^\dagger\Phi_m)(\Phi_m^\dagger\Phi_n) + \frac{\Lambda_{5,nm}}{2}\Big[(\Phi_n^\dagger\Phi_m)^2 + (\Phi_m^\dagger\Phi_n)^2\Big]\Big] \tag{7.58}$$

and

$$\mathcal{L}^{(3)}_{\text{temporal}} = \frac{1}{2}(D_rA_0^a)^2 + \frac{1}{2}(\partial_rB_0)^2 + \frac{1}{2}m_D^2(A_0^a)^2 + \frac{1}{2}m_D'^2B_0^2 + \frac{1}{4}\kappa_1(A_0^a)^4 + \frac{1}{4}\kappa_2B_0^4 + \frac{1}{4}\kappa_3(A_0^a)^2B_0^2 + \sum_n\Big(h_{1,n}\Phi_n^\dagger\Phi_n(A_0^a)^2 + h_{2,n}\Phi_n^\dagger\Phi_nB_0^2 + h_{3,n}B_0\Phi_n^\dagger\vec{A}_0\cdot\vec{\tau}\Phi_n\Big). \tag{7.59}$$

The procedure for determining the parameters of the three-dimensional theory in terms of the parameters of the original theory is identical to the procedure in chapter 6. Therefore, we simply list the results

$$\mu_{3,n}^2 = \mu_n^2 - \frac{T^2}{12}\Big[\frac{9}{4}g^2 + \frac{3}{4}g'^2 + 3\lambda_{1,n} + \sum_{k\neq n}(2\lambda_{3,nm} + \lambda_{4,nm}) + \delta_{n1}g_{Y,1}^2\Big] \tag{7.60}$$

$$h_{1,n} = T\Big\{\frac{g^2(\mu)}{4} + \frac{g^2}{4(4\pi)^2}\Big[\Big(\frac{44-N_h}{6}L_b + \frac{53-2N_h}{6} - \frac{4}{3}N_f(L_f-1)\Big)g^2 + \frac{1}{2}g'^2 - 2\delta_{n1}g_{Y,1}^2 + 6\lambda_{1,n} + \sum_{k\neq n}(4\lambda_{3,kn} + 2\lambda_{4,kn})\Big]\Big\} \tag{7.61}$$

$$
\begin{aligned}
h_{2,n} =& T\Big\{\frac{g'^2(\mu)}{4} - \frac{g'^2}{4(4\pi)^2}\Big[\Big(\frac{1}{6}N_h L_b + \frac{2N_h-3}{6} + \frac{20}{9}N_f(L_f-1)\Big)g'^2 - \frac{3}{2}g^2 + 2\delta_{n1}g_{Y,2}^2 \\
&- 6\lambda_{1,n} - \sum_{n\neq k}(4\lambda_{3,kn} + 2\lambda_{4,kn})\Big]\Big\} && (7.62)\\
h_{3,n} =& T\Big\{\frac{g(\mu)g'(\mu)}{2} + \frac{gg'}{2(4\pi)^2}\Big[2(\lambda_{1,n} + \sum_{k\neq n}\lambda_{4,kn}) \\
&+ g^2\Big(\frac{44-N_h}{12}L_b - \frac{5+N_h}{6} - \frac{2}{3}N_f(L_f-1)\Big) \\
&+ g'^2\Big(-\frac{1}{12}N_h L_b + \frac{3-N_h}{6} - \frac{10}{9}N_f(L_f-1) + 2\delta_{n1}g_{Y,3}^2\Big)\Big]\Big\}. && (7.63)
\end{aligned}
$$

We use the effective potential to determine the scalar couplings. The couplings take the form

$$
\begin{aligned}
\Lambda_{1,n} =& T\Big[\lambda_{1,n}(\mu) + \frac{1}{4(4\pi)^2}\Big(3g^4 + g'^4 + 2g^2g'^2 + 8L_f(G_{Y,1}^4 - \lambda_{1,n}g_{Y,1}^2)\Big) \\
&- \frac{2L_b}{(4\pi)^2}\Big(\frac{9}{16}g^4 + \frac{3}{16}g'^4 + \frac{3}{8}g^2g'^2 + 3\lambda_{1,n}^2 \\
&+ \frac{1}{4}\sum_{k\neq n}(2\lambda_{3,kn}^2 + (\lambda_{3,kn} + \lambda_{4,kn} + \lambda_{kn})^2 + (\lambda_{3,kn} + \lambda_{4,kn} - \lambda_{kn})^2) \\
&- \frac{3}{4}\lambda_{1,n}(3g^2 + g'^2)\Big)\Big] && (7.64a)\\
\Lambda_{3,nm} =& T\Big[\lambda_{3,nm}(\mu) + \frac{1}{4(4\pi)^2}\Big(3g^4 + g'^4 - 2g^2g'^2 - 4L_f\lambda_{3,nm}g_{Y,1}^2\Big) \\
&- \frac{L_b}{(4\pi)^2}\Big(\frac{9}{8}g^4 + \frac{3}{8}g'^4 - \frac{3}{4}g^2g'^2 \\
&+ (\lambda_{1,n} + \lambda_{1,m})(3\lambda_{3,nm} + \lambda_{4,nm}) + 2\lambda_{3,nm}^2 + \lambda_{4,nm}^2 + \lambda_{5,nm}^2 \\
&+ \sum_{k\neq n,m}(\lambda_{3,kn}\lambda_{3,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}) - \frac{3}{2}\lambda_{3,nm}(3g^2 + g'^2)\Big)\Big] && (7.64b)\\
\Lambda_{4,nm} =& T\Big[\lambda_{4,nm}(\mu) + \frac{1}{(4\pi)^2}\Big(g^2g'^2 - L_f\lambda_{4,nm}g_{Y,1}^2 \\
&- L_b\Big(\frac{3}{2}g^2g'^2 + (\lambda_{1,n} + \lambda_{1,m} + 4\lambda_{3,nm} + 2\lambda_{4,nm})\lambda_{4,nm} + 4\lambda_{5,nm}^2 \\
&+ \sum_{k\neq n,m}(\lambda_{3,kn}\lambda_{3,km} + \lambda_{4,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,kn}\lambda_{4,km} + \frac{1}{2}\lambda_{3,km}\lambda_{4,kn}) \\
&- \frac{3}{2}\lambda_{4,nm}(3g^2 + g'^2)\Big)\Big)\Big] && (7.64c)\\
\Lambda_{5,nm} =& T\Big[\lambda_{5,nm}(\mu) - \frac{1}{(4\pi)^2}\Big(L_f\lambda_{5,nm}g_{Y,1}^2 + L_b\Big(\lambda_{5,nm}(\lambda_{1,n} + \lambda_{1,n} + 4\lambda_{3,nm} + 6\lambda_{4,nm}) \\
&+ \sum_{k\neq n,m}\lambda_{5,kn}\lambda_{5,km} - \frac{3}{2}\lambda_{5,nm}(3g^2 + g'^2)\Big)\Big)\Big]. && (7.64d)
\end{aligned}
$$

The remaining couplings are the same as in chapter 6.

We can also include a mass-mixing term which softly breaks the Z_2 symmetries

$$\mathcal{L}_{\text{mix}} = -\sum_{m<n} \frac{1}{2}(m_{nm}^2 \Phi_n^\dagger \Phi_m + \text{h.c.}). \tag{7.65}$$

The terms in $\mathcal{L}_{\text{mix}}$ can be treated as perturbations, and we can calculate the counterterms as we did in eq. (4.16). The counterterms become

$$\delta m_{nm}^2 = \frac{1}{(4\pi)^2\epsilon}\Big[(\lambda_{3,nm} + 2\lambda_{4,nm})m_{nm}^2 + 3\lambda_{5,nm} m_{nm}^{*2}\Big] \tag{7.66}$$

and the β-functions take the form

$$\beta_{m_{nm}^2} = \frac{1}{(4\pi)^2}\Big[2(\lambda_{3,nm} + 2\lambda_{4,nm})m_{nm}^2 + 6\lambda_{5,nm} m_{nm}^{*2} - m_{nm}^2\Big(\frac{9}{2}g^2 + \frac{3}{2}g'^2 - (\delta_{n1} + \delta_{m1})g_{Y,1}^2\Big)\Big]. \tag{7.67}$$

We also include mass-mixing terms in the effective three-dimensional theory. The three-dimensional mass-mixing parameters become

$$\begin{aligned}\mu_{nm}^2 = m_{nm}^2 &- \Big[(\lambda_{3,nm} + 2\lambda_{4,nm})m_{nm}^2 + 3\lambda_{5,nm} m_{nm}^{*2} - \Big(\frac{9}{4}g^2 + \frac{3}{4}g'^2\Big)m_{nm}^2\Big]\frac{L_b}{(4\pi)^2} \\ &- (\delta_{n1} + \delta_{m1})\frac{g_{Y,1}^2 m_{nm}^2}{2(4\pi)^2} L_f. \end{aligned} \tag{7.68}$$

All the three-dimensional parameters are independent of the renormalisation scale. This serves as an independent check for the correctness of our results.

