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Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology. The eleven papers contained within represents the results of my research activities in the period spanning from January 2014 to February 2017.

I thank my co-authors for sharing their ideas and efforts with me. Additional gratitude is extended to Kristian Seip for highly effective advice, to Frédéric Bayart for inviting me to spend the spring term of 2015 at the Université Blaise Pascal and to Karl-Mikael Perfekt for his continuous mathematical collaboration.

Ole Fredrik Brevig
Trondheim, 2017

Contents

Introduction

The research contained in the papers that constitute the present thesis is primarily related to questions arising from operator-related function theory in spaces of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad s = \sigma + it. \tag{1}$$

This subject combines different fields of mathematics such as analytic number theory, complex analysis in one and several variables, ergodic theory, functional analysis, harmonic analysis and probability theory. The modern treatment of the subject was initiated in a 1997 paper by Hedenmalm, Lindqvist and Seip [**13**], which relies on a century-old insight of Harald Bohr [**4**] on the interaction between Dirichlet series and analysis in polydiscs. Essentially, Bohr's point of view is that each prime number p_j^{-s} should be considered an independent complex variable z_j, which associates the Dirichlet series (1) to a power series in an infinite number of variables.

In the first decade following [**13**], the elements of various spaces of Dirichlet series viewed as analytic functions in the half-plane

$$\mathbb{C}_{1/2} = \{s = \sigma + it \,:\, \sigma > 1/2\}$$

were studied intensively by several authors [**14, 24, 25, 26, 30, 32**]. Among the investigated topics were boundary limits, convergence properties, interpolation problems, local embeddings, partial sums and zero sets. Independently of these developments, pseudomoments of the Riemann zeta function were introduced in [**6**]. Intended as tractable analogues to the classical moments investigated by Hardy and Littlewood [**12**] and Ingham [**19**], we now interpret these pseudomoments as Hardy space norms of the partial sums of the Riemann zeta function on the critical line $\sigma = 1/2$.

It has consistently been found that the *additive* structure of the integers plays a role whenever questions and problems arising from the half-plane point of view are investigated. Papers 1, 4, 5 and 11 in the present thesis contain contributions to this aspect of the theory.

Concurrently with the investigations discussed above, a nascent operator theory in spaces of Dirichlet series was being developed. Composition operators were first studied by Gordon and Hedenmalm in the pioneering paper [**11**], which incited a flurry of activity [**2, 3, 9, 20, 28, 29**]. Papers 2, 3 and 4 in the present thesis are concerned with composition operators.

In another direction, Helson initiated the study of multiplicative Hankel forms [**15, 16, 17**]. For a sequence $\varrho \in \ell^2$, consider the bilinear form

$$\varrho(a,b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \varrho_{mn}, \tag{2}$$

where the subindex mn denotes the product of m and n. Helson observed that from Bohr's point of view, the form (2) is naturally realized as a (small) Hankel operator on the infinite polydisc, and enquired whether a classical theorem of Nehari [**23**] holds in this setting. He provided a positive answer in the case where the bilinear form (2) is Hilbert–Schmidt, but it was demonstrated by Ortega-Cerdà and Seip [**27**] that Nehari's theorem for multiplicative Hankel forms does not hold in full generality. Papers 6, 7, 8 and 9 in the present thesis are devoted to the study of multiplicative Hankel forms.

It is of interest to note that the positive result from [**16**] and the negative result from [**27**] both are obtained by multiplicatively iterating a suitable finite dimensional result. This phenomenon is typical for function spaces on the infinite polydisc, which are infinite products of their classical one-dimensional counterparts. In particular, questions and problems arising from the polydisc often enjoy multiplicative properties, which through Bohr's point of view can be viewed as statements about the *multiplicative* structure of the integers.

Function and operator theory in the unit disc clearly constitutes a major source of inspiration for the development of the corresponding theory for Dirichlet series. However, we also observe many new and interesting phenomena, which are often related to analytic number theory or appear as a consequence of the infinite dimensional nature of the spaces considered. Many of the classical objects exhibit different and often surprising properties. Moreover, several of the powerful tools employed in the development of the classical theory, such as duality arguments and inner-outer factorization, do not exist in the Dirichlet series setting. In practice, this means that results are often proved by novel combinations of the various fields of mathematics mentioned above. We refer to [**31**] for an overview of some of the recent developments and related open problems.

We have found that interactions between number theory and operator theory appear naturally when investigating operators that rely on both the *additive* and the *multiplicative* structures of the integers. This is featured most prominently in Paper 10, where Volterra operators defined by multiplication, differentiation and integration are studied. Other examples are found in Papers 7 and 11.

This introduction contains four additional sections. The first two sections contain a minimal amount of background material in order to prepare the reader for the papers that follow. We first recall the definitions of Bergman and Hardy spaces of the unit disc and exemplify the deep connections between function theory and operator theory. In the second section, we explain how spaces of Dirichlet series can be defined and their connection to analysis on the polydisc through Bohr's point of view. The third section contains an overview of the thesis, while the final section is comprised of some editorial remarks.

1. Function spaces in the unit disc

Function spaces in the unit disc and the related operator theory constitutes a classical topic, and we refer generally to the monographs **[7, 8, 33]**. In this introductory section we present suitable definitions of Bergman spaces and Hardy spaces, which will serve as a backdrop for the exposition of function spaces of Dirichlet series contained in the following section.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ its boundary. For $\alpha > 1$, consider the measure

$$dm_\alpha(z) = (\alpha - 1)\left(1 - |z|^2\right)^{\alpha-2}\,\frac{dxdy}{\pi}. \tag{3}$$

Note that $m_\alpha(\mathbb{D}) = 1$. For $0 < p < \infty$ and $\alpha > 1$, we take the Bergman space A^p_α to be the closure of the set of analytic polynomials in the (quasi)-norm

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p\,dm_\alpha(z).$$

For the case $\alpha = 1$, we let m_1 denote the normalized Lebesgue measure on $\mathbb{T}$ and define the Hardy space H^p, for $0 < p < \infty$, as the closure of analytic polynomials in $L^p(\mathbb{T})$. This definition is justified by the fact that

$$\lim_{\alpha \to 1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}.$$

Among the many interesting properties of Bergman spaces and Hardy spaces are their rich function-related operator theory. In many cases, one can characterize properties such as boundedness and compactness by investigating the *symbol* generating an operator, rather than the operator itself. Let us now look at two specific examples, compactness of composition operators on Bergman spaces and boundedness of Hankel forms on Hardy spaces.

Composition operators on Bergman spaces and Hardy spaces are generated by analytic self-maps ϕ of the unit disc, for which the composition operator is defined by $\mathscr{C}_\phi(f) = f \circ \phi$. It follows from the closed graph theorem that every composition operator is bounded and the following sharp upper bound for the norm is easily deduced from Littlewood's subordination principle [**21**].

$$\|\mathscr{C}_\phi f\|_{A^p_\alpha} \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{\alpha/p} \|f\|_{A^p_\alpha}, \tag{4}$$

where we in (4) include the case $\alpha = 1$ by letting A^p_1 denote H^p. For the Bergman spaces A^2_α, it can be shown that the operator $\mathscr{C}_\phi$ is compact if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\phi(z)|^2} = 0. \tag{5}$$

For the Hardy space H^2 an analogous statement to (5) can be made in terms of the Nevanlinna counting function N_ϕ.

Given a symbol $\varphi \in H^2$, we can define a (possibly unbounded) Hankel form on $H^2 \times H^2$ by

$$H_\varphi(fg) = \langle fg, \varphi\rangle_{L^2(\mathbb{T})} = \sum_{m=0}^\infty \sum_{n=0}^\infty a_m b_n \varrho_{m+n},$$

where we in the final equality assumed that the coefficients of f, g and φ are a_m, b_n and $\overline{\varrho_k}$, respectively. Clearly, if $\varphi \in L^\infty(\mathbb{T})$, then $\|H_\varphi\| \leq \|\varphi\|_{L^\infty(\mathbb{T})}$. However, the classical Hilbert inequality

$$\left|\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_m b_n}{m+n+1}\right| \leq \pi \left(\sum_{m=0}^\infty |a_m|^2\right)\left(\sum_{n=0}^\infty |b_n|^2\right)^{\frac{1}{2}}, \tag{6}$$

gives an example of a bounded Hankel form with the unbounded symbol

$$\varphi(z) = \frac{1}{z}\,\mathrm{Log}\left(\frac{1}{1-z}\right). \tag{7}$$

By orthogonality, we can replace φ by any symbol $\psi \in L^2(\mathbb{T})$ with $\widehat{\psi}(n) = \widehat{\varphi}(n)$ for $n = 0, 1, 2, \ldots$ while retaining the property that $\langle fg, \varphi\rangle_{L^2(\mathbb{T})} = \langle fg, \psi\rangle_{L^2(\mathbb{T})}$. In particular, for the symbol (7) we may choose

$$\psi(z) = -i\,\overline{z}\,\mathrm{Arg}(z),$$

for which clearly $\|\psi\|_{L^\infty(\mathbb{T})} = \pi$. Nehari's theorem [**23**] states that if $\varphi \in H^2$ generates a bounded Hankel form on $H^2 \times H^2$, we can always find a symbol $\psi \in L^\infty(\mathbb{T})$ such that $H_\varphi = H_\psi$ and $\|H_\varphi\| = \|\psi\|_{L^\infty(\mathbb{T})}$. By the Hanh–Banach theorem, this is equivalent to

$$\|H_\varphi\| = \|\varphi\|_{(H^1)^*}.$$

The statement holds true also for Hankel forms on Bergman spaces, but the norms of the Hankel form and the symbol are no longer identical, but merely equivalent (see Paper 9).

2. Function spaces of Dirichlet series

We begin with the Hardy spaces $\mathscr{H}^p$, which we for $0 < p < \infty$ define as the closure of the set of Dirichlet polynomials with respect to the Besicovitch norm

$$\|f\|_{\mathscr{H}^p}^p = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(it)|^p\,dt. \tag{8}$$

Let $\mathbb{C}_\theta = \{s = \sigma + it : \sigma > \theta\}$. The spaces $\mathscr{H}^p$ are Dirichlet series analogues to the classical Hardy spaces $H^p(\mathbb{D})$. A simple computation shows that if f denotes the Dirichlet series (1), then

$$\|f\|_{\mathscr{H}^2}^2 = \sum_{n=1}^{\infty} |a_n|^2.$$

It therefore follows from the Cauchy–Schwarz inequality that $\mathscr{H}^2$ is a space of (absolutely) convergent Dirichlet series in the half-plane $\mathbb{C}_{1/2}$, where its elements enjoy the pointwise estimate

$$|f(s)|^2 \leq \zeta(2\sigma)\|f\|_{\mathscr{H}^2}^2.$$

To see that $\theta = 1/2$ cannot be improved, consider $f(s) = \zeta(1/2 + \varepsilon + s)$ where ζ denotes the Riemann zeta function and $\varepsilon > 0$. These statements carry over to $\mathscr{H}^p$ for every $0 < p < \infty$. Thus we observe that while the norm of $\mathscr{H}^p$ is computed at the boundary of $\mathbb{C}_0$, its elements are analytic functions in the smaller half-plane $\mathbb{C}_{1/2}$.

Let us now turn to the Bohr correspondence. Every positive integer n can be factored uniquely into a product of prime factors

$$n = \prod_{j=1}^{\infty} p_j^{\kappa_j(n)}.$$

The factorization defines a bijection between the positive integers and the set of finite non-negative multi-indeces, by $\kappa(n) = (\kappa_1(n), \kappa_2(n), \kappa_3(n), \ldots)$. The Dirichlet series (1) now corresponds to its Bohr lift, which is the power series

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}, \qquad z = (z_1, z_2, z_3, \ldots).$$

The Birkhoff ergodic theorem for the Kronecker flow on $\mathbb{T}^\infty$ gives that if f is a Dirichlet polynomial, then

$$\|f\|_{\mathscr{H}^p} = \|\mathscr{B}f\|_{L^p(\mathbb{T}^\infty)}. \tag{9}$$

Here $L^p(\mathbb{T}^\infty)$ is defined with respect to the (countably) infinite product measure

$$\mathbf{m}_1(z) = m_1(z_1) \times m_1(z_2) \times m_1(z_3) \times \cdots, \tag{10}$$

which is equal to the Haar measure of $\mathbb{T}^\infty$ considered as a compact group. Observe that if $p = 2$, then it is easily verified that (9) holds. This leads to a simple proof of (9) through the Weierstrass approximation theorem, which is found in [**30**].

Through (9), we find that $\mathscr{B}$ defines a multiplicative isometric isomorphism from $\mathscr{H}^p$ to the Hardy space of the infinite polydisc $H^p(\mathbb{D}^\infty)$, where the latter space is defined as the closure of the set of analytic polynomials in $L^p(\mathbb{T}^\infty)$. The Bohr correspondence also allows us to define Bergman spaces of Dirichlet series as the closure of Dirichlet polynomials with respect to the norm

$$\|f\|_{\mathscr{A}^p_\alpha}^p = \int_{\mathbb{D}^\infty} |\mathscr{B}f(z)|^p \, d\mathbf{m}_\alpha(z).$$

Here $\mathbf{m}_\alpha$ denotes the infinite product measure generated from (3) in the same way as (10) was generated from m_1. Let $d_\alpha(n)$ denote the coefficients of the Dirichlet series defined by $[\zeta(s)]^\alpha$. For instance $d_2(n)$ is the well-known divisor function $d(n)$. It now follows that if f is the Dirichlet series (1), then

$$\|f\|_{\mathscr{A}^2_\alpha}^2 = \sum_{n=1}^\infty \frac{|a_n|^2}{d_\alpha(n)}.$$

By Cauchy–Schwarz inequality we conclude that $\mathscr{A}^2_\alpha$ is a space of (absolutely) convergent Dirichlet series in the half-plane $\mathbb{C}_{1/2}$ for every $\alpha > 1$, a statement which also holds for $\mathscr{A}^p_\alpha$.

3. Overview of the thesis

This thesis is organized into three parts. The first part contains five papers concerned with the properties of elements in spaces of Dirichlet series viewed as functions in the half-plane $\mathbb{C}_{1/2}$. The second part deals with multiplicative Hankel forms and contains four papers. The last part consists of two relatively unrelated papers, the first dealing with Volterra operators and the second with pseudomoments.

It should be pointed out that there are several connections between the papers in the different parts. For instance, Section 4 of Paper 4 builds on Paper 7, while Theorem 3.4 in Paper 11 is an extension of Theorem 1 in Paper 4. Moreover, Paper 10 relies crucially on certain results from Paper 8. Here follows a brief summary of the contents of the three parts.

Part 1: Composition operators and local embeddings. One of the most important problems when considering $\mathscr{H}^p$ as a space of analytic functions in the half-plane $\mathbb{C}_{1/2}$ is whether they are locally embedded in the usual Hardy spaces of $\mathbb{C}_{1/2}$. Equivalently, is there a constant $C_p \geq 1$ such that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |f(1/2+it)|^p \, \frac{dt}{1+t^2} \leq C_p \|f\|_{\mathscr{H}^p}^p \tag{11}$$

for every $f \in \mathscr{H}^p$? Ever since [**13**] it has been known that the answer is yes when $p = 2, 4, 6, \ldots$, but the problem remains open for other $0 < p < \infty$. In Paper 5, we give a simple and direct proof of (11) for $p = 2, 4, 6, \ldots$ and identify the optimal constant $C_p = 2$. A similar question can be asked for Bergman spaces of Dirichlet series, but nothing is known unless p is an even integer [**24**]. In Paper 1, we show that the bounded zero sequences of certain Hilbert spaces of Dirichlet series are the same as the spaces they are optimally embedded into.

In Paper 2, we use embeddings from [**24**] to extend the Gordon–Hedenmalm theorem on composition operators [**11**] to Bergman spaces. The arguments of [**11**] and Paper 2 rely on suitably applying the one-dimensional result (4). As explained in [**3**, Sec. 3], we cannot generally use one-dimensional results such as (5) to study compactness of composition operators on spaces of Dirichlet series. This means that the question of compactness is considerably more difficult than the question of boundedness.

Paper 3 is devoted to the study of compact composition operators on $\mathscr{H}^2$. We are able to completely describe the compact composition operators generated by polynomial symbols of degree 1 or 2 through analysis on the polydisc and a result from [**29**]. In Paper 4, we observe that the Gordon–Hedenmalm theorem for $\mathscr{H}^p$ in fact is equivalent to the embedding (11). We also prove a weaker version of (11), which is employed to give the first non-trivial examples of bounded composition operators on $\mathscr{H}^p$ generated by polynomial symbols.

Part 2: Multiplicative Hankel forms. In Paper 6, we improve on the construction from [**27**] to show that there are multiplicative Hankel forms (2) in the Schatten class S_p for every $p > p_0$, where

$$p_0 = \left(1 - \frac{\log \pi}{\log 4}\right)^{-1} = 5.7388\ldots,$$

that do not satisfy Nehari's theorem. The result from [**16**] states that Nehari's theorem holds in the Hilbert–Schmidt class S_2. Hence it remains an interesting problem to find the biggest p such that Nehari's theorem holds for multiplicative Hankel forms in S_p. We know that that the optimal p satisfies $2 \leq p \leq p_0$.

In Paper 7, we identify and study a multiplicative analogue of the Hilbert inequality (6) whose analytic symbol is the primitive of the Riemann zeta function

$$\varphi(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}\, n^{-s}.$$

We are unable to verify if the multiplicative Hilbert matrix satisfies Nehari's theorem, that is whether $\varphi \in (\mathscr{H}^1)^*$. However, we demonstrate that if (11) holds for $p = 1$, then this is the case. Note also that in Paper 4 it is shown that $\varphi \in (\mathscr{H}^p)^*$ for every $p > 1$.

In Paper 8, we discuss certain technical details about weak product spaces of Dirichlet series. We prove a square function characterization of $\mathscr{H}^p$ and formulate a Schur multiplier problem related to skew products. The main goal of Paper 9 is to investigate multiplicative Hankel forms on Bergman spaces, but we also find several counter-examples on the infinite polydisc to well-known finite dimensional results regarding Carleson measures.

Part 3: Volterra operators and pseudomoments. In Paper 10, we study Volterra operators

$$(\mathbf{T}_g f)(s) = -\int_s^\infty f(w)g'(w)\,dw$$

acting on Hardy spaces of Dirichlet series for some symbol $g \in \mathscr{H}^2$. The space of symbols generating bounded Volterra operators on $\mathscr{H}^2$ serves as a promising candidate for a BMOA–type space of Dirichlet series. We prove that it lies between the classical spaces $\mathrm{BMOA}(\mathbb{C}_0)$ and $\mathrm{BMOA}(\mathbb{C}_{1/2})$ and that its elements satisfy a John–Nirenberg type inequality. When investigating symbols such that $1 + g'(s)$ can be represented by an Euler product, we find connections to two number theoretic papers of Hilberdink [**18**] and Gál [**10**]. We are able to find a symbol of this type which generates a Volterra operator which is bounded, but not compact. This symbol converges in $\mathbb{C}_0$, but fails spectacularly to have bounded mean oscillation on the imaginary axis. We also study m-homogeneous symbols, and prove results related to those obtained for multipliers in [**1, 22**].

The pth pseudomoment of the Dirichlet series (1) is the sequence

$$\left\|\sum_{n=1}^N a_n n^{-s}\right\|_{\mathscr{H}^p}^p.$$

In [**6**], precise asymptotics as $N \to \infty$ for the pseudomoments of $\zeta(1/2+s)$ were computed for even integers p. For general values of $0 < p < \infty$, upper and lower bounds were found in [**5**], and we improve these estimates for $p \geq 2$ in Paper 11. It is interesting to note that the improved estimates are obtained after replacing an additive technique (partial sums of Euler products) with a multiplicative technique (Hardy–Littlewood inequalities).

In the opposite direction, Paper 11 also contains an example related to the zeta function which shows that for small $0 < p < 1$, the multiplicative technique does not always provide the correct asymptotics. In fact, Paper 11 contains several results regarding the Hardy spaces $\mathscr{H}^p$ for $0 < p < 1$ such as estimates for coefficients and partial sums. When applying multiplicative techniques to investigate these additive problems, we encounter again the intriguing interplay between the additive and multiplicative structures of the integers.

4. Editorial remarks

The papers included in this thesis represents their final preprinted version, with one notable exception. An additional section has been added to Paper 4. This section contains a sharper version of the necessary and sufficient conditions for bounded zero sequences for functions in $\mathscr{H}^p$ found in Section 3 of Paper 1 and [**32**, Sec. 4], respectively. The new results are the best possible we can expect to obtain from Hilbert space methods.

Several typographical adjustments has been made in order to accommodate the change to the `B5` format of the thesis. The bibliographies have also been revised and updated. Some effort has been made to make the notation employed in the different papers as uniform as possible, but there are still some lingering inconsistencies and constant vigilance is advised.

In particular, we would like to make clear some facts about the two scales of Bergman–type Hilbert spaces appearing in the various papers. For the Dirichlet series (1) consider the spaces defined by the norms

$$\|f\|_{\mathscr{D}_\alpha}^2 = \sum_{n=1}^\infty \frac{|a_n|^2}{[d(n)]^\alpha},$$
$$\|f\|_{\mathscr{A}_\beta^2}^2 = \sum_{n=1}^\infty \frac{|a_n|^2}{d_\beta(n)}.$$

Here $d(n) = d_2(n)$ denotes the usual divisor function, $\alpha \geq 0$ and $\beta \geq 1$. In particular $\mathscr{D}_1 = \mathscr{A}_2^2$ and $\mathscr{D}_0 = \mathscr{A}_1^2 = \mathscr{H}^2$, and these are the only cases of equality. Note that $\mathscr{D}_\alpha$ appear in Papers 1, 2, and 4, and also in [**5, 32**], while $\mathscr{A}_\beta^2$ appear in Paper 9 and (implicitly) in Paper 11. Both scales of spaces appear as examples in [**24**].

The scale $\mathscr{A}_\beta^2$ has several advantages compared to $\mathscr{D}_\alpha$. First, its reproducing kernels are simply $[\zeta(s)]^\beta$, while the reproducing kernels of $\mathscr{D}_\alpha$ are perturbations of $[\zeta(s)]^{2^\alpha}$. This is the reason why the scale $\mathscr{A}_\beta^2$ appears naturally when studying multiplicative Hankel forms in Paper 9. Note also that the results of this paper cannot generally be reproved for $\mathscr{D}_\alpha$. Conversely, every result for $\mathscr{D}_\alpha$ proved in Papers 1, 2, 4 and in [**5, 32**] can with only minor modification be reproved for $\mathscr{A}_\beta^2$, replacing α with $\beta - 1$ and 2^α with β in the various statements. Moreover, the necessity of a Möbius factor in an inequality for $\mathscr{D}_\alpha$ discussed on [**5**, p. 203] does not apply to the corresponding statement for $\mathscr{A}_\beta^2$.

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Part 1

Composition operators and local embeddings

Paper 1

Zeros of functions in Bergman–type Hilbert spaces of Dirichlet series

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ZEROS OF FUNCTIONS IN BERGMAN–TYPE HILBERT SPACES OF DIRICHLET SERIES

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ABSTRACT. For a real number α the Hilbert space $\mathscr{D}_\alpha$ consists of those Dirichlet series $\sum_{n=1}^\infty a_n/n^s$ for which

$$\sum_{n=1}^\infty \frac{|a_n|^2}{[d(n)]^\alpha} < \infty,$$

where $d(n)$ denotes the number of divisors of n. We extend a theorem of Seip on the bounded zero sequences of functions in $\mathscr{D}_\alpha$ to the case $\alpha > 0$. Generalizations to other weighted spaces of Dirichlet series are also discussed, as are partial results on the zeros of functions in the Hardy spaces of Dirichlet series $\mathscr{H}^p$, for $1 \leq p < 2$.

1. INTRODUCTION

Let $d(n)$ denote the divisor function let α be a real number. We are interested in the following Hilbert spaces of Dirichlet series:

$$\mathscr{D}_\alpha = \left\{ f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s} \ : \ \|f\|_{\mathscr{D}_\alpha}^2 = \sum_{n=1}^\infty \frac{|a_n|^2}{[d(n)]^\alpha} < \infty \right\}.$$

The functions of $\mathscr{D}_\alpha$ are analytic in $\mathbb{C}_{1/2} = \{s = \sigma + it \ : \ \sigma > 1/2\}$. Bounded Dirichlet series are almost periodic, and this implies that they have either no zeros or infinitely many zeros, as observed by Olsen and Seip in [10]. This leads us to restrict our investigations to bounded zero sequences for spaces of Dirichlet series. In [13], Seip studied bounded zero sequences for $\mathscr{D}_\alpha$, when $\alpha \leq 0$. This includes the Hardy–type ($\alpha = 0$) and Dirichlet–type ($\alpha < 0$) spaces. The topic of the present work is the Bergman–type spaces ($\alpha > 0$).

Let us therefore introduce the weighted Bergman spaces in the half-plane, A_β. For $\beta > 0$, these spaces consists of functions F which are analytic in $\mathbb{C}_{1/2}$ and satisfy

$$\|F\|_{A_\beta} = \left(\int_{\mathbb{C}_{1/2}} |F(s)|^2 \left(\sigma - \frac{1}{2}\right)^{\beta - 1} dm(s) \right)^{\frac{1}{2}} < \infty.$$

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It was shown by Olsen in [9] that the local behavior of the spaces $\mathscr{D}_\alpha$ are similar to the spaces A_β, where $\beta = 2^\alpha - 1$. This relationship between α and β will be retained throughout this paper.

For a class of analytic functions $\mathscr{C}$ on some domain $\Omega \subseteq \mathbb{C}$, we will say that a sequence S of not necessarily distinct numbers in Ω is a zero sequence for $\mathscr{C}$ if there is some non-trivial $F \in \mathscr{C}$ vanishing on S, taking into account multiplicities. We will let $Z(\mathscr{C})$ denote the set of all zero sequences for $\mathscr{C}$.

A result proved by Horowitz in [6] shows that if $\mathscr{C} = A_\beta$ we may assume that F vanishes precisely on $S \in Z(A_\beta)$, i.e. F has no extraneous zeros in $\mathbb{C}_{1/2}$. We will exploit this fact to prove our main result.

Theorem 1. *Suppose $S = (\sigma_j + it_j)$ is a bounded sequence of points in $\mathbb{C}_{1/2}$ and that $\alpha > 0$. Then there is a non-trivial function in $\mathscr{D}_\alpha$ vanishing on S if and only if $S \in Z(A_\beta)$.*

The "only if" part follows from the local embedding of $\mathscr{D}_\alpha$ into A_β of Theorem 1 and Example 4 from [9]. To prove the "if" part, we will adapt the methods of [13], where an analogous result for $\alpha \leq 0$ was obtained.

The "if" part can essentially be split into two steps. The first step is a discretization lemma, which depends on the properties of $\mathscr{D}_\alpha$ — or rather the weights $[d(n)]^\alpha$. The second step is an iterative scheme, where the properties of A_β become more prominent.

Comparing this with [13], the first step is somewhat harder, since we require very precise estimates on the weights as α grows to infinity. The second step is considerably easier, mainly due to the fact that the norms of A_β are easier to work with than those of the Dirichlet spaces used in [13].

We will use the notation $f(x) \ll g(x)$ to indicate that there is some constant $C > 0$ so that $|f(x)| \leq Cg(x)$. Sometimes the constant C may depend on certain parameters, and this will be specified in the text. Moreover, we write $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

2. Proof of Theorem 1

We begin with the Paley–Wiener representation of functions $F \in A_\beta$, and seek to construct a Dirichlet series $f \in \mathscr{D}_\alpha$ which approximates F.

Lemma 2 (Paley–Wiener Representation). *A_β is isometrically isomorphic to*

$$L^2_\beta = \left\{ \phi \text{ measurable on } [0,\infty) \,:\, \|\phi\|^2_{L^2_\beta} = \frac{2\pi\Gamma(\beta)}{2^\beta} \int_0^\infty |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} < \infty \right\},$$

under the Laplace transformation

$$F(s) = \int_0^\infty \phi(\xi) e^{-(s-1/2)\xi}\, d\xi.$$

Proof. A proof can be found in [2]. □

The other ingredient needed for the discretization lemma is estimates on the growth of $[d(n)]^\alpha$. We will partition the integers into blocks and use an average order type estimate. To prove this estimate, we will need the precise form of a formula stated by Ramanujan [11] and proved by Wilson [15]: For any real number α and any integer $\nu > 2^\alpha - 2$, we have

$$(1) \qquad D_\alpha(x) = \sum_{n \le x} [d(n)]^\alpha = x(\log x)^{2^\alpha - 1} \left(\sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^\lambda} + O\left(\frac{1}{(\log x)^{\nu+1}} \right) \right).$$

Wilson's proof of (1) can be considered at special case of Selberg–Delange method. For more about the Selberg–Delange method, we refer to Chapter II.5 of [14]. However, we mention that the coefficients A_λ depend on the coefficients of the Dirichlet series ϕ_α, which we implicitly define through the relation

$$(2) \qquad \zeta_\alpha(s) = \sum_{n=1}^{\infty} [d(n)]^\alpha n^{-s} = \prod_{j=1}^{\infty} \left(1 + \sum_{k=1}^{\infty} (k+1)^\alpha p_j^{-sk} \right) = [\zeta(s)]^{2^\alpha} \phi_\alpha(s).$$

The partial sums of the coefficients of ζ_α are estimated through Perron's formula and the residue theorem. While (2) is only valid for $\mathrm{Re}(s) > 1$, a simple computation using Euler products shows that ϕ_α converges for $\mathrm{Re}(s) > 1/2$, and thus Theorem 5 of [14] may be applied. In particular, the coefficients A_λ depend on the coefficients of ϕ_α, and since the coefficients of ϕ_α depend continuously on α, so does A_λ in (1).

Lemma 3. *Let α be a real number and $0 < \gamma < 1$. Then*

$$(3) \qquad \sum_{j^\gamma \le \log n \le (j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \asymp j^{\gamma 2^\alpha - 1},$$

as $j \to \infty$. The implied constants may depend on α and γ.

Proof. We will first assume that 2^α is not an integer. Fix ν such that $\nu > 2^\alpha - 1$ and $\nu > 1/\gamma - 1$. We use Abel summation to rewrite

$$(4) \qquad \sum_{y < n \le x} \frac{[d(n)]^\alpha}{n} = \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} + \int_y^x \frac{D_\alpha(z)}{z^2}\, dz.$$

By using (1) and the fact that $2^\alpha - 1 - \nu < 0$ we perform some standard calculations to estimate

$$\begin{aligned}\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} &= \sum_{\lambda=0}^{\nu} A_\lambda \left((\log x)^{2^\alpha - 1 - \lambda} - (\log y)^{2^\alpha - 1 - \lambda} \right) \\ &\quad + O\left((\log y)^{2^\alpha - 2 - \nu} \right), \\ \int_y^x \frac{D_\alpha(z)}{z^2}\, dz &= \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{2^\alpha - \lambda} \left((\log x)^{2^\alpha - \lambda} - (\log y)^{2^\alpha - \lambda} \right) \\ &\quad + O\left((\log y)^{2^\alpha - 1 - \nu} \right).\end{aligned}$$

Let us now take $x = \exp\left((j+1)^\gamma\right)$ and $y = \exp\left(j^\gamma\right)$. For any exponent η it is clear that

$$(\log x)^\eta - (\log y)^\eta = \gamma\eta j^{\gamma\eta - 1} \left(1 + O\left(\frac{1}{j} \right) \right).$$

Hence we have

$$\begin{aligned}\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} &\asymp \sum_{\lambda=0}^{\nu} A_\lambda (\gamma(2^\alpha - 1 - \lambda)) j^{\gamma(2^\alpha - 1 - \lambda) - 1} + O\left(j^{\gamma(2^\alpha - 2 - \nu)} \right), \\ \int_y^x \frac{D_\alpha(z)}{z^2}\, dz &\asymp \sum_{\lambda=0}^{\nu} A_\lambda j^{\gamma(2^\alpha - \lambda) - 1} + O\left(j^{\gamma(2^\alpha - 1 - \nu)} \right).\end{aligned}$$

We combine these estimates with (4) to obtain

$$\begin{aligned}&\sum_{j^\gamma \le \log n \le (j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \\ &\qquad \asymp j^{\gamma 2^\alpha - 1} \left(A_0 + \sum_{\lambda=1}^{\nu} \frac{B_\lambda}{j^{\gamma\lambda}} + O\left(\frac{1}{j^{\gamma 2^\alpha - 1 - \gamma(2^\alpha - 1 - \nu)}} \right) \right),\end{aligned} \tag{5}$$

where $B_\lambda = A_\lambda + A_{\lambda - 1}\gamma\left(2^\alpha - \lambda\right)$. This proves (3) since $\nu > 1/\gamma - 1$. By continuity on both sides of (5), the assumption that 2^α is not an integer may be dropped. □

The parameter $0 < \gamma < 1$ will be used to control the "block size" in our partition of the integers. It will become apparent that as α grows to infinity, we must be able to let γ tend to 0. In [13] it was sufficient to have a similar estimate only for $1/2 < \gamma < 1$.

Lemma 4 (Discretization Lemma)**.** *Let $\alpha > 0$ and let N be a sufficiently large positive integer. Then there exists positive constants A and B (depending on α,* *but not N) such that the following holds: For every function $\phi \in L^2_\beta$ supported on $[\log N, \infty)$, there is a function of the form*

$$f(s) = \sum_{n=N}^{\infty} \frac{a_n}{n^s}$$

in $\mathscr{D}_\alpha$ such that $\|f\|_{\mathscr{D}_\alpha} \le A\|\phi\|_{L^2_\beta}$. Moreover, f may be chosen so that

$$\Phi(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s-1/2)\xi}\, d\xi - f(s)$$

enjoys the estimate

$$|\Phi(s)| \le B|s-1/2| N^{-\sigma+1/2} (\log N)^{-1} \|\phi\|_{L^2_\beta},$$

in $\mathbb{C}_{1/2}$.

Proof. Let $\gamma = 2/(4+2^\alpha)$ and let J be the largest integer smaller than $(\log N)^{1/\gamma}$. For $j \ge J$, let n_j be the smallest integer n such that $e^{j^\gamma} \le n$. When γ is small it is possible that $n_j = n_{j+1}$. This can be avoided by taking N sufficiently large. Set $\xi_{n_j} = j^\gamma$ and for $n_j < n \le n_{j+1}$ iteratively choose ξ_n such that

$$\frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta+1} = A_j \frac{[d(n)]^\alpha}{n}, \tag{6}$$

where A_j is chosen so that $\xi_{n_{j+1}} = (j+1)^\gamma$. Clearly, Lemma 3 implies that A_j is bounded as $j \to \infty$. Let us set

$$a_n = \sqrt{n} \int_{\xi_n}^{\xi_{n+1}} \phi(\xi)\, d\xi.$$

A simple computation using the Cauchy–Schwarz inequality shows that

$$|a_n|^2 = n \left| \int_{\xi_n}^{\xi_{n+1}} \phi(\xi)\, d\xi \right|^2 \le n \cdot \frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta+1} \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta}.$$

In view of (6) it is clear that $\|f\|_{\mathscr{D}_\alpha} \le A\|\phi\|_{L^2_\beta}$. Now, if $n_j \le n \le n_{j+1}$ and $\xi \in [\xi_{n_j}, \xi_{n_{j+1}}]$ we see that

$$\left| e^{-(s-1/2)} - n^{-(s-1/2)} \right| \le N^{-\sigma+1/2} |s-1/2| j^{\gamma-1}. \tag{7}$$

Then, by (7) and the Cauchy–Schwarz inequality

$$|\Phi(s)| \le N^{-\sigma+1/2}|s-1/2| \sum_{j=J}^{\infty} j^{\gamma-1} \sum_{n=n_j}^{n_{j+1}-1} \left(\frac{\xi_{n+1}^\beta - \xi_n^\beta}{\beta} \right)^{\frac{1}{2}} \left(\int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

By using the Cauchy–Schwarz inequality again with (6) we get

$$|\Phi(s)| \ll N^{-\sigma+1/2}|s-1/2| \sum_{j=J}^{\infty} j^{\gamma-1} \left(\sum_{n=n_j}^{n_{j+1}-1} \frac{[d(n)]^\alpha}{n} \right)^{\frac{1}{2}} \left(\int_{\xi_{n_j}}^{\xi_{n_{j+1}}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

Now Lemma 3 and the Cauchy–Schwarz inequality yield

$$|\Phi(s)| \ll N^{-\sigma+1/2}|s-1/2| \left(\sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma-3} \right)^{\frac{1}{2}} \left(\int_{\log N}^{\infty} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta} \right)^{\frac{1}{2}}.$$

The series converges since $\gamma < 2/(2+2^\alpha)$. The proof is completed by a standard estimate of the convergent series,

$$\left(\sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma-3} \right)^{\frac{1}{2}} \ll (\log N)^{((2+2^\alpha)\gamma-2)/(2\gamma)} = (\log N)^{-1},$$

where we used that $J \asymp (\log N)^{1/\gamma}$. □

The final result needed for the iterative scheme is the following simple lemma on the $\overline{\partial}$-equation. We omit the proof, which is obvious.

Lemma 5. *Suppose g is a continuous function on $\mathbb{C}_{1/2}$, supported on*

$$\Omega(R,\tau) = \{s = \sigma + it \,:\, 1/2 \le \sigma \le 1/2+\tau,\ -R \le t \le R\},$$

for some positive real numbers τ and R. Then

$$u(s) = \frac{1}{\pi} \int_\Omega \frac{g(w)}{s-w}\, dm(w)$$

solves $\overline{\partial} u = g$ in $\mathbb{C}_{1/2}$ and satisfies $\|u\|_\infty \le C_\Omega \|g\|_\infty$.

We have now collected all our preliminary results and are ready to begin the proof of Theorem 1. For any positive integer N we set $E_N(s) = N^{-s+1/2}$ and consider the space $E_N A_\beta$. By a substitution it is evident that any $F \in E_N A_\beta$ can be represented as

$$F(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s-1/2)\xi}\, d\xi$$

for some $\phi \in L^2_\beta[\log N, \infty)$, in view of Lemma 2.

Final step in the proof of Theorem 1. Let us fix $\alpha > 0$ and a bounded sequence $S = (\sigma_j + it_j) \in Z(A_\beta)$. From this point all constants may depend on α and S. Since S is bounded we may assume $S \subset \Omega(R-2, \tau-2)$ for some R, $\tau > 2$. Let Θ be some smooth function defined on $\overline{\mathbb{C}_{1/2}}$ with the following properties:

- Θ is supported on $\Omega(R,\tau)$,
- $\Theta(s) = 1$ for $s \in \Omega(R-1, \tau-1)$,
- $|\overline{\partial}\Theta(s)| \le 2$.

Let $G \in A_\beta$ vanish precisely on S and assume furthermore that $\|G\|_{A_\beta} = 1$. Now, suppose that $F \in E_N A_\beta$, and let $f \in \mathscr{D}_\alpha$ be the function obtained by applying Lemma 4 to F, and $\Phi = F - f$. Moreover, let u denote the solution to the equation

$$\overline{\partial} u = \frac{\overline{\partial}(\Theta\Phi)}{GE_N}. \tag{8}$$

The right hand side of (8) is a smooth function compactly supported on $\Omega(R,\tau)$ since $|G(s)|$ is bounded from below where $\overline{\partial}\Theta(s) \neq 0$. We can use Lemma 5 and Lemma 2 to estimate

$$\|u\|_\infty \ll \left\| \frac{\overline{\partial}(\Theta\Phi)}{GE_N} \right\|_\infty \ll (\log N)^{-1} \|\phi\|_{L^2_\beta} = (\log N)^{-1} \|F\|_{A_\beta}. \tag{9}$$

We set $T_N F = \Theta\Phi - GE_N u$. The function $T_N F$ has the following properties:

- $T_N F(s) = \Phi(s)$ for $s \in S$,
- $T_N F$ is analytic in $\mathbb{C}_{1/2}$ since $\overline{\partial} T_N F(s) = 0$ for $s \in \mathbb{C}_{1/2}$,
- $T_N F \in E_N A_\beta$, by the compact support of Θ and the estimate (9).

Hence T_N defines an operator on $E_N A_\beta$. By the triangle inequality, Lemma 4 and the fact that Θ has compact support, it is clear that

$$\|T_N F\|_{A_\beta} \le \|\Theta\Phi\|_{A_\beta} + \|GE_N u\|_{A_\beta} \ll (\log N)^{-1} \|\phi\|_{L^2_\beta} + \|u\|_\infty \|G\|_{A_\beta}.$$

Since $\|G\|_{A_\beta} = 1$ and $\|\phi\|_{L^2_\beta} = \|F\|_{A_\beta}$ we have $\|T_N\| \ll (\log N)^{-1}$ in view of (9). Let N be large, but arbitrary, and define $F_0(s) = E_N(s)G(s)$. Then $F_0 \in E_N A_\beta$ and its norm in this space is ≤ 1. Set

$$F_j = T_N^j F_0.$$

Let f_j be the Dirichlet series of Lemma 4 obtained from F_j. Then $f_0 + F_1$ vanishes on S, since

$$f_0(s) + F_1(s) = f_0(s) + T_N F_0(s) = f_0(s) + F_0(s) - f_0(s) = F_0(s) = 0,$$

for $s \in S$, by the fact that $T_N F(s) = \Phi(s)$ for $s \in S$. Iteratively, the function $f_0 + f_1 + \cdots + f_j + F_{j+1}$ also vanishes on S. Define

$$f(s) = \sum_{j=0}^{\infty} f_j(s)$$

and choose N so large that $\|T_N\| < 1$ so that $\|F_j\|_{A_\beta} \to 0$ and, say

$$|f(1)| > \sum_{j=1}^{\infty} |f_j(1)|,$$

so that f is non-trivial in $\mathscr{D}_\alpha$ and vanishing on S. $\square$

By again following [13], we can modify the iterative scheme in the following way: Let $F \in A_\beta$ be arbitrary, and set $F_0 = F$. Using the algorithm in the same manner as above, we see that $F_1(s) + f_0(s) = F_0(s)$ for $s \in S$. Moreover,

$$F_{j+1}(s) + f_j(s) + f_{j-1}(s) + \cdots + f_0(s) = F(s),$$

for $s \in S$. Continuing as above, we obtain the following result:

Corollary 6. *Suppose $S = (\sigma_j + it_j) \in Z(A_\beta)$ is bounded. For every function $F \in A_\beta$ there is some $f \in \mathscr{D}_\alpha$ such that $f(s) = F(s)$ on S.*

We can extend Theorem 1 and Corollary 6 by considering different weights. Let $w = (w_1, w_2, \dots)$ be a non-negative weight. Define the Hilbert space of Dirichlet series $\mathscr{D}_w$ in the same manner as above, with the added convention that the basis vector n^{-s} is excluded if $w_n = 0$. Theorem 1 in [9] states that $\mathscr{D}_w$ embeds locally into A_β if and only if

$$\sum_{n \le x} w_n \ll x(\log x)^\beta, \tag{10}$$

where $\beta > 0$. By modifying the proof of our Theorem 1, we can obtain a similar result for $\mathscr{D}_w$ with respect to A_β provided we additionally have

$$\sum_{j^\gamma \le \log n \le (j+1)^\gamma} \frac{w_n}{n} \asymp j^{\gamma(\beta+1)-1}, \tag{11}$$

as $j \to \infty$, for some $0 < \gamma < 2/(3+\beta)$. Several of the weights considered in [9] are possible, but we only mention the case $w_n = (\log n)^\beta$ for $\beta > 0$. These spaces were introduced by McCarthy in [8]. It is easy to show that these weights satisfy (10) and (11) for any $0 < \gamma < 1$, and similar results with respect to A_β are obtained.

Remark. The embeddings of [9] extend to any $\beta \le 0$, in view of (10), and we get the Hardy space ($\beta = 0$) and Dirichlet spaces ($\beta < 0$) in the half-plane. We can extend the results in [13] in a similar manner as above. However, this is only possible for $-1 \le \beta < 0$. The method of [13] breaks down for $\beta < -1$ due to the fact that the norms of the corresponding Dirichlet spaces in the half-plane uses higher order derivatives and different estimates are needed.

