Regularity of the p-Poisson equation in the plane

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Abstract

We study the regularity of the p-Poisson equation

$$\Delta_p u = h, \quad h \in L^q$$

in the plane, when $p \geq 2$. In the case $2 < q < \infty$ we obtain the sharp Hölder exponent for the gradient. In the case $q = \infty$ we come arbitrary close to the sharp exponent.

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1 Introduction

In the plane, the theory of many partial differential equations is more explicit than in higher dimensions. Sometimes the theory of quasiregular mappings and other devices are available, see [Ber58]. Our object is to study the so-called p-Poisson equation

$$\Delta_p v\,(x,y) \equiv \operatorname{div}(|\nabla v(x,y)|^{p-2}\nabla v(x,y)) = h(x,y), \tag{1}$$

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in a bounded domain $\Omega \subset \mathbb{R}^2$, where $p \geq 2$. This equation arises as the Euler-Lagrange equation of the variational integral

$$\iint_\Omega \left(\frac{1}{p}|\nabla v|^p + hv\right) dxdy.$$

We are interested in *the sharp Hölder exponent for the gradient* of the solution. The weak solutions are known to be of class $C^{1,\kappa}_{\text{loc}}$ for some $\kappa \in (0,1)$. We record a well known result:

Proposition 1. *Suppose that v is a solution of* (1) *in the disc B_{2R} and that $h \in L^q(B_{2R})$ for some fixed $2 < q \leq \infty$. Then $v \in C^{1,\kappa}_{loc}(B_{2R})$, for some $\kappa = \kappa(p,q)$. More exactly, we have the estimate*

$$\|v\|_{C^{1,\kappa}(B_R)} \leq A,$$

where $A = A(p, q, R, \|h\|_{L^q(B_{2R})}, \|v\|_{L^\infty(B_{2R})})$.

Here and in the rest of the paper, we use the notation

$$[u]_{C^s,D} = \left\|\frac{u(x)-u(y)}{|x-y|^s}\right\|_{L^\infty(D\times D)}, \quad \|u\|_{C^s(D)} = [u]_{C^s(D)} + \|u\|_{L^\infty(D)}$$

and

$$\|u\|_{C^{1,s}(D)} = \|\nabla u\|_{C^s(D)} + \|u\|_{L^\infty(D)}$$

when $s \in (0,1)$ and is D a bounded domain. The proof of the above theorem can for $q = \infty$ be found in [Tol84] and for $2 < q < \infty$ in [Lie93].

In the homogeneous case, $\Delta_p v = 0$, the optimal Hölder exponent

$$\kappa = \frac{1}{6}\left(\frac{p}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right) > \frac{1}{p-1}, \quad (p > 2)$$

has been determined by Iwaniec and Manfredi in [IM89]. They used the hodograph transform. However, the "torsional creep equation"

$$\Delta_p v = 2,$$

has a weak solution given by

$$v(x) = \frac{p-1}{p}|x|^{\frac{p}{p-1}},$$

so that $|\nabla v(x)| = |x|^{\frac{1}{p-1}}$, exhibiting the fact that, in general one must have $\kappa \leq \frac{1}{p-1}$. The example

$$v(x) = \int_0^{|x|} \left(\frac{\rho^{1-\frac{2}{q}}}{(\ln \rho)^{\frac{2}{q}}} \right)^{\frac{1}{p-1}} d\rho$$

solves the p-Poisson equation with the right-hand side in L^q, showing that, also $\kappa \leq \frac{1-\frac{2}{q}}{p-1}$. Our result determines the *optimal* Hölder exponent: it is given by $\frac{1-\frac{2}{q}}{p-1}$ for $1 < q < \infty$ and is arbitrary close to $1/(p-1)$ for $q = \infty$.