CHAPTER 8

CONCLUSION AND OUTLOOK

We have calculated the β-functions of the gauge couplings g and g', and the scalar couplings λ_i, where $i = 1, \dots, 5$, for the Two-Higgs Doublet Model in the Landau gauge at one-loop. The results are in agreement with previous calculations of the β-functions [78]. In addition, the four-dimensional theory has been matched to an effective three-dimensional theory through the method called dimensional reduction, where the non-zero Matsubara modes have been integrated out. As the fermions have only non-zero Matsubara modes, the effective theory is purely bosonic. The masses and couplings of the effective theory have been determined as a function of the the masses and couplings of the original four-dimensional theory, and the temperature. This is done by calculating correlators in the three- and four-dimensional theory, and requiring that the long distance behaviour of the two theories should be the same, i.e. the correlators with zero external momentum should be matched. The scalar correlators were calculated using the effective potential, with different choices for the background fields. When comparing the results with previous calculations of the effective couplings, we discovered a discrepancy in the results by Losada [25]. There is a factor of 2 difference in the three-dimensional scalar couplings Λ_4 and Λ_5. As our results for the β-functions for the scalar couplings agree with the review article by Branco *et al.* [78], and the factors trivially go through in the calculation of the three-dimensional couplings Λ_4 and Λ_5, we believe the error is in the paper by Losada. Apart from the factor of 2, our results agreed with the calculations by Losada. We extended the calculation to include both the hypercharge gauge boson and the fermions.

We also extended the calculation to the N-Higgs Doublet Model. This is the first calculation of dimensional reduction for the general N-Higgs Doublet Model with softly broken Z_2 symmetries at one-loop. In the limiting cases of $N_h = 1$ and $N_h = 2$ the results agree with the SM and 2HDM results. The method of using the effective potential, with different choices for the background field, turned out to easily generalise to N_h doublets. Also, the β-functions was calculated for the NHDM.

At first we calculated the dimensional reduction with a strict Z_2 symmetry imposed, i.e. with no mass-mixing. Later, we relaxed this restriction, and treated the mass-mixing term as a perturbation. The corrections to the mass-mixing term in the effective theory turned out to be of order g^4, while the corrections to the other mass terms are of order g^2. The corrections to the couplings are also of order g^4. To be consistent with the order of accuracy in our calculations, we should have extended the calculations for the mass terms to two-loops and order g^4. We reserve this calculation as a natural extension of the calculations in this thesis.

Also, the gluonic sector of the 2HDM has been neglected. This could be incorporated in the calculations with minimal difficulties.

In the dimensional reduction step we integrated out the superheavy modes, i.e. the modes with a mass of order T. A natural second step is to also integrate out the heavy modes, i.e. the modes with a mass of order gT. Thus, the temporal part of the gauge bosons would be integrated out, possibly along with some scalar doublets.

The 2HDM has been proposed as a possible candidate for explaining baryogenesis. In the collaboration between NTNU, the University of Stavanger and the University of Helsinki we do numerical simulation of the effective three-dimensional theory to find the region of parameter space where the electroweak phase transition is a sufficiently strong first order phase transition. The calculations of dimensional reduction will be a major part of the project of determining the potential of the 2HDM for explaining baryogenesis at the electroweak phase transition. With the extension to the general NDHM we are in the reach of investigating a large family of theories as possible candidates for the theory describing Nature. However, the parameter space increases rapidly when additional scalar doublets are included, and thus makes numerical simulations impractial.

APPENDIX A

NOTATION AND CONVENTIONS

Here we establish the notation used in this thesis. Both Euclidean and Minkowski space play a role in thermal field theory. We will for the most time be working in Euclidean space, where we write

$$X = (\tau, x_i), \quad x \equiv |\boldsymbol{x}|, \quad S = \int_X \mathcal{L}, \tag{A.1}$$

where $i = 1, \ldots, d$,

$$\int_X \equiv \int_0^\beta d\tau \int_{\boldsymbol{x}} \equiv \int_0^\beta d\tau \int d^d x, \quad \beta \equiv \frac{1}{T}, \tag{A.2}$$

and d is the dimensionality of space. When we move to Minkowski space we will explicitly label the action and Lagrangian accordingly, S_M and $\mathcal{L}_M$. Going to momentum space, we have

$$K \equiv (\omega_n, k_i), \quad k \equiv |\boldsymbol{k}|, \quad \phi(X) = \sum\!\!\!\!\!\!\!\int_K \tilde{\phi}(K) e^{iK\cdot X} \tag{A.3}$$

where

$$\sum\!\!\!\!\!\!\!\int_K \equiv T \sum_{\omega_n^B} \int_{\boldsymbol{k}} \tag{A.4a}$$

$$\sum\!\!\!\!\!\!\!\int_{\{K\}} \equiv T \sum_{\omega_n^F} \int_{\boldsymbol{k}}, \tag{A.4b}$$

where ω_n^B and ω_n^F are the discrete Matsubara frequencies. The square of a four-vector in Euclidean space is $K^2 = \omega_n^2 + k^2$. We will use dimensional regularisation to regularise both ultraviolet and infrared divergences, and the dimensionality of space is $d = 3 - 2\epsilon$, while the dimensionality of space-time is $D = 4 - 2\epsilon$. The integral is defined in eq. (C.3).

We employ the natural units, where the speed of light c, the Boltzmann constant k_B and the reduced Planck constant $\hbar$ all have been set to unity, $c = k_B = \hbar = 1$.

APPENDIX B

FEYNMAN RULES

The Feynman rules are given in Euclidean spacetime in the unbroken phase, i.e. they are valid in the high temperature limit, where the Higgs field expectation values vanish. The gluon sector is not included, as we will not include the gluons in the dimensional reduction step.

B.1 Propagators

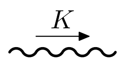
We define the projection operators

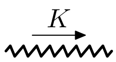
$$\mathcal{P}_T(K)_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{K_\mu K_\nu}{K^2}, \tag{B.1}$$

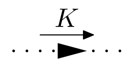
$$\mathcal{P}_R \equiv \frac{1}{2}(1+\gamma_5), \tag{B.2}$$

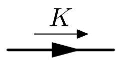
$$\mathcal{P}_L \equiv \frac{1}{2}(1-\gamma_5). \tag{B.3}$$

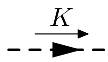
The propagators take the form

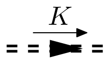
U(1)$_Y$ gauge boson: μ  $\nu = \dfrac{\mathcal{P}_T(K)_{\mu\nu}}{K^2}$ (B.4)

SU(2)$_L$ gauge boson: $a\mu$  $b\nu = \dfrac{\delta_{ab}\mathcal{P}_T(K)_{\mu\nu}}{K^2}$ (B.5)

SU(2)$_L$ ghost: a  $b = \dfrac{\delta_{ab}}{K^2}$ (B.6)

Fermions:  $= P_{L/R}\dfrac{i}{\not{K}}$ (B.7)

Φ_1 Higgs doublet: i  $j = \dfrac{\delta_{ij}}{K^2}$ (B.8)

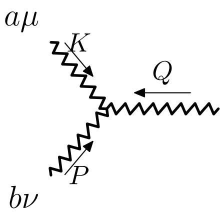
Φ_2 Higgs doublet: i 

$j = \dfrac{\delta_{ij}}{K^2}$ (B.9)

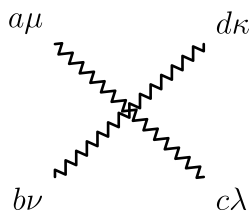
B.2 Interactions

The interaction vertices take the form

SU(2)$_L$ gauge boson self-interaction

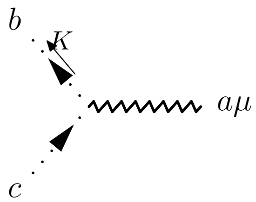


$$= -ig\epsilon^{abc}\Big[(P-Q)_\mu\delta_{\nu\lambda} + (Q-K)_\nu\delta_{\lambda\mu} + (K-P)_\lambda\delta_{\mu\nu}\Big] \tag{B.10}$$



