NTNU
Norwegian University of
Science and Technology

Colourful Cohomology

Paul André Dillon Trygsland

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Supervisor: Sverre Olaf Smalø, IMF

Norwegian University of Science and Technology
Department of Mathematical Sciences

Forord

Dette er et arbeid gjennomført vårsemesteret 2017 under veiledning av professor Sverre Olaf Smalø. Den offisielle emnekoden er TMA4900 med tittel "Matematikk, masteroppgave" og den har et omfang på 30 studiepoeng.

Motivasjonen bak oppgaven ligger i at jeg liker å trekke røde tråder mellom ulike deler av matematikken, noe jeg i aller høyeste grad har fått mulighet til å gjøre i løpet av denne våren. Videre er jeg interessert i hvordan homologi dukker opp i ulike problemstillinger; det har vært meget interessant å undersøke hvordan dette utspiller seg for firefargeleggingsproblemet. Oppgaven får også et historisk preg som følge av at firefargeleggingsproblemet blomstret rundt 1900-tallet.

Av forkunnskaper antar jeg at eventuelle lesere er kjent med algebra opp til ringer og moduler, punkt-sett-topologi og grunnleggende kategoriteori. Utover disse forkunnskapene blir alt av teori motivert og/eller utledet, spesielt gjelder dette for relevante konsept innen algebraisk topologi. Veien fra abstrakt til geometrisk har vært meget lærerik.

Referanser inkluderer relevante artikler og lærebøker jeg har benyttet og/eller blitt inspirert av under skrivingen. Alle diagram er produsert med Ti*k*Z og figurer er tegnet i Inkscape.

Jeg ønsker å takke Marit Funderud for retting av språk og trivelige kaffepauser. Jeg vil også takke Sverre Olaf Smalø for hans engasjement, gode tilbakemeldinger og hyggelige samtaler gjennom semesteret. Utover dette har jeg lært utrolig mye matematikk av Sverre i løpet av det siste året, noe jeg virkelig setter pris på.

Til slutt må jeg si at det har vært en fornøyelse å skrive denne oppgaven og at jeg gleder meg til å fortsette som stipendiat til høsten.

Paul André Dillon Trygsland, juli 2017

Sammendrag

Oppgaven starter med å gjøre rede for sammenhenger mellom grafteori, kategoriteori og homologi. Deretter blir det veldig abstrakte oversatt til geometriske konsept, spesielt utledes simplisiell kohomologi. Her blir opplysende teori inkludert sammen med relevante eksempler. Å studere firefargeleggingsproblemet med simplisiell kohomologi gir en reformulering uttrykt ved ligninger som inkluderer korand-operatoren. Ligningene gir en direkte sammenheng mellom historiske oppdagelser av P. G. Tait og O. Veblen. Løsninger fra Hamiltonske sykluser i tillegg til sammenhengenhet av trianguleringer diskuteres. Avslutningsvis presenteres et kort bevis av det svakere femfargeleggingsproblemet.

Abstract

The thesis starts out by explaining connections between graph theory, category theory and homology. Thereafter, the very abstract is translated into geometrical concepts, simplicial cohomology is especially derived. Enlightening theory is included along with relevant examples. Studying the four colour problem with simplicial cohomology gives a reformulation in terms of equations involving the coboundary operator. The equations give a direct connection of historical discoveries by P. G. Tait and O. Veblen. Solutions by Hamiltonian cycles as well as connectedness of triangulations are discussed. In the end, a short proof of the weaker five colour problem is presented.

Contents

1 Introduction

The famous four colour problem was conjectured by Francis Guthrie in 1852: When colouring a map in such a way that no regions separated by a single border have the same colours, it always suffices to use four colours [Saa72]. The conjecture caught the attention of many mathematicians, such as Arthur Cayley, who presented it to the London Mathematical Society in 1878 [Die00, Saa72]. Nevertheless, almost a century would pass until Kenneth Appel and Wolfgang Haken came up with an acceptable proof in 1977 [AHK77a, AHK77b]. In short, this proof reduced the problem to 1482 special cases verified by a computer. The proof was met with a lot of discontent, hence an approximately 700 page amended version was published as a book in 1989 [AH89]. Several proofs were submitted before 1977, but they all turned out to be false. In 1890, Percy John Heawood modified a false proof by A. B. Kempe [Kem79, Saa72] to prove that five colours suffice [Hea90, Saa72].

Topology is used in the investigation of the four colour problem. Examples include reductions involving the Euler Characteristic and triangulations of the sphere [Saa72]. The early work of Henri Poincaré in Analysis Situs and its five supplements [Die09, Veb31] dating back to 1899-1904 marks the beginning of modern topology. Oswald Veblen uses the boundary matrices of Poincaré to study the four colour problem in 1912 [Veb12]. More precisely, Veblen studies boundary matrices over GF(4), the Galois field of four elements, to give an equivalent formulation of the problem. We study the four colour problem using simplicial cohomology, which results in an extension of Veblen's reformulation together with clarifying implications. Before this view on the four colour problem, we spend time to develop the modern language needed and make the very abstract geometrical.

The second section, Categories, develop connections between graph theory, category theory and homology. Graphs and edge-preserving maps make up a category, **Grph**, resulting in a categorical overview as well as geometrical properties of **Grph**-morphism. On the other hand, graph theory is useful in category theory, therefore the adjointness of quivers and graphs is discussed for completion. Lastly, quivers give a short and precise definition of the homology associated with simplicial objects.

The third section, Homology and Complexes, is dedicated to process the abstract notions from the second section. The singular homology of a topological space and the simplicial homology of a simplicial complex are derived over the integers. The focus lies in geometrical aspects to be used in the study of the four colour problem, thus there is an emphasis on CW-complexes, triangulations and the Euler characteristic.

The fourth section, Cohomology and Changing Coefficients, develop the formal concepts surrounding cohomology and extending scalars from $\mathbb{Z}$ to arbitrary commutative rings. In particular, properties of Hom and tensor functors, the derived functors Ext and Tor, and compatibility with simplicial objects are discussed. Examples are given to separate the concepts from ordinary homology over $\mathbb{Z}$. Lastly, the geometry of simplicial cohomology is treated in detail.

The fifth section, The Four Colour Problem, presents a reformulation of the four colour problem involving equations in cohomology. The equivalence of P. G. Tait's 3-colouring of edges and Veblen's formulation over GF(4) is shown directly using the equations in simplicial cohomology. Solutions obtained by Hamiltonian cycles in the dual cell structure are discussed together with the insufficiency of this approach. Morphisms in **Grph** and the Euler characteristic give insight in the connectedness of the triangulation together with an easy proof of how five colours suffice when colouring maps.

2 Categories

We will look at the foundation of some specific categories that highlight some connections between category theory, graph theory and homology.

Note that for any category, we will assume the Hom-sets to be actual sets. This agrees with the definition by S. Mac Lane and J. J. Rotman [ML71, Rot08], while every category is locally small according to T. Leinster and S. Awodey [Awo10, Lei14]. Some standard examples of categories include:

- **Set**, the category of sets and functions.
- **Top**, the category of topological spaces and continuous functions.
- **Gp**, the category of groups and group homomorphisms.
- **Ab**, the category of Abelian groups and group homomorphisms.
- $\mathbf{Vect}_{\mathrm{K}}$, the category of vector spaces over a field K with linear transformations.
- More generally, $\operatorname{Mod} R$, the category of left R-modules over a ring R with R-homomorphisms. The Hom-sets, $\operatorname{Hom}_{\operatorname{Mod} R}(M, N)$, will be denoted $\operatorname{Hom}_R(M, N)$.
- A partially ordered set, $(A, \leq)$, is a category with elements of A as objects and morphisms given by $\leq$, i.e. for any objects a and b there is a unique morphism $a \to b$ if and only if $a \leq b$.
- **Cat**, the category of small categories and functors.
- Given a small category $\mathscr{A}$ and a category $\mathscr{C}$, we have the category of $\mathscr{C}$-valued presheaves, $\operatorname{Presh}_{\mathscr{C}} \mathscr{A}$ as the collection of contravariant functors[1] $\mathscr{A} \to \mathscr{C}$ and natural transformations.

Note that $\operatorname{Presh}_{\mathscr{C}} \mathscr{A}$ gives **Cat** the structure of a 2-category, i.e. a category for which we have morphisms of morphisms. We give a brief discussion about possible paradoxes before proceeding. In 1903, Bertrand Russell released "The principles of mathematics" which includes a famous set-theoretic paradox known as Russell's paradox [Rus96]. Simply put, under certain assumptions one may form the set of all sets not an element in themselves, which implies membership and not membership, i.e. a contradiction. Similar problems arise

[1] That is, functors $\mathscr{A}^{\mathrm{op}} \to \mathscr{C}$.

in category theory, and one should be aware of this when defining categories. As an example, smallness guarantee that the Hom-sets of **Cat** and $\mathrm{Presh}_{\mathscr{C}}\,\mathscr{A}$ are actual sets. Indeed,

$$\mathrm{Hom}_{\mathrm{Presh}_{\mathscr{C}}\,\mathscr{A}}(F,G) \subset \prod_{A\in \mathrm{Ob}\mathscr{A}} \mathrm{Hom}_{\mathscr{C}}(F(A),G(A))$$

as natural transformations are special cases of such products,[2] and

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathscr{A},\mathscr{B}) \subset \prod_{A_1,A_2\in \mathrm{Ob}\mathscr{A},B_1,B_2\in \mathrm{Ob}\mathscr{B}} \mathrm{Hom}_{\mathscr{B}}(B_1,B_2)^{\mathrm{Hom}_{\mathscr{A}}(A_1,A_2)}$$

as functors are special cases of such products.[3] If we extend **Cat** to include every large category, then considering **Set** and the trivial category, $\mathbf{1} = \{x, 1_x : x \mapsto x\}$, we get that each set gives rise to a functor $\mathbf{1} \to \mathbf{Set}$ so that $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{1},\mathbf{Set}) = \mathrm{Ob}\,\mathbf{Set}$. That is, **Cat** is no longer a category according to our definition.

2.1 The Category of Graphs

What we will be referring to as a graph is often called a simple graph, i.e. the following definition is a special case of a more general definition.[4]

Definition 2.1. An (abstract) graph is a pair $G = (V(G), E(G))$, or simply $G = (V, E)$, of sets where $E \subset 2^V$ satisfies $|e| = 2\ \forall e \in E$. The elements of V and E are called vertices and edges, respectively.[5]

Graphs are often represented by node networks; nodes represent vertices and straight lines represent edges. The next definition contains some standard terminology regarding graphs.

Definition 2.2. Let $G = (V, E)$ be a graph.

- If $V = \emptyset$, then G is the empty graph.
- The size of G, $|G|$, is equal to the cardinality of V. If $|V| < \infty$ G is said to be finite.
- Two vertices, $v, u \in V$, are adjacent if $\{v, u\} \in E$.
- A vertex, v, and an edge, e, are mutually incident if $v \in e$.

[2] There is an added requirement of commutativity among the components.

[3] There is an added requirement of preserving identities and composition, note that the object assignment is included as $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ whenever a functor is applied to an identity.

[4] This is to ease language throughout the text.

[5] We will only refer to an abstract graph if there is a possibility for confusion.

- Let $X, Y \subset V$ then $E(X,Y) = \{\{x,y\} \in E | x \in X \text{ and } y \in Y\}$, i.e. the set of edges between X and Y. In the case $X = Y$ we simply write $E(X)$.

- G is complete if any pair of vertices are adjacent, and the complete graph of size n is denoted K_n.

- The vertex $u \in V$ is a neighbour of $v \in V$ if they are adjacent.[6]

- The degree of a vertex v in G, $\deg_G(v)$, equals $|E(v,V)|$. Whenever G is finite, $d(G)$ equals the average degree, which is easily seen to be $2\frac{|E|}{|V|}$.

Remark 2.3. A graph is to be considered as a finite graph if nothing else is stated.

The next definition will make up the morphisms in the upcoming category.

Definition 2.4. Let G and G' be two graphs. A (graph) homomorphism from G to G' is a function $\phi : V(G) \to V(G')$ such that $\forall u, v \in G \;\; \{u,v\} \in E(G) \Rightarrow \{\phi(u), \phi(v)\} \in E(G')$, and it is customary to write $\phi : G \to G'$ in this case.

Proposition 2.5. *The composition of two graph homomorphisms is a homomorphism.*

Proof. Let A, B and C be graphs, and let $a : A \to B$, $b : B \to C$ be homomorphisms. Take $\{u,v\} \in E(A)$, then $\{a(u), a(u)\} \in E(B)$ and $\{b(a(u)), b(a(u))\} \in E(C)$ as a and b are homomorphims. □

By the above proposition, it is straightforward to check that the following definition does make up a category.

Definition 2.6. Grph is the category of (abstract) graphs and graph homomorphisms.

Example 2.7. As we know, the inverse of a bijective continuous map is not necessarily continuous. An explicit example is given by equipping $[0,1)$ and S^1 with standard subspace topologies from $\mathbb{R}$ and $\mathbb{C}$, respectively. The geometrical setting makes it clear that the function $[0,1) \to S^1$, $t \mapsto e^{2\pi it}$ is a continuous bijection, but the pre-image of $[0,\epsilon)$ under the inverse is not open in S^1 for any ϵ in $(0,1)$. Hence, the inverse is not continuous. One cannot expect categories with actual functions as morphisms to satisfy that inverse functions are automatically morphisms; an isomorphism, in any category, is defined by a morphism that admits an inverse morphism. Several of our favourite categories does satisfy that inverse functions are morphisms. Some examples include algebraic structures such as **Gp**

[6] There is an implicit notion of uniqueness here, "the" will be justified by the upcoming category.

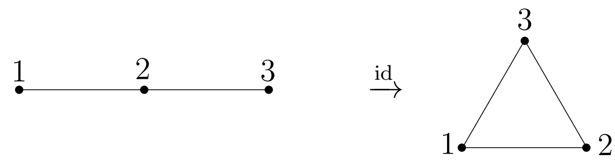


Figure 1: A bijective graph homomorphism induced by the identity on $\{1,2,3\}$ that does not make up an isomorphism.

and $\mathbf{Vect}_{\mathrm{K}}$ for a given field K.[7] As for continuous maps between topological spaces, graph homomorphisms that are injective need not restrict to isomorphisms. An easy example is given by an obvious bijection from a graph on two distinct vertices without edges to K_2.

The coproduct in **Set** is easily proven to agree with the disjoint union [Awo10], and extends to **Grph** by choosing the correct definition of adjacency with respect to the universal property:

Proposition 2.8. *The coproduct of two graphs G_1 and G_2 is the graph $G_1 \amalg G_2$ with vertex set $V(G_1) \amalg V(G_2) = \cup\{(x_i, i) | \ x_i \in G_i\}$ and induced edge set from G_1 and G_2, i.e. two vertices (v, i) and (u, j), are adjacent in $G_1 \amalg G_2$ if and only if $i = j$ and u and v are adjacent in G_i.*

Proof. The proof is straightforward; use the construction from **Set** and show that it extends to **Grph**. Let $i_1 : V(G_1) \to V(G_1 \amalg G_2),\ i_1(v) = (v, 1)$ and $i_2 : V(G_2) \to V(G_1 \amalg G_2),\ i_2(v) = (v, 2)$ be the coprojections. Given $j = 1$ or 2 and adjacent vertices u and v in G_j, we have that $i_j(u) = (u, j)$ and $i_j(v) = (v, j)$ are adjacent by definition of adjacency in $G_1 \amalg G_2$. Take any graph X and homomorphisms $f_1 : G_1 \to X,\ f_2 : G_2 \to X$, such that we have **Set**-morphisms $f_1 : V(G_1) \to V(X),\ f_2 : V(G_2) \to V(X)$. Define a function $f : V(G_1 \amalg G_2) \to V(X)$ by

$$f(v, i) = \begin{cases} f_1(v) & \text{if } i = 1 \\ f_2(v) & \text{if } i = 2 \end{cases} .$$

If u and v are adjacent in G_j for $j = 1$ or 2, we have that $f(u, j) = f_j(u)$ and $f(v, j) = f_j(v)$ are adjacent as f_1 and f_2 are homomorphisms, so f is a homomorphism again. By the construction of the coproduct in **Set**, we have that f is the unique homomorphism satisfying $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$ [Awo10]. Hence, $G_1 \amalg G_2$ satisfies the universal property and therefore agrees with the coproduct. □

Whenever the vertex sets of two graphs are both contained in a common larger set, the union, intersection and inclusion carry over to graphs in a natural way. If G and G' are two

[7] For the group case, given a bijective group homomorphism $f : G \to H$ and two elements $h_1 = f(g_1)$, $h_2 = f(g_2)$, we have that $f^{-1}(h_1 h_2) = f^{-1}(f(g_1) f(g_2)) = f^{-1}(f(g_1 g_2)) = g_1 g_2 = f^{-1}(h_1) f^{-1}(h_2)$. This argument easily replicates to other algebraic structures.

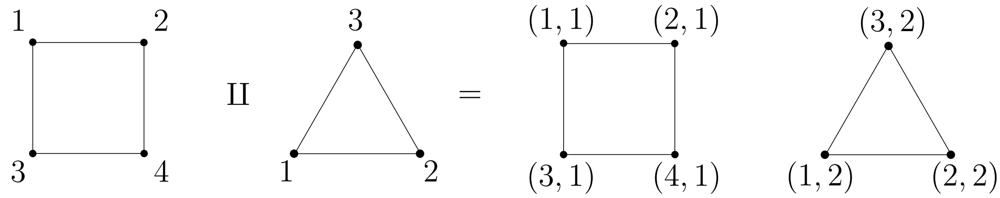


Figure 2: The coproduct/disjoint union in **Grph**.

such graphs, then $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$, $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$ and $G' \subset G$ if and only if $V(G') \subset V(G)$ and $E(G') \subset E(G)$.

Definition 2.9. Let G and G' be graphs.

- G' is a subgraph of G if $G' \subset G$.
- G' is an induced subgraph of G if $V(G') \subset G$ and $E(G') = \{\{u, v\} \in E(G) |\ u, v \in V(G')\}$. This is denoted $G' = G[V(G')]$.
- An inclusion, or (graph) embedding, $i : G' \hookrightarrow G$ of G' into G is an isomorphism of G' onto a subgraph of G.

Definition 2.9 extend the set difference to graphs. If G and G' are graphs, then $G - G' = G[V(G) - V(G')]$.

It is customary to define a sequence in a set X as a function $f : \mathbb{N}_+ \to X$, where f is identified with the infinite tuple $(x_1, x_2, ...)$, $x_n = f(n)$ for any $n \in \mathbb{N}_+$. We will, however, also refer to functions $f : \{1, 2, ..., n\} \to X$ as sequences. The following definitions partially follow C. Godsil in [GR01].

Definition 2.10. Let G be a graph and k a positive natural number.

- A walk in G is a sequence in $V(G)$ satisfying that there are edges between successive vertices.
- A path in G is a walk for which any vertex appears at most once.
- A walk (and therefore also a path) is finite if it is defined over a finite set as a sequence, otherwise it is infinite.
- Let $W : D \to V(G)$ be a walk with $|D| \geq 2$, then the length of W is $|D| - 1$.
- A cycle, $C = (v_1, ..., v_n)$, is a finite walk of length greater or equal to three such that $(v_1, ..., v_{n-1})$ is a path and $v_1 = v_n$.

- A cycle is induced if the corresponding subgraph is induced.[8]
- A cycle is even (odd) if the length of the cycle is even (odd).
- An edge, e, is traversed in a walk W if there are successive vertices v_i, v_{i+1} in W such that $e = \{v_i, v_{i+1}\}$.
- A path is Hamiltonian if every vertex appears once. A Hamiltonian cycle is a hamiltonian path, $(v_1, ..., v_n)$, such that v_1 and v_n are adjacent.
- G is connected if there is a path between any pair of vertices, otherwise it is disconnected.
- G is k-connected if $|G| > k$ and $G - X$ is connected for any $X \subset V(G)$ with $|X| < k$.
- A forest is an acylcic graph, i.e. there are no cycles, and a tree is a connected forest.

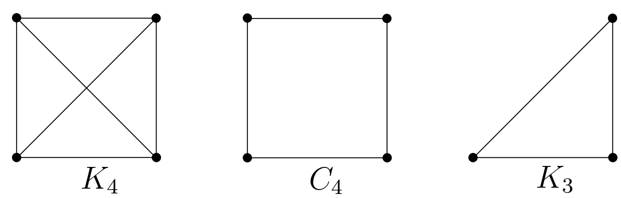


Figure 3: The graph K_4 together with subgraphs C_4 (even cycle) and K_3.

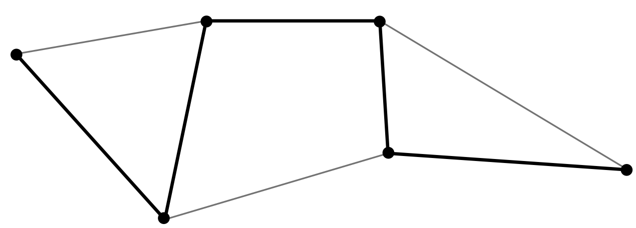


Figure 4: A graph together with a highlighted path of length five.

Definition 2.11. Let G be a graph [Die00].

- Let $C = \{c_1, ..., c_k\}$ be a set. A function $f : V(G) \to C$ is said to be a k-colouring of G if $\{u, v\} \in E(G) \Rightarrow f(u) \neq f(v)$. We refer to C as a colour set, and elements of C as colours.

[8] A walk, $W = (v_1, ..., v_n)$, is identified with the subgraph $(\{v_1, ..., v_n\}, \{\{v_1, v_2\}, ..., \{v_{n-1}, v_n\}\})$.

- G is k-colourable if there exist a k-colouring of G.

The concept of colouring graphs may be stated in more categorical terms.

Proposition 2.12. *Let G be a graph, then G is k-colourable if and only if there is a homomorphism from G to K_k.*

Proof. The statement is trivial for $|G| \leq k$, so assume $|G| > k$. Let $V(K_k) = \{1, ..., k\}$, and consider the partition $\{f^{-1}(1), ..., f^{-1}(k)\}$ of G determined by a homomorphism $f : G \to K_k$. Note that some of the preimages may be empty, as a homomorphism does not need to be surjective. Within each partition class there cannot be any edges as f is a homomorphism, hence, we can safely assign equal colours to equivalent vertices. Contrary, a vertex colouring $f : V(G) \to \{c_1, ..., c_k\}$ is a homomorphism from G to K_k by defining $V(K_k) = \{c_1, ..., c_k\}$ [GR01]. □

Corollary 2.13. *Let G and G' be graphs. If G' is k-colourable and there is a homomorphism $G \to G'$, then G is k-colourable.*

Proof. There is a homomorphism $G' \to K_k$ by Proposition 2.12, so that we have a homomorphism $G \to K_k$ by assumption. Apply Proposition 2.12 again. □

Observation 2.14. In the setting of Corollary 2.13 the colouring of G is determined by the preimage of the homomorphism $G \to G'$.

Proof. The proof of Proposition 2.12 shows that the colouring of G' is determined by the preimage of the homomorphism $G' \to K_k$. Consequently the homomorphism $G \to K_k$ that factor as $G \to G' \to K_k$ admits colours determined by the preimage of $G \to G'$. □

Example 2.15. Given a cycle $C = (c_0, ..., c_{n-1})$, there is a homomorphism $C \to K_2$ (2-colouring) if n is even and a homomorphism $C \to K_3$ (3-colouring) if n is odd. First, assume that n is even, say $n = 2m$, and define a function $\phi_{2m} : c_i \mapsto c_{i \bmod 2}$. This is a graph homomorphism onto K_2 with vertex set $\{c_0, c_1\}$: The case $m = 2$ is trivial, so, assuming that the assertion holds for $m = k$, we notice how $\phi_{2(k+1)} = \phi_2 \circ \phi_{2k}$, where ϕ_2 acts on the vertex set for which c_2 is replaced with c_{2k} and c_3 is replaced with c_{2k+1}. Consequently, the claim follows by induction. In the case of n odd, say $n = 2m + 1$, we apply ϕ_{2m} from the even case and notice that this is a homomorphism to K_3 with $\{c_0, c_1, c_{2m}\}$ as vertex set.

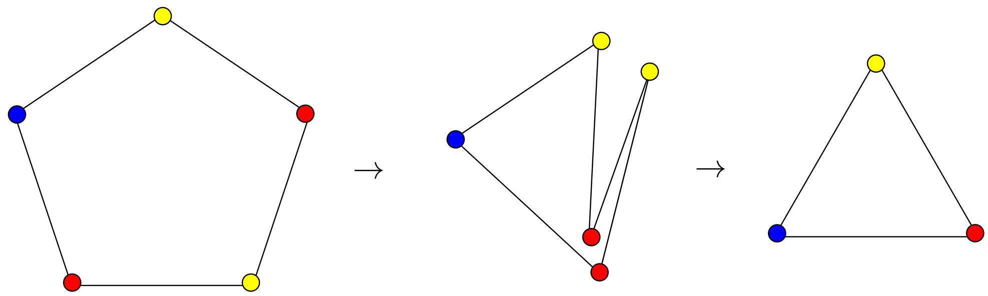


Figure 5: An illustration of how homomorphisms are connected to colouring graphs.