Theorem 2. *Suppose $\Delta_p v = h$ in Ω and that $h \in L^q(\Omega)$, where $2 < q \leq \infty$. Then $\nabla v \in C^{\beta-1}_{loc}(\Omega)$ for any $\beta < \frac{p}{p-1}$ when $q = \infty$ and for $\beta = \frac{p-\frac{2}{q}}{p-1}$ if $q < \infty$. In particular, for any compact $K \subset \Omega$, we have the estimate*

$$[\nabla v]_{C^{\beta-1}}(K) \leq C(q, p, \beta, K) \max \left(\|h\|_{L^q(\Omega)}^{\frac{1}{p-1}}, \|v\|_{L^\infty(\Omega)} \right).$$

Our method of proof is based on universal estimates for the p-Laplace equation $\Delta_p u = 0$, which come from the fact that the complex gradient, $f = u_x - i u_y$ is a quasiregular mapping. A balanced perturbation of the p-Poisson equation leads to the p-Laplace equation at the limit so that the universal estimates can be employed.

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2 Auxiliary results for the homogeneous equation $\Delta_p u = 0$

It was proved by Bojarski and Iwaniec that the complex gradient

$$f = u_x - i u_y$$

of a solution to the p-Laplace equation $\Delta_p u = 0$ is a quasiregular mapping; see [BI87]. We need the following consequence of this fundamental result.

Lemma 3. *Suppose $u \in C(B_{2R}) \cap W^{1,p}(B_{2R})$ is a solution to $\Delta_p u = 0$ in the disc B_{2R}. Then there is a constant $\Lambda = \Lambda(p)$ such that*

$$[\nabla u]_{C^\alpha(B_R)} \leq \frac{\Lambda}{R^{1+\alpha}} \|u\|_{L^\infty(B_{2R})},$$

where $\alpha = \frac{1}{p-1}$.

It is of utmost importance that the same Λ will do for all solutions u. We sketch the proof of this known result.

Sketch of the proof. First, by [BI87] the complex gradient $f(z)$, which belongs to $W^{1,2}_{\text{loc}}(\Omega)$ and is continuous, satisfies the inequality

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \frac{p-2}{p}\left|\frac{\partial f}{\partial z}\right|,$$

a.e. in the Ω. Here it is decisive that $(p-2)/p < 1$. As in the proof of Lemma 12.1 in [GT01] page it follows that the Dirichlet integral

$$I(r) = \iint_{B_r} |Df|^2 dxdy$$

satisfies the inequality

$$I(r) \leq I(r_0) \left(\frac{r}{r_0}\right)^{2\alpha}, \quad \alpha = \frac{1}{p-1},$$

when $r \leq r_0 < 2R$. Then Morrey's lemma implies

$$|f(z_2) - f(z_1)| \leq 2 \left(\frac{|z_2 - z_1|}{r_0}\right)^{\alpha} \sqrt{\frac{1}{\alpha} I(r_0)},$$

when $|z_2 - z_1| \leq r_0 < 2R$; see Lemma 12.2 in [GT01].

We also have the standard estimate

$$I(r) = \iint_{B_r} |Df|^2 dxdy \leq \frac{C_1}{r^2} \iint_{B_{\frac{3r}{4}}} |f|^2 dxdy = \frac{C_1}{r^2} \iint_{B_{\frac{3r}{4}}} |\nabla u|^2 dxdy, \quad (2)$$

for a quasiregular mapping, sometimes called Mikljukow's inequality. There C_1 depends on the dilatation $1/(p-1)$, hence only on p. Now

$$\left(\frac{1}{r^2}\iint_{B_{\frac{3r}{4}}}|\nabla u|^2 dxdy\right)^{\frac{1}{2}} \le \left(\frac{1}{r^2}\iint_{B_{\frac{3r}{4}}}|\nabla u|^p dxdy\right)^{\frac{2}{p}}$$
$$\le \left(\frac{C_2}{r^{p+2}}\iint_{B_{2r}}|u|^p dxdy\right)^{\frac{2}{p}} \le \frac{C_2^{\frac{2}{p}}}{r^2}\|u\|_{L^\infty(B_{2r})}, \tag{3}$$

by Hölder's inequality and a standard Caccioppoli estimate. The new constant C_2 depends only on p. Combining (2) and (3) we arrive at

$$|f(z_2)-f(z_1)| \le \frac{\Lambda}{r_0}\left(\frac{|z_2-z_1|}{r_0}\right)^{\alpha}\|u\|_{L^\infty(B_{2r_0})},$$

whenever $|z_2 - z_1| \le r_0 < R$. The various constants have been joined in Λ. This is the desired result. □

The above lemma has the following immediate consequence.