3. Blaschke-type conditions for $\mathscr{D}_\alpha$ and $\mathscr{H}^p$

Now that we have identified the bounded zero sequences of $\mathscr{D}_\alpha$ as those of A_β, let us consider necessary and sufficient conditions for bounded zero sequences of A_β. The zero sequences of Bergman spaces in the unit disc $\mathbb{D}$ have attracted considerable attention. We refer to the monograph [3]. For $\beta > 0$, these are the spaces

$$A_\beta(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) \,:\, \|F\| = \int_{\mathbb{D}} |F(z)|^2 (1-|z|)^{\beta-1} dm(z) < \infty \right\}.$$

Results pertaining to zero sequences of $A_\beta(\mathbb{D})$ are relevant to our case since

$$\phi(s) = \frac{s-3/2}{s+1/2}$$

is a conformal mapping from $\mathbb{C}_{1/2}$ to $\mathbb{D}$, and

$$F \mapsto (s+1/2)^{-2(\beta+1)} F\left(\frac{s-3/2}{s+1/2}\right)$$

defines an isometric isomorphism from $A_\beta(\mathbb{D})$ to A_β. This implies that $S \in Z(A_\beta)$ if and only if $\phi(S) \in Z(A_\beta(\mathbb{D}))$. Since the Hardy space $H^2(\mathbb{D})$ is included in $A_\beta(\mathbb{D})$ for every $\beta > 0$, it is clear that the Blaschke condition

$$\sum_j (\sigma_j - 1/2) < \infty \tag{12}$$

is sufficient for bounded zero sequences of A_β. Moreover, Theorem 4.1 of [3] shows that the Blaschke condition (12) is both necessary and sufficient provided the bounded sequence S is contained in any cone $|t-t_0| \leq c(\sigma - 1/2)$. Unfortunately, the situation becomes more complicated in the general case and we do not have a precise Blaschke-type condition for bounded zero sequences. In fact, for every $\epsilon > 0$ and every A_β a necessary condition for bounded zero sequences is

$$\sum_j (\sigma_j - 1/2)^{1+\epsilon} < \infty, \tag{13}$$

by Corollary 4.8 of [3]. Clearly, this condition does not offer any insight into what happens as $\beta \to 0^+$. However, using the notion of density introduced by Korenblum in [7] it is possible to provide a generalized condition describing the geometrical information of the zero sequences of $A_\beta(\mathbb{D})$. The most precise results on Korenblum's density are obtained by Seip in [12]. We omit the details, only mentioning that this generalized condition in a certain sense tends to (12) when $\beta \to 0^+$.

The Hardy spaces of Dirichlet series $\mathscr{H}^p$, $1 \leq p < \infty$, can be defined as the closure of the set of all Dirichlet polynomials with respect to the norms

$$\left\| \sum_{n=1}^N \frac{a_n}{n^s} \right\|_{\mathscr{H}^p} = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^N \frac{a_n}{n^{it}} \right|^p dt \right)^{\frac{1}{p}}.$$

For the basic properties of these spaces we refer to [4] and [1]. However, we immediately observe that $\mathscr{H}^2 = \mathscr{D}_0$. In [13], the bounded zero sequences of the spaces $\mathscr{H}^p$, for $2 \leq p < \infty$, are studied. In particular, for $\mathscr{H}^2$ the Blaschke condition (12) is shown to be both necessary and sufficient. Results for $2 < p < \infty$ are obtained through embeddings $\mathscr{D}_\alpha \subset \mathscr{H}^p \subset \mathscr{H}^2$, where $\alpha < 0$ depends on p. The embedding of $\mathscr{H}^p$ into $\mathscr{H}^2$ implies that the Blaschke condition (12) is necessary for $\mathscr{H}^p$.

The sufficient conditions are obtained through a similar result as Theorem 1: For $\alpha < 0$, the spaces $\mathscr{D}_\alpha$ have the same bounded zero sequences as certain weighted Dirichlet spaces in $\mathbb{C}_{1/2}$. In particular, for $2 < p < \infty$ there is some $0 < \gamma < 1$ such that a sufficient condition for bounded zero sequences of $\mathscr{H}^p$ is

$$\sum_j (\sigma_j - 1/2)^{1-\gamma} < \infty, \tag{14}$$

and moreover $\gamma \to 0$ as $p \to 2^-$. We omit the details, which can be found in [13].

We will now consider the case $1 \leq p < 2$. That $\mathscr{H}^2 \subset \mathscr{H}^p \subseteq \mathscr{H}^1$ for $1 \leq p < 2$ is trivial, and this shows that (12) is a sufficient condition for bounded zero sequences of $\mathscr{H}^p$. In [5], Helson proved the beautiful inequality

$$\|f\|_{\mathscr{D}_1} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{H}^1}, \tag{15}$$

which implies that $\mathscr{H}^p \subset \mathscr{D}_1$. This shows that the Blaschke-type condition (13) is necessary for bounded zero sequences of $\mathscr{H}^p$, for every $\epsilon > 0$. Regrettably, this means we are unable to specify how the situation changes as $p \to 2^-$, in a manner similar to (14). However, if we again restrict S to the cone $|t - t_0| \leq c(\sigma - 1/2)$, the Blaschke condition (12) is both necessary and sufficient for bounded zero sequences of $\mathscr{H}^p$.

Remark. The Blaschke condition (12) is well-known to be necessary and sufficient for bounded zero sequences of the Hardy spaces $H^p(\mathbb{C}_{1/2})$. By a theorem in [4], $\mathscr{H}^2$ embeds locally into $H^2(\mathbb{C}_{1/2})$. This trivially extends to even integers p. Whether the local embedding extends to every $p \geq 1$ is an open question. Observe that if (12) is not the optimal necessary condition for bounded zero sequences of $\mathscr{H}^p$, when $1 \leq p < 2$, then the local embedding would be impossible for these p. However, since (14) is a sufficient condition for bounded zero sequences of $\mathscr{H}^p$ when $p \geq 2$, its optimality would not contradict the local embedding for these p.

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Paper 2

Composition operators on Bohr–Bergman spaces of Dirichlet series

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Paper 3

Compact composition operators with non-linear symbols on the H^2 space of Dirichlet series

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COMPACT COMPOSITION OPERATORS WITH NON-LINEAR SYMBOLS ON THE H^2 SPACE OF DIRICHLET SERIES

FRÉDÉRIC BAYART AND OLE FREDRIK BREVIG

Abstract. We investigate the compactness of composition operators on the Hardy space of Dirichlet series induced by a map $\varphi(s) = c_0 s + \varphi_0(s)$, where φ_0 is a Dirichlet polynomial. Our results depend heavily on the characteristic c_0 of φ and, when $c_0 = 0$, on both the degree of φ_0 and its local behaviour near a boundary point. We also study the approximation numbers for some of these operators. Our methods involve geometric estimates of Carleson measures and tools from differential geometry.

1. Introduction

A theorem of Gordon and Hedenmalm [9] describes the bounded composition operators on the Hilbert space $\mathscr{H}^2$ of Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

with square summable coefficients endowed with the norm $\|f\|_{\mathscr{H}^2}^2 := \sum_{n=1}^{\infty} |a_n|^2$. We let $\mathbb{C}_\theta$ denote the half-plane of complex numbers $s = \sigma + it$ with $\sigma > \theta$. The Dirichlet series in $\mathscr{H}^2$ represent analytic functions in $\mathbb{C}_{1/2}$ and a mapping φ of $\mathbb{C}_{1/2}$ into itself defines a function $\mathscr{C}_\varphi(f) := f \circ \varphi$ on $\mathbb{C}_{1/2}$, if $f \in \mathscr{H}^2$. The operator $\mathscr{C}_\varphi : \mathscr{H}^2 \to \mathscr{H}^2$ is well-defined and bounded if and only if φ is a member of the following class:

Definition. The *Gordon–Hedenmalm class*, denoted $\mathscr{G}$, is the set of functions $\varphi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ of the form

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \varphi_0(s), \tag{1}$$

where c_0 is a non-negative integer called the *characteristic* of φ, the Dirichlet series φ_0 converges uniformly in $\mathbb{C}_\varepsilon$ ($\varepsilon > 0$) and has the following mapping properties:

(a) If $c_0 = 0$, then $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.

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(b) If $c_0 \geq 1$, then either $\varphi_0 \equiv 0$ or $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_0$.

Since the paper of Gordon and Hedenmalm, several authors have studied the properties of composition operators acting on $\mathscr{H}^2$ or on similar spaces of Dirichlet series (see for instance [1, 2, 7, 8, 11]). In the present work, we are interested in the study of the compactness of $\mathscr{C}_\varphi$ when φ is a polynomial symbol, say

$$\varphi(s) = c_0 s + c_1 + \sum_{n=2}^{N} c_n n^{-s}, \tag{2}$$

and we implicitly assume that $\varphi \in \mathscr{G}$. The symbol φ is said to have *unrestricted range* if

$$\inf_{s \in \mathbb{C}_0} \operatorname{Re}(\varphi(s)) = \begin{cases} 1/2 & \text{if } c_0 = 0, \\ 0 & \text{if } c_0 \geq 1. \end{cases}$$

Correspondingly, if $\varphi(\mathbb{C}_0)$ is strictly contained in any smaller half-plane, we say that $\mathscr{C}_\varphi$ has *restricted range.* It is well-known that the composition operator $\mathscr{C}_\varphi$ is compact when φ has restricted range [1, Thm. 21]. In what follows, we will assume that φ has unrestricted range.

Definition. A set of integers $\Lambda \subseteq \mathbb{N} - \{1\}$ is called $\mathbb{Q}$-*independent* if the set $\{\log n : n \in \Lambda\}$ is linearly independent over $\mathbb{Q}$.

Symbols of the form (2) have been extensively studied in the *linear case*,

$$\varphi(s) = c_0 s + c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s}, \tag{3}$$

where the set $\{q_j\}$ is $\mathbb{Q}$-independent and $c_{q_j} \neq 0$. When $c_0 \geq 1$, it is proven in [2] that the operator $\mathscr{C}_\varphi$ is compact if and only if φ has restricted range. Our first result extends this to the case of an arbitrary polynomial:

Theorem 1. *Let φ be a Dirichlet polynomial of the form* (2) *with $c_0 \geq 1$. Then $\mathscr{C}_\varphi$ is compact if and only if φ has restricted range.*

As is to be expected when investigating composition operators on $\mathscr{H}^2$, the symbols with $c_0 = 0$ are more difficult to handle and require different techniques. In this case, it is proven independently in [2] and [8] that composition operators induced by linear symbols (3) with $c_0 = 0$ are compact if and only if φ has restricted range or $d \geq 2$.

The main effort of this paper is dedicated to extending this result to general polynomials. We rely crucially on a geometric description of such compact composition operators found in [11] (see Lemma 5 below). Our second result is:

Theorem 2. *Suppose that $\{q_j\}_{j=1}^d$ are $\mathbb{Q}$-independent and that*

$$\varphi(s) = \sum_{j=1}^{d} P_j(q_j^{-s})$$

is in $\mathscr{G}$, and that the polynomials P_j are non-constant. Then $\mathscr{C}_\varphi$ is compact if and only if φ has restricted range or $d \geq 2$.

Theorem 2 is truly a non-linear extension of the results for linear symbols, however it fails to handle the relatively simple cases

$$\begin{aligned} \varphi_1(s) &= \frac{9}{2} - 2^{-s} - 3^{-s} - 2 \cdot 6^{-s}, \\ \varphi_2(s) &= \frac{13}{2} - 4 \cdot 2^{-s} - 4 \cdot 3^{-s} + 2 \cdot 6^{-s}, \end{aligned} \tag{4}$$

where "mixed terms" are present. However, the compactness of the associated operators can be decided by our main result. Before this result can be stated, we need to introduce some additional definitions.

Definition. Let $\Lambda \subseteq \mathbb{N} - \{1\}$. We let the *complex dimension* of Λ, denoted $\mathscr{D}(\Lambda)$, be the infimum of $\operatorname{card}(\Lambda_0)$ where $\Lambda_0 \subset \mathbb{N} - \{1\}$ is $\mathbb{Q}$-independent and multiplicatively generates Λ.

At this point, we should mention that the set Λ_0 attaining such an infimum is not necessarily unique. This is easily seen by considering

$$\Lambda = \left\{2^2 \cdot 3^2,\, 2^4 \cdot 3^2,\, 2^2 \cdot 3^4,\, 2^4 \cdot 3^4\right\},$$

where Λ_0 can be chosen as any of the following sets:

$$\{2,\,3\} \qquad \left\{2^2,\,3\right\} \qquad \left\{2,\,3^2\right\} \qquad \left\{2^2,\,3^2\right\} \qquad \left\{2^2 \cdot 3,\,3\right\} \qquad \left\{2,\,2 \cdot 3^2\right\}$$

Now, we will rewrite (2) as

$$\varphi(s) = c_1 + \sum_{n \in \Lambda} c_n n^{-s} \tag{5}$$

with $c_n \neq 0$ for every $n \in \Lambda$. We pick some $\Lambda_0 = \{q_1, q_2, \ldots, q_d\}$ where $d = \mathscr{D}(\Lambda)$. Since Λ_0 generates Λ, any $n \in \Lambda$ can be written uniquely as a product of elements in Λ_0,

$$n = \prod_{j=1}^{d} q_j^{\alpha_j}.$$

This associates to n the d-dimensional multi-index $\alpha(n)$. Clearly, $\alpha(n)$ depends on the choice of Λ_0 as the example considered above illustrates.

Definition. The *degree of φ with respect to* Λ_0 is defined by

$$\deg(\varphi, \Lambda_0) = \sup\big\{|\alpha(n)| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \,:\, n \in \Lambda\big\}.$$

Among the different Λ_0 which generate Λ and with $\mathrm{card}(\Lambda_0) = \mathscr{D}(\Lambda)$, we choose an optimal Λ_0 in the sense that it minimizes $\deg(\varphi, \Lambda_0)$. The *degree* of φ is then equal to the value of $\deg(\varphi, \Lambda_0)$ where Λ_0 is optimal in the previous sense.

It is clear that there can be more than one optimal Λ_0, as the example considered above again demonstrates, where the three final possibilities all have $\deg(\varphi, \Lambda_0) = 4$ if φ is given by (5).

Remark. For maps of the form (3) as considered before, the complex dimension is equal to d and the degree is equal to 1, which justifies our terminology "linear case".

The study of the Hardy space of Dirichlet series $\mathscr{H}^2$ is intimately related to function theory on polydiscs. In our concerns, the main tool will be the so-called Bohr lift. Indeed, consider an optimal Λ_0 and use the substitution $q_j^{-s} \mapsto z_j$. To simplify the expressions in what follows, we will also subtract $1/2$. Hence we obtain a polynomial in d variables with the same degree as φ,

$$\Phi(z) = \left(c_1 - \frac{1}{2}\right) + \sum_{n\in\Lambda} c_n z^{\alpha(n)}. \tag{6}$$

The polynomial Φ will be called an *optimal Bohr lift* of φ. Using Kronecker's theorem (see for instance [10, Ch. 13]), the $\mathbb{Q}$-independence of Λ_0 implies that Φ maps $\mathbb{D}^d$ into $\mathbb{C}_0$. The polynomial Φ induces a map, denoted by ϕ, on $\mathbb{R}^d$ defined by

$$\phi(\theta_1, \theta_2, \ldots, \theta_d) = \Phi\big(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}\big).$$

Remark. We will sometimes need to define the Bohr lift when the map $\varphi(s) = \sum_{n\geq 1} c_n n^{-s}$ is not a Dirichlet polynomial. It is then defined as

$$\Phi(z) = \left(c_1 - \frac{1}{2}\right) + \sum_{n\geq 2} c_n z^{\alpha(n)}$$

where we use the substitution $p_j^{-s} \mapsto z_j$. If we assume that $\varphi \in \mathscr{G}$, its Bohr lift Φ is now well-defined on $\mathbb{D}^\infty \cap c_0$, and Kronecker's theorem shows that this set is mapped by Φ into $\mathbb{C}_0$.

Let us come back to a polynomial $\varphi \in \mathscr{G}$. If we assume that φ has unrestricted range, there exists at least one point $w \in \mathbb{T}^d$ so that $\operatorname{Re}\Phi(w) = 0$, by the compactness of $\mathbb{T}^d$. Let $w = \big(e^{i\vartheta_1}, e^{i\vartheta_2}, \ldots, e^{i\vartheta_d}\big)$. Then $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_d)$ has to be a critical point of $\operatorname{Re}\phi$ since this last map admits a minimum at ϑ. Moreover, the mapping properties of φ implies that the Hessian matrix of $\operatorname{Re}\phi$ at ϑ should be non-negative.

Definition. We define the *boundary index of* Φ *at* w as the non-negative integer $J(\Phi, w)$ such that the signature of the Hessian matrix of $\operatorname{Re}\phi$ at ϑ is equal to $\big(J(\Phi, w), 0\big)$.

With these definitions at hand, we are able to state our main theorem which shows that, when there are mixed terms, the complex dimension does not give enough information and that we need a more careful study of φ.

Theorem 3. *Let $\varphi(s) = c_1 + \sum_{n\geq 2} c_n n^{-s}$ be a Dirichlet polynomial in $\mathscr{G}$ with unrestricted range. Suppose that its complex dimension d is greater than or equal to 2, and let Φ be a minimal Bohr lift of φ. Assume that*

- *either the degree of φ is equal to 1 or 2,*
- *or the degree of φ is at least 3 and for any $w \in \mathbb{T}^d$, either $\operatorname{Re}\Phi(w) > 0$ or $\operatorname{Re}\Phi(w) = 0$ and $J(\Phi, w) \geq 2$.*

Then $\mathscr{C}_\varphi$ is compact on $\mathscr{H}^2$. Moreover, the result is optimal in the following sense:

- *If the complex dimension of φ is equal to 1, then $\mathscr{C}_\varphi$ is never compact.*
- *There exist polynomials $\varphi \in \mathscr{G}$ of arbitrary complex dimension and of arbitrary degree greater than or equal to 3 such that $\mathscr{C}_\varphi$ is not compact.*

At this point we should mention that Theorem 3 does not encompass Theorem 2, and we will return to this point later (see Section 7). However, Theorem 3 allows us to conclude that for the Dirichlet polynomials φ given by (4), which have complex dimension and degree equal to 2, the induced composition operators are compact.

We are also interested in the degree of compactness of our operators, which may be estimated using their approximation numbers.

Definition. Let H be a Hilbert space and let $T \in \mathfrak{L}(H)$. The nth approximation number of T, denoted $a_n(T)$, is the distance of T to the operators of rank $< n$.

The study of the behaviour of $a_n(\mathscr{C}_\varphi)$ when $\varphi \in \mathscr{G}$ is a linear symbol (3) has been done in [11]. In particular, it is shown there that

$$\left(\frac{1}{n}\right)^{(d-1)/2} \ll a_n(\mathscr{C}_\varphi) \ll \left(\frac{\log n}{n}\right)^{(d-1)/2}$$

where d is the complex dimension of φ. We will extend this result to a general context. To keep this introduction sufficiently short, we refer to Section 8 for our statement, and give only one striking consequence of it: We may distinguish the Schatten classes of linear operators on $\mathscr{H}^2$ using composition operators induced by polynomial symbols. By definition, a compact linear operator T belongs to the Schatten class S_p, for $0 < p < \infty$, if

$$\|T\|_p^p := \operatorname{Tr}(|T|^p) = \sum_{n=1}^{\infty} a_n(T)^p < \infty.$$

Corollary 4. *Let $0 < p < q$. There exists a Dirichlet polynomial $\varphi \in \mathscr{G}$ such that $\mathscr{C}_\varphi \in S_q \backslash S_p$.*

Let us end this introduction by mentioning that the the composition operators induced by the maps φ_1 and φ_2 have different degrees of compactness. Indeed, we will show that

$$\begin{aligned} \left(\frac{1}{n}\right)^{1/2} &\ll a_n(\mathscr{C}_{\varphi_1}) \ll \left(\frac{\log n}{n}\right)^{1/2}, \\ \left(\frac{1}{n}\right)^{1/3} &\ll a_n(\mathscr{C}_{\varphi_2}) \ll \left(\frac{\log n}{n}\right)^{1/3}. \end{aligned}$$

Organization. The remainder of this paper is divided into seven sections.

- Section 2 contains the proof of Theorem 1. The content of this section is independent from that of the following sections.
- In Section 3 we introduce some necessary tools and results needed for the proof of Theorem 2 and Theorem 3.
- Section 4 is devoted to the proof of Theorem 2.
- Section 5 contains the proof of Theorem 3.
- In Section 6 we prove Lemma 12, which is the most technical part of Theorem 3.
- In Section 7 we discuss the case $\deg(\varphi) \geq 3$ and $J(\Phi, w) = 0$, its connection to Theorem 2 and some related examples.
- Finally, in Section 8, we discuss the decay of the sequence of approximation numbers for some of our operators.

Notation. The notation $f(\varepsilon) \ll g(\varepsilon)$ will mean that $f(\varepsilon) \leq Cg(\varepsilon)$ for some constant C which does not depend on ε. We will sometimes write $f(\varepsilon) \ll_a g(\varepsilon)$ to emphasize that C depends on a. As usual, we let $\{p_j\}$ denote the sequence of prime numbers written in increasing order. We let $\mathbf{m}_d$ denote the normalized Lebesgue measure on $\mathbb{T}^d$. This measure is invariant under rotations. If we do not have a priori knowledge of the complex dimension d, we will often call this measure $\mathbf{m}_\infty$. For a point $z = e^{i\theta}$ on the unit circle $\mathbb{T}$, we will always assume that $\theta \in (-\pi, \pi]$. Finally, $\mathbf{0}$ will denote the point $(0, \ldots, 0) \in \mathbb{C}^d$, and $\mathbf{1}$ will similarly denote the point $(1, \ldots, 1)$.

2. Proof of Theorem 1

Let $\varphi(s) = c_0 s + c_1 + \sum_{n=2}^{N} c_n n^{-s} \in \mathscr{G}$ such that $c_0 \geq 1$. We already know that if φ has restricted range, then $\mathscr{C}_\varphi$ is compact. Let us therefore assume that $\mathscr{C}_\varphi$ is compact and also assume that φ has unrestricted range, to argue by contradiction.

By [2, Thm. 3], we know that

$$\frac{\operatorname{Re}\varphi(s)}{\operatorname{Re}(s)} \xrightarrow{\operatorname{Re}(s)\to 0} +\infty. \tag{7}$$

Now, since φ has unrestricted range there exists a sequence $\{s_k = \sigma_k + it_k\}_{k\geq 1}$ in $\mathbb{C}_0$ such that $\operatorname{Re}\varphi(s_k) \to 0$. It is well-known that this forces that $\sigma_k \to 0$ (see [2]). Then

$$\operatorname{Re}\varphi(s_k) = c_0\sigma_k + \operatorname{Re}(c_1) + \sum_{n=2}^{N} n^{-\sigma_k}\big(\operatorname{Re}(c_n)\cos(t_k\log(n)) + \operatorname{Im}(c_n)\sin(t_k\log(n))\big).$$

By successive extraction of subsequences, we may assume that there exist real numbers a_n and b_n so that for $2 \leq n \leq N$ we have, as $k \to \infty$,

$$\cos(t_k\log(n)) \to a_n \qquad \text{and} \qquad \sin(t_k\log(n)) \to b_n.$$

Hence, we may write

$$\operatorname{Re}\varphi(s_k) = c_0\sigma_k + \operatorname{Re}(c_1) + \sum_{n=2}^{N} n^{-\sigma_k}\big(\operatorname{Re}(c_n)a_n + \operatorname{Im}(c_n)b_n\big) + \sum_{n=2}^{N} n^{-\sigma_k}F_n(t_k),$$

where each $F_n(t_k) \to 0$ as $k \to \infty$. Since $\operatorname{Re} s_k = \sigma_k$ also goes to 0, we may deduce that

$$\operatorname{Re}(c_1) + \sum_{n=2}^{N}\big(\operatorname{Re}(c_n)a_n + \operatorname{Im}(c_n)b_n\big) = 0,$$

so that we have

$$\operatorname{Re}\varphi(s) = c_0\sigma + \sum_{n=2}^{N}\big(n^{-\sigma} - 1\big)\big(\operatorname{Re}(c_n)a_n + \operatorname{Im}(c_n)b_n\big) + \sum_{n=2}^{N} n^{-\sigma}F_n(t).$$

We will now choose another sequence $\{s'_k = \sigma'_k + it_k\}_{k\geq 1}$ where $\operatorname{Re}(s'_k) \to 0$ in order to obtain a contradiction with (7). More precisely, let $\{\sigma'_k\}_{k\geq 1}$ be any sequence of positive real numbers tending to 0 such that, for any $n = 2, \dots, N$ and every $k \geq 1$, we have $n^{-\sigma'_k}|F_n(t_k)| \leq \sigma'_k$. Then we obtain

$$\operatorname{Re}\varphi(s'_k) = c_0\sigma'_k + \sum_{n=2}^{N}\Big(n^{-\sigma'_k} - 1\Big)\big(\operatorname{Re}(c_n)a_n + \operatorname{Im}(c_n)b_n\big) + \sum_{n=2}^{N} n^{-\sigma'_k}F_n(t_k),$$

so $\operatorname{Re}\varphi(s'_k) = O(\sigma'_k) = O\big(\operatorname{Re}(s'_k)\big)$ and this contradicts (7). The assumption that φ has unrestricted range must be wrong. $\square$

Remark. An inspection of the proof reveals that the statement of Theorem 1 remains true if we assume that $\varphi(s) = c_0 s + c_1 + \sum_{n=2}^{\infty} c_n n^{-s} \in \mathscr{G}$ with $c_0 \geq 1$, $\sum_{n=1}^{\infty} |c_n| < +\infty$ and that the complex dimension of φ is finite. The latter assumption is needed to use (7).

3. Preliminaries

As explained in the introduction, our main tool for proving or disproving compactness is a result from [11]. We formulate it in a more general context than for polynomials since it will be used under this form in Section 8. Recall that a *Carleson square* in $\mathbb{C}_0$ is a closed square in $\overline{\mathbb{C}_0}$ with one of its sides lying on the vertical line $i\mathbb{R}$; the side length of Q is denoted by $\ell(Q)$. A non-negative Borel measure μ on $\overline{\mathbb{C}_0}$ is called a *vanishing Carleson measure* if

$$\limsup_{\ell(Q)\to 0} \frac{\mu(Q)}{\ell(Q)} = 0.$$

Lemma 5. *Suppose that* $\varphi(s) = \sum_{n\geq 1} c_n n^{-s} \in \mathscr{G}$ *and that* $\varphi(\mathbb{C}_0)$ *is bounded. The corresponding composition operator* $\mathscr{C}_\varphi$ *is compact on* $\mathscr{H}^2$ *if and only if the measure*

$$\mu_\varphi(E) := \mathbf{m}_\infty\left(\{z \in \mathbb{T}^\infty \,:\, \Phi(z) \in E\}\right), \qquad E \subseteq \mathbb{C}_0.$$

is vanishing Carleson in $\mathbb{C}_0$, *where* Φ *denotes a Bohr lift of* φ.

Proof. This is Corollary 4.1 in [11]. □

Hence we consider squares

$$Q = Q(\tau, \varepsilon) = [0, \varepsilon] \times [\tau - \varepsilon/2, \tau + \varepsilon/2],$$

and want to investigate whether $\mu_\varphi(Q) = o(\varepsilon)$ uniformly in $\tau \in \mathbb{R}$. Our next lemma points out that this depends only on the local behaviour of Φ.

Lemma 6. *Let* φ *be a Dirichlet polynomial* (2) *with* $c_0 = 0$ *mapping* $\mathbb{C}_0$ *into* $\mathbb{C}_{1/2}$ *and let* Φ *be a minimal Bohr lift of* φ. *If for every* $w \in \mathbb{T}^d$ *with* $\operatorname{Re}\Phi(w) = 0$ *there exists a neighbourhood* $\mathscr{U}_w \ni w$ *in* $\mathbb{T}^d$, *constants* $C_w > 0$ *and* $\kappa_w > 1$ *such that for every* $\tau \in \mathbb{R}$ *and every* $\varepsilon > 0$ *we have*

$$\mathbf{m}_d\left(\{z \in \mathscr{U}_w \,:\, \Phi(z) \in Q(\tau, \varepsilon)\}\right) \leq C_w \varepsilon^{\kappa_w}, \tag{8}$$

then $\mathscr{C}_\varphi$ *is compact.*

Proof. Since φ is a Dirichlet polynomial, it has finite complex dimension d.

We first observe that (8) is always satisfied for those $w \in \mathbb{T}^d$ with $\operatorname{Re}\Phi(w) > 0$. Indeed, by continuity of Φ, we may always find a neighbourhood $\mathscr{U}_w \ni w$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\tau \in \mathbb{R}$, $\big\{z \in \mathscr{U}_w \,:\, \Phi(z) \in Q(\tau, \varepsilon)\big\}$ is empty. We may then take $\kappa_w > 1$ be arbitrary and choose C_w with $C_w \varepsilon_0^{\kappa_w} \geq 1$.

We will then use a compactness argument and Lemma 5. Indeed, there exists a finite number of points $w_1, \dots, w_N$ such that $\mathbb{T}^d$ is covered by $\mathscr{U}_{w_1}, \dots, \mathscr{U}_{w_N}$. Now, we may take $C = C_{w_1} + \cdots + C_{w_N}$ and $\kappa = \min(\kappa_{w_1}, \dots, \kappa_{w_N})$. Hence, for all $\tau \in \mathbb{R}$ and all $\varepsilon > 0$,

$$\mathbf{m}_d \left(\left\{ z \in \mathbb{T}^d \, : \, \Phi(z) \in Q(\tau, \varepsilon) \right\} \right) \leq C\varepsilon^{\kappa}$$

which achieves the proof of the compactness of $\mathscr{C}_\varphi$ on $\mathscr{H}^2$. $\square$

Hence, we will require more information about the Taylor coefficients of Φ at a boundary point. Assume that $\Phi(w) = 0$ where $w = \mathbf{1}$. In this case, we will rewrite

$$\Phi(z) = \sum_{n \in \Lambda} \widetilde{c_n} \prod_{j=1}^{d} (1 - z_j)^{\alpha_j} = \sum_{\alpha \in \mathbb{N}^d} c_\alpha (1 - z)^\alpha, \tag{9}$$

where we have adopted the convention $c_\alpha = \widetilde{c_n}$, which is not generally equal to c_n. We shall need a kind of Julia–Caratheodory theorem for Φ of the form (9).

Lemma 7. *Let $\Phi : \mathbb{D}^d \to \mathbb{C}_0$ be of the form (9) and let $|\alpha| = 1$. Then $c_\alpha \geq 0$. Moreover, there exists at least one multi-index α with $|\alpha| = 1$ and $c_\alpha > 0$, unless $\Phi \equiv 0$.*

Proof. We may assume that $\alpha = (1, 0, \dots, 0)$. Consider the one-variable polynomial

$$\psi(w) = \Phi(w, 1, \dots, 1).$$

Clearly, ψ maps $\mathbb{D}$ to $\mathbb{C}_0$, and $\psi(1) = 0$. We write

$$\psi(w) = a(1 - w) + b(1 - w)^2 + O\left((1 - w)^3\right).$$

We set $w = e^{i\theta}$ and obtain

$$\psi\big(e^{i\theta}\big) = a\left(\frac{\theta^2}{2} - i\theta\right) - b\theta^2 + O\left(\theta^3\right).$$

In particular,

$$\operatorname{Re}\left(\psi\big(e^{i\theta}\big)\right) = \theta \operatorname{Im}(a) + \theta^2 \left(\frac{\operatorname{Re}(a)}{2} - \operatorname{Re}(b)\right) + O\left(\theta^3\right).$$

Since this should be non-negative, clearly $\operatorname{Im}(a) = 0$. We now set $w = 1 - \delta$ for $0 < \delta < 1$ and consider $\psi(\delta) = a\delta + O\left(\delta^2\right)$. Since the real part of this also should be non-negative as $\delta \to 0^+$ we must have $a \geq 0$. Hence $c_\alpha \geq 0$ when $|\alpha| = 1$.

Now, consider the mapping

$$\alpha \mapsto n(\alpha) = \prod_{j=1}^{d} p_j^{\alpha_j}.$$

It defines a total order on $\mathbb{N}^d$ by setting $\alpha \le \beta$ if and only if $n(\alpha) \le n(\beta)$. Assume that $\Phi \not\equiv 0$ and that $c_\alpha = 0$ whenever $|\alpha| = 1$. Consider

$$\beta = \inf\{\alpha \,:\, c_\alpha \neq 0\},$$

which exists since $\Phi \not\equiv 0$. There is $\theta \in (-\pi, \pi]$ so that $c_\beta = |c_\beta| e^{i\theta}$. Fix $\theta_j \in (-\pi/2, \pi/2)$ and define

$$z_j = 1 - p_j^{-\sigma} e^{i\theta_j},$$

where $\sigma > 0$. For large enough σ, clearly $z = (z_1, \dots, z_d) \in \mathbb{D}^d$. Moreover,

$$\Phi(z_1, \dots, z_d) = |c_\beta| e^{i\theta} [n(\beta)]^{-\sigma} e^{i(\beta_1\theta_1 + \cdots + \beta_d\theta_d)} + o\left([n(\beta)]^{-\sigma}\right),$$

as $\sigma \to \infty$. This implies that

$$\operatorname{Re}\left(\Phi(z_1, \dots, z_d)\right) = |c_\beta| [n(\beta)]^{-\sigma} \cos(\theta + \beta_1\theta_1 + \cdots \beta_d\theta_d) + o\left([n(\beta)]^{-\sigma}\right).$$

Since $|\beta| \ge 2$, we can always choose $\theta_j \in (-\pi/2, \pi/2)$ such that $\cos(\theta + \beta_1\theta_1 + \cdots + \beta_d\theta_d) < 0$. This contradicts the mapping properties of Φ, and hence the assumption that $c_\alpha = 0$ whenever $|\alpha| = 1$ is wrong. □

We will also need two lemmas from differential geometry. The first one is the parametrized Morse lemma (see for instance [5, Sec. 4.44]).

Lemma (Parametrized Morse Lemma)**.** *Let $\mathscr{U} \subset \mathbb{R}^J \times \mathbb{R}^{d-J}$ be a neighbourhood of $\mathbf{0} \in \mathbb{R}^d$ and let $F : \mathscr{U} \to \mathbb{R}$, $(u, v) \mapsto F(u, v)$ be a smooth function. Assume that $F(\mathbf{0}) = 0$, that $\partial F/\partial u_i(\mathbf{0}) = 0$ for all $i = 1, \dots, J$ and that the matrix*

$$\left(\frac{\partial^2 F}{\partial u_i \partial u_j}(\mathbf{0})\right)_{1 \le i,j \le J}$$

is positive definite. Then there exist a neighbourhood $\mathscr{V} \ni \mathbf{0}$ with $\mathscr{V} \subset \mathscr{U}$, a smooth diffeomorphism $\Gamma : \mathscr{V} \to \mathbb{R}^d$, $(u, v) \mapsto (\gamma(u, v), v)$ with $\Gamma(\mathbf{0}) = \mathbf{0}$ and a smooth map $h : \mathbb{R}^{d-J} \to \mathbb{R}$ such that, for any $(u, v) \in \mathscr{V}$,

$$F(u, v) = \sum_{j=1}^{J} \gamma_j(u, v)^2 + h(v).$$

The second lemma reads as follows.

Lemma 8. *Let $p \ge 1$ be an integer, and let $f : I \to \mathbb{R}$ be a smooth function where I is an open interval containing 0 and $f(x) \sim_0 x^p$. Then there exist $C > 0$ and an open interval $I' \ni 0$ inside I such that, for any $\tau \in \mathbb{R}$ and any $\delta > 0$, the set $\{x \in I' \,:\, |f(x) - \tau| < \delta\}$ has Lebesgue measure less than $C\delta^{1/p}$.*

Proof. Assume first that $f(x) = x^p$. If $|\tau| \le 2\delta$, then the result is clear. Otherwise, if $\tau \ge 2\delta$, then x has to live in $\left[(\tau - \delta)^{1/p}, (\tau + \delta)^{1/p}\right]$ and the length of this interval may be easily estimated using the mean value theorem.

The general case reduces to this one. For small values of x, set $y = [f(x)]^{1/p}$ if p is odd or $y = [f(x)]^{1/p}$ for $x > 0$, $y = -[f(x)]^{1/p}$ for $x < 0$ if p is even. In both cases, y is differentiable at 0 and $dy/dx > 0$. Hence, $x = \gamma(y)$ where γ is a smooth diffeomorphism. Now, for some small open interval $I' \ni 0$, we have

$$\{x \in I' : |f(x) - \tau| < \delta\} = \left\{x \in I' : \left|\left(\gamma^{-1}(x)\right)^p - \tau\right| < \delta\right\}.$$

Since γ is a diffeomorphism, the latter has Lebesgue measure less than $C\delta^{1/p}$. □

4. Proof of Theorem 2

We intend to apply Lemma 6. Hence, let $w \in \mathbb{T}^d$ with Re $\Phi(w) = 0$. By the rotational invariance of $\mathbf{m}_d$, we may always assume that $w = \mathbf{1}$. Moreover, since the conditions in Lemma 6 are invariant by vertical translations, we may also assume that $\Phi(w) = 0$. In this case we have

$$\Phi(z_1, z_2, \ldots, z_d) = \sum_{j=1}^{d} \Phi_j(z_j) = \sum_{j=1}^{d} \sum_{k} a_k^{(j)} (1 - z_j)^k.$$

Since Φ is a minimal Bohr lift of φ, inspecting the proof of Lemma 7, we may conclude that in this case $a_1^{(j)} > 0$ for every $j = 1, 2, \ldots, d$. This means we have

$$\operatorname{Re} \Phi\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}\right) = \sum_{j=1}^{d} b_j \theta_j^{k_j} + o\left(\theta_j^{k_j}\right),$$

where the coefficients $b_j \neq 0$ are real numbers and the exponents $k_j \geq 2$ are integers. The fact that this quantity is supposed to be non-negative implies that $b_j > 0$ and that k_j is even, by similar considerations as those in the proof of Lemma 7. Moreover

$$\operatorname{Im} \Phi\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}\right) = -\sum_{j=1}^{d} a_1^{(j)} \theta_j + o(\theta_j).$$

Proof of the first part of Theorem 2. Let $\tau \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. The preceding discussion means there is some neighbourhood $\mathscr{U} \ni (1, 1, \ldots, 1)$ in $\mathbb{T}^d$ so that

$$\frac{1}{2} \sum_{j=1}^{d} b_j \theta_j^{k_j} \leq \operatorname{Re} \Phi\left(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}\right) \leq 2 \sum_{j=1}^{d} b_j \theta_j^{k_j},$$

when $e^{i\theta} \in \mathscr{U}$. Hence if $\Phi\left(e^{i\theta}\right) \in Q(\tau, \varepsilon)$ and $e^{i\theta} \in \mathscr{U}$, we conclude from the real part that $|\theta_j| \ll \varepsilon^{1/k_j}$, for $j = 1, 2, \ldots, d$. Now, fixing θ_j for $j = 2, \ldots, d$ we conclude from the imaginary part and Lemma 8 that θ_1 can live in an interval of size at most $C\varepsilon$. Hence we have

$$\mathbf{m}_d \left(\{z \in \mathscr{U}_w : \Phi(z) \in Q(\tau, \varepsilon)\}\right) \ll_w \varepsilon^{1 + 1/k_2 + \cdots + 1/k_d}.$$

In fact, we may choose

$$\kappa_w = 1 + \sum_{j=1}^{d} \frac{1}{k_j} - \min_{1 \le j \le d} \frac{1}{k_j},$$

and conclude by Lemma 6, since $d \geq 2$. □

Proof of the second part of Theorem 2. In this case $d = 1$, and the polynomial $\Phi(z)$ is of only one variable. We again consider some neighbourhood $\mathscr{U} \ni 1$ in $\mathbb{T}$, so that when $e^{i\theta} \in \mathscr{U}$ we have

$$0 \le \operatorname{Re}\Phi\big(e^{i\theta}\big) \le 2b\theta^k \qquad \text{and} \qquad \big|\operatorname{Im}\Phi\big(e^{i\theta}\big)\big| \le 2a|\theta|,$$

where $a = a_1$, $b = b_1$ and $k \geq 2$ is even. Now, we choose $\tau = 0$ and observe that $\varphi\big(e^{i\theta}\big)$ belongs to $Q(\tau, \varepsilon)$ provided $|\theta| \ll \varepsilon$. Hence

$$\mathbf{m}_1\left(\{z \in \mathbb{T} \,:\, \Phi(z) \in Q(\tau,\varepsilon)\}\right) \geq \mathbf{m}_1\left(\{z \in \mathscr{U}_w \,:\, \Phi(z) \in Q(\tau,\varepsilon)\}\right) \gg \varepsilon,$$

and $\mathscr{C}_\varphi$ cannot be compact by Lemma 5. □

Remark. Inspecting the proof of Theorem 2, we see that we may replace the polynomials P_j, by corresponding power series

$$P_j\big(q_j^{-s}\big) = \sum_{k=0}^{\infty} c_k^{(j)} q_j^{-ks},$$

provided $\sum_{k=0}^{\infty} \big|c_k^{(j)}\big| < \infty$. However, we still require the complex dimension d to be finite.

5. Proof of Theorem 3

We begin by observing that the penultimate point of Theorem 3 follows from the second part of Theorem 2. Regarding the final part of Theorem 3, it is contained in the following result:

Lemma 9. *There are polynomials $\varphi \in \mathscr{G}$ of any complex dimension and of any degree ≥ 3 for which the corresponding composition operator $\mathscr{C}_\varphi$ is non-compact.*

Proof. Let $P(z) = P(z_1, z_2, \ldots, z_d)$ be any polynomial in d variables and define

$$\Phi(z) = (1 - z_1) + \delta(1 - z_1)^2 P(z),$$

for some $\delta > 0$ to be decided later. We compute

$$\begin{aligned}\operatorname{Re}\Phi\big(e^{i\theta_1}, \ldots, e^{i\theta_d}\big) &= (1 - \cos\theta_1) \\ &\quad\times \left(1 - 2\delta\left(\cos(\theta_1)\operatorname{Re}P\big(e^{i\theta}\big) - \sin(\theta_1)\operatorname{Im}P\big(e^{i\theta}\big)\right)\right).\end{aligned}$$

Pick δ small enough so that we have

$$\frac{1 - \cos\theta_1}{2} \le \operatorname{Re}\Phi\big(e^{i\theta_1}, \ldots, e^{i\theta_d}\big) \le 2(1 - \cos\theta_1).$$

The first inequality tells us that Φ is a minimal Bohr lift of

$$\varphi(s) = \big(1 - p_1{}^{-s}\big) + \delta\big(1 - p_1{}^{-s}\big)^2 P\big(p_1^{-s}, \dots, p_d^{-s}\big),$$

with $\varphi \in \mathscr{G}$ having unrestricted range. Using the second inequality and a Taylor expansion of $\operatorname{Im}\Phi$, we also get that near $\mathbf{1}$,

$$\begin{aligned}\operatorname{Re}\Phi\big(e^{i\theta_1}, \dots, e^{i\theta_d}\big) &= O\left(\theta_1^2\right),\\ \operatorname{Im}\Phi\big(e^{i\theta_1}, \dots, e^{i\theta_d}\big) &= O(\theta_1).\end{aligned}$$

Similar considerations as in the proof of the second part of Theorem 2 allow us to conclude that $\mathscr{C}_\varphi$ is not compact. □

Remark. The key point of Lemma 9 is that even if Φ involves d variables, its local behaviour near $\mathbf{1}$ depends too heavily on z_1 to ensure compactness.