2.2 The Category of Quivers

Definition 2.16. A directed multigraph consists of an indexing set I together with a pair $G = (V(G), E(G))$, or simply $G = (V, E)$, of sets where $E \subset I \times V \times V$. The elements of V and E are called vertices and edges, respectively. Edges in the diagonal $I \times \Delta(V)$, $\Delta(V) = \{(v, v)|\ v \in V\}$, are called loops.[9]

Notice how the above definition reformulate Definition 2.1 with two added features; subsets of 2^V are replaced by elements of $V \times V$ which correspond to adding a direction to each edge, and an indexing set I is added to allow several ordered edges between pairs of vertices. So, the above definition is really two definitions, directed graphs and multigraphs, incorporated in one definition.

Definition 2.17. A quiver is a quadruple $Q = (V, E, s, t)$ consisting of sets V and E called vertices and edges/arrows, respectively, as well as functions $s, t : E \to V$ called source and target, respectively.

Observe how Definition 2.16 and 2.17 are equivalent, which essentially follows as the functions s and t applied to an arrow correspond to the first and second component of an edge in a directed multigraph.

Observation 2.18. Definition 2.16 and Definition 2.17 are equivalent.

Proof. Given a quiver $Q = (V, E, s, t)$, we construct Q from a directed multigraph by taking $G = (V, E(G))$ where $E(G) = \{(e, v, u) \in E \times V \times V|\ s(e) = v \text{ and } t(e) = u\}$. Contrary, given a directed multigraph $G = (V, E)$, consider the functions $s, t : E \to V$, $s(e) = v_2$ and $t(e) = v_3$ whenever $e = (i_1, v_2, v_3)$. □

Historically, the name quiver originates from Peter Gabriel's article "Unzerlegbare Darstellungen I" where he proposed the name "Köcher", which directly translates to quiver, for a

[9]The notation here comes from the diagonal functor which is often denoted Δ.

directed multigraph [Gab72]. Gabriel imagined nodes as archers pulling arrows out of quivers and shooting them at different targets/nodes.

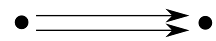


Figure 6: The mother of all quivers.

Consider the category $\mathscr{I}$ given by $\mathrm{Ob}\mathscr{I} = \{0, 1\}$ and morphisms $a, b : 0 \to 1$ in addition to identities, and notice how any quiver $Q = (V, E, s, t)$ determines a contravariant functor $F : \mathscr{I} \to \mathbf{Set}$ by setting $V = F(0)$, $E = F(1)$, $s = F(a)$ and $t = F(b)$.[10] Contrary, a contravariant functor $F : \mathscr{I} \to \mathbf{Set}$ determines a quiver $(F(0), F(1), F(a), F(b))$. So, we can identify quivers with functors, and therefore we give the following definition [ML71].[11]

Definition 2.19. We denote **Quiv** as the category $\mathrm{Presh}_{\mathbf{Set}}\,\mathscr{I}$.

In this context, objects are quivers and morphisms are natural transformations of corresponding functors. A natural transformation $\eta : F \to G$ of contravariant functors $F, G : \mathscr{I} \to \mathbf{Set}$ is, by definition, a collection $\eta = (\eta_0, \eta_1)$ of functions $\eta_0 : F(0) \to G(0)$ and $\eta_1 : F(1) \to G(1)$ satisfying commutative diagrams:

$$\begin{array}{ccc} F(1) & \xrightarrow{\eta_1} & G(1) \\ {\scriptstyle F(a)}\downarrow & & \downarrow{\scriptstyle G(a)} \\ F(0) & \xrightarrow{\eta_2} & G(0) \end{array}$$

$$\begin{array}{ccc} F(1) & \xrightarrow{\eta_1} & G(1) \\ {\scriptstyle F(b)}\downarrow & & \downarrow{\scriptstyle G(b)} \\ F(0) & \xrightarrow{\eta_2} & G(0) \end{array}$$

That is, taking source and target commutes with the functions that send vertices to vertices and edges to edges, i.e. arrows in the quiver associated with F are preserved. So, the morphisms in **Quiv** are indeed morphisms in **Grph** extended to quivers.

[10]Notice that the morphisms $a, b : 0 \to 1$ are flipped by a contravariant functor.

[11]The terminology here is inconsistent with that of Saunders Mac Lane in "Categories for the working Mathematician", where he refers to a quiver as a graph.

Note how quivers almost constitutes categories, and categories define quivers by taking objects as vertices and arrows as morphisms. The latter is reflected by the forgetful functor $U : \mathbf{Cat} \to \mathbf{Quiv}$, where a (small) category $\mathscr{A}$ is sent to the underlying quiver $U(\mathscr{A})$ consisting of:

- Vertices as objects of $\mathscr{A}$.
- Arrows as morphisms.[12]
- Source and target as domain and codomain, respectively.

Moreover, a functor of small categories sends objects to objects and morphisms to morphisms, so U simply restricts the functor to the underlying quiver [ML71]. Given a field K, a finite quiver determines a K-algebra where multiplication is given by concatenating arrows and the vector space structure is given by taking every finite product as a basis.[13] More precisely, given a quiver $Q = (V, E, s, t)$, add trivial arrows e_v for each $v \in V$ satisfying $s(e_v) = v = t(e_v)$ and

$$e \cdot e_v = \begin{cases} e & \text{if } s(e) = v \\ 0 & \text{if } s(e) \neq v \end{cases},$$

$$e_v \cdot e = \begin{cases} e & \text{if } t(e) = v \\ 0 & \text{if } t(e) \neq v \end{cases}$$

for any arrow e in E. Include arrows whenever concatenating is possible, i.e. add arrows $e \circ f$ according to the multiplication rule

$$e \cdot f = \begin{cases} e \circ f & \text{if } s(e) = t(f) \\ 0 & \text{if } s(e) \neq t(f) \end{cases}.$$

The trivial arrows (e_v) sums up to the multiplicative identity in the algebra [ARS97]. This process basically extend quivers to categories in a functorial way, i.e. there is a functor $C : \mathbf{Quiv} \to \mathbf{Cat}$ sending a quiver to the corresponding quiver that includes trivial arrows and non-zero concatenation of arrows; identity morphisms are trivial arrows and composition is given by multiplication. A natural transformation, $\eta = (\eta_0, \eta_1)$, between two quivers is sent to the functor $C(\eta)$ given by the assignment η_0 on objects and extending η_1 to trivial arrows and concatenations functorially; define $C(\eta)(e) = \mathrm{id}_1(e)$ whenever e is in the domain of η_1, extend it to be functorial by setting $C(\eta)(ef) = C(\eta)(e) \circ C(\eta)(f)$ and let $C(\eta)(e_v) = e_{C(v)}$. As in the case of U, one can easily verify C to be a functor.

[12] Smallness ensures that the set of arrows is an actual set.

[13] As for graphs, a quiver is finite if its set of vertices is finite.

The morale of how a quivers and small categories are connected is summed up in the following proposition [ML71].

Proposition 2.20. *C is left adjoint to U, i.e. there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Cat}}(C(-),-) \simeq \mathrm{Hom}_{\mathbf{Quiv}}(-,U(-))$$

of functors $\mathbf{Quiv}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Set}$.

2.3 Simplicial Objects and Homology

We present a category that will associate a chain complex, and thus homology, to certain contravariant functors.

Definition 2.21. The simplex category, $\mathbf{\Delta}$, is the full subcategory of **Cat** consisting of objects $[n]$, $[n] = C(0 \to 1 \to \cdots \to n)$ where $n \in \mathbb{N}_0$.[14]

Equivalently, we may define $\mathbf{\Delta}$ as every poset $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ which replaces $\to$ with $\leq$ and functors with order-preserving maps. We define two important types of morphisms:

- Face maps, for a given $n \in \mathbb{N}_0$ and $0 \leq i \leq n$ we have a functor $\delta_i^n : [n-1] \to [n]$, $(0 \to 1 \to \cdots \to n-1) \mapsto (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$.
- Degeneracy maps, for a given $n \in \mathbb{N}_0$ and $0 \leq i \leq n-1$ we have a functor $\sigma_i^n : [n] \to [n-1]$, $(0 \to 1 \to \cdots \to n) \mapsto (0 \to 1 \to \cdots \to i \xrightarrow{\mathrm{id}_i} i \to \cdots \to n-1)$ where $i \to i+1$ is mapped to $i \xrightarrow{\mathrm{id}_i} i$.

It is intuitively clear that face maps and degeneracy maps determine every morphism due to functoriality [ML71].[15]

Proposition 2.22. *Any morphism in $\mathbf{\Delta}$ is a unique composition of face maps and degeneracy maps.*

Definition 2.23. A simplicial object in a category $\mathscr{C}$ is an object in $\mathrm{Presh}_{\mathscr{C}}\,\mathbf{\Delta}$. In the special case for which $\mathscr{C} = \mathbf{Set}$, we refer to simplicial objects as simplicial sets.

[14] C is adjoint to the forgetful functor between **Cat** and **Quiv**.

[15] Functoriality restricts to composing forwards or restricting an arrow to the identity as there are no arrows $m \to n$ for $m > n$.

Notice that the objects of $\mathbf{\Delta}$ can be considered as simplicial sets through the Yoneda embedding that identifies $[n]$ with $\mathrm{Hom}_{\mathbf{\Delta}}(-,[n])$ [Opp16].

Recall that a pre-additive category is a category where the Hom-sets are Abelian groups satisfying that composition is bilinear, an additive category is a pre-additive category that admits the following two features.

- Zero object. An object 0 such that given any object A, both $\mathrm{Hom}_{\mathscr{A}}(A,0)$ and $\mathrm{Hom}_{\mathscr{A}}(0,A)$ contain a unique morphism. This map is always denoted 0.
- Biproduct. Given any two objects A and B, there is an object $A\oplus B$ with morphisms i_A, i_B, π_A and π_B satisfying $\mathrm{id}_A=\pi_A\circ i_A$, $\mathrm{id}_B=\pi_B\circ i_B$ and $\mathrm{id}_{A\oplus B}=i_A\circ\pi_A+i_B\circ\pi_B$.

$$A \underset{i_A}{\overset{\pi_A}{\leftrightarrows}} A\oplus B \underset{i_B}{\overset{\pi_B}{\rightleftarrows}} B$$

The biproduct (if it exists) can be seen to agree with both the limit and the colimit of the objects, and consequently it is unique. As the notation suggests, the 0 object is the 0-module in the case of modules, and the biproduct is the direct sum [Awo10, Opp16]. Let $f:A\to B$ be a morphism in an additive category, then the kernel of f is the pullback/limit (if it exists) of the diagram:

$$\begin{array}{ccc} & & A \\ & & \big\downarrow{\scriptstyle f} \\ 0 & \xrightarrow{\quad 0 \quad} & B \end{array}$$

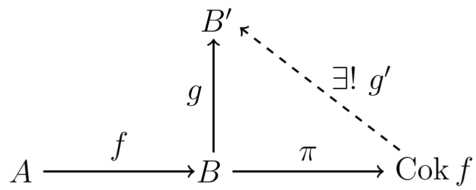
That is to say an object $\mathrm{Ker}\, f$ with a monomorphism $\mathrm{Ker}\, f \xrightarrow{i} A$ such that $f\circ i=0$ and any $B' \xrightarrow{g} A$ satisfying $f\circ g=0$ factor uniquely through $\mathrm{Ker}\, f$:

$$\begin{array}{ccccc} & & B' & & \\ & {\scriptstyle \exists!\, g'} \swarrow & \big\downarrow{\scriptstyle g} & & \\ \mathrm{Ker}\, f & \xrightarrow{\quad i \quad} & A & \xrightarrow{\quad f \quad} & B \end{array}$$

Dually, the cokernel of $f:A\to B$ is the pushout/colimit (if it exists) of the diagram:

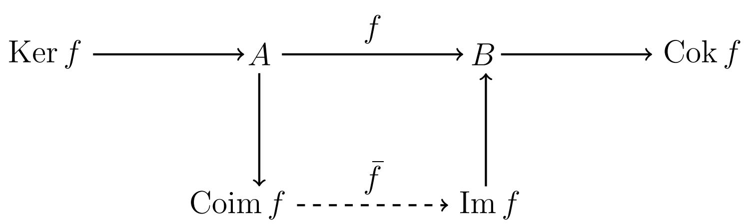
$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle 0}\big\downarrow & & \\ 0 & & \end{array}$$

That is to say an object $\operatorname{Cok} f$ with an epimorphism $B \xrightarrow{\pi} \operatorname{Cok} f$ such that $\pi \circ f = 0$ and any $B \xrightarrow{g} B'$ satisfying $g \circ f = 0$ factor uniquely through π:



Moreover, with i and π as in the above diagrams, we define the image of f, $\operatorname{Im} f$, as the kernel of π (if it exists) and the coimage of f, $\operatorname{Coim} f$, as the cokernel of i (if it exists).

Recall that a pre-Abelian category is an additive category where kernels and cokernels always exist. Composing f with its kernel induces a map through the coimage, say $f = f' \circ (A \to \operatorname{Coim} f)$. Further, f composed with $B \to \operatorname{Cok} f$ is zero so that $f = f' \circ (A \to \operatorname{Coim} f)$ implies that $f' \circ (B \to \operatorname{Cok} f) = 0$ as $A \to \operatorname{Coim} f$ is an epimorphism. Hence, f', factor uniquely through the image of f and we have a uniquely induced map $\bar{f} : \operatorname{Coim} f \to \operatorname{Im} f$.



An Abelian category is a pre-Abelian category where this induced map is an isomorphism [Opp16]. In the special case of $\operatorname{Mod} R$ for a given ring R, the kernel and cokernel of an R-homomorphism $f : M \to N$ is reduced to $\operatorname{Ker} f = \{x \in M |\ f(x) = 0\}$ with inclusion and $\operatorname{Cok} f = {}^{N}\!/\!_{\operatorname{Im} f}$ with the canonical map, where $\operatorname{Im} f = \{y \in N |\ \exists\, x \in M,\, y = f(x)\}$ [Rot08].[16]

Observation 2.24. As the Hom-sets are Abelian groups and $\circ$ is bilinear, Hom functors from Abelian categories may be considered as functors into **Ab** rather than **Set**.

Given a chain complex

$$A = \cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots ,$$

[16]The coimage is the cokernel of the kernel, so that the image being isomorphic to the coimage is reduced to the first isomorphism theorem in the case of modules; $f : M \to N$ in $\operatorname{Mod} R$ for a given ring R satisfy $\operatorname{Coim} f \simeq {}^{M}\!/\!_{\operatorname{Ker} f}$.

i.e. $\partial^2 = 0$, in an Abelian category, we have that the monomorphism $\operatorname{Im}\partial_{n+1} \hookrightarrow A_n$ factor through the monomorphism $\operatorname{Ker}\partial_n \hookrightarrow A_n$. The definition of a kernel gives a unique monomorphism $\operatorname{Im}\partial_{n+1} \hookrightarrow \operatorname{Ker}\partial_n$ whose cokernel is defined as the n'th homology, $H_n(A)$.[17] In the case of modules this definition is reduced to the factor module ${\operatorname{Ker}\partial_n}/{\operatorname{Im}\partial_{n+1}}$ [Opp16, Rot08]. Note that a chain complex is an underlying quiver of a category by including identities and compositions, i.e. applying the functor C. As a result, there is a category $\mathbf{C}(\mathscr{A})$ consisting of chain complexes and chain maps that correspond to natural transformations; given chain complexes

$$A = \cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots ,$$

and

$$B = \cdots \to B_{n+1} \xrightarrow{\delta_{n+1}} B_n \xrightarrow{\delta_n} B_{n-1} \to \cdots ,$$

a chain map $f : A \to B$ is simply a collection of maps $f_n : A_n \to B_n$ that commute with boundary operators; $f_{n-1} \circ \partial_n = \delta_n \circ f_n$ for any n. As H_n is a cokernel, and more generally a colimit, it is a functor $\mathbf{C}(\mathscr{A}) \to \mathscr{A}$ [Opp16].

Proposition 2.25. *Let $\mathscr{A}$ be an additive category, and let S be any simplicial object in $\mathscr{A}$. Then we have a chain complex*

$$\cdots \xrightarrow{\partial_{n+1}} S([n]) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S([1]) \xrightarrow{\partial_1} S([0]) \xrightarrow{0} 0,$$

where $\partial_n = \sum_{i=0}^{n}(-1)^i S(\delta_i^n)$ is the corresponding boundary operator.

Proof. Take any $n \geq 2$, we want to show that $\partial_{n-1} \circ \partial_n = 0$. Notice that the face maps satisfy the relation $\delta_j^{m+1}\delta_i^m = \delta_i^{m+1}\delta_{j-1}^m$ whenever $i < j$ for any $m \geq 1$;

$$\delta_j^{m+1}\delta_i^m(0 \to 1 \to \cdots \to n-1) = 1 \to \cdots \to j-1 \to j+1 \to \cdots \to i-1 \to i+1 \cdots \to m+1,$$

and

$$\delta_i^{m+1}\delta_{j-1}^m(0 \to 1 \to \cdots \to n-1) = 1 \to \cdots \to j-1 \to j+1 \to \cdots \to i-1 \to i+1 \cdots \to m+1.$$

In particular, using that S is contravariant, we have that $S(\delta_i^m)S(\delta_j^{m+1}) = S(\delta_{j-1}^m)S(\delta_i^{m+1})$

[17]The assumption of Abelian gives a dual and equivalent definition of homology through cokernels and coimages.

for i, j and m as above. Hence the composition is zero:

$$\begin{aligned}
\partial_{n-1} \circ \partial_n &= (\sum_{i=0}^{n-1} (-1)^i S(\delta_i^{n-1})) \circ (\sum_{j=0}^{n} (-1)^j S(\delta_j^n)) \\
&= \sum_{i,j} (-1)^{i+j} S(\delta_i^{n-1}) S(\delta_j^n) \\
&= \sum_{i<j} (-1)^{i+j} S(\delta_i^{n-1}) S(\delta_j^n) + \sum_{i \geq j} (-1)^{i+j} S(\delta_i^{n-1}) S(\delta_j^n) \\
&= \sum_{i<j} (-1)^{i+j} S(\delta_{j-1}^{n-1}) S(\delta_i^n) + \sum_{i \geq j} (-1)^{i+j} S(\delta_i^{n-1}) S(\delta_j^n) \\
&= 0,
\end{aligned}$$

where $(-1)^{i+j} S(\delta_{j-1}^{n-1}) S(\delta_i^n)$ in the first sum cancel with $(-1)^{i+j-1} S(\delta_{j-1}^{n-1}) S(\delta_i^n)$ in the second sum. □

The summation trick in the above proof is standard when proving $\partial^2 = 0$ in homology theory, and is usually specialized to the case of singular homology for a given space [Rot08]. This approach however, illustrates the combinatorial nature of the boundary operator. Moreover, Proposition 2.25 associates a chain complex to a simplicial object whenever $\mathscr{A}$ is Abelian.

3 Homology and Complexes

3.1 Singular Homology

Definition 3.1. The standard p-simplex is defined as the topological subspace

$$\Delta_p = \{(t_0, ..., t_p) \in \mathbb{R}^{p+1} | \ \sum t_i = 1, \ \ t_i \geq 0 \ \ \forall i = 0, ..., p\}$$

of $\mathbb{R}^{p+1}$ equipped with the standard topology. Equivalently, Δ_p is the convex hull of the standard basis vectors in $\mathbb{R}^{p+1}$.

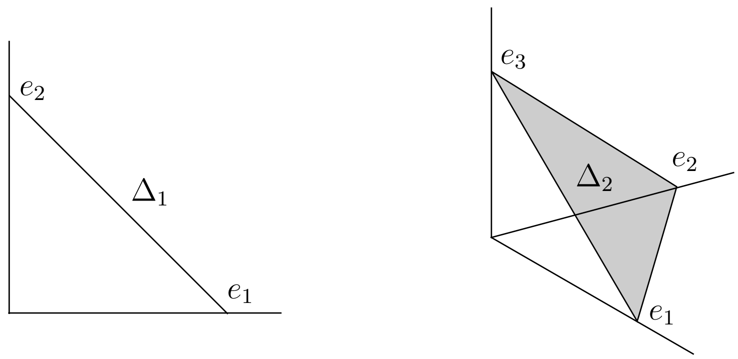


Figure 7: An illustration of the standard 1- and 2-simplex. The standard basis vectors in Euclidean space are denoted e_i.

This gives a geometrical interpretation of $\mathbf{\Delta}$ through the functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$, determined by $[p] \mapsto \Delta[p] = \Delta_p$ and a morphism $f : [p] \to [m]$ maps to $\Delta(f)(t_0, ..., t_p) = (t'_0, ..., t'_m)$ where $t'_j = \sum_{f(i)=j} t_i$.

Proposition 3.2. $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ *is a functor.*

Proof. We clearly have a well-defined mapping $\text{Ob}\,\mathbf{\Delta} \mapsto \text{Ob}\,\mathbf{Top}$ given by $[p] \mapsto \Delta_p$, but we need to verify that $\Delta(f)$ is a continuous function, i.e. a morphism in **Top**, whenever f is a morphism in $\mathbf{\Delta}$. We have equipped the standard simplexes with the subspace topology of Euclidean space, so that given $f : [p] \to [m]$ and an $\epsilon > 0$, it suffices to find a $\delta > 0$ such that $|(t_0, ..., t_p) - (s_0, ..., s_p)| < \delta$ implies $|\Delta(f)(t_0, ..., t_p) - \Delta(f)(s_0, ..., s_p)| < \epsilon$. Let $\Delta(f)(t_0, ..., t_p) = (t_{1,1} + ... + t_{1,r_1}, ..., t_{m,1} + ... + t_{m,r_m})$ and $\Delta(f)(s_0, ..., s_p) = (s_{1,1} + ... + s_{1,r_1}, ..., s_{m,1} + ... + s_{m,r_m})$, where $r_1 + ... + r_m = p$, according to the definition of Δ. Observe how

$$
\begin{aligned}
&|\Delta(f)(t_0,...,t_p) - \Delta(f)(s_0,...,s_p)| \\
&= |((t_{1,1}-s_{1,1})+...+(t_{1,r_1}-s_{1,r_1}),...,(t_{m,1}-s_{m,1})+...+(t_{m,r_m}-s_{m,r_m}))| \\
&= \sqrt{((t_{1,1}-s_{1,1})+...+(t_{1,r_1}-s_{1,r_1}))^2+...+((t_{m,1}-s_{m,1})+...+(t_{m,r_m}-s_{m,r_m}))^2} \\
&\leq \sqrt{(t_0-s_0)^2+...+(t_p-s_p)^2+(r_1(r_1-1)+...+r_m(r_m-1))(t_i-s_i)(t_j-s_j)},
\end{aligned}
$$

where $r_1(r_1-1)+...+r_m(r_m-1)$ are the number of cross terms from the squares, and i and j satisfy that $(t_i-s_i)(t_j-s_j)$ is the largest cross term. Without loss of generality, we may assume that $|(t_i-s_i)(t_j-s_j)| \leq (t_i-s_i)^2$, giving the estimate

$$
\begin{aligned}
&|\Delta(f)(t_0,...,t_p) - \Delta(f)(s_0,...,s_p)| \\
&\leq \sqrt{(t_0-s_0)^2+...+(t_p-s_p)^2+(r_1(r_1-1)+...+r_m(r_m-1))(t_i-s_i)(t_j-s_j)} \\
&\leq \sqrt{(t_0-s_0)^2+...+(t_p-s_p)^2+(r_1(r_1-1)+...+r_m(r_m-1))(t_i-s_i)^2} \\
&\leq \sqrt{(t_0-s_0)^2+...+(t_p-s_p)^2+(r_1(r_1-1)+...+r_m(r_m-1))((t_0-s_0)^2+...+(t_p-s_p)^2)} \\
&= \sqrt{1+r_1(r_1-1)+...+r_m(r_m-1)}\sqrt{(t_0-s_0)^2+...+(t_p-s_p)^2} \\
&= \sqrt{1+r_1(r_1-1)+...+r_m(r_m-1)}|(t_0,...,t_p)-(s_0,...,s_p)|.
\end{aligned}
$$

It suffices to take $\delta = \frac{\epsilon}{\sqrt{1+r_1(r_1-1)+...+r_m(r_m-1)}}$, and $\Delta(f)$ is indeed continuous. Moreover, it is clear that Δ send $\mathrm{id}_{[n]}$ to id_{Δ_n}, and composition is preserved by noting the following. Given $f:[p]\to[m]$ and $g:[m]\to[q]$, it is clear that if $t'_j = \sum_{f(i)=j} t_i$, then $t''_j = \sum_{g(i)=j} t'_i = \sum_{g(i)=j}\sum_{f(k)=i} t_k = \sum_{g\circ f(k)=j} t_k$. □

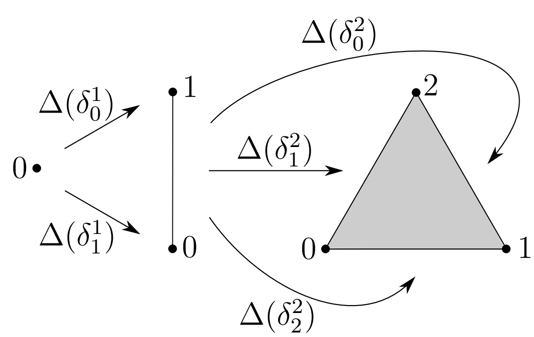


Figure 8: The geometry of Δ.