Corollary 4 (Liouville)**.** *If $\Delta_p u = 0$ in $\mathbb{R}^2$ and if*

$$\|u\|_{L^\infty(B_{R_j})} \le CR_j^{1+\alpha-\varepsilon}, \quad \alpha = \frac{1}{p-1}$$

for some fixed constant C, some subsequence $R_j \to \infty$ and $\varepsilon > 0$, then ∇u must be constant.

3 The oscillation of the solution when the gradient is small

In this chapter we assume

$$\Delta_p v = h \text{ in } B_1$$

where

$$\|h\|_{L^q(B_1)} \le 1,\ q > 2, \quad \|v\|_{L^\infty(B_1)} \le 1.$$

In this normalized situation, our aim is to prove the following estimate:

Proposition 5. *If $q = \infty$ let β be any number less than $p/(p-1)$ and if $q < \infty$ let*

$$\beta = \frac{p - \frac{2}{q}}{p-1}.$$

If the inequality $|\nabla v(x)| \leq r^{\beta - 1}$, where $r < 1/4$, hold at some fixed point $x \in B_{1/2}$, then

$$S_r = \sup_{y \in B_r(x)} |v(y) - v(x)| \leq C r^{\beta},$$

where $C = C(p, q, \beta)$.

The difficulty is that the gradient constraint is only assumed at the point x, otherwise result would be trivial. The proof is based on rescaled functions and a blow-up argument. At the end, the limiting function turns out to be a solution of the p-Laplace equation in the whole plane, which satisfies the Liouville theorem. We begin with the key lemma.

Lemma 6. *Assume the hypotheses of Proposition 5. Then there is a constant $C = C(p, q, \beta)$ such that for every fixed $r \in (0, 1/4)$, at least one of the following alternatives hold:*

(i) $S_r = \sup_{y \in B_r(x)} |v(y) - v(x)| \leq C r^{\beta}$,

(ii) There is an integer $k \geq 1$ such that $2^k r < 1/4$ and $S_r \leq 2^{-k\beta} S_{2^k r}$.

Proof. The proof is indirect, starting from the antithesis that no constant C will ever do. Thus, giving C the successive values $j = 1, 2, 3, \ldots$, we can select solutions v_j, radii $r_j < 1/4$ and points $x_j \in B_{1/4}$ so that the three conditions

1) $S_{r_j} = \sup_{y \in B_{r_j}(x_j)} |v_j(y) - v_j(x_j)| \geq j r_j^{\beta}$,

2) $S_{r_j} \geq 2^{-k\beta} S_{2^k r_j}$ for all integers k such that $2^k r_j < 1/4$, or $r_j \geq \frac{1}{8}$,

3) $|\nabla v_j(x_j)| \leq r_j^{\beta - 1}$,

all hold. By 1) and the assumed bound on v_j, we have $j r_j^{\beta} \leq 2$, which forces $r_j \to 0$, as $j \to \infty$. This excludes the alternative $r_j \geq 1/8$ in 2). Notice that 2) is perfectly designed for iteration. We define the rescaled functions

$$V_j(x) = \frac{v_j(x + r_j x) - v_j(x_j)}{S_{r_j}}, \quad j = 1, 2, 3, \ldots$$

which, as we will see, solve a p-Poisson equation. By the chain rule

$$\nabla V_j(x) = \frac{r_j}{S_{r_j}} \nabla v_j(y)\Big|_{y=x_j+r_jx}.$$

The following properties are now immediate:

$$\begin{cases} V_j(0) = 0, \\ |\nabla V_j(0)| = \frac{r_j}{S_{r_j}} |\nabla v_j(x_j)| \le \frac{r_j^\beta}{S_{r_j}} \le \frac{1}{j} \to 0, \\ \sup_{B_{2^k}} |V_j| = \frac{S_{2^k r_j}}{S_{r_j}} \le 2^{k\beta}, \text{ for all integers } k \text{ such that } 2^k < \frac{1}{4r_j}, \\ \Delta_p V_j(x) = \frac{r_j^p}{S_{r_j}^{p-1}} h(x_j + r_j x) \equiv H_j(x), \text{ when } |x| < \frac{1}{4r_j}. \end{cases}$$

In particular, the rescaled functions v_j solve a p-Poisson equation in the disc $|x| < 1/(4r_j)$, which is expanding to the whole plane as $j \to \infty$. Note that the use of second derivatives can be totally avoided if one just writes the equations in their weak form, using test functions under the integral sign.