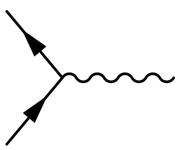
$$= g^2\Big[\delta_{ab}\delta_{cd}(\delta_{\mu\lambda}\delta_{\nu\kappa} + \delta_{\mu\kappa}\delta_{\nu\lambda} - 2\delta_{\mu\nu}\delta_{\lambda\kappa}) \tag{B.11}$$
$$+\delta_{ac}\delta_{bd}(\delta_{\mu\nu}\delta_{\lambda\kappa} + \delta_{\mu\kappa}\delta_{\nu\lambda} - 2\delta_{\mu\lambda}\delta_{\nu\kappa})$$
$$+\delta_{ad}\delta_{bc}(\delta_{\mu\nu}\delta_{\lambda\kappa} + \delta_{\mu\lambda}\delta_{\nu\kappa} - 2\delta_{\mu\kappa}\delta_{\nu\lambda})$$

SU(2)$_L$ gauge boson - ghost interaction

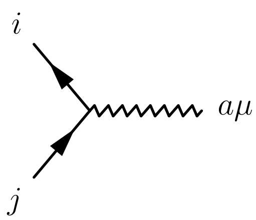


$$= ig\epsilon^{abc}K_\mu \tag{B.12}$$

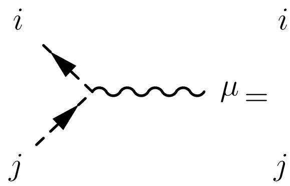
Gauge boson - matter interaction



$$= \frac{i}{2}g'Y\gamma_\mu \qquad \text{(all fermions)} \tag{B.13}$$



$$= \frac{i}{2}g(\tau_a)_{ij}\gamma_\mu \qquad \text{(left-handed fermions)} \tag{B.14}$$



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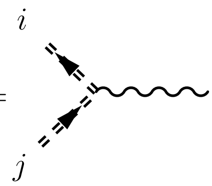
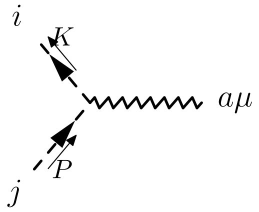


$$= -\frac{1}{2}g'\delta_{ij}(K+P)_\mu \tag{B.15}$$



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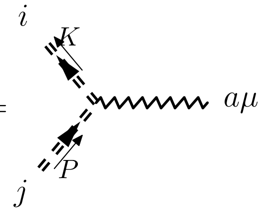
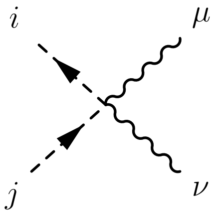


$$= -\frac{1}{2} g (\tau_a)_{ij} (K + P)_\mu \tag{B.16}$$



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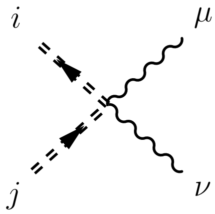
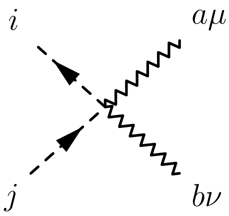


$$= -\frac{1}{2} g'^2 \delta_{ij} \delta_{\mu\nu} \tag{B.17}$$



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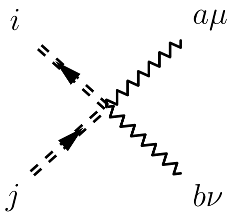
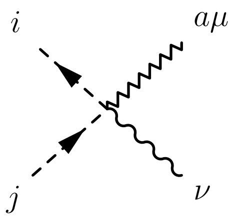


$$= -\frac{1}{2} g^2 \delta_{ij} \delta_{ab} \delta_{\mu\nu} \tag{B.18}$$



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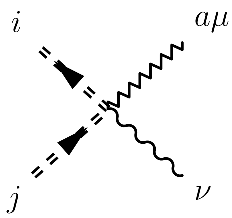
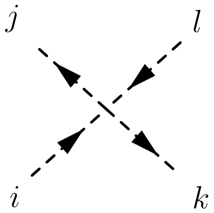
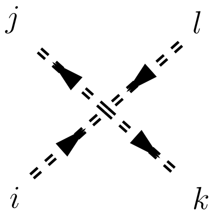


$$= -\frac{1}{2} g g' (\tau_a)_{ij} \delta_{\mu\nu} \tag{B.19}$$

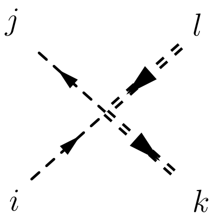
Scalar interactions



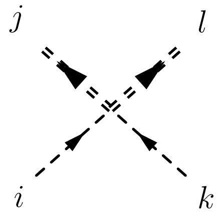
$$= -\lambda_1 (\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}) \tag{B.20}$$



$$= -\lambda_2 (\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}) \tag{B.21}$$



$$= -(\lambda_3 \delta_{ij}\delta_{kl} + \lambda_4 \delta_{ik}\delta_{jl}) \tag{B.22}$$



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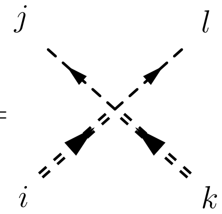
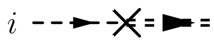
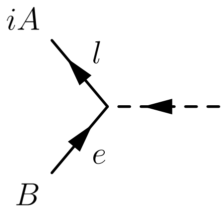


$$= -\lambda_5(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \qquad \text{(B.23)}$$

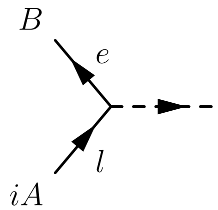


$$j = \frac{1}{2}\delta_{ij} m_{12}^2 \qquad \text{(B.24)}$$

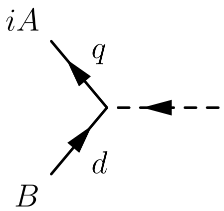
Yukawa interactions



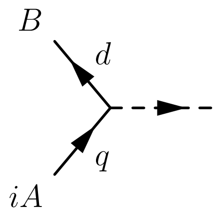
$$j = -\delta_{ij} h_{AB}^{(e)}$$



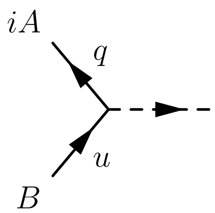
$$j = -\delta_{ij} h_{AB}^{(e)*} \qquad \text{(B.25)}$$



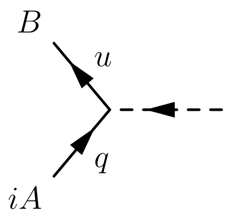
$$j = -\delta_{ij} h_{AB}^{(d)}$$



$$j = -\delta_{ij} h_{AB}^{(d)*} \qquad \text{(B.26)}$$



$$j = -i(\tau_2)_{ij} h_{AB}^{(u)}$$



$$j = -i(\tau_2)_{ij} h_{AB}^{(u)*} \qquad \text{(B.27)}$$

APPENDIX C

SUM-INTEGRALS

We use the imaginary-time formalism for quantum field theories at finite temperature. The 4-momentum $K = (\omega_n, \boldsymbol{k})$ is Euclidean, $K^2 = \omega_n^2 + \boldsymbol{k}^2$, where the Matsubara frequencies take discrete values, $\omega_n = \omega_n^B = 2\pi nT$ and $\omega_n = \omega_n^F = (2n+1)\pi T$ for bosons and fermions, respectively, with $n \in \mathbb{Z}$. Loop diagrams involve sums over ω_n and integrals over $\boldsymbol{k}$. We will employ dimensional regularisation to regularise both ultraviolet and infrared divergences. We follow the common notation for the regularised sum-integrals

$$\sum\!\!\!\!\!\!\!\int_K \equiv \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon T \sum_{\omega_n = 2\pi nT} \int \frac{d^d k}{(2\pi)^d} \tag{C.1}$$

$$\sum\!\!\!\!\!\!\!\int_{\{K\}} \equiv \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon T \sum_{\omega_n = (2n+1)\pi T} \int \frac{d^d k}{(2\pi)^d} \tag{C.2}$$

$$\int_k \equiv \left(\frac{e^\gamma \mu^2}{4\pi}\right)^\epsilon \int \frac{d^d k}{(2\pi)^d}, \tag{C.3}$$

where $d = 3 - 2\epsilon$ is the dimensionality of space and μ is the renormalisation scale. The factor $(e^\gamma/4\pi)^\epsilon$ is introduced so that μ coincides with the $\overline{\mathrm{MS}}$ renormalisation scale after minimal subtraction of the poles in ϵ due to ultraviolet divergences.