Having now concluded the negative parts of Theorem 3, we turn to the positive parts. Let us fix a polynomial $\varphi \in \mathscr{G}$ and let Φ denote a minimal Bohr lift of φ. We can simplify how to write Φ around a point $w \in \mathbb{T}^d$ such that $\operatorname{Re}\Phi(w) = 0$. Without loss of generality, we may again assume that $w = \mathbf{1}$ and that $\Phi(w) = 0$. Then we may write

$$\begin{aligned}\Phi(z) = \sum_{j=1}^d a_j(1 - z_j) &+ \sum_{j=1}^d b_j(1 - z_j)^2\\ &+ \sum_{1\le j<k\le d} c_{j,k}(1 - z_j)(1 - z_k) + o\left(\sum_{1\le j\le d} |1 - z_j|^2\right).\end{aligned}$$

We let $z_j = e^{i\theta_j}$ and since $a_j \geq 0$ by Lemma 7 we get

$$\operatorname{Re}\left(\Phi(z)\right) = \sum_{j=1}^d \Big(\frac{a_j}{2} - \operatorname{Re}(b_j)\Big)\,\theta_j^2 - \sum_{1\le j<k\le d} \operatorname{Re}(c_{j,k})\theta_j\theta_k + o\left(\sum_{1\le j\le d} \theta_j^2\right).$$

The quadratic form appearing above is brought to standard form by a linear change of variables,

$$\operatorname{Re}\left(\Phi(z)\right) = \sum_{j=1}^d \left(\ell_j(\theta)\right)^2 + o\left(\sum_{1\le j\le d} \theta_j^2\right).$$

Next, we write

$$\operatorname{Im}\left(\Phi(z)\right) = -\sum_{j=1}^d a_j\theta_j + o\left(\sum_{j=1}^d |\theta_j|\right) = -\ell_{d+1}(\theta) + o\left(\sum_{j=1}^d |\theta_j|\right),$$

and by Lemma 7 we know that $\ell_{d+1} \not\equiv 0$, since at least one $a_j > 0$. The last step to finish the proof of Theorem 3 is the following result:

Lemma 10. *Let $\Phi : \mathbb{D}^d \to \mathbb{C}_0$ be an optimal Bohr lift of $\varphi \in \mathscr{G}$, where φ has unrestricted range and complex dimension $d \geq 2$. Suppose that $w \in \mathbb{T}^d$ is such that $\operatorname{Re}\Phi(w) = 0$. Then there exist a neighbourhood $\mathscr{U}_w \ni w$ in $\mathbb{T}^d$, $\kappa = \kappa_w > 1$ and $C = C_w > 0$ such that, for any $\tau \in \mathbb{R}$ and for every $\varepsilon > 0$,*

$$\mathbf{m}_d\left(\{z \in \mathscr{U}_w \,:\, \Phi(z) \in Q(\tau, \varepsilon)\}\right) \leq C\varepsilon^{\kappa}$$

in the following cases:

- *$J(\Phi, w) \geq 1$ and ℓ_{d+1} is independent from $(\ell_1, \ldots, \ell_J)$. We may choose $\kappa = 1 + J(\Phi, w)/2$.*
- *$J(\Phi, w) \geq 2$ and ℓ_{d+1} belongs to $\operatorname{span}(\ell_1, \ldots, \ell_J)$. We may choose $\kappa = (1 + J(\Phi, w))/2$.*
- *$J(\Phi, w) = 1$, ℓ_{d+1} is a multiple of ℓ_1 and Φ has degree 2. We may choose $\kappa = 9/8$.*
- *$J(\Phi, w) = 0$ and Φ has degree 2. We may choose $\kappa = (d + 3)/4$.*

Before we prove the different cases of this lemma, let us make some comments. Firstly, it is clear that Lemma 10 and Lemma 6 imply Theorem 3 when the degree of φ is at least 2. When the degree of φ is equal to 1, then

$$\Phi(z) = \sum_{j=1}^{d} a_j(1 - z_j)$$

so that each a_j is positive. This implies that $J(\Phi, w) = d$ so that we may again apply Lemma 10 and Lemma 6.

It is also important to notice that Φ cannot be an arbitrary polynomial mapping of $\mathbb{D}^d$ into $\mathbb{C}_0$. It is an optimal Bohr lift of some $\varphi \in \mathscr{G}$ with complex dimension d. In particular, we shall use that $\frac{\partial \Phi}{\partial z_j} \not\equiv 0$ for every $1 \leq j \leq d$. Moreover, the polynomial $\Phi(z) = \lambda(1 - z_1 z_2)$ is not an optimal Bohr lift. Otherwise, it would arise from $\varphi(s) = \lambda(1 - q_1^{-s} q_2^{-s})$, but the optimal Bohr lift of φ is $\lambda(1 - z)$.

We are now ready for the proof of Lemma 10. By similar considerations as before, we may again assume that $w = \mathbf{1}$ and that $\Phi(w) = 0$. We write J for $J(\Phi, w)$.

The case $J = 0$. This implies that

$$\frac{a_j}{2} - \operatorname{Re}(b_j) = \operatorname{Re}(c_{j,k}) = 0,$$

for $j, k = 1, \ldots, d$. We set $z_j = e^{i\theta_j}$ and compute

$$\begin{aligned} \operatorname{Re}(a_j(1-z_j)) &= a_j(1-\cos\theta_j) \\ \operatorname{Re}\left(b_j(1-z_j)^2\right) &= -a_j\cos\theta_j(1-\cos\theta_j) + 2\operatorname{Im}(b_j)\sin\theta_j(1-\cos\theta_j) \\ \operatorname{Re}(c_{j,k}(1-z_j)(1-z_k)) &= \operatorname{Im}(c_{j,k})\left(\sin\theta_j(1-\cos\theta_k) + \sin\theta_k(1-\cos\theta_j)\right) \end{aligned}$$

This means that

$$\operatorname{Re}(\Phi(z)) = \sum_{j=1}^{d} \operatorname{Im}(b_j)\theta_j^3 + \sum_{1\le j<k\le d} \frac{\operatorname{Im}(c_{j,k})}{2}\left(\theta_j\theta_k^2 + \theta_k\theta_j^2\right) + o\left(\sum_{j=1}^{d}|\theta_j|^3\right).$$

However, the non-negativity of $\operatorname{Re}\Phi$ then implies that $\operatorname{Im}(b_j) = \operatorname{Im}(c_{j,k}) = 0$. Hence we in total have $b_j = a_j/2$ and $c_{j,k} = 0$, which means

$$\Phi(z) = \sum_{j=1}^{d}\left(a_j(1-z_j) + \frac{a_j}{2}(1-z_j)^2\right).$$

In fact, this means that $a_j > 0$ for every j, by the assumption that the complex dimension is d and Lemma 7. We may now use (the proof of) Theorem 2 to conclude that there exists a neighbourhood $\mathscr{U}_w \ni w$ such that

$$\mathbf{m}_d\left(\{z \in \mathscr{U}_w \,:\, \Phi(z) \in Q(\tau,\varepsilon)\}\right) \ll \varepsilon \times \varepsilon^{\frac{d-1}{4}} = \varepsilon^{\frac{d+3}{4}},$$

since we now have

$$\operatorname{Re}\Phi\left(e^{i\theta_1},\ldots,e^{i\theta_d}\right) = \frac{1}{4}\sum_{j=1}^{d} a_j\theta_j^4 + o\left(\theta_j^4\right),$$

and we are done with this case. □

The case $J \geq 1$ and independence. After a linear change of variables, we may write $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ as

$$\begin{aligned} \operatorname{Re}\phi(\theta_1,\ldots,\theta_d) &= u_1^2 + \cdots + u_J^2 + o\left(\sum_{j=1}^{d} u_j^2\right) \\ \operatorname{Im}\phi(\theta_1,\ldots,\theta_d) &= u_d + o\left(\sum_{j=1}^{d}|u_j|\right) \end{aligned}$$

Since a linear change of variables does not change the value of the volume up to constants, we may assume that ϕ depends on (u, v) with $u = (u_1, u_2, \ldots, u_J)$ and $v = (u_{J+1}, \ldots, u_d)$. Applying the parametrized Morse lemma to $\operatorname{Re}\phi$, we may write

$$\operatorname{Re}\phi(u,v) = \gamma_1(u,v)^2 + \cdots + \gamma_J(u,v)^2 + h(v).$$

We also apply the change of variables $(u, v) \mapsto \Gamma(u, v)$ to $\operatorname{Im}\phi$ and since $\Gamma_d(u, v) = u_d$, we find

$$\operatorname{Im}\phi(u, v) = u_d + g(\Gamma(u, v)),$$

where g is a smooth function defined on $\mathscr{V}$ such that $\partial g/\partial u_d(\mathbf{0}) = 0$.

Now, we know that $\operatorname{Re}\phi(u, v) \geq 0$ for every $(u, v) \in \mathbb{R}^d$. Since Γ is a diffeomorphism, v can take any value in some neighbourhood of zero in $\mathbb{R}^{d-J}$ even if we require that

$$\gamma_1(u, v) = \gamma_2(u, v) = \cdots = \gamma_J(u, v) = 0,$$

and hence $h(v) \geq 0$.

This implies that we may find some neighbourhood $\mathscr{W} \ni \mathbf{0}$ in $\mathscr{V}$ such that, for every $\tau \in \mathbb{R}$ and every $\varepsilon > 0$,

$$(u, v) \in \mathscr{W} \quad \text{and} \quad \phi(u, v) \in Q(\tau, \varepsilon) \qquad \Longrightarrow \qquad |\gamma_j(u, v)| \leq \varepsilon^{1/2}.$$

Now, for if we fix $\gamma_1(u, v), \ldots, \gamma_{d-1}(u, v)$, it follows from Lemma 8 with $p = 1$ that $\gamma_d(u, v) = u_d$ has to belong to some interval of size $C\varepsilon$, provided that (u, v) is sufficiently close to $\mathbf{0}$. This means that there exists a neighbourhood $O \subset \mathscr{W}$ of $\mathbf{0}$ such that

$$\{(u, v) \in O \,:\, \phi(u, v) \in Q(\tau, \varepsilon)\} \subset \{(u, v) \in O \,:\, \Gamma(u, v) \in R(\tau, \varepsilon)\},$$

where the volume of $R(\tau, \varepsilon)$ is less than $C\varepsilon^{1+\frac{J}{2}}$. Since Γ is a diffeomorphism, we are done. □

The case $J \geq 2$ and dependence. With a similar linear change of variables as in the previous case, we may write

$$\begin{aligned} \operatorname{Re}\phi(u_1, \ldots, u_d) &= u_1^2 + \cdots + u_J^2 + o\left(\sum_{j=1}^{d} u_j^2\right) \\ \operatorname{Im}\phi(u_1, \ldots, u_d) &= \sum_{j=1}^{J} \alpha_j u_j + o\left(\sum_{j=1}^{d} |u_j|\right) \end{aligned}$$

We use again the parametrized Morse lemma with $\operatorname{Re}\phi$, and it is again easy to show that $\gamma_j(u, v) = u_j + o\left(\sum_{j=1}^{d} |u_j|\right)$ so that

$$\operatorname{Im}\phi(u, v) = \sum_{j=1}^{J} \alpha_j \gamma_j(u, v) + g(\Gamma(u, v))$$

with $\partial g/\partial u_j(\mathbf{0}) = 0$ for $j = 1, \ldots, d$.

We argue as in the previous case. For every $j = 2, \ldots, J$, for any $\tau \in \mathbb{R}$ and every $\varepsilon > 0$,

$$(u, v) \in \mathscr{W} \subset \mathscr{V} \quad \text{and} \quad \phi(u, v) \in Q(\tau, \varepsilon) \qquad \Longrightarrow \qquad |\gamma_j(u, v)| \leq \varepsilon^{1/2}.$$

Now, for a fixed value of $\gamma_2(u,v),\dots,\gamma_d(u,v)$, it is again clear that $\gamma_1(u,v)$ has to belong to some interval of size $C\varepsilon$, provided (u,v) is sufficiently close to $\mathbf{0}$. This means that there exists a neighbourhood $O \ni \mathbf{0}$ in $\mathscr{W}$ such that

$$\big\{(u,v) \in O \,:\, \phi(u,v) \in Q(\tau,\varepsilon)\big\} \subset \big\{(u,v) \in O \,:\, \Gamma(u,v) \in R(\tau,\varepsilon)\big\},$$

where the volume of $R(\tau,\varepsilon)$ is less than $C\varepsilon^{1+\frac{J-1}{2}}$. We conclude as in the previous step. □

The case $J = 1$ and dependence, $d = 2$. This is the most difficult case. At first, we do not assume that $d = 2$ but we always assume that the degree of φ is equal to 2. We know that there is constant $\lambda \in \mathbb{R}^*$ so that $\ell_1(\theta) = \lambda \ell_{d+1}(\theta)$, which means

$$\sqrt{\frac{a_j}{2} - \operatorname{Re}(b_j)} = \lambda a_j, \qquad 1 \le j \le d$$

and that $\lambda > 0$ by the computations in the beginning of this section. We normalize $\Phi(z)$ as $\lambda^{-2}\Phi(z)$, so that we may assume that $\lambda = 1$. Hence

$$\ell_1(\theta) = \sum_{j=1}^{d} a_j\theta_j,$$

and this immediately implies that

$$\text{(10)} \qquad \operatorname{Re}(b_j) = \frac{a_j}{2} - a_j^2 \qquad \text{and} \qquad \operatorname{Re}(c_{j,k}) = -2a_ja_k, \qquad 1 \le j,k \le d.$$

Suppose that $a_1 = 0$. Then $\operatorname{Re}(b_1) = 0$ and $\operatorname{Re}(c_{1,k}) = 0$ for $2 \le k \le d$. We compute

$$\operatorname{Re}\big(\Phi(e^{ix},1,\dots,1)\big) = -2\operatorname{Im}(b_1)\sin x(1-\cos x) \ge 0,$$

which means that $\operatorname{Im}(b_1) = 0$, so that $b_1 = 0$. Next we compute

$$\begin{aligned}\Phi\big(e^{ix},e^{iy},1\dots,1\big) &= a_2(1-\cos y) + \Big(\frac{a_2}{2} - a_2^2\Big)(1 - 2\cos y + \cos 2y)\\ &\qquad - \operatorname{Im}(c_{1,2})\big(-\sin x - \sin y + \sin(x+y)\big)\\ &= (1-\cos y)\big(a_2(1-\cos y) + 2a_2^2\cos y + \operatorname{Im}(c_{1,2})\sin x\big)\\ &\qquad + \operatorname{Im}(c_{1,2})\sin y(1-\cos x).\end{aligned}$$

Taking $y = \pm\delta$ for small enough δ, we obtain that $\operatorname{Im}(c_{1,2}) = 0$. There is nothing special about z_2, and hence we conclude that $\operatorname{Im}(c_{1,k}) = 0$, for $2 \le k \le d$. In particular, $c_{1,k} = 0$ for the same values of k. But this is impossible, since the variable z_1 no longer appear in our polynomial. Hence the assumption that $a_1 = 0$ must be wrong.

Arguing in the same way, we have that $a_j > 0$ for $1 \le j \le d$. Moreover, after renaming the variables, we may suppose $a_1 \ge a_2 \ge \cdots a_d > 0$. Finally,

$$0 \le \operatorname{Re}\left(\Phi(-1,1,\dots,1)\right) = 2a_1 + 4\left(\frac{a_1}{2} - a_1^2\right) \qquad \Longrightarrow \qquad a_1 \le 1,$$

so without loss of generality, we may assume that

$$1 \geq a_1 \geq a_2 \geq \cdots \geq a_d > 0.$$

From now on, we assume that $d = 2$ and that $1 \geq a_1 \geq a_2 > 0$. We need the following lemma.

Lemma 11. *We have* $a_2 \leq 1 - a_1$.

Proof. We compute

$$\Phi(-1,-1) = -4a_1^2 - 4a_2^2 - 8a_1a_2 + 4a_1 + 4a_2 = 4(a_1 + a_2)(1 - a_1 - a_2).$$

Since this has to be non-negative, we get the result. □

Remark. Lemma 11 immediately implies that $a_1 \in (0,1)$ and $a_2 \in (0, 1/2]$ by the assumptions that $0 < a_2 \leq a_1 \leq 1$.

Let us apply the change of variables $\theta_1 = a_2u + a_2v$, $\theta_2 = a_1u - a_1v$ to ϕ:

$$\text{(11)} \qquad \operatorname{Re}\phi(u,v) = -4a_1^2a_2^2u^2 + o(u^2 + v^2)$$

$$\text{(12)} \qquad \operatorname{Im}\phi(u,v) = 2a_1a_2u + o(|u| + |v|).$$

As before, we intend to apply the parametrized Morse lemma to $\operatorname{Re}\phi$. Setting $\Psi = \Gamma^{-1}$, we get that, around $\mathbf{0}$,

$$\operatorname{Re}\phi \circ \Psi(u,v) = u^2 + h(v)$$
$$\operatorname{Im}\phi \circ \Psi(u,v) = u + g(u,v)$$

with h and g smooth functions which have no terms of order 1 at $\mathbf{0}$.

Assume first that $h \not\equiv 0$. Let $p \geq 2$ be such that $h(v) \sim_0 \alpha_p v^p$ with $\alpha_p \neq 0$. Because $\phi \circ \Psi$ maps $\mathbb{R}^2$ into $\overline{\mathbb{C}_0}$, we must have that $\alpha_p > 0$ and that p is even. Now, if $\phi \circ \Psi(u,v) \in Q(\tau, \varepsilon)$ with (u,v) sufficiently close to $\mathbf{0}$, then $0 \leq h(v) \leq \varepsilon$ which implies by Lemma 8 that v belongs to some set of measure less than $C\varepsilon^{1/p}$. Moreover, for a fixed value of v, a look at the imaginary part and Lemma 8 yield that u has to belong to some interval of size $C\varepsilon$ and thus we are done with $\kappa = 1 + 1/p$.

Thus, we are lead to study what happens if $h \equiv 0$. The situation is easier if the Taylor expansion of $g(u,v)$ admits some term in v^p. In that case, we may write

$$\operatorname{Im}\phi \circ \Psi(u,v) = ug_1(u,v) + v^p g_2(v)$$

with smooth functions g_1 and g_2, such that $g_1(0,0) = 1$ and $g_2(0) \neq 0$. If $\phi \circ \Psi(u,v)$ belongs to $Q(\tau,\varepsilon)$, we conclude from the real part that then $|u| \leq \varepsilon^{1/2}$, and from the imaginary part, we get that, near $\mathbf{0}$,

$$|v^p g_2(v) - \tau| \leq C\varepsilon^{1/2}.$$

By appealing again to Lemma 8, we conclude that v belongs to some set of Lebesgue measure less than $C\varepsilon^{1/2p}$. For a fixed value of v, we look once more at the imaginary part, and obtain that u must belongs to some interval of size $C\varepsilon$. Hence, we are done with $\kappa = 1 + 1/(2p)$.

Therefore, it remains to show that we will always fall into one of the previous cases and compute the value of p. We again recall that the polynomial

$$\Phi(z) = \lambda(1 - z_1 z_2),$$

is a contradiction to the fact that Φ is a minimal Bohr lift of $\varphi \in \mathscr{G}$. More precisely, we are reduced to proving the following lemma.

Lemma 12. *Let $0 < a_2 \leq a_1 \leq 1$ and $a_2 \leq 1 - a_1$. Suppose that $\Phi : \mathbb{D}^2 \to \mathbb{C}_0$ is the polynomial*

$$\Phi(z) = a_1(1-z_1) + a_2(1-z_2) + b_1(1-z_1)^2 + b_2(1-z_2)^2 + c(1-z_1)(1-z_2), \tag{13}$$

where

$$\operatorname{Re}(b_1) = \frac{a_1}{2} - a_1^2, \qquad \operatorname{Re}(b_2) = \frac{a_2}{2} - a_2^2 \qquad \text{and} \qquad \operatorname{Re}(c) = -2a_1a_2.$$

Set $\theta_1 = a_2u + a_2v$, $\theta_2 = a_1u - a_1v$ and

$$\phi(u,v) = \Phi\big(e^{i\theta_1}, e^{i\theta_2}\big).$$

Then there does not exist smooth maps $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\operatorname{Re}\phi(u,v) = \gamma(u,v)^2 \tag{14}$$

$$\operatorname{Im}\phi(u,v) = \gamma(u,v)h(u,v) \tag{15}$$

except if $\Phi(z) = \frac{1}{2}(1-z_1z_2)$. More precisely, if $\Phi(z) \neq \frac{1}{2}(1-z_1z_2)$, for any smooth maps $\gamma : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, then

- *either the Taylor series of $\operatorname{Re}\phi - \gamma^2$ at $\mathbf{0}$ has a non-zero term of order ≤ 5,*
- *or the Taylor series of $\operatorname{Im}\phi - \gamma \cdot h$ at $\mathbf{0}$ has a non-zero term of order ≤ 4.*

The proof of this lemma is rather delicate and will be postponed in Section 6 in order to keep a clearer exposition of the proof of Lemma 10. However, using Lemma 12 we are able to finish this case. Indeed, if $\Gamma(u,v) = (\gamma(u,v), v)$ is the map given by the parametrized Morse lemma and if f_1, f_2 and f_3 are smooth functions such that

$$\operatorname{Re}\phi(u,v) = \gamma(u,v)^2 + f_1(v) \qquad \text{and} \qquad \operatorname{Im}\phi(u,v) = \gamma(u,v)f_2(u,v) + f_3(v),$$

then Lemma 12 implies that either $f_1(v) \sim_0 \alpha_p v^p$ with $p \leq 5$ or $f_3(v) \sim_0 \beta_p v^p$ with $p \leq 4$. By the considerations above we may conclude that $\kappa = 9/8$ is possible. $\square$

The case $J = 1$ and dependence, $d \geq 3$. It remains to consider the case $J = 1$, $d \geq 3$ and ℓ_1 is a multiple of ℓ_{d+1}. We shall deduce this case from the case $d = 2$ using the following lemma.

Lemma 13. *Let $\varepsilon > 0$, $\tau \in \mathbb{R}$ and $z_3, \ldots, z_d \in \mathbb{T}^{d-2}$. Consider the set*

$$A_{z_3,\ldots,z_d}(\tau,\varepsilon) = \left\{(z_1, z_2) \in \mathbb{T}^2 \,:\, \Phi(z) \in Q(\tau,\varepsilon)\right\}.$$

Then, for every $w_3, \ldots, w_d \in \mathbb{T}^{d-2}$, there exists a neighbourhood

$$\mathscr{W} \ni (w_3, \ldots, w_d)$$

in $\mathbb{T}^{d-2}$ such that, for all $(z_3, \ldots, z_d) \in \mathscr{W}$ we have

$$A_{z_3,\ldots,z_d}(\tau,\varepsilon) \subset A_{w_3,\ldots,w_d}(\tau, 2\varepsilon).$$

Proof. Assume that this is not the case. Then there exists a sequence

$$\left(z_1^{(k)}, \ldots, z_d^{(k)}\right)$$

in $\mathbb{T}^d$ such that $z_j^{(k)} \to w_j$ for $3 \leq j \leq d$ and

$$\left(z_1^{(k)}, z_2^{(k)}\right) \in A_{z_3^{(k)},\ldots,z_d^{(k)}}(\tau,\varepsilon) \backslash A_{w_3,\ldots,w_d}(\tau, 2\varepsilon).$$

Extracting a subsequence if necessary, we may assume that $z_1^{(k)} \to w_1$ and $z_2^{(k)} \to w_2$ for some $(w_1, w_2) \in \mathbb{T}^2$. By the continuity of Φ,this implies that $\Phi(w) \in \overline{Q(\tau,\varepsilon)} \backslash Q(\tau, 2\varepsilon)$, which is a contradiction. $\square$

We now set

$$\Psi_{1,2}(z_1, z_2) = \Phi(z_1, z_2, 1, \ldots, 1).$$

Since $J(\Phi, w) = 1$, we already know that $a_j > 0$ for all $j = 1, \ldots, d$ and hence the variables z_1 and z_2 both appear in the polynomial $\Psi_{1,2}$. Provided $\Psi_{1,2}(z) \neq \lambda_{1,2}(1 - z_1 z_2)$ for some $\lambda_{1,2} \in \mathbb{R}^*$, we know from the case $d = 2$ that there exists a neighbourhood $\mathscr{V} \ni (z_1, z_2)$ in $\mathbb{T}^2$ and $C > 0$ such that, for any $\tau \in \mathbb{R}$ and every $\varepsilon > 0$,

$$\mathbf{m}_2\left(\left\{(z_1, z_2) \in \mathscr{V} \,:\, \Phi(z_1, z_2, 1, \ldots, 1) \in Q(\tau, 2\varepsilon)\right\}\right) \leq C\varepsilon^{\kappa}$$

with $\kappa = 9/8$. By Lemma 13, there exists a neighbourhood $\mathscr{W} \ni \mathbf{1}$ in $\mathbb{T}^{d-2}$ such that, for any $(z_3, \ldots, z_d) \in \mathscr{W}$,

$$\left\{(z_1, z_2) \in \mathscr{V} \,:\Phi(z_1, \ldots, z_d) \in Q(\tau,\varepsilon)\right\} \subset \left\{(z_1, z_2) \in \mathscr{V} \,:\, \Phi(z_1, z_2, 1, \ldots, 1) \in Q(\tau, 2\varepsilon)\right\}.$$

This yields

$$\mathbf{m}_d\left(\{z \in \mathscr{V} \times \mathscr{W} \,:\, \Phi(z) \in Q(\tau,\varepsilon)\}\right) \leq C\varepsilon^{\kappa}.$$

So the result is proved except if, for every $j < k$, there exists some $\lambda_{j,k} > 0$ such that

$$\Phi(1, \ldots, 1, z_j, 1, \ldots, 1, z_k, 1, \ldots, 1) = \lambda_{j,k}(1 - z_j z_k). \tag{16}$$

Writing $1 - z_j z_k = (1 - z_j) + (1 - z_k) - (1 - z_j)(1 - z_k)$ and comparing this with the expansion of Φ near $\mathbf{1}$, we get

$$a_j = a_k = \lambda_{j,k}, \qquad b_j = 0, \qquad c_{j,k} = -\lambda_{j,k}.$$

Using (10), we may conclude that $a_j = 1/2$ and $c_{j,k} = -1/2$. In total this means that

$$\Phi(z_1, z_2, \dots, z_d) = \frac{1}{2}\sum_{j=1}^{d}(1 - z_j) - \frac{1}{2}\sum_{1 \le j < k \le d}(1 - z_j)(1 - z_k).$$

However,

$$\Phi(-1, -1, \dots, -1) = \frac{2d - 2d(d-1)}{2} = d(2 - d) < 0,$$

since $d \geq 3$. Hence (16) is not possible for every $j < k$ and we are done. $\square$

6. Proof of Lemma 12

We intend to prove this result by contradiction. We require several tedious computations, which can be done either by hand or by a computer algebra system. We have used `Xcas`, and our file is available for download [3]. In the proof below, we will skip certain computations such as computing Taylor coefficients, simplifying algebraical expressions and solving simple equations. The proof consists of three steps, and in each step we refer to the lines in [3] where the computations are performed.

The idea of the argument is rather easy. We assume that we may factorize $\operatorname{Re}\phi(u, v)$ and $\operatorname{Im}\phi(u, v)$ as (14) and (15) and we write

$$\begin{aligned}\gamma(u, v) &= -2a_1a_2u + \gamma_{2,0}u^2 + \gamma_{1,1}uv + \gamma_{0,2}v^2 + \gamma_{3,0}u^3 + \gamma_{2,1}u^2v + \gamma_{1,2}u^2v \\ &\quad + \gamma_{0,3}v^3 + \gamma_{4,0}u^4 + \gamma_{3,1}u^3v + \gamma_{2,2}u^2v^2 + \gamma_{1,3}uv^3 + \gamma_{0,4}v^4 + o\big(|u|^5 + |v|^5\big), \\ h(u, v) &= 1 + h_{1,0}u + h_{0,1}v + h_{2,0}u^2 + h_{1,1}uv + h_{0,2}v^2 + o\big(|u|^2 + |v|^2\big).\end{aligned}$$

We already know the first coefficients of γ and h by (11) and (12). We will then compare the Taylor expansions of $\operatorname{Re}\phi(u, v)$ and $\operatorname{Im}\phi(u, v)$ obtained using (13) or using (14) and (15). Looking at all coefficients of a given order, we will get first the value of the coefficients of the Taylor expansions of γ and h of a certain order and also an equation for $\operatorname{Im}(b_1)$, $\operatorname{Im}(b_2)$ and $\operatorname{Im}(c)$.

At one point, we will have more equations than variables. These equations will have to be compatible, and will force $\Phi(z_1, z_2) = (1 - z_1z_2)/2$, which is equivalent to saying $a_1 = a_2 = 1/2$ and $\operatorname{Im}(b_1) = \operatorname{Im}(b_2) = \operatorname{Im}(c) = 0$. This will imply the desired result.

Step 1. The goal of the first step is to show that if we have $a_1 = a_2 = 1/2$, then we also have $\operatorname{Im}(b_1) = \operatorname{Im}(b_2) = \operatorname{Im}(c) = 0$. In addition to this, we obtain some useful equations for the following steps. [3, Lin. 1–14]

We begin by looking at the coefficients of uv^2 in the real part of $\Phi(u, v)$. Using on the one hand (13) and on the other hand (14) we conclude that

$$\gamma_{0,2} = \frac{-6a_1^3 \operatorname{Im}(b_2) - 6a_2^3 \operatorname{Im}(b_1) + a_1a_2(a_1 + a_2) \operatorname{Im}(c)}{8a_1a_2}.$$

We then obtain the first equation for $\operatorname{Im}(b_1)$, $\operatorname{Im}(b_2)$ and $\operatorname{Im}(c)$ by looking at the coefficients of v^3 in the real part:

$$a_2^3 \operatorname{Im}(b_1) - a_1^3 \operatorname{Im}(b_2) + \frac{a_1a_2(a_2 - a_1)}{2} \operatorname{Im}(c) = 0. \tag{17}$$

Since we know the value of $\gamma_{0,2}$, we can get a second equation for $\operatorname{Im}(b_1)$, $\operatorname{Im}(b_2)$ and $\operatorname{Im}(c)$ by looking at the coefficients of v^2 in the imaginary part. Hence we get

$$\frac{4a_1a_2^2 - 3a_2^2}{4a_1} \operatorname{Im}(b_1) + \frac{4a_1^2a_2 - 3a_1^2}{4a_2} \operatorname{Im}(b_2) + \frac{-8a_1a_2 + a_1 + a_2}{8} \operatorname{Im}(c) = 0. \tag{18}$$

By the assumptions on a_1 and a_2, we know that $2(a_1 + a_2) < 3$ and hence we can solve (17) and (18) with respect to $\operatorname{Im}(b_1)$ and $\operatorname{Im}(b_2)$ to obtain

$$\operatorname{Im}(b_1) = \frac{a_1(2a_2^2 + 2a_1a_2 + a_1 - 2a_2)}{2a_2^2(2a_1 + 2a_2 - 3)} \operatorname{Im}(c)$$
$$\operatorname{Im}(b_2) = \frac{a_2(2a_1^2 + 2a_1a_2 - 2_1 + a_2)}{2a_1^2(2a_1 + 2a_2 - 3)} \operatorname{Im}(c).$$

In particular, we may conclude that if $\operatorname{Im}(c) = 0$, then we also have $\operatorname{Im}(b_1) = \operatorname{Im}(b_2) = 0$. If we substitute these values into the expression for $\gamma_{0,2}$, we obtain

$$\gamma_{0,2} = \frac{-(a_1 + a_2)^2}{2(2a_1 + 2a_2 - 3)} \operatorname{Im}(c).$$

Now, looking at the coefficient of v^4 in the real part shows that

$$(\gamma_{0,2})^2 = -\frac{a_1a_2(a_1 + a_2)(a_1a_2^2 + a_1^2a_2 - a_1^2 - a_2^2 + a_1a_2)}{4},$$

and this yields our first expression for $\operatorname{Im}(c)^2$,

$$\operatorname{Im}(c)^2 = \frac{-a_1a_2(2a_1 + 2a_2 - 3)^2(a_1a_2^2 + a_1^2a_2 - a_1^2 - a_2^2 + a_1a_2)}{(a_1 + a_2)^3}. \tag{19}$$

From (19), it is evident that if $a_1 = a_2 = 1/2$, then $\operatorname{Im}(c) = 0$. □

Step 2. In this step, we want to show that $a_1 = a_2$. Our first goal is to compute another equation to compare with (19). [3, Lin. 15–21]

We begin by successively looking at the coefficients of u^2v in the real part, uv in the imaginary part and uv^3 in the real part, to obtain

$$\begin{aligned}
\gamma_{1,1} &= \frac{a_1 - a_2}{2}\,\mathrm{Im}(c),\\
h_{0,1} &= \frac{(a_1 - a_2)(4a_1 + 4a_2 - 3)}{4a_1a_2(2a_1 + 2a_2 - 3)}\,\mathrm{Im}(c),\\
\gamma_{0,3} &= \frac{-a_1 + a_2}{24a_1a_2(2a_1 + 2a_2 - 3)} \times\\
&\quad \big(20{a_1}^2{a_2}^4 + 40{a_1}^3{a_2}^3 + 20{a_1}^4{a_2}^2 - 12a_1{a_2}^4 - 54{a_1}^2{a_2}^3 - 54{a_1}^3{a_2}^2\\
&\quad\ -12{a_1}^4a_2 + 18a_1{a_2}^3 + 18{a_1}^2{a_2}^2 + 18{a_1}^3a_2 + 3(a_1 + a_2)^2\,\mathrm{Im}(c)^2\big).
\end{aligned}$$

Using these values, we will investigate the coefficient of v^3 in the imaginary part. This term depends indeed only on $\gamma_{1,1}$, $h_{0,1}$ and $\gamma_{0,3}$. Using the above expression, we obtain our second equation on $\mathrm{Im}(c)^2$:

$$\begin{aligned}
&\frac{-3(a_1 - a_2)(a_1 + a_2)^2(a_1 + a_2 - 1)}{4a_1a_2(2a_1 + 2a_2 - 3)^2}\,\mathrm{Im}(c)^2 = \\
&\qquad\qquad \frac{-(a_1 - a_2)(3a_1a_2^2 + 3a_1^2a_2 - a_1^2 - a_2^2 - a_1a_2)}{4}.
\end{aligned} \tag{20}$$

At this stage, we have to consider several cases. Assuming that $a_1 - a_2 \neq 0$ and $a_1 + a_2 - 1 \neq 0$, we may compute

$$\mathrm{Im}(c)^2 = \frac{-a_1a_2(2a_1 + 2a_2 - 3)^2(3a_1a_2^2 + 3a_1^2a_2 - a_1^2 - a_2^2 - a_1a_2)}{3(a_1 + a_2)^2(a_1 + a_2 - 1)}. \tag{21}$$

The only possibility is that the two values for $\mathrm{Im}(c)^2$ have to coincide. Equating (19) and (21) and simplifying, we obtain

$$\frac{a_1a_2(2a_1 + 2a_2 - 3)^2P(a_1, a_2)}{3(a_1 + a_2)^2(a_1 + a_2 - 1)} = 0,$$

where

$$P(a_1, a_2) = 2a_1^3 + 2a_2^3 + a_1a_2^2 + a_1^2a_2 - 3a_1^2 - 3a_2^2 + 3a_1a_2.$$

Since $2(a_1 + a_2) < 3$, the only possibility is that $P(a_1, a_2)$ vanishes somewhere in the domain

$$\Omega = \{(a_1, a_2) \in (0, 1)^2 \;:\; a_2 < a_1,\ a_2 < 1 - a_1\}.$$

We first look at what happens on the boundary, where we have

$$\begin{aligned}
P(a_1, 0) &= 2a_1^3 - 3a_1^2 < 0 \quad \text{provided } a_1 \in (0, 1),\\
P(a_1, a_1) &= 3a_1^2(2a_1 - 1) < 0 \quad \text{provided } a_1 \in (0, 1/2),\\
P(a_1, 1 - a_1) &= -(2a_1 - 1) < 0 \quad \text{provided } a_1 \in (1/2, 1).
\end{aligned}$$

Hence, P is negative on the boundary of , except at $(1/2, 1/2)$. Hence, if P vanishes in Ω, then it admits a critical point there. Now, we consider the system

$$\begin{cases} 0 = \frac{\partial P}{\partial a_1}(a_1, a_2) &= 6a_1^2 + a_2^2 + 2a_1a_2 - 6a_1 + 3a_2, \\ 0 = \frac{\partial P}{\partial a_2}(a_1, a_2) &= a_1^2 + 6a_2^2 + 2a_1a_2 + 3a_1 - 6a_2. \end{cases}$$

The solutions of this system are easily found to be at the intersection of two distinct ellipses. We cannot have more than two points of intersection and we have two trivial solutions, $(0, 0)$ and $(1/3, 1/3)$. Hence, none of the critical points of P are inside . Hence we get a contradiction, and we have finished this case.

Hence we must have $a_1 + a_2 = 1$ or $a_1 = a_2$. Let us now investigate the case $a_1 + a_2 = 1$. Looking at (20), this means that either $a_1 = a_2 = 1/2$ (and we are done) or

$$3a_1a_2^2 + 3a_1^2a_2 - a_1^2 - a_2^2 - a_1a_2 = 0.$$

Taking into account that $a_2 = 1 - a_1$, we get the equation $-4a_1^2 + 4a_1 - 1 = 0$ which admits the single solution $a_1 = 1/2$ and we get the same conclusion. Hence, the only remaining possibility is that $a_1 = a_2$. □

Step 3. It remains to deal with the case $a_1 = a_2 = a \in (0, 1/2]$. We can no longer use (21) and need to find another equation for $\operatorname{Im}(c)^2$. [3, Lin. 21–30]

We will be looking at the coefficient before v^4 in the imaginary part. By considering $\gamma \times h$, we see that this coefficient is equal to

$$\gamma_{0,4} + \gamma_{0,3}h_{0,1} + \gamma_{0,2}h_{0,2}.$$

Hence, it remains to compute $\gamma_{0,4}$ and $h_{0,2}$. First, we compute some auxiliary values. By looking at u^3 in the real part, u^2 in the imaginary part and u^2v^2 in the real part, respectively, we obtain

$$\begin{aligned} \gamma_{2,0} &= -\frac{a(2a-1)}{3a-4}\operatorname{Im}(c), \\ h_{1,0} &= \frac{(2a-1)(4a-1)}{2a(4a-3)}\operatorname{Im}(c), \\ \gamma_{1,2} &= \frac{a(2a-1)\left(48a^4 - 72a^3 + 27a^2 + 4\operatorname{Im}(c)^2\right)}{4(4a-3)^2}. \end{aligned}$$

Knowing these values, we look at the coefficients of uv^4 in the real part and uv^2 and the imaginary part, respectively, to obtain

$$\begin{aligned} \gamma_{0,4} &= -a\operatorname{Im}(c)\frac{32a^5 - 96a^4 + 90a^3 + 12a\operatorname{Im}(c)^2 - 27a^2 - 6\operatorname{Im}(c)^2}{6(4a-3)^3}, \\ h_{0,2} &= (-2a+1)\frac{128a^5 - 240a^4 + 144a^3 + 16a\operatorname{Im}(c)^2 - 27a^2 - 8\operatorname{Im}(c)^2}{8a(4a-3)^2}. \end{aligned}$$

Finally, we investigate the coefficient of v^4 in the imaginary part to obtain the equation

$$a(2a-1)\frac{16a^4-32a^3+15a^2+4\,\mathrm{Im}(c)^2}{4(4a-3)^2}\,\mathrm{Im(c)}=0.$$

Now, if $\mathrm{Im}(c)=0$ and $a_1=a_2=a$, it follows at once from (19) that $a=1/2$, since $a\in(0,1/2]$. Conversely, we divide away $\mathrm{Im}(c)$ and solve for $\mathrm{Im}(c)^2$ to obtain the equation

$$\mathrm{Im}(c)^2=-a^2\frac{(4a-3)(4a-5)}{4}.$$

Now, this has to be equal to (19), and we find

$$-\frac{a(2a-1)(4a-3)^2}{8}=-\frac{a^2(4a-3)(4a-5)}{4}.$$

Here the only solutions are $a=0$ and $a=3/4$, neither of which belong to $(0,1/2]$. Hence the assumption $\mathrm{Im}(c)\neq 0$ must be wrong and we conclude $a_1=a_2=1/2$. □

7. Remarks and Further Examples

If we look more closely at the map Φ defined in Lemma 9 (the negative part of Theorem 3), then we may observe that these counterexamples all satisfy

$$\begin{aligned}\mathrm{Re}\,\Phi(e^{i\theta_1},\ldots,e^{i\theta_d}) &= \frac{\theta_1^2}{2}+o(\theta_1^2)\\ \mathrm{Im}\,\Phi(e^{i\theta_1},\ldots,e^{i\theta_d}) &= -\theta_1+o(|\theta_1|).\end{aligned}$$

Hence, $J\big(\Phi,\mathbf{1}\big)=1$ and, using the terminology of the Section 5, we have dependence. Our next results shows that we may also have non-compactness if $J(\Phi,w)=0$ for some $w\in\mathbb{T}^d$.

Theorem 14. *There are polynomials φ with unrestricted range, of any complex dimension $d\geq 2$ and of any complex degree ≥ 4 for which the corresponding composition operator $\mathscr{C}_\varphi$ is non-compact and such that they admit a minimal Bohr lift Φ satisfying $J(\Phi,w)=0$ for any $w\in\mathbb{T}^d$ with $\mathrm{Re}\,\Phi(w)=0$.*

Proof. Let $\delta>0$ and define

$$\Phi(z_1,z_2)=2(1-z_1)+(1-z_1)^2\big(1-\delta(1-z_2)-\delta(1-z_1)(1-z_2)\big).$$

Let $\varphi(s)=\Phi(p_1^{-s},p_2^{s})$. Clearly Φ is a minimal Bohr lift of φ. Then a computation shows that

$$\begin{aligned}\mathrm{Re}\,\Phi(z_1,z_2)=2(1-\cos x)\Big((1-\cos x)\big(1+2\delta(\sin x\sin y+(1-\cos y)\cos x)\big)\\ +\delta(1-\cos y)\Big).\end{aligned}$$

Clearly, for small enough $\delta > 0$ this quantity is non-negative. Hence $\varphi \in \mathscr{G}$ and $J(\Phi, (1,1)) = 0$ because of the relation between a_1 and b_1. Considerations similar to those of Lemma 9 show that $\mathscr{C}_\varphi$ cannot be compact. To produce a counterexample with degree 3 and a bigger complex dimension d, we may simply replace $(1 - z_2)$ by

$$\frac{1}{d-1} \cdot \sum_{j=2}^{d} (1 - z_j)$$

in the definition of Φ. The production of examples with degree ≥ 5 is easier. Set

$$\Phi(z) = (1 - z_1) + \frac{1}{2}(1 - z_1)^2 + \delta(1 - z_1)^4 P(z),$$

where $P(z) = P(z_1, z_2, \ldots, z_d)$ is any polynomial. The proof follows now that of Lemma 9. $\square$

The reason the first counterexample in Theorem 14 works is that we have a cancellation of the term $(1-\cos x)\sin x \sin y$. It seems difficult to obtain the same cancellation if we restrict ourself to degree 3 and require $J = 0$.

Question. Is it possible to construct a counterexample of degree 3 with $J = 0$?

An answer to the question would in a certain sense improve the optimality of Lemma 10, but it would not yield the complete answer to which Dirichlet polynomials in $\mathscr{G}$ induce compact composition operators. Indeed, the natural next point of investigation would be this: What happens when the "quartic form" is degenerate?

In this case, terms of degree 5 also have to disappear. This follows by the mapping properties and the argument is identical to the one used to show that degree 3 terms disappear in the case $J = 0$ given above. Hence we are reduced to studying a "sextic form".

Our counterexamples can be modified to work in this case, but they now have degree 6 and 7. Degree ≤ 3 will also easily reduce to the case of Theorem 2 in the same manner as $J = 0$ did for degree ≤ 2. However, the cases with degree 4 and 5 would need further investigation. Even if we could solve this case, we would need to investigate the case when the "sextic form" is degenerate and this leads to the "octic form" and so on.

Remark. The previous counterexample shows that we cannot deduce Theorem 2 from Theorem 3. Indeed, it is easy to construct symbols $\varphi \in \mathscr{G}$ which may be written $\varphi(s) = \sum_{j=1}^{d} P_j(p_j^{-s})$ and such that $J(\Phi, \mathbf{1}) = 0$ for Φ a minimal Bohr lift of φ. Indeed, we may consider

$$P_j(z) = (1 - z) + \frac{1}{2}(1 - z)^2 + \delta(1 - z)^4 Q_j(z)$$

where Q_j is an arbitrary polynomial and $\delta > 0$ is sufficiently small. Then $\mathscr{C}_\varphi$ is compact by Theorem 2 if $d \geq 2$ but this cannot be deduced from Lemma 10.