The functor Δ is often referred to as the geometric realization of $\mathbf{\Delta}$, and not only do we get a geometrical interpretation of the objects, but also of the morphisms. The face map δ^n_i is sent to $\Delta(\delta_i^n)$ that embeds Δ_{n-1} on the i'th face of Δ_n; $\{(t_0, ..., t_n) \in \Delta_n| \ \ t_i = 0\}$. Similarly, the geometric image of the degeneracy map σ_i^n collapse Δ_n onto Δ_{n-1} by combining the successive coordinates i and $i+1$.

In Section 2.2, we looked at a forgetful functor $U : \mathbf{Cat} \to \mathbf{Quiv}$, we also looked at how this is adjoint to equipping a quiver with identities and composition of arrows. Such adjointness occurs in several situations, another example is the construction of free modules over arbitrary sets. There is a forgetful functor $U_R : \operatorname{Mod} R \to \mathbf{Set}$ that simply forgets the module structure; a module is a set with additional structure, and an R-homomorphism is a function. Moreover, U_R admits an adjoint. Consider the functor $R^{(-)} : \mathbf{Set} \to \operatorname{Mod} R$ that sends a set, X, to the free R-module with X as a basis, i.e. $X \mapsto R^{(X)} = \{f : X \to R \mid \ |f^{-1}(R - \{0\})| < \infty\}$ where addition and scalar multiplication is inherited from R; $(f+g)(x) = f(x) + g(x)$ and $(rf)(x) = rf(x)$. This module is free and admits a basis given by the indicator functions[18]

$$\chi_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

that are identified with the elements of X. Moreover, a function $\phi : X \to Y$ is sent to $R^{(\phi)} : R^{(X)} \to R^{(Y)}$ defined by

$$(f : X \to R) \mapsto (y \mapsto \sum_{x \in \phi^{-1}(y)} f(x)).$$

U_R is obviously functorial, $R^{(-)}$ is easily checked to be functorial, and they are mutually adjoint; $\operatorname{Hom}_{\mathbf{Set}}(-, U_R(-)) \simeq \operatorname{Hom}_{\operatorname{Mod} R}(R^{(-)}, -)$ [Opp16]. This gives a short definition of singular homology.

Definition 3.3. Let X be any topological space, then we define the singular homology of X with coefficients in $\mathbb{Z}$ as the homology of the corresponding chain complex from the simplicial object $\mathbb{Z}^{(-)} \circ \operatorname{Hom}_{\mathbf{Top}}(\Delta(-), X)$.[19] The i'th face of a basis element $\phi : \Delta_n \to X$ is given by $\mathbb{Z}^{(\operatorname{Hom}_{\mathbf{Top}}(\Delta(\delta_i^n), X))}(\phi)$.[20]

Let us make this construction explicit. Applying $\mathbb{Z}^{(-)} \circ \operatorname{Hom}_{\mathbf{Top}}(\Delta(-), X)$ to $[p]$ yields $\mathbb{Z}^{(\operatorname{Hom}_{\mathbf{Top}}(\Delta_p, X))}$, i.e. the free $\mathbb{Z}$-module with continuous functions $\Delta_p \to X$ as a basis, which we will denote $S_p(X) = \mathbb{Z}^{(\operatorname{Hom}_{\mathbf{Top}}(\Delta_p, X))}$. Applying $\operatorname{Hom}_{\mathbf{Top}}(\Delta(-), X)$ to $\delta_i^p : [p-1] \to [p]$ gives $\partial_i^p = \operatorname{Hom}_{\mathbf{Top}}(\Delta(\delta_i^p), X) : \operatorname{Hom}_{\mathbf{Top}}(\Delta_p, X) \to \operatorname{Hom}_{\mathbf{Top}}(\Delta_{p-1}, X)$, $\partial_i^p f = f \circ \Delta(\delta_i^p)$ defined by $\partial_i^p(f)(t_0, ..., t_{p-1}) = f \circ \Delta(\delta_i^p)(t_0, ..., t_{p-1}) = f(t_0, ..., t_{i-1}, 0, t_i, ..., t_{p-1})$, the i'th

[18]The functions have finite support so that $f(y) = \sum_{x \in X} f(y)\chi_x(y)$, i.e. χ_x's generate $R^{(X)}$, and linear independence follows as $\sum a_i \chi_{x_i} = 0$ implies that $a_i = 0$ by applying x_i's.

[19]Note that we could have chosen any (commutative) ring R instead of $\mathbb{Z}$.

[20]Observe how this generalizes the geometrical i'th face of Δ_n.

face of f. In particular, the i'th face of Δ_p corresponds to the image of its geometrical i'th face homeomorphic to Δ_{p-1}; $\{(t_0, ..., t_p) \in \Delta_p | \ \ t_i = 0\}$. Applying $\mathbb{Z}^{(-)}$ to ∂_i^p extend it to a homomorphism which we also denote ∂_i^p. Now, $\partial_p = \sum_{i=0}^{p}(-1)^i \partial_i^p$ correspond to restricting continuous functions to the faces of Δ_p with a sign convention that yields $\partial^2 = 0$.

H_n is a functor $\mathbf{C}(\mathbf{Ab}) \to \mathbf{Ab}$, but $\mathrm{Hom}_{\mathbf{Top}}(\Delta(-), -)$ is a bifunctor so that we may in fact see H_n as a functor $\mathbf{Top} \to \mathbf{Ab}$. We ease notation by setting $H_n(X) = H_n(\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{0} 0)$. Moreover, it immediately follows that homeomorphic spaces have equal homology groups; if $h : X \to Y$ is a homeomorphism, then $H_n(h^{-1}) \circ H_n(h) = H_n(h^{-1} \circ h) = H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$ and similarly $H_n(h) \circ H_n(h^{-1}) = \mathrm{id}_{H_n(Y)}$. This can be generalized further by introducing homotopy theory.

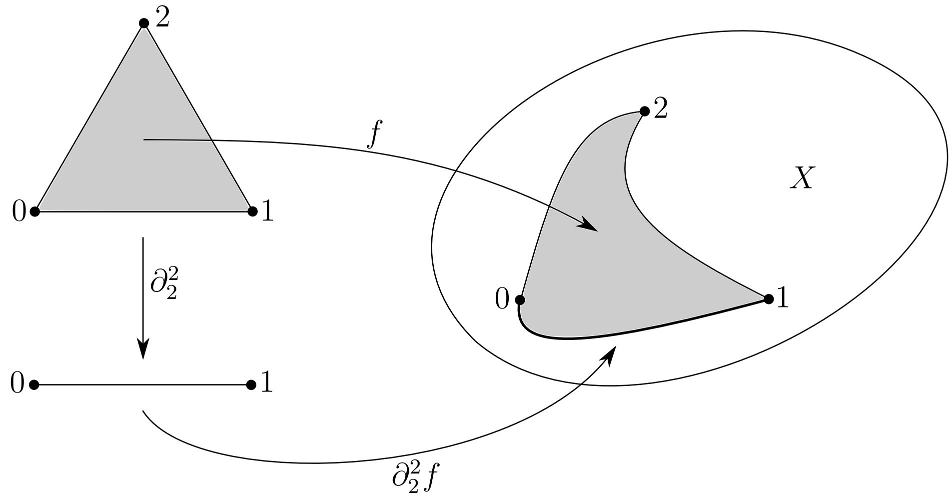


Figure 9: The geometry of ∂_i^n; ∂_2^2 is applied to a map $f : \Delta_2 \to X$.

We end this section with a simple and illustrative example taken from [Vic12].

Example 3.4. Let $X = \{x\}$ be a space consisting of a single point x. There is only one topology on this set as both the empty set and the whole set needs to be open. Moreover, given another space Y there is only one function $Y \to X$, namely $y \mapsto x$ for any $y \in Y$, and this is continuous as the preimage of x is all of Y which is open. Thus, $S_p(X) = (\Delta_p \to X) \simeq \mathbb{Z}$ for any p, and $\partial_i^p(\Delta_p \to X) = (\Delta_{p-1} \to X)$.[21] Considering the chain complex

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{0} 0,$$

we have that $\partial_1(X \to \Delta_1) = 0$ as ∂_0^1 and ∂_1^1 cancel, and this argument generalizes to ∂_p giving 0 if p is odd and $\Delta_{p-1} \to X$ if p is even. Now, we can inductively determine H_n;

[21] Given a module M and a subset S, the notation (S) reads the smallest submodule containing S.

$H_0(X) = {}^{\operatorname{Ker} 0}\!/_{\operatorname{Im} \partial_1} \simeq {}^{\mathbb{Z}}\!/_{0} \simeq \mathbb{Z}$, while the others cancel giving $H_n(X) = 0$ for $n \geq 1$.

3.2 Simplicial Complexes

Definition 3.5. Given a set, S, we define an abstract simplicial complex, A, as a subset of 2^S satisfying that $\sigma' \subset \sigma \in A$ implies $\sigma' \in A$. The elements that satisfy $|\sigma| = k$ are referred to as $(k-1)$-simplexes,[22] and the set of all k-simplexes is denoted A^k, the k-skeleton of A.

The set, S, in this definition is best seen as a discrete set in a topological space, i.e. a subset of a topological space that carry the discrete topology as its subspace topology.[23] Intuitively, one thinks of 0-simplexes as vertices or points in a space, 1-simplexes as edges/lines in a space, 2-simplexes as triangles in a space, etc.

Observation 3.6. A graph, possibly infinite, is an abstract simplicial complex, A, for which $|\sigma| = 1$ or 2 for any $\sigma \in A$.

Proof. Any graph, $G = (V, E)$, can be defined as the abstract simplicial complex $A = V \cup E \subset 2^V$; given an edge $\sigma = \{\sigma_1, \sigma_2\} \in E$, we know that σ_1 and σ_2 are both contained in V and therefore they are also contained in A. Hence A does indeed make up an abstract simplex. Definition 3.5 immediately gives the opposite direction. □

Graphs and graph homomorphisms gives a category, **Grph**, which extends to abstract simplicial complexes by not only preserving 1-simplexes, but k-simplexes in general.

Definition 3.7. Given two simplicial complexes A_1 and A_2, a morphism between them is a function $f : A_1^0 :\to A_2^0$ such that a k-simplex $\{v_0, ..., v_k\} \in A_1^k$ is sent to $\{f(v_0), ..., f(v_k)\} \in A_2^k$.

The exact same reasoning as in Proposition 2.5 shows that abstract simplicial complexes with the above morphisms forms a category.

We will use standard simplexes (Definition 3.1) as building blocks to embed abstract simplicial complexes into topological spaces. Note that the standard simplexes are both compact and Hausdorff; it is well known that $[0, 1]^p$ is compact in the standard topology, and the standard p-simplex is homeomorphic to $[0, 1]^p$.[24] Following the construction by A. Hatcher in [Hat01], we have the following definition.

[22]The k-simplexes are abstract simplexes by definition.

[23]Recall that the discrete topology on a set is the power set, i.e. the discrete topology on a set S is given by 2^S.

[24]Compactness is to be understood as every open cover admitting a finite subcover.

Definition 3.8. A simplicial complex in a Hausdorff space, X, is a collection of maps, $\{\sigma_\alpha : \Delta_{f(\alpha)} \to X\}_{\alpha \in I}$, where I is an indexing set, and $f : I \to \mathbb{N}_0$ is a function,[25]. The collection respect the following properties:

- The restriction $\sigma_\alpha|\text{Int}(\Delta_{f(\alpha)})$ is injective for any α.[26]
- For any restriction of σ_α to a face of $\Delta_{f(\alpha)}$, there is a β such that the restriction agrees with σ_β.
- For any $\alpha, \beta \in I$ either $\sigma_\alpha(\Delta_{f(\alpha)}) \cap \sigma_\beta(\Delta_{f(\beta)}) = \emptyset$ or they agree on a unique lower simplex, σ_γ, in $\{\sigma_\eta | \; f(\eta) = f(\gamma)\}$.[27]

The map σ_α is an $f(\alpha)$-simplex, the k-simplexes are the elements in $\{\sigma_\alpha | \; f(\alpha) = k\}$ and $\cup_{f(\alpha)\leq k}\sigma_\alpha(\Delta_{f(\alpha)})$ is the k-skeleton. Finally, a simplicial complex is said to be finite whenever the indexing set is finite.

The Hausdorff assumption gives nice topological properties. More precisely, an injective continuous map is not necessarily an embedding of topological spaces in general, however this is the case for continuous injections between compact spaces and Hausdorff spaces. That is, an injective and continuous map, σ, from a compact space, X, to a Hausdorff space, Y, restricts to a homeomorphism from X to the image of σ in the subspace topology from Y. The proof of this statement is quite easy. Indeed, take any closed V in X, then it is compact by the compactness of X, the continuous image $\sigma(V)$ is compact by the continuity of σ, and finally $\sigma(V)$ is closed as it is compact in Y which is Hausdorff. Thus, for a simplicial complex $\{\sigma_\alpha\}$, we know that $\sigma_\alpha(\Delta_{f(\alpha)})$ is homeomorphic to $\Delta_{f(\alpha)}$, and in particular closed, compact and Hausdorff. If, in addition, the simplicial complex is finite, we know that $\cup_\alpha \sigma_\alpha(\Delta_{f(\alpha)})$ is a finite union of closed/compact and hence closed/compact.

Example 3.9. In Section 2.1 we looked at a continuous bijection $[0,1) \to S^1$ that did not make up a homeomorphism, and consequently not a topological embedding. This gives a proof of the non-compactness of $[0,1)$ by showing that S^1 is Hausdorff.

Remark 3.10. We will ambiguously refer to the simplicial complex $\{\sigma_\alpha\}$ as $\cup_\alpha \sigma_\alpha(\Delta_{f(\alpha)})$, and vice versa.[28]

---

[25]The function, f, associated with $\{\sigma_\alpha\}$ keeps track of the geometrical dimension of the standard simplex associated with a given simplex.

[26]Int denotes the topological interior, i.e. the union of every open set contained in a given subset.

[27]Hatcher does not include this requirement, however it meets our goal of embedding abstract simplicial complexes.

[28]This corresponds to thinking of an embedding as its image.

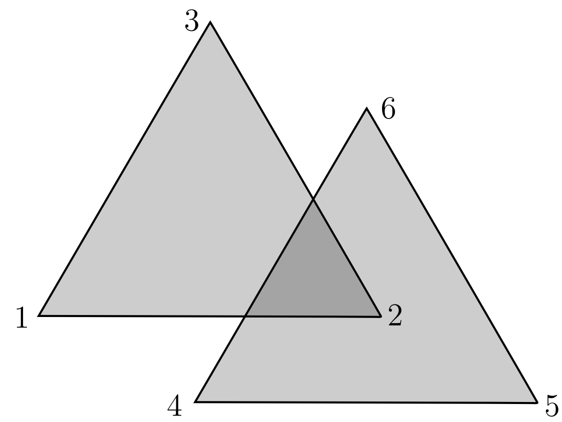


Figure 10: Not an embedding of the abstract simplicial complex $\{1, 2, 3, 4, 5, 6, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 3\}, \{4, 5, 6\}\}$.

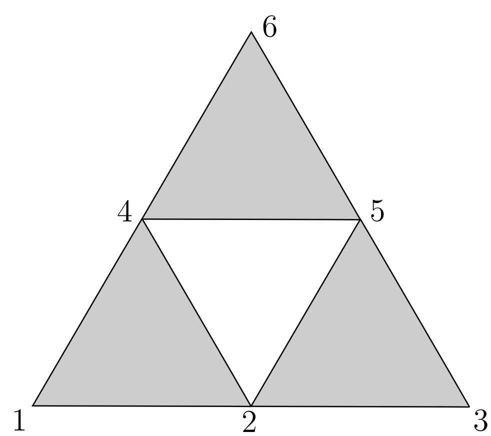


Figure 11: An embedding of the abstract simplicial complex $\{1, 2, 3, 4, 5, 6, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 4\}, \{2, 3, 5\}, \{4, 5, 6\}\}$.

Definition 3.11. An embedding of an abstract simplicial simplex, A, in a Hausdorff space, X, consist of a simplicial complex $\{\sigma_\alpha\}$ in X and bijections $b_k : A^k \to \{\sigma_\alpha| \ f(\alpha) = k\}$. The bijections respect the abstract simplicial complex, i.e. if $a_k \subset a_{k+1}$, then $b_k(a_k)$ is a face of $b_{k+1}(a_{k+1})$.

The above definition identifies an abstract simplicial complex with a simplicial complex in a natural way. The second property in Definition 3.8 ensures that a simplicial complex is an abstract simplicial complex by identifying a k-simplex with the $(k + 1)$ underlying 0-simplexes. In terms of geometry this means that we associate 0-simplexes with points, 1-simplexes with lines, 2-simplexes with triangles, etc. Contrary, it is not possible to embed abstract simplicial complexes in general as we shall see later.[29]

Remark 3.12. We will deliberately not distinguish a simplicial complex from the abstract simplicial complex it defines.

Given a finite simplicial complex, $\{\sigma_\alpha\}$, one often labels the 0-simplexes $\{v_0, ..., v_n\}$ and denote $[v_{i_0}, ..., v_{i_p}]$ as the p-simplex with $v_{i_0}, ..., v_{i_p}$ as underlying 0-simplexes.

Beware that there is one possible confusion, the k-skeleton of a simplicial complex is defined as the union of t-simplexes for $0 \leq t \leq k$, while the k-skeleton of the corresponding abstract simplicial complex is given by the collection of all combinatorial k-simplexes. The information of lower simplexes is, however, encoded in an abstract simplex.

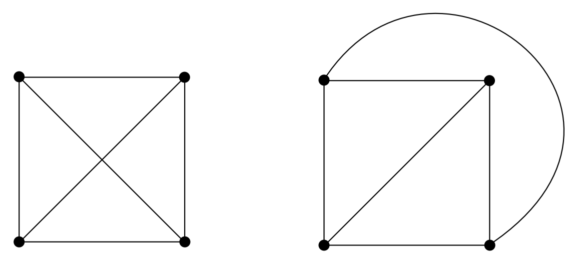


Figure 12: Two representations of K_4 for which only one is an embedding into $\mathbb{R}^2$.

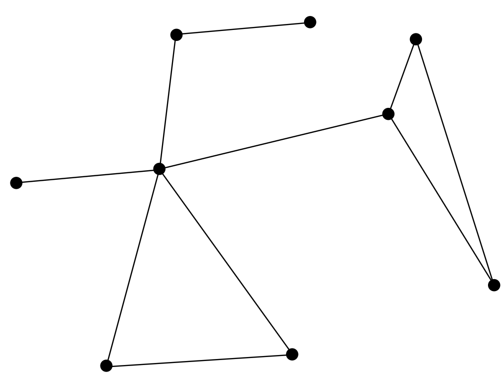


Figure 13: A representation of a graph as a 1-skeleton.

Example 3.13. By Observation 3.6, a graph is an abstract simplex only consisting of 0- and 1-simplexes, and thus an embedding of a graph is a simplicial complex $\{\sigma_\alpha\}$ for which $f(\alpha) = 0$ or 1 for every α. Note that the third property in Definition 3.8 ensures that the

[29]The complete graph on five vertices cannot be embedded in $\mathbb{R}^2$.

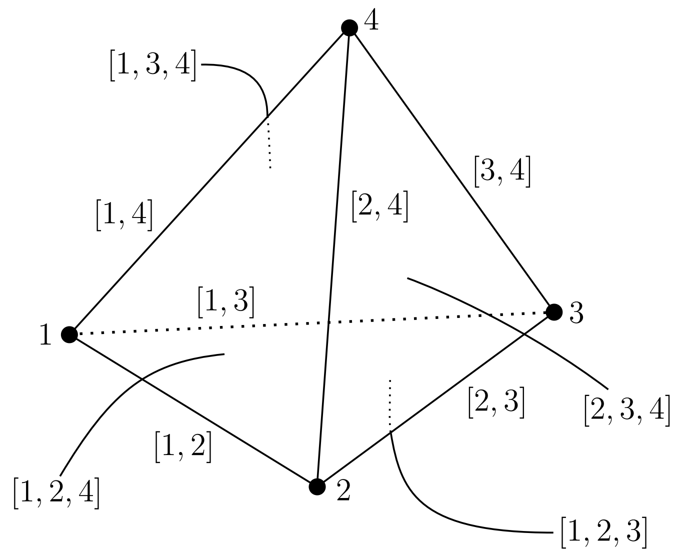


Figure 14: The tetrahedron as a simplicial complex.

interior of different simplexes do not intersect, i.e. edges/1-simplexes do not intersect. Also note that given a simplicial complex, its underlying 1-skeleton defines the embedding of some graph.

Example 3.14. Considering the tetrahedron as a subspace of $\mathbb{R}^3$, we may identify it with a simplicial complex in $\mathbb{R}^3$; four maps from the standard 0-simplex, six maps from the standard 1-simplex and four maps from the standard 2-simplex. Also note that the 1-skeleton is an embedding of K_4 in $\mathbb{R}^3$ as every vertex is adjacent in the tetrahedron. The tetrahedron is homeomorphic to the sphere, S^2, and therefore gives an embedding of $T^0 \cup T^1 = K_4$ in S^2.

3.3 Simplicial Homology

Let $\{\sigma_\alpha\}$ be a simplicial complex in a Hausdorff space, X. From Definition 3.3 we have an associated chain complex and singular homology of $X_0 = \cup_\alpha \sigma_\alpha(\Delta_{f(\alpha)})$. The submodules, $\bar{S}_n(X_0)$ of $S_n(X_0)$, generated by n-simplexes in $\{\sigma_\alpha\}$ gives an induced chain complex, and thus homology. This homology will be referred to as the simplicial homology of X_0.

Singular homology was first introduced by S. Eilenberg in 1944 [Eil44], and together with N. Steenrod the Eilenberg-Steenrod axioms were introduced [ES45]. CW-complexes, short for closure weak complexes, generalizes simplicial complexes, and was introduced by J. H. C. Whitehead [Whi49]. The idea is to attach n-disks inductively by maps from the boundary.[30] Given two topological spaces X and Y, as well as a map $f : A \to Y$ with $A \subset X$,

[30] The boundary of D^n is given by $\partial D^n = \{x \in \mathbb{R}^n | \ |x| = 1\} = S^{n-1}$.

we may construct the space $X \coprod Y/\sim$, where $\sim$ is generated by $a \sim f(a)$ for any $a \in A$. Geometrically, A is attached to its image in Y. A CW-complex is constructed inductively. Starting out with a non-empty set of points, the 0-skeleton, X^0 attach 1-disks to X^0 via maps from the boundary to construct X^1, the 1-skeleton. Proceed inductively by attaching $(k+1)$-disks to the k-skeleton, X^k, via maps from the boundary. The images of the interior of n-disks, homeomorphic to $\mathbb{R}^n$, are called n-cells. The attaching map must admit the finite closure property. That is, the closure of each n-cell only intersect a finite number of lower

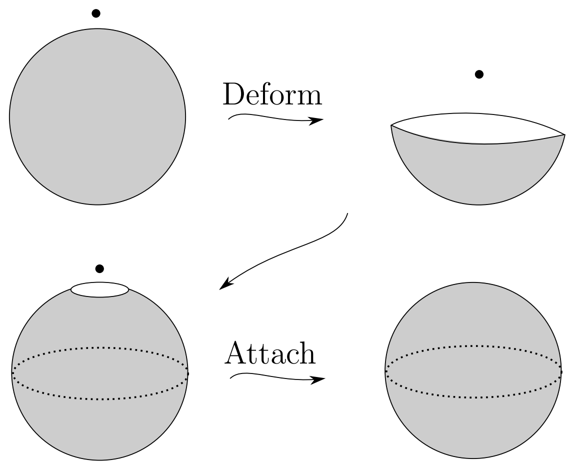


Figure 15: The 2-sphere as a CW-complex obtained by attaching a point to the boundary of the 2-disk.

cells. An example is given by attaching D^2 to a point along the boundary, giving the sphere, S^2. Moreover, CW-complexes are not equipped with the quotient topology from the above construction, but with the weak topology; a subset is closed if and only if the intersection with any closure of a cell is closed. Clearly, closed in the (strong) original topology implies closed in the weak topology, and so the weak topology is in fact coarser (or weaker). A CW-complex is finite if it is constructed by a finite set of cells. Observe how a finite CW-complex is necessarily compact; a finite union of continuous images of compact disks.