We list the one-loop sum-integrals needed in the thesis, which have been calculated in [25; 28; 29; 71]:

$$I_1^{4b} \equiv \sum\!\!\!\!\!\!\!\int_K \frac{1}{K^2} = \frac{T^2}{12}\Big[1 + \mathcal{O}(\epsilon)\Big] \tag{C.4}$$

$$I_2^{4b} \equiv \sum\!\!\!\!\!\!\!\int_K \frac{1}{(K^2)^2} = \frac{1}{(4\pi)^2}\Big[\frac{1}{\epsilon} + L_b + \mathcal{O}(\epsilon)\Big] \tag{C.5}$$

$$I_{3,1}^{4b} \equiv \sum\!\!\!\!\!\!\!\int_K \frac{K_0^2}{(K^2)^3} = \frac{1}{4(4\pi)^2}\Big[\frac{1}{\epsilon} + L_b + 2 + \mathcal{O}(\epsilon)\Big] \tag{C.6}$$

$$I_{4,2}^{4b} \equiv \sum\!\!\!\!\!\!\!\int_K \frac{K_0^4}{(K^2)^4} = \frac{1}{8(4\pi)^2}\Big[\frac{1}{\epsilon} + L_b + \frac{8}{3} + \mathcal{O}(\epsilon)\Big] \tag{C.7}$$

$$\sum\!\!\!\!\!\!\!\int_K \frac{K_\mu K_\nu}{(K^2)^3} = \frac{1}{2}\Big[\delta_{\mu i}\delta_{\nu j}\delta_{ij} - (1-2\epsilon)\delta_{\mu 0}\delta_{\nu 0}\Big]\sum\!\!\!\!\!\!\!\int_K \frac{1}{K^2} \tag{C.8}$$

where $L_b = 2\log\frac{\mu}{4\pi T} + 2\gamma_E$ and γ_E is the Euler-Mascheroni constant, as defined in [28]. The corresponding fermionic one-loop sum-integrals are

$$I_1^{4f} \equiv \sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{1}{K^2} = -\frac{T^2}{24}\Big[1+\mathcal{O}(\epsilon)\Big] \tag{C.9}$$

$$I_2^{4f} \equiv \sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{1}{(K^2)^2} = \frac{1}{(4\pi)^2}\Big[\frac{1}{\epsilon} + L_f + \mathcal{O}(\epsilon)\Big] \tag{C.10}$$

$$I_{3,1}^{4f} \equiv \sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{K_0^2}{(K^2)^3} = \frac{1}{4(4\pi)^2}\Big[\frac{1}{\epsilon} + L_f + 2 + \mathcal{O}(\epsilon)\Big] \tag{C.11}$$

$$I_{4,2}^{4f} \equiv \sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{K_0^4}{(K^2)^4} = \frac{1}{8(4\pi)^2}\Big[\frac{1}{\epsilon} + L_f + \frac{8}{3} + \mathcal{O}(\epsilon)\Big] \tag{C.12}$$

$$I_{\alpha,\beta}^{4f} \equiv \sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{(K_0^2)^\beta}{(K^2)^\alpha} = \Big(2^{2\alpha-2\beta-d} - 1\Big)\sum\!\!\!\!\!\!\!\int_K \frac{(K_0^2)^\beta}{(K^2)^\alpha} \tag{C.13}$$

$$\sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{K_\mu K_\nu}{(K^2)^2} = \frac{1}{2}\Big[\delta_{\mu i}\delta_{\nu j}\delta_{ij} - (1-2\epsilon)\delta_{\mu 0}\delta_{\nu 0}\Big]\sum\!\!\!\!\!\!\!\int_{\{K\}} \frac{1}{K^2} \tag{C.14}$$

where $L_f = L_b + 4\log 2$. For the effective potential we need the integrals

$$J_b(m) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_K \log(K^2+m^2) = \Big[\frac{m^2T^2}{24} - \frac{m^3T}{12\pi} - \frac{m^4}{64\pi^2}\Big(\frac{1}{\epsilon} + L_b\Big)\Big] + \mathcal{O}\Big(\frac{m^6}{T^2}\Big) \tag{C.15}$$

$$J_f(m) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\{K\}} \log(K^2+m^2) = \Big[-\frac{m^2T^2}{48} + \frac{m^4}{64\pi^2}\Big(\frac{1}{\epsilon} + L_f\Big)\Big] + \mathcal{O}\Big(\frac{m^6}{T^2}\Big). \tag{C.16}$$

In the three-dimensional effective theory we need the one-loop integrals:

$$\int_k \frac{1}{p^2+m^2} = -\frac{m}{4\pi}\Big[1+\mathcal{O}(\epsilon)\Big] \tag{C.17}$$

$$\int_k \frac{1}{(p^2+m_1^2)(p^2+m_2^2)} = \frac{1}{4\pi(m_1+m_2)}\Big[1+\mathcal{O}(\epsilon)\Big]. \tag{C.18}$$

C.1 Derivation of some sum-integrals

We will in this section derive some formulas for sum-integrals later used in loop calculations. The calculations are performed for bosonic fields, but can easily be extended to include fermionic fields as well.

We start with the sum-integral

$$J(m,T) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_K \log(K^2+m^2). \tag{C.19}$$

From this integral we can obtain many other sum-integrals by taking the derivative with respect to the mass m,

$$I(m,T) = \frac{1}{m}\frac{d}{dm}J(m,T) = \sum\!\!\!\!\!\!\!\int_K \frac{1}{K^2+m^2}. \tag{C.20}$$

For future reference we will calculate a generic integral in detail,

$$\Phi(m,d,A) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)^A} \tag{C.21}$$

where d and A are left unspecified. By changing the integration variable thrice, $z = k^2$, $m^2t = z$, $s = (t+1)^{-1}$, we get

$$\Phi(m,d,A) = \frac{\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\int_0^\infty dz z^{\frac{d-2}{2}}(z+m^2)^{-A} \tag{C.22}$$

$$= \frac{m^{d-2A}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dt t^{d/2-1}(1+t)^{-A} \tag{C.23}$$

$$= \frac{m^{d-2A}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^1 ds s^{A-d/2-1}(1-s)^{d/2-1}. \tag{C.24}$$

We recognise that the last integral is simply the product of gamma functions,

$$\int_0^1 ds s^{A-d/2-1}(1-s)^{d/2-1} = \frac{\Gamma(A-d/2)\Gamma(d/2)}{\Gamma(A)} \tag{C.25}$$

so we get the formula

$$\Phi(m,d,A) = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(A-d/2)}{\Gamma(A)}\frac{1}{(m^2)^{A-d/2}}. \tag{C.26}$$

Now we are in a position to evaluate the sum-integral

$$I = \SumInt_K \frac{1}{K^2+m^2}. \tag{C.27}$$

Firstly, we divide the sum into the zero and the non-zero parts

$$I = I^{n=0} + I^{n\neq 0}. \tag{C.28}$$

Using eq. (C.26) with $d = 3-2\epsilon$ and $A = 1$ we can directly evaluate the zero mode integral

$$I^{n=0} = T\Phi(m, 3-2\epsilon, 1) = \frac{T}{(4\pi)^{3/2-\epsilon}}\frac{\Gamma(-1/2+\epsilon)}{\Gamma(1)}\frac{1}{(m^2)^{-1/2+\epsilon}} = -\frac{Tm}{4\pi} + \mathcal{O}(\epsilon). \tag{C.29}$$