Recall that $2 < q \le \infty$. We need to treat the case $q = \infty$ separately in the following formal computations.

Case $q = \infty$: Now $p - \beta(p-1) > 0$ and thus for any $R > 0$ we have

$$\begin{aligned} \|\Delta_p V_j\|_{L^\infty(B_R)} &= \frac{r_j^p}{S_{r_j}^{p-1}} \|h\|_{L^\infty(B_{Rr_j}(x_j))} \\ &\le \frac{r_j^p}{(j r_j^\beta)^{p-1}} \to 0, \end{aligned}$$

as $j \to \infty$, since sooner or later $Rr_j < 1/2$, as required.

Case $q < \infty$: Now $q(p - \beta(p-1)) - 2 > 0$ and

$$\begin{aligned} \|\Delta_p V_j\|^q_{L^q(B_R)} &= \frac{r_j^{pq}}{S_{r_j}^{(p-1)q}} \int_{B_R} |h(x_j + r_j x)|^q \, dx \\ &= \frac{r_j^{pq-2}}{S_{r_j}^{(p-1)q}} \int_{B_{Rr_j}(x_j)} |h(y)|^q \, dy \le \frac{r_j^{pq-2}}{S_{r_j}^{(p-1)q}} \\ &\le \frac{r_j^{q(p-\beta(p-1)-2}}{(j)^{(p-1)q}} \to 0, \end{aligned}$$

as $j \to \infty$, since as above, $Rr_j < 1/2$ sooner or later, as required.

Now we go back the equation for the V_js:

$$\Delta_p V_j = H_j.$$

In order to be able to pass to the limit as $j \to \infty$, we need some compactness. We recall Proposition 1 in the introduction. It yields an estimate of the form

$$\|V_j\|_{C^{1,\kappa}(B_{\frac{R}{2}})} \leq A(p, q, R, \|h\|_{L^q(B_R)}, \|v\|_{L^\infty(B_R)}), \tag{4}$$

for some $\kappa = \kappa(p, q)$. Recall also that

$$\|V_j\|_{L^\infty(B_{2^k})} \leq 2^{\beta k}$$

and that

$$\|H_j\|_{L^q(B_R)} < 1, \text{ for } j \text{ large enough.}$$

Thus, the bound in (4) is uniform in j. It follows that, up to extracting a subsequence, V_j converges locally uniformly in $C^{1,\kappa/2}(\mathbb{R}^2)$ to some limit function V. The limit function inherits many properties. We obtain that

$$\begin{cases} V(0) = 0, \quad |\nabla V(0)| = 0, \\ \sup_{B_{2^k}} |V| \leq 2^{k\beta} \text{ for all integers } k \geq 1, \\ \sup_{B_1} |V| = 1, \\ \Delta_p V = 0 \text{ in } \mathbb{R}^2. \end{cases}$$

Thus, V is an entire solution of the p-Laplace equation. Since in any case, $\beta < p/(p-1)$, it follows from Liouville's theorem (Corollary 4) with $R_j = 2^j$, that ∇V reduces to a constant. Thus, V is an affine function and since $V(0) = |\nabla V(0)| = 0$, we must have $v \equiv 0$. This contradicts the fact that

$$\sup_{B_1} |V| = 1.$$

We conclude that the antithesis is false. The lemma follows. $\square$

In order to prove Proposition 5 we have to show that the first alternative in Lemma 6 is always valid.