Constructing a simplified chain complex for CW-complexes requires a lot of machinery and will be skipped as it is a standard construction included in e.g. [Hat01], but note that the basis in the n'th dimension is given by the n-cells.[31] CW-complexes generalize simplicial complexes; the n-disk is homeomorphic to the standard n-simplex, and embeddings gives strict rules for gluing together simplexes. Further, the resulting homology can be shown to agree with both the singular homology and possibly the simplicial homology of the constructed space via the Eilenberg-Steenrod axioms.[32] This is shown in most standard texts on algebraic topology

[31] We only need to know the generators for future arguments.

[32] The constructed space need not be a simplicial complex, and therefore simplicial homology need not

[Vic12, Hat01]. In Section 3.2, there was no notion of a weak topology. However, we are mainly concerned with finite simplicial complexes where the two topologies agree. Indeed, given a simplicial complex X and a closed subset $V \subset X$, the intersection $V \cap \bar{e}$ with the closure of a simplex e is closed as it is the finite intersection of closed, consequently V is closed in the weak topology. Contrary, if X is a finite simplicial complex and V is closed in the weak topology, simply observe how

$$X = \bigcup_{\text{simplexes}} \bar{e}$$

such that

$$V = \bigcup_{\text{simplexes}} V \cap \bar{e}$$

is closed as it is a finite union of closed. The same argument works for CW-complexes in general by replacing simplex with cell.

It is possible to compute the homology directly by applying the Smith normal form in the finite case.[33]

Example 3.15. Let T be the embedding of the tetrahedron into $\mathbb{R}^3$, defined by Figure 14. Numerate the 0-skeleton of the tetrahedron with $\{1, 2, 3, 4\}$. Then we have that $\partial_2[i, j, k] = [i, j] + [j, k] - [i, k]$ whenever $1 \leq i < j < k \leq 4$, and $\partial_1[i, j] = j - i$ whenever $1 \leq i < j \leq 4$. Fixing ordered bases 1, 2, 3, 4 for $\mathbb{Z}^4$, $[1, 2]$, $[1, 3]$, $[1, 4]$, $[2, 3]$, $[2, 4]$, $[3, 4]$ for $\mathbb{Z}^6$ and $[1, 2, 3]$, $[1, 2, 4]$, $[1, 3, 4]$, $[2, 3, 4]$ for $\mathbb{Z}^4$, we get matrix representations

$$\partial_1 = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and

$$\partial_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

make sense in general.

[33]It is crucial that $\mathbb{Z}$ is a PID here.

Applying row and column operations, we easily find the Smith normal forms to be

$$\partial_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\partial_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, we calculate the homology of

$$0 \to \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^4 \to 0$$

by changing bases according to the Smith normal form, giving

$$H_0(T) = {}^{\operatorname{Ker} 0}\!\big/_{\operatorname{Im} \partial_1} \simeq \mathbb{Z},$$

$$H_1(T) = {}^{\operatorname{Ker} \partial_1}\!\big/_{\operatorname{Im} \partial_2} \simeq 0$$

and

$$H_2(T) = {}^{\operatorname{Ker} \partial_2}\!\big/_{\operatorname{Im} 0} \simeq \mathbb{Z}.$$

3.4 Manifolds and Triangulations

Simplicial homology is particularly easy to compute over PIDs (principal ideal domains) and fields, a computational aspect of homology that transfer to topological spaces homeomorphic to simplicial complexes.

Definition 3.16. An n-dimensional manifold, or n-manifold, M is a Hausdorff space with a second countable basis that is locally homeomorphic to $\mathbb{R}^n$; the topology admits a countable basis, and given any $p \in M$ there is an open neighbourhood U_p of p and a homeomorphism $h_p : U_p \to \mathbb{R}^n$. The local homeomorphisms, h_p, are referred to as charts.

Note that the open n-disk, $B_1(0) = \operatorname{Int}(D^n)$, is homeomorphic to $\mathbb{R}^n$, therefore we may replace locally homeomorphic to $\mathbb{R}^n$ with locally homeomorphic to $B_1(0)$. The open n-disk

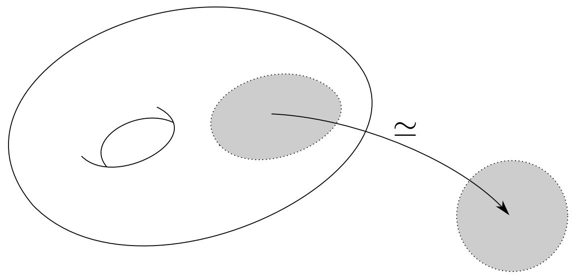


Figure 16: A 2-manifold.

is in fact diffeomorphic (to be defined) to $\mathbb{R}^n$ by scaling with arctan.

Example 3.17. Equip $\mathbb{R}^{n+1}$, $n \geq 0$, with the standard topology and consider the sphere $S^n = \{x \in \mathbb{R}^{n+1} | \ \ |x| = 1\}$ in the subspace topology. The space is clearly second countable and Hausdorff as $\mathbb{R}^{n+1}$ is second countable and Hausdorff. Indeed, the standard topology is generated by ϵ-balls, $B_\epsilon(x) = \{z \in \mathbb{R}^{n+1} | \ \ |x - z| < \epsilon\}$ where $\epsilon > 0$ and $x \in \mathbb{R}^{n+1}$. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. for any open neighbourhood U of a point in $\mathbb{R}$, we have that $\mathbb{Q} \cap U \neq \emptyset$. Hence, we may take the collection $B_r((q_1, ..., q_{n+1}))$, where r is a positive rational number and q_i's are rational numbers, as a basis for the standard topology. This basis is countable as $\mathbb{Q}$ is countable and a finite product of countable sets is countable again [Mun00]. Given any distinct points x, y in $\mathbb{R}^{n+1}$, we have that $\epsilon = \frac{|x-y|}{2}$ gives two disjoint open balls $B_\epsilon(x)$ and $B_\epsilon(y)$; $z \in B_\epsilon(x)$ gives $2\epsilon = |x - y| \leq |x - z| + |z - y| < \epsilon + |z - y|$ so that $|z - y| > \epsilon$, i.e. $z \notin B_\epsilon(y)$. That is to say that points are separated by open neighbourhoods. Altogether, S^n is Hausdorff and second countable in the subspace topology. The open subsets $U_i^- = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \ \ x_i < 0\}$ and $U_i^+ = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \ \ x_i > 0\}$ for $i = 1, ..., n+1$ gives open subsets $V_i^- = U_i^- \cap S^n = \{(x_1, ..., x_{n+1}) \in S^n | \ \ x_i < 0\}$ and $V_i^+ = U_i^+ \cap S^n = \{(x_1, ..., x_{n+1}) \in S^n | \ \ x_i > 0\}$ of S^n. This gives homeomorphisms $\phi_i^\pm : V_i^\pm \to \mathrm{Int}(D^n)$, $(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1})$ with inverses given by $(\phi_i^\pm)^{-1}(x_1, ..., x_n) = (x_1, ..., x_{i-1}, \pm\sqrt{1 - |x|^2}, x_i, ..., x_n)$ [Lee03]. Moreover, any point x in S^n has a non-zero component, say x_i, and hence x is in U_i^- or U_i^+ depending on whether $x_i > 0$ or $x_i < 0$; S^n is locally homeomorphic to $\mathbb{R}^n$.

Example 3.18. From the manifold S^n, the real projective space $\mathbb{R}P^n$ may be constructed as follows. Let $\sim$ be the equivalence relation generated by $x \sim -x$, then we define $\mathbb{R}P^n$ as $S^n/\sim$ with the quotient topology from the canonical map $\pi : S^n \to S^n/\sim$. Note that the equivalence classes may be interpreted as the collection of all lines through the origin in $\mathbb{R}^{n+1}$; thinking of S^n as a subset of $\mathbb{R}^{n+1}$, the lines through the origin are defined by antipodal points on S^n. As S^n is an n-manifold, it follows canonically that $\mathbb{R}P^n$ is an n-manifold. Indeed,

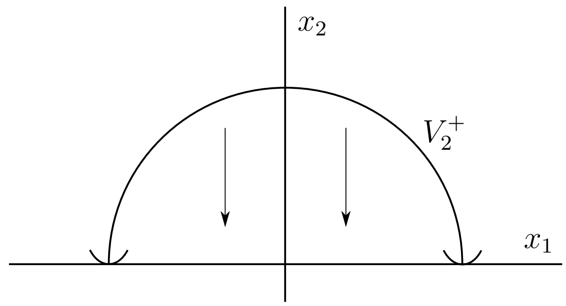


Figure 17: The upper hemisphere V_2^+ is homeomorphic to $(-1, 1)$.

the basis for the topology in $\mathbb{R}P^n$ is countable as it is induced by the countable basis of S^n. A space X is Hausdorff if and only if the diagonal, $\Delta(X) = \{(x, x) \in X \times X\}$, is closed [Vic12]. Hence, the real projective space is Hausdorff as the pre-image of the diagonal under π is given by the union

$$\{(x, x) \in S^n \times S^n\} \cup \{(x, -x) \in S^n \times S^n\}$$

which is a finite union of closed as S^n is Hausdorff. Finally, it is locally homeomorphic to $\mathbb{R}^n$ by composing the charts from S^n with π. Moreover, it is compact as it is the continuous image of the compact space S^n.

Example 3.19. A simplicial complex need not be a manifold. A simple example is given by considering three 0-simplexes, 0,1,2, and a convex 1-simplex, $[1, 2]$, in $\mathbb{R}^2$, i.e. the disjoint union of a point and a straight line. This cannot be a manifold for two reasons. First, the point can only be homeomorphic to $\mathbb{R}^0$, i.e. a point, while an interior point of $[1, 2](\Delta_1) = \{[0, 1](1, 0)t + [0, 1](0, 1)(1 - t)| \quad 0 \leq t \leq 1\}$ does not admit a local homeomorphism to a point. Given $x \in \text{Int}([1, 2](\Delta_1))$, take $\epsilon = \min\{|x - [0, 1](0, 1)|, |x - [0, 1](1, 0)|\}$ such that there is an open neighbourhood $U = B_\epsilon(x) \cap [0, 1](\Delta_1) = \{[0, 1](1, 0)t + [0, 1](0, 1)(1 - t)| \quad |x| - \epsilon < t < |x| + \epsilon\}$ that is homeomorphic to $(-\epsilon, \epsilon)$ by rotation and translation. As $(-\epsilon, \epsilon)$ is homeomorphic to $\mathbb{R}$ it cannot be homeomorphic to a point. The second reason is that a point on the geometrical boundary of $[0, 1](\delta_1)$, i.e. $\partial_0^1[0, 1](\Delta_1) \cup \partial_1^1[0, 1](\Delta_1) = \{[0, 1](0, 1), [0, 1](1, 0)\}$, gives open neighbourhoods homeomorphic to $[0, \infty)$ by a similar argument as above. Removing a point, x, from $\mathbb{R}$ gives two connected components, $(-\infty, x)$ and (x, ∞), but removing 0 from $[0, \infty)$ yields $(0, \infty)$, which is connected. As connectedness is preserved under continuity, $[0, \infty)$ cannot be homeomorphic to $\mathbb{R}$ [Mun00]. This illustrates two general problems. First, simplicial complexes need not have a consistent geometrical dimension (locally homeomorphic to $\mathbb{R}^n$ for n fixed). Secondly, simplicial complexes can have boundaries of lower dimension.[34]

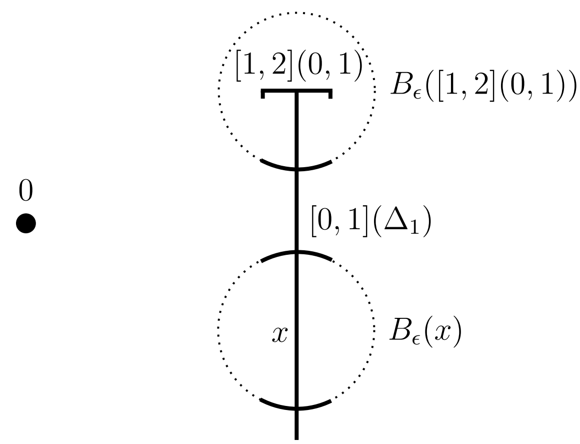


Figure 18: The disjoint union of a 0-simplex and a 1-simplex do not make up a manifold.

Definition 3.20. Let M be an n-dimensional manifold. An atlas is a collection of charts, $\mathcal{A} = \{h_\alpha : U_\alpha \to \mathbb{R}^n\}$, for which $\{U_\alpha\}$ is an open cover of M. The atlas is smooth if the transition maps, $h_{\alpha_2} \circ h_{\alpha_1}^{-1} : h_{\alpha_1}(U_{\alpha_1} \cap U_{\alpha_2}) \to h_{\alpha_2}(U_{\alpha_1} \cap U_{\alpha_2})$, are smooth (as functions between open subsets of $\mathbb{R}^n$).

If, for a given manifold, the set of all smooth atlases is non-empty, we easily deduce that there is a maximal smooth atlas; a chain $\mathcal{A}_1 \subset \mathcal{A}_2 \subset ...$ admits an upper bound, $\mathcal{U} = \cup \mathcal{A}_i$, which clearly is smooth again, and hence there is a maximal smooth atlas by Zorn's Lemma.

Definition 3.21. A smooth structure on a manifold, M, is a maximal smooth atlas $\mathcal{A}$. In this case, the pair $(M, \mathcal{A})$ is referred to as a smooth manifold.

Note that the existence of a single smooth atlas gives the existence of a maximal smooth atlas by the argument involving Zorn's Lemma above. Nonetheless, the argument does not guarantee uniqueness of maximal smooth atlases in any way, and consequently one cannot expect smooth structures to be unique. More traditional approaches however, reveals that a single smooth atlas is included in a unique maximal smooth atlas [Lee03]. Smooth structures enables the use of analysis on manifolds. Given two smooth manifolds, $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$, we say that a map $f : M_1 \to M_2$ is smooth if for any $x \in M_1$ and charts $h_1 : U_x \to \mathbb{R}^n$, $h_2 : U_{f(x)} \to \mathbb{R}^m$ the local map $h_2 \circ f|_{U_x \cap f^{-1}(U_{f(x)})} \circ h_1|^{-1}_{U_x \cap f^{-1}(U_{f(x)})}$ is smooth. Consider a smooth map $f : M \to N$, a point $x \in M$, two charts, h_1 and $\tilde{h}_1$ about x, and two charts,

[34]This property is allowed in manifolds with boundary.

h_2 and $\bar{h_2}$, about $f(x)$. Similarly, to the definition of smoothness, there are neighbourhoods where $\bar{h_2} \circ f \circ \bar{h_1}^{-1} = (\bar{h_2} \circ h_2^{-1}) \circ (h_2 \circ f \circ h_1^{-1}) \circ (h_1 \circ \bar{h_1}^{-1})$; smoothness of $h_2 \circ f \circ h_1^{-1}$ implies smoothness of $\bar{h_2} \circ f \circ \bar{h_1}^{-1}$ as transition charts are smooth [Lee03]. As a result, smoothness about a point is independent of the choice of charts and it suffices to consider sub-atlases $\mathcal{B}_1 \subset \mathcal{A}_1$ and $\mathcal{B}_2 \subset \mathcal{A}_2$ when verifying smoothness.[35]

The composition of smooth maps is easily seen to be smooth again by arguments similar to the above.

Definition 3.22. Let **Diff** denote the category consisting of smooth manifolds and smooth maps. An isomorphism in **Diff** is called a diffeomorphism.

Example 3.23. We looked at how the n-sphere, S^n, is a manifold in Example 3.17. As a consequence, we obtained the atlas $\mathcal{A} = \{\phi_i^{\pm} : V_i^{\pm} \to \text{Int}(D^n)\}$. The transition maps are given by $\phi_i^{\pm} \circ \phi_j^{\pm}(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, \pm\sqrt{1 - |x|^2}, x_j, ..., x_n)$, and known to be smooth for $|x| = 1$ by calculus. Hence, $\mathcal{A}$ is included in a maximal smooth atlas, and therefore gives a smooth structure on S^n. Similarly, $\mathbb{R}P^n$ is smooth [Lee03].

Based on the discussion preceding Definition 3.22, one cannot expect maximal smooth atlases to be unique, i.e. objects in **Diff** with homeomorphic (isomorphic in **Top**) underlying manifolds are not diffeomorphic (isomorphic in **Diff**) in general. In 1956, J. Milnor discovered smooth 7-manifolds homeomorphic to S^7, but not mutually diffeomorphic [Mil56]. Together with M. Kervaire, Milnor classified these exotic 7-spheres to 28 distinct objects in **Diff** [KM63]. It immediately follows that **Diff** cannot be a subcategory of **Top**; isomorphic objects in **Top** does not need to be isomorphic in **Diff**.

We will direct our discussion from smooth/analytic to combinatorial.

Definition 3.24. A triangulation of a manifold, M, is a simplicial complex equipped with the weak topology homeomorphic to M.

The name triangulation is best understood in the case of 2-dimensional manifolds, e.g. covering S^2 with a union of triangles that obey the rules of Definition 3.8. Higher dimensions take simplexes as generalizations of triangles. Triangulations of manifolds gives a combinatorial view of manifolds through abstract simplicial simplexes. In the finite case, this does not only gives a simple way of calculating homology given the triangulation, but also makes it possible to represent the manifold on a computer. In Example 3.19 we looked at how a simplicial complex does not need to be a manifold, and it is only natural to ask the reverse question. Does every manifold admit a triangulation? By the work of S. S. Cairns, supplemented by Whitehead, the answer is yes for smooth manifolds [Cai35, Whi40].

[35] A sub-atlas is a subset that also makes up an atlas.

Theorem 3.25. *If $(M, \mathcal{A})$ is a smooth manifold, then M admits a triangulation.*

Example 3.26. Let $(M, \mathcal{A})$ be a smooth manifold, then M admits a triangulation, T, by Theorem 3.25. Whitehead proves that a compact CW-complex is necessarily finite in [Whi49]; otherwise, one may pick an infinite collection of points in disjoint cells, giving an infinite open cover with no finite subcover in the weak topology. As such, if M is compact, any triangulation of M is finite.[36]

Remark 3.27. Simplicial complexes, and therefore triangulations, will be assumed finite if nothing else is stated.[37]

Example 3.28. Consider the tetrahedron, T, from Example 3.15. The tetrahedron is clearly homeomorphic to the sphere, S^2, by smoothing out the 2-simplexes. Consequently, we immediately have that

$$H_0(S^2) \simeq \mathbb{Z},$$

$$H_1(S^2) \simeq 0$$

and

$$H_2(S^2) \simeq \mathbb{Z}.$$

Example 3.29. Figure 19 determines a triangulation of $\mathbb{R}P^2$ [Bar82]. Similarly as in Figure 14, we order bases $1, 2, 3, 4, 5, 6$ for $\mathbb{Z}^6$, $[1,2], [1,3], [1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6]$ for $\mathbb{Z}^{15}$ and $[1,2,4], [1,2,6], [1,3,4], [1,3,5], [1,5,6], [2,3,5], [2,3,6], [2,4,5], [3,4,6], [4,5,6]$ for $\mathbb{Z}^{10}$. This gives a chain complex

$$0 \to \mathbb{Z}^{10} \xrightarrow{\partial_2} \mathbb{Z}^{15} \xrightarrow{\partial_1} \mathbb{Z}^6 \to 0.$$

The boundary maps are explicitly given by

$$\partial_1 = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

[36] An infinite simplicial complex cannot be compact in the weak topology, but this does not hold true for the strong topology.

[37] We are interested in compact manifolds.

and

$$\partial_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

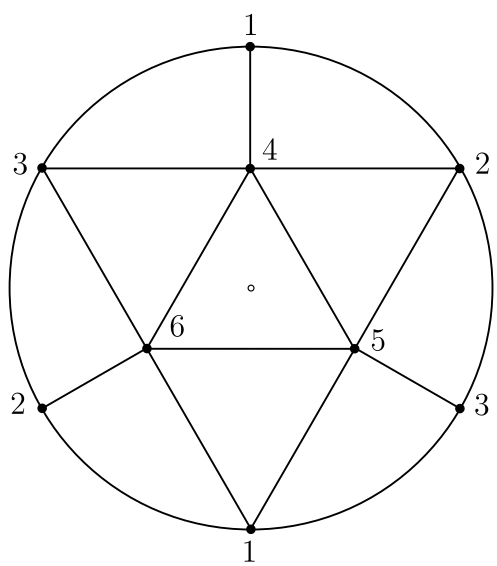


Figure 19: The sphere seen from above. Antipodal points are identified, and the illustration is reflected across the origin giving a triangulation of $\mathbb{R}P^2$ [Bar82].

With some effort, the Smith normal forms are calculated as

$$\partial_1 \sim \begin{bmatrix} \mathrm{diag}(1,1,1,1,1,0) & 0 \end{bmatrix}$$

and

$$\partial_2 \sim \begin{bmatrix} \text{diag}(1,1,1,1,1,1,1,1,1,2) \\ 0 \end{bmatrix}.$$

Changing bases according to the Smith normal forms immediately gives the kernel and images of the boundary operators. We have to take care as one basis element in the new basis for $\mathbb{Z}^{10}$, corresponding to the Smith normal form of ∂_2, is sent to 2 times a basis element in the new basis for $\mathbb{Z}^{15}$, which results in non-trivial torsion for H_1. Consequently,

$$H_0(\mathbb{R}P^2) = {}^{\text{Ker}\,0}\!\big/_{\text{Im}\,\partial_1} \simeq \mathbb{Z},$$

$$H_1(\mathbb{R}P^2) = {}^{\text{Ker}\,\partial_1}\!\big/_{\text{Im}\,\partial_2} \simeq \mathbb{Z}_2$$

and

$$H_2(\mathbb{R}P^2) = {}^{\text{Ker}\,\partial_2}\!\big/_{\text{Im}\,0} \simeq 0.$$

3.5 The Euler Characteristic

Recall that a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

splits if $B \simeq A \oplus C$. Equivalently the sequence splits if and only if there is a $g' : C \to B$ with $g \circ g' = \text{id}_C$, and if and only if there is an $f' : B \to A$ with $f' \circ f = \text{id}_A$. Indeed, the forward implication is trivial. If we assume the existence of $g' : C \to B$ such that $g \circ g' = \text{id}_C$, following the standard trick for the module version, we observe that $\text{id}_B = (\text{id}_B - g' \circ g) + g' \circ g$. Using that f is the kernel of g and $g \circ (\text{id}_B - g' \circ g) = 0$ we find an $f' : B \to A$ such that $\text{id}_B = f \circ f' + g' \circ g$. Finally, applying f from the right and using that f is mono, we find that $\text{id}_A = f \circ f'$ and the result follows as B agree with the biproduct of A and C. The last implication, the existence of $\bar{f}$ gives biproduct, follows dually.

Proposition 3.30. *A short exact sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

in Mod R *is split if* M'' *is free.*

Proof. Let $\{m_1'', ..., m_t''\}$ be a basis, choose m_i in the pre-image of m_i'' under g and define $g' : M'' \to M$ accordingly. □

Proposition 3.31. *Let*

$$\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}^{a_n} \xrightarrow{\partial_n} \mathbb{Z}^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots$$

be a chain complex in $\mathbf{C}(\mathbb{Z})$*. Then, both the kernel and the image of* ∂_n *are free on less or equal number of generators than* $\mathbb{Z}^{a_n}$ *and the sequence splits everywhere;* $\mathbb{Z}^{a_n} \simeq \operatorname{Ker}\partial_n \oplus \operatorname{Im}\partial_n$ *for every* n*. Moreover, the free part of* $H_n(\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}^{a_n} \xrightarrow{\partial_n} \mathbb{Z}^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots)$ *is generated on* $\# \operatorname{Ker}\partial_n - \# \operatorname{Im}\partial_{n+1}$ *generators, where* $\#$ *denotes the minimal number of generators.*

Proof. Consider the short exact sequences

$$0 \to \operatorname{Ker}\partial_n \to \mathbb{Z}^{a_n} \to \operatorname{Im}\partial_n \to 0.$$

Recall that a submodule of a free module, on n generators, over a PID is free on $s \leq n$ generators [BJN94], and so we have that the kernels, $\operatorname{Ker}\partial_n$, and the images, $\operatorname{Im}\partial_n$, are both free. The sequence splits everywhere by Proposition 3.30. Consider the exact sequence

$$0 \to \operatorname{Im}\partial_{n-1} \to \operatorname{Ker}\partial_n \to H_n(\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}^{a_n} \xrightarrow{\partial_n} \mathbb{Z}^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots) \to 0,$$

and take s and n such that $\operatorname{Im}\partial_{n-1}$ is free on s generators and $\operatorname{Ker}\partial_n$ is free on n generators, where $s \leq n$ as $\operatorname{Im}\partial_{n-1}$ is a submodule of $\operatorname{Ker}\partial_n$. Finally, $H_n(\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}^{a_n} \xrightarrow{\partial_n} \mathbb{Z}^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots)$ is the quotient, and therefore has a free part generated by $n - s$ elements [BJN94].[38] □

Definition 3.32. The *n′th* Betti number of a CW-complex (e.g. simplicial complex), X, is the minimal number of generators for the free part of $H_n(X)$, denoted $b_n(X)$. The Euler characteristic of X is (if it exists) the alternating sum

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

From here, an exercise in [Vic12] by Vick gives the following corollary.