This evaluation clearly shows the unintuitive behaviour of dimensional regularisation; a linearly divergent, positive definite integral turns out to give a finite and negative result. We can also from this get the zero mode contribution to eq. (C.19) by integrating our result,

$$J^{n=0} = -\frac{Tm^3}{12\pi} + \mathcal{O}(\epsilon) \tag{C.30}$$

Now we turn our attention to the non-zero modes. We start by Taylor expanding the integrand, using eq. (C.26) with $m = 2\pi nT$ and $A = l+1$, and using the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$,

$$I^{n\neq 0} = \sum\!\!\!\!\!\!\!\int \frac{1}{\omega_n^2 + k^2 + m^2} = 2T \sum_{n=1}^{\infty} \int \frac{d^d k}{(2\pi)^2} \sum_{l=0}^{\infty} (-1)^l \frac{(m^2)^l}{\left[(2\pi n T)^2 + k^2\right]^{l+1}} \tag{C.31}$$

$$= 2T \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{(m^2)^l}{(4\pi)^{d/2}} \frac{\Gamma(l+1-d/2)}{\Gamma(l+1)} \frac{1}{(2\pi n T)^{2l+2-d}} \tag{C.32}$$

$$= \frac{2T}{(4\pi)^{d/2}(2\pi T)^{2-d}} \sum_{l=0}^{\infty} \left[\frac{-m^2}{(2\pi n T)}\right]^l \frac{\Gamma(l+1-d/2)}{\Gamma(l+1)} \zeta(2l+2-d). \tag{C.33}$$

Expanding the first few terms, with $d = 3 - 2\epsilon$, we get

$$I^{n\neq 0} = \frac{T^2}{12} - \frac{2m^2\mu^{-\epsilon}}{(4\pi)^2}\left[\frac{1}{2\epsilon} + \log\left(\frac{\overline{\mu} e^{\gamma_E}}{4\pi T}\right)\right] + \frac{2m^4\zeta(3)}{(4\pi)^4 T^2} + \mathcal{O}\left(\frac{m^6}{T^4}\right) + \mathcal{O}(\epsilon). \tag{C.34}$$

Combining the zero-mode and non-zero mode contributions, we get

$$I = \frac{T^2}{12} - \frac{Tm}{4\pi} - \frac{2m^2\mu^{-\epsilon}}{(4\pi)^2}\left[\frac{1}{2\epsilon} + \log\left(\frac{\overline{\mu} e^{\gamma_E}}{4\pi T}\right)\right] + \frac{2m^4\zeta(3)}{(4\pi)^4 T^2} + \mathcal{O}\left(\frac{m^6}{T^4}\right) + \mathcal{O}(\epsilon), \tag{C.35}$$

where we have used eqs. (C.28), (C.29) and (C.34). By setting $m = 0$ we get eq. (C.4).

APPENDIX D

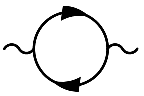
DETAILED RESULTS FOR LOOP DIAGRAMS

The one-loop diagrams needed for dimensional reduction are provided, expressed in terms of the master integrals from appendix C.

D.1 Self-energy diagrams

The diagrams needed for wave function renormalisation and Debye mass for the bosons are presented. The diagrams with only a quartic vertex contribute to the Debye mass, while the wave function renormalisation is extracted from the part quadratic in momentum.

$\mathrm{U}(1)_Y$ gauge boson self-energy

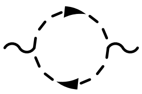


$$= -\frac{g'^2}{2}(d-1)\Big[(1-2^{2-d})I_1^{4b} + \frac{1}{6}(2^{4-d}-1)P^2 I_2^{4b}\Big]\sum_f Y_f^2 \tag{D.1}$$

for $\mu = 0, \nu = 0,$

$$= \frac{g'^2}{6}(2^{4-d}-1)\Big[P_i P_j - \delta_{ij}P^2\Big] I_2^{4b} \sum_f Y_f^2$$

for $\mu = i, \nu = j$

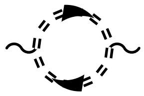


$$= -2g'^2\Big[\frac{1}{2}(d-2)I_1^{4b} + \frac{1}{12}(4-d)P^2 I_2^{4b}\Big] \tag{D.2}$$

for $\mu = 0, \nu = 0,$

$$= g'^2\Big[\delta_{ij}I_1^{4b} + \frac{1}{6}(P_i P_j - \delta_{ij}P^2) I_2^{4b}\Big]$$

for $\mu = i, \nu = j$



$$= -2g'^2\Big[\frac{1}{2}(d-2)I_1^{4b} + \frac{1}{12}(4-d)P^2 I_2^{4b}\Big] \tag{D.3}$$

for $\mu = 0, \nu = 0,$

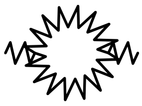
$$= g'^2\Big[\delta_{ij} I_1^{4b} + \frac{1}{6}(P_i P_j - \delta_{ij} P^2) I_2^{4b}\Big]$$

for $\mu = i, \nu = j$

$$= -g'^2 \delta_{\mu\nu} I_1^{4b} \tag{D.4}$$

$$= -g'^2 \delta_{\mu\nu} I_1^{4b} \tag{D.5}$$

$\mathbf{SU(2)}_L$ gauge boson self-energy

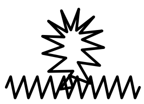


$$= 2g^2 \delta_{ab}\Big[-d(d-2)I_1^{4b} + \frac{1}{12}(16 - 3d + 2d^2)P^2 I_2^{4b}\Big] \tag{D.6}$$

for $\mu = 0, \nu = 0,$

$$= g^2 \delta_{ab}\Big[2d\delta_{ij} I_1^{4b} + \Big(\frac{1}{6}(31-2d)\delta_{ij}P^2 - \frac{1}{3}(17-d)P_i P_j\Big) I_2^{4b}\Big]$$

for $\mu = i, \nu = j$

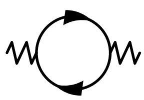


$$= -g^2 \delta_{ab} d I_1^{4b} \tag{D.7}$$

for $\mu = 0, \nu = 0,$

$$= -g^2 \delta_{ab}(2d-1)\delta_{ij} I_1^{4b}$$

for $\mu = i, \nu = j$

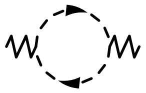


$$= -g^2 \delta_{ab}(d-1)\Big[(1 - 2^{2-d})I_1^{4b} + \frac{1}{6}(2^{4-d} - 1)P^2 I_2^{4b}\Big]\sum_{\text{left}} \tag{D.8}$$

for $\mu = 0, \nu = 0,$

$$= \frac{1}{3} g^2 \delta_{ab}(2^{4-d} - 1)\Big[P_i P_j - \delta_{ij} P^2\Big] I_2^{4b} \sum_{\text{left}}$$

for $\mu = i, \nu = j$

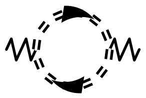


$$= -2g^2\delta_{ab}\Big[\frac{1}{2}(d-2)I_1^{4b} + \frac{1}{12}(4-d)P^2 I_2^{4b}\Big] \tag{D.9}$$

for $\mu = 0, \nu = 0$,

$$=g^2\delta_{ab}\Big[\delta_{ij}I_1^{4b} + \frac{1}{6}(P_iP_j - \delta_{ij}P^2)I_2^{4b}\Big]$$

for $\mu = i, \nu = j$



$$= -2g^2\delta_{ab}\Big[\frac{1}{2}(d-2)I_1^{4b} + \frac{1}{12}(4-d)P^2 I_2^{4b}\Big] \tag{D.10}$$

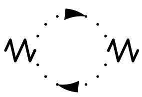
for $\mu = 0, \nu = 0$,

$$=g^2\delta_{ab}\Big[\delta_{ij}I_1^{4b} + \frac{1}{6}(P_iP_j - \delta_{ij}P^2)I_2^{4b}\Big]$$

for $\mu = i, \nu = j$

$$= -g^2\delta_{ab}\delta_{\mu\nu}I_1^{4b} \tag{D.11}$$

$$= -g^2\delta_{ab}\delta_{\mu\nu}I_1^{4b} \tag{D.12}$$



$$=2g^2\delta_{ab}\Big[\frac{1}{2}(d-2)I_1^{4b} + \frac{1}{12}(4-d)P^2 I_2^{4b}\Big] \tag{D.13}$$

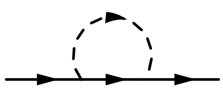
for $\mu = 0, \nu = 0$,

$$= -g^2\delta_{ab}\Big[\delta_{ij}I_1^{4b} - \frac{1}{6}(2P_iP_j + \delta_{ij}P^2)I_2^{4b}\Big]$$

for $\mu = i, \nu = j$

Lepton doublet self-energy

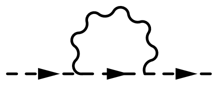
We calculate the divergent momentum dependent part of the lepton doublet self-energy in order to determine the wave function renormalisation.