Proof of Proposition 5. If alternative (i) holds for all $r < 1/4$ we are done. If not, we pick a radius r for which, by alternative (ii),

$$S_r \leq 2^{-k_1\beta} S_{2^{k_1} r},$$

for some integer k_1 with $2^{k_1}r < 1/4$. If (i) holds for $S_{2^{k_1}r}$, then

$$S_r \le 2^{-k_1\beta} S_{2^{k_1}r} \le 2^{-k_1\beta} C(2^{k_1}r)^\beta = Cr^\beta$$

and again we are done. If not, we continue with

$$S_{2^{k_1}r} \le 2^{-k_2\beta} S_{2^{k_2}2^{k_1}r},$$

where $2^{k_1}2^{k_1}r < 1/4$. Iterating this as long as alternative (i) fails, we obtain

$$S_r \le 2^{-k_n\beta} \cdots 2^{-k_1\beta} S_{2^{k_n}\cdots 2^{k_1}r} = 2^{-(k_1+\cdots k_n)\beta} S_{2^{k_1+\cdots k_n}r},$$

where $2^{k_1+\cdots k_n}r < 1/4$. Since every $k_j \ge 1$, the procedure must stop after a finite number of steps (depending on r), at its latest when

$$2^{k_1+\cdots k_n}r \ge \frac{1}{8}.$$

Then the alternative (i) holds for the radius $2^{k_1+\cdots k_n}r$ and so, finally,

$$S_r \le 2^{-(k_1+\cdots k_n)\beta} S_{2^{k_1+\cdots k_n}r} \le 2^{-(k_1+\cdots k_n)\beta} C(2^{k_1+\cdots k_n}r)^\beta = Cr^\beta.$$

This proves the claim. □

4 Proof of the main theorem

We are now ready to give the proof of our main result. The idea is that when the gradient is small, we can apply the result of the previous section to obtain the desired estimates. On the other hand, when the gradient is large, then the equation becomes non-degenerate so that classical results apply. We first prove the following intermediate result.

Theorem 7. *Assume that*

$$\Delta_p v = h \text{ in } B_1, \quad \|v\|_{L^\infty(B_1)} \le 1, \quad \|h\|_{L^q(B_1)} \le 1,$$

for some $q > 2$. If $q = \infty$ let β be any number less than $p/(p-1)$ and if $q < \infty$ let

$$\beta = \frac{p - \frac{2}{q}}{p-1}.$$

Then for any $x \in B_{1/2}$,

$$\sup_{B_r(x)} |v(y) - v(x) - (y-x)\cdot\nabla v(x)| \le Lr^\beta, \tag{5}$$

when $0 < r < 1/4$ and where $L = L(p,q,\beta)$.

Proof. Fix $x \in B_{1/2}$. If $|\nabla v(x)| \le r^{\beta-1} \le \left(\frac{1}{4}\right)^{\beta-1}$, then by Proposition 5,

$$\sup_{B_r(x)} |v(y) - v(x) - (y-x)\cdot \nabla v(x)| \le Cr^\beta + r \cdot r^{\beta-1} = (C+1)r^\beta, \tag{6}$$

where C depends on p, q and β. We need the estimate also for $r^{\beta-1} < |\nabla v(x)|$. To this end, let $\rho = |\nabla v(x)|^{\frac{1}{\beta-1}} > 0$ and

$$w(y) = \frac{v(x+\rho y) - v(x)}{\rho^\beta},$$

so that $\nabla w(y) = \rho^{1-\beta} \nabla v(x+\rho y)$ and

$$\sup_{B_1} |w(y)| = \rho^{-\beta} \sup_{B_\rho(x)} |v(y) - v(x)| \le C,$$

since ρ is the largest radius for which Proposition 5 is available. Moreover, by calculation

$$\|\Delta_p w\|_{L^q(B_1)} \le \|h\|_{L^q(B_\rho(x))} \rho^{p-\beta(p-1)-\frac{2}{q}} \le 1.$$

Hence, we can once more apply Proposition 1 to obtain the estimate

$$\|w\|_{C^{1,\kappa}(B_{\frac{1}{2}})} \le A = A(p,q,\beta), \quad \kappa = \kappa(p,q).$$