This construction can be generalized to show that Theorem 2 can handle a variety of different interesting cases not covered by Theorem 3. In fact, given any d positive integers k_j, we may find a polynomial $\Phi(z) = \sum_{j=1}^d \Phi_j(z_j)$ which is a minimal Bohr lift of some $\varphi \in \mathscr{G}$, with $\operatorname{Re}\Phi(z_1, \ldots, z_d) = 0$ if and only if $z = \mathbf{1}$ and here we have the expansion

$$\operatorname{Re}\Phi\big(e^{i\theta_1}, \ldots, e^{i\theta_d}\big) = \sum_{j=1}^d \theta_j^{2k_j} + o\left(\sum_{j=1}^d \theta_j^{2k_j}\right),$$

$$\operatorname{Im}\Phi\big(e^{i\theta_1}, \ldots, e^{i\theta_d}\big) = \sum_{j=1}^d a_1^{(j)}\theta_j + o\left(\sum_{j=1}^d |\theta_j|\right).$$

As remarked upon in the proof of Theorem 2, we must have $a_1^{(j)} > 0$. The construction of such a polynomial is immediate from our next result.

Lemma 15. *There is a polynomial $\Phi : \mathbb{C} \to \mathbb{C}$ which satisfy $\operatorname{Re}\Phi\big(e^{ix}\big) = (1 - \cos x)^k$, for any $k \in \mathbb{N}$.*

Proof. Fix N with $k \leq 2N$, and for real numbers a_n and b_n consider

$$\Phi(z) = \sum_{n=1}^N \frac{(-1)^{n-1}}{2^n}\big(a_n(1-z)^{2n-1} - b_n(1-z)^{2n}\big).$$

Our first goal is to expand the real part of $\Phi\big(e^{ix}\big)$ as degree $2N$ polynomial in $(1 - \cos x)$ with no constant term. To this end, we compute

$$\begin{aligned}\big(1 - e^{ix}\big)^{2n-1} &= e^{i(2n-1)x/2}\big(e^{-ix/2} - e^{ix/2}\big)^{2n-1} \\ &= e^{i(2n-1)x/2}2^{2n-1}(-1)^n i \sin^{2n-1}\left(\frac{x}{2}\right).\end{aligned}$$

We use $2\sin^2(x/2) = 1 - \cos x$, and obtain

$$\begin{aligned}\operatorname{Re}\big(1 - e^{ix}\big)^{2n-1} &= 2^{2n-1}(-1)^{n-1}\sin^{2n-1}\left(\frac{x}{2}\right)\sin\left(nx - \frac{x}{2}\right) \\ &= (-1)^{n-1}2^n(1-\cos x)^n \frac{\sin\left(nx - \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.\end{aligned} \tag{22}$$

Similarly, we obtain

$$\begin{aligned}\operatorname{Re}\big(1 - e^{ix}\big)^{2n} &= 2^{2n}(-1)^n \sin^{2n}\left(\frac{x}{2}\right)\cos(nx) \\ &= (-1)^n 2^n (1-\cos x)^n \cos(nx).\end{aligned} \tag{23}$$

To continue the computations, we introduce the Chebyshev polynomials

$$U_n(y) = \sum_{j=0}^{n} (-2)^j \frac{(n+j+1)!}{(n-j)!(2j+1)!} (1-y)^j,$$
$$T_n(y) = n \sum_{j=0}^{n} (-2)^j \frac{(n+j-1)!}{(n-j)!(2j)!} (1-y)^j.$$

The Chebyshev polynomials are relevant due to the formulas

$$\sin nx = \sin(x) U_{n-1}(\cos x) \qquad \text{and} \qquad \cos nx = T_n(\cos x).$$

We record the following coefficients.

$$u_{n-2}^{(n)} = (-2)^{n-1} \left(-\frac{(2n-1)(n-2)}{2} \right)$$
$$u_{n-1}^{(n)} = (-2)^{n-1}\,(2n), \qquad t_{n-1}^{(n)} = (-2)^{n-1}\,(n),$$
$$u_{n}^{(n)} = (-2)^{n-1}\,(-2), \qquad t_{n}^{(n)} = (-2)^{n-1}\,(-1).$$

Now, we rewrite (23) as

$$\operatorname{Re}\left(1-e^{ix}\right)^{2n} = (-1)^n 2^n (1-\cos x)^n T_n(\cos x),$$

which is then clearly a degree $2n$ polynomial in $(1-\cos x)$ with no constant term. For (22) we have to work a bit more, so we first compute

$$\frac{\sin\left(nx - \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{\sin(nx)\cos\left(\frac{x}{2}\right) - \cos(nx)\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \cos^2\left(\frac{x}{2}\right) U_{n-1}(\cos x) - \frac{T_n(\cos x)}{2},$$

which implies that we may rewrite (22) as

$$\operatorname{Re}\left(1-e^{ix}\right)^{2n-1} = (-1)^{n-1} 2^n (1-\cos x)^n \times \left(\left(1 - \frac{1-\cos x}{2}\right) U_{n-1}(\cos x) - \frac{T_n(\cos x)}{2} \right).$$

Again we observe that this is a polynomial of degree $2n$ in $(1-\cos x)$ with no constant term. In total, we have

$$\operatorname{Re}\Phi\left(1-e^{ix}\right) = \sum_{m=1}^{2N} c_m (1-\cos x)^m = \sum_{n=1}^{N} \left(a_n P_n(1-\cos x) + b_n Q_n(1-\cos x)\right),$$

where

$$P_n(y) = \sum_{j=0}^{n} d_j^{(n)} y^{n+j} = y^n \left(\left(1 - \frac{y}{2}\right) U_{n-1}(1-y) - \frac{T_n(1-y)}{2} \right),$$

$$Q_n(y) = \sum_{j=0}^{n} e_j^{(n)} y^{n+j} = y^n T_n(1-y).$$

Given any choice of c_m (for instance $c_m = 0$ for $m \neq k$ and $c_k = 1$), we now have $2N$ linear equations and $2N$ unknowns, a_n and b_n for $1 \leq n \leq N$. We will now show that this system can always be solved.

We first observe that a_n and b_n only have an effect on c_m when $n \leq m \leq 2n$. Ordering the unknowns as $a_N, b_N, a_{N-1}, b_{N-1}, \ldots, a_1, b_1$ and the datas as $c_{2N}, c_{2N-1}, \ldots, c_1$, this means that the matrix of our system can be written in upper triangular block form, where the blocks on the diagonal are

$$\begin{pmatrix} e_n^{(n)} & d_n^{(n)} \\ e_{n-1}^{(n)} & d_{n-1}^{(n)} \end{pmatrix}, \qquad n = N, N-1, \ldots, 1.$$

We know that $e_{n-1}^{(n)} = t_{n-1}^{(n)}$ and $e_n^{(n)} = t_n^{(n)}$, which we recorded above. It is now easy to verify that

$$d_{n-1}^{(n)} = u_{n-1}^{(n)} - \frac{u_{n-2}^{(n)}}{2} - \frac{t_{n-1}^{(n)}}{2} = (-2)^{n-1} \left(\frac{3n}{2} + \frac{(2n-1)(n-2)}{4} \right),$$

$$d_n^{(n)} = u_n^{(n)} - \frac{u_{n-1}^{(n)}}{2} - \frac{t_n^{(n)}}{2} = (-2)^{n-1} \left(-\frac{3}{2} - n \right).$$

Hence we are reduced to considering the equation

$$0 = \frac{d_{n-1}^{(n)} e_n^{(n)} - d_n^{(n)} e_{n-1}^{(n)}}{4^{n-1}} = \left(\frac{3n}{2} + \frac{(2n-1)(n-2)}{4} \right)(-1) - \left(-\frac{3}{2} - n \right) n$$
$$= n^2 - \frac{(2n-1)(n-2)}{4},$$

which has no integer solutions, and we are done. □

The construction of Φ with specific expansion facilitated by Lemma 15 will be used in the next section to prove Corollary 4.

8. Approximation numbers

In this section, we consider only the case $c_0 = 0$. We intend to estimate the decay of $a_n(\mathscr{C}_\varphi)$ for maps φ which are, in a certain sense, regular at their boundary points. For this we need as previously a careful inspection of the behaviour of the Bohr lift Φ near these boundary points.

Definition. Suppose that $\varphi(s) = c_1 + \sum_{n=2}^{N} c_n n^{-s} \in \mathscr{G}$, and that φ has complex dimension d and unrestricted range. Let Φ be a minimal Bohr lift of φ and let $w \in \mathbb{T}^d$ be such that $\operatorname{Re}\Phi(w) = 0$. We say that φ is *boundary regular* at w if there exist independent linear forms $\ell_1, \ldots, \ell_d$ on $\mathbb{C}^d$, even integers $k_1 \geq k_2 \geq \ldots \geq k_d$ and real numbers $b_1, \ldots, b_d, \tau$ with $b_1 \neq 0$ such that

$$\operatorname{Re}\Phi\big(e^{i\theta_1}w_1, \ldots, e^{i\theta_d}w_d\big) = \ell_1(\theta)^{k_1} + \cdots + \ell_d(\theta)^{k_d} + \sum_{j=1}^{d} o\left(\ell_j^{k_j}(\theta)\right) \tag{24}$$

$$\operatorname{Im}\Phi\big(e^{i\theta_1}w_1, \ldots, e^{i\theta_d}w_d\big) = \tau + b_1\ell_1(\theta) + \cdots + b_d\ell_d(\theta) + o\left(\sum_{j=1}^{d} |\ell_j(\theta)|\right). \tag{25}$$

We define the *compactness index* of φ at w as

$$\eta_{\varphi,w} = \left(\sum_{j=2}^{d} \frac{1}{k_j}\right) \times \frac{k_1}{2(k_1-1)}.$$

If every boundary point is boundary regular, we say that φ is *boundary regular.*

The proof of Theorem 2 then shows that given a boundary regular map φ, the composition operator $\mathscr{C}_\varphi$ is compact if and only if $d \geq 2$. We shall now assume that there is only one point $w \in \mathbb{T}^d$ such that $\operatorname{Re}\Phi(z) = 0$. In this case, we let the *compactness index* of φ be $\eta_\varphi := \eta_{\varphi,w}$.

The main theorem of this section now reads.

Theorem 16. *Let $\varphi(s) = c_1 + \sum_{n=2}^{N} c_n n^{-s} \in \mathscr{G}$ have unrestricted range and complex dimension d. Let Φ be a minimal Bohr lift and assume that there exists a unique $w \in \mathbb{T}^d$ such that* $\operatorname{Re}\Phi(w) = 0$. *Suppose moreover that φ is boundary regular at w. Then*

$$\left(\frac{1}{n}\right)^{\eta_\varphi} \ll a_n(\mathscr{C}_\varphi) \ll \left(\frac{\log n}{n}\right)^{\eta_\varphi}.$$

This statement may be applied to several cases.

Corollary 17. *Let $\varphi(s) = c_1 + \sum_{n=2}^{N} c_n n^{-s} \in \mathscr{G}$ have unrestricted range and complex dimension d. Let Φ be a minimal Bohr lift of φ and assume that there exists a unique $w \in \mathbb{T}^d$ such that* $\operatorname{Re}\Phi(w) = 0$ *and that $J(\Phi, w) = d$. Then*

$$\left(\frac{1}{n}\right)^{(d-1)/2} \ll a_n(\mathscr{C}_\varphi) \ll \left(\frac{\log n}{n}\right)^{(d-1)/2}.$$

Proof. Under these assumptions, φ is boundary regular at w with $k_1 = \cdots = k_d = 2$. □

In particular, this corollary covers the result of Queffélec and Seip for linear symbols (3), as well as the map φ_1 given in (4). We may also apply Theorem 16 to the maps considered in Theorem 2. In this case, one has simply $\ell_j(\theta) = \theta_j$ (up to a reordering of the terms).

Another interesting application of Theorem 16 is that we may distinguish the Schatten classes of bounded linear operators on $\mathscr{H}^2$ using composition operators, as mentioned in the introduction.

Proof of Corollary 4. Let $p' < q'$ and $\varepsilon > 0$ be such that

$$p \leq p'/2 \quad \text{and} \quad \left(\frac{1}{2} + \varepsilon\right) q' \leq q.$$

Then let $d \geq 2$ and $k \geq 2$ even such that

$$p' < \frac{d-1}{k} < q' \quad \text{and} \quad \frac{1}{2} < \frac{k}{2(k-1)} < \frac{1}{2} + \varepsilon.$$

By Lemma 15, we know that there exists a boundary regular polynomial $\Phi : \mathbb{T}^d \to \mathbb{C}_0$ such that $\operatorname{Re} \Phi(w) = 0$ if and only if $w = \mathbf{1}$ and

$$\Phi\big(e^{i\theta_1}, \dots, e^{i\theta_d}\big) = \theta_1^k + \dots + \theta_d^k + o\big(\theta_1^k\big) + \dots + o\big(\theta_d^k\big).$$

Letting $\varphi \in \mathscr{G}$ any map such that Φ is a minimal Bohr lift of φ, we immediately get

$$\left(\frac{1}{n}\right)^{\frac{d-1}{k} \times \frac{k}{2(k-1)}} \ll a_n(\mathscr{C}_\varphi) \ll \left(\frac{\log n}{n}\right)^{\frac{d-1}{k} \times \frac{k}{2(k-1)}},$$

which completes the proof. □

Theorem 16 may be also applied to many other maps. We will consider here the map φ_2 given in (4). Its boundary regularity is different than that of φ_1, and hence the degree of compactness is also different.

Example. Let $\varphi_2(s) = 13/2 - 4 \cdot 2^{-s} - 4 \cdot 3^{-s} + 2 \cdot 6^{-s}$ as in (4) and let Φ be its minimal Bohr lift. It can be shown that $\operatorname{Re} \Phi(w) = 0$ for $w \in \mathbb{T}^2$ if and only if $w = (1,1)$, and

$$\begin{aligned} \operatorname{Re} \Phi\big(e^{i\theta_1}, e^{i\theta_2}\big) &= \ell_1(\theta)^4 + \ell_2(\theta)^2 + o\big(\ell_1^4(\theta)\big) + o\big(\ell_2^2(\theta)\big), \\ \operatorname{Im} \Phi\big(e^{i\theta_1}, e^{i\theta_2}\big) &= -2\ell_1(\theta) + o\big(|\ell_1(\theta)| + |\ell_2(\theta)|\big), \end{aligned}$$

where $\ell_1(\theta) = \theta_1 + \theta_2$ and $\ell_2(\theta) = \theta_1 - \theta_2$. Hence $\eta_{\varphi_2} = (1/2) \times (4/6) = 1/3$.

The remaining part of this section is devoted to the proof of Theorem 16. We use the scheme introduced by Queffélec and Seip in [11] in the context of Dirichlet series (see also [12] for similar works on the classical Hardy space of the disk). Their method is based on Carleson measures, interpolation sequences and model spaces. In Subsection 8.1, we survey these tools and give a couple of lemmas.

Subsection 8.2 is devoted to the proof of the upper bound, in a more general context, whereas Subsection 8.3 will be devoted to the lower bound.

8.1. **Tools.**

The Hyperbolic Metric. The pseudo-hyperbolic metric on the half-plane $\mathbb{C}_0$ is defined by

$$\rho(z, w) = \left|\frac{z - w}{z + \overline{w}}\right| = \frac{1 - e^{-d(z,w)}}{1 + e^{d(z,w)}}$$

where $d(z, w)$ is the hyperbolic distance between two points z and w in $\mathbb{C}_0$. The hyperbolic length of a curve $\Gamma \subset \mathbb{C}_0$ is given by the integral

$$L_p(\Gamma) = \int_\Gamma \frac{|dz|}{\operatorname{Re} z}.$$

Carleson Measures and Interpolating Sequences. Let H be a Hilbert space of functions defined on some measurable set Ω in $\mathbb{C}$. A non-negative Borel measure μ on Ω is a *Carleson measure* for H if there exists some constant $C > 0$ such that

$$\int_\Omega |f(z)|^2 d\mu(z) \leq C\|f\|_H^2,$$

for every f in H. The smallest possible C will be called the *Carleson norm* of μ with respect to H and will be denoted by $\|\mu\|_{\mathscr{C},H}$.

We also assume that the linear point evaluation is bounded at any $z \in \Omega$. Then H admits a reproducing kernel $K_z^H \in H$ for any $z \in \Omega$ which satisfies $f(z) = \langle f, K_z^H \rangle$ for every $f \in H$. We then say that a sequence $Z = (z_m)$ of distinct points in Ω is a *Carleson sequence* for H if the measure

$$\mu_{Z,H} := \sum_m \left\|K_{z_m}^H\right\|_H^{-2} \delta_{z_m}$$

is a *Carleson measure* for H.

We say that a sequence $Z = (z_m)$ of distinct points in Ω is an *interpolating sequence* for H if the interpolation problem $f(z_m) = a_m$ has a solution $f \in H$ whenever the admissibility condition

$$\sum_m |a_m|^2 \left\|K_{z_m}^H\right\|_H^{-2} < \infty$$

is satisfied. By the open mapping theorem, if Z is an interpolating sequence for H, there is a constant $C > 0$ such that we can solve $f(z_m) = a_m$ with f satisfying

$$\|f\|_H \leq C \left(\sum_m |a_m|^2 \left\|K_{z_m}^H\right\|_H^{-2}\right)^{1/2}.$$

The smallest constant C with this property will be called the *constant of interpolation* of Z and will be denoted by $M_H(Z)$.

We shall consider the two spaces $H = \mathscr{H}^2$ and $H = H^2\left(\mathbb{T}^d\right)$. Then we have, respectively, $\Omega = \mathbb{C}_{1/2}$ and $\Omega = \mathbb{D}^d$, and moreover

$$\left\|K_s^{\mathscr{H}^2}\right\|^{-2} = [\zeta(2\operatorname{Re} s)]^{-1} \qquad \text{and} \qquad \left\|K_z^{H^2(\mathbb{T}^d)}\right\|^{-2} = \prod_{j=1}^{d}(1-|z_j|^2).$$

We will need the three following lemmas.

Lemma 18. *Let μ be a Borel measure on $\overline{\mathbb{C}_0}$, let $\sigma \in (0,1)$ and $R > 0$. Assume that μ is supported on the rectangle $0 \le \operatorname{Re} s \le \sigma$, $|\operatorname{Im} s| \le R$. Then*

$$\|\mu\|_{\mathscr{C},\mathscr{H}^2} \ll_R \sup_{\varepsilon>0,\ \tau\in\mathbb{R}} \frac{\mu\big(Q(\tau,\varepsilon)\big)}{\varepsilon} \le 2 \sup_{\varepsilon\in(0,\sigma),\ \tau\in\mathbb{R}} \frac{\mu\big(Q(\tau,\varepsilon)\big)}{\varepsilon}.$$

Proof. The first inequality is Lemma 2.3 in [11] (the involved constant does not depend on $\sigma \in (0,1)$). The second follows from the inequality

$$\sup_{\tau\in\mathbb{R}} \frac{\mu\big(Q(\tau,2^{k+1}\sigma)\big)}{2^{k+1}\sigma} \le \sup_{\tau\in\mathbb{R}} \frac{\mu\big(Q(\tau,2^{k}\sigma)\big)}{2^{k}\sigma},$$

valid for any $k \ge 0$. Indeed, for any $\tau \in \mathbb{R}$ and any $k \ge 0$, we may find $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$\mu\big(Q(\tau,2^{k+1}\sigma)\big) = \mu\big(Q(\tau_1,2^{k}\sigma)\big) + \mu\big(Q(\tau_2,2^{k}\sigma)\big),$$

since the support of μ is contained in $0 \le \operatorname{Re} s \le \sigma$. □

Lemma 19. *Let $\nu > 0$. There exists $C > 0$ such that, for any $\delta \in (0, 1/\nu)$, $M_{\mathscr{H}^2}(S_\delta) \le C$ where $S_\delta = (s_m)_{m=1}^{1/\delta}$ with $s_m = \frac{1}{2} + \nu\delta + im\delta$.*

Proof. The proof of this lemma can be found in [11, Sec 8.2]. □

Lemma 20. *Let $C_1, C_2 > 0$. There exists $D > 0$ such that for any $\delta > 0$ and any (finite) sequence*

$$Z = \big(Z(\alpha)\big) = \big((1-\rho_1(\alpha))e^{i\theta_1(\alpha)},\ \dots,\ (1-\rho_d(\alpha))e^{i\theta_d(\alpha)}\big)$$

in $\mathbb{D}^d$ satisfying

- $\sup_{j=1,\dots,d} |\theta_j(\alpha) - \theta_j(\beta)| \ge C_1\delta$, *when $\alpha \ne \beta$,*
- $\rho_j(\alpha) \le C_2\delta$, *for any α and $j = 1, \dots, d$,*

we have $\left\|\mu_{Z,H^2(\mathbb{T}^d)}\right\|_{\mathscr{C},H^2(\mathbb{T}^d)} \le D$.

Proof. To each point $Z(\alpha)$, we associate a rectangle R_α on the distinguish boundary $\mathbb{T}^d$ centered at

$$\left(\frac{z_1(\alpha)}{|z_1(\alpha)|}, \dots, \frac{z_d(\alpha)}{|z_d(\alpha)|}\right),$$

with side lengths $2(1-|z_1(\alpha)|)$, ..., $2(1-|z_d(\alpha)|)$. By Chang's characterization of Carleson measures on the polydisc (see [4] or [6]), it is enough to show that we for all open sets $\mathscr{U}$ of $\mathbb{T}^d$ have

$$\sum_{R_\alpha \subset \mathscr{U}} \mathbf{m}_d(R_\alpha) \le D\mathbf{m}_d(\mathscr{U}).$$

If R is some rectangle in $\mathbb{T}^d$ and $\lambda > 0$, denote by λR the rectangle with the same center and side lengths multiplied by λ. Then our assumptions on Z imply that there exists some $\lambda \in (0,1)$ depending only on C_1 and C_2 such that the rectangles R_α are pairwise disjoint. Thus

$$\sum_{R_\alpha \subset \mathscr{U}} \mathbf{m}_d(\mathscr{U}) \le \sum_{R_\alpha \subset \mathscr{U}} \frac{1}{\lambda^d}\mathbf{m}_d(\lambda R_\alpha) \le \frac{1}{\lambda^d}\mathbf{m}_d\left(\bigcup_{R_\alpha \subset \mathscr{U}} \lambda R_\alpha\right) \le \frac{1}{\lambda^d}\mathbf{m}_d(\mathscr{U}),$$

which completes the proof with $D = 1/\lambda^d$. □

The Queffélec–Seip Method. We have to introduce additional conventions. For $\varphi \in \mathscr{G}$ and Ω a compact subset of $\mathbb{C}_0$, we denote by $\mu_{\varphi,\Omega}$ the non-negative Borel measure on $\overline{\mathbb{C}_0}$ defined by

$$\mu_{\varphi,\Omega}(E) := \mathbf{m}_\infty\left(\{z \in \mathbb{T}^\infty \,:\, \Phi(z) \in E\backslash\Omega\}\right).$$

Next, assume that φ has complex dimension d and Bohr lift $\Phi : \mathbb{C}^d \to \mathbb{C}$. Let $S = (s_m)$ be a sequence of n points in $\mathbb{C}_{1/2}$ and let Z be a finite sequence of points in $\mathbb{D}^d$ such that $\Phi(Z) = S - \frac{1}{2}$. We set

$$N_\Phi(s_m; Z) := \sum_{z \in Z \cap \Phi^{-1}(s_m - 1/2)} \left\|K_z^{H^2(\mathbb{T}^d)}\right\|^{-2}.$$

We state Theorem 4.1 of [11] as the forthcoming lemma (we have modified it slightly to take into account our normalization).

Lemma 21. *Let $\varphi(s) = \sum_{n=1}^\infty c_n n^{-s} \in \mathscr{G}$ such that $\varphi(\mathbb{C}_0)$ is bounded.*

(a) *Let $\sigma > 0$ and Ω be a compact subset of $\overline{\mathbb{C}_\sigma}$. Let B be a Blaschke product of degree n on $\mathbb{C}_0$ whose zeros lie in Ω. Then*

$$a_n(\mathscr{C}_\varphi) \le \left(\sup_{s\in\Omega}|B(s)|^2\zeta(1+2\sigma) + \sup_{\varepsilon>0,\tau\in\mathbb{R}} \frac{\mu_{\varphi,\Omega}(Q(\tau,\varepsilon))}{\varepsilon}\right)^{1/2}.$$

(b) *Assume that φ has complex dimension d. Let S and Z be finite sets in respectively $\mathbb{C}_{1/2}$ and $\mathbb{D}^d$ such that $\Phi(Z) = S - \frac{1}{2}$. Then*

$$a_n(\mathscr{C}_\varphi) \ge [M_{\mathscr{H}^2}(S)]^{-1} \left\|\mu_{Z,H^2(\mathbb{T}^d)}\right\|_{\mathscr{C},H^2(\mathbb{T}^d)}^{-1/2} \inf_m \left[N_\Phi(s_m;Z)\zeta(2\operatorname{Re} s_m)\right]^{1/2}.$$

8.2. **The Upper Bound.**

Let $\varphi(s)=\sum_{n=1}^{\infty}c_n n^{-s}\in\mathscr{G}$ and suppose that $\varphi(\mathbb{C}_0)$ bounded. By Lemma 5, $\mathscr{C}_\varphi$ is compact if and only if $\mu_\varphi\big(Q(\tau,\varepsilon)\big)=o(\varepsilon)$ uniformly in $\tau\in\mathbb{R}$. We are planning to get an upper bound of $a_n(\mathscr{C}_\varphi)$ depending on the behaviour of $\sup_{\tau\in\mathbb{R}}\mu_\varphi\big(Q(\tau,\varepsilon)\big)$ with respect to ε and on the size of the image of φ near a boundary point.

Thus, let Φ be a Bohr lift of φ. We define κ_φ as the infimum of those $\kappa\geq 1$ such that there exists a constant $C>0$ such that, for every $\tau\in\mathbb{R}$ and every $\varepsilon>0$,

$$\mathbf{m}_\infty\left(\{z\in\mathbb{T}^\infty\,:\,\Phi(z)\in Q(\tau,\varepsilon)\}\right)\leq C\varepsilon^{\kappa}.$$

Assume now that there exists a unique $w\in\mathbb{T}^\infty$ such that $\operatorname{Re}\Phi(w)=0$ and write $\Phi(w)=i\tau$. Let ω_φ be the infimum of the positive ω such that, for any $s\in\mathbb{C}_0$,

$$|\operatorname{Im}\varphi(s)-\tau|^{\omega}\leq C\left(\operatorname{Re}\varphi(s)-\frac{1}{2}\right).$$

Theorem 22. *Let $\varphi(s)=\sum_{n=1}^{\infty}c_n n^{-s}\in\mathscr{G}$ with $\varphi(\mathbb{C}_0)$ bounded, let Φ be a Bohr lift of φ and assume that there is a unique $w\in\mathbb{T}^\infty$ such that $\operatorname{Re}\Phi(w)=0$. Then*

$$a_n(\mathscr{C}_\varphi)\ll\begin{cases}\exp\left(-\lambda n^{-1/2}\right) & \text{if } \omega_\varphi\leq 1,\\ \left(\frac{\log n}{n}\right)^{(\kappa_\varphi-1)\times\frac{\omega_\varphi}{2(\omega_\varphi-1)}} & \text{if } \omega_\varphi>1.\end{cases}$$

Here λ is some positive constant depending on φ.

This theorem illustrates the following general principle for composition operators (valid beyond $\mathscr{H}^2$): The more restricted the image of the symbol is, the more compact the associated composition operator is. In particular, the case $\omega_\varphi=1$ (the range of φ is contained in an angle) is reminiscent from [12, Thm. 1.2] where a similar result was obtained for composition operators on $H^2(\mathbb{D})$.

Before we embark upon the proof of Theorem 22, we first employ it to deduce the upper bound of Theorem 16.

Final part in the proof of the upper bound of Theorem 16. Suppose that $\varphi\in\mathscr{G}$ is a boundary regular Dirichlet polynomial, and assume that $\operatorname{Re}\Phi(\mathbf{1})=0$. We write

$$\begin{aligned}\operatorname{Re}\Phi\big(e^{i\theta_1},\dots,e^{i\theta_d}\big) &= \ell_1(\theta)^{k_1}+\dots+\ell_d(\theta)^{k_d}+\sum_{j=1}^{d}o\big(\ell_j^{k_j}(\theta)\big),\\ \operatorname{Im}\Phi\big(e^{i\theta_1},\dots,e^{i\theta_d}\big) &= \tau+b_1\ell_1(\theta)+\dots+b_d\ell_d(\theta)+o\left(\sum_{j=1}^{d}|\ell_j(\theta)|\right),\end{aligned}$$

with $k_1\geq\ldots\geq k_d$ and $b_1\neq 0$. The proof of Theorem 2 shows that we have $\kappa_\varphi\geq 1+\sum_{j=2}^{d}1/k_j$.

Now, let us write the Taylor expansion of $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ near $\mathbf{1}$, but also now for a point belonging to the unit polydisc. Writing

$$\Phi(z)=\sum_{j=1}^{d}a_j(1-z_j)+o\left(\sum_{j=1}^{d}|1-z_j|\right)$$

and $z=\big((1-\rho_1)e^{i\theta_1},\dots,(1-\rho_d)e^{i\theta_d}\big)$, it is easy to get

$$\operatorname{Re}\Phi(z)=a_1\rho_1+\cdots+a_d\rho_d+\ell_1(\theta)^{k_1}+\cdots+\ell_d(\theta)^{k_d}+o\left(\sum_{j=1}^{d}\Big(\rho_j+\ell_j^{k_j}(\theta)\Big)\right),$$

$$\operatorname{Im}\Phi(z)=\tau+b_1\ell_1(\theta)+\cdots+b_d\ell_d(\theta)+o\left(\sum_{j=1}^{d}|\ell_j(\theta)|\right).$$

Recalling that $a_j\geq 0$ for $j=1,\,\dots,\,d$, it is easy to conclude that there exists a neighbourhood $\mathscr{U}\ni\mathbf{1}$ in $\overline{\mathbb{D}^d}$ and $C>0$ such that, for all $z\in\mathscr{U}$,

$$|\operatorname{Im}\Phi(z)-\tau|^{k_1}\leq C\operatorname{Re}\Phi(z).$$

Outside $\mathscr{U}$, $\operatorname{Re}\Phi(z)$ is bounded away from 0, and $|\operatorname{Im}\Phi(z)-\tau|$ is here trivially majorized. Hence, the upper bound of Theorem 16 follows from Theorem 22. □

Let us now turn to the proof of Theorem 22. The proof will be preceded by two lemmas. The first one is inspired by Lemma 3.1 in [12].

Lemma 23. *Let Ω be a bounded domain in $\mathbb{C}_0$ whose boundary is a piecewise regular Jordan curve Γ, with $L_p(\Gamma)\geq 1$. Let $s_1,\,\dots,\,s_n$ be points in Γ such that the hyperbolic length of the curve between any two points s_j and s_{j+1} is equal to $L_p(\Gamma)/n$, $1\leq j\leq n$, where $s_{n+1}=s_1$. Let B be the Blaschke product of degree n whose zeros are precisely $s_1,\,\dots,\,s_n$. Then, for any $s\in\Omega$,*

$$|B(s)|\leq\exp\left(-C\frac{n}{L_p(\Gamma)}\right).$$

Proof. By the maximum principle, it is sufficient to prove this inequality for $s\in\Gamma$. In this case, we know that there exists some $j\in\{1,\,\dots,\,n\}$ such that $d(s,s_j)\leq L_p(\Gamma)/n$, from which we deduce that

$$d(s,s_k)\leq\frac{L_p(\Gamma)}{n}(1+|k-j|)$$

for any $k=1,\,\dots,\,n$. Using the link between the pseudo hyperbolic distance and the hyperbolic distance, we deduce that

$$|B(s)|\leq\prod_{j=1}^{n}\left(\frac{1-e^{-j\frac{L_p(\Gamma)}{n}}}{1+e^{-j\frac{L_p(\Gamma)}{n}}}\right).$$

By a Riemann sum argument, this means that

$$\begin{aligned}|B(s)| &\le \exp\left(-n\int_0^1 \ln\left(\frac{1-e^{-xL_p(\Gamma)}}{1+e^{xL_p(\Gamma)}}\right)dx\right)\\ &\le \exp\left(-\frac{n}{L_p(\Gamma)}\int_{e^{-L_p(\Gamma)}}^1 \frac{1}{y}\ln\left(\frac{1+y}{1-y}\right)dy\right),\end{aligned}$$

and we get the desired conclusion, since by assumption $L_p(\Gamma)\ge 1$. □

Hence, we require estimates of the hyperbolic length of some curves which are linked to the way that φ touches the boundary. Such estimates are contained in the following result.

Lemma 24. *Let $\omega\ge 1$, $\sigma\in(0,1/2)$ and $C>1$. Consider*

$$\Omega_{\omega,\sigma,C}=\big\{s\in\mathbb{C}_0\,:\,|\operatorname{Im}s|^\omega\le C\operatorname{Re}(s),\ \sigma\le\operatorname{Re}s\le C\big\}.$$

Let $\Gamma_{\omega,\sigma,C}$ denote the boundary of $\Omega_{\omega,\sigma,C}$. Then

$$L_p(\Gamma_{\omega,\sigma,C})\ll_{\omega,C}\begin{cases}\left(\frac{1}{\sigma}\right)^{\frac{\omega-1}{\omega}} & \text{if } \omega>1,\\ -\ln(\sigma) & \text{if } \omega=1.\end{cases}$$

Proof. Consider the curves

$$\begin{aligned}\Gamma_1 &= \left\{s\in\mathbb{C}_0\,:\,\operatorname{Re}s=\sigma,\ |\operatorname{Im}s|\le C^{1/\omega}(\operatorname{Re}s)^{1/\omega}\right\},\\ \Gamma_2 &= \left\{s\in\mathbb{C}_0\,:\,\operatorname{Re}s=C,\ |\operatorname{Im}s|\le C^{1/\omega}(\operatorname{Re}s)^{1/\omega}\right\},\\ \Gamma_3 &= \left\{s\in\mathbb{C}_0\,:\,\sigma\le\operatorname{Re}s\le C,\ |\operatorname{Im}s|= C^{1/\omega}(\operatorname{Re}s)^{1/\omega}\right\}.\end{aligned}$$

Clearly, $\Gamma_{\omega,\sigma,C}\subset\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and it is sufficient to prove the corresponding inequalities for Γ_j, $j=1,2,3$. Firstly, $L_p(\Gamma_2)\ll_{\omega,C}1$. Regarding Γ_1,

$$L_p(\Gamma_1)=\int_{-C^{1/\omega}\sigma^{1/\omega}}^{C^{1/\omega}\sigma^{1/\omega}}\frac{dy}{\sigma}\ll_{\omega,C}\left(\frac{1}{\sigma}\right)^{\frac{\omega-1}{\omega}}$$

which is even a stronger inequality than required when $\omega=1$. Finally,

$$L_p(\Gamma_3)\ll_{\omega,C}\int_\sigma^C\frac{\sqrt{1+x^{\frac{2}{\omega}-1}}}{x}dx\ll_{\omega,C}\begin{cases}-\ln(\sigma) & \text{if } \omega=1,\\ \left(\frac{1}{\sigma}\right)^{\frac{\omega-1}{\omega}} & \text{if } \omega>1.\end{cases}$$

The last estimate follows from inspecting the integrand near $x=0$, since $\sigma\in(0,1)$. □

Proof of Theorem 22. Let $\sigma\in(0,1)$ and $n\ge 1$. Without loss of generality, we may assume that $\Phi(\mathbf{1})=0$. Keeping the notations of Lemma 24, there exists $C>0$ such that

$$\varphi(\mathbb{C}_0)-\frac{1}{2}\subset\big\{s\in\mathbb{C}_0\,:\,0\le\operatorname{Re}s\le\sigma\big\}\cup\Omega_{\omega_\varphi,\sigma,C}.$$

Let B be a Blaschke product of degree n defined as in Lemma 23 with $\Omega_{\omega_\varphi,\sigma,C}$. Enlarging C if necessary, we may always assume that $L_p(\Gamma_{\omega_\varphi,\sigma,C}) \geq 1$, so that the assumptions of Lemma 23 are satisfied. The set

$$\Omega = \overline{\left\{\varphi(s) - \frac{1}{2} \,:\, \operatorname{Re}\varphi(s) \geq \frac{1}{2} + \sigma\right\}}$$

is a compact subset of $\mathbb{C}_0$, and we may apply part (a) of Lemma 21. Since $\Omega \subset \Omega_{\omega_\varphi,\sigma,C}$, we obtain

$$\sup_{s\in\Omega} |B(s)|^2 \leq \exp\left(-2C' \frac{n}{L_p(\Gamma_{\omega_\varphi,\sigma,C})}\right).$$

Moreover, $\zeta(1+2\sigma) \ll 1/\sigma$. Finally, using Lemma 18, we obtain

$$\|\mu_{\varphi,\Omega}\|_{\mathscr{C},\mathscr{H}^2} \ll \sup_{\varepsilon\in(0,\sigma),\ \tau\in\mathbb{R}} \frac{\mu_{\varphi,\Omega}\big(Q(\tau,\varepsilon)\big)}{\varepsilon} \ll \sigma^{\kappa_\varphi - 1}.$$

We will now optimize the choice of σ with respect to n. When $\omega_\varphi > 1$, we set

$$\sigma = \rho \left(\frac{\log n}{n}\right)^{\frac{\omega_\varphi}{\omega_\varphi - 1}},$$

where ϱ is some numerical parameter to be chosen later. Then

$$\sup_{s\in\Omega} |B(s)|^2 \zeta(1+2\sigma) \leq \exp\left(-2C'\rho^{\frac{\omega_\varphi-1}{\omega_\varphi}} \log n\right) \cdot \frac{1}{\sigma} \ll \left(\frac{\log n}{n}\right)^{\frac{\omega_\varphi}{\omega_\varphi-1}(\kappa_\varphi - 1)},$$

provided $\rho > 0$ is sufficiently large. When $\omega_\varphi \leq 1$, we set $\sigma = \exp\big(-\rho n^{-1/2}\big)$, so that

$$\sup_{s\in\Omega} |B(s)|^2 \zeta(1+2\sigma) \leq \exp\left(-\frac{C''}{\rho} n^{1/2} + \rho n^{1/2}\right),$$

and the result is proved provided $\rho > 0$ is sufficiently small. □

Remark. Our method of proof also shows, provided $\varphi(\mathbb{C}_0)$ is bounded and $\kappa_\varphi > 1$, that

$$a_n(\mathscr{C}_\varphi) \leq \left(\frac{\log n}{n}\right)^{\frac{\kappa_\varphi - 1}{2}}.$$

Indeed, we apply the same method with $\Omega_{\sigma,C} = \{s \in \mathbb{C}_0 \,:\, \sigma \leq \operatorname{Re} s \leq C,\ |\operatorname{Im} s| \leq C\}$ which satisfies $L_p(\Gamma) \ll_C \sigma^{-1}$. The rest of the proof remains unchanged.

8.3. **The Lower Bound.**

Let $\varphi \in \mathscr{G}$ satisfying the assumptions of Theorem 16 and let us assume that around $\mathbf{1}$, Φ satisfies (24) and (25). Let $\nu > 0$. For $\delta \in (0, 1/\nu)$, we consider the sequence $S_\delta = (s_m)$, given by

$$s_m = \frac{1}{2} + \nu\delta + im\delta, \qquad \text{where} \quad 1 \le m \le \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_1}}.$$

We intend to apply part (b) of Lemma 21. We will require the construction of preimages of $S_\delta - 1/2$ by Φ, and the inverse function theorem will provide the solution.

Lemma 25. *Let $\varphi \in \mathscr{G}$ satisfy the assumptions of Theorem 16. Then there exist ν_0, $C_1, C_2 > 0$ such that for all $\nu \ge \nu_0$ and every $\delta \in (0, 1/\nu)$, there exists a finite sequence $Z_\delta = (Z(\alpha))$ in $\mathbb{D}^d$ with*

$$Z(\alpha) = \left[(1-\rho_1(\alpha))e^{i\theta_1(\alpha)}, \dots, (1-\rho_d(\alpha))e^{i\theta_d(\alpha)}\right]$$

such that

- *for any $\alpha \neq \beta$, we have $\sup_{j=1,\dots,d} |\theta_j(\alpha) - \theta_j(\beta)| \ge C_1\delta$,*
- *for any α and any $j = 1, \dots, d$, we have $C_2^{-1}\delta \le \rho_j(\alpha) \le C_2\delta$,*
- *$\Phi(Z_\delta) = S_\delta - 1/2$ and, for any $1 \le m \le \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_1}}$, the equation $\Phi(Z(\alpha)) = s_m - \frac{1}{2}$ has at least $\prod_{j=2}^d \left\lfloor \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_j}} \right\rfloor$ solutions.*

Proof. We start as in the deduction of the upper bound in Theorem 16 from Theorem 22, writing

$$\operatorname{Re}\Phi(z) = a_1\rho_1 + \cdots + a_d\rho_d + \ell_1(\theta)^{k_1} + \cdots + \ell_d(\theta)^{k_d} + o\left(\sum_{j=1}^d \left(\rho_j + \ell_j^{k_j}(\theta)\right)\right),$$

$$\operatorname{Im}\Phi(z) = \tau + b_1\ell_1(\theta) + \cdots + b_d\ell_d(\theta) + o\left(\sum_{j=1}^d |\ell_j(\theta)|\right),$$

for $z = \left((1-\rho_1)e^{i\theta_1}, \dots, (1-\rho_d)e^{i\theta_d}\right)$. To simplify the notations, we use the (linear) change of variables $u_j = \ell_j(\theta)$. We also set

$$\Lambda = \mathbb{N}^d \cap \prod_{j=1}^d \left[1, \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_j}}\right]$$

and, for $\alpha \in \Lambda$ and $j = 2, \dots, d$ we let $\rho_j(\alpha) = \delta$ and $u_j(\alpha) = \alpha_j\delta$.

Setting $m = \alpha_1$, we want to find $Z(\alpha)$ such that $\operatorname{Re}\Phi(Z(\alpha)) = \nu\delta$ and $\operatorname{Im}\Phi(Z(\alpha)) = m\delta$. It remains to determine $\rho_1(\alpha)$ and $u_1(\alpha)$. We rewrite this system as

$$
\left\{\begin{aligned} f_\alpha(\rho_1,u_1)\rho_1+g_\alpha(\rho_1,u_1)u_1^{k_1} &= \nu\delta+d_\alpha\\ h_\alpha(\rho_1,u_1)u_1 &= m\delta+e_\alpha\end{aligned}\right. \tag{26}
$$

where f_α, g_α and h_α are smooth functions depending only on $\alpha_2, \ldots, \alpha_d$ and there exists a neighbourhood $\mathscr{U} \ni (0,0)$ so that for every $(\rho,u)\in\mathscr{U}$,

$$
|f_\alpha(\rho,u)-a_1|\ll\delta,\qquad |g_\alpha(\rho,u)-1|\ll\delta\qquad\text{and}\qquad |h_\alpha(\rho,u)-1|\ll\delta.
$$

Here, the open set $\mathscr{U}$ and the involved constants are uniform with respect to α, $\nu\geq 1$ and $\delta\in(0,1/\nu)$. Moreover, the real numbers d_α and e_α satisfy

$$
d_\alpha \ll \sum_{j=2}^d \delta + \sum_{j=2}^d \left(\left(\frac{1}{\delta}\right)^{1-\frac{1}{k_j}}\right)^{k_j} \ll \delta
$$

$$
e_\alpha \ll \delta \sum_{j=2}^d \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_j}} \ll \delta^{\frac{1}{k_1}}.
$$

We now apply the inverse function theorem to solve the system (26). Provided ν is large enough, we get a solution $(\rho_1(\alpha),u_1(\alpha))$ satisfying $\sup(\rho_1(\alpha),|u_1(\alpha)|)\ll\delta^{1/k_1}$. In this case, the involved constant depends on ν, but it is uniform with respect to α and δ.

Now, a look at the first equation of (26) shows that we in fact have the more precise inequality $\delta\ll\rho_1(\alpha)\ll\delta$, provided ν is sufficiently large, and this is independent of α and $\delta\in(0,1)$. Looking now at the second equation of (26), if $\alpha\neq\beta\in\Lambda$ satisfy $\alpha_j=\beta_j$ for $j\geq 2$, so that $e_\alpha=e_\beta$ and $h_\alpha=h_\beta$, then $|u_1(\alpha)-u_1(\beta)|\gg\delta$.