Corollary 3.33. *Let* X *be a finite CW-complex (e.g. simplicial complex) with associated chain complex*

$$\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}^{a_n} \xrightarrow{\partial_n} \mathbb{Z}^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots .$$

Then the Euler characteristic of X *is given by*

$$\chi(X) = \sum_i (-1)^i a_i,$$

i.e. the alternating sum of the number of cells in each dimension.

[38] The s quotients may, of course, result in torsion.

Proof. We know that $b_i(X) = \# \operatorname{Ker} \partial_n - \# \operatorname{Im} \partial_{n+1}$ by Proposition 3.31. Now,

$$\begin{aligned}
\chi(X) &= \sum_i (-1)^i b_i(X) \\
&= \sum_i (-1)^i (\# \operatorname{Ker} \partial_i - \# \operatorname{Im} \partial_{i+1}) \\
&= (\# \operatorname{Ker} 0 - \# \operatorname{Im} \partial_1) - (\# \operatorname{Ker} \partial_1 - \# \operatorname{Im} \partial_2) + (\# \operatorname{Ker} \partial_2 - \# \operatorname{Im} \partial_3) - \cdots \\
&= \# \operatorname{Ker} 0 - (\# \operatorname{Ker} \partial_1 + \# \operatorname{Im} \partial_1) + (\# \operatorname{Ker} \partial_2 + \# \operatorname{Im} \partial_2) - \cdots \\
&= \sum_i (-1)^i a_i,
\end{aligned}$$

where the last equality follows by using Porposition 3.31 again. □

From this corollary, we recover the classical formulae of Euler for convex polyhedrons and more generally, the Euler characteristic of a graph that can be embedded on S^2.

Definition 3.34. A graph is planar if it can be embedded on the 2-manifold S^2.

The name is best seen in the light of the stereographic projection that gives a homeomorphsm between $\mathbb{R}^2$ and S^2 minus a point; embed $\mathbb{R}^2$ so that the intersection with S^2 is the south pole and draw straight lines from the north pole through the sphere [Vic12].

Observation 3.35. Let G be a planar graph, then an embedding defines a CW-complex homeomorphic to S^2.

Proof. Take 0- and 1-skeletons according to Definition 3.11. Cycles in the embedding, X^1, separates the sphere into a finite set of open components by applying the Jordan curve theorem [Vic12] a finite number of times. The components may intersect trees (if there is a vertex that is not included in any cycle), but the embedding of a finite tree is closed. Hence we may pick 2-cells as the (open) components of $S^2 - X^1$; the closure defines a gluing process to the 1-skeleton. □

Corollary 3.36. *Let G be a planar graph whose embedding on S^2 consist of v vertices (0-cells), e edges (1-cells) and f 2-cells, then*

$$v - e + f = 2.$$

Proof. Simply calculate the Euler characteristic of S^2, $\chi(S^2) = 1 - 0 + 1$, and apply Corollary 3.33.[39] □

[39] Note that we are using CW-complexes and not simplicial complexes - a planar graph need not be homeomorphic to the 1-skeleton of a simplicial complex, but it is always a CW-complex.

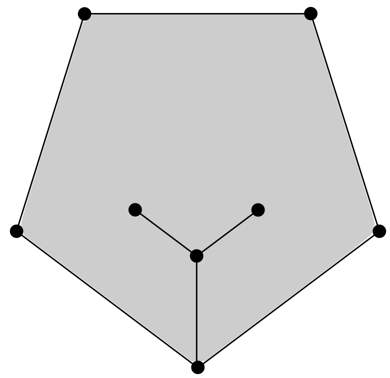


Figure 20: The components defined by a (finite) planar graph gives 2-cells.

From here it is an easy exercise to show that K_n, for $n \geq 5$, cannot be embedded into $\mathbb{R}^2$.

Example 3.37. The complete graph on five vertices, K_5, is not embeddable in $\mathbb{R}^2$. Indeed, assume it is, then

$$2 = v - e + f = 5 - (4 + 3 + 2 + 1) + f$$

such that $f = 7$. But there are at least

$$\frac{4 \cdot (4-1)}{2} + \frac{3 \cdot (3-1)}{2} + \frac{2 \cdot (2-1)}{2} = 10$$

faces, i.e. a contradiction.

Following an exercise by Fulton in [Ful13], the Euler characteristic gives combinatorial bounds for triangulations.

Example 3.38. Let $(M, \mathcal{A})$ be a smooth 2-manifold which is also compact, then Theorem 3.25 guarantee the existence of a triangulation of M, say T. Moreover, by Example 3.26, T is finite. Let v, e and f denote the number of vertices (0-simplexes), edges (1-simplexes) and triangles (2-simplexes), respectively. The second property in Definition 3.8 gives $2e = 3f$ as there are three edges per triangle, each shared by two triangles. In particular $2|3f$ so that $2|f$, i.e. there is an even number of triangles. Moreover, the number of edges in K_v equals $\frac{1}{2}v(v-1)$ so that we have the bound $e \leq \frac{1}{2}v(v-1)$. Combining these with the Euler characteristic, $v - e + f = 2$, the bound

$$v^2 - 7v + 6\chi(T) \geq 0$$

follows, which reduces to

$$v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(T)}),$$

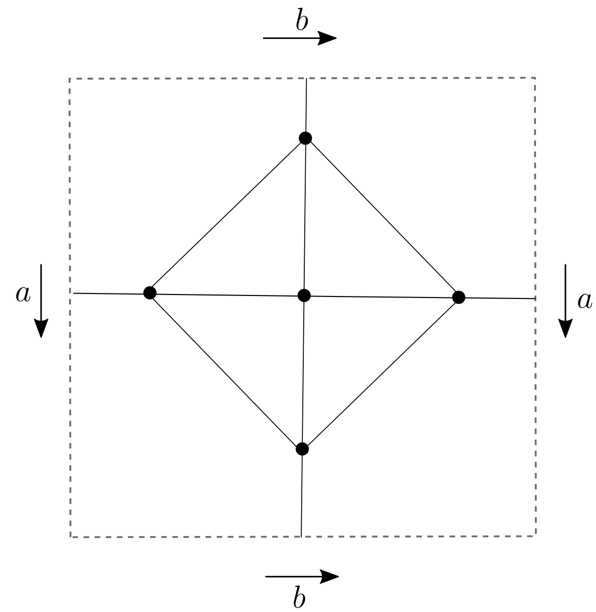


Figure 21: Although K_5 is not embeddable in $\mathbb{R}^2$, it is embeddable on the torus; constructed by identifying opposite edges of the square.

as v is necessarily positive. Together with $v - e + f = 2$ and $2e = 3f$ this bound completely determines the minimal number of generators in each dimension for a triangulation. Some easy calculations show that taking $M = S^2$ yields $v = 4$, $e = 6$ and $f = 4$, while $M = \mathbb{R}P^2$ gives $v = 6$, $e = 15$ and $f = 10$. We used minimal triangulations to calculate the homology of S^2 and $\mathbb{R}P^2$ in Example 3.28 and 3.29, respectively.

Example 3.39. Let G be a planar graph only consisting of cycles, i.e. every vertex of G lies on some cycle, let $X^0 \subset X^1 \subset X^2 = X$ denote a CW-complex as described in Observation 3.35 and set $v = |X^0|$, $e = |X^1|$ and $f = |X^2|$. The cycle assumption ensures that X^1 partition S^2 into n-gons for $n \geq 3$, hence the equality $2e = 3f$ from Example 3.38 takes the weaker form $2e \geq 3f$ as the closure of a 2-cell consists of at least three edges. Combined with the Euler characteristic, $v - e + f = 2$, we deduce $e \leq 3(v-2)$ so that the average degree of G is bounded by

$$d(G) = 2\frac{e}{v} \leq 6\frac{v-2}{v} < 6.$$

4 Cohomology and Changing Coefficients

Let $\mathscr{A}$ and $\mathscr{B}$ denote Abelian categories throughout this section.

4.1 The Adjunction of Hom and Tensor Functors

Given a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathscr{A}$, recall that a functor $F : \mathscr{A} \to \mathscr{B}$ is:

- Left exact if $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact in $\mathscr{B}$, i.e. preserve kernels.[40]
- Right exact if $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact in $\mathscr{B}$, i.e. preserve cokernels.
- Exact if $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact in $\mathscr{B}$.

Equivalently, a functor is exact if and only if it preserves any exact sequence,[41] if and only if it is left exact and preserve epimorphisms, and if and only if it is right exact and preserve monomorphisms. It is well known that the Hom functors are left exact:

Proposition 4.1. *Given any object X in $\mathscr{A}$, then both $\operatorname{Hom}_{\mathscr{A}}(X, -)$ and $\operatorname{Hom}_{\mathscr{A}}(-, X)$ are left exact as functors $\mathscr{A} \to \mathbf{Ab}$ and $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, respectively.*

Proof. We show the statement for $\operatorname{Hom}_{\mathscr{A}}(-, X)$ and the case of $\operatorname{Hom}_{\mathscr{A}}(X, -)$ follows dually. Given an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathscr{A}$ and any object X. Apply $\operatorname{Hom}_{\mathscr{A}}(X, -)$ to obtain a sequence

$$0 \to \operatorname{Hom}_{\mathscr{A}}(X, A) \xrightarrow{\operatorname{Hom}_{\mathscr{A}}(X,f)} \operatorname{Hom}_{\mathscr{A}}(X, B) \xrightarrow{\operatorname{Hom}_{\mathscr{A}}(X,g)} \operatorname{Hom}_{\mathscr{A}}(X, C)$$

in $\mathbf{Ab}$. If $\operatorname{Hom}_{\mathscr{A}}(X, f)(h_1) = \operatorname{Hom}_{\mathscr{A}}(X, f)(h_2)$, i.e. $f \circ h_1 = f \circ h_2$, it follows that $h_1 = h_2$ as f is mono, so $\operatorname{Hom}_{\mathscr{A}}(X, f)$ is injective. If h is in the kernel of $\operatorname{Hom}_{\mathscr{A}}(X, g)$, then $\operatorname{Hom}_{\mathscr{A}}(X, g)(h) = 0$, i.e. $g \circ h = 0$. As f is the kernel of g, it follows that h factor through f; there is an $\bar{h}$ such that $h = f \circ \bar{h}$. Contrary, if h is in the image of $\operatorname{Hom}_{\mathscr{A}}(X, f)$, say $h = f \circ \bar{h}$, we have that $g \circ h = g \circ f \circ \bar{h} = 0 \circ \bar{h} = 0$ such that h is in the kernel of $\operatorname{Hom}_{\mathscr{A}}(X, g)$. $\square$

Before proceeding, we recall some elementary properties of the tensor product and the Hom-sets [Opp16, Rot08].

[40] It is easily verified that $A \to B$ is the kernel of $B \to C$; as $\operatorname{Ker} f = 0$ we have that $A \simeq \operatorname{Im}(A \to B) \simeq \operatorname{Ker}(B \to C)$.

[41] The sequence $\cdots \xrightarrow{\partial_{i-1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i+1}} \cdots$ is exact if and only if every induced sequence $0 \to \operatorname{Im} \partial_{i-1} \to M_i \to \operatorname{Im} \partial_i \to 0$ is short exact. That is, a long exact sequence splits into short exact sequences.

Proposition 4.2. *Given a ring R, a left R-module M and a right R-module N, we have the following isomorphisms in* **Ab**.

- $R \otimes_R M \simeq M$, $r \otimes_R m \mapsto rm$.
- $N \otimes_R M \simeq M \otimes_{R^{\mathrm{op}}} N$, $m \otimes_R n \mapsto n \otimes_{R^{\mathrm{op}}} m$.

If, in addition, R is commutative and $(N_i)_{i \in I}$ is a family of R-modules, we have the following isomorphisms in $\operatorname{Mod} R$.

- $R \otimes_R M \simeq M$, $r \otimes_R m \mapsto rm$.
- $N \otimes_R (\oplus_{i \in I} N_i) \simeq \oplus_{i \in I}(N \otimes_R N_i)$, $n \otimes_R (n_{i_1} + \cdots + n_{i_t}) \mapsto (n \otimes_R n_{i_1} + \cdots + n \otimes_R n_{i_t})$.
- $N \otimes_R M \simeq M \otimes_R N$, $m \otimes_R n \mapsto n \otimes_R m$.

Proposition 4.3. *Let $A_1, ..., A_n, B_1, ..., B_m$ be objects in $\mathscr{A}$. Then, there is an isomorphism*

$$\operatorname{Hom}_{\mathscr{A}}(\oplus_{k=1}^n A_k, \oplus_{j=1}^m B_j) \simeq \begin{bmatrix} \operatorname{Hom}_{\mathscr{A}}(A_1, B_1) & \operatorname{Hom}_{\mathscr{A}}(A_2, B_1) & \dots & \operatorname{Hom}_{\mathscr{A}}(A_n, B_1) \\ \operatorname{Hom}_{\mathscr{A}}(A_1, B_2) & \operatorname{Hom}_{\mathscr{A}}(A_2, B_2) & \dots & \operatorname{Hom}_{\mathscr{A}}(A_n, B_2) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_{\mathscr{A}}(A_1, B_m) & \operatorname{Hom}_{\mathscr{A}}(A_2, B_m) & \dots & \operatorname{Hom}_{\mathscr{A}}(A_n, B_m) \end{bmatrix},$$

given by

$$f \mapsto (\pi_{B_j} \circ f \circ i_{A_k}),$$

where i and π denotes the biproduct inclusions and projections.

A left R-module is an Abelian group M together with a ring homomorphism from R into $\operatorname{End}(M)$, the endomorphism ring of M. If R^{op} denotes the opposite ring, i.e. multiplication given by $a \cdot_{\mathrm{op}} b = ba$,[42] a right R-module is obtained by a ring homomorphism $R^{\mathrm{op}} \to \operatorname{End}(M)$. That is, a right module over R is a left module over R^{op}, and vice versa. Given rings R and S as well as an R-S-bimodule M, $\operatorname{Hom}_{S^{\mathrm{op}}}(M, -)$ becomes a functor $\operatorname{Mod}(S^{\mathrm{op}}) \to \operatorname{Mod}(R^{\mathrm{op}})$ by defining $f \cdot r = f(- \cdot_{\mathrm{op}} r)$. Similarly, $- \otimes_R M$ becomes a functor $\operatorname{Mod}(R^{\mathrm{op}}) \to \operatorname{Mod}(S^{\mathrm{op}})$ by defining multiplication in $L \otimes_R M$ as $(l \otimes_R m) \cdot s = l \otimes_R (ms)$.

Proposition 4.4. *Given rings S and R, as well as an R-S-bimodule M, the functor $- \otimes_R M$ is left adjoint to* $\operatorname{Hom}_{S^{\mathrm{op}}}(M, -)$.

[42] A ring is a category consisting of a single object with elements as morphisms, multiplication as composition and unity as identity. The opposite ring is simply the opposite category in this setting.

Whenever R is commutative, left and right modules agree so that tensoring two R-modules automatically defines a functor $\operatorname{Mod} R \to \operatorname{Mod} R$ rather than $\operatorname{Mod} R \to \mathbf{Ab}$.[43] Similarly, $\operatorname{Hom}_R(M, -)$ becomes a functor $\operatorname{Mod} R \to \operatorname{Mod} R$.

Remark 4.5. Every ring is assumed to be commutative.

Recall that left adjoints preserve colimits, and right adjoints preserve limits [Opp16].

Proposition 4.6. *Given a ring R and an R-module M, the functor $- \otimes_R M : \operatorname{Mod} R \to \operatorname{Mod} R$ is right exact.*

Proof. Apply Proposition 4.4 to $- \otimes_R M$. □

The motivation of the next example is due to standard calculations of Ext and Tor (to be discussed later), and shows that neither Hom nor tensor functors are exact in general.

Example 4.7. Take n and m in $\mathbb{N}_+$ such that $n, m \geq 2$ and $d = \gcd(n, m) > 0$. Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0,$$

where $\cdot n$ is defined by multiplication with n. Apply $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, -)$ and use that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \cdot n) = \cdot n$, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \cdot n)(f) = \cdot n \circ f = nf$, to obtain

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) \xrightarrow{\cdot n} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n),$$

which is exact by Proposition 4.1. We want to check if $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$ is surjective. Given f in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, observe how

$$0 = f(\bar{0}) = f(\bar{m}) = mf(1),$$

such that $f(1) = 0$ and hence $f = 0$, i.e. $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) = 0$.[44] But $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$, on the other hand, is not zero. Indeed, an f in $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$ is uniquely determined by $f(1)$ and is well-defined if and only if $mf(1) = f(\bar{m}) = 0 \bmod n$. Using that d is the greatest common divisor this reduces to the requirement of $df(1) = 0 \bmod n$, which is solved by

$$f(1) = 0, \frac{n}{d}, 2\frac{n}{d}, ..., (d-1)\frac{n}{d} \mod n.$$

This constitutes an isomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \simeq \mathbb{Z}_d,$$

[43] The tensor product, $M \otimes_R N$, of R-modules becomes an R-module by defining $r \cdot (m \otimes n) = (rm) \otimes_R n$ whenever R is commutative.

[44] A bar above an element is its coset in the adequate quotient.

but there are no surjections $0 \to \mathbb{Z}_d$. Consequently, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, -)$ is not exact. Going back to the start and applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_m)$ to

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0,$$

and using that $\mathrm{Hom}_{\mathbb{Z}}(\cdot n, \mathbb{Z}_m)(f) = f \circ \cdot n = nf$ as f is a $\mathbb{Z}$-homomorphism, yields

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_m) \xrightarrow{\cdot n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_m)$$

which is exact by Proposition 4.1.[45] Reusing arguments from above, we have isomorphisms $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_m) \simeq \mathbb{Z}_m$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \simeq \mathbb{Z}_d$. But $\cdot n : \mathbb{Z}_m \to \mathbb{Z}_m$ is no surjection; we simply observe how $0 < x < d$ in $\mathbb{Z}_m$ cannot be in the image of $\cdot n$ as d is the greatest common divisor. The non-exactness of $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_m)$ follows. Analogous arguments show that neither $\mathbb{Z}_m \otimes_{\mathbb{Z}} -$ nor $- \otimes_{\mathbb{Z}} \mathbb{Z}_m$ are exact by first proving

$$\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \simeq \mathbb{Z}_d.$$

A functor between Abelian categories is additive if it preserves the biproduct and consequently an additive functor preserve split sequences.

Proposition 4.8. *Let $F : \mathscr{A} \to \mathscr{B}$ be additive and assume that*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is split exact. Then

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$$

is split exact again.

Corollary 4.9. *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is split exact, then given any object X

$$0 \to \mathrm{Hom}_{\mathscr{A}}(C, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(g,X)} \mathrm{Hom}_{\mathscr{A}}(B, X) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(f,X)} \mathrm{Hom}_{\mathscr{A}}(A, X) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathscr{A}}(X, A) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,f)} \mathrm{Hom}_{\mathscr{A}}(X, B) \xrightarrow{\mathrm{Hom}_{\mathscr{A}}(X,g)} \mathrm{Hom}_{\mathscr{A}}(X, C) \to 0$$

[45] A contravariant functor is by definition a functor from the opposite category, so that $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_m)$ preserve cokernels, i.e. kernels in the opposite category.

are both exact. If $\mathscr{A} = \mathrm{Mod}\, R$*, then given any* R*-module* M*,*

$$0 \to A \otimes_R M \xrightarrow{f \otimes_R id_M} B \otimes_R M \xrightarrow{f \otimes_R id_M} C \otimes_R M \to 0$$

and

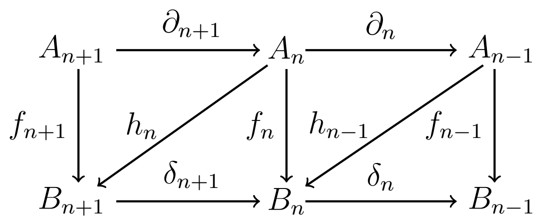
$$0 \to M \otimes_R A \xrightarrow{id_M \otimes_R f} M \otimes_R B \xrightarrow{id_M \otimes_R g} M \otimes_R C \to 0$$

are both exact.

Proof. By Proposition 4.2 and 4.3, Hom and tensor functors are additive. Apply Proposition 4.8. □

4.2 Projective and Injective Resolutions

Definition 4.10. Let $A = \{A_n, \partial_n\}$ and $B = \{B_n, \delta_n\}$ be chain complexes, and $f : A \to B$ a morphism in $\mathbf{C}(\mathscr{A})$. Then, f is null-homotopic if there are morphisms $h_n \in \mathrm{Hom}_{\mathscr{A}}(A_n, B_{n+1})$ such that $f_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ for every n.



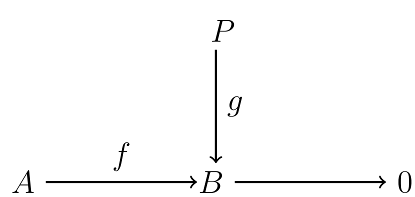
Two morphisms, f and g, in $\mathbf{C}(\mathscr{A})$ are homotopic if $f - g$ is null-homotopic.

One may show that identifying homotopic maps in $\mathbf{C}(\mathscr{A})$ defines a category $\mathbf{K}(\mathscr{A})$, the homotopy category, and that homology is preserved under null-homotopic maps. Therefore, H_n becomes a functor $\mathbf{K}(\mathscr{A}) \to \mathscr{A}$ [Opp16, Rot08].[46]

Definition 4.11. An object, P, is projective if $\mathrm{Hom}_{\mathscr{A}}(P, -)$ is exact. Dually, an object, I, is injective if $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is exact.

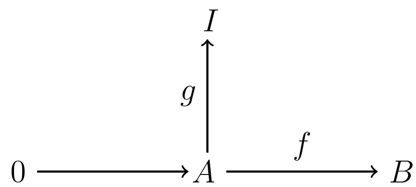
Writing out what this means, the following Proposition follows [Rot08].

Proposition 4.12. *An object* P *is projective if and only if given a diagram*



[46]The name is due to the fact that homotopy of chain complexes relates to homotopy of topological spaces.

with exact row, there is a $g' : P \to A$ satisfying $g = f \circ g'$. Dually, an object I is injective if and only if given a diagram



with exact row, there is a $g' : B \to I$ satisfying $g = g' \circ f$.

An immediate corollary is that injective resolutions in $\mathscr{A}$ are equivalent to projective resolutions in $\mathscr{A}^{\mathrm{op}}$ and vice versa.

Definition 4.13. $\mathscr{A}$ has enough projectives if for any object A there is a projective P and an epimorphism $P \to A$. Dually, $\mathscr{A}$ has enough injectives if for any object A there is an injective I and a monomorphism $A \to I$.

Example 4.14. The Abelian category $\operatorname{Mod} R$ has both enough projectives and enough injectives for any ring R [Rot08]. A module is projective if and only if it is the direct summand of a free module, a statement that is quite straightforward to prove. Indeed, if P is projective, then we have a canonical map $R^{(P)} \xrightarrow{p} P$. Apply Proposition 4.12 to $\mathrm{id}_P : P \to P$; the sequence splits, that is $R^{(P)} \simeq P \oplus \operatorname{Ker}(p)$. Contrary, given any indexing set I, consider the adjunction preceding Definition 3.3; $\operatorname{Hom}_{\mathbf{Set}}(-, U_R(-)) \simeq \operatorname{Hom}_{\operatorname{Mod} R}(R^{(-)}, -)$. It suffices to show that $\operatorname{Hom}_{\mathbf{Set}}(I, -)$ sends surjections to surjections. Given a surjection $f : A \to B$ in **Set** and a function $g_2 : I \to B$, define $g_1 : I \to A$ by $g_2(i) = a$ if and only if $a \in f^{-1}(g_1(i))$. Then, $\operatorname{Hom}_{\mathbf{Set}}(I, f)(g_1) = g_2$ as f is surjective. The argument of injectives requires more work [Rot08].

Definition 4.15. Given an object A, a projective resolution of A is (if it exists) a chain complex,

$$\cdots P_1 \to P_0 \to 0,$$

of projectives that is exact everywhere, except in position 0, where the cokernel is given by A. Dually, an injective resolution of A is (if it exists) an exact sequence,

$$0 \to I_0 \to I_{-1} \to \cdots,$$

of injectives that is exact everywhere, except in position 0, where the kernel is given by A.

It is easily observed that the assumption of enough projectives and injectives gives the existence of resolutions. Indeed, take a projective P_0 such that

$$0 \to \mathrm{Ker}(P_0 \to A) \to P_0 \to A \to 0$$

is exact. Now, take a projective P_1 such that

$$0 \to \mathrm{Ker}(P_1 \to \mathrm{Ker}(P_0 \to A)) \to P_1 \to \mathrm{Ker}(P_0 \to A) \to 0$$

is exact and define $P_1 \to P_0 = (\mathrm{Ker}(P_0 \to A) \to P_0) \circ (P_1 \to \mathrm{Ker}(P_0 \to A))$. Proceed inductively to construct a sequence of projectives, (P_n), such that

$$\cdots \to P_1 \to P_0 \to A \to 0$$

is exact. The argument for injectives is dual. Hence, we make the following assumption.