Therefore we can fix a small radius $\tau = \tau(p,q,\beta)$ so that

$$\operatorname*{osc}_{B_\tau} (\nabla w) < \frac{1}{2}.$$

Since

$$|\nabla w(0)| = \rho^{1-\beta} \underbrace{|\nabla v(x)|}_{\rho^{\beta-1}} = 1,$$

we must have $|\nabla w(y)| > 1/2$ in B_τ. Thus w solves an equation which is uniformly elliptic with uniformly Hölder continuous coefficients in B_τ. Recall also that $|w| \le C$ in B_1 and hence also in B_τ. Then, from Theorem 9.11 in [GT01] and the Sobolev embedding, there are uniform $C^{1,\gamma}$-estimates available for w with $\gamma = 1 - 2/q$, so that

$$\|w\|_{C^{1,\gamma}(B_{\tau/2})} \le B = B(p,q,\beta).$$

In particular

$$\sup_{y \in B_s} |w(y) - w(0) - (y-0)\cdot \nabla w(0)| \leq B|y-0|^{2-\frac{2}{q}},$$

when $s < \tau/2$. In terms of v this means

$$\sup_{y \in B_s} \left| \frac{v(x+\rho y) - v(x)}{\rho^\beta} - y \cdot \rho^{1-\beta} \nabla v(x) \right| \leq B|y|^{2-\frac{2}{q}}.$$

Write $z = x + \rho y$ and recall that $\beta \leq 2 - \frac{2}{q}$. Then the above estimate reads

$$\sup_{z \in B_{s\rho}} |v(z) - v(x) - (z-x)\cdot \nabla v(x)| \leq B|y|^{2-\frac{2}{q}} \rho^\beta = B(s\rho)^\beta s^{2-\beta-\frac{2}{q}} \leq B(s\rho)^\beta,$$

whenever

$$r = s\rho < \frac{\tau\rho}{2} = \frac{\tau}{2} |\nabla v(x)|^{\frac{1}{\beta-1}}.$$

This is the same as saying that

$$\sup_{z \in B_r} |v(z) - v(x) - (z-x)\cdot \nabla v(x)| \leq B|y|^{2-\frac{2}{q}} \rho^\beta = B(s\rho)^\beta s^{2-\beta-\frac{2}{q}} \leq Br^\beta,$$

whenever

$$r < \frac{\tau\rho}{2} = \frac{\tau}{2} |\nabla v(x)|^{\frac{1}{\beta-1}}.$$

It remains to verify estimate (5) also when r is in the interval $\tau\rho/2 < r < \rho$. This is easy. Take such an r. Then estimate (6) is available for the radius ρ and we obtain

$$\begin{aligned}\sup_{z \in B_r} |v(z) - v(x) - (z-x)\cdot \nabla v(x)| &\leq \sup_{z \in B_\rho} |v(z) - v(x) - (z-x)\cdot \nabla v(x)| \\ &\leq (C+1) r^\beta \left(\frac{\rho}{r}\right)^\beta \leq \left(\frac{2}{\tau}\right)^\beta (C+1) r^\beta.\end{aligned}$$

Hence, we finally obtain the estimate (5) for all $r < 1/4$ with the constant

$$L = \max\left(C+1, (C+1)\left(\frac{2}{\tau}\right)^\beta, B\right),$$

which only depends on p, q and β. $\square$

We now conclude the proof of our main result.

Proof of Theorem 2. It is enough to prove the result for $\Omega = B_1$ and $K = B_{1/4}$. By the normalization

$$\tilde{v} = \frac{v}{\max\left(\|v\|_{L^\infty(B_1)}, \|h\|_{L^q(B_1)}^{\frac{1}{p-1}} \right)},$$

we see that $\tilde{v}$ satisfies the assumption of Theorem 7. Hence, the esimate (5) holds true for $\tilde{v}$. Then the same estimate holds true for v with L replaced by

$$L \max\left(\|v\|_{L^\infty(B_1)}, \|h\|_{L^q(B_1)}^{\frac{1}{p-1}} \right).$$

This implies immediately

$$|\nabla v(x) - \nabla v(y)| \leq 2L \max\left(\|v\|_{L^\infty(B_1)}, \|h\|_{L^q(B_1)}^{\frac{1}{p-1}} \right) |x-y|^{\beta-1},$$

whenever $x, y \in B_{1/4}$. This ends the proof of the theorem. □

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