Hence, we have obtained $\prod_{j=2}^d \left\lfloor \left(\frac{1}{\delta}\right)^{1-\frac{1}{k_j}} \right\rfloor$ solutions to the equation $\Phi(Z(\alpha))=s_m$, and they satisfy the conclusions of Lemma 25 since the inequalities on $u_j(\alpha)$ are also valid for $\theta_j(\alpha)$ up to a constant depending only of Φ. □

Final part in the proof of the lower bound of Theorem 16. We apply Lemma 21 to S_δ and Z_δ given by the previous lemma, for

$$
\delta=\left(\frac{1}{n}\right)^{\frac{k_1}{k_1-1}},
$$

so that S_δ has cardinal number equal to n. Since $M_{\mathscr{H}^2}(S_\delta)\ll 1$ and

$$
\left\|\mu_{Z,H^2(\mathbb{T}^d)}\right\|_{\mathscr{C},H^2(\mathbb{T}^d)}\ll 1
$$

by Lemma 19 and Lemma 20, it remains to estimate the sum $N_\Phi(s_m;Z)\zeta(2\operatorname{Re}s_m)$ for any m. Using the fact that $\rho_j(\alpha)\gg\delta$ for any $j=1,\ldots,d$ and any α, we obtain

$$N_\Phi(s_m;Z)\zeta(2\operatorname{Re}s_m) \gg \left(\frac{1}{\delta}\right)^{\sum_{j=2}^d \left(1-\frac{1}{k_j}\right)} \cdot \delta^d \cdot \delta^{-1}$$
$$\gg \delta^{\sum_{j=2}^d \frac{1}{k_j}} \gg \left(\frac{1}{n}\right)^{\left(\sum_{j=2}^d \frac{1}{k_j}\right)\times\frac{k_1}{k_1-1}},$$

and we are done. □

Remark. We may modify the proof of Theorem 16 so that we do not assume that there exists a unique $w \in \mathbb{T}^d$ such that $\operatorname{Re}\Phi(w) = 0$. Suppose that φ is boundary regular at any $w \in \mathbb{T}^d$ such that $\operatorname{Re}\Phi(w) = 0$. Define now the *compactness index* of φ as the real number

$$\eta_\varphi(s) = \inf\left\{\eta_{\varphi,w} \,:\, \operatorname{Re}\Phi(w) = 0\right\}.$$

It should be observed that this infimum is in fact a minimum. Indeed, our assumptions imply that the points $w \in \mathbb{T}^d$ such that $\operatorname{Re}\Phi(w) = 0$ are isolated. Theorem 16 remains true with this new definition of η_φ.

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Paper 4

Composition operators and embedding theorems for some function spaces of Dirichlet series

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COMPOSITION OPERATORS AND EMBEDDING THEOREMS FOR SOME FUNCTION SPACES OF DIRICHLET SERIES

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Abstract. We observe that local embedding problems for certain Hardy and Bergman spaces of Dirichlet series are equivalent to boundedness of a class of composition operators. Following this, we perform a careful study of such composition operators generated by polynomial symbols φ on a scale of Bergman–type Hilbert spaces $\mathscr{D}_\alpha$. We investigate the optimal β such that the composition operator $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ boundedly into $\mathscr{D}_\beta$. We also prove a new embedding theorem for the non-Hilbertian Hardy space $\mathscr{H}^p$ into a Bergman space in the half-plane and use it to consider composition operators generated by polynomial symbols on $\mathscr{H}^p$, finding the first non-trivial results of this type. The embedding also yields a new result for the functional associated to the multiplicative Hilbert matrix.

1. Introduction

A paper by Gordon and Hedenmalm [11] initiated the study of composition operators acting on function spaces of Dirichlet series, $f(s) = \sum_{n\geq 1} a_n n^{-s}$. Their object of study was the Hilbert space of Dirichlet series with square-summable coefficients, $\mathscr{H}^2$. In this paper, we consider composition operators acting on various scales of function spaces of Dirichlet series.

For $1 \leq p < \infty$, we follow [3] and define the Hardy space $\mathscr{H}^p$ as the Banach space completion of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ in the Besicovitch norm

$$\|P\|_{\mathscr{H}^p} := \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt \right)^{\frac{1}{p}}. \tag{1}$$

The spaces $\mathscr{H}^p$ are Dirichlet series analogues of the classical Hardy spaces in unit disc. We refer to [19] and to [20, Ch. 6] for basic properties of $\mathscr{H}^p$, mentioning for the moment only that their elements are absolutely convergent in the half-plane $\mathbb{C}_{1/2}$, where $\mathbb{C}_\theta := \{s \in \mathbb{C} : \mathrm{Re}(s) > \theta\}$.

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For $\alpha \in \mathbb{R}$, we let $\mathscr{D}_\alpha$ denote the Hilbert space consisting of Dirichlet series f satisfying

$$\|f\|_{\mathscr{D}_\alpha} := \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^\alpha} \right)^{\frac{1}{2}} < \infty. \tag{2}$$

Here $d(n)$ denotes the number of divisors of the positive integer n. Note that $\mathscr{D}_0 = \mathscr{H}^2$. We are interested in the range $\alpha \geq 0$ and, as explained in [1], these spaces may be thought of as Dirichlet series analogues of the classical scale of weighted Bergman spaces in the unit disc. Since $d(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$, it follows from the Cauchy–Schwarz inequality that Dirichlet series in $\mathscr{D}_\alpha$ also are absolutely convergent in $\mathbb{C}_{1/2}$.

Due to an insight of H. Bohr (see Section 2), both $\mathscr{H}^p$ and $\mathscr{D}_\alpha$ can be identified with certain function spaces in countably infinite number of complex variables, and — consequently — the norms (1) and (2) can be computed as integrals on the polytorus $\mathbb{T}^\infty$ or in the polydisc $\mathbb{D}^\infty$, respectively.

In an attempt to better understand these spaces, their composition operators $\mathscr{C}_\varphi(f) = f \circ \varphi$ have recently been investigated in a series of papers. It is well-known (see [1, 3, 11, 21]) that any function $\varphi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ defining a bounded composition operator from $\mathscr{H}^p$ to $\mathscr{H}^q$, for some $p, q \geq 1$, or from $\mathscr{D}_\alpha$ to $\mathscr{D}_\beta$, for some $\alpha, \beta \geq 0$, necessarily is a member of the following class.

Definition. The *Gordon–Hedenmalm class*, denoted $\mathscr{G}$, is the set of functions $\varphi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ of the form

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \varphi_0(s),$$

where c_0 is a non-negative integer called the *characteristic* of φ and is denoted $\operatorname{char}(\varphi)$, the Dirichlet series φ_0 converges uniformly in $\mathbb{C}_\varepsilon$ ($\varepsilon > 0$) and has the following mapping properties:

(a) If $c_0 = 0$, then $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$, then either $\varphi_0 \equiv 0$ or $\varphi_0(\mathbb{C}_0) \subset \mathbb{C}_0$.

Regarding sufficient conditions, the case (b) is the most well-understood. It was shown in [3] that (b) is sufficient for boundedness of $\mathscr{C}_\varphi$ from $\mathscr{H}^p$ to $\mathscr{H}^p$ and in [1] that the same holds for boundedness of $\mathscr{C}_\varphi$ from $\mathscr{D}_\alpha$ to $\mathscr{D}_\alpha$.

The case $\operatorname{char}(\varphi) = 0$, which is the topic of this paper, is more difficult. Here it is only known that (a) is sufficient for boundedness of $\mathscr{C}_\varphi$ from $\mathscr{H}^p$ to $\mathscr{H}^p$ if p is an even integer. In [1], it was shown that if $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$, then $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ into $\mathscr{D}_{2^\alpha - 1}$ (which is smaller than $\mathscr{D}_\alpha$ if $0 < \alpha < 1$ and larger than $\mathscr{D}_\alpha$ if $\alpha > 1$). It was left open whether the value $2^\alpha - 1$ is optimal or not.

The sticking point seems to be that in order to prove sufficient conditions for boundedness of composition operators with $\operatorname{char}(\varphi) = 0$, we require an embedding of the function spaces of Dirichlet series into certain classical function spaces in the half-plane $\mathbb{C}_{1/2}$. The existence of such embeddings in the non-Hilbertian case is a well-known open problem in the field.

This paper is initiated by the observation that such embeddings are in fact equivalent to the sufficiency of condition (a). Our approach is related to the transference principle introduced in [21]. As a corollary, we obtain that the parameter $2^\alpha - 1$ discussed above is sharp, since it was demonstrated in [15] that the corresponding embedding is optimal.

We also discuss embeddings of $\mathscr{H}^p$ when $1 \leq p < 2$. Although we were unable to prove that $\mathscr{H}^p$ embeds into the corresponding conformally invariant Hardy space of $\mathbb{C}_{1/2}$, we show that it embeds into an optimal conformally invariant Bergman space.

Theorem 1. *Let $1 \leq p < 2$. There exists a constant $C_p > 0$ such that*

$$\left(\int_{\mathbb{R}}\int_{1/2}^{\infty} |f(s)|^2 \left(\sigma - \frac{1}{2}\right)^{\frac{2}{p}-2} \frac{d\sigma dt}{|s+1/2|^{4/p}}\right)^{\frac{1}{2}} \leq C_p\|f\|_{\mathscr{H}^p},$$

for every $f \in \mathscr{H}^p$. The exponent $\frac{2}{p} - 2$ is the smallest possible.

We then perform a careful study of composition operators with polynomial symbols mapping $\mathscr{D}_\alpha$ to $\mathscr{D}_\beta$, in the spirit of [5]. We show that for certain polynomial symbols, $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ into $\mathscr{D}_\beta$ with $\beta < 2^\alpha - 1$ and that the optimality of $\beta = 2^\alpha - 1$ also can be decided by investigating the most simple non-trivial symbol, namely $\varphi(s) = 3/2 - 2^{-s}$.

Consequently, we consider boundedness of this simple composition operator an interesting necessary condition for the embedding problem for $\mathscr{H}^p$. This leads us to an in-depth study of composition operators with linear symbols on $\mathscr{H}^p$. By using Theorem 1 and estimates of Carleson measures, we prove the following result.

Theorem 2. *Let $\varphi(s) = c_1 + \sum_{j=1}^{d} c_{p_j} p_j^{-s}$ be a Dirichlet polynomial belonging to $\mathscr{G}$ such that $c_{p_j} \neq 0$ for $j = 1, \ldots, d$ and suppose that $d \geq 2$. Then $\mathscr{C}_\varphi$ is bounded on $\mathscr{H}^p$, for $p \in [1, \infty)$.*

Observe that the case $d = 1$ corresponds to the simple symbol discussed above. It should also be mentioned that very few non-trivial composition operators of characteristic 0 on $\mathscr{H}^p$ are known when p is not an even integer, and none involving two or more prime numbers. Moreover, it is possible to generate more examples from our method and results in [5].

We show that if $\varphi(s) = 3/2 - 2^{-s}$ generates a bounded composition operator on $\mathscr{H}^1$, then Nehari's theorem holds for the multiplicative Hilbert matrix introduced in [9]. Furthermore, we apply Theorem 1 to demonstrate that the associated functional is bounded on $\mathscr{H}^p$ for $p \in (1, \infty)$. We also explain how Theorem 1 and the techniques used in its proof can be used to improve the necessary and sufficient conditions for bounded zero sequences for $\mathscr{H}^p$ from [8, 23].

Organization. This paper is divided into seven sections. Section 2 contains an exposition of our observation that the embedding problem is equivalent to boundedness of certain composition operators for $\mathscr{H}^p$ and $\mathscr{D}_\alpha$, in addition to the proof of Theorem 1. In Section 3, we collect some results regarding Carleson measures in the half-plane and on the polydisc. Section 4 is devoted to a study of composition operators from $\mathscr{D}_\alpha$ to $\mathscr{D}_\beta$ generated by polynomial symbols. In Section 5, we discuss composition operators with linear symbols on $\mathscr{H}^p$. The penultimate section contains some connections from the results obtained in this paper to the validity of Nehari's theorem for the multiplicative Hilbert matrix, while the final section is dedicated to bounded zero sequences for $\mathscr{H}^p$.

Notation. We will use the notation $f(x) \ll g(x)$ when there is some constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all (appropriate) x. If both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold, we will write $f(x) \asymp g(x)$. As usual, $\{p_j\}_{j\geq 1}$ will denote the increasing sequence of prime numbers.

2. Composition operators and the embedding problem

2.1. **Hardy spaces.** As mentioned in the introduction, functions in $\mathscr{H}^p$ are holomorphic in the half-plane $\mathbb{C}_{1/2}$. It is therefore interesting to investigate how they behave on the line $1/2 + it$. In this context, the most important question is the embedding problem (see [22, Sec. 3]), which can be formulated as follows. Is there a constant C_p such that

$$\sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} |P(1/2+it)|^p \, dt \leq C_p \|P\|_{\mathscr{H}^p}^p \tag{3}$$

for every Dirichlet polynomial P? It follows from an inequality of Montgomery and Vaughan (see [14, pp. 140–141]) that (3) holds for $p = 2$, and hence for every even integer p, but its validity for other values remains open. Now, from (1) it is clear that the $\mathscr{H}^p$-norm is invariant under vertical translations, so it is enough to check (3) for a fixed τ, say $\tau = 0$.

A typical application of the local embedding is to deduce that $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$ is a sufficient condition for boundedness of the composition operator $\mathscr{C}_\varphi$ on $\mathscr{H}^p$. This is usually done through the following equivalent formulation of (3).

The conformally invariant Hardy space in the half-plane $\mathbb{C}_{1/2}$, which we denote H^p_{i}, consists of those functions f such that $f \circ \mathscr{T} \in H^p(\mathbb{T})$, where $\mathscr{T}$ is the following mapping from $\mathbb{D}$ to $\mathbb{C}_{1/2}$,

$$\mathscr{T}(z) = \frac{1}{2} + \frac{1-z}{1+z}.$$

The mapping $\mathscr{T}$ appeared in the transference principle of [21], where it was used to transfer certain results about composition operators on $H^2(\mathbb{T})$ to results about composition operators on $\mathscr{H}^2$. Now, the norm of H^p_{i} can be computed as

$$\begin{aligned} \|f\|^p_{H^p_{\mathrm{i}}} := \|f \circ \mathscr{T}\|^p_{H^p(\mathbb{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(1/2 + i\tan(\theta/2))|^p \, d\theta \\ &= \frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+it)|^p \, \frac{dt}{1+t^2}. \end{aligned} \tag{4}$$

The inequality (3) is equivalent to $\|P\|_{H^p_{\mathrm{i}}} \leq C'_p \|P\|_{\mathscr{H}^p}$, since evidently

$$\int_0^1 |P(1/2+it)|^p \, dt \ll \|P\|^p_{H^p_{\mathrm{i}}} \ll \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} |P(1/2+it)|^p \, dt.$$

Our observation is that not only does the embedding (3) imply a sufficient condition for boundedness of certain composition operators, it is in fact equivalent to boundedness of all composition operators of this type.

Theorem 3. *Fix $1 \leq p < \infty$. The following are equivalent.*

(a) *The local embedding* (3) *holds for p.*
(b) *For every $\varphi \in \mathscr{G}$ with* $\mathrm{char}(\varphi) = 0$, *the composition operator $\mathscr{C}_\varphi$ acts boundedly on $\mathscr{H}^p$.*
(c) *Let $\psi(s) = \mathscr{T}(2^{-s})$. The composition operator $\mathscr{C}_\psi$ acts boundedly on $\mathscr{H}^p$.*

As explained in [3], the proof of (a) $\implies$ (b) can be adapted from the proof given for $p = 2$ in [11]. This argument relies on approximating the Besicovitch norm (1) by taking a limit in a family of conformal mappings. A simpler proof of this implication, based on a trick from [1], is included below.

To facilitate this, let us recall the Bohr lift. Every positive integer n can be written uniquely as a product of prime numbers,

$$n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

This factorization associates the finite multi-index $\kappa(n) = (\kappa_1, \kappa_2, \ldots)$ to n. Consider a Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$. Its Bohr lift $\mathscr{B}f$ is the power series

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}.$$

It is well-known (see [3, 20]) that the Bohr lift defines an isometric isomorphism between $\mathscr{H}^p$ and the Hardy space of the countably infinite polytorus, $H^p(\mathbb{T}^\infty)$.

The polytorus $\mathbb{T}^\infty$ is a compact abelian group, which we endow with its normalized Haar measure ν, so that

$$\|f\|_{\mathscr{H}^p}^p = \|\mathscr{B}f\|_{H^p(\mathbb{T}^\infty)}^p := \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f(z)|^p \, d\nu(z)\right)^{\frac{1}{p}}.$$

It is important to note that the Haar measure $\nu = \nu_0$ of the polytorus $\mathbb{T}^\infty$ is simply the product of the normalized Lebesgue measure on $\mathbb{T}$, denoted $m = m_0$, in each variable. The subscript is included to indicate the connection to $\mathscr{D}_0 = \mathscr{H}^2$.

Proof of Theorem 3. For (a) $\implies$ (b), we first suppose that Φ is a holomorphic function mapping $\mathbb{D}$ to $\mathbb{C}_{1/2}$. Using Littlewood's subordination principle (see [27, Ch. 11]), we find that

$$\|f \circ \Phi\|_{H^p(\mathbb{T})}^p \leq \frac{1 + |\mathscr{T}^{-1}(\Phi(0))|}{1 - |\mathscr{T}^{-1}(\Phi(0))|} \|f\|_{H^p_{\mathrm{i}}}^p, \tag{5}$$

for $f \in H^p_{\mathrm{i}}$. For $G \in H^p(\mathbb{T}^\infty)$ and $w \in \mathbb{C}$, set $G_w(z) = G(wz_1, wz_2, \ldots)$. By Fubini's theorem,

$$\|G\|_{H^p(\mathbb{T}^\infty)}^p = \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} |G_w(z)|^p \, dm(w) d\nu(z).$$

Let P be a Dirichlet polynomial and assume that $\varphi \in \mathscr{G}$ with $\mathrm{char}(\varphi) = 0$. The latter assumption implies that $\mathscr{B}(P \circ \varphi) = P \circ (\mathscr{B}\varphi)$. Thus, by setting $G = \mathscr{B}(P \circ \varphi)$, we obtain

$$\|P \circ \varphi\|_{\mathscr{H}^p}^p = \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} |P \circ (\mathscr{B}\varphi)_w(z)|^p \, dm(w) d\nu(z).$$

Fixing for a moment $z \in \mathbb{T}^\infty$, we notice that $\Phi(w) = (\mathscr{B}\varphi)_w(z)$ maps $\mathbb{D}$ to $\mathbb{C}_{1/2}$ with $\Phi(0) = c_1$. Considering therefore P a member of H^p_{i}, we apply (5) and conclude that

$$\|P \circ \varphi\|_{\mathscr{H}^p}^p \leq \int_{\mathbb{T}^\infty} \left(\frac{1 + |\mathscr{T}^{-1}(c_1)|}{1 - |\mathscr{T}^{-1}(c_1)|} \|P\|_{H^p_{\mathrm{i}}}^p\right) d\nu(z) = \frac{1 + |\mathscr{T}^{-1}(c_1)|}{1 - |\mathscr{T}^{-1}(c_1)|} \|P\|_{H^p_{\mathrm{i}}}^p,$$

seeing as the constant in this instantiation of Littlewood's subordination principle does not involve z.

The implication (b) $\implies$ (c) is obvious, seeing as it is easy to verify that $\psi \in \mathscr{G}$. To prove that (c) $\implies$ (a), assume that $\mathscr{C}_\psi$ acts boundedly on $\mathscr{H}^p$, say that

$$\|\mathscr{C}_\psi P\|_{\mathscr{H}^p} \leq C_p \|P\|_{\mathscr{H}^p}$$

holds for every Dirichlet polynomial P. Arguing as above, we find that $\mathscr{B}(P \circ \psi) = P \circ (\mathscr{B}\psi)$ and that, in this case, $\mathscr{B}\psi(z) = \mathscr{T}(z_1)$. In particular, using the Bohr lift, this means that

$$\|\mathscr{C}_\psi P\|_{\mathscr{H}^p} = \|P \circ \mathscr{T}\|_{H^p(\mathbb{T})},$$

so we are done by (4). $\square$

2.2. **Bergman spaces.** Let us now explain how to do the same for the Bergman–type spaces $\mathscr{D}_\alpha$. Let $\alpha, \beta > 0$, and consider the following probability measures on $\mathbb{D}$.

$$dm_\alpha(z) = \frac{1}{\Gamma(\alpha)}\left(\log\frac{1}{|z|^2}\right)^{\alpha-1} dm_1(z), \tag{6}$$

$$d\widetilde{m}_\beta(z) = \beta\left(1-|z|^2\right)^{\beta-1} dm_1(z). \tag{7}$$

Here m_1 (which is the only case where $m = \widetilde{m}$) is taken to be the standard Lebesgue measure on $\mathbb{C}$, normalized so that $m_1(\mathbb{D}) = 1$. For $\alpha > 0$, the Bergman space $D_\alpha(\mathbb{D})$ can be defined as the L^2-closure of polynomials with respect to either measure, yielding equivalent norms. We will for simplicity use the measure (7) in most cases.

However, in an infinite number of variables, the norms are no longer equivalent. We use (6) to compute the norm of $\mathscr{D}_\alpha$ as an integral over $\mathbb{D}^\infty$ to ensure that (2) is satisfied. Therefore, we define $d\nu_\alpha(z) = dm_\alpha(z_1) \times dm_\alpha(z_2) \times \cdots$. It is straightforward to verify that

$$\|f\|^2_{\mathscr{D}_\alpha} = \int_{\mathbb{D}^\infty} |\mathscr{B}f(z)|^2\, d\nu_\alpha(z).$$

Set $S_\tau = [1/2, 1] \times [\tau, \tau+1]$. For the Bergman spaces $\mathscr{D}_\alpha$, the local embedding problem takes on the following form: Given $\alpha > 0$, what is the smallest $\beta > 0$ such that

$$\sup_{\tau\in\mathbb{R}} \int_{S_\tau} |P(s)|^2 \left(\sigma - \frac{1}{2}\right)^{\beta-1} dm_1(s) \leq C_{\alpha,\beta}\|P\|^2_{\mathscr{D}_\alpha} \tag{8}$$

for every Dirichlet polynomial P? Again, it is clear that the norm of $\mathscr{D}_\alpha$ is invariant under vertical translations, so arguing as above, we find that (8) is equivalent to $\|P\|_{D_{\beta,\mathrm{i}}} \leq C'_{\alpha,\beta}\|P\|^2_{\mathscr{D}_\alpha}$, setting

$$\|f\|^2_{D_{\beta,\mathrm{i}}} := \|f\circ\mathscr{T}\|^2_{D_\beta(\mathbb{D})} = 4^\beta\beta\int_{\mathbb{C}_{1/2}} |f(s)|^2\left(\sigma-\frac{1}{2}\right)^{\beta-1}\frac{dm_1(s)}{|s+1/2|^{2\beta+2}}, \tag{9}$$

since any f in $\mathscr{D}_\alpha$ is uniformly bounded in $\mathbb{C}_1$ by its $\mathscr{D}_\alpha$-norm. For the next result, (a) $\implies$ (b) is part of the main result in [1]. The other steps are identical to the proof of Theorem 3 in view of the discussion above.

Theorem 4. *Fix $\alpha, \beta > 0$. The following are equivalent.*

(a) *The local embedding* (8) *holds for α and β.*
(b) *For every $\varphi \in \mathscr{G}$ with* $\operatorname{char}(\varphi) = 0$*, the composition operator $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded.*
(c) *Let $\psi(s) = \mathscr{T}(2^{-s})$. The composition operator $\mathscr{C}_\psi$ maps $\mathscr{D}_\alpha$ boundedly into $\mathscr{D}_\beta$.*

It was shown in [15] that $\beta = 2^\alpha - 1$ is the optimal exponent in (8). We will touch upon the reason behind this value in the next section, see in particular (17). From this optimality, we obtain at once the following result, clarifying the optimal β in the main result of [1], which states that if $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$, then $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ boundedly into $\mathscr{D}_\beta$ if $\beta \geq 2^\alpha - 1$.

Corollary 5. *Let $\alpha \geq 0$. There is $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$ such that $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded if and only if $\beta \geq 2^\alpha - 1$.*

2.3. **Embedding of $\mathscr{H}^p$ into $D_{\beta,\mathrm{i}}$.** Even if one is unable to prove the embedding inequality (3) for $1 \leq p < 2$, it is natural to ask whether it is possible to embed $\mathscr{H}^p$ into some Bergman space $D_{\beta,\mathrm{i}}$. For the Hardy spaces of the unit disc, this type of result goes back to the function theoretic version of the isoperimetric inequality due to Carleman, which asserts that

$$\|f\|_{D_1(\mathbb{D})} \leq \|f\|_{H^1(\mathbb{D})}. \tag{10}$$

Iterating the inequality (its contractivity is crucial) and using the Bohr lift, Helson [12] found that $\|f\|_{\mathscr{D}_1} \leq \|f\|_{\mathscr{H}^1}$. Combining Helson's inequality with the results from [15] discussed above, one finds that $\mathscr{H}^1$ is embedded in $D_{1,\mathrm{i}}$, thereby reclaiming (10) in the context of Hardy spaces of Dirichlet series and weighted Bergman spaces in $\mathbb{C}_{1/2}$.

If we seek to extend Helson's inequality to $1 < p < 2$, we are required to use the measure (6) when defining the spaces $D_\alpha(\mathbb{D})$, to ensure that we get $\mathscr{D}_\alpha$ after the iterative procedure. By a standard interpolation argument between (10) and $H^2(\mathbb{D})$, one find that for $p \in (1,2)$,

$$\|f\|_{D_{\frac{2}{p}-1}(\mathbb{D})} \leq C_p \|f\|_{H^p(\mathbb{D})}. \tag{11}$$

Nevertheless, the constant C_p arising from interpolation between Hardy spaces is strictly bigger than 1 (see [7]). Without contractivity, we cannot argue as Helson, starting from (11), to prove that $\mathscr{H}^p$ embeds into $\mathscr{D}_{2/p-1}$. It turns out that this embedding is false, since it can be proved (see [7] or the argument at the end of the proof of Theorem 1) that if $\mathscr{H}^p$ embeds into $\mathscr{D}_\alpha$, then $\alpha \geq 1 - \log p/\log 2$ which is stricly bigger than $2/p - 1$ when $p \in (1,2)$.

On the other hand, such an embedding is not known to exist, unless $p \in \{1,2\}$. If we could prove that $\mathscr{H}^p$ embeds into $\mathscr{D}_\alpha$, with $\alpha = 1 - \log p/\log 2$, then the embedding (8), which is valid with $\beta = 2^\alpha - 1$, would imply that

$$\|f\|_{D_{\frac{2}{p}-1,\mathrm{i}}} \ll \|f\|_{\mathscr{H}^p}, \tag{12}$$

again reclaiming (11) for Hardy spaces of Dirichlet series and weighted Bergman spaces in $\mathbb{C}_{1/2}$. Similarly, the embedding (3) also implies (12), in this case by first translating (11) to $\mathbb{C}_{1/2}$ with $\mathscr{T}$. We have been able to prove (12) by different methods, which is our Theorem 1.

The proof uses several tools from harmonic analysis and analytic number theory. The first is a special case of a result of Weissler [25], who studied the hypercontractivity of the Poisson kernel.

Lemma 6. *Let $p \in [1,2]$. For any $f(z) = \sum_{k\geq 0} a_k z^k$, we have the contractive estimate*

$$\left(\sum_{k=0}^{\infty} |a_k|^2 \left(\frac{p}{2}\right)^k\right)^{1/2} \leq \|f\|_{H^p(\mathbb{D})}.$$

The second tool is a way to iterate this inequality multiplicatively, first devised in [3] and later used in [7, 12]. We formulate it in an abstract context and we give a brief account of the proof.

Lemma 7. *Let $p \in [1,2]$ and assume that there exists a sequence $\{\gamma_k\}_{k\geq 0}$ of positive real numbers with $\gamma_0 = 1$, such that for every $f(z) = \sum_{k\geq 0} a_k z^k \in H^p(\mathbb{D})$,*

$$\left(\sum_{k=0}^{\infty} |a_k|^2 \gamma_k\right)^{1/2} \leq \|f\|_{H^p(\mathbb{D})}.$$

Let Γ denote the multiplicative function defined on the prime powers by $\Gamma(p_j^k) = \gamma_k$. Then,

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \Gamma(n)\right)^{1/2} \leq \|f\|_{\mathscr{H}^p},$$

for every $f(s) = \sum_{n\geq 1} a_n n^{-s} \in \mathscr{H}^p$.

Proof. Fix $d \geq 1$ and $f(z) = \sum_{\kappa\in\mathbb{N}^d} a_\kappa z^\kappa \in H^p(\mathbb{T}^d)$. By the Bohr lift, it is sufficient to prove that

$$\left(\sum_{\kappa\in\mathbb{N}^d} |a_\kappa|^2 \gamma_{\kappa_1}\cdots\gamma_{\kappa_d}\right)^{1/2} \leq \|f\|_{H^p(\mathbb{T}^d)}. \tag{13}$$

The assumption of the lemma is that (13) holds for $d = 1$. We will argue by induction on d and assume that (13) is true for $d-1$. Then, fixing $z_1,\ldots,z_{d-1} \in \mathbb{T}^{d-1}$ and considering f a function only of z_d, we use (13) with $d = 1$ to get

$$\left(\int_{\mathbb{T}} \left|\sum_{\kappa\in\mathbb{N}^d} a_\kappa \gamma_{\kappa_d}^{1/2} z_1^{\kappa_1}\cdots z_d^{\kappa_d}\right|^2 dm(z_d)\right)^{p/2} \leq \int_{\mathbb{T}} \left|\sum_{\kappa\in\mathbb{N}^d} a_\kappa z_1^{\kappa_1}\cdots z_d^{\kappa_d}\right|^p dm(z_d).$$

We integrate over the remaining coordinates $z_1, \dots, z_{d-1}$ and use Minkowski inequality in the following form: For measure spaces X and Y, a measurable function g on $X \times Y$ and $r \geq 1$,

$$\left(\int_X \left(\int_Y |g(x,y)|dy\right)^r dx\right)^{1/r} \leq \int_Y \left(\int_X |g(x,y)|^r dx\right)^{1/r} dy.$$

This yields, with $X = \mathbb{T}$, $Y = \mathbb{T}^{d-1}$ and $r = 2/p$, that

$$\left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}^{d-1}} \left|\sum_{\kappa \in \mathbb{N}^d} a_\kappa \gamma_{\kappa_d}^{1/2} z_1^{\kappa_1} \cdots z_d^{\kappa_d}\right|^p dm(z_1) \cdots dm(z_{d-1})\right)^{2/p} dm(z_d)\right)^{p/2} \leq \|f\|_{H^p(\mathbb{T}^d)}^p.$$

The induction hypothesis allows us to conclude. □

Our final tool is a number theoretic estimate on the average order of a multiplicative function. Let $\Omega(n)$ be the total number of prime divisors of n, say $\Omega(p_1^{\kappa_1} \cdots p_d^{\kappa_d}) = \kappa_1 + \cdots + \kappa_d$. For $0 < y < 2$ we refer to Selberg–Delange method (see [24, Thm. II.6.2]) and for $y = 2$ we refer to [2].

Lemma 8. *Let $0 < y \leq 2$. Then*

$$\frac{1}{x}\sum_{n \leq x} y^{\Omega(n)} \asymp \begin{cases} (\log x)^{y-1} & \text{if } 0 < y < 2, \\ (\log x)^2 & \text{if } y = 2. \end{cases} \tag{14}$$

Observe the phase change at $y = 2$, which occurs since 2 is the first prime number. We are now ready to proceed with the proof of (12).

Proof of Theorem 1. Combining Lemma 6 and Lemma 7, we get the inequality

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \left(\frac{p}{2}\right)^{\Omega(n)}\right)^{1/2} \leq \|f\|_{\mathscr{H}^p}, \tag{15}$$

for every $f(s) = \sum_{n \geq 1} a_n n^{-s} \in \mathscr{H}^p$, since in this case $\Gamma(n) = (p/2)^{\Omega(n)}$. In other words, following the conventions of [15], the space $\mathscr{H}^p$ is continuously embedded into

$$\mathscr{H}_{w_p} := \left\{\sum_{n=1}^{\infty} a_n n^{-s} \,:\, \|f\|_{w_p} := \left(\sum_{n=1}^{\infty} |a_n|^2/w_p(n)\right)^{\frac{1}{2}} < \infty\right\},$$

where $w_p(n) = (2/p)^{\Omega(n)}$. The main result of [15] relates the average order of the weight $w(n)$ with the optimal embedding of $\mathscr{H}_w$ into $D_{\beta,\mathrm{i}}$, the relation being the two-sided estimate

$$\frac{1}{x}\sum_{n \leq x} w(n) \asymp (\log x)^\beta. \tag{16}$$

Now, the case $p = 1$ was discussed and resolved above, using Helson's inequality. For $1 < p < 2$, we have $1 < y < 2$, so we conclude using (14) that $\mathscr{H}_{w_p}$ is continuously embedded in $\mathscr{D}_{2/p-1,\mathrm{i}}$ and that the parameter $2/p - 1$ is optimal, with respect to $\mathscr{H}_{w_p}$. This proves (12), using (15).

It remains only to verify that the optimality of the parameter $2/p-1$ extends to $\mathscr{H}^p$. Fix $\varepsilon > 0$ and consider

$$f_\varepsilon(s) = \frac{[\zeta(s+1/2+\varepsilon)]^{2/p}}{[\zeta(1+2\varepsilon)]^{1/p}},$$

which, as shown in [3, Thm. 3], satisfies $\|f_\varepsilon\|_{\mathscr{H}^p} = 1$. For $s = \sigma + it$ satisfying, say, $1 < \sigma < 3/2$ and $0 < t < 1$, we have that $\zeta(s) \asymp (s-1)^{-1}$. Assume now that $\mathscr{H}^p$ embed continuously into $D_{\beta,\mathrm{i}}$. Then, for $1 \le p < 2$ and $0 < \beta \le 1$, we estimate

$$\begin{aligned} 1 \gg \|f_\varepsilon\|_{D_{\beta,\mathrm{i}}} &\gg \int_{1/2}^{1}\int_0^1 \frac{|\zeta(s+1/2+\varepsilon|]^{4/p}}{[\zeta(1+2\varepsilon)]^{2/p}} \left(\sigma - \frac{1}{2}\right)^{\beta-1} dt d\sigma \\ &\gg \varepsilon^{2/p} \int_{1/2}^{1}\int_0^1 \frac{(\sigma-1/2)^{\beta-1}}{((\sigma-1/2+\varepsilon)^2+t^2)^{2/p}}\, dt d\sigma \\ &\asymp \varepsilon^{2/p}\int_{1/2}^{1} \frac{(\sigma-1/2)^{\beta-1}}{(\sigma-1/2+\varepsilon)^{4/p-1}}\, d\sigma \gg \varepsilon^{2/p+\beta-4/p+1}, \end{aligned}$$

which means that if $\mathscr{H}^p$ is continuously embedded in $D_{\beta,\mathrm{i}}$, then necessarily $\beta \ge 2/p - 1$. □

Let us compare the space $\mathscr{H}_{w_p}$ to the space $\mathscr{D}_\alpha$ for $\alpha = 1 - \log p/\log 2$. It turns out that if n is square-free, then $(p/2)^{\Omega(n)} = 1/[d(n)]^\alpha$. For other values, $w_p(n)$ is strictly smaller than $1/[d(n)]^\alpha$, and it can be significantly smaller, most easily seen by considering $n = 2^k$. Thus, the space $\mathscr{H}_{w_p}$ is (strictly) bigger than $\mathscr{D}_\alpha$. However, when $1 < p < 2$, the weights $w_p(n)$ are dominated by their square-free parts, so $\mathscr{D}_\alpha$ and $\mathscr{H}_{w_p}$ are embedded into the same $\mathscr{D}_\beta$.

To explain why this happens, let ξ be any positive multiplicative function with $\xi(p_j) = \beta$ and $\xi(p_j^k) \ll (2-\delta)^k$ for some $0 < \delta < 2$. Then, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned} \sum_{n=1}^\infty \xi(n) n^{-s} &= \prod_{j=1}^\infty \left(1 + \beta p_j^{-s} + \sum_{k=2}^\infty \xi(p_j^k) p_j^{-ks}\right) \\ &= [\zeta(s)]^\beta \prod_{j=1}^\infty \left(1+\beta p_j^{-s} + O(p_j^{-2s})\right)\left(1-\beta p_j^{-s}+O(p_j^{-2s})\right) \\ &= [\zeta(s)]^\beta \prod_{j=1}^\infty \left(1+O(p_j^{-2s})\right), \end{aligned}$$

so by the Selberg–Delange method, we find $\sum_{n\leq x}\xi(n) \asymp x(\log x)^{\beta-1}$. Observe again the phase change at $\delta = 0$, leading to different embeddings for $\mathscr{H}_{w_1}$ and $\mathscr{D}_1$ in view of (16), since the latter weight satisfies the assumption $\xi(p_j^k) \ll (2-\delta)^k$, while the former does not.

3. Carleson measures in the half-plane and on the polydisc

3.1. Carleson measures in the half-plane. For $\beta > 0$, the non-conformal Bergman space $D_\beta(\mathbb{C}_{1/2})$ consists of the holomorphic functions f in $\mathbb{C}_{1/2}$ which satisfy

$$\|f\|^2_{D_\beta(\mathbb{C}_{1/2})} := \int_{\mathbb{C}_{1/2}} |f(s)|^2 \left(\sigma - \frac{1}{2}\right)^{\beta-1} ds < \infty.$$

If $\beta = 0$, then $D_\beta(\mathbb{C}_{1/2})$ is taken to be the non-conformal Hardy space, $H^2(\mathbb{C}_{1/2})$, with norm

$$\|f\|^2_{H^2(\mathbb{C}_{1/2})} := \sup_{\sigma>1/2} \int_{\mathbb{R}} |f(\sigma+it)|^2\, dt < \infty.$$

For $\alpha, \beta \geq 0$, let X denote either $\mathscr{D}_\alpha$ or $D_\beta(\mathbb{C}_{1/2})$. A positive Borel measure μ on $\mathbb{C}_{1/2}$ is called a *Carleson measure* for X provided there is a constant $C = C(X,\mu)$ such that for every $f \in X$,

$$\int_{\mathbb{C}_{1/2}} |f(s)|^2\, d\mu(s) \leq C\|f\|_X^2.$$

The smallest such constant $C(X,\mu)$ is called the *Carleson constant* for μ with respect to X. A Carleson measure μ is said to be a *vanishing Carleson measure* for X provided

$$\lim_{k\to\infty} \int_{\mathbb{C}_{1/2}} |f_k(s)|^2\, d\mu(s) = 0$$

for every weakly compact sequence $\{f_k\}_{k\geq 1}$ in X. In this case, weakly compact means that $\phi(f_k) \to 0$ for every $\phi \in X^*$. Since both $X = D_\beta(\mathbb{C}_{1/2})$ and $X = \mathscr{D}_\alpha$ are reproducing kernel spaces, it is clear that $\{f_k\}_{k\geq 1}$ in X is weakly compact if and only if $\|f_k\|_X \leq C$ and $f_k(s) \to 0$ on every compact subset K of $\mathbb{C}_{1/2}$.

Lemma 9. *Let $\alpha \geq 0$. Suppose that μ is a Borel measure on $\mathbb{C}_{1/2}$ with bounded support. Then μ is a Carleson measure for $\mathscr{D}_\alpha$ if and only if μ is a Carleson measure for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$. Moreover, μ is vanishing Carleson for $\mathscr{D}_\alpha$ if and only if μ is vanishing Carleson for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$.*

The first part of this result can be extracted from [15, 16]. In preparation for the part regarding vanishing Carleson measures, let us collect some preliminary results. The following geometric characterization of Carleson measures for Bergman spaces can be found in [27, Sec. 7.2].

Lemma 10. *Let $\beta \geq 0$ and let μ be a Borel measure on $\mathbb{C}_{1/2}$. Then μ is a Carleson measure for $D_\beta(\mathbb{C}_{1/2})$ if and only if*

$$\mu\big(Q(\tau,\varepsilon)\big) = O\big(\varepsilon^{\beta+1}\big)$$

for every Carleson square $Q(\tau,\varepsilon) = [1/2, 1/2+\epsilon]\times[\tau-\varepsilon/2, \tau+\varepsilon/2]$. Additionally, μ is vanishing Carleson for $D_\beta(\mathbb{C}_{1/2})$ if and only if

$$\mu\big(Q(\tau,\varepsilon)\big) = o\big(\varepsilon^{\beta+1}\big),$$

as $\varepsilon \to 0^+$, uniformly for $\tau \in \mathbb{R}$.

The reproducing kernels of $\mathscr{D}_\alpha$ are given by $K_\alpha(s,w) = \zeta_\alpha\left(s+\overline{w}\right)$, where

$$\zeta_\alpha(s) = \sum_{n=1}^\infty [d(n)]^\alpha\, n^{-s}.$$

It is clear that $\|K_\alpha(\cdot,w)\|_{\mathscr{D}_\alpha} = \sqrt{\zeta_\alpha(2\operatorname{Re} w)}$. We extract from [26, pp. 240–241] that
(17)

$$\zeta_\alpha(s) := \sum_{n=1}^\infty [d(n)]^\alpha\, n^{-s} = [\zeta(s)]^{2^\alpha} \prod_{j=1}^\infty \left(1 + \sum_{m=2}^\infty b_m p_j^{-ms}\right) =: [\zeta(s)]^{2^\alpha}\, \phi_\alpha(s),$$

where the Euler product $\phi_\alpha(s)$ converges absolutely in $\mathbb{C}_{1/2}$ with $\phi_\alpha(1) \neq 0$.

Proof of Lemma 9. As stated above, the first part regarding Carleson measures can be extracted from [15, 16]. We will only consider the part pertaining to vanishing Carleson measures here.

We argue first by contradiction. Assume that μ is vanishing Carleson for $\mathscr{D}_\alpha$, and that μ is not vanishing Carleson for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$. By Lemma 10, the latter assumption implies that there is some sequence of Carleson squares $\{Q_k(\tau_k,\varepsilon_k)\}_{k\geq 1}$, where $\varepsilon_k \to 0$, satisfying

$$\mu(Q_k) \gg \varepsilon_k^{2^\alpha}.$$

Let $s_k = 1/2 + \varepsilon_k + i\tau_k$ and consider

$$f_k(s) = \frac{K_\alpha(s,s_k)}{\|K_\alpha(\cdot,s_k)\|_{\mathscr{D}_\alpha}} = \frac{\zeta_\alpha\left(s+\overline{s_k}\right)}{\sqrt{\zeta_\alpha(1+2\varepsilon_k)}}.$$

It is easy to see that f_k is weakly compact in $\mathscr{D}_\alpha$, since $\|f_k\|_{\mathscr{D}_\alpha} = 1$ and $f_k(s) \to 0$ uniformly in $\sigma \geq 1/2+\delta$ for every $\delta > 0$. Since μ is assumed to be vanishing Carleson for $\mathscr{D}_\alpha$, this means that

$$\lim_{k\to\infty} \int_{Q_k} |f_k(s)|^2\, d\mu(s) \leq \lim_{k\to\infty} \int_{\mathbb{C}_{1/2}} |f_k(s)|^2\, d\mu(s) = 0.$$

Now, let $s = \sigma + it \in Q_k$. Then $1/2 \leq \sigma \leq 1/2 + \varepsilon_k$ and $\tau_k - \varepsilon_k/2 \leq t \leq \tau_k + \varepsilon_k/2$. Recalling the simple pole of the zeta function and using (17), we obtain

$$\zeta_\alpha\left(s + \overline{s_k}\right) \asymp \left(s + \overline{s_k} - 1\right)^{-2^\alpha} \gg (1 + 2\varepsilon_k + i\varepsilon_k/2 - 1)^{-2^\alpha} \asymp \varepsilon_k^{-2^\alpha}.$$

Similarly, $\sqrt{\zeta_\alpha(1 + 2\varepsilon_k)} \asymp \varepsilon_k^{-2^{\alpha-1}}$. Hence, by the assumption that μ is not vanishing Carleson for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$, we estimate

$$0 = \lim_{k\to\infty} \int_{Q_k} |f_k(s)|^2 \, d\mu(s) \gg \lim_{k\to\infty} \mu(Q_k)\varepsilon_k^{-2^\alpha} \gg 1,$$

and the desired contradiction is obtained.