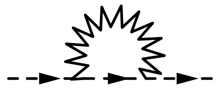
$$=\frac{1}{2}i\Big[h^{(e)\dagger}h^{(e)}\Big]_{AB}\delta_{ij}\not{P}I_2^{4b} \tag{D.14}$$

Higgs doublet self-energies

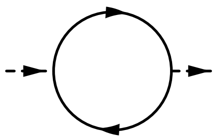
We include only those diagrams which contribute to the wave function renormalisation of the Higgs fields, i.e. those diagrams with a divergence proportional to the external momentum squared. The mass renormalisation will be extracted from the effective potential.



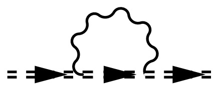
$$=\frac{3}{4}g'^2\delta_{ij}P^2I_2^{4b} \tag{D.15}$$



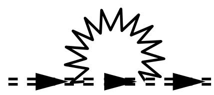
$$=\frac{9}{4}g^2\delta_{ij}P^2I_2^{4b} \tag{D.16}$$



$$= g_{Y,1}^2\delta_{ij}(2I_1^{4f} - P^2I_2^{4f}) \tag{D.17}$$



$$=\frac{3}{4}g'^2\delta_{ij}P^2I_2^{4b} \tag{D.18}$$

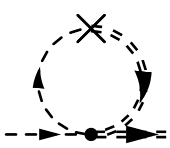


$$=\frac{9}{4}g^2\delta_{ij}P^2I_2^{4b} \tag{D.19}$$

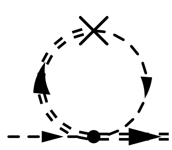
where $g_{Y,1}^2 = \mathrm{Tr}[h^{(e)\dagger}h^{(e)} + N_c h^{(d)\dagger}h^{(d)} + N_c h^{(u)\dagger}h^{(u)}]$.

The $\Phi_1^\dagger\Phi_2$ correlator

To extract the mass mixing counterterm, the diagrams for the $\Phi_1^\dagger\Phi_2$ correlator have been calculated. The momentum-independent divergences are absorbed by the mass mixing counterterm.

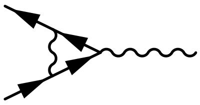


$$= -\frac{1}{2}(\lambda_3 + 2\lambda_4)\delta_{ij}m_{12}^2I_2^{4b} \tag{D.20}$$

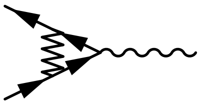


$$= -\frac{3}{2}\lambda_5\delta_{ij}m_{12}^{*2}I_2^{4b} \tag{D.21}$$

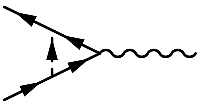
Lepton doublet - U$(1)_Y$ gauge boson vertex



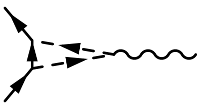
$$=\frac{1}{8} i g'^3 Y_l^3 \delta_{ij} \gamma_0 (d-3) I_2^{4b} \quad \text{for } \mu = 0$$
$$=0 \quad \text{for } \mu = r \tag{D.22}$$



$$=\frac{3}{8} i g^2 g' Y_l \delta_{ij} \gamma_0 (d-3) I_2^{4b} \quad \text{for } \mu = 0$$
$$=0 \quad \text{for } \mu = r \tag{D.23}$$

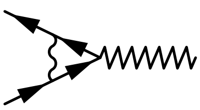


$$=\frac{1}{4} i g' Y_e \delta_{ij} \left[h^{(e)\dagger} h^{(e)}\right]_{AB} \gamma_0 (d-2) I_2^{4b} \quad \text{for } \mu = 0$$
$$=\frac{1}{4} i g' Y_e \delta_{ij} \left[h^{(e)\dagger} h^{(e)}\right]_{AB} \gamma_r I_2^{4b} \quad \text{for } \mu = r \tag{D.24}$$

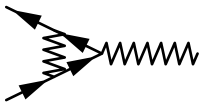


$$=\frac{1}{4} i g' \delta_{ij} \left[h^{(e)\dagger} h^{(e)}\right]_{AB} \gamma_0 (4-d) I_2^{4b} \quad \text{for } \mu = 0$$
$$=\frac{1}{4} i g' \delta_{ij} \left[h^{(e)\dagger} h^{(e)}\right]_{AB} \gamma_r I_2^{4b} \quad \text{for } \mu = r \tag{D.25}$$

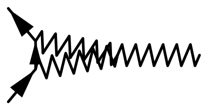
Lepton doublet - SU$(2)_L$ gauge boson vertex



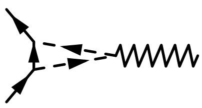
$$=\frac{1}{8} i g g'^2 Y_l^2 (\tau_a)_{ij} \gamma_0 (d-3) I_2^{4b} \quad \text{for } \mu = 0$$
$$=0 \quad \text{for } \mu = r \tag{D.26}$$



$$= -\frac{1}{8} i g^3 (\tau_a)_{ij} \gamma_0 (d-3) I_2^{4b} \quad \text{for } \mu = 0 \qquad (D.27)$$
$$= 0 \quad \text{for } \mu = r$$



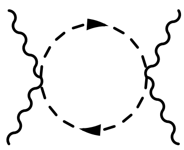
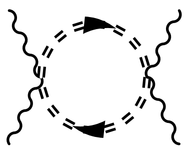
$$= \frac{3}{4} i g^3 (\tau_a)_{ij} \gamma_0 (4-d) I_2^{4b} \quad \text{for } \mu = 0 \qquad (D.28)$$
$$= \frac{3}{4} i g^3 (\tau_a)_{ij} \gamma_i I_2^{4b} \quad \text{for } \mu = r$$



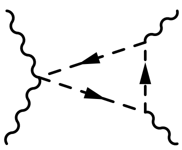
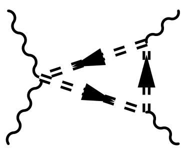
$$= \frac{1}{4} i g (\tau_a)_{ij} \left[h^{(e)\dagger} h^{(e)} \right]_{AB} \gamma_0 (4-d) I_2^{4b} \quad \text{for } \mu = 0 \qquad (D.29)$$
$$= \frac{1}{4} i g (\tau_a)_{ij} \left[h^{(e)\dagger} h^{(e)} \right]_{AB} \gamma_i I_2^{4b} \quad \text{for } \mu = r$$

$U(1)_Y$ gauge boson coupling

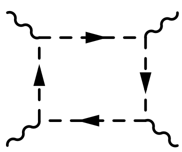
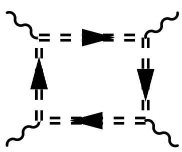
The four-point functions for the temporal component of the gauge fields are found at zero external momentum.

 = 

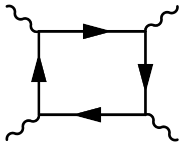
$$= \frac{3}{2} g'^4 I_2^{4b} \qquad (D.30)$$

 = 

$$= -3(4-d) g'^4 I_2^{4b} \qquad (D.31)$$

 = 

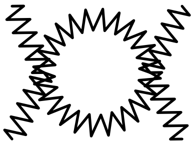
$$= \frac{1}{2} (6-d)(4-d) g'^4 I_2^{4b} \qquad (D.32)$$



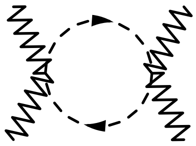
$$= -\frac{1}{4}(d-1)(d-3)(2^{4-d}-1)N_f\left(18 + N_c\frac{274}{81}\right)g'^4 I_2^{4b} \tag{D.33}$$

$\mathbf{SU}(2)_L$ gauge boson coupling

The $A_0^a A_0^b A_0^c A_0^d$ correlator at zero external momentum is



$$= \frac{1}{6}d(14+d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.34}$$



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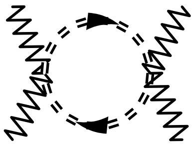
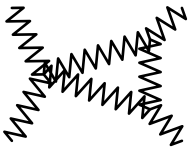
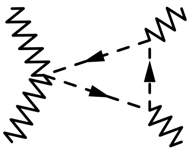


$$= \frac{1}{2}g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.35}$$



$$= -\frac{20}{3}d(4-d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.36}$$



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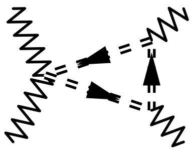
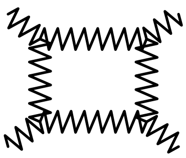
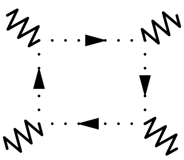