Remark 4.16. Any Abelian category is assumed to have both enough projectives and enough injectives.

In light of Proposition 4.12, it is obvious that given projective resolutions

$$P = \cdots P_1 \to P_0 \to 0$$

and

$$P' = \cdots P'_1 \to P'_0 \to 0$$

of objects A and B, respectively, as well as a morphism $f : A \to B$, there is an induced chain map $\mathbf{p}(f) : P \to P'$ determined by $f_n : P_n \to P'_n$ induced successively starting with $n = 0$. The chain map $\mathbf{p}(f)$ also extends to a chain map (f_n, f) between $P \to A \to 0$ and $P' \to B \to 0$ by construction, and is in fact unique in $\mathbf{K}(\mathscr{A})$ with respect to this property; a morphism in $\mathscr{A}$ determines a unique morphism in $\mathbf{K}(\mathscr{A})$ [Rot08].

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

In the case of $B = A$ and $f = \mathrm{id}_A$, the induced map $\mathbf{p}(\mathrm{id}_A)$ can be shown to be an isomorphism in $\mathbf{K}(\mathscr{A})$ [Rot08].[47] Thus, taking projective resolutions defines a functor $\mathbf{p} : \mathscr{A} \to \mathbf{K}(\mathscr{A})$. Dually, one may define a functor $\mathbf{i} : \mathscr{A} \to \mathbf{K}(\mathscr{A})$ by taking injective resolutions.

[47] So any resolutions of a given object agree up to homotopy.

4.3 Tor and Ext

The functor $H_n : \mathbf{K}(\mathscr{A}) \to \mathscr{A}$ measure exactness in position n of a given chain complex, an interpretation that will be extended to measure exactness of functors. Note that any functor $F : \mathscr{A} \to \mathscr{B}$ extend to a functor $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ by using functoriality,

$$F(\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots) = (\cdots \xrightarrow{F(\partial_{n+1})} F(A_n) \xrightarrow{F(\partial_n)} F(A_{n-1}) \xrightarrow{F(\partial_{n-1})} \cdots),$$

deliberately denoted F again.

Definition 4.17. Given a right exact functor $F : \mathscr{A} \to \mathscr{B}$, we define the n'th left derived functor of F as

$$L_n(F) = H_n \circ F \circ \mathbf{p} : \mathscr{A} \to \mathscr{B}$$

for any $n \in \mathbb{N}_0$. Dually, given a left exact functor $F : \mathscr{A} \to \mathscr{B}$ between Abelian categories, we define the n'th right derived functor of F as

$$R^n(F) = H_{-n} \circ F \circ \mathbf{i} : \mathscr{A} \to \mathscr{B}$$

for any $n \in \mathbb{N}_0$.

Using that a right (left) exact functor $F : \mathscr{A} \to \mathscr{B}$ preserves cokernels (kernels), we immediately deduce that $L_0(F)(A) = F(A)$ ($R^0(F)(A) = F(A)$). Further, if $F : \mathscr{A} \to \mathscr{B}$ is exact, then $L_n(F)$ and $R^n(F)$ vanish for $n > 0$. An observation that follows as exact functors preserve exact sequences.[48] Given a short exact sequence,

$$0 \to A \to B \to C \to 0,$$

in $\mathscr{A}$, there are induced long exact sequences

$$\cdots L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

when F is right exact and

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots$$

when F is left exact [Opp16].

Let A be an object in $\mathscr{A}$. Consider a right exact functor $F : \mathscr{A} \to \operatorname{Mod} R$ and a projective

[48] A projective (injective) resolution is exact for $n > 0$, according to Definition 4.15.

resolution

$$\cdots P_1 \to P_0 \to 0,$$

of A. Then, we know that the higher ($n > 0$) left derived functors are given by

$$L_n F(A) = \frac{\operatorname{Ker}(F(P_n) \to F(P_{n-1}))}{\operatorname{Im}(F(P_{n+1}) \to F(P_n))}.$$

Dually, for a left exact functor $F : \mathscr{A} \to \operatorname{Mod} R$ and an injective resolution

$$0 \to I_0 \to I_{-1},$$

of A, we get the higher ($n > 0$) right derived functors

$$R^n F(A) = \frac{\operatorname{Ker}(F(I_{-n}) \to F(I_{-n-1}))}{\operatorname{Im}(F(I_{-n+1}) \to F(I_{-n}))}.$$

Hom and tensor functors are left exact and right exact, respectively, by Proposition 4.1 and 4.6.

Definition 4.18. We define

$$\operatorname{Ext}^n_{\mathscr{A}}(A, -) = R^n \operatorname{Hom}_{\mathscr{A}}(A, -)$$

and

$$\operatorname{Ext}^n_{\mathscr{A}}(-, B) = R^n \operatorname{Hom}_{\mathscr{A}}(-, B).$$

When $\mathscr{A}$ is given by $\operatorname{Mod} R$ for some ring R, we also define

$$\operatorname{Tor}^R_n(M, -) = L_n(M \otimes_R -)$$

and

$$\operatorname{Tor}^R_n(-, N) = L_n(- \otimes_R N).$$

It can be shown that Tor and Ext are bifunctors in the sense that $\operatorname{Ext}^n_{\mathscr{A}}(A, -)(B) = \operatorname{Ext}^n_{\mathscr{A}}(-, B)(A)$ and $\operatorname{Tor}^R_n(M, -)(N) = \operatorname{Tor}^R_n(-, N)(M)$, which we will denote $\operatorname{Ext}^n_{\mathscr{A}}(A, B)$ and $\operatorname{Tor}^R_n(M, N)$, respectively [Opp16, Rot08].

As the names suggests, Tor and Ext are connected to torsion and extensions, respectively. The details will be omitted, but the following example illustrates the relationship of Tor with torsion.

Example 4.19. Let R be a PID and M a finitely generated R-module. If $M = (m_1, ..., m_n)$, then there is a canonical map $R^n \simeq R^{(\{m_1,...,m_n\})} \xrightarrow{p}, M$ depending on $m_1, ..., m_n$, and hence a short exact sequence

$$0 \to \mathrm{Ker}(p) \to R^n \to M \to 0.$$

A submodule of a free module, on n generators, over a PID is free on $s \le n$ generators [BJN94], such that $\mathrm{Ker}(p) \simeq R^s$ for some $s \le n$ resulting in an exact sequence

$$0 \to R^s \to R^n \to M \to 0.$$

Notice how the sequence is a projective resolution of M by Example 4.14. Further, Proposition 4.2 shows that the sequence is in fact $(- \otimes R) \circ \mathbf{p}$ applied to M, i.e. $\mathrm{Tor}_n^R(M, R)$ vanishes for $n > 0$ by going to homology. More generally, by applying $- \otimes_R N$, we see that $\mathrm{Tor}_n^R(M, N)$ vanishes for $n > 1$. Let $\mathrm{Quot}(R)$ denote the field of fractions over R and consider the short exact sequence

$$0 \to R \to \mathrm{Quot}(R) \to K \to 0,$$

where K is the cokernel. The induced long exact sequence when applying $M \otimes_R -$ reduces to

$$0 \to \mathrm{Tor}_1^R(M, K) \to M \to \mathrm{Quot}(R) \otimes_R M \to K \otimes_R M \to 0$$

by Proposition 4.2 and vanishing higher Tor for R and $\mathrm{Quot}(R)$; $- \otimes_R \mathrm{Quot}(R)$ is exact as it corresponds to localizing in 0 [AM94]. Consequently, $\mathrm{Tor}_1^R(M, K)$ is the torsion part of M (the elements for which there is an $r \neq 0$ and $rm = 0$). By the structure theorem of finitely generated modules over PIDs, M can be decomposed as $R^s \oplus \frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_r)}$ where $a_i | a_{i+1}$, and therefore the torsion part of M is a finite product of quotients; $\frac{R}{(a_1)} \oplus \cdots \oplus \frac{R}{(a_r)}$ [BJN94].[49]

Example 4.20. Take n, m and d as in Example 4.7. A projective resolution (injective in the opposite category) of $\mathbb{Z}_n$ is given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to 0.$$

Apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_m)$ and reuse isomorphisms from Example 4.7 to obtain the chain complex

$$0 \to \mathbb{Z}_m \xrightarrow{\cdot n} \mathbb{Z}_m \to 0$$

with kernel $\mathbb{Z}_d$. The cokernel is given by ${\mathbb{Z}_m}/{n \cdot \mathbb{Z}_m} \simeq \mathbb{Z}_d$, i.e.

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) = \begin{cases} \mathbb{Z}_d & \text{if } i = 0, 1 \\ 0 & \text{if } i > 1 \end{cases}.$$

[49]The decomposition follows by applying the Smith normal form to $R^s \to R^n$ in $0 \to R^s \to R^n \to M \to 0$.

Analogous arguments give the same answer for $\operatorname{Tor}_i^R(\mathbb{Z}_n, \mathbb{Z}_m)$. For the case $n = 1$ $(d = m)$ consider the projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to 0$$

of $\mathbb{Z}_m$. Apply $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$, and notice how $\operatorname{Hom}_{\mathbb{Z}}(\cdot m, \mathbb{Z})(f) = f \circ (\cdot m) = mf$ as f is a $\mathbb{Z}$-homomorphism, to obtain

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to 0$$

whose kernel is zero and 1'st homology is $\mathbb{Z}_m$, i.e.

$$\operatorname{Ext}_{\mathbb{Z}}^i(\mathbb{Z}_m, \mathbb{Z}) = \begin{cases} \mathbb{Z}_m & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}.$$

Finally, for the case $m = 1$ $(d = n)$, simply use the projective resolution

$$0 \to \mathbb{Z} \to 0$$

for $\mathbb{Z}$ to deduce

$$\operatorname{Ext}_{\mathbb{Z}}^i(\mathbb{Z}, \mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}.$$

4.4 Extension of Scalars

Let R denote a (commutative) ring throughout this section. Given an R-algebra B and an R-module M, we may construct the B-module $B \otimes_R M$, where multiplication is given by $b \cdot (1 \otimes m) = b \otimes m$; an extension of scalars. Thus, $B \otimes_R -$ becomes a functor $\operatorname{Mod} R \to \operatorname{Mod} B$. This extends to chain complexes, if

$$\cdots \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

is a chain complex in $\operatorname{Mod} R$, apply $B \otimes_R -$ to obtain a chain complex

$$\cdots \xrightarrow{\mathrm{id}_B \otimes_R \partial_{n+1}} B \otimes_R M_n \xrightarrow{\mathrm{id}_B \otimes_R \partial_n} B \otimes_R M_{n-1} \xrightarrow{\mathrm{id}_B \otimes_R \partial_{n-1}} \cdots$$

in $\operatorname{Mod} B$. Notice that the Abelian group structure of a ring makes every ring into a $\mathbb{Z}$-algebra by defining multiplication with $n \in \mathbb{Z}$ as the repeated sum n times.

Definition 4.21. Let X be any topological space, then we define the singular homology of X with coefficients in R as the homology of the corresponding chain complex from the simplicial object $R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(-)} \circ \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X))$.

We will use the notation $H_n(X; R)$ for the homology associated with $R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(-)} \circ \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X))$. Moreover, this is compatible with our earlier construction through simplicial objects:

Proposition 4.22. *The chain complexes associated with the simplicial objects* $R^{(-)} \circ \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X)$ *and* $R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(-)} \circ \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X))$ *are isomorphic.*

Proof. Let $\tilde{\partial}_n = \sum_{i=0}^{n} (-1)^i \tilde{\partial}_i^n$ denote the boundary operator associated with $R^{(\cdot)} \circ \mathrm{Hom}_{\mathbf{Top}}(\Delta(\cdot), X)$ and notice how $\tilde{\partial}_i^n(f) = \partial_i^n(f)$, i.e. $\tilde{\partial}_n(f) = \partial_n(f)$, when applied to a basis element $f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)$.[50] Proposition 4.2 gives isomorphisms

$$\phi_n : R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X))}) \to R^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X))}$$

for each n. Therefore, it suffices to show that the following diagram commute.

$$\begin{array}{ccc}
R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X))}) & \xrightarrow{\mathrm{id}_R \otimes_{\mathbb{Z}} \partial_n} & R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_{n-1}, X))}) \\
\downarrow \phi_n & & \downarrow \phi_{n-1} \\
R^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X))} & \xrightarrow{\tilde{\partial}_n} & R^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_{n-1}, X))}
\end{array}$$

As $R \otimes_{\mathbb{Z}} (\mathbb{Z}^{(\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X))})$ is generated by elements of the form $r \otimes_{\mathbb{Z}} f$ where f is of the form $\sum_{i=1}^{t} z_i f_i$ with $z_i \in \mathbb{Z}$ and $f_i \in \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)$, the bilinearity of $\otimes_{\mathbb{Z}}$ shows that it suffices to check elements of the form $r \otimes_{\mathbb{Z}} f$ with $r \in R$ and $f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)$. A direct calculation reveals that

$$\phi_{n-1} \circ (\mathrm{id}_R \otimes_{\mathbb{Z}} \partial_n)(r \otimes_{\mathbb{Z}} f) = \phi_{n-1}(r \otimes_{\mathbb{Z}} \partial_n(f)) = r\partial_n(f) = r\tilde{\partial}_n(f) = \tilde{\partial}_n(rf) = \tilde{\partial}_n \circ \phi_n(r \otimes_{\mathbb{Z}} f).$$

□

Example 4.23. The homology of $\mathbb{R}P^2$ was calculated as $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $H_2(\mathbb{R}P^2) = 0$ in Example 3.29. Doing the same calculations as for the Smith normal form over $\mathbb{Z}$, we may replace 2 with 1 by multiplying with $\frac{1}{2}$, so that $H_0(\mathbb{R}P^2; \mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{R}P^2; \mathbb{Q}) = 0$ and $H_2(\mathbb{R}P^2; \mathbb{Q}) = 0$.

4.5 Cohomology

In category theory "co" is to be understood in a dual context. An example is given by monomorphisms and epimorphisms. A morphism f is mono if given any composable g_1 and

[50] The functors $\mathbb{Z}^{(-)}$ and $R^{(-)}$ simply extend $\mathrm{Hom}_{\mathbf{Top}}(\Delta(\partial_i^n), X)$ to homomorphisms.

g_2 such that $f \circ g_1 = f \circ g_2$, we have that $g_1 = g_2$. Dually, h is an epimorphism if given any composable g_1 and g_2 such that $g_1 \circ h = g_2 \circ h$, we have that $g_1 = g_2$. Transitioning to the opposite category, these concepts are swapped so that mono is epi and vice versa; an epimorphism is a comonomorphism. Other examples include kernels and cokernels, and more generally, limits and colimits.

Given a chain complex,

$$\cdots \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

in $\operatorname{Mod} R$, how can we revert the order? Simply passing to the opposite category does not produce anything new, but applying the contravariant functor $\operatorname{Hom}_R(-, R)$ gives a cochain complex

$$\cdots \xleftarrow{\operatorname{Hom}_R(\partial_{n+1},R)} \operatorname{Hom}_R(M_n, R) \xleftarrow{\operatorname{Hom}_R(\partial_n,R)} \operatorname{Hom}_R(M_{n-1}, R) \xleftarrow{\operatorname{Hom}_R(\partial_{n-1},R)} \cdots$$

as $\operatorname{Hom}_R(\partial_n, R) \circ \operatorname{Hom}_R(\partial_{n-1}, R) = \operatorname{Hom}_R(\partial_{n-1} \circ \partial_n, R) = \operatorname{Hom}_R(0, R) = 0$.

Definition 4.24. Let X be any topological space. The singular cohomology of X with coefficients in R is given by the homology corresponding to the simplicial object $\operatorname{Hom}_R(-, R) \circ (R^{(-)} \circ \operatorname{Hom}_{\mathbf{Top}}(\Delta(-), X))$.

We will use the notation $H^n(X; R)$ for the n'th cohomology of X with coefficients in R, and $d_n = \operatorname{Hom}_R(\partial_n, R)$ for the coboundary operator. In the case of $R = \mathbb{Z}$, we simply write $H^n(X)$.

There is an important connection between homology and cohomology, namely the universal coefficient theorem for cohomology [Rot08].

Theorem 4.25. *Let R be a* PID *(or more generally hereditary), B be an R-algebra and K be a chain complex in $\mathbf{K}(\operatorname{Mod} R)$. Then, for all $n > 0$ there is a split exact sequence*

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(K), B) \to H^n(\operatorname{Hom}_R(K, B)) \to \operatorname{Hom}_R(H_n(K), B) \to 0.$$

Example 4.26. The homology of $\mathbb{R}P^2$ was calculated as $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$ and $H_2(\mathbb{R}P^2) = 0$ in Example 3.29. From Example 4.20 it is known that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) \simeq \mathbb{Z}_2$, $\operatorname{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z})$ is obviously zero and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is zero by considering the resolution $0 \to \mathbb{Z} \to 0$. Thus, applying the universal coefficient theorem for cohomology (Theorem 4.25), the cohomology of $\mathbb{R}P^2$ is given by

$$H^0(\mathbb{R}P^2) \simeq \operatorname{Hom}_{\mathbb{Z}}(H_0(\mathbb{R}P^2), \mathbb{Z}) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(0, \mathbb{Z}) \simeq \mathbb{Z},$$

$$H^1(\mathbb{R}P^2) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_0(\mathbb{R}P^2), \mathbb{Z}) \simeq 0$$

and

$$H^2(\mathbb{R}P^2) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_2(\mathbb{R}P^2), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) \simeq \mathbb{Z}_2.$$

In particular, the homology and cohomology of $\mathbb{R}P^2$ do not agree.

Let simplicial cohomology denote the resulting homology from applying $\mathrm{Hom}_R(-, R)$ to simplicial homology over a (commutative) ring R. An immediate corollary of the universal coefficient theorem is that homology completely determines cohomology, such that simplicial cohomology agree with singular cohomology. In simplicial homology, applying the boundary operator to an edge (1-simplex), results in a formal linear combination of the vertices (0-simplexes). More generally, when applying the boundary operator to a k-simplex, the result is a formal linear combination of its faces consisting of $(k-1)$-simplexes. This is a higher dimensional analogue of adjacency.

Definition 4.27. Let A be an abstract simplicial complex. Two k-simplexes in A, σ_1 and σ_2, are adjacent if there is a $(k+1)$-simplex in A that admit both σ_1 and σ_2 as faces. A k-simplex, σ_k, and a $(k+1)$-simplex, σ_{k+1}, are mutually incident if σ_k is a face of σ_{k+1}.

Observation 4.28. Adjacency/incidence in the 1-skeleton of an abstract simplicial complex agree with Definition 2.2.

To give a geometrical interpretation of simplicial cohomology, we take a simplicial complex with a finite number of generators in each dimension and order the bases such that we have a corresponding chain complex

$$\cdots \xrightarrow{\partial_{n+1}} R^{a_n} \xrightarrow{\partial_n} R^{a_{n-1}} \xrightarrow{\partial_{n-1}} \cdots .$$

Applying $\mathrm{Hom}_R(-, R)$ gives the transition to the cochain complex

$$\cdots \xrightarrow{d_{n-1}} \mathrm{Hom}_R(R^{a_n}, R) \xrightarrow{d_n} \mathrm{Hom}_R(R^{a_{n+1}}, R) \xrightarrow{d_{n+1}} \cdots .$$

Proposition 4.3 gives canonical isomorphisms $\mathrm{Hom}_R(R^{a_n}, R) \simeq \mathrm{Hom}_R(R, R)^{a_n}$. The isomorphism from Example 4.7, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$, generalizes to $\mathrm{Hom}_R(R, R) \simeq R$ as the isomorphism only rely on the existence of unity; $f \mapsto f(1)$. Altogether $\mathrm{Hom}_R(R^{a_n}, R) \simeq R^{a_n}$. Furthermore, the isomorphism gives a natural way of identifying the standard basis of R^{a_n} with the standard basis of $\mathrm{Hom}_R(R^{a_n}, R)$. Or in terms of geometry, given an n-simplex, σ_n, identify it with the projection, π_{σ_n}, that send σ_n to 1 and other n-simplexes to 0. Considering the coboundary operator, we know that $d = \mathrm{Hom}_R(\partial, R)$, which applied to a homomorphism,

f, yields $d(f) = f \circ \partial$. Let σ_k be a k-simplex with associated co-simplex π_{σ_k}, and similarly σ_{k+1} a $(k+1)$-simplex corresponding to $\pi_{\sigma_{k+1}}$. Then, $d_{k+1}(\sigma_k)(\sigma_{k+1}) = \sigma_k(\partial_{k+1}(\sigma_{k+1}))$ equals ± 1, if and only if σ_k is incident to σ_{k+1}. This shows that applying d_{k+1} to a k-simplex, σ, results in an alternating linear combination of the $(k+1)$-simplexes that are incident to σ.

There is another dual concept of geometrical nature, obtained by interchanging the roles of faces (adjacency) and incidence, i.e. dualizing the dimension of a simplicial complex:

The plan is to partition n-simplexes using barycentric coordinates, and then embed them into triangulations to obtain CW-complexes. Consider the standard n-simplex, $n \geq 2$, and take $p_{0,1} = e_1, \ldots, p_{0,n+1} = e_{n+1}$ as the standard basis elements of $\mathbb{R}^{n+1}$, i.e. the 0-simplexes. For k-simplex number i in Δ_n, $\sigma_{k,i}$, let $p_{0,i_1}, \ldots p_{0,i_{k+1}}$ denote the $k+1$ 0-simplexes that meet $\sigma_{k,i}$ and take

$$p_{k,i} = \frac{p_{0,i_1} + \cdots + p_{0,i_{k+1}}}{k+1},$$

i.e. its centroid ($\sigma_{k,i}$ is the convex hull of $p_{0,i_1}, \ldots, p_{0,i_{k+1}}$). Pick $q_0 = p_{n,1}$ and $q_{1,i}$ as the convex hull of $\{p_{n-1,i}, q_0\}$ for every $(n-1)$-simplex, i.e. $i = 1, \ldots, n+1$. Proceed inductively by choosing $q_{k,i}$ as the convex hull of

$$\{p_{n-k,i}\} \cup \{q_{k-1,j} \mid \ \sigma_{n-k,i} \text{ is incident to } \sigma_{n-(k-1),j}\}.$$

Observe how a $q_{k,i}$ is the convex hull of a finite set of points, and increasing from $k-1$ to k adds one affine dimension, so it is in fact homeomorphic to Δ_k. The result decomposes Δ_n,

$$\Delta_n = q_{n,1} \cup \cdots \cup q_{n,k+1}.$$

Indeed, the case $n = 2$ is easily seen by explicitly calculating the q's as shown in Figure 22, so assume that the assertion holds for $(n-1) \geq 2$. Given any $x \in \Delta_n$, the straight line defined by q_0 and x is contained in Δ_n by convexity. As Δ_n is bounded, the line intersects a face of Δ_n in the direction from q_0 to x, say $\sigma_{n-1,i}$, and therefore a $\tilde{q}$ in the decomposition of $\sigma_{n-1,i}$ by the induction hypothesis. But $\tilde{q}$ is constructed by taking convex hulls of centroids belonging to simplexes in Δ_n, and consequently $\tilde{q}$ is contained in some $q_{n,j}$ (by construction). Finally, the convexity of $q_{n,j}$ implies that the straight line, and in particular x, is contained in $q_{n,j}$. Observe how this shows that the decomposition of Δ_n project down to give decompositions of its faces. Recall that a continuous injection from a compact space into a Hausdorff space is necessarily a topological embedding. In particular, given a triangulation of an n-manifold M, each k-simplex is an embedding of Δ_k into M, so that the above decomposition of intersecting faces may be identified for every k. The result is a CW-complex homeomorphic to M satisfying that the k-cells are in bijection with the $(n-k)$-cells of the triangulation; for a given $(n-k)$-

simplex, σ, glue together the homeomorphic images of $q_{k,i}$'s that intersect σ and originates from n-simplexes containing σ. Indeed, the case $n = 2$ is geometrically clear, glue together triangles as in Figure 22 following the rules of Definition 3.8, so assume the assertion holds for $(n-1) \geq 2$. Take 0-cells as centroids of n-simplexes. For a given 0-simplex, v, consider

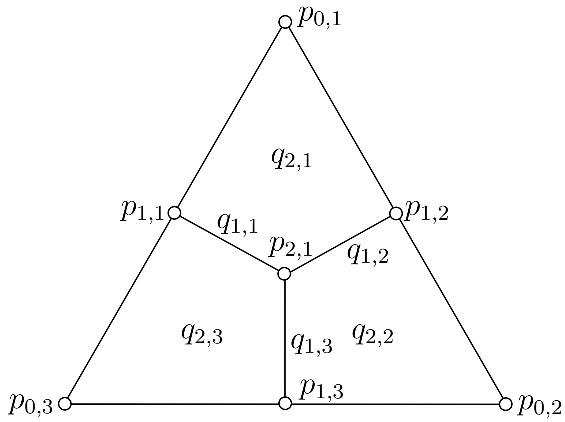


Figure 22: The building blocks for the dual cell in the 2-dimensional case.

the n-simplexes that contain v, say $\sigma_1, ..., \sigma_l$. For a fixed i, the unique (top dimensional) q_{n,i_j} containing v is homeomorphic to D^n (homeomorphic to Δ_n as discussed above). Intersections of $\sigma_i(q_{n,i_j})$'s agree with homeomorphic images of $q_{n-1,k}$'s, and thus become homeomorphic to D^n again, i.e. $\cup_i \sigma_i(q_{n,i_j}) \simeq D^n$. Finally, the boundary homeomorphic to S^{n-1} contain every centroid of lower simplexes in the $\sigma_i(q_{n,i_j})$'s and therefore admit a CW-structure by the induction hypothesis; take the interior of $\cup_i \sigma_i(q_{n,i_j})$ as the n-cell corresponding to v. The result is homeomorphic to M as the decomposition into q's reconstruct each n-simplex by the previous induction.