In the other direction, assume that μ is vanishing Carleson for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$. Let $\{f_k\}_{k\geq 1}$ be a weakly compact sequence in $\mathscr{D}_\alpha$. Since μ has bounded support, there is some constant $M > 0$ so that

$$\int_{\mathbb{C}_{1/2}} |f_k(s)|^2 \, d\mu(s) \leq M \int_{\mathbb{C}_{1/2}} \left| \frac{f_k(s)}{(s + 1/2)^{2^\alpha}} \right|^2 d\mu(s). \tag{18}$$

Let $F_k(s) = f_k(s)/(s+1/2)^{2^\alpha}$. Clearly $F_k(s) \to 0$ on compact subsets K of $\mathbb{C}_{1/2}$ since this is true for f_k. From (9) and the discussion following Theorem 4, we conclude that $\|F_k\|_{D_{2^\alpha-1}} \ll \|f_k\|_{\mathscr{D}_\alpha}$. In particular, this implies that $\{F_k\}_{k\geq 1}$ is a weakly compact sequence in $D_{2^\alpha-1}(\mathbb{C}_{1/2})$ and hence by (18), the measure μ is vanishing Carleson for $\mathscr{D}_\alpha$. □

Remark. The first part of the proof of Lemma 9 does not use that μ has bounded support, so a vanishing Carleson measure for $\mathscr{D}_\alpha$ is always vanishing Carleson for $D_{2^\alpha-1}(\mathbb{C}_{1/2})$.

3.2. **Carleson measures on the polydisc.** Let $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$, and let Φ denote the Bohr lift of φ. For $\beta \geq 0$ we will consider the following measures on $\mathbb{C}_{1/2}$.

$$\mu_{\beta,\varphi}(E) = \begin{cases} \nu_\beta\big(\{z \in \mathbb{D}^\infty : \Phi(z) \in E\}\big), & \text{if } \beta > 0, \\ \nu_\beta\big(\{z \in \mathbb{T}^\infty : \Phi(z) \in E\}\big), & \text{if } \beta = 0, \end{cases} \qquad E \subset \mathbb{C}_{1/2}.$$

The following necessary and sufficient Carleson conditions for boundedness and compactness of $\mathscr{C}_\varphi$ when $\varphi \in \mathscr{G}$ with $\operatorname{char}(\varphi) = 0$ and $\varphi(\mathbb{C}_0)$ is a bounded set will be our main technical tool for the study of composition operators between the spaces $\mathscr{D}_\alpha$.

Lemma 11. *Let $\alpha, \beta \geq 0$. Suppose that $\varphi \in \mathscr{G}$ with* $\operatorname{char}(\varphi) = 0$ *and suppose that $\varphi(\mathbb{C}_0)$ is a bounded subset of $\mathbb{C}_{1/2}$. Then $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded if and only if*

$$\mu_{\beta,\varphi}\big(Q(\tau,\varepsilon)\big) = O\big(\varepsilon^{2^\alpha}\big) \tag{19}$$

for every Carleson square $Q(\tau,\varepsilon) = [1/2, 1/2+\epsilon] \times [\tau - \varepsilon/2, \tau + \varepsilon/2]$. *Moreover,* $\mathscr{C}_\varphi$ *is compact from* $\mathscr{D}_\alpha$ *to* $\mathscr{D}_\beta$ *if and only if*

$$\mu_{\beta,\varphi}\big(Q(\tau,\varepsilon)\big) = o\big(\varepsilon^{2^\alpha}\big),$$

as $\varepsilon \to 0^+$*, uniformly for* $\tau \in \mathbb{R}$*.*

Proof. We begin with the proof of the boundedness criterion (19). Assume at first that $\alpha, \beta > 0$. Let P be a Dirichlet polynomial. Since $c_0 = 0$, we observe as in the proof of Theorem 3 that $\mathscr{B}(P \circ \varphi) = P \circ \mathscr{B}\varphi$, so

$$\|\mathscr{C}_\varphi P\|_\beta^2 = \int_{\mathbb{D}^\infty} |P\left(\Phi(z)\right)|^2 \, d\nu_\beta(z). \tag{20}$$

Now, since $\mu_{\beta,\varphi} = \nu_{\beta,\varphi} \circ \Phi^{-1}$ and since Dirichlet polynomials are dense in $\mathscr{D}_\alpha$, it is easy to deduce from (20) that $\mathscr{C}_\varphi$ is bounded from $\mathscr{D}_\alpha$ to $\mathscr{D}_\beta$ if and only if

$$\int_{\mathbb{C}_{1/2}} |f(s)|^2 \, d\mu_{\beta,\varphi}(s) \ll \|f\|_{\mathscr{D}_\alpha}^2.$$

Using Kronecker's theorem and the maximum modulus principle on the polydisc, we find that $\operatorname{supp}\left(\mu_{\beta,\varphi}\right) = \overline{\varphi(\mathbb{C}_0)}$. By assumption, $\varphi(\mathbb{C}_0)$ is a bounded subset of $\mathbb{C}_{1/2}$, so $\mu_{\beta,\varphi}$ has bounded support. Hence, by Lemma 9 and Lemma 10, $\mu_{\beta,\varphi}$ is a Carleson measure for $\mathscr{D}_\alpha$ if and only if

$$\mu_{\beta,\varphi}\big(Q(\tau,\varepsilon)\big) = O\left(\varepsilon^{2^\alpha}\right).$$

The argument for compactness follows by similar considerations. If $\alpha = 0$, these arguments work line for line. If $\beta = 0$, we appeal directly to [21, Lem. 4.1]. Clearly $\operatorname{supp}\left(\mu_{\beta,\varphi}\right) \subseteq \overline{\varphi(\mathbb{C}_0)}$, so the measure is still boundedly supported. The remaining deliberations apply directly. ☐

This lemma can be combined with a compactness argument as in [5, Lem. 6], to obtain the next result. But first, note that if $\varphi \in \mathscr{G}$ is a Dirichlet polynomial with $\operatorname{char}(\varphi) = 0$, its Bohr lift $\Phi = \mathscr{B}\varphi$ is always a polynomial of $d < \infty$ variables. We call d the *complex dimension* of φ and write $d = \dim(\varphi)$.

Corollary 12. *Let* $\varphi \in \mathscr{G}$ *be a Dirichlet polynomial with* $\dim(\varphi) = d$ *and Bohr lift* Φ*. If for every* $w \in \mathbb{T}^d$ *with* $\operatorname{Re}\Phi(w) = 1/2$ *there exist a neighborhood* $\mathscr{U}_w \ni w$ *in* $\overline{\mathbb{D}^d}$*, constants* $C_w > 0$ *and* $\kappa_w \geq 2^\alpha$ *such that, for every* $\tau \in \mathbb{R}$ *and every* $\varepsilon > 0$*,*

$$\nu_\beta\big(\{z \in \mathscr{U}_w \colon \Phi(z) \in Q(\tau,\varepsilon)\}\big) \leq C_w \varepsilon^{\kappa_w},$$

then $\mathscr{C}_\varphi$ *maps* $\mathscr{D}_\alpha$ *boundedly into* $\mathscr{D}_\beta$*. If moreover* $\kappa_w > 2^\alpha$ *for every* $w \in \mathbb{T}^d$ *with* $\operatorname{Re}\Phi(w) = 1/2$*, then* $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ *is compact.*

3.3. **Measures of some sets in $\mathbb{D}^d$.** Corollary 12 indicates that we need to estimate the measure of some sets in $\mathbb{D}^d$. Let us collect some estimates for some particular subsets of $\mathbb{D}^d$. To simplify the computations, we will replace the measure ν_β with the new measure $\widetilde{\nu}_\beta$ associated to $\widetilde{m}_\beta$ as defined in (7). Now, if $\dim(f) = d$, then clearly

$$\int_{\mathbb{D}^d} |\mathscr{B}f(z)|^2 \, d\widetilde{\nu}_\beta(z) \asymp_{d,\beta} \int_{\mathbb{D}^d} |\mathscr{B}f(z)|^2 \, d\nu_\beta(z).$$

In particular, we can replace ν_β by $\widetilde{\nu}_\beta$ in Corollary 12. We should also point out that for $\beta = 0$, we do not change the measure and adopt the convention $\nu_0 = \widetilde{\nu}_0$.

For $\delta, \varepsilon > 0$, let $S(\delta, \varepsilon) = \left\{ z = (1-\rho)e^{i\theta} \in \mathbb{D} : 0 \le \rho \le \delta, \, |\theta| \le \varepsilon \right\}$. As usual, $B(w, r)$ will denote the open ball centered at $w \in \mathbb{C}$ with radius $r > 0$. Geometric considerations show that there exist absolute constants $c, C > 0$ such that, for every $\varepsilon > 0$ and every $w \in \mathbb{T}$, we have

$$S(c\varepsilon, c\varepsilon^{1/2}) \subset \{z \in \mathbb{D} : \operatorname{Re}(1-z) < \varepsilon\} \subset S(C\varepsilon, C\varepsilon^{1/2}) \tag{21}$$

$$wS(c\varepsilon, c\varepsilon) \subset B(w, \varepsilon) \cap \mathbb{D} \subset wS(C\varepsilon, C\varepsilon). \tag{22}$$

The following lemmas are inspired by [4], and for the sake of clarity we include a brief account of their proofs.

Lemma 13. *For any $\beta > 0$, $\widetilde{m}_\beta\big(S(\delta, \varepsilon)\big) \asymp_\beta \delta^\beta \varepsilon$.*

Proof. This follows from an integration in polar coordinates. □

Lemma 14. *For any $\beta > 0$, $\widetilde{m}_\beta\big(\{z \in \mathbb{D} : \operatorname{Re}(1-z) < \varepsilon\}\big) \asymp_\beta \varepsilon^{\beta + \frac{1}{2}}$.*

Proof. The result follows from Lemma 13 and (21). □

Lemma 15. *Let $\beta > 0$ and $v \in \mathbb{C}$. Then*

$$\widetilde{m}_\beta\big(\{z \in \mathbb{D} : \operatorname{Re}(1-z) < \varepsilon, \, |\operatorname{Im}(v - z)| < \varepsilon\}\big) \ll_\beta \varepsilon^{1+\beta}.$$

Proof. This follows again from an integration in polar coordinates. □

Lemma 16. *Let $\beta > 0$. There exists $c > 0$ such that, for any $v \in \mathbb{C}$ satisfying*

$$|\operatorname{Re}(v) - 1| \le c\varepsilon \qquad \textit{and} \qquad |\operatorname{Im}(v)| \le (c\varepsilon)^{1/2},$$

then

$$\widetilde{m}_\beta\big(\{z \in \mathbb{D} : \operatorname{Re}(1-z) < \varepsilon, \, |v - z| < \varepsilon\}\big) \asymp_\beta \varepsilon^{1+\beta}.$$

Proof. The upper bound is Lemma 15. For the lower bound, observe that, provided $c \in (0, 1/2)$, then $\{z \in \mathbb{D} : |z - v| < \varepsilon/2\} \subset \{z \in \mathbb{D} : \operatorname{Re}(1-z) < \varepsilon\}$. Hence, we just need to minorize $\widetilde{m}_\beta\big(B(v, \varepsilon/2) \cap \mathbb{D}\big)$. Now, it is easy to check that upon the conditions $c \in (0, 1/2)$ and $\varepsilon \in (0, 1)$,

$$-8c\varepsilon \le 1 - |v| \le 8c\varepsilon.$$

Writing

$$|z - v| \leq \left| z - \frac{v}{|v|} \right| + \big|1 - |v|\big|$$

we get that $B(v/|v|, \varepsilon/4) \subset B(v, \varepsilon/2)$ provided $c < 1/32$. We finish the proof as in Lemma 15. □

Remark. When $\delta = \varepsilon$, the sets $S(\delta, \varepsilon)$ are the classical Carleson windows of the disc. However, we are required to handle inhomogeneous Carleson windows in what follows.

4. Composition operators with polynomial symbols on $\mathscr{D}_\alpha$

Let us consider a polynomial symbol in $\mathscr{G}$ of characteristic $c_0 = 0$, say $\varphi(s) = \sum_{n=1}^{N} c_n n^{-s}$. We are only interested in symbols having *unrestricted range*, which means that $\varphi(\mathbb{C}_0)$ is not contained in $\mathbb{C}_{1/2+\delta}$, for any $\delta > 0$. If the symbol has restricted range, it is trivial to deduce from [1, Thm. 1] that $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ compactly into $\mathscr{D}_\beta$, for any choice of $\alpha, \beta \geq 0$.

Let us now look at the Bohr lift of φ, denoted Φ. As in the previous section, we will let $\dim(\varphi)$ denote the *complex dimension* of φ, which is equal to the number of variables in the polynomial $\Phi(z_1, \ldots, z_d)$. Now, the *degree* of φ will be the degree of Φ, and we will write $\deg(\varphi)$. When the complex dimension is big and the degree is small, we can improve $\beta = 2^\alpha - 1$ from the main result of [1] substantially.

Theorem 17. *Fix $\alpha > 0$ and consider a Dirichlet polynomial φ in $\mathscr{G}$ with unrestricted range.*

- (i) *If $d = \dim(\varphi) \geq 2$ and $\deg(\varphi) \in \{1, 2\}$, then $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ boundedly into $\mathscr{D}_\beta$ for some $\beta < 2^\alpha - 1$. More precisely, $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_{(2^\alpha - 1)/d}$ is bounded.*

The result is optimal in the following sense.

- (ii) *If $\dim(\varphi) = 1$, then $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is not bounded for any $\beta < 2^\alpha - 1$.*
- (iii) *There are polynomials $\varphi \in \mathscr{G}$ of any complex dimension and with arbitrary $\deg(\varphi) \geq 3$ for which $\mathscr{C}_\varphi$ is not bounded from $\mathscr{D}_\alpha$ to $\mathscr{D}_\beta$ for any $\beta < 2^\alpha - 1$.*

From the proof of Theorem 17 (and Corollary 12) it is possible to deduce the following result regarding compactness. However, before we state the result, let us stress that the inclusion $\mathscr{D}_\alpha \subset \mathscr{D}_\beta$ is *not* compact for $\alpha < \beta$. To realize this one needs only consider the weakly compact sequence generated by the prime numbers, $\{p_j^{-s}\}_{j \geq 1}$, since $d(p_j) = 2$.

Corollary 18. *Fix $\alpha > 0$ and consider a Dirichlet polynomial φ in $\mathscr{G}$ with unrestricted range.*

(i) *If* $\dim(\varphi) \geq 2$ *and* $\deg(\varphi) \in \{1, 2\}$*, then* $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_{2^\alpha - 1}$ *is compact.*

The result is optimal in the following sense.

(ii) *If* $\dim(\varphi) = 1$*, then* $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_{2^\alpha - 1}$ *is never compact.*

(iii) *There are polynomials* $\varphi \in \mathscr{G}$ *of any complex dimension and with arbitrary* $\deg(\varphi) \geq 3$ *for which* $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_{2^\alpha - 1}$ *is not compact.*

It is interesting to compare Corollary 18 to its version for $\alpha = 0$ which is [5, Thm. 3]. Ignoring the technical part of [5, Thm. 3] regarding minimal Bohr lift and boundary index, we observe that the results match up. However, going into the details, we observe that this correspondance is not completely true. We shall give later (see Theorem 21) simple examples of polynomial symbols φ such that $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ compactly into $\mathscr{D}_{2^\alpha - 1}$ for $\alpha > 0$, but does not map $\mathscr{H}^2$ compactly into $\mathscr{H}^2$. This phenomenon is due to the necessity to introduce the minimal Bohr lift in the context of $\mathscr{H}^2$.

Observe also that it is possible to deduce a version of Theorem 17 for the case $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{H}^2$ from [5, Lem. 10] using Lemma 11 and Corollary 12. However, the result would be cumbersome to state, due to the above mentioned technical parts, so we avoid it here.

We need one final lemma to prove Theorem 17, which can easily be deduced from the Julia–Caratheodory theorem (or from elementary considerations as in the proof of [5, Lem. 7]).

Lemma 19. *Let* $P(z) = \sum_{k=1}^{K} a_k (1-z)^k$ *be a polynomial mapping* $\mathbb{D}$ *into* $\overline{\mathbb{C}_0}$*. Then* $P \equiv 0$ *or* $a_1 > 0$*.*

We split the proof of Theorem 17 into two parts, and begin with the easiest part.

Proof of Theorem 17 — (ii) and (iii). We begin with (ii). Fix $\alpha > 0$ and assume that $\varphi \in \mathscr{G}$ is a Dirichlet polynomial with $\dim(\varphi) = 1$ and unrestricted range. By Corollary 12 we investigate some $w \in \mathbb{T}$ such that $\Phi(w) = 1/2 + i\tau$, where Φ denotes the Bohr lift of φ. We may assume that $w = 1$ and $\tau = 0$ after, if necessary, a (complex) rotation and a (vertical) translation. Hence, Φ is a polynomial of the form

$$\Phi(z) = \frac{1}{2} + \sum_{k=1}^{K} a_k (1-z)^k.$$

By Lemma 19 we know that $a_1 > 0$. In view of Corollary 12, it suffices to prove that for $\beta > 0$ and every small enough $\varepsilon > 0$,

$$\mu_{\beta,\varphi}\big(Q(0,\varepsilon)\big) \gg \varepsilon^{\beta+1}.$$

Using Lemma 13, we see that it is sufficient to prove that the homogeneous Carleson window $S(\varepsilon, \varepsilon)$ is included in the pre-image of $Q(0, c\varepsilon)$ under Φ for some fixed $c \in (0,1)$ and for every small enough $\varepsilon > 0$. Now, note that if $z \in S(\varepsilon,\varepsilon)$, then

$$\max\left\{\operatorname{Re}\left((1-z)^k\right),\, \operatorname{Im}\left((1-z)^k\right)\right\} \leq \varepsilon^k.$$

In particular, since Φ is a polynomial and $a_1 > 0$, we find that if $z \in S(\varepsilon,\varepsilon)$, then

$$1/2 \leq \operatorname{Re}\Phi(z) \leq 1/2 + a_1\varepsilon + O(\varepsilon^2),$$
$$|\operatorname{Im}\Phi(z)| \leq a_1\varepsilon + O(\varepsilon^2).$$

Hence any $c > a_1/2$ will do. Part (iii) can be deduced from this argument in the following way. Let $\delta > 0$ and let $\Psi(z) = \Psi(z_1, \ldots, z_d)$ be any polynomial in d variables and define

$$\Phi(z) = \frac{1}{2} + (1-z_1) + \delta(1-z_1)^2\Psi(z).$$

Clearly Φ is the Bohr lift of

$$\varphi(s) = \frac{1}{2} + (1-p_1^{-s}) + \delta(1-p_1^{-s})^2\Psi(p_1^{-s}, \ldots, p_d^{-s}).$$

It is proved in [5, Lem. 9] that by choosing $\delta > 0$ sufficiently small, we can guarantee that $\varphi \in \mathscr{G}$, that φ has unrestricted range and furthermore that if Φ touches the boundary of $\mathbb{C}_{1/2}$ at some point $z \in \overline{\mathbb{D}^d}$, then necessarily $z_1 = 1$. The argument given above works line for line with one minor modification. Suppose $z_1 \in S(\varepsilon,\varepsilon)$. Then for every choice of $z_2, \ldots, z_d$ in $\overline{\mathbb{D}}$ we have

$$\max\left\{\operatorname{Re}\left(\delta(1-z_1)^2\Psi(z)\right),\, \operatorname{Im}\left(\delta(1-z_1)^2\Psi(z)\right)\right\} \leq \delta\|\Psi\|_\infty\varepsilon^2,$$

so we conclude again by Corollary 12 and Lemma 13. □

Proof of Theorem 17 — (i). Let $\varphi \in \mathscr{G}$ be a Dirichlet polynomial and assume that $\dim(\varphi) = d \geq 2$ and $\deg(\varphi) \in \{1,2\}$. Let Φ be the Bohr lift of φ. We will again apply Corollary 12. Hence, let $w \in \mathbb{T}^d$ be such that $\operatorname{Re}\Phi(w) = 1/2$. Without loss of generality, we may assume that $w = \mathbf{1} = (1,\ldots,1)$ and that $\Phi(\mathbf{1}) = 1/2$. We may write Φ as

$$\Phi(z) = \frac{1}{2} + \sum_{j=1}^{d} a_j(1-z_j) + \sum_{j=1}^{d} b_j(1-z_j)^2 + \sum_{1\leq j<k\leq d} c_{j,k}(1-z_j)(1-z_k).$$

We first claim that $a_j > 0$ for any $j = 1, \ldots, d$. Indeed, applying Lemma 19 to $\Phi(\mathbf{1}, z_j, \mathbf{1}) - 1/2$, we know that either $a_j > 0$ or $a_j = b_j = 0$. Assume that the latter case holds. Since φ has complex dimension d, there exists $k \neq j$ so that $c_{j,k} \neq 0$. Let us consider $\Psi(z_j, z_k) = \Phi(\mathbf{1}, z_j, \mathbf{1}, z_k, \mathbf{1})$. Then a Taylor expansion of $\Psi(e^{i\theta_j}, e^{i\theta_k})$ shows that

$$\operatorname{Re}\Psi\left(e^{i\theta_j}, e^{i\theta_k}\right) = \frac{1}{2} + \left(\frac{a_k}{2} - \operatorname{Re}(b_k)\right)\theta_k^2 - \operatorname{Re}(c_{j,k})\theta_j\theta_k + o(\theta_j^2) + o(\theta_k^2).$$

Choosing $\theta_j = \delta$ and $\theta_k = \delta^2$ and letting δ to 0, this implies that $\mathrm{Re}(c_{j,k}) = 0$ since by assumption $\mathrm{Re}\,\Psi \geq 1/2$. On the other hand, for $\rho_j \in (0,1)$,

$$\mathrm{Re}\,\Psi(1-\rho_j, e^{i\theta_k}) = \frac{1}{2} + \left(\frac{a_k}{2} - \mathrm{Re}(b_k)\right)\theta_k^2 + \mathrm{Im}(c_{j,k})\rho_j\theta_k + o(\rho_j^2) + o(\theta_k^2).$$

This in turn yields that $\mathrm{Im}(c_{j,k}) = 0$, a contradiction.

We come back to Φ and, for $j = 1, \ldots, d$, we write $z_j = (1-\rho_j)e^{i\theta_j}$ where $\rho_j \in (0,1)$ and $\theta_j \in [-\pi, \pi)$. We shall use the local diffeomorphism between a neighborhood of $\mathbf{1}$ in $\mathbb{C}^d$ and a neighborhood of $\mathbf{0}$ in $\mathbb{R}^{2d}$ given by

$$(\rho,\theta) \mapsto \big((1-\rho_1)e^{i\theta_1}, \ldots, (1-\rho_d)e^{i\theta_d}\big).$$

A Taylor expansion of $\mathrm{Re}\,\Phi$ near $\mathbf{1}$ shows that

$$\mathrm{Re}\,\Phi(z) = \frac{1}{2} + \sum_{j=1}^{d} \rho_j F_j(\rho,\theta) + G(\theta)$$

where $F_j(\mathbf{0}) = a_j$. Taking all ρ_j equal to zero, we get that $G(\theta) \geq 0$. Hence, there exists a (fixed) neighborhood $\mathscr{U} \ni \mathbf{1}$ in $\overline{\mathbb{D}^d}$ such that for all $\varepsilon > 0$ and all $\tau \in \mathbb{R}$,

$$0 \leq \sum_{j=1}^{d} \rho_j F_j(\rho,\theta) \leq \varepsilon \qquad \text{and} \qquad F_j(\rho,\theta) \geq \frac{a_j}{2}$$

provided $z \in \mathscr{U}$ and $\Phi(z) \in Q(\tau,\varepsilon)$. This implies that $|\rho_j| \leq 2\varepsilon/a_j$ for any $j = 1, \ldots, d$. We now look at $\mathrm{Im}\,\Phi$ and let us write it under the following form:

$$\mathrm{Im}\,\Phi(z) = \gamma_{(\rho,\theta_2,\ldots,\theta_d)}(\theta_1) = a_1\theta_1 + o(\theta_1).$$

The map $(\rho,\theta) \mapsto \gamma_{(\rho,\theta_2,\ldots,\theta_d)}(\theta_1)$ is smooth and satisfies $\gamma'_{(\rho,\theta_2,\ldots,\theta_d)}(0) = a_1$. Then there exists $\mathscr{V}$ a neighborhood of $\mathbf{1}$ in $\mathbb{C}^d = \mathbb{R}^{2d}$ such that, for any $z \in \mathscr{V}$,

$$\gamma'_{(\rho,\theta_2,\ldots,\theta_d)}(\theta_1) \geq \frac{a_1}{2}. \tag{23}$$

Now, if $(\rho,\theta_2,\ldots,\theta_d)$ are fixed and θ_1 is such that z belongs to $\mathscr{V}$, the condition $\Phi(z) \in Q(\tau,\varepsilon)$ implies that $\gamma_{(\rho,\theta_2,\ldots,\theta_d)}(\theta_1)$ belongs to some interval of length ε. By (23), this implies that θ_1 belongs to some interval of length $C\varepsilon$, where C does not depend on $(\rho,\theta_2,\ldots,\theta_d)$ provided $z \in \mathscr{V}$.

Let us summarize the previous computations. We have shown that there exist a (fixed) neighborhood $\mathscr{W} = \mathscr{U} \cap \mathscr{V}$ of $\mathbf{1}$ in $\mathbb{C}^d$ and a constant $D > 0$ such that, for any $z \in \mathscr{W} \cap \mathbb{D}^d$ and any $\varepsilon > 0$ satisfying $\Phi(z) \in Q(\tau,\varepsilon)$, then $\rho_j \leq D\varepsilon$ and $\rho, \theta_2, \ldots, \theta_d$ being fixed, θ_1 belongs to some fixed interval of length $D\varepsilon$. By Fubini's theorem and polar integration as in Lemma 13, we get that

$$\widetilde{\nu}_\beta\left(\left\{z \in \mathscr{W} \cap \mathbb{D}^d \,:\, \Phi(z) \in Q(\tau,\varepsilon)\right\}\right) \ll \varepsilon^{d\beta+1}.$$

We conclude by Corollary 12. $\square$

Let us focus our attention on part (ii) of Theorem 17, which implies that it is sufficient to consider the most simple non-trivial symbol,

$$\varphi(s) = 3/2 - 2^{-s} = 1/2 + (1 - 2^{-s}), \tag{24}$$

to conclude that the sharp β for $\mathscr{C}_\varphi\colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is $\beta = 2^\alpha - 1$. This is perhaps not so surprising, since we can consider (24) a local version of the symbol associated to the transference map,

$$\mathscr{T}(2^{-s}) = \frac{1}{2} + \frac{1-2^{-s}}{1+2^{-s}},$$

as considered in Section 2. We will devote the remainder of this section to investigating two classes of examples that generalize (24).

The first extension of (24) are the *linear symbols*, namely symbols which are of the form

$$\varphi(s) = c_1 + \sum_{j=1}^{d} c_{p_j} p_j^{-s}. \tag{25}$$

Observe in particular that (24) is just the case $d = 1$. We have the following result.

Theorem 20. *Let $\alpha, \beta \geq 0$. Let φ of the form (25) with unrestricted range and $c_{p_j} \neq 0$ for every j. Then $\mathscr{C}_\varphi\colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded if and only if*

$$\frac{1}{2} + d\left(\frac{1}{2} + \beta\right) \geq 2^\alpha. \tag{26}$$

Moreover, $\mathscr{C}_\varphi\colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is compact if and only if the inequality in (26) is strict.

Proof. If $\beta = 0$, this can be extracted from [21, Lem. 8.2] in combination with Corollary 12.

Assume therefore that $\beta > 0$. Arguing as in [21], we may assume that $c_1 > 0$ and that $c_{p_j} < 0$ for every j. Since φ has unrestricted range, we know that

$$c_1 = \frac{1}{2} + \sum_{j=1}^{d} |c_{p_j}|.$$

We will represent the Bohr lift of φ in the following way.

$$\Phi(z) = c_1 + \sum_{j=1}^{d} c_{p_j} - \sum_{j=1}^{d} c_{p_j}(1 - z_j) = \frac{1}{2} + \sum_{j=1}^{d} |c_{p_j}|(1 - z_j). \tag{27}$$

Let $\tau \in \mathbb{R}$ and $\varepsilon > 0$. If $\Phi(z) \in Q(\tau, \varepsilon)$, we inspect (27) to conclude, for any $j = 1, \ldots, d-1$, that

$$\operatorname{Re}(1 - z_j) \leq \frac{\varepsilon}{|c_{p_j}|}.$$

Hence, for any $j = 1, \dots, d-1$, by Lemma 14 we know that z_j belongs to some set $R_j(\varepsilon)$ satisfying $\widetilde{m}_\beta(R_j(\varepsilon)) \ll \varepsilon^{\frac{1}{2}+\beta}$. Moreover, for a fixed value of $z_1, \dots, z_{d-1}$, we also have

$$\operatorname{Re}(1 - z_d) \le \frac{\varepsilon}{|c_{p_d}|} \qquad \text{and} \qquad |\operatorname{Im}(v - z_d)| \le \frac{\varepsilon}{2|c_{p_d}|}$$

for $v \in \mathbb{C}$ depending on $\tau, z_1, \dots, z_{d-1}$. By Lemma 15, z_d belongs to some set $R_d(z_1, \dots, z_{d-1})$ satisfying $\widetilde{m}_\beta(R_d(z_1, \dots, z_{d-1})) \ll \varepsilon^{1+\beta}$. Using Fubini's theorem, we get

$$\mu_{\beta,\varphi}\big(Q(\tau, \varepsilon)\big) \ll \varepsilon^{\frac{1}{2}+d\left(\frac{1}{2}+\beta\right)}$$

and we compare this with the sufficient condition for continuity.

Conversely, assume that z_j belongs to $D(\eta) = \{z \in \mathbb{D} : \operatorname{Re}(1 - z) \ge \eta\varepsilon\}$ for some small $\eta > 0$ and for any $j = 1, \dots, d-1$. Observe that $\widetilde{m}_\beta\big(D(\eta)\big) \gg \varepsilon^{\frac{1}{2}+\beta}$. Then, setting

$$v = \big(|c_{p_1}|(1 - z_1) + \cdots + |c_{p_{d-1}}|(1 - z_{d-1})\big)/|c_{p_d}|$$

we get

$$\begin{aligned}
&\Phi(z_1, \dots, z_d) \in Q(0, \varepsilon)\\
&\qquad \iff \begin{cases} 0 \le \operatorname{Re}\big(c_1 - |c_{p_1}|z_1 - \cdots - |c_{p_d}|z_d\big) \le \varepsilon \\ \big|\operatorname{Im}\big(|c_{p_1}|z_1 + \cdots + |c_{p_d}|z_d\big)\big| \le \varepsilon/2 \end{cases}\\
&\qquad \iff \begin{cases} 0 \le |c_{p_1}| \operatorname{Re}(1 - z_1) + \cdots + |c_{p_d}| \operatorname{Re}(1 - z_d) \le \varepsilon \\ \big|\operatorname{Im}\big(|c_{p_1}(1 - z_1) + \cdots + |c_{p_{d-1}}|(1 - z_{d-1}) - |c_{p_d}|z_d\big)\big| \le \varepsilon/2 \end{cases}\\
&\qquad \iff \begin{cases} 0 \le \operatorname{Re}\big(v + (1 - z_d)\big) \le \varepsilon/|c_{p_d}| \\ \big|\operatorname{Im}\big(v - z_d\big)\big| \le \varepsilon/2|c_{p_d}|. \end{cases}
\end{aligned}$$

Now, $\operatorname{Re}(v) \le C\eta\varepsilon$ and $|\operatorname{Im}(v)| \le (2C\eta\varepsilon)^{1/2}$ for

$$C = \frac{|c_{p_1}| + \cdots + |c_{p_{d-1}}|}{|c_{p_d}|}.$$

Hence, provided η is small enough, then $\Phi(z_1, \dots, z_d) \in Q(0, \varepsilon)$ as soon as $\operatorname{Re}(1 - z_d) < \eta\varepsilon$ and $|v - z_d| < \eta\varepsilon$. By Fubini's theorem and Lemma 16,

$$\mu_{\beta,\varphi}\big(Q(0, \varepsilon)\big) \gg \varepsilon^{\frac{1}{2}+d\left(\frac{1}{2}+\beta\right)}.$$

We conclude by Corollary 12. The same proof shows that $\mathscr{C}_\varphi$ maps $\mathscr{D}_\alpha$ compactly into $\mathscr{D}_\beta$ if and only if $1/2 + d(1/2 + \beta) > 2^\alpha$. □

It is clear that in $\mathscr{H}^2$, the monomials n^{-s} all have norm 1. This is of course no longer the case in $\mathscr{D}_\alpha$ when $\alpha > 0$. Thus we have more flexibility in choosing the Bohr lift for $\mathscr{H}^2$, since we may use any sequence of independent integers $(q_1, \dots, q_d)$ instead of $(p_1, \dots, p_d)$. This lead us to introduce the notion of *minimal* Bohr lift in [5]. For the Bergman spaces, we are by definition required to consider the *canonical* Bohr lift, since it is used to compute the norm. In this sense the situation is less subtle. To further emphasize the difference between $\alpha = 0$ and $\alpha > 0$, we have the following result.

Theorem 21. *Let $\alpha \geq 0$ and consider $\varphi(s) = 3/2 - n^{-s}$ for some fixed integer $n \geq 2$. Set $d = \dim(\varphi)$, which in this case is equal to the number of distinct prime factors of n. Then $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded if and only if $\beta \geq (2^\alpha - 1)/d$. If $\alpha = 0$, then $\mathscr{C}_\varphi$ is not compact on $\mathscr{H}^2$. If $\alpha > 0$, then $\mathscr{C}_\varphi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is compact if and only if $\beta > (2^\alpha - 1)/d$.*

Observe that for every $\alpha > 0$, we can make $\mathscr{C}_\varphi$ map $\mathscr{D}_\alpha$ into $\mathscr{D}_\beta$ for any $\beta > 0$, by increasing the number of prime factors in n. However, we can never obtain $\beta = 0$ in this case.

Proof. Assume first that $\alpha = 0$. As explained in [5], the minimal Bohr lift is simply $\Phi(z) = 3/2 - z$ for every integer $n \geq 2$, and by the results in [5], this means that $\mathscr{C}_\varphi \colon \mathscr{H}^2 \to \mathscr{H}^2$ is bounded, but not compact.

Assume now that $\alpha > 0$. Let p be any prime number that does not divide n and consider $\psi(s) = 3/2 - p^{-s}$. By Theorem 17 (ii) and Corollary 18 (ii) we know that $\mathscr{C}_\psi \colon \mathscr{D}_\alpha \to \mathscr{D}_\beta$ is bounded if and only if $\beta \geq 2^\alpha - 1$ and compact if and only if $\beta > 2^\alpha - 1$. Now, define the operator T on $\mathscr{C}_\psi(\mathscr{D}_\alpha)$ by $T(p^{-s}) = n^{-s}$ so that $\mathscr{C}_\varphi = T \circ \mathscr{C}_\psi$. A trivial estimate with the divisor function shows that if $g \in \mathscr{C}_\psi(\mathscr{D}_\alpha)$, then

$$\|g\|_{\mathscr{D}_\beta} \leq \|T(g)\|_{\mathscr{D}_{\beta/d}} \ll_{n,\beta} \|g\|_{\mathscr{D}_\beta},$$

so we are done. $\square$

Remark. It is natural to ask whether the space $\mathscr{D}_{(2^\alpha-1)/d}$ in Theorem 17 (i) is optimal. We found that this is not the case for linear symbols in Theorem 20. By Theorem 21, it is optimal if $\dim(\varphi) = 2$. For $\dim(\varphi) \geq 3$, we conjecture that $(2^\alpha - 1)/d$ is not optimal, but our results do not further substantiate this claim.

5. Composition operators with linear symbols on $\mathscr{H}^p$

Let us reiterate that the results of the previous section show that the optimal β for the local embedding of $\mathscr{D}_\alpha$ can, through the results of Section 2, be decided simply by considering the symbol $\varphi(s) = 3/2 - 2^{-s}$. The embedding problem is in general open for $\mathscr{H}^p$, so it is therefore interesting to investigate how the composition operator generated by this symbol acts on $\mathscr{H}^p$.

As previously mentioned, composition operators with characteristic 0 acting on $\mathscr{H}^p$ are not well understood when p is not an even integer. In particular, very few examples are known. To our knowledge, the only known non-trivial examples appear in [6]. The symbols of these operators are given by

$$\varphi(s) = \frac{1}{2} + \left(\frac{1 - \omega(2^{-s})}{1 + \omega(2^{-s})}\right)^{1-\varepsilon} \tag{28}$$

where ω is an analytic self-map of $\mathbb{D}$ and $\varepsilon \in (0,1)$. Observe that the fact that we are not allowed to set $\varepsilon = 0$ restricts the range of φ in $\mathbb{C}_{1/2}$. Symbols of this type are a type of lens maps from $\mathbb{C}_0$ to $\mathbb{C}_{1/2}$. Observe also that the most simple case $\omega(z) = z$ yields a restricted version of the "transference map" from Theorem 3 (iii).

Now, it is clear that $\varphi(s) = 3/2 - 2^{-s}$, or indeed any Dirichlet polynomial, is not of the form (28). We are not able to settle the boundedness of the composition operator induced by this symbol on $\mathscr{H}^p$, but we will again consider symbols of linear type. Using Theorem 1, we will be able to prove boundedness when the complex dimension is bigger than or equal to 2.

Our last main tool for this will be the so-called *p/q–Carleson measures*. Let $1 \leq p, q < \infty$ and let X be one of the spaces considered in this paper, for instance $X = \mathscr{H}^q$ or $X = H^q(\mathbb{C}_{1/2})$. If $X = \mathscr{D}_\alpha$ or $X = D_\beta(\mathbb{C}_{1/2})$ then $q = 2$. We require that a measure μ satisfies

$$\left(\int_{\mathbb{C}_{1/2}} |f(s)|^p \, d\mu(s)\right)^{\frac{1}{p}} \leq C\|f\|_X, \tag{29}$$

for some constant $C = C(p,q,X)$ to be p/q–Carleson for X. For $X = H^q(\mathbb{C}_{1/2})$ and $q \leq p$, the following description can be found in [10, Thm. 9.4].

Lemma 22. *Let $1 \leq q \leq p < \infty$. A positive Borel measure μ on $\mathbb{C}_{1/2}$ is p/q–Carleson for $H^q(\mathbb{C}_{1/2})$ if and only if*

$$\mu\big(Q(\tau,\varepsilon)\big) = O\big(\varepsilon^{p/q}\big)$$

for every Carleson square $Q(\tau,\varepsilon) = [1/2, 1/2+\epsilon] \times [\tau - \varepsilon/2, \tau + \varepsilon/2]$.

Let us now extend a result from [16] to the case $p < q$, which will be needed in the proof of Theorem 2 for the range $2 < p < \infty$.

Lemma 23. *Fix $1 \leq q \leq p < \infty$ and let μ be a positive Borel measure on $\mathbb{C}_{1/2}$.*

(i) *If μ is p/q–Carleson for $\mathscr{H}^q$, then μ is p/q–Carleson for $H^q(\mathbb{C}_{1/2})$.*
(ii) *If the embedding* (3) *holds for q and μ has bounded support, then the converse is true.*

Proof. To prove part (i), we use Lemma 22 and argue by contradiction as in the first part of the proof of Lemma 9. In particular, assume that μ is a p/q–Carleson measure $\mathscr{H}^q$. Consider a sequence of Carleson squares $Q_k = Q(\tau_k, \varepsilon_k)$ and the Dirichlet series

$$f_k(s) = [\zeta(s + 1/2 + \varepsilon_k + i\tau_k)]^{2/q},$$

which satisfies $\|f_k\|_{\mathscr{H}^q} = [\zeta(1+2\varepsilon_k)]^{1/q}$. We deduce from (29) that $\mu(Q_k) \ll \varepsilon_k^{p/q}$ as $\varepsilon_k \to 0$ and conclude as in Lemma 9. Part (ii) follows from a routine application of the embedding. We proceed as in the proof of the second part of Lemma 9, setting now $F(s) = f(s)/(s+1/2)^{2/q}$ and using the same trick as in (18). $\square$

To make the statement of our final lemma more convenient, we will move to $\mathbb{C}_0$ as in [5]. This change is easily carried out when working with composition operators, since it corresponds to the changes $f(s) \mapsto f(s+1/2)$ and $\varphi(s) \mapsto \varphi(s+1/2)-1/2$. In particular, the translated $f \in \mathscr{D}_\alpha$ is embedded in $D_{2^\alpha-1,\mathrm{i}}(\mathbb{C}_0)$.

Let $d_{\mathrm{H}}(z,w)$ be the hyperbolic distance in the half-plane $\mathbb{C}_0$ which is defined by

$$\frac{1-e^{-d_{\mathrm{H}}(z,w)}}{1+e^{-d_{\mathrm{H}}(z,w)}} = \left|\frac{z-w}{z+\overline{w}}\right|$$

and let $B_{\mathrm{H}}(s,r)$ be the hyperbolic disc of centre s and radius $r \in (0,1)$. It is well-known that $B_{\mathrm{H}}(s,r)$, for $s = \sigma + it$, is simply the Euclidean disc of centre $(\sigma \cosh r, t)$ and radius $\sigma \sinh r$. In particular, we shall use that if r is not too big, then $B_{\mathrm{H}}(s,r)$ is contained in $[\sigma/2, 2\sigma] \times [t-\sigma, t+\sigma]$.

Luecking in [13] has characterized the p/q–Carleson measures of the (unweighted) Bergman spaces in the unit disc when $p < q$. As observed in [18], his proof carries on the weighted Bergman spaces. The next lemma is simply [18, Thm. B] with $p = 2$ and $n = 0$, translated from $D_\beta(\mathbb{D})$ to $D_{\beta,\mathrm{i}}(\mathbb{C}_0)$ using $\mathscr{T} - 1/2$.

Lemma 24. *Let $1 \leq p < 2$ and let μ be a positive Borel measure on $\mathbb{C}_0$. Then μ is $p/2$–Carleson for $D_{\beta,\mathrm{i}}(\mathbb{C}_0)$ if and only if for some (any) $r > 0$,*

$$\int_{\mathbb{C}_0} \big(\mu\big(B_{\mathrm{H}}(s,r)\big)\big)^{\frac{2}{2-p}} \, \sigma^{\frac{(p-4)-p\beta}{2-p}} \, |s+1|^{\frac{2p(\beta+1)}{2-p}} \, dm_1(s) < \infty. \tag{30}$$

We are finally in a position to prove Theorem 2.

Proof of Theorem 2. We first assume $p > 2$. We begin by fixing some positive integer k and consider $q = 2k < p < 2k+2$. We want to investigate when $\mathscr{C}_\varphi$ maps $\mathscr{H}^q$ to $\mathscr{H}^p$. Since $p > q$, this also means that $\mathscr{C}_\varphi$ acts boundedly on $\mathscr{H}^p$. Setting $\mu_\varphi := \mu_{0,\varphi}$, we argue as in the proof of Lemma 11 to find that boundedness of $\mathscr{C}_\varphi \colon \mathscr{H}^q \to \mathscr{H}^p$ is equivalent to

$$\left(\int_{\mathbb{C}_{1/2}} |f(s)|^p \, d\mu_\varphi(s)\right)^{\frac{1}{p}} \ll \|f\|_{\mathscr{H}^q}. \tag{31}$$

Using Lemma 22 and Lemma 23, while keeping in mind that the embedding (3) holds for $q = 2k$, we find that (31) is equivalent to

$$\mu_\varphi\big(Q(\tau,\epsilon)\big) \ll \varepsilon^{p/q},$$

for every Carleson square Q. However, from [21, Lem. 8.2] we know that

$$\mu_\varphi\big(Q(\tau,\epsilon)\big) \ll \varepsilon^{(d+1)/2}.$$

Hence we require of d that

$$d \geq \frac{2p}{q} - 1 = \frac{p}{k} - 1.$$

It is easy to check that $d \geq 2$ is sufficient if $p \in (2,3] \cup (4,\infty)$ and $d \geq 3$ is sufficient if $3 < p < 4$.

We now consider $1 \leq p < 2$. First, we use Theorem 20 with $\alpha = 1$ and $\beta = 0$ to conclude that if $d \geq 3$, then $\mathscr{C}_\varphi$ maps $\mathscr{D}_1$ boundedly into $\mathscr{D}_0 = \mathscr{H}^2$. To conclude that $\mathscr{C}_\varphi \colon \mathscr{H}^p \to \mathscr{H}^p$ is bounded, we use the inequalities

$$\|f\|_{\mathscr{D}_1} \leq \|f\|_{\mathscr{H}^1} \leq \|f\|_{\mathscr{H}^p} \leq \|f\|_{\mathscr{H}^2},$$

where the first one is Helson's inequality.