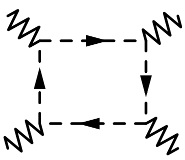
$$= -(4-d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.37}$$



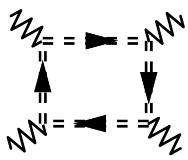
$$= \frac{4}{3}d(4-d)(6-d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.38}$$



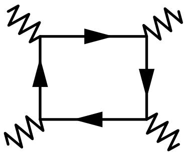
$$= -\frac{1}{6}(4-d)(6-d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.39}$$



$=$



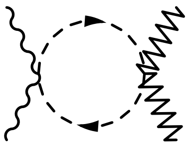
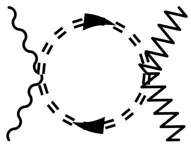
$$= \frac{1}{6}(4-d)(6-d)g^4(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_2^{4b} \tag{D.40}$$



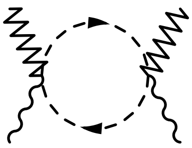
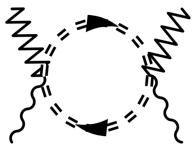
$$= -\frac{1}{6}(d-1)(d-3)(2^{4-d}-1)N_f(1+N_c)g^4(\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})I_2^{4b} \tag{D.41}$$

The $A_0^a A_0^b B_0^2$ correlator

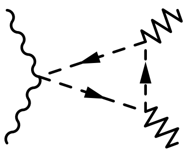
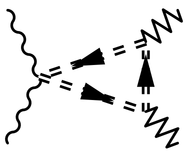
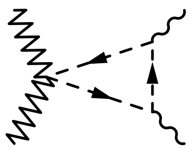
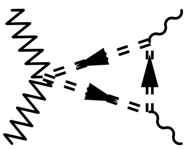
The four-point functions for the temporal component of the gauge fields are found at zero external momentum.

 = 

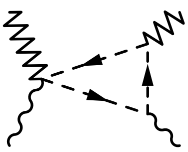
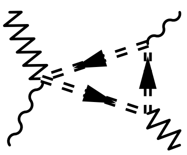
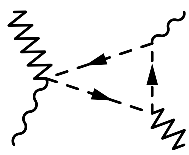
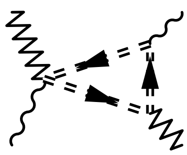
$$= \frac{1}{2}g^2 g'^2 \delta_{ab} I_2^{4b} \tag{D.42}$$

 = 

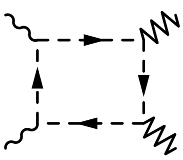
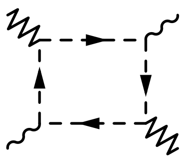
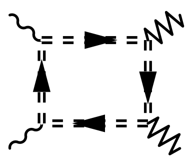
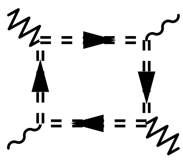
$$= g^2 g'^2 \delta_{ab} I_2^{4b} \tag{D.43}$$

 =  =  = 

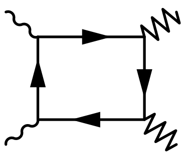
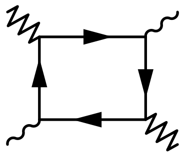
$$= -\frac{1}{2}(4-d)g^2 g'^2 \delta_{ab} I_2^{4b} \tag{D.44}$$

 =  =  = 

$$= -(4-d)g^2 g'^2 \delta_{ab} I_2^{4b} \tag{D.45}$$

 +  =  + 

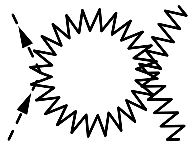
$$= \frac{1}{2}(4-d)(6-d)g^2 g'^2 \delta_{ab} I_2^{4b} \tag{D.46}$$

 + 

$$= -\frac{1}{2}(d-1)(d-3)N_f(Y_l^2+N_c Y_q^2)g^2 g'^2 \delta_{ab} I_2^{4f} \tag{D.47}$$

The $\Phi_1^{\dagger i}\Phi_1^j A_\mu^a A_\nu^b$ correlator

The diagrams contributing to the correlator with two scalar legs and two $SU(2)_L$ gauge boson legs at zero external momentum are presented below. The diagrams with two Φ_2 legs instead of Φ_1 take the same form, with the substitution $\lambda_1 \to \lambda_2$, and the absence of fermion loops (box diagrams).

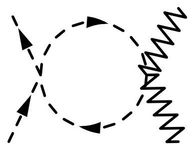


$$= \frac{3}{4} d g^4 \delta_{ij}\delta_{ab} I_2^{4b} \quad \text{(D.48a)}$$

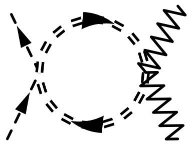
for $\mu = 0, \nu = 0$

$$= \Big(d - \frac{3}{4}\Big) g^4 \delta_{ij}\delta_{ab}\delta_{rs} I_2^{4b} \quad \text{(D.48b)}$$

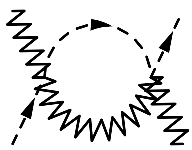
for $\mu = r, \nu = s$



$$= \frac{3}{2}\lambda_1 g^2 \delta_{ij}\delta_{ab}\delta_{\mu\nu} I_2^{4b} \quad \text{(D.49)}$$



$$= \Big(\lambda_3 + \frac{1}{2}\lambda_4\Big) g^2 \delta_{ij}\delta_{ab}\delta_{\mu\nu} I_2^{4b} \quad \text{(D.50)}$$

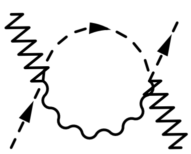


$$= \frac{1}{8} d g^4 \delta_{ij}\delta_{ab} I_2^{4b} \quad \text{(D.51a)}$$

for $\mu = 0, \nu = 0$

$$= \frac{3}{8} g^4 \delta_{ij}\delta_{ab}\delta_{rs} I_2^{4b} \quad \text{(D.51b)}$$

for $\mu = r, \nu = s$

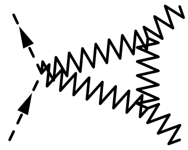


$$= \frac{1}{8} d g^2 g'^2 \delta_{ij}\delta_{ab} I_2^{4b} \quad \text{(D.52a)}$$

for $\mu = 0, \nu = 0$

$$= \frac{3}{8} g^2 g'^2 \delta_{ij}\delta_{ab}\delta_{rs} I_2^{4b} \quad \text{(D.52b)}$$

for $\mu = r, \nu = s$

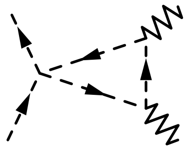


$$= -(4-d)dg^4\delta_{ab}\delta_{ij}I_2^{4b} \tag{D.53a}$$

for $\mu = 0, \nu = 0$

$$= -dg^4\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4b} \tag{D.53b}$$

for $\mu = r, \nu = s$

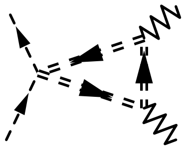


$$= -\frac{3}{2}(4-d)\lambda_1 g^2\delta_{ab}\delta_{ij}I_2^{4b} \tag{D.54a}$$

for $\mu = 0, \nu = 0$

$$= -\frac{3}{2}\lambda_1 g^2\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4b} \tag{D.54b}$$

for $\mu = r, \nu = s$

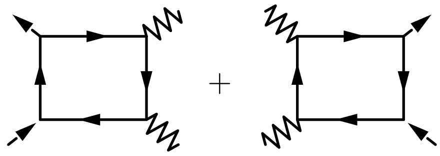


$$= -(4-d)\Big(\lambda_3 + \frac{1}{2}\lambda_4\Big)g^2\delta_{ab}\delta_{ij}I_2^{4b} \tag{D.55a}$$

for $\mu = 0, \nu = 0$

$$= -\Big(\lambda_3 + \frac{1}{2}\lambda_4\Big)g^2\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4b} \tag{D.55b}$$

for $\mu = r, \nu = s$



$$= -\frac{1}{2}(d-2)g^2g_{Y,1}^2\delta_{ab}\delta_{ij}I_2^{4f} \tag{D.56a}$$

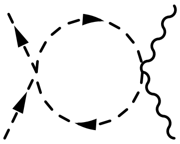
for $\mu = 0, \nu = 0$

$$= -\frac{1}{2}g^2g_{Y,1}^2\delta_{ab}\delta_{ij}\delta_{rs}I_2^{4f} \tag{D.56b}$$