Given a finite triangulation, T, of an n-manifold, the constructed CW-complex is the dual simplicial complex, denoted T^*. This construction is well-defined; combinatorial relations determine the dual simplicial complex, not the choice of embeddings. Note that Definition 3.11 implies that the 1-skeleton of T^* is a graph as every $(n-1)$-simplex is the face of exactly two distinct n-simplexes, i.e. adjacency is defined by incidence in $T^{n-1} \cup T^n$.

Definition 4.29. Let T be a triangulation of a manifold M. The dual graph of $T^0 \cup T^1$ is the graph defined by the 0- and 1-cells in T^*.

In particular, the above construction for $M = S^2$ shows the following as the dual is automatically embedded in S^2.

Proposition 4.30. *Let T be a triangulation of S^2, then the dual graph of $T^0 \cup T^1$ is planar.*

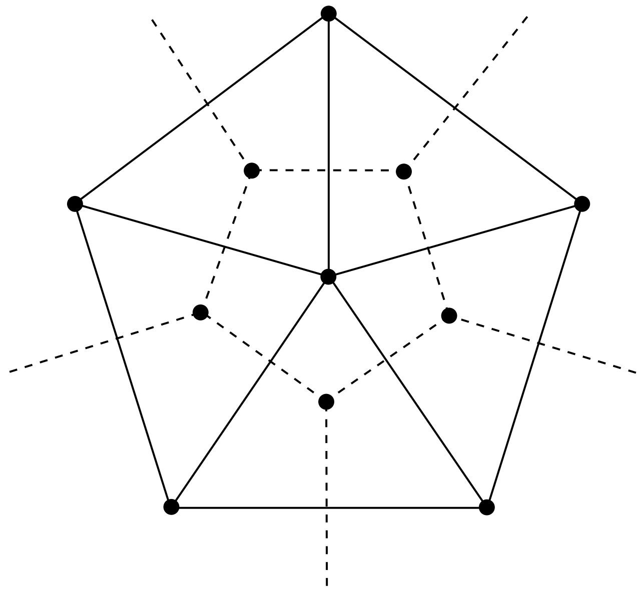


Figure 23: The dual about a vertex of degree five in some triangulation.

The above construction dates to the work of Henri Poincaré in Analysis Situs [Die09, Veb31], where it is used to prove the first version of Poincaré duality; the above discussion gives an outline of why it is reasonable to expect isomorphisms $H^{n-k}(M) \simeq H_k(M)$ under certain conditions, i.e. a categorical duality from a geometric construction.[51]

[51] Although we have not defined the CW-boundary operator, the basis in each dimension is given by n-cells for n fixed.

5 The Four Colour Problem

Recall that we are going to colour (actual) maps in such a way that no regions separated by a single border have the same colour, and want to achieve more understanding of why four colours suffice. It clearly suffices to colour planar graphs; let vertices represent regions and edges represent borders. The result by Appel and Haken may be stated in the following way [AHK77a, AHK77b].

Theorem 5.1 (Appel, Haken 1977)**.** *Every planar graph is* 4*-colourable.*

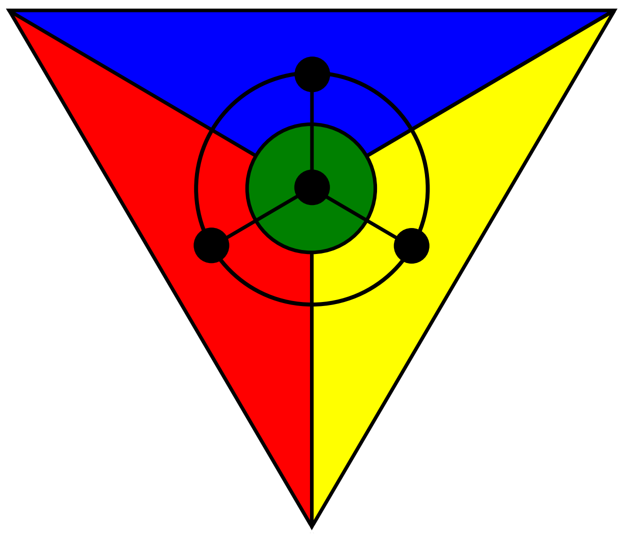


Figure 24: A map that requires four colours together with the underlying graph (K_4).

5.1 A Simple Reduction

By considering morphisms in **Grph** we use some rather geometrical properties to show that it suffices to verify Theorem 5.1 for triangulations of the sphere. A typical intuitive view of continuous maps between Euclidean spaces is how they act as deformations without ripping or tearing apart. An example of this intuition is Brouwer fixed point theorem that dates back to a paper by L. E. J. Brouwer in 1911 [Bro11]. The theorem states that any continuous map $f : D^n \to D^n$ admits a fixed point, i.e. there is an $x \in D^n$ such that $f(x) = x$. A modern proof assumes that this is not the case, which makes it possible to construct a continuous map from D^n to S^{n-1} by normalizing the difference $\mathrm{id}_{D^n} - f$. This contradict the intuition of continuity as the map essentially poke a hole in the disk, an argument that is made formal by using of homology [Vic12]. In the same manner, thinking about graphs as 1-skeletons,

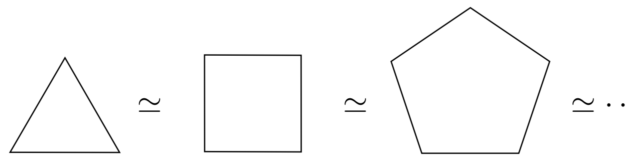


Figure 25: An infinite chain of homeomorphic 1-skeletons representing non-isomorphic graphs.

a graph homomorphism is a map that cannot rip the vertices of a graph. Hence, graph homomorphisms somehow fold graphs into subsets of other graphs. Note that there is an added combinatorial complexity of graph homomorphisms, in contrast to continuous maps, e.g. calculating the automorphisms of a given graph.

Lemma 5.2. *Let C be a cycle and T a tree such that they have exactly one vertex in common. Then there is a graph homomorphism $T \cup C \to C$.*

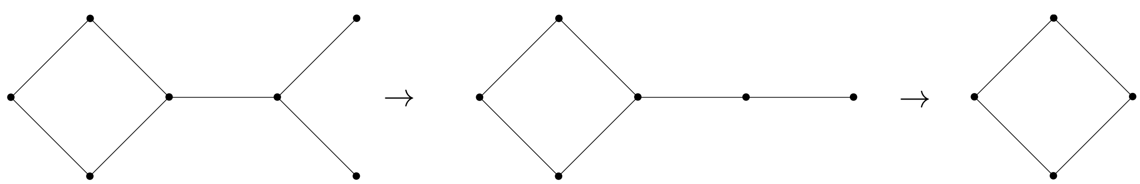


Figure 26: A geometrical view on a graph homomorphism onto a cycle.

Proof. Let $V(C) \cap V(T) = \{r_0\}$. First, fix r_1 adjacent to r_0 in T and let $f_1 : V(C \cup T) \to V(C \cup T)$ map $r' \mapsto r_1$ for any r' adjacent to r_0 in $T[V(T) \backslash r_1]$, while keeping the rest of the vertices fixed. Clearly, f_1 is a homomorphism as any pair of vertices adjacent to r_0 in T cannot be adjacent by the definition of a tree. Take any vertex, say r_2, different from r_0 and adjacent to r_1 and construct a homomorphism, f_2, with the same scheme as for f_1. Repeating this process gives a sequence of homomorphisms, $f_1, f_2, f_3, ...$, but the process will end as the tree is finite, say $f_1, f_2, ..., f_n$. After applying $f_n \circ ... \circ f_1$, we are left with C and a path $P \subset T$. By construction, we have that $P = (r_0, r_1, ..., r_n)$. Letting $C = (c_0, c_2, ..., c_m)$ with $c_1 = c_m$, define $g : V(C \cup P) \to V(C)$ by $r_i \mapsto c_{i \text{ mod } m}$ for $i = 0, ..., n$ and $c_j \mapsto c_j$ for $j = 0, ..., m$. Now, $f = g \circ f_n \circ ... \circ f_1$ is a homomorphism of $C \cup T$ onto C. $\square$

Theorem 5.3. *Given any connected planar graph, we can find a homomorphism to some triangulation of the sphere.*

Proof. First, note that any cycle on the sphere separates it in two open components by the Jordan curve theorem [Vic12]. Let X^1 denote the embedding of the graph on S^2. Given any cycle for which one of the component has an empty intersection with X^1, add a vertex adjacent to every vertex in the cycle [Sto79]. This results in an inclusion into a connected planar graph consisting of unions of triangles and trees (Figure 27). Given a tree that intersects with a single cycle, apply Lemma 5.2 to safely remove it. Take a maximal union of triangles M with respect to Definition 3.8, i.e. cannot be extended and still be a simplicial complex, and apply Lemma 5.2 to any tree, T, intersecting it. The other cycles intersecting T will intersect M in exactly one vertex post homomorphism and can be embedded inside a triangle in M by the assumption of planar. Repeat the above actions iteratively and note that the process will end as graphs are assumed finite. Hence, the result is a single union of triangles without trees and cycles of length greater than three. The resulting space must be a triangulation of the sphere, otherwise the above process may continue. □

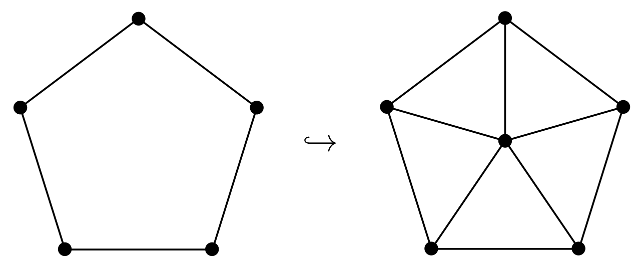


Figure 27: An inclusion of a cycle to a refinement consisting only of 3-cycles that preserve the planar property.

This theorem tells us that it suffices to prove Theorem 5.1 for triangulations of the sphere. Indeed, given any planar graph, G, there is a triangulation, T, and a homomorphism $G \to T$. If every triangulation is 4-colourable, then Corollary 2.13 gives the assertion.

5.2 The Galois Field with Four Elements

We construct the Galois field with four elements explicitly and recall some arithmetic properties.

The maximal ideals in a PID are given by the ideals generated by irreducible elements, i.e. the elements p for which $p = p_1 p_2$ implies that p_1 or p_2 is a unit [BJN94]. Standard examples of PIDs includes $\mathbb{Z}$ and $K[x]$ for a field K, such that maximal ideals in $\mathbb{Z}$ and $K[x]$ are generated by prime numbers and irreducible polynomials, respectively. The ideal generated by 2 in $\mathbb{Z}$, (2), gives the field $\mathbb{Z}_2 = \mathbb{Z}/(2)$ consisting of two elements. Now, we need an irreducible polynomial of degree two to obtain a field with four elements. Checking the $2^2 = 4$ monic polynomials (1 as leading coefficient) of degree two in $\mathbb{Z}_2$, we see that the only irreducible is x^2+x+1. Thus, we have the field $\mathrm{GF}(4) = {}^{\mathbb{Z}_2[x]}\!/\!_{(x^2+x+1)}$ with four elements.[52]

The characteristic of GF(4) is clearly two and the elements are given by $\overline{0}$, $\overline{1}$, $\overline{x}$ and $\overline{x+1}$ with the notation $\overline{f} = f + (x^2+x+1)$. Noticing how $\overline{x^2+x+1} = 0$, we have that $\overline{x+1} = \overline{x}^2$, such that the elements of GF(4) may be written $\overline{0}$, $\overline{1}$, $\overline{x}$ and $\overline{x}^2$. There are two arithmetic properties that will be important to us in the discussion of the four colour problem:

- $a + b \neq 0$ for any distinct elements $a, b \in \mathrm{GF}(4)$.
- $\overline{1} + \overline{x} + \overline{x}^2 = 0$.

The first follow as the characteristic is two (each element is its own additive inverse), and the second by the construction of GF(4) as a quotient.

5.3 A Reformulation Using Homology

Consider a finite triangulation of S^2, say T. Let v be the number of vertices (0-simplexes), e be the number of edges (1-simplexes) and f the number of triangles (2-simplexes). By the discussion in Example 3.38, we know that f is even, say $f = 2n$, and $2e = 3f$, so that applying the Euler characteristic (Corollary 3.36) gives $e = 3n$ and $v = n+2$. Ordering bases results in a chain complex

$$0 \to \mathbb{Z}^{2n} \xrightarrow{\partial_2} \mathbb{Z}^{3n} \xrightarrow{\partial_1} \mathbb{Z}^{n+2} \to 0.$$

The homology of S^2 was calculated in Example 3.28, and reveals that we have exactness in $\mathbb{Z}^{3n}$, kernel $\operatorname{Ker}\partial_2 = \mathbb{Z}$ and cokernel $\operatorname{Cok}\partial_1 = \mathbb{Z}$, which results in an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{2n} \xrightarrow{\partial_2} \mathbb{Z}^{3n} \xrightarrow{\partial_1} \mathbb{Z}^{n+2} \to \mathbb{Z} \to 0.$$

This sequence splits everywhere by Proposition 3.31, such that applying $- \otimes_{\mathbb{Z}} \mathrm{GF}(4)$ results in the exact sequence

$$0 \to \mathrm{GF}(4) \to \mathrm{GF}(4)^{2n} \xrightarrow{\tilde{\partial_2}} \mathrm{GF}(4)^{3n} \xrightarrow{\tilde{\partial_1}} \mathrm{GF}(4)^{n+2} \to \mathrm{GF}(4) \to 0,$$

by Proposition 4.2 and Corollary 4.9. Similarly, applying $\operatorname{Hom}_{\mathrm{GF}(4)}(-, \mathrm{GF}(4))$ is exact, and thus Proposition 4.3 together with the isomorphism $\operatorname{Hom}_{\mathrm{GF}(4)}(\mathrm{GF}(4), \mathrm{GF}(4)) \simeq \mathrm{GF}(4)$ yields

$$0 \to \mathrm{GF}(4) \to \mathrm{GF}(4)^{n+2} \xrightarrow{d_1} \mathrm{GF}(4)^{3n} \xrightarrow{d_2} \mathrm{GF}(4)^{2n} \to \mathrm{GF}(4) \to 0.$$

[52]By the division algorithm, any polynomial in $\mathbb{Z}_2[x]$ can be written as $(x^2+x+1)q(x) + r(x)$, where the degree of r is less than two.

Consequently, the cohomology of S^2 over GF(4) is given by

$$H^0(S^2; \mathrm{GF}(4)) \simeq \mathrm{GF}(4),$$

$$H^1(S^2; \mathrm{GF}(4)) \simeq 0$$

and

$$H^2(S^2; \mathrm{GF}(4)) \simeq \mathrm{GF}(4).$$

We are ready to present a reformulation of the four colour problem.[53]

Proposition 5.4. *Let T be a finite triangulation of S^2, v be the number of vertices (0-simplexes), e be the number of edges (1-simplexes) and f the number of triangles (2-simplexes). Colouring T with four colours is equivalent to finding an element $\alpha \in (GF(4)^*)^e$ such that $d_2\alpha = 0$.*

Proof. Assume that T is 4-colourable. Then, we can colour T with elements of GF(4), say $\beta = (\beta_1, ..., \beta_v)$. Picking $\alpha = d_1(\beta)$, it immediately follows that $d_2\alpha = d_2 \circ d_1\beta = 0$ as $d_2 \circ d_1 = 0$. Moreover, each coefficient α_i comes from the coefficients of the vertices that makes up its 0-skeleton. That is, given any $1 \leq i \leq e$, there are two components/colours β_{i_1} and β_{i_2} in β satisfying $\alpha_i = \beta_{i_1} + \beta_{i_2}$, but β is a 4-colouring of T such that $\beta_{i_1} \neq \beta_{i_2}$, and hence $\alpha_i \neq 0$ by the arithmetic properties of GF(4). Contrary, assume that $\alpha = (\alpha_1, ..., \alpha_e)$ satisfies that $d_2(\alpha) = 0$ and $\alpha_i \neq 0$ for any i. Using the exactness of the free GF(4)-module over the edges, we also have that $\alpha \in \operatorname{Im} d_1$, i.e. there is a $\beta \in \mathrm{GF}(4)^v$ such that $\alpha = d_1(\beta)$. We claim that β is a 4-colouring of T: As above, each component $\alpha_i \neq 0$ is the sum $\beta_{i_1} + \beta_{i_2}$, such that $\beta_{i_1} \neq \beta_{i_2}$, otherwise $\alpha_i = 0$ as GF(4) has characteristic two. □

Remark 5.5. Given any triangulation of the sphere, the equation $d\alpha = 0$ will always refer to the equation in Proposition 5.4 that determine a 4-colouring of the 1-skeleton.

Consequently, colouring a graph with four colours may be stated as an equation involving the coboundary operator. In the context of deRham cohomology, d is often referred to as a differential and the above equation may be interpreted as a discrete differential equation. The problem of solving $d\alpha = 0$ also somehow captures the philosophy of cohomology; easy to solve locally, but hard to glue together into a global solution.

Graph homomorphisms, adjacency and incidence were all extended to abstract simplicial complexes. Similarly, one may extend graph colouring to abstract simplicial complexes.

[53] Given any ring R, we denote $R^* = R - \{0\}$.

Definition 5.6. Let A be an abstract simplicial complex and $C = \{c_1, ..., c_n\}$ a set. A function $f : A^k \to C$ is an n-colouring of A^k if any adjacent k-simplexes, σ_1 and σ_2, satisfy $f(\sigma_1) \neq f(\sigma_2)$. A is n-colourable if there is an n-colouring.

Observation 5.7. Colouring the $(k+1)$-skeleton of an abstract simplicial complex, A, agree with colouring the graph with A^k as vertices and edge set according to adjacency in A.

Proof. Setting $V = A^k$ and $E = \{\{\sigma_1, \sigma_2\} \subset A^k | \quad \sigma_1 \text{ and } \sigma_2 \text{ are adjacent}\}$, we see that colouring A^k is equivalent to colour the graph (V, E). □

Given an abstract simplicial complex A, we recover Definition 2.11 by colouring A^0, further colouring A^1 correspond to colour the edges (1-simplexes).

Proposition 5.8. *Let T be a finite triangulation of S^2. Solving $d\alpha = 0$ is equivalent to colouring the edges of T with three colours.*

Proof. Take $C = \mathrm{GF}(4)^*$ in Definition 5.6, notice how each component of $d\alpha = 0$ correspond to the sum of non-zero coefficients of the edges incident to the corresponding triangle. This occurs if and only if the edges incident to a triangle admits pairwise distinct coefficient by the arithmetic properties of GF(4). □

Corollary 5.9. *The following are equivalent.*

- *Colouring any planar graphs with four colours.*
- *The equation $d\alpha = 0$ may be solved for any triangulation of the sphere.*
- *The edges of any triangulation of the sphere may be coloured by three colours.*

Proof. By Theorem 5.3, it suffices to consider graphs that corresponds to triangulations of the sphere. Given an arbitrary triangulation of the sphere, say T, T^0 is 4-colourable if and only if $d\alpha = 0$ by Proposition 5.4, if and only if T^1 is 3-colourable by Proposition 5.8. □

We give a simple example to illustrate the connections in Corollary 5.9.

Example 5.10. Let T be the tetrahedron as illustrated in Figure 14 and order bases as in Example 3.15. Let $\mathrm{GF}(4) = \{\overline{0}, \overline{1}, \overline{x}, \overline{x}^2\}$ and consider the 4-colouring $\alpha = (\overline{0}, \overline{1}, \overline{x}, \overline{x}^2)$ of T^0. Applying d_1 and using that $\overline{x}^2 = \overline{x+1}$ yields a 3-colouring of T^1;

$$\begin{aligned} d_1(\overline{0}, \overline{1}, \overline{x}, \overline{x}^2) &= (\overline{0}+\overline{1}, \overline{0}+\overline{x}, \overline{0}+\overline{x}^2, \overline{1}+\overline{x}, \overline{1}+\overline{x}^2, \overline{x}+\overline{x}^2) \\ &= (\overline{1}, \overline{x}, \overline{x}^2, \overline{x}^2, \overline{x}, \overline{1}). \end{aligned}$$

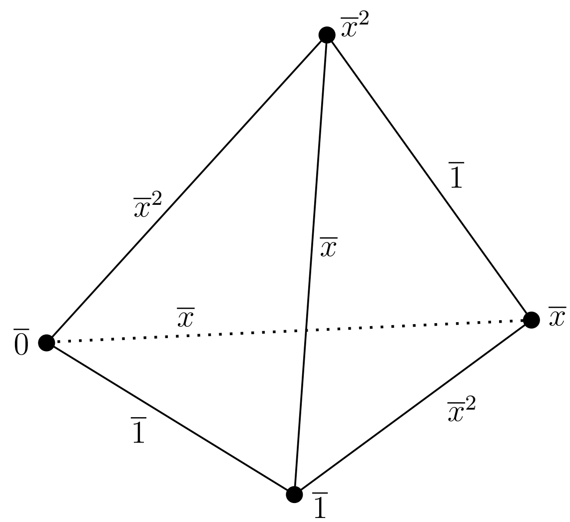


Figure 28: A 4-colouring of the vertices gives a 3-colouring of the edges.

Of course, $d_2 \circ d_1\alpha = 0$, by the properties of the coboundary operator, explicitly

$$\begin{aligned} d_2(\overline{1}, \overline{x}, \overline{x}^2, \overline{x}^2, \overline{x}, \overline{1}) &= (\overline{1} + \overline{x} + \overline{x}^2, \overline{1} + \overline{x} + \overline{x}^2, \overline{1} + \overline{x} + \overline{x}^2, \overline{1} + \overline{x} + \overline{x}^2) \\ &= (\overline{0}, \overline{0}, \overline{0}, \overline{0}). \end{aligned}$$

5.4 Solutions Given by Hamiltonian Cycles

Whenever the dual graph associated with a triangulation admits a Hamiltonian cycle (Definition 2.10), there is an easy strategy to colour the edges with three colours, i.e. solve the equation $d\alpha = 0$.

Proposition 5.11. *Let T be a triangulation such that the dual graph in T^* admits a Hamiltonian cycle. Then T^0 is 4-colourable.*

Proof. Let G denote the 1-skeleton of T with G^* as the dual graph and let H denote a Hamiltonian cycle in G^*. The number of triangles (2-simplexes) in T is even by Example 3.38, such that the number of 0-cells in T^* is even, and consequently H takes the form $(v_0, v_1, ..., v_{2n})$. Define $f : E(H) \to \mathrm{GF}(4)$,

$$\{v_i, v_{i+1}\} \mapsto \begin{cases} 1 & \text{if } i \text{ even} \\ \overline{x} & \text{if } i \text{ odd} \end{cases},$$

notice how this label the edges of H with 1 and $\overline{x}$ successively as H is even. Each edge in H intersect exactly one edge in T by the construction of T^*, such that we may define an injection

$i : H \to T^1$ according to this property. This gives a well-defined function $g : T^1 \to \mathrm{GF}(4)^*$,

$$e \mapsto \begin{cases} f(i^{-1}(e)) & \text{if } e \in i(H) \\ \overline{x}^2 & \text{if } e \notin i(H) \end{cases},$$

which constitutes a 3-colouring of T^1. Indeed, any triangle (2-simplex) in T admits exactly three faces for which two are given distinct colours in $\{1, \overline{x}\}$ by the construction of f and the last face is coloured with $\overline{x}^2$. $\square$

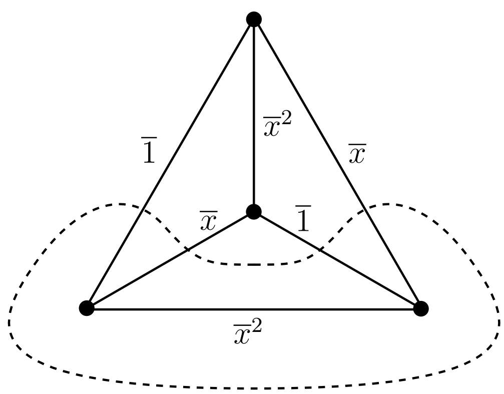


Figure 29: Colouring the edges of K_4 using a Hamiltonian cycle in the dual graph (isomorphic to K_4).