It remains to prove that $d \geq 2$ is sufficient when $1 \leq p < 2$ and $p \in (3,4)$. The trivial identity

$$\|f \circ \varphi\|_{\mathscr{H}^{2p}}^{2p} = \left\|f^2 \circ \varphi\right\|_{\mathscr{H}^p}^p$$

shows that it is enough to conclude for $p \in [1,2)$. Assume that $\varphi(s) = c_1 + c_{p_1}p_1^{-s} + c_{p_2}p_2^{-s}$ has unrestricted range. Using Theorem 1, we find that it is sufficient to verify that

$$\left(\int_{\mathbb{T}^2} |f \circ \Phi(z)|^p \, d\nu(z)\right)^{\frac{1}{p}} = \left(\int_{\mathbb{C}_{1/2}} |f(s)|^p \, d\mu_{0,\varphi}(s)\right)^{\frac{1}{p}} \ll \|f\|_{D_{\frac{2}{p}-1,\mathrm{i}}(\mathbb{C}_{1/2})}.$$

We now move to $\mathbb{C}_0$ to use Lemma 24, and subtract $1/2$ from φ. Arguing as in (27) we may assume that $\widetilde{\Phi}(z) = |c_{p_1}|(1-z_1) + |c_{p_2}|(1-z_2)$, and we consider the measure $\widetilde{\mu}$ defined on $\mathbb{C}_0$ by

$$\widetilde{\mu}(E) = \nu\big(\big\{(z_1,z_2) \in \mathbb{T}^2 \,:\, \widetilde{\Phi}(z_1,z_2) \in E\big\}\big), \qquad E \subset \mathbb{C}_0.$$

We need to investigate for which $1 \leq p < 2$ the measure $\widetilde{\mu}$ satisfies the condition of Lemma 24 with $\beta = 2/p - 1$. Recall that, for $s = \sigma + it$ and some suitably small $r > 0$,

$$\begin{aligned}\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big) &\leq \widetilde{\mu}\big([\sigma/2, 2\sigma] \times [t-\sigma, t+\sigma]\big) \\ &= \nu\big(\big\{(z_1,z_2) \in \mathbb{T}^2 \\ &\qquad : |c_{p_1}|(1-z_1) + |c_{p_2}|(1-z_2) \in [\sigma/2, 2\sigma] \times [t-\sigma, t+\sigma]\big\}\big).\end{aligned}$$

Since $\widetilde{\Phi}(\mathbb{T}^2)$ is a bounded subset of $\overline{\mathbb{C}_0}$, it is clear that $\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big) = 0$ when $\mathrm{Re}(s)$ is large enough, say $\sigma > \sigma_0$, or when $|\mathrm{Im}(s)|$ is large enough, say $|t| > t_0$. This means that the integral (30) in our case is equal to

$$\begin{aligned}&\int_0^{\sigma_0} \int_{|t| \leq t_0} \big(\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big)\big)^{\frac{2}{2-p}} \, \sigma^{\frac{2p-6}{2-p}} \, |s+1|^{\frac{4}{2-p}} \frac{dt d\sigma}{\pi} \\ &\qquad \asymp \int_0^{\sigma_0} \int_{|t| \leq t_0} \big(\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big)\big)^{\frac{2}{2-p}} \, \sigma^{\frac{2p-6}{2-p}} \, dt \, d\sigma =: I.\end{aligned}$$

This means we only need to prove that $I < \infty$ for any fixed pair (σ_0, t_0). Because $\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big)$ is bounded, we may in fact assume that σ_0 is very small. Now, let us fix $s \in \mathbb{C}_0$ with $\mathrm{Re}(s) \leq \sigma_0$ and let us consider $(\theta_1, \theta_2) \in [-\pi, \pi)^2$ such that $\widetilde{\Phi}(e^{i\theta_1}, e^{i\theta_2}) \in B_{\mathrm{H}}(s,r)$. Writing

$$\mathrm{Re}\,\widetilde{\Phi}(e^{i\theta_1}, e^{i\theta_2}) = |c_{p_1}|\,(1-\cos(\theta_1)) + |c_{p_2}|\,(1-\cos(\theta_2))\,,$$

it is clear that θ_1 and θ_2 are close to 0, so that

$$\theta_1^2 + \theta_2^2 \ll \mathrm{Re}\,\widetilde{\Phi}(e^{i\theta_1}, e^{i\theta_2}) \leq 2\sigma,$$

and hence we conclude that $|\theta_1|, |\theta_2| \ll \sigma^{1/2}$. On the other hand, this implies that

$$\left|\mathrm{Im}\,\widetilde{\Phi}(e^{i\theta_1}, e^{i\theta_2})\right| = \big||c_{p_1}|\sin(\theta_1) + |c_{p_2}|\sin(\theta_2)\big| \ll \sigma^{1/2},$$

which yields that $\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big) = 0$ provided $|t| \gg \sigma^{1/2}$. Otherwise, for a fixed value of θ_2, we note that θ_1 belongs to some interval with length dominated by $C\sigma$. Therefore, by Fubini's theorem, $\widetilde{\mu}\big(B_{\mathrm{H}}(s,r)\big) \ll \sigma^{3/2}$ where the involved constant does not depend on t. In total, this means that we require

$$I \ll \int_0^{\sigma_0} \int_{|t| \ll \sigma^{1/2}} \sigma^{\frac{2p-3}{2-p}}\,dt\,d\sigma \asymp \int_0^{\sigma_0} \sigma^{\frac{2p-3}{2-p}+\frac{1}{2}}\,d\sigma < \infty.$$

This last integral is convergent for $p \geq 1$. ☐

Remark. It is possible to generate more examples from the results in [5] or from the results of Section 4 in combination with Theorem 1. If $2k < p < 2k+2$, we can choose any Dirichlet polynomial with $\kappa \geq p/2k$, where κ as defined in [5, Lem. 10]. However, this also illustrates the disadvantage of this interpolation method, since the natural condition is $\kappa \geq 1$, which corresponds to the case $d = 1$ in (25).

We end this section by emphasizing that results with $d = 1$, or results for Dirichlet polynomials $\varphi \in \mathscr{G}$ with unrestricted range and $\dim(\varphi) = 1$, cannot be obtained from the type of Carleson measure arguments employed in this section and the local embedding (3) seems to be completely unavoidable in this setting.

6. The multiplicative Hilbert matrix

It was asked in [9, Sec. 6] whether the multiplicative Hilbert matrix introduced in the same paper has a bounded symbol on the polytorus $\mathbb{T}^\infty$, or, equivalently, whether the functional

$$L(f) = \int_{1/2}^{\infty} (f(s) - a_1)\,ds \tag{32}$$

is bounded on $\mathscr{H}^1$. It follows by standard Carleson measure techniques that if the embedding (3) holds for $\mathscr{H}^p$, then the functional (32) acts boundedly on

$\mathscr{H}^p$. It is therefore only known that the functional is bounded on $\mathscr{H}^p$ when $2 \leq p < \infty$.

Returning to the composition operator with symbol $\varphi(s) = 3/2 - 2^{-s}$, we write out explicitly the associated Carleson measure, finding that boundedness of $\mathscr{C}_\varphi$ on $\mathscr{H}^p$ is equivalent to the inequality

$$\int_{3/2+\mathbb{T}} |P(z)|^p \, dm(z) \ll \|P\|_{\mathscr{H}^p}^p,$$

for every Dirichlet polynomial P. If we apply the characterization of Carleson measures for $H^p(\mathbb{T})$, we find furthermore that

$$\int_{1/2}^{1} |P(z)|^p \, dz \ll \int_{3/2+\mathbb{T}} |P(z)|^p \, dm(z).$$

From this and the results in [9, Sec. 6] we observe that if $\mathscr{C}_\varphi$ acts boundedly on $\mathscr{H}^1$, then the multiplicative Hilbert matrix considered in [9] has a bounded symbol on the polytorus $\mathbb{T}^\infty$. In [9] it is only shown that the embedding (3) implies that the multiplicative Hilbert matrix has a bounded symbol, so this observation is in some sense an improvement.

Using Theorem 1, we can prove boundedness of L on $\mathscr{H}^p$ for $p \in (1, \infty)$.

Theorem 25. *The functional L defined by* (32) *is bounded on $\mathscr{H}^p$ for any $p > 1$.*

Proof. We may restrict ourselves to $p \in (1, 2)$. By Theorem 1, it is sufficient to verify that the functional of integration from $1/2$ to 1 is bounded on $D_{2/p-1,\mathrm{i}}(\mathbb{C}_{1/2})$ or, equivalently, that the functional of integration from 0 to 1 is bounded on $D_{2/p-1}(\mathbb{D})$. By duality, this is true since

$$f_\alpha(z) = \sum_{k=0}^{\infty} (k+1)^{\alpha-1} z^k$$

is in $D_\alpha(\mathbb{D})$ if and only if $\alpha < 1$, so that $p > 1$ is sufficient for L to act boundedly on $\mathscr{H}^p$. $\square$

This theorem has an interesting corollary. Write $L(f) = \langle f, g \rangle_{\mathscr{H}^2}$, where

$$g(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} n^{-s}.$$

We first have the following computation.

Lemma 26. *$g \in \mathscr{H}^p$ if and only if $p < 4$.*

Proof. From the estimate $\sum_{n \leq x} [d(n)]^\alpha \asymp x(\log x)^{2^\alpha - 1}$ (see [26]) and a standard computation with Abel summation, we find that

$$\sum_{n=2}^{\infty} \frac{[d(n)]^\alpha}{n(\log n)^\beta} < \infty$$

if and only if $2^\alpha < \beta$. Assume first that $2 < p < 4$. Using [23, Thm. 3] we have that

$$\|g\|_{\mathscr{H}^p} \leq \left(\sum_{n=2}^{\infty} \frac{[d(n)]^{2-\frac{4}{p}}}{n(\log n)^2}\right)^{\frac{1}{2}} < \infty,$$

since $\alpha = 2 - 4/p < 1$ when $2 < p < 4$. For $p = 4$, we compute

$$\|g\|_{\mathscr{H}^4}^4 = \left\|g^2\right\|_{\mathscr{H}^2}^2 = \sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{\substack{d|n \\ 1<d<n}} (\log(d)\log(n/d))^{-1} \right)^2 \gg \sum_{n=2}^{\infty} \frac{[d(n)]^2}{n(\log n)^4} = \infty,$$

so we are done. □

Theorem 25 and Lemma 26 yield an explicit and natural example of the observation that $\mathscr{H}^q \subsetneq (\mathscr{H}^p)^*$ for Hölder conjugates $1 < p, q < \infty$, as discussed in [22, Sec. 3].

Corollary 27. *Let $1 < p \leq 4/3$ and set $1/p + 1/q = 1$. The Dirichlet series g is in $(\mathscr{H}^p)^* \backslash \mathscr{H}^q$.*

7. Bounded zero sequences for $\mathscr{H}^p$

Let us show how Theorem 1 and the technique used in its proof can be used to improve the known partial results for zeros of functions in $\mathscr{H}^p$. As explained in [17], the almost periodicity of absolutely convergent Dirichlet series implies that $f \in \mathscr{H}^p$ either has no zeros or infinite many zeros in $\mathbb{C}_{1/2}$. One is therefore led to consider bounded zero sequences. A sequence $S \subset \mathbb{C}_{1/2}$ of (not necessarily distinct) points is a bounded zero sequence of some space of analytic functions X if it is bounded and there is some nontrivial $f \in X$ such that $f(s) = 0$ for $s \in S$.

If X is a Hilbert space of Dirichlet series, the situation is relatively clear thanks to results in [8, 15, 17, 23]. For example, if $X = \mathscr{H}^2$, it is known that the Blaschke condition

$$\sum_{s \in S} (\mathrm{Re}(s) - 1/2) < \infty \tag{33}$$

is both necessary and sufficient. In both [8] and [23], embeddings between Hilbert spaces of Dirichlet series and $\mathscr{H}^p$ is used to obtain necessary and sufficient conditions for $1 \leq p < 2$ and $2 < p < \infty$, respectively. In particular, Theorem 1 of the present paper improves the results discussed in [8, Sec. 3], and allows us to conclude that the necessary condition for bounded zero sequences of $\mathscr{H}^p$ in a certain sense converges to the Blaschke condition when $p \to 2^-$.

However, the main point of this section is to sharpen the sufficient condition from [23] when $2 < p < \infty$. We begin by taking the dual of (11) in the H^2-pairing, we find for $f(z) = \sum_{k\geq 0} a_k z^k$ that

$$\|f\|_{H^p(\mathbb{D})} \leq C_p \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^{2/p-1} \right)^{\frac{1}{2}} \tag{34}$$

for $2 \leq p < \infty$. Here we introduce $D_\alpha(\mathbb{D})$ for $\alpha < 0$ as the Hilbert space of functions f that satisfy $\sum_{k\geq 0} |a_k|^2 (k+1)^{-\alpha} < \infty$. The above inequality can be written as

$$\|f\|_{H^p(\mathbb{D})} \leq C_p \|f\|_{D_{1-2/p}(\mathbb{D}))}.$$

As before we are interested in contractive inequalities. We test with $f_\varepsilon(z) = 1+\varepsilon z$ to find that if

$$\|f\|_{H^p(\mathbb{D})} \leq \|f\|_{D_\alpha(\mathbb{D})},$$

then $\alpha \leq 1 - (\log p)/(\log 2)$. Indeed, it was shown in [23] that if $p = 2^k$ this is also sufficient which yields for the Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ that

$$\|f\|_{\mathscr{H}^p} \leq \left(\sum_{n=1}^{\infty} |a_n|^2 [d(n)]^{\frac{\log p}{\log 2} - 1} \right)^{\frac{1}{2}}, \qquad p = 2^k. \tag{35}$$

Following this, Riesz–Thorin interpolation between $p = 2^k$ and the results for bounded zero sequences for Hilbert spaces of Dirichlet series gave a sufficient condition for bounded zero sequences of $\mathscr{H}^p$. Let us now show how to improve this result by replacing interpolation with Weissler's inequality as in (the proof of) Theorem 1.

Note that for $0 < \alpha < 1$, we have

$$\|f\|^2_{D_\alpha(\mathbb{D})} \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{2-\alpha} \, dA(z).$$

The corresponding space in $\mathbb{C}_{1/2}$ can be given the norm

$$\|f\|_{D_\alpha(\mathbb{C}_{1/2})} = |f(3/2)|^2 + \int_{\mathbb{C}_{1/2}} |f'(s)|^2 \left(\sigma - \frac{1}{2}\right)^{2-\alpha} \frac{dA(s)}{|s+1/2|^2}.$$

The result in [23] discussed above was that every bounded zero sequences of $D_\alpha(\mathbb{C}_{1/2})$ is also a zero sequence for $\mathscr{H}^p$, but due to the interpolation step, unless $p = 2^k$ then $\alpha > 2/p - 1$. (We refer to [23] for the precise value.) Again, we will find that a "worse" inequality than (35) will yield a better result locally.

Theorem 28. *Let $2 < p < \infty$. Every bounded zero sequence in $D_{2/p-1}(\mathbb{C}_{1/2})$ is also a bounded zero sequence for $\mathscr{H}^p$.*

Proof. Let $f(z) = \sum_{k\geq 0} a_k z^k$. Set $P_r f(z) = f(rz)$. Weissler's inequality [25] states that if $0 < p \leq q < \infty$ and $r \leq \sqrt{p/q}$, then

$$\|P_r f\|_{H^q} \leq \|f\|_{H^p}.$$

We set $p = 2$, $q > 2$, $r = \sqrt{p/q}$ and recast the inequality as

$$\|f\|_{H^q} \leq \|P_{1/r}f\|_{H^2} = \left(\sum_{k=0}^{\infty} |a_k|^2 \left(\frac{q}{2}\right)^k \right)^{\frac{1}{2}}.$$

Iterating the inequality using Minkowski's inequality, we find that

$$\|f\|_{\mathscr{H}^q} \leq \left(\sum_{n=0}^{\infty} |a_n|^2 \left(\frac{q}{2}\right)^{\Omega(n)} \right)^{\frac{1}{2}}.$$

Using Lemma 8, the techniques from [15] and [23] show that the bounded zero sequences of the Hilbert space of Dirichlet series weighted by $(q/2)^{\Omega(n)}$ weighted space and $D_{1-2/q}(\mathbb{C}_{1/2})$ are the same. □

In view of (34), Theorem 28 is the best possible result we can hope to obtain by Hilbert space techniques. While the Weissler–type weights $(q/2)^{\Omega(n)}$ gives rise to a rather small space when $2 < q < \infty$ (since the weights have the "wrong" average order (compared to what is expected from (34)) when $q > 4$), the local embeddings and interpolation results used for zero sequences depends on the average order of $1/(q/2)^{\Omega(n)}$ which has the "correct" value compared to (34). Hence it is a small space, but big enough to have all the bounded zero sequences of $D_{1-2/q}(\mathbb{C}_{1/2})$.

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Paper 5

An embedding constant for the Hardy space of Dirichlet series

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AN EMBEDDING CONSTANT FOR THE HARDY SPACE OF DIRICHLET SERIES

OLE FREDRIK BREVIG

Abstract. A new and simple proof of the embedding of the Hardy–Hilbert space of Dirichlet series into a conformally invariant Hardy space of the half-plane is presented, and the optimal constant of the embedding is computed.

Let $\mathscr{H}^2$ denote the Hardy–Hilbert space of Dirichlet series,

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

with square summable coefficients, and set

$$\|f\|_{\mathscr{H}^2} := \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}.$$

Using the Cauchy–Schwarz inequality, we find that a Dirichlet series $f \in \mathscr{H}^2$ is absolutely convergent in the half-plane $\mathbb{C}_{1/2} := \{s : \mathrm{Re}(s) > 1/2\}$. To see that $\mathbb{C}_{1/2}$ is the largest half-plane of convergence for $\mathscr{H}^2$, consider $f(s) = \zeta(1/2+\varepsilon+s)$, where ζ denotes the Riemann zeta function and $\varepsilon > 0$.

When studying function and operator theoretic properties of $\mathscr{H}^2$, it has proven fruitful to employ the embedding of $\mathscr{H}^2$ into the conformally invariant Hardy space of $\mathbb{C}_{1/2}$ (see e.g. [6, Sec. 9]). The embedding inequality takes on the form

$$\|f\|_{H^2_{\mathrm{i}}} := \left(\frac{1}{\pi}\int_{-\infty}^{\infty} |f(1/2+it)|^2 \, \frac{dt}{1+t^2}\right)^{\frac{1}{2}} \leq C\|f\|_{\mathscr{H}^2}. \tag{1}$$

Observe that the embedding inequality (1) implies that Dirichlet series in $\mathscr{H}^2$ are locally L^2-integrable on the line $\mathrm{Re}(s) = 1/2$. Indeed, the proofs of (1) in the literature go via the local (but equivalent) formulation

$$\sup_{\tau\in\mathbb{R}} \left(\int_{\tau}^{\tau+1} |f(1/2+it)|^2 \, dt\right)^{\frac{1}{2}} \leq \widetilde{C}\|f\|_{\mathscr{H}^2}. \tag{2}$$

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To prove (2), one can use a general Hilbert–type inequality due to Montgomery and Vaughan [3] or a version of the classical Plancherel–Polya inequality [2, Thm. 4.11]. It is also possible to give Fourier analytic proofs of (2), the reader is referred to [4, pp. 36–37] and [5, Sec. 1.4.4]. It should be pointed out that these proofs do not give a precise value for either of the constants C and $\widetilde{C}$.

This note contains a new and simple proof of (1), which additionally identifies the optimal constant C. The proof is based on the observation that the H^2_{i}-norm of a Dirichlet series is associated to a Hilbert–type bilinear form which is easy to estimate precisely.

Theorem. *Suppose that* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *is in* $\mathscr{H}^2$*. Then*

$$\left(\frac{1}{\pi}\int_{-\infty}^{\infty} |f\,(1/2+it)|^2\,\frac{dt}{1+t^2}\right)^{\frac{1}{2}} < \sqrt{2}\|f\|_{\mathscr{H}^2},$$

and the constant $\sqrt{2}$ *is optimal.*

Proof. Let x be a positive real number. We begin by computing

$$\begin{aligned} I(x) &:= \frac{1}{\pi}\int_{-\infty}^{\infty} x^{it}\,\frac{dt}{1+t^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \cos(|\log x|\,t)\,\frac{dt}{1+t^2} = e^{-|\log x|} = \frac{1}{\max(x,1/x)}. \end{aligned}$$

Expanding $|f(1/2+it)|^2$, we find that

$$\|f\|^2_{H^2_{\mathrm{i}}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m\overline{a_n}}{\sqrt{mn}} I(n/m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m\overline{a_n}\frac{\sqrt{mn}}{[\max(m,n)]^2}. \tag{3}$$

The identity (3) will serve as the starting point for both the proof of the inequality $\|f\|_{H^2_{\mathrm{i}}} < \sqrt{2}\|f\|_{\mathscr{H}^2}$, and for the proof that $\sqrt{2}$ cannot be improved.

Let us first consider the Hilbert–type (see [1, Ch. IX]) bilinear form associated to (3). Given sequences $a, b \in \ell^2$, we want to estimate

$$B(a,b) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n \frac{\sqrt{mn}}{[\max(m,n)]^2}.$$

By the Cauchy–Schwarz inequality, we find that

$$|B(a,b)| \leq \left(\sum_{m=1}^{\infty} |a_m|^2 \sum_{n=1}^{\infty} \frac{m}{[\max(m,n)]^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |b_n|^2 \sum_{m=1}^{\infty} \frac{n}{[\max(m,n)]^2}\right)^{\frac{1}{2}}.$$

Then $|B(a,b)| < 2\|a\|_{\ell^2}\|b\|_{\ell^2}$, since

$$\sum_{n=1}^{\infty} \frac{m}{[\max(m,n)]^2} = \sum_{n=1}^{m} \frac{m}{m^2} + \sum_{n=m+1}^{\infty} \frac{m}{n^2} < 1 + m\int_m^{\infty} \frac{dx}{x^2} = 2.$$

Setting $b = \overline{a}$, we obtain the desired inequality $\|f\|_{H^2_{\mathrm{i}}} < \sqrt{2}\|f\|_{\mathscr{H}^2}$.

For the optimality of $\sqrt{2}$, we again let $f(s) = \zeta(1/2 + \varepsilon + s)$ for some $\varepsilon > 0$. Clearly, $\|f\|^2_{\mathscr{H}^2} = \zeta(1 + 2\varepsilon)$. We insert f into (3) and estimate the inner sums using integrals, which yields

$$\begin{aligned}
\|f\|^2_{H^2_{\mathrm{i}}} &= \sum_{m=1}^{\infty} m^{-\varepsilon} \left(\frac{1}{m^2} \sum_{n=1}^{m} n^{-\varepsilon} + \sum_{n=m+1}^{\infty} \frac{n^{-\varepsilon}}{n^2} \right) \\
&> \sum_{m=1}^{\infty} m^{-\varepsilon} \left(\frac{1}{m^2} \frac{m^{1-\varepsilon} - 1}{1 - \varepsilon} + \frac{(m+1)^{-1-\varepsilon}}{1 + \varepsilon} \right) \\
&> \frac{\zeta(1+2\varepsilon) - \zeta(2+\varepsilon)}{1 - \varepsilon} + \frac{\zeta(1+2\varepsilon) - 1}{1 + \varepsilon}.
\end{aligned}$$

Letting $\varepsilon \to 0^+$, we conclude that if $\|f\|_{H^2_{\mathrm{i}}} \le C\|f\|_{\mathscr{H}^2}$, then $C^2 \ge 2$. □

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Part 2

Multiplicative Hankel forms

Paper 6

Failure of Nehari's theorem for multiplicative Hankel forms in Schatten classes

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FAILURE OF NEHARI'S THEOREM FOR MULTIPLICATIVE HANKEL FORMS IN SCHATTEN CLASSES

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Abstract. Ortega-Cerdà–Seip demonstrated that there are bounded multiplicative Hankel forms which do not arise from bounded symbols. On the other hand, when such a form is in the Hilbert–Schmidt class $\mathcal{S}_2$, Helson showed that it has a bounded symbol. The present work investigates forms belonging to the Schatten classes between these two cases. It is shown that for every $p > (1 - \log \pi / \log 4)\ ^{1}$ there exist multiplicative Hankel forms in the Schatten class $\mathcal{S}_p$ which lack bounded symbols. The lower bound on p is in a certain sense optimal when the symbol of the multiplicative Hankel form is a product of homogeneous linear polynomials.

1. Introduction

For a sequence $\varrho = (\varrho_1,\, \varrho_2,\, \varrho_3,\, \dots) \in \ell^2$ its corresponding *multiplicative Hankel form* on $\ell^2 \times \ell^2$ is given by

$$\varrho(a, b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varrho_{mn} a_m b_n, \tag{1}$$

which initially is defined at least for finitely supported $a, b \in \ell^2$. Such forms are naturally understood as small Hankel operators on the Hardy space of the infinite polydisc, $H^2(\mathbb{D}^\infty)$. Therefore, one is led to investigate the relationship between the symbol — a function on the polytorus $\mathbb{T}^\infty$ generating the Hankel form — and the properties of the corresponding Hankel operator.

In the classical setting, (additive) Hankel forms are realized as Hankel operators on the Hardy space in the unit disc, $H^2(\mathbb{D})$. Nehari's theorem [8] states that every bounded Hankel form is generated by a bounded symbol on the torus $\mathbb{T}$.

On the infinite polydisc, the study of the corresponding statement was initiated by H. Helson [4, pp. 52–54], who raised the following questions.

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Question 1. Does every bounded multiplicative Hankel form have a bounded symbol ψ on the polytorus $\mathbb{T}^\infty$?

Question 2. Does every multiplicative Hankel form in the Hilbert–Schmidt class $\mathcal{S}_2$ have a bounded symbol?

Helson himself [5] gave a positive answer to Question 2. Ortega-Cerdà and Seip [9] proved that there are bounded multiplicative Hankel forms that do not have bounded symbols, using an idea of Helson [6], and hence gave a negative answer to Question 1. Furthermore, their argument also quickly produces that there are compact Hankel forms without bounded symbols (see Lemma 1). In light of these results, a next natural question to ask is:

Question 3. Does there exist a Hankel form belonging to a Schatten class $\mathcal{S}_p$, $2 < p < \infty$, without a bounded symbol? If so, for which values of p does such a form exist?

We will answer the first part of this question, by showing that for every

$$p > p_0 = \left(1 - \frac{\log \pi}{\log 4}\right)^{-1} \approx 5.738817179,$$

there are multiplicative Hankel forms in $\mathcal{S}_p$ which do not have bounded symbols.

Our construction relies on independent products of homogeneous linear symbols and is optimal when testing against products of linear homogeneous polynomials, see Theorem 4. It is quite tempting to further conjecture that forms without bounded symbols can be found in $\mathcal{S}_p$ for every $p > 2$, but our method does not substantiate this claim.

The paper is organized into two further sections. Section 2 reviews the connection between multiplicative Hankel forms, the Hardy space of Dirichlet series, and the Hardy space of the infinite polydisc. In Section 3 the main results are proven.

2. Preliminaries

We let $\mathscr{H}^2$ denote the Hilbert space of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{2}$$

with square summable coefficients. If g and φ are Dirichlet series in $\mathscr{H}^2$ with coefficients b_n and $\overline{\varrho_n}$, respectively, a computation shows that

$$\langle fg, \varphi \rangle_{\mathscr{H}^2} = \varrho(a, b).$$

A key tool in the study of Hardy spaces of Dirichlet series is the *Bohr lift* [1]. For any $n \in \mathbb{N}$, the fundamental theorem of arithmetic yields the prime factorization

$$n = \prod_{j=1}^{\infty} p_j^{\kappa_j},$$

which associates the finite non-negative multi-index $\kappa(n) = (\kappa_1, \kappa_2, \kappa_3, \dots)$ to n. The Bohr lift of the Dirichlet series (2) is the power series

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}, \tag{3}$$

where $z = (z_1, z_2, z_3, \dots)$. Hence (3) is a power series in countably infinite number of variables, but each term contains only a finite number of variables.

Under the Bohr lift, $\mathscr{H}^2$ corresponds to the infinite dimensional Hardy space $H^2(\mathbb{D}^\infty)$, which we view as a subspace of $L^2(\mathbb{T}^\infty)$. We refer to [3] for the details, mentioning only that the Haar measure of the compact abelian group $\mathbb{T}^\infty$ is simply the product of the normalized Lebesgue measures of each variable. In particular, $H^2(\mathbb{D}^d)$ is a natural subspace of $H^2(\mathbb{D}^\infty)$.

A formal computation shows that

$$\langle \mathscr{B}f\mathscr{B}g, \mathscr{B}\varphi\rangle_{L^2(\mathbb{T}^\infty)} = \langle fg, \varphi\rangle_{\mathscr{H}^2},$$

allowing us to compute the multiplicative Hankel form (1) on $\mathbb{T}^\infty$. In the remainder of this paper we work exclusively in the polydisc, with no reference to Dirichlet series. Therefore, we drop the notation $\mathscr{B}$ and study Hankel forms

$$H_\varphi(fg) = \langle fg, \varphi\rangle_{L^2(\mathbb{T}^\infty)}, \qquad f, g \in H^2(\mathbb{D}^\infty). \tag{4}$$

In the previous considerations we had that $\varphi \in H^2(\mathbb{D}^\infty)$, but there is nothing to prevent us from considering arbitrary symbols from $L^2(\mathbb{T}^\infty)$. Hence, each $\varphi \in L^2(\mathbb{T}^\infty)$ induces by (4) a (possibly unbounded) Hankel form H_φ on $H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty)$. Of course, this is not a real generalization. Each form H_φ is also induced by a symbol $\psi \in H^2(\mathbb{D}^\infty)$; letting $\psi = P\varphi$ we have $H_\varphi = H_\psi$, where P denotes the orthogonal projection of $L^2(\mathbb{T}^\infty)$ onto $H^2(\mathbb{D}^\infty)$.

Note that if $\psi \in L^\infty(\mathbb{T}^\infty)$, then the corresponding multiplicative Hankel form is bounded, since

$$|H_\psi(fg)| = |\langle fg, \psi\rangle| \le \|f\|_2 \, \|g\|_2 \|\, \psi\|_\infty.$$

We say that H_φ has a bounded symbol if there exists a $\psi \in L^\infty(\mathbb{T}^\infty)$ such that $H_\varphi = H_\psi$. As mentioned in the introduction, it was shown in [9] that not every bounded multiplicative Hankel form has a bounded symbol.

On the polydisc the Hankel form H_φ is naturally realized as a (small) Hankel operator $\mathbf{H}_\varphi$, which when bounded acts as an operator from $H^2(\mathbb{D}^\infty)$ to the anti-analytic space $\overline{H^2}(\mathbb{D}^\infty)$. Letting $\overline{P}$ denote the orthogonal projection of $L^2(\mathbb{T}^\infty)$

onto $\overline{H^2}(\mathbb{D}^\infty)$, we have at least for polynomials $f \in H^2(\mathbb{D}^\infty)$ that

$$\mathbf{H}_\varphi f = \overline{P}(\overline{\varphi} f). \tag{5}$$

It is clear that when written in standard bases, the form H_φ and the operator $\mathbf{H}_\varphi$ both correspond to the same infinite matrix

$$M_\varrho = \begin{pmatrix} \varrho_1 & \varrho_2 & \varrho_3 & \cdots \\ \varrho_2 & \varrho_4 & \varrho_6 & \cdots \\ \varrho_3 & \varrho_6 & \varrho_9 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Finally, we briefly recall the definition of the Schatten classes $\mathcal{S}_p$, $0 < p < \infty$. Assume that the Hankel form H_φ is compact. Let $\Lambda = \{\lambda_k\}_{k=1}^\infty$ denote the singular value sequence of M_ϱ, which of course is the same as the singular value sequence of the operator $\mathbf{H}_\varphi$. The form H_φ, or equivalently the operator $\mathbf{H}_\varphi$, is in the *Schatten class* $\mathcal{S}_p$ if $\Lambda \in \ell^p$, and

$$\|H_\varphi\|_{\mathcal{S}_p} = \|\mathbf{H}_\varphi\|_{\mathcal{S}_p} = \|\Lambda\|_{\ell^p}.$$

3. Results

To prove that there for each $p > p_0$ exist multiplicative Hankel forms in $\mathcal{S}_p$ without bounded symbols, we will assume that every $H_\varphi \in \mathcal{S}_p$ has a bounded symbol and derive a contradiction. We begin with the following routine lemma.

Lemma 1. *Let $p \geq 1$. Assume that every $H_\varphi \in \mathcal{S}_p$ has a bounded symbol on $\mathbb{T}^\infty$. Then there is a constant $C_p \geq 1$ with the property that every $H_\varphi \in \mathcal{S}_p$ has a symbol $\psi \in L^\infty(\mathbb{T}^\infty)$ with $H_\varphi = H_\psi$ and such that $\|\psi\|_\infty \leq C_p \|H_\varphi\|_{\mathcal{S}_p}$.*

Proof. We will define a lifting operator and show that it has to be continuous by appealing to the closed graph theorem.

Let BH denote the space of bounded multiplicative Hankel forms. By a standard argument it is isomorphic to the dual space of the weak product $\mathscr{H}^2 \odot \mathscr{H}^2$ [6]. In particular BH is a Banach space under the operator norm. It follows that $\mathcal{S}_p$H is also a Banach space, where $\mathcal{S}_p$H denotes the space of multiplicative Hankel forms in $\mathcal{S}_p$ equipped with the norm of $\mathcal{S}_p$.

Now we define

$$X = L^\infty(\mathbb{T}^\infty) \cap \left(L^2(\mathbb{T}^\infty) \ominus H^2(\mathbb{T}^\infty)\right),$$
$$Y = L^\infty(\mathbb{T}^\infty)/X.$$

Y is a Banach space under the norm $\|\varphi\|_Y = \inf\{\|\psi\|_\infty : \psi - \varphi \in X\}$, seeing as X is a closed subspace of $L^\infty(\mathbb{T}^\infty)$. Since by assumption every $H_\varphi \in \mathcal{S}_p$H has a symbol $\psi \in L^\infty(\mathbb{T}^\infty)$, we can define a map $T\colon \mathcal{S}_p\mathrm{H} \to Y$ by $T(H_\varphi) = \psi$. This is a well-defined linear map since $H_\varphi = 0$ for a symbol $\varphi \in L^\infty(\mathbb{T}^\infty)$ if and only if $\varphi \in X$. An obvious computation verifies that T is a closed operator, hence continuous. Therefore, there is a $C_p \geq 1$ such that

$$\|T(H_\varphi)\|_Y \leq C_p \|H_\varphi\|_{\mathcal{S}_p}.$$

The statement of the lemma follows immediately. □

Given the assumption of the lemma, we hence have for each polynomial f and form $H_\varphi \in \mathcal{S}_p$ that

$$|\langle f, \varphi\rangle| = |H_\varphi(f \cdot 1)| = |H_\psi(f \cdot 1)| = |\langle f, \psi\rangle| \leq \|\psi\|_\infty \|f\|_1 \leq C_p \|H_\varphi\|_{\mathcal{S}_p} \|f\|_1,$$

where $\|\cdot\|_1$ denotes the norm of $L^1(\mathbb{T}^\infty)$. We thus obtain

$$\frac{|\langle f, \varphi\rangle|}{\|H_\varphi\|_{\mathcal{S}_p} \|f\|_1} \leq C_p \tag{6}$$

for every polynomial f and every $H_\varphi \in \mathcal{S}_p$. To prove our main result we will construct a sequence of polynomials and finite rank forms to show that no finite constant C_p satisfying (6) exists for $p > p_0$, thus obtaining a contradiction to the assumption of Lemma 1. We will require the following lemma.

Lemma 2. *Suppose that $\varphi_1, \varphi_2, \ldots, \varphi_m$ are symbols that depend on mutually separate variables and which generate the multiplicative Hankel forms $H_{\varphi_j} \in \mathcal{S}_p$, $1 \leq j \leq m$. Then*

$$\|H_\varphi\|_{\mathcal{S}_p} = \|H_{\varphi_1}\|_{\mathcal{S}_p} \|H_{\varphi_2}\|_{\mathcal{S}_p} \cdots \|H_{\varphi_m}\|_{\mathcal{S}_p}, \tag{7}$$

where $\varphi = \varphi_1\varphi_2\cdots\varphi_m$.

Proof. For $1 \leq j \leq m$, we let X_j denote the Hardy space of precisely the variables that the symbol φ_j depends on, and if necessary let X_0 denote the Hardy space of the remaining variables, so that — as tensor products of Hilbert spaces — we have

$$H^2(\mathbb{D}^\infty) = X_0 \otimes X_1 \otimes X_2 \otimes \cdots X_m.$$

We set $\varphi_0 = 1$ and consider the small Hankel operators $\widetilde{\mathbf{H}}_{\varphi_j} : X_j \to \overline{X_j}$, defined similarly to (5) for $0 \leq j \leq m$. Now, if $f_j \in X_j$, $0 \leq j \leq m$, we observe that

$$\mathbf{H}_\varphi(f_0 f_1 \cdots f_m) = \widetilde{\mathbf{H}}_{\varphi_0}(f_0)\, \widetilde{\mathbf{H}}_{\varphi_1}(f_1) \cdots \widetilde{\mathbf{H}}_{\varphi_m}(f_m),$$

and hence $\mathbf{H}_\varphi = \widetilde{\mathbf{H}}_{\varphi_0} \otimes \widetilde{\mathbf{H}}_{\varphi_1} \otimes \cdots \otimes \widetilde{\mathbf{H}}_{\varphi_m}$.

Note that $\widetilde{\mathbf{H}}_{\varphi_0}$ has the sole singular value 1, of multiplicity 1. It follows that all singular values λ of $\mathbf{H}_\varphi$ are obtained as products $\lambda = \lambda_1\lambda_2\cdots\lambda_m$, where λ_j is a singular value of $\widetilde{\mathbf{H}}_{\varphi_j}$, see [2]. The multiplicity of λ is also obtained in the expected way. From this, a short computation shows that

$$\|\mathbf{H}_\varphi\|_{\mathcal{S}_p} = \|\widetilde{\mathbf{H}}_{\varphi_1}\|_{\mathcal{S}_p} \|\widetilde{\mathbf{H}}_{\varphi_2}\|_{\mathcal{S}_p} \cdots \|\widetilde{\mathbf{H}}_{\varphi_m}\|_{\mathcal{S}_p}.$$

Finally, we have $\mathbf{H}_{\varphi_j} = \widetilde{\mathbf{H}}_{\varphi_0} \otimes \widetilde{\mathbf{H}}_{\varphi_j}$, where we now regard $\widetilde{\mathbf{H}}_{\varphi_0}$ as an operator on the Hardy space of the variables of which φ_j is independent. Arguing as above, it follows that $\|\mathbf{H}_{\varphi_j}\|_{\mathcal{S}_p} = \|\widetilde{\mathbf{H}}_{\varphi_j}\|_{\mathcal{S}_p}$, completing the proof. $\square$

If $f_1, f_2, \ldots, f_m$ are polynomials depending on the same separate variables as $\varphi_1, \varphi_2, \ldots, \varphi_m$, respectively, and we set $f = f_1 f_2 \cdots f_m$, then

$$|\langle f, \varphi\rangle| = |\langle f_1, \varphi_1\rangle|\,|\langle f_2, \varphi_2\rangle| \cdots |\langle f_m, \varphi_m\rangle|,$$
$$\|f\|_1 = \|f_1\|_1\,\|f_2\|_1 \cdots \|f_m\|_1.$$

Let S be the shift operator $\mathrm{S}f(z_1, z_2, \ldots) = f(z_2, z_3, \ldots)$. Suppose that we can find polynomials f and φ, both depending on the first d variables $z_1, z_2, \ldots, z_d$, satisfying

$$\frac{|\langle f, \varphi\rangle|}{\|H_\varphi\|_{\mathcal{S}_p}\,\|f\|_1} > 1. \tag{8}$$

Then, for $1 \le j \le m$, consider the functions

$$\varphi_j(z) = \mathrm{S}^{d(j-1)}\varphi(z) \quad \text{and} \quad f_j(z) = \mathrm{S}^{d(j-1)} f(z).$$

With $\Phi = \varphi_1\varphi_2\cdots\varphi_m$ and $F = f_1 f_2 \cdots f_m$, Lemma 2 yields

$$\frac{|\langle F, \Phi\rangle|}{\|H_\Phi\|_{\mathcal{S}_p}\,\|F\|_1} = \left(\frac{|\langle f, \varphi\rangle|}{\|H_\varphi\|_{\mathcal{S}_p}\,\|f\|_1}\right)^m \to \infty, \qquad m \to \infty,$$

giving us the sought contradiction to (6). We realize this scheme in the next theorem.

Theorem 3. *For every $p > p_0$ there is a multiplicative Hankel form $H_\varphi \in \mathcal{S}_p$ which does not have a bounded symbol.*

Proof. Let d be a large positive integer to be chosen later. Consider the symbol

$$\varphi(z) = \frac{z_1 + z_2 + z_3 + \cdots + z_d}{\sqrt{d}}.$$

It is clear that the sequence $\varrho = (\varrho_n)_{n=1}^\infty$ for the matrix of H_φ is given by

$$\varrho_n = \begin{cases} 1/\sqrt{d} & \text{if } n = p_j \text{ and } 1 \le j \le d \\ 0 & \text{otherwise} \end{cases},$$

where p_j denotes the jth prime. In other terms, the matrix M_ϱ of H_φ, with all zero rows and columns omitted, is the $(d+1)\times(d+1)$ matrix

$$\frac{1}{\sqrt{d}}\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This matrix is easily seen to have the singular values 1 (with multiplicity 2) and 0 (with multiplicity $d-1$), and thus

$$\|H_\varphi\|_{\mathcal{S}_p} = 2^{\frac{1}{p}}.$$

We choose $f(z) = \varphi(z)$. Then $\langle f, \varphi \rangle = 1$, and, moreover, the central limit theorem for Steinhaus variables gives us that

$$\lim_{d\to\infty} \|f\|_1 = \lim_{d\to\infty} \mathbb{E}\left(\frac{|z_1+z_2+z_3+\cdots+z_d|}{\sqrt{d}}\right) = \frac{\sqrt{\pi}}{2}.$$

In particular, for each $\delta > 0$ we have for sufficiently large d that

$$\|f\|_1 \le \frac{\sqrt{\pi}}{2} + \delta.$$

We now observe that $p = p_0$ is the solution of the equation $2^{1/p} \cdot \sqrt{\pi}/2 = 1$, and hence if $p > p_0$ we may find $\delta > 0$ small enough that

$$\|H_\varphi\|_{\mathcal{S}_p} \cdot \|f\|_1 \le 2^{1/p} \cdot \left(\frac{\sqrt{\pi}}{2} + \delta\right) < 1.$$

This implies that if d is large enough, f and φ satisfy (8). This completes the proof by appealing to the discussion preceding the statement of the theorem. □

Our result is optimal for symbols which are independent products of linear homogeneous polynomials and test functions of the same form, as shown by the following result.

Theorem 4. *Suppose $p \le p_0$ and consider*

$$\varphi(z) = a_1z_1 + a_2z_2 + \cdots + a_dz_d \quad \text{and} \quad f(z) = b_1z_1 + b_2z_2 + \cdots + b_dz_d,$$

for a_j, $b_j \in \mathbb{C}$. Then $|\langle f, \varphi\rangle| \le \|H_\varphi\|_{\mathcal{S}_p}\|f\|_1$.