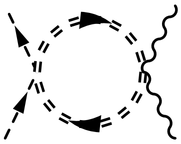
for $\mu = r, \nu = s$

The $\Phi_1^{\dagger i}\Phi_1^j B_\mu B_\nu$ correlator

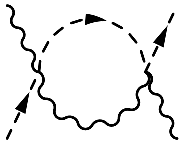
The contribution to the correlator with two Φ_2 scalar legs instead of Φ_1 is similar the diagrams listed below, except that no fermion loops are present (no box diagrams). Also, the substitution $\lambda_1 \to \lambda_2$ should be made wherever appropriate. The diagrams contributing to the correlator with two scalar legs and two $\mathrm{U}(1)_Y$ gauge boson legs at zero external momentum are



$$= \frac{3}{2}\lambda_1 g'^2 \delta_{\mu\nu}\delta_{ij} I_2^{4b} \tag{D.57}$$



$$= \left(\lambda_3 + \frac{1}{2}\lambda_4\right) g'^2 \delta_{\mu\nu}\delta_{ij} I_2^{4b} \tag{D.58}$$

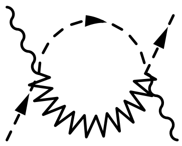


$$= \frac{1}{8} d g'^4 \delta_{ij} \tag{D.59a}$$

for $\mu = 0, \nu = 0$

$$= \frac{3}{8} g'^4 \delta_{ij}\delta_{rs} \tag{D.59b}$$

for $\mu = r, \nu = s$

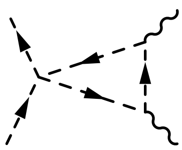


$$= \frac{3}{8} d g'^2 g^2 \delta_{ij} \tag{D.60a}$$

for $\mu = 0, \nu = 0$

$$= \frac{9}{8} g'^2 g^2 \delta_{ij}\delta_{rs} \tag{D.60b}$$

for $\mu = r, \nu = s$

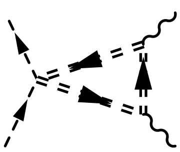


$$= -\frac{3}{2}(4-d) g'^2 \lambda_1 \delta_{ij} I_2^{4b} \tag{D.61a}$$

for $\mu = 0, \nu = 0$

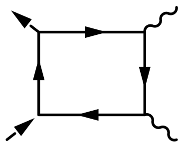
$$= -\frac{3}{2} g'^2 \lambda_1 \delta_{ij}\delta_{rs} I_2^{4b} \tag{D.61b}$$

for $\mu = r, \nu = s$



$$= -(4-d) g'^2 \left(\lambda_3 + \frac{1}{2}\lambda_4\right) \delta_{ij} I_2^{4b} \tag{D.62a}$$

$$\begin{aligned} &\text{for } \mu = 0, \nu = 0 \\ &= -g'^2\Big(\lambda_3 + \frac{1}{2}\lambda_4\Big)\delta_{ij}\delta_{rs}I_2^{4b} \\ &\text{for } \mu = r, \nu = s \end{aligned} \tag{D.62b}$$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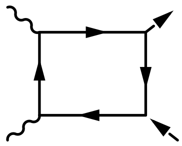
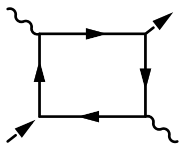


$$= -\frac{1}{2}(d-2)g'^2\delta_{ij}\,\mathrm{Tr}[(Y_l^2 + Y_e^2)h^{(e)}h^{(e)\dagger} + N_c(Y_q^2 + Y_d^2)h^{(d)}h^{(d)\dagger} + N_c(Y_q^2 + Y_u^2)h^{(u)}h^{(u)\dagger}]I_2^{4f} \tag{D.63a}$$

$$\begin{aligned} &\text{for } \mu = 0, \nu = 0 \\ &= -\frac{1}{2}g'^2\delta_{ij}\delta_{rs}\,\mathrm{Tr}[(Y_l^2 + Y_e^2)h^{(e)}h^{(e)\dagger} \\ &+ N_c(Y_q^2 + Y_d^2)h^{(d)}h^{(d)\dagger} + N_c(Y_q^2 + Y_u^2)h^{(u)}h^{(u)\dagger}]I_2^{4f} \end{aligned} \tag{D.63b}$$

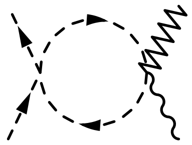
for $\mu = r, \nu = s$



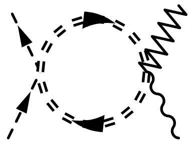
$$= g'^2\,\mathrm{Tr}[Y_lY_eh^{(e)}h^{(e)\dagger} + N_cY_qY_dh^{(d)}h^{(d)\dagger} + N_cY_qY_uh^{(u)}h^{(u)\dagger}]\delta_{ij}\delta_{\mu\nu}I_2^{4f} \tag{D.64}$$

The $\Phi_1^{\dagger i}\Phi_1^jA_0^aB_0$ correlator

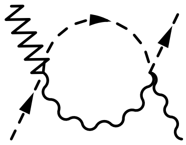
Lastly, we list the diagrams contributing to the correlator with two scalar legs and two different gauge field legs at zero external momentum. Again, the diagrams with two Φ_2 external legs instead of Φ_1 take the same form, with the substitution $\lambda_1 \to \lambda_2$, with the absence of fermion loops (box diagrams).



$$= \frac{1}{2}gg'\lambda_1(\tau_a)_{ij}I_2^{4b} \tag{D.65}$$



$$= \frac{1}{2}gg'\lambda_4(\tau_a)_{ij}I_2^{4b} \tag{D.66}$$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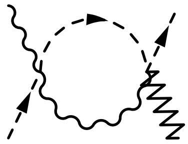
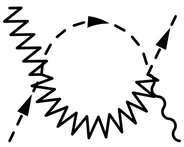
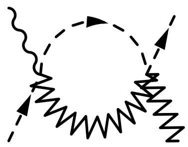
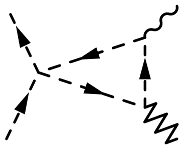
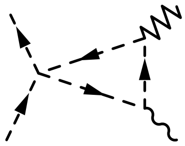
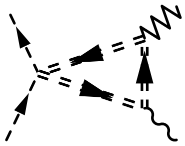
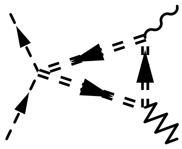
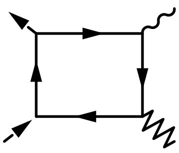
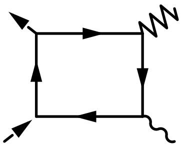
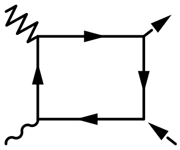
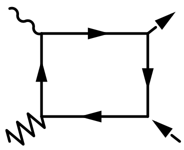


$$= \frac{1}{16}dgg'^3(\tau_a)_{ij}I_2^{4b} \tag{D.67}$$

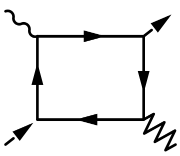
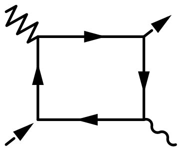
 $=$  $= \frac{1}{16} d g^3 g' (\tau_a)_{ij} I_2^{4b}$ (D.68)

 $=$  $= -\frac{1}{4}(4-d) g g' \lambda_1 (\tau_a)_{ij} I_2^{4b}$ (D.69)

 $=$  $= -\frac{1}{4}(4-d) g g' \lambda_4 (\tau_a)_{ij} I_2^{4b}$ (D.70)

 $+$  $+$  $+$ 

$$= -\frac{1}{2}(d-2) g g' (\tau_a)_{ij} \operatorname{Tr}[Y_l h^{(e)} h^{(e)\dagger} + Y_q N_c h^{(d)} h^{(d)\dagger} - Y_q N_c h^{(u)} h^{(u)\dagger}] I_2^{4f} \quad \text{(D.71)}$$

 $+$ 

$$= \frac{1}{2} (\tau_a)_{ij} g g' \operatorname{Tr}[Y_e h^{(e)} h^{(e)\dagger} + Y_d N_c h^{(d)} h^{(d)\dagger} - Y_u N_c h^{(u)} h^{(u)\dagger}] I_2^{4f} \quad \text{(D.72)}$$

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