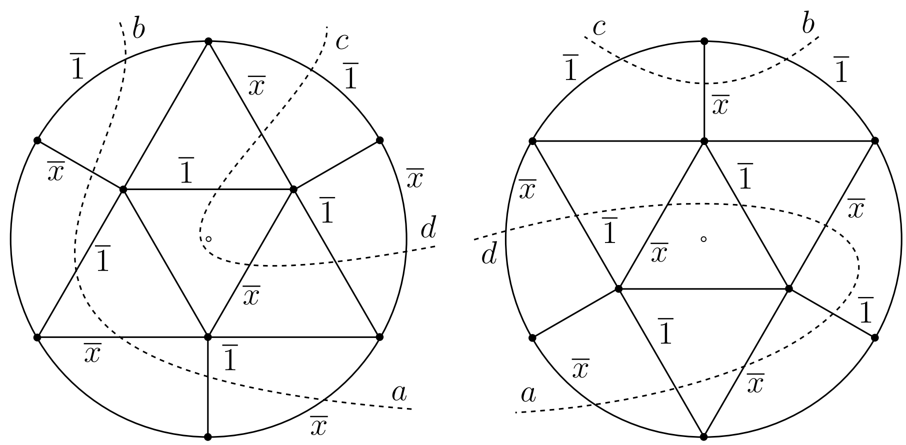


Figure 30: Colouring the edges of the triangulation used to construct a minimal triangulation of $\mathbb{R}P^2$. The left part represents the north of equator, while the right part represents the south of equator. The edges that are identified along the equator is indicated by a, b, c and d.

The proposition is illustrated in Figure 29 and 30.

It is only natural to ask when a planar graph admits a Hamiltonian cycle. Whitney proved that every 4-connected (Definition 2.10) triangulation of the sphere admits a Hamiltonian cycle in 1931 [Whi31].

Theorem 5.12 (Whitney, 1931)**.** *A triangulation whose* 1*-skeleton is* 4*-connected admits an Hamiltonian cycle.*

Notice that the dual of a triangulation is a triangulation if and only if every vertex in the original triangulation has degree three. Indeed, $(2-k)$-simplexes and k-cells are in bijection, such that a 0-simplex with degree greater than three gives a cycle of length greater than three and vice versa. Hence, Theorem 5.12 cannot guarantee Theorem 5.1. W. T. Tutte proved that the triangulation assumption is superfluous in 1956 [Tut56].

Theorem 5.13 (Tutte, 1956)**.** *Any* 4*-connected planar graph admits a Hamiltonian cycle on the subgraph induced by cycles.*

The awareness of Proposition 5.11 is due to P. G. Tait [Saa72, Tai31], who also conjectured the existence of Hamiltonian cycles in planar graphs for which every vertex has degree three. This is, in particular, the case for dual graphs of triangulations as a triangle consists of three edges. The conjecture was proven to be false in 1946 by Tutte himself, who gave an explicit counterexample on 46 vertices illustrated in Figure 31 [Tut46]. Consequently, Tutte already knew that Theorem 5.13 could not guarantee Theorem 5.1 at the time. However, he also noticed that Theorem 5.13 is somewhat useless in the situation of Theorem 5.1. Indeed, the dual graph of a triangulation cannot be 4-connected; removing three adjacent 0-cells of a given 0-cell renders the dual graph disconnected whenever the triangulation admits more than four 2-simplexes.

Let T be a triangulation of the sphere. Any vertex in T^0 has degree greater than two; otherwise, the third property in Definition 3.11 will be violated. Moreover, the vertices may be assumed to have degree greater than three. If the triangulation consists of four vertices, colouring with four colours is trivial, otherwise any vertex of degree three is a refinement of a triangulation. More precisely, if $\deg(v) = 3$, take the three edges in T^1 incident to v, say $[v, v_1]$ $[v, v_2]$ and $[v, v_3]$, and consider the triangle, t, defined by v_1, v_2 and v_3. The Jordan curve theorem implies that t separates S^2 in two disjoint open components, U and V, such that one intersection with T^1 is $\cup([v, v_i](\Delta_1) - \{v_i\})$. Without loss of generality, we assume $U \cap T^1 = \cup([v, v_i](\Delta_1) - \{v_i\})$, while $V \cap T$ is non-empty by the assumption of $|T^0| > 4$. Consequently, U agree with the interior of a face that makes up a triangulation of S^2; if triangulations are assumed to be 4-colourable, v may safely be given a fourth colour. If there are "enough"

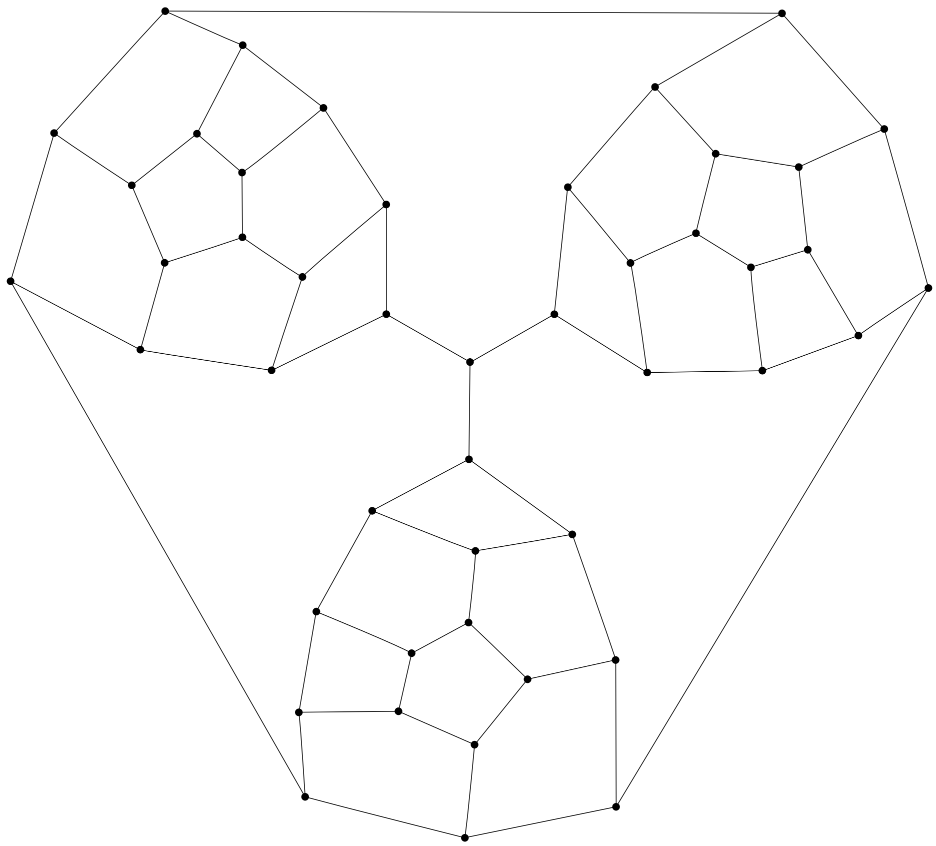


Figure 31: The counterexample by Tutte [Tut46].

triangles, one may even assume that every vertex has degree greater than four. Consider a 4-cycle, (v_1, v_2, v_3, v_4), embedded into $\mathbb{R}^2$ by identifying $v_1 = (-1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 0)$ and $v_4 = (0, -1)$. Triangulate the bounded component of $\mathbb{R}^2 - (v_1, v_2, v_3, v_4)$ by identifying a vertex, v, with $(0, 0)$ and adding edges to every v_i (similar as in the proof of Lemma 5.2). Denote the resulting graph G and define a homomorphism $\phi : G \to G[V(G) - \{v_3\}]$ by $v_3 \mapsto v_1$, while keeping the other vertices fixed. Geometrically, we fold the square defined by the convex hull of $\{v_1, v_2, v_3, v_4\}$ into the triangle given by the convex hull of $\{v_1, v_2, v_4, v\}$. Alternatively, we divide out by the orbit of the obvious group action of $\mathbb{Z}_2$ given by reflecting graph about the $(0, 1)$-axis, i.e. generated by the automorphism $v_1 \mapsto v_3$, $v_3 \mapsto v_1$, while keeping the other vertices fixed. Whenever there is a vertex of degree four on T, it defines a 1-skeleton isomorphic to G in **Grph**, and thus ϕ may be extended to T by fixing the other vertices. Intuitively, ϕ induces a continuous map, ϕ^*, that quotient points by squeezing the surface. The result is a space constructed by gluing one edge of a triangle to the sphere; the

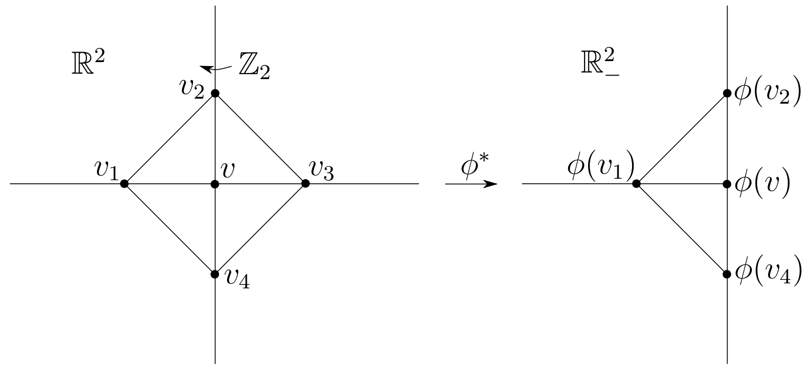


Figure 32: The group action given by the identity and the automorphism $v_1 \mapsto v_3$, $v_3 \mapsto v_1$ to the left, and the quotient of the orbit to the right.

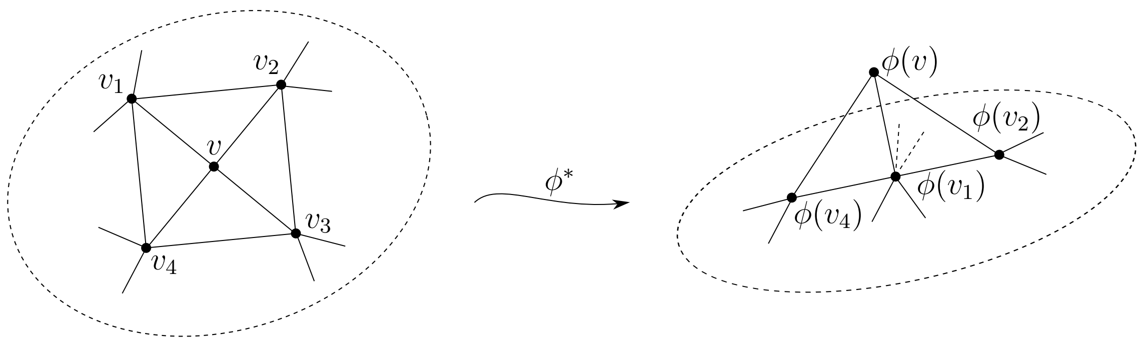


Figure 33: The graph homomorphism $v_3 \mapsto v_1$ induces a continuous map from the sphere to the space constructed by gluing a triangle to the sphere along a face.

triangle contains $\phi(v)$ as well as the edges $[\phi(v), \phi(v_1)]$, $[\phi(v), \phi(v_2)]$ and $[\phi(v), \phi(v_4)]$ while the rest of $\phi(T^0 \cup T^1)$ is contained in S^2 (Figure 33). The subgraph of $\phi(T^0 \cup T^1)$ induced by $\phi(T^0) - \{\phi(v)\}$ is especially embedded on S^2, and if, in addition, v is the only vertex of degree less than five, it is in fact a triangulation of S^2 again. Indeed, each v_i has degree greater than four before applying ϕ and therefore lies in more than four triangles. When ϕ identify $[v_1, v_2]$ with $[v_2, v_3]$ and $[v_4, v_1]$ with $[v_3, v_4]$, the third property in Definition 3.11 is preserved; no triangles that share edges have other edges identified. Under the assumption that the subgraph induced by $\phi(T^0) - \{\phi(v)\}$ is 4-colourable, we immediately deduce a 4-colouring of $\phi(T^0)$, as $\phi(v)$ is only adjacent to three vertices in $\phi(T^0 \cup T^1)$. Corollary 2.13 gives a 4-colouring of T^0 for which v_1 and v_3 are given equal colours by Observation 2.14. The following observation is justified.

Observation 5.14. Under the assumption that 0-skeletons of triangulations for which every vertex has degree less or equal to five may be coloured with four colours, we can ensure that the 0-skeleton of any triangulation for which there is only one vertex of degree less than five may be coloured with four colours.

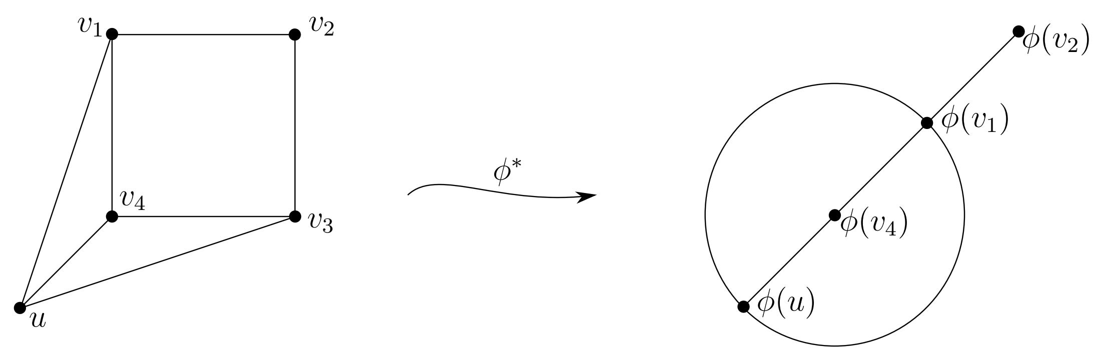


Figure 34: Degree greater or equal to five is necessary for two non-adjacent vertices both adjacent to a vertex of degree four; the image is not a graph (nor part of a triangulation).

The counterexample by Tutte, in Figure 31, contain cycles of length four, and based on Observation 5.14 one may argue that this is somewhat insufficient. This matter is however settled by G. B. Faulkner and D. H. Younger who found a non-Hamiltonian planar graph with no cycles of length less than five [FY74]. As a result, it seems like one cannot expect to prove Theorem 5.1 using the Hamiltonian connection given by Proposition 5.11.

Relaxing the triangulation assumption to planar, we give another reason why vertices of degree five are particularly interesting. Consider the set

$$\Gamma = \{n \in \mathbb{N}_+ |\ \exists \text{ a non-4-colourable planar graph of size } n\}.$$

Theorem 5.1 obviously imply $\Gamma = \emptyset$, but assume not, for the purpose of contradiction. We may safely assume the graphs resulting in non-empty Γ to only consist of cycles by Lemma 5.2, i.e. every vertex lies on a cycle. That is, the graphs in question partition S^2 into n-gons for $n \geq 3$ via the Jordan curve theorem [Vic12]. The existence of a minimal $m \in \Gamma$ implies the existence of a planar graph, G, with $|G| = m$ that is not 4-colourable. Example 3.39 gives the existence of a vertex, say v, of degree less than six. If $\deg_G(v) = 4$, consider the four 0-cells (vertices) adjacent to v, say v_1, v_2, v_3 and v_4, that constitute four 2-cells, R_1, R_2, R_3 and R_4, whose closure contain v. Removing v, the underlying embedding of $G[V(G) - \{v\}]$ connects region R_1, R_2, R_3 and R_4 to one 2-cell. Firstly, $G[V(G) - \{v\}]$ is 4-colourable as $|G[V(G) - \{v\}]| = m - 1$ and we may define a graph homomorphism, say ϕ, sending one

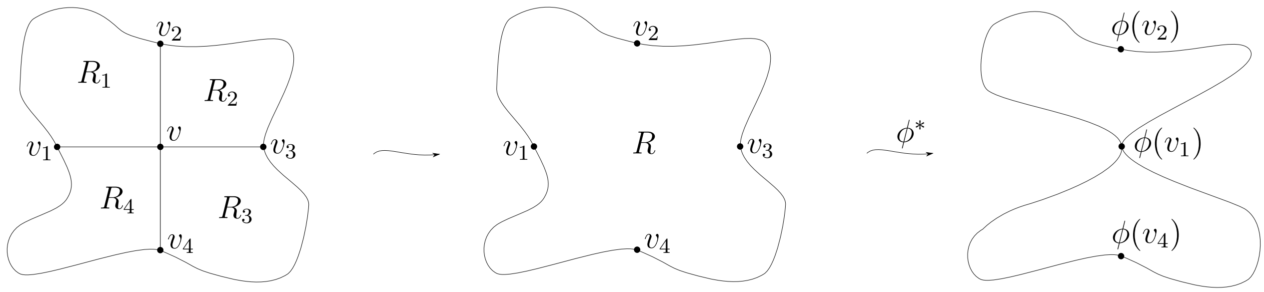


Figure 35: A minimal planar non-4-colourable graph has five as minimal degree.

of the v_i's to a distinct v_j.[54] The image graph, $\phi(G[V(G) - \{v\}])$, is also 4-colourable as it is planar and $|\phi(G[V(G) - \{v\}])| = m - 2$; the image is a continuous deformation on the sphere as illustrated in Figure 36. A 4-colouring $\phi(G[V(G) - \{v\}]) \to K_4$ gives a 4-colouring $G[V(G) - \{v\}] \to K_4$, and Observation 2.14 reveals that the distinct v_i and v_j are coloured with the same colour. As a result, we see that a 4-colouring of $G[V(G) - \{v\}]$ gives a 4-colouring of G by safely colouring v with the fourth colour. A contradiction.

Observation 5.15. If there is a non-4-colourable planar graph, then any minimal, with respect to Γ, such graph satisfy that every vertex has degree greater than four.

5.5 An Application of Modular Equations in Analysis Situs

In the early days, the problem of colouring maps was not defined as a graph-colouring problem, but as a colouring of partitions into bounded regions (map colouring). The graph formulation is somewhat dual, but a planar graph does not need not be a suitable partition into regions, e.g. trees do not make sense before applying Lemma 5.2. Hence, the existence of a dual construction in the language of graph theory is non-trivial. Considering Theorem 5.3, however, the equivalence is easily checked. The transition from maps to graphs caused some confusion, but was treated in detail by Whitney [Saa72, Whi31]. Older papers study the map problem rather than the equivalent graph problem.

P.G. Tait proved the equivalence of Theorem 5.1 and 3-colouring the 1-skeleton of triangulations of the sphere [Saa72, Tai80].[55] Tait also proved the map-version of Proposition 5.11 and conjectured the existence of Hamiltonian cycles as discussed in Section 5.4. About 32 years later, in 1912, O. Veblen published a paper called "An Application of Modular Equations in Analysis Situs" as a response to Poincaré's "Analysis Situs" and its second supplement [Die09, Poi00, Veb12, Veb31]. At the time, there was no notion of homology groups

[54] In the case where every R_i is a triangle, we have to pick v_i and v_j that do not constitute a triangle before removing v.

[55] The classic language here would be a cubic map rather than a 1-skeleton.

and everything was expressed in terms of Betti numbers, e.g. the first version of Poincaré duality as motivated in Section 4.5. There was, in particular, no notion of neither changing coefficients nor going to cohomology. The early work of Poincaré focus on polyhedrons and their generalized geometry, in particular a boundary map was introduced. Given a manifold partitioned into a finite number of polyhedrons (instead of triangles), one may define the following boundary matrices from the associated CW-structure: $B = (b_{ij})$ and $A = (a_{ij})$, where b_{ij} equals 1 if and only if the i'th 1-cell is in the closure of the j'th 2-cell, otherwise b_{ij} is zero, and similarly, for a_{ij} between 1-cells and 0-cells. In the case where the entries are considered as elements in a field of characteristic two, we know that these are the boundary operators. In his paper, Veblen first studies these matrices over $\mathbb{Z}_2$, where he spends some time discussing kernels and cokernels, before considering the entries as elements of GF(4). He notices how the colours of maps may be considered as elements of GF(4), and use characteristic two to prove the following. Let $X^0 \subset X^1 \subset X^2$ be the cell decomposition of a map partitioned into polyhedrons, then colouring X^2 with four colours (in terms of the map problem) is equivalent to finding a $Z \in \mathrm{GF}(4)^{|X^2|}$ such that $BZ \in (\mathrm{GF}(4)^*)^{|X^1|}$. The special case where each polyhedron is a triangle follows from part of the proof of Proposition 5.4, and the argument is in fact the same; the image on an edge (1-cell), which is a sum of the elements corresponding to the two incident polyhedrons (2-cells), is non-zero if and only if the elements on the incident polyhedrons are distinct as the characteristic of GF(4) equals two. Of course, the boundary matrices of Poincaré relates to boundary maps in homology; the building blocks can be any choice of n-gons [Hat01].

The reformulations of Tait and Veblen seems to be directly related as both associate three distinct elements with edges of triangulations. Nonetheless, they have only been proven equivalent through Theorem 5.1. The proof of Proposition 5.4 shows how trivial first homology of S^2 directly associate the reformulations; $BZ \in (\mathrm{GF}(4)^*)^{|X^1|}$ is a 3-colouring of the edges and conversely a 3-colouring $Y \in \mathrm{GF}(4)^{|X^1|}$ is in the image as $AY = 0$. Veblen probably did not notice this due to the incomplete state of homology theory. Or put differently, the paper is an early motivation of changing coefficients in homology.

5.6 Topology and Colouring Graphs

We have seen that Theorem 5.1 is associated with triangulations of the sphere through Theorem 5.3, and the motivation for studying colourings originates from colouring actual maps. Further, when Heawood proved that five colours suffice to colour maps by reusing

Kempe's arguments, he also derived the Heawood numbers

$$H_g = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor, \quad g = 1, 2, 3, ...$$

The numbers give a sufficient number of colours when colouring graphs embeddable on a surface of genus g [Hea90, Saa72]. The proof is topological in nature as it relies on the Euler characteristic, but the argument does not extend to the case $g = 0$ (a sphere). Note that inserting $g = 0$ into the formulae gives $H_0 = 4$, such that Theorem 5.1 completes the Heawood numbers. Hence, it is reasonable to associate Theorem 5.1 with S^2. Combining arguments similar to those in Section 5.4, there is an easy proof of how five colours suffice to colour maps/planar graphs, and the argument rely heavily on the Euler characteristic.

Proposition 5.16. *Any planar graph is* 5*-colourable.*

Proof. Observation 5.15 is eligible in the case of five colours. Our strategy is to replicate the arguments preceding Observation 5.15 for 5-cycles. Assume

$$\Gamma = \{n \in \mathbb{N}_+ | \ \exists \text{ a non-5-colourable planar graph of size } n\} \neq \emptyset.$$

Take a minimal G, and consider a vertex, v, of minimal degree. Observation 5.15 gives $\deg_G(v) = 5$. The removal of v enables a homomorphism between two of the five vertices adjacent to v, showing how four colours is enough to colour the vertices adjacent to v; apply the argument involving Observation 2.14. Consequently, we may extend the colouring to a 5-colouring of G. A contradiction.[56] □

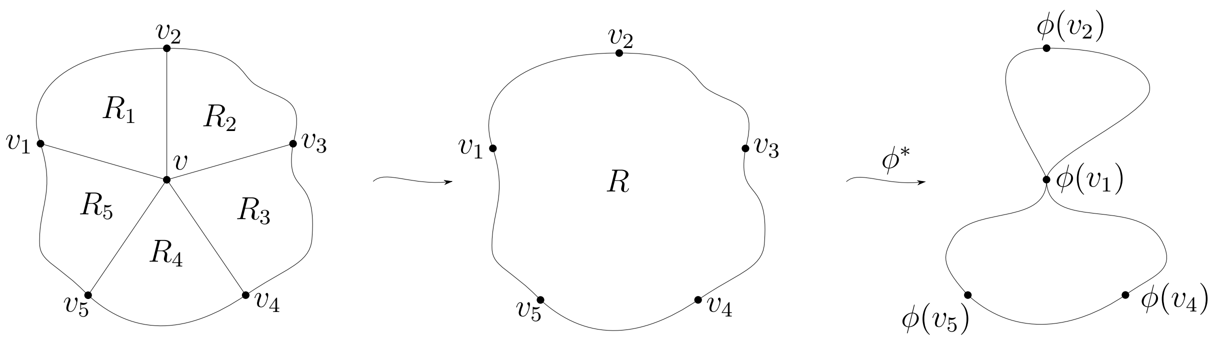


Figure 36: Planar graphs are 5-colourable.

The topology of S^2 is not fully utilized in the discussion of Theorem 5.1. More precisely, one may restate the above theory in $\mathbb{R}^2$ through the stereographic projection; they are not

[56] Note that these arguments cannot be extended to show Theorem 5.1; the vertex of degree five is coloured by a fifth colour.

homeomorphic, they do not even have the same homology ($H_2(\mathbb{R}^2) = 0$). In particular, $H_2(S^2) \simeq \mathbb{Z}$ does not seem to be helpful. We are interested in an element in the kernel of d_2, the coboundary operator, and thus surjectivity does not seem to be important.

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