Proof. By the Cauchy–Schwarz inequality and Parseval's formula, it is clear that

$$|\langle f, \varphi\rangle| \le \|a\|_{\ell^2}\|b\|_{\ell^2}.$$

Straightforward computations with the matrix M_ϱ of H_φ show that

$$M_\varrho M_\varrho^* = \begin{pmatrix} \|a\|_{\ell^2}^2 & 0 & 0 & \cdots & 0 \\ 0 & a_1\overline{a_1} & a_1\overline{a_2} & \cdots & a_1\overline{a_d} \\ 0 & a_2\overline{a_1} & a_2\overline{a_2} & \cdots & a_2\overline{a_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_d\overline{a_1} & a_d\overline{a_2} & \cdots & a_d\overline{a_d} \end{pmatrix}.$$

Here we have again omitted zero rows and columns. Note that the lower right block has rank 1. By considering the vector $(0, a_1, a_2, \ldots, a_d)$ it is clear that it has the sole eigenvalue $\|a\|_{\ell^2}^2$. Thus, the singular value sequence of M_ϱ is $\Lambda = \{\|a\|_{\ell^2},\, \|a\|_{\ell^2},\, 0,\, \ldots,\, 0\}$, and hence

$$\|H_\varphi\|_{\mathcal{S}_p} = 2^{1/p}\|a\|_{\ell^2}.$$

We use the optimal Khintchine inequality for Steinhaus variables [7, 10], $p = 1$, and obtain

$$\|f\|_1 \geq \frac{\sqrt{\pi}}{2}\|b\|_{\ell^2}.$$

The hypothesis that $p \leq p_0$ implies that $2^{1/p}\sqrt{\pi}/2 \geq 1$, and the proof is finished by the following chain of inequalities.

$$\|H_\varphi\|_{\mathcal{S}_p} \cdot \|f\|_1 \geq 2^{1/p} \cdot \|a\|_{\ell^2} \cdot \frac{\sqrt{\pi}}{2} \cdot \|b\|_{\ell^2} \geq \|a\|_{\ell^2} \cdot \|b\|_{\ell^2} \geq |\langle f, \varphi\rangle|. \qquad \square$$

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Paper 7

The multiplicative Hilbert matrix

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THE MULTIPLICATIVE HILBERT MATRIX

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Abstract. It is observed that the matrix with entries $(\sqrt{mn}\log(mn))^{-1}$ for $m, n \geq 2$ appears as the matrix of the integral operator

$$\mathbf{H}f(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w+s)-1)\,dw$$

with respect to the basis $(n^{-s})_{n\geq 2}$; here $\zeta(s)$ is the Riemann zeta function and $\mathbf{H}$ is defined on the Hilbert space $\mathscr{H}_0^2$ of Dirichlet series vanishing at $+\infty$ and with square-summable coefficients. This infinite matrix defines a multiplicative Hankel operator according to Helson's terminology or, alternatively, it can be viewed as a bona fide (small) Hankel operator on the infinite-dimensional torus $\mathbb{T}^\infty$. By analogy with the standard integral representation of the classical Hilbert matrix, this matrix is referred to as the multiplicative Hilbert matrix. It is shown that its norm equals π and that it has a purely continuous spectrum which is the interval $[0, \pi]$; these results are in agreement with known facts about the classical Hilbert matrix. It is shown that the matrix $(m^{1/p}n^{(p-1)/p}\log(mn))^{-1}$ has norm $\pi/\sin(\pi/p)$ when acting on ℓ^p for $1 < p < \infty$. However, the multiplicative Hilbert matrix fails to define a bounded operator on $\mathscr{H}_0^p$ for $p \neq 2$, where $\mathscr{H}_0^p$ are H^p spaces of Dirichlet series. It remains an interesting problem to decide whether the analytic symbol $\sum_{n\geq 2}(\log n)^{-1}n^{-s-1/2}$ of the multiplicative Hilbert matrix arises as the Riesz projection of a bounded function on the infinite-dimensional torus $\mathbb{T}^\infty$.

1. Introduction

The classical Hilbert matrix

$$A := \left(\frac{1}{m+n+1}\right)_{m,n\geq 0}$$

is the prime example of an infinite Hankel matrix, i.e., a matrix whose entries $a_{m,n}$ only depend on the sum $m+n$. The Hilbert matrix can be viewed as the

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matrix of the integral operator

$$\mathbf{H}_a f(z) := \int_0^1 f(t)(1-zt)^{-1}dt \tag{1}$$

with respect to the standard basis $(z^n)_{n\geq 0}$ for the Hardy space $H^2(\mathbb{D})$. This representation was first used by Magnus [14] who found that the Hilbert matrix has no eigenvalues and that its continuous spectrum is $[0,\pi]$. It was also used in [5] and [6] to study the Hilbert matrix as an operator on Hardy and Bergman spaces of the disc and in particular to obtain its norm on those spaces.

The purpose of this paper is to identify and study a multiplicative analogue of A. This means that we seek an infinite matrix with entries $a_{m,n}$ that depend only on the product mn and with properties that parallel those of A. Our starting point is the multiplicative counterpart to (1) which we have found to be the integral operator

$$\mathbf{H}f(s) := \int_{1/2}^{+\infty} f(w)(\zeta(w+s)-1)dw \tag{2}$$

acting on Dirichlet series $f(s) = \sum_{n\geq 2} a_n n^{-s}$. Here $\zeta(s)$ denotes the Riemann zeta function, and we assume that f is in $\mathscr{H}_0^2$, which means that

$$\|f\|^2_{\mathscr{H}_0^2} := \sum_{n=2}^{\infty} |a_n|^2 < \infty.$$

By the Cauchy–Schwarz inequality, every f in $\mathscr{H}_0^2$ represents an analytic function in the half-plane $\sigma = \operatorname{Re} s > 1/2$. The same calculation shows that point evaluations $f \mapsto f(s)$ are bounded linear functionals on $\mathscr{H}_0^2$ for s in this half-plane. As is readily seen, the reproducing kernel K_w of $\mathscr{H}_0^2$ is $K_w(s) = \zeta(s+\overline{w})-1$. This implies that

$$\langle \mathbf{H}f, g\rangle_{\mathscr{H}_0^2} = \int_{1/2}^{\infty} f(w)\overline{g(w)}dw \tag{3}$$

when f and g are Dirichlet polynomials. Now observe that arc length measure on the half-line $(1/2,+\infty)$ is a Carleson measure for $\mathscr{H}_0^2$ (the contribution from $1/2 < s < 3/2$ is handled by [19, Theorem 4], while the contribution from $s > 3/2$ is handled by a pointwise estimates). We therefore get that (3) in fact holds for arbitrary functions f and g in $\mathscr{H}_0^2$, and hence $\mathbf{H}$ is well defined and bounded on $\mathscr{H}_0^2$. Taking into account that every f in $\mathscr{H}_0^2$ is analytic when $\sigma > 1/2$, we find that $\langle \mathbf{H}f, f\rangle_{\mathscr{H}_0^2} = 0$ if and only if $f \equiv 0$. Hence (3) also implies that $\mathbf{H}$ is a strictly positive operator. Now an explicit computation of the integral on the right-hand side of (2) shows that the matrix of $\mathbf{H}$ with respect to the orthonormal basis $(n^{-s})_{n\geq 2}$ is

$$M := \left(\frac{1}{\sqrt{mn}\log(mn)}\right)_{m,n\geq 2}.$$

We will refer to this matrix as the multiplicative Hilbert matrix. We will be interested in understanding M as an operator on $\ell^2 = \ell^2(\mathbb{N}\setminus\{1\})$, which means that, equivalently, we will be concerned with the properties of the integral operator $\mathbf{H}$ acting on $\mathscr{H}_0^2$.

Our main result reads as follows.

Theorem 1. *The operator* $\mathbf{H}$ *is a bounded and strictly positive operator on* $\mathscr{H}_0^2$ *with* $\|\mathbf{H}\| = \pi$. *It has no eigenvalues, and its continuous spectrum is* $[0,\pi]$.

This theorem, which is in agreement with what is known about the classical Hilbert matrix, should be seen as an outgrowth of Helson's last two papers [12, 13]. In these works, a study of multiplicative Hankel matrices was initiated, mainly focused on the question of to which extent Nehari's theorem [17, 21] extends to the multiplicative setting. We will return to this interesting question in the final section of this paper. At this point, we just wish to point out that the existence of a canonical operator like $\mathbf{H}$, closely related to the Riemann zeta function, clearly demonstrates that multiplicative Hankel matrices may arise quite naturally.

The computation of the norm of $\mathbf{H}$ is straightforward, by a simple adaption of the classical proof of [10, pp. 226–229]. In fact, this adaption leads us to consider an ℓ^p version of the multiplicative Hilbert matrix M, namely

$$M_p := \left(\frac{1}{m^{(p-1)/p}n^{1/p}\log(mn)}\right)_{m,n\geq 2},$$

where $1 < p < \infty$. We will see that M_p has norm $\pi/\sin(\pi/p)$, viewed as an operator on ℓ^p, which is analogous to the classical fact that A has norm $\pi/\sin(\pi/p)$ when it acts on ℓ^p. We will explain this link in Section 2. This result was actually first obtained by Mulholland [16], as a corollary to certain related integral estimates.

The identification of the spectrum is the hardest part of the proof of Theorem 1. Inspired by Magnus's work [14], it is split into two main parts. First, in Section 3, we establish estimates near the singular point $s = 1/2$ for the anticipated solutions f to equations of the form

$$(\mathbf{H}-\lambda)f = c\cdot\psi,$$

where c is a constant and ψ is the analytic symbol of $\mathbf{H}$. This means that ψ is the primitive of $-(\zeta(s+1/2)-1)$ belonging to $\mathscr{H}_0^2$. The point of this estimation is to show that $f'(w)$ must be square integrable on $(1/2,\infty)$. Here we make use of the fact that $\zeta(s)-(s-1)^{-1}$ is an entire function, which allows us to relate $\mathbf{H}$ to a classical operator studied by Carleman. This analysis requires a fair amount of classical-type computations involving Mellin transforms. In Section 4, we may then finish the proof by resorting to the following commutation relation, obtained by integration by parts, between $\mathbf{H}$ and the differentiation operator $\mathbf{D}$:

$$\mathbf{DH}f(s) = -f(1/2)(\zeta(s+1/2)-1) - \mathbf{HD}f(s).$$

After finishing the proof of Theorem 1, we turn to two questions related to Helson's viewpoint, namely that multiplicative Hankel operators are bona fide (small) Hankel operators on the infinite-dimensional torus $\mathbb{T}^\infty$. The first question is whether there is a counterpart to the result of [5, 6] saying that the norm of $\mathbf{H}_a$ viewed as an operator on $H^p(\mathbb{D})$ is again $\pi/\sin(\pi/p)$. We will show in Section 5 that the analogy with $\mathbf{H}_a$ breaks down at this point, or, more precisely, that $\mathbf{H}$ does not extend to a bounded operator on the H^p analogues of $\mathscr{H}_0^2$, which by Bayart's work [1] can be associated with $H^p(\mathbb{T}^\infty)$. This negative result is related to, though not a trivial consequence of, the fact that $H^p(\mathbb{T}^\infty)$ is not complemented in $L^p(\mathbb{T}^\infty)$ [8].

The final question to be discussed concerns the analytic symbol

$$\psi(s) := \sum_{n=2}^{\infty} \frac{n^{-s}}{\sqrt{n}\log n} \tag{4}$$

of the multiplicative Hankel matrix. Since $-\psi$ is, up to a linear term, a primitive of the Riemann zeta function, it appears to be of interest to investigate it more closely. While it is known from [20] that Nehari's theorem does not hold in the multiplicative setting, it could still be true that ψ is the Riesz projection of a bounded function. In the final Section 6, we will explain the exact meaning of this statement and show how this question relates to a long-standing embedding problem for H^p spaces of Dirichlet spaces.

A word on notation: Throughout this paper, the notation $U(z) \ll V(z)$ (or equivalently $V(z) \gg U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. We write $U(z) \asymp V(z)$ to signify that both $U(z) \ll V(z)$ and $V(z) \ll U(z)$ hold.

2. The norm of the matrix M_p

In this section, $\|M_p\|_p$ will denote the norm of M_p viewed as an operator on ℓ^p. Our aim is to prove the following theorem, which in particular shows that $\|\mathbf{H}\| = \pi$.

Theorem 2. *We have* $\|M_p\|_p = \pi/\sin(\pi/p)$ *for* $1 < p < \infty$.

Proof. The proof relies, as in [10, pp. 226–234], on the following homogeneity property of the kernel $(x+y)^{-1}$:

$$\int_0^\infty x^{-1/p}\frac{1}{1+x}dx = \int_0^\infty x^{-(p-1)/p}\frac{1}{1+x}dx = \frac{\pi}{\sin(\pi/p)}. \tag{5}$$

The exact computation of the integral can be found in [24, p. 254, Example 4] or [7, Section 9.5].

We prove first that $\|M_p\|_p \le \pi/\sin(\pi/p)$. We write $q = p/(p-1)$ and assume that $(a_m)_{m\ge 2}$ is in ℓ^p and $(b_n)_{n\ge 2}$ is in ℓ^q. By Hölder's inequality, we find that

$$\sum_{m,n=2}^\infty |a_m||b_n|m^{-1/q}n^{-1/p}(\log(mn))^{-1} \le P\cdot Q,$$

where

$$P := \left(\sum_{m=2}^\infty |a_m|^p \sum_{n\ge 2} n^{-1}\left(\frac{\log m}{\log n}\right)^{1/q}\frac{1}{\log(mn)}\right)^{1/p} \tag{6}$$

and

$$Q := \left(\sum_{n=2}^\infty |b_n|^q \sum_{m\ge 2} m^{-1}\left(\frac{\log n}{\log m}\right)^{1/p}\frac{1}{\log(mn)}\right)^{1/q}. \tag{7}$$

By a change of variables argument, each of the inner sums is dominated by the integral in (5), and hence we obtain the desired bound by duality.

To prove that the norm is bounded below by $\pi/\sin(\pi/p)$, we use the sequences defined by

$$a_m = m^{-1/p}(\log m)^{-(1+\varepsilon)/p} \qquad \text{and} \qquad b_n = n^{-1/q}(\log n)^{-(1+\varepsilon)/q}$$

for which we have

$$\|(a_m)\|_p^p = \frac{1}{\varepsilon} + O(1) \qquad \text{and} \qquad \|(b_n)\|_q^q = \frac{1}{\varepsilon} + O(1) \tag{8}$$

when $\varepsilon \to 0^+$. We see that

$$\begin{aligned}\sum_{m,n=2}^\infty \frac{a_m b_n}{m^{1/q}\,n^{1/p}\,\log(mn)} &= \sum_{m,n=2}^\infty (\log m)^{-(1+\varepsilon)/p}(\log n)^{-(1+\varepsilon)/q}m^{-1}n^{-1}\frac{1}{\log(mn)}\\ &\ge \int_{\log 3}^\infty\int_{\log 3}^\infty x^{-(1+\varepsilon)/p}y^{-(1+\varepsilon)/q}\frac{1}{x+y}dxdy.\end{aligned}$$

This iterated integral can computed as the corresponding integral in [10, p. 233, Equation 9.5.2] so that we get

$$\sum_{m,n=2}^\infty a_m b_n m^{-1/q}n^{-1/p} = \frac{1}{\varepsilon}\left(\frac{\pi}{\sin(\pi/p)} + o(1)\right)$$

when $\varepsilon \to 0^+$. Combining this estimate with (8), we get the desired bound $\|M_p\|_p \geq \pi/\sin(\pi/p)$. $\square$

It is of interest to observe that when we replace the inner sums in (6) and (7) by the respective integrals in (5), we get a strict inequality. In particular, we get that

$$\|\mathbf{H}f\|_{\mathscr{H}_0^2} < \pi\|f\|_{\mathscr{H}_0^2}$$

for every nontrivial function f in $\mathscr{H}_0^2$. This means that we have already shown that π is not an eigenvalue for $\mathbf{H}$.

Another observation is that the matrix M_p fails to be bounded on $\ell^{p'}$ when $p' \neq p$. This is most easily seen when $p' > p$ because we can find a sequence a in $\ell^{p'}$ for which the entries in $M_p a$ become infinite. When $p' < p$, we can apply the same argument to the conjugate exponents q and q' and the matrix M_q.

In preparation for the proof of the second part of Theorem 1, we now clarify the relationship between $\mathscr{H}_0^2$ and $L^2(1/2, \infty)$ implied by Theorem 2.

Corollary 1. *If f is in $\mathscr{H}_0^2$, then $\|f\|_{L^2(1/2,\infty)} \leq \sqrt{\pi}\|f\|_{\mathscr{H}_0^2}$. Additionally,* $\mathbf{H}$ *extends to an operator from $L^2(1/2, \infty)$ to $\mathscr{H}_0^2$ and $\|\mathbf{H}f\|_{\mathscr{H}_0^2} \leq \sqrt{\pi}\|f\|_{L^2(1/2,\infty)}$.*

Proof. The first statement follows from Theorem 2 with $p = 2$ and the fact that

$$\langle \mathbf{H}f, f\rangle_{\mathscr{H}_0^2} = \int_{1/2}^{+\infty} |f(w)|^2\, dw.$$

Given $f \in L^2(1/2, \infty)$, clearly $\mathbf{H}f$ is a Dirichlet series vanishing at $+\infty$. If $g(s) = \sum_{n\geq 2} b_n n^{-s}$, it follows from Fubini's theorem that

$$\langle \mathbf{H}f, g\rangle_{\mathscr{H}_0^2} = \sum_{n=2}^{\infty} \left(\int_{1/2}^{\infty} f(w) n^{-w}\, dw \right) \overline{b_n} = \int_{1/2}^{\infty} f(w)\overline{g(w)}\, dw,$$

so that (3) extends to hold for $f \in L^2(1/2, \infty)$ and Dirichlet polynomials g. The second statement now follows from the first, since

$$\begin{aligned} \|\mathbf{H}f\|_{\mathscr{H}_0^2} &= \sup_{\|g\|_{\mathscr{H}_0^2}=1} \left| \langle \mathbf{H}f, g\rangle_{\mathscr{H}_0^2} \right| \\ &\leq \sup_{\|g\|_{\mathscr{H}_0^2}=1} \|f\|_{L^2(1/2,\infty)}\|g\|_{L^2(1/2,\infty)} \leq \sqrt{\pi}\|f\|_{L^2(1/2,\infty)}. \end{aligned}$$

$\square$

3. Estimates for solutions of $(\mathbf{H} - \lambda)f = c\psi$

In preparation for the characterization of the spectrum of $\mathbf{H}$, we will in this section prove precise asymptotics as $s \to 1/2$ for solutions f in $\mathscr{H}_0^2$ of the equation $(\mathbf{H}-\lambda)f = c\psi$, where c is a constant and ψ is the analytic symbol of $\mathbf{H}$ defined by (4). The considerations to come are in fact of a rather general nature, providing a spectral decomposition of f in terms of generalized eigenvectors of the (shifted) Carleman operator [3, p. 169] defined by

$$\mathbf{C}f(s) = \int_{1/2}^{\infty} \frac{f(w)}{s+w-1}\,dw, \qquad s > 1/2.$$

We choose to focus on $\mathbf{H}$ for simplicity, but it will be clear from the proof of the next theorem that minor modifications yield similar results for other integral operators whose kernels are perturbations $K(s+w)$, K analytic, of the Carleman kernel.

Theorem 3. *Suppose that* $0 < \lambda < \pi$, *and let* ψ *denote the analytic symbol of* $\mathbf{H}$, *that is*

$$\psi(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} n^{-s}, \qquad \operatorname{Re} s > 1/2.$$

If f *in* $\mathscr{H}_0^2$ *satisfies* $(\mathbf{H}-\lambda)f = c\psi$, *then there exists a complex number* d *and polynomially bounded sequences of complex numbers* $(c_k)_{k\geq 1}$ *and* $(d_k)_{k\geq 1}$ *such that* f *has the following series representation for* $1/2 < s < 3/2$,

$$f(s) = cd + \sum_{k=1}^{\infty} (s-1/2)^{2k-1/2}\left(c_k(s-1/2)^{-i\theta} + d_k(s-1/2)^{i\theta}\right), \tag{9}$$

where θ *is a real number dependent on* λ, *namely*

$$\theta = \frac{1}{\pi}\log\left(\frac{\pi}{\lambda} - \sqrt{\left(\frac{\pi}{\lambda}\right)^2 - 1}\right).$$

In particular, if f *in* $\mathscr{H}_0^2$ *solves* $(\mathbf{H}-\lambda)f = c\psi$ *then* $f' \in L^2(1/2, \infty)$.

Remark. Note that for each k, the functions $s \mapsto (s-1/2)^{2k-1/2\pm i\theta}$ are generalized eigenvectors of the Carleman operator $\mathbf{C}$ belonging to the eigenvalue λ, $0 < \lambda < \pi$; see Lemma 1. The constant function $s \mapsto cd$ is not such an eigenfunction, and its appearance in (9) will allow us to derive a contradiction in the case that $c \neq 0$.

It is also possible to treat the case $\lambda = \pi$ with the methods below, although we choose not to since we do not need it. Carrying out the details, one obtains for $\lambda = \pi$ a decomposition of f in terms of the eigenfunctions $s \mapsto (s-1/2)^{2k-1/2}$ and $s \mapsto (s-1/2)^{2k-1/2}\log(s-1/2)$ of the Carleman operator.

To simplify the computations and to align our proof with the classical representation of the Carleman operator, we will in this section shift everything to $\mathbb{R}_+ = (0,\infty)$, and prove Theorem 3 on this ray. Shifting the representation back to $(1/2, +\infty)$ will then give (9). This means that we consider $\mathscr{H}_0^2$ the space of Dirichlet series

$$f(s) = \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n}} n^{-s},$$

with coefficients $(a_n)_{n\geq 2} \in \ell^2$, and the operator

$$\mathbf{H}f(s) = \int_0^\infty f(w)\,(\zeta(s+w+1)-1)\,ds.$$

We let $\{x\}$ denote the fractional part of x, and use the well-known formula

$$\begin{aligned}\zeta(s+1)-1 &= \frac{1}{s} - (s+1)\int_1^\infty \{x\}x^{-(s+1)}\,\frac{dx}{x} \\ &= \frac{1}{s} - (s+1)\int_0^\infty \{e^x\}e^{-(s+1)x}\,dx =: \frac{1}{s} - K(s).\end{aligned}$$

The function $1/s$ is the kernel of Carleman's operator, defined on $L^2(\mathbb{R}_+)$ as

$$\mathbf{C}f(s) = \int_0^\infty \frac{f(w)}{s+w}\,dw.$$

We will let $\mathbf{K}$ denote the similarly defined integral operator with kernel $(s,w) \mapsto K(s+w)$, so that $\mathbf{H} = \mathbf{C} - \mathbf{K}$. For $0 < \lambda < \pi$ and f in $\mathscr{H}_0^2$, we consider the equation $(\mathbf{H}-\lambda)f = c\psi$, where ψ denotes

$$\psi(s) = \sum_{n=2}^\infty \frac{1}{n\log n} n^{-s}.$$

(Note that this function also differs by a 1/2 shift from the actual symbol appearing in Theorem 3.) It is convenient to rewrite this equation in the form

$$(\mathbf{C}-\lambda)f = \mathbf{K}f + c\psi. \tag{10}$$

To analyze the equation (10), we will use the Mellin transform, which is defined by

$$\mathscr{M}f(z) = \int_0^\infty s^z f(s)\,\frac{ds}{s}. \tag{11}$$

By the Cauchy–Schwarz inequality and Corollary 1, taking into account the rapid decay near infinity, we obtain that if f is in $\mathscr{H}_0^2$, then the integral (11) converges absolutely when $\operatorname{Re} z > 1/2$. This means that the function $\mathscr{M}f(z)$ is analytic in (at least) $\operatorname{Re} z > 1/2$. Our first goals are thus to compute $\mathscr{M}\mathbf{C}f$ and $\mathscr{M}\mathbf{K}f$ for f in $\mathscr{H}_0^2$, as well as the special transform $\mathscr{M}\psi$.

Lemma 1. *Suppose that f is in $\mathscr{H}_0^2$. Then*

$$(\mathscr{M}\mathbf{C}f)(z) = \frac{\pi}{\sin(\pi z)}\,(\mathscr{M}f)(z), \tag{12}$$

has a meromorphic continuation to $\operatorname{Re} z > 1/2$.

Proof. When $\operatorname{Re} z < 1$, $z \notin \mathbb{Z}$ and $w > 0$, we have

$$\int_0^\infty \frac{s^{z-1}}{s+w}\,ds = \frac{\pi}{\sin(\pi z)} w^{z-1},$$

which is the same integral (5) which was used in the proof of Theorem 2. By this formula and Fubini's theorem, we obtain (12) in the strip $1/2 < \operatorname{Re} z < 1$. However, the right hand side of (12) has a meromorphic continuation to the domain $\operatorname{Re} z > 1/2$. □

Remark. Note that the choice of θ is such that $\pi/\sin(\pi(i\theta+1/2)) = \lambda$. This motivates the appearance of the functions $s \mapsto s^{2k-1/2\pm i\theta}$ in (9) as generalized eigenfunctions to the Carleman operator. Compare with the remark following Theorem 3.

Lemma 2. *Let f be a function in $\mathscr{H}_0^2$. Then $(\mathscr{M}\mathbf{K}f)(z)$ has a meromorphic continuation to $\operatorname{Re} z < 1$ with simple poles at the non-positive integers. If $\operatorname{Re} z \le 1-\varepsilon$ and $|\operatorname{Im} z| \ge \varepsilon$, for some positive ε, then*

$$(\mathscr{M}\mathbf{K}f)(z) \ll \|f\|_{\mathscr{H}_0^2} |z| e^{-\pi|\operatorname{Im} z|/2}. \tag{13}$$

Proof. We begin by computing

$$\mathbf{K}f(s) = \int_0^\infty f(w)K(s+w)\,dw = \sum_{n=2}^\infty \frac{a_n}{\sqrt{n}\log n}\left(\alpha_n(s)+\beta_n(s)\right), \tag{14}$$

where

$$\alpha_n(s) = \int_0^\infty A_n(x)\, s e^{-sx}\, x dx \qquad \text{and} \qquad \beta_n(s) = \int_0^\infty 2B_n(x)\, e^{-sx}\, x dx,$$

with

$$A_n(x) = \frac{1}{1+x/\log n}\,\frac{\{e^x\}}{x} e^{-x},$$
$$B_n(x) = \frac{1}{2}\left(\frac{1}{(1+x/\log n)^2} + \frac{1}{1+x/\log n}\right)\frac{\{e^x\}}{x} e^{-x}.$$

We will only need the estimates $A_n(x),\, B_n(x) \le e^{-x}$, which imply that $\mathbf{K}f(s)$ is analytic in $\operatorname{Re} s > -1$, since $(a_n/(\sqrt{n}\log n))_{n\ge 2}$ is in ℓ^1. We apply the Mellin transform of (14), initially with $0 < \operatorname{Re} z < 1$, obtaining

$$(\mathscr{M}\mathbf{K}f)(z) = \sum_{n=2}^\infty \frac{a_n}{\sqrt{n}\log n}\left(\Gamma(1+z)\widetilde{\alpha}_n(z) + \Gamma(z)\widetilde{\beta}_n(z)\right),$$

where Γ denotes the Gamma function and

$$\widetilde{\alpha}_n(z) = \int_0^\infty A_n(x)\, x^{1-z}\,\frac{dx}{x} \qquad \text{and} \qquad \widetilde{\beta}_n(z) = \int_0^\infty 2B_n(x)\, x^{2-z}\,\frac{dx}{x}.$$

When $\operatorname{Re} z < 1$, we use the estimates $A_n(x),\, B_n(x) \le e^{-x}$ along with the triangle inequality to obtain

$$|\widetilde{\alpha}_n(z)| \le \Gamma(1-\operatorname{Re} z) \qquad \text{and} \qquad |\widetilde{\beta}_n(z)| \le 2\Gamma(2-\operatorname{Re} z).$$

Hence $\mathscr{M}\mathbf{K}f$ has a meromorphic continuation to $\operatorname{Re} z < 1$, with simple poles at the poles of $\Gamma(z)$. Moreover, by the Cauchy–Schwarz inequality, we obtain that

$$|(\mathscr{M}\mathbf{K}f)(z)| \ll \|f\|_{\mathscr{H}_0^2}\big(|\Gamma(1+z)|\Gamma(1-\operatorname{Re} z) + 2|\Gamma(z)|\Gamma(2-\operatorname{Re} z)\big).$$

When $|\operatorname{Im} z| \geq \varepsilon$, we may use the functional equation and reflection formula for the Gamma function, and estimate further that

$$\begin{aligned} |(\mathscr{M}\mathbf{K}f)(z)| &\ll \|f\|_{\mathscr{H}_0^2}\,\big(|\Gamma(1+z)|\,\Gamma(1-\operatorname{Re} z)\big) \\ &= \|f\|_{\mathscr{H}_0^2}\,\frac{\pi}{|\sin(\pi z)|}\,\frac{\Gamma(1-\operatorname{Re} z)}{|\Gamma(-z)|}. \end{aligned} \tag{15}$$

By our restriction that $\operatorname{Re} z \leq 1-\varepsilon$ and $|\operatorname{Im} z| \geq \varepsilon$, Stirling's formula (see [15, p. 525]) now yields that

$$\frac{\Gamma(1-\operatorname{Re} z)}{|\Gamma(-z)|} \ll \frac{|1-\operatorname{Re} z|^{1/2-\operatorname{Re} z}}{|z|^{-\operatorname{Re} z-1/2}}e^{\pi|\operatorname{Im} z|/2} \ll |z|e^{\pi|\operatorname{Im} z|/2},$$

where the implicit constants depend only on ε. Hence returning to (15), we find that

$$|(\mathscr{M}\mathbf{K}f)(z)| \ll \|f\|_{\mathscr{H}_0^2}|z|e^{-\pi|\operatorname{Im} z|/2}$$

as claimed. □

Lemma 3. *For* $\operatorname{Re} z > 0$*, we have*

$$\mathscr{M}\psi(z) = -\frac{1}{z^2} + \sum_{n=0}^{\infty}\frac{b_n}{z+n} + E_\psi(z), \tag{16}$$

where $|b_n|$ *decays super-exponentially, and* $E_\psi(z)$ *is an entire function that, for every real number* R*, is bounded in the half-plane* $\operatorname{Re} z < R$*. Hence* $\mathscr{M}\psi(z)$ *has a meromorphic continuation to* $\mathbb{C}$ *with a double pole at* $z=0$ *and simple poles at the negative integers.*

Proof. Set $h(s) := \psi(s) - \log s$. Since $h'(s) = \zeta(s+1) - 1 - 1/s$, $h(s) = \sum_{n\geq 0} b_n s^n$ is an entire function. Note now that for $\operatorname{Re} z > 0$ we have

$$\int_0^1 s^{z-1}\log s\,ds = -\frac{1}{z^2},$$

while

$$\int_0^1 s^{z-1}h(s)\,ds = \sum_{n=0}^{\infty}\frac{b_n}{z+n}.$$

We finish the proof by setting $E_\psi(z) := \int_1^\infty s^{z-1}\psi(s)\,ds$. □

Proof of Theorem 3. Suppose that $0 < \lambda < \pi$. Transforming the equation (10) by the Mellin transform and solving for $\mathscr{M}f$, we obtain

$$\mathscr{M}f(z) = \left(\frac{\pi}{\sin(\pi z)} - \lambda\right)^{-1} (\mathscr{M}\mathbf{K}f(z) + c\mathscr{M}\psi(z)) . \tag{17}$$

Initially this formula is only valid for $1/2 < \operatorname{Re} z < 1$, but we note that the left hand side can be analytically continued to $\operatorname{Re} z > 1/2$ and the right hand side can be meromorphically continued to $\operatorname{Re} z < 1$.

The inverse Mellin transform is given by

$$\mathscr{M}^{-1}h(s) = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} s^{-z}h(z)\,dz \tag{18}$$

for a suitable κ. For (17) the Mellin inversion theorem allows us to choose $\kappa \in (1/2, 1)$. Our expressions for $\mathscr{M}\mathbf{K}f$ and $\mathscr{M}\psi$ show that the right-hand side of (17) is meromorphic in $\operatorname{Re} z < 1$ with (possible) simple poles at the solutions of $\sin(\pi z) = \pi/\lambda$ as well as at $z = 0$. Note here that the factor in front of $\mathscr{M}\mathbf{K}f(z) + c\mathscr{M}\psi(z)$ has simple zeroes at the integers. Note also that there actually are no poles in $\operatorname{Re} z > 1/2$, since $\mathscr{M}f(z)$ is analytic there. Hence we are left with the pole $z = 0$ (if $c \neq 0$) and those given by

$$1 - \frac{\lambda}{\pi}\sin(\pi z) = 0, \qquad \operatorname{Re} z \le 1/2 \qquad \Longleftrightarrow \qquad z = \pm i\theta + (2k + 1/2),$$

where $k = 0, -1, -2, \ldots$

We now compute (18) for $h = \mathscr{M}f$ and $\kappa = 2/3$ by the method of residues. Let $J_n = [\theta] + n$ and form the rectangular contour $\mathscr{J}_n$ with corners in $2/3 \pm iJ_n$ and $-(2J_n + 3/2) \pm iJ_n$, traversed counter-clockwise. Using (13) and (16), straightforward estimates show that for $0 < s < 1$ we have

$$\lim_{n\to\infty}\int_{\mathscr{J}_n} s^{-z}\mathscr{M}f(z)\,dz = \frac{1}{2\pi i}\int_{2/3 - i\infty}^{2/3 + i\infty} s^{-z}\mathscr{M}f(z)\,dz.$$

Evaluating the left-hand side by residues, we obtain

$$f(s) = cd + \sum_{k=0}^{\infty} s^{2k-1/2}\left(c_k s^{-i\theta} + d_k s^{i\theta}\right), \qquad 0 < s < 1,$$

where cd, c_k, and d_k are obtained as the residues of the right-hand side of (17) at $z = 0$, $z = i\theta - 2k + 1/2$ and $z = -i\theta - 2k + 1/2$, respectively. In fact, it is clear that c_k and d_k grow at most polynomially in k, as seen from the estimates of Lemma 2 and Lemma 3.

It remains to show that $c_0 = d_0 = 0$. However, either of them assuming a non-zero value contradicts the fact that f is in $L^2(\mathbb{R}_+)$. Moving back to $(1/2, +\infty)$, we obtain (9).

The final statement follows from the fact that $f'(s)$ is bounded in $1/2 < s < 1$ due to (9), the contribution from $s > 1$ is easily estimated by the fact that f is a Dirichlet series in $\mathscr{H}_0^2$. $\square$

Note that in the excluded case $\lambda = \pi$ one may use the same argument, but the representation of f is different because all poles of the right-hand side of (17) except $z = 0$ are double. We also note that a more careful analysis would show that the sequences $(c_k)_{k\geq 0}$ and $(d_k)_{k\geq 0}$ are in fact bounded, but since we do not need this, we have not made an effort to optimize this part of the theorem.

4. The spectrum of the multiplicative Hilbert matrix

In this section we establish that $\mathbf{H}$ has the purely continuous spectrum $[0, \pi]$ on $\mathscr{H}_0^2$. Our argument is based on a commutation relation between $\mathbf{H}$ and the operator $\mathbf{D}$ of differentiation, $\mathbf{D}f(s) = f'(s)$. To establish this relation, we observe that

$$\mathbf{DH}f(s) = \int_{1/2}^{\infty} f(w)\mathbf{D}(\zeta(w+s)-1)dw, \qquad s > 1/2.$$

Supposing that f' is integrable on the segment $(1/2, 1)$, we get that

$$\begin{aligned}\mathbf{DH}f(s) &= -f(1/2)(\zeta(s+1/2)-1) - \int_{1/2}^{\infty} f'(w)(\zeta(w+s)-1)dw \\ &= -f(1/2)(\zeta(s+1/2)-1) - \mathbf{HD}f(s), \qquad s > 1/2,\end{aligned}$$

where we have defined $f(1/2) = f(1) - \int_{1/2}^{1} f'(w)dw$. Thus, $\mathbf{D}$ and $\mathbf{H}$ anti-commute up to an (unbounded) rank-one term. This observation is crucial for the characterization of the spectrum of $\mathbf{H}$.

To demonstrate that $\mathbf{H}$ has the purely continuous spectrum $[0, \pi]$, it suffices to show that $\mathbf{H}$ has no eigenvalues and that $H - \lambda$ does not have full range for λ in $(0, \pi)$. Indeed, $\mathbf{H}$ is a positive operator with norm π, and so it follows that its spectrum is $[0, \pi]$. Since any λ in the spectrum of a self-adjoint operator must either be an eigenvalue or part of the continuous spectrum, we can conclude that $\mathbf{H}$ has purely continuous spectrum. With this in mind we now finish the proof of Theorem 1.

Theorem 4. *The operator $\mathbf{H} : \mathscr{H}_0^2 \to \mathscr{H}_0^2$ has no point spectrum. Furthermore, if f in $\mathscr{H}_0^2$ solves the equation $(\mathbf{H} - \lambda)f = c\psi$, where c is a complex number and*

$$\psi(s) = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} n^{-s},$$

then $f = c = 0$. In particular, the spectrum of $\mathbf{H}$ is $[0, \pi]$ and purely continuous.

Proof. We have already proved that $\lambda = 0$ and $\lambda = \pi$ are not eigenvalues, since we have shown in Section 2 that $\mathbf{H}$ is a strictly positive operator for which $\|\mathbf{H}f\|_{\mathscr{H}_0^2} < \pi\|f\|_{\mathscr{H}_0^2}$, $f \neq 0$. It is hence sufficient to verify the second part of Theorem 4, since it shows simultaneously that no λ in $(0, \pi)$ is an eigenvalue, and that $\mathbf{H} - \lambda$ does not have full range.

Accordingly, we suppose that f in $\mathscr{H}_0^2$ satisfies $(\mathbf{H} - \lambda)f = c\psi$. By Theorem 3, we have the series representation (9). In particular f' is square-integrable on $(1/2, \infty)$ and $f(1/2) = cd$. But noting that $\psi'(s) = \zeta(s + 1/2) - 1$ and using the commutation relation of $\mathbf{H}$ and $\mathbf{D}$, we then get that

$$-(\mathbf{H} + \lambda)f' - cd(\zeta(s + 1/2) - 1) = c(\zeta(s + 1/2) - 1).$$

Since f' is in $L^2(1/2, \infty)$ we use Corollary 1 to conclude that $\mathbf{H}f'$ is also in $L^2(1/2, \infty)$. Since $\zeta(s + 1/2)$ has a pole of order 1 at $s = 1/2$, it follows that $d = -1$. Hence, we have obtained that

$$(\mathbf{H} + \lambda)f' = 0. \tag{19}$$

From (19) and Corollary 1, we get that f' is $\mathscr{H}_0^2$. But since $\mathbf{H}$ is a positive operator on $\mathscr{H}_0^2$, applying (19) again, we find that $f' \equiv 0$. □

5. Failure of boundedness of $\mathbf{H}$ on $\mathscr{H}_0^p$ when $p \neq 2$

We follow [1] and define $\mathscr{H}^p$ as the completion of the set of Dirichlet polynomials $P(s) = \sum_{n \leq N} a_n n^{-s}$ with respect to the norm

$$\|P\|_{\mathscr{H}^p} := \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T |P(it)|^p dt \right)^{1/p}.$$

The Dirichlet series of a function f in $\mathscr{H}^p$ converges uniformly in each half-plane $\operatorname{Re} s > 1/2 + \varepsilon$, $\varepsilon > 0$, so f is analytic in the half-plane $\operatorname{Re} s > 1/2$ (see [1, 22]). The space $\mathscr{H}_0^p$ is the subspace of $\mathscr{H}^p$ consisting of Dirichlet series of the form $\sum_{n \geq 2} a_n n^{-s}$, which means that series in $\mathscr{H}_0^p$ vanish at $+\infty$.

Theorem 5. $\mathbf{H}$ *does not act boundedly on* $\mathscr{H}_0^p$ *for* $1 \leq p < \infty$, $p \neq 2$.

The proof of this theorem requires us to associate $\mathscr{H}^p$ with $H^p(\mathbb{T}^\infty)$. This means that we need to invoke the so-called Bohr lift, which we now recall (see [11, 22] for further details). For every positive integer n, the fundamental theorem of arithmetic allows the prime factorization

$$n = \prod_{j=1}^{\pi(n)} p_j^{\kappa_j},$$

which associates n to the finite non-negative multi-index $\kappa(n) = (\kappa_1, \kappa_2, \kappa_3, \dots)$. The Bohr lift of the Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is the power series

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}, \tag{20}$$

where $z = (z_1, z_2, z_3, \dots)$. Hence (20) is a power series in infinitely many variables, but each term contains only a finite number of these variables. An important example is the Bohr lift of the Riemann zeta function. Let $f_w(s) = \zeta(s+w)$ for $\mathrm{Re}(w) > 1/2$. Using the Euler product of the Riemann zeta function, we find that

$$\mathscr{B}f_w(z) = \sum_{n=1}^{\infty} n^{-w} z^{\kappa(n)} = \prod_{j=1}^{\infty} \left(1 - p_j^{-w} z_j\right)^{-1}. \tag{21}$$

Indeed, any Dirichlet series with an Euler product has a Bohr lift that separates the variables in the same way.

Under the Bohr lift, $\mathscr{H}^p$ corresponds to the Hardy space $H^p(\mathbb{T}^\infty)$, which we view as a subspace of $L^p(\mathbb{T}^\infty)$. This means that $\mathscr{B}$ is a multiplicative and isometric map from $\mathscr{H}^p$ onto $H^p(\mathbb{T}^\infty)$. We refer to [1, 4, 11, 22] for the details, mentioning only a few important facts. Functions in $H^p(\mathbb{T}^\infty)$ are analytic at the points $\xi \in \mathbb{D}^\infty \cap \ell^2$. Indeed the reproducing kernel at ξ is given by

$$K_\xi(z) = \prod_{j=1}^{\infty} \left(1 - \overline{\xi_j} z_j\right)^{-1},$$

compare with (21). The Haar measure of the compact abelian group $\mathbb{T}^\infty$ is simply the product of the normalized Lebesgue measures for each variable. In particular, $H^p(\mathbb{T}^d)$ is a natural subspace of $H^p(\mathbb{T}^\infty)$. We denote the orthogonal projection (Riesz projection) from $L^2(\mathbb{T}^\infty)$ onto $H^2(\mathbb{T}^\infty)$ by P_+. Even though $H^p(\mathbb{T}^\infty)$ is uncomplemented in $L^p(\mathbb{T}^\infty)$ when $p \neq 2$ [8], we can still identify its dual with the Riesz projection of $L^q(\mathbb{T}^\infty)$ for $1/p + 1/q = 1$ using the Hahn–Banach theorem, $(H^p(\mathbb{T}^\infty))^* = P_+ L^q(\mathbb{T}^\infty)$, $1 \leq p < \infty$.

We require the following lemma which is established by direct computation. Here and in what follows, the L^p norm with respect to normalized Lebesgue measure on $\mathbb{T}$ (or $\mathbb{T}^\infty$) is denoted by $\|\cdot\|_p$.

Lemma 4. *Let λ be a real parameter and suppose that $0 < \varepsilon(1+|\lambda|) < 1/4$, $1 \leq p < \infty$. Then*

$$\|1 + \varepsilon(z + \lambda\overline{z})\|_p^p = 1 + \frac{p}{4}\left[(p-1)(1+\lambda)^2 + (1-\lambda)^2\right]\varepsilon^2 + O(\varepsilon^3).$$

The norm is minimal when $\lambda = (2-p)/p$:

$$\left\|1 + \varepsilon\left(z + \frac{(2-p)}{p}\overline{z}\right)\right\|_p^p = 1 + (p-1)\varepsilon^2 + O(\varepsilon^3).$$

Proof. We write $z = e^{i\theta}$ so that we have

$$
\begin{aligned}
|1+\varepsilon(z+\lambda\overline{z})|^p &= \big(1+2\varepsilon(1+\lambda)\cos\theta+\varepsilon^2(1+\lambda)^2\cos^2\theta+\varepsilon^2(1-\lambda)^2\sin^2\theta\big)^{p/2} \\
&= 1+p\varepsilon(1+\lambda)\cos\theta+\frac{p}{2}\varepsilon^2\Bigg(\Big[1+2\Big(\frac{p}{2}-1\Big)\Big](1+\lambda)^2\cos^2\theta \\
&\qquad + (1-\lambda)^2\sin^2\theta\Bigg)+O(\varepsilon^3).
\end{aligned}
$$

Integrating, we get

$$
\|1+\varepsilon(z+\lambda\overline{z})\|_p^p = 1+\frac{p}{4}\left[(p-1)(1+\lambda)^2+(1-\lambda)^2\right]\varepsilon^2+O(\varepsilon^3). \qquad \square
$$

The point of the lemma is that $p^2/4 > p-1$ whenever $p \neq 2$, so that (one-dimensional) Riesz projection acts expansively on $g(z) = 1+\varepsilon(z+\lambda\overline{z})$, since $\|P_+g\|_p^p = 1+(p/2)^2\varepsilon^2+O(\varepsilon^4)$.

Proof of Theorem 5. Assume first that $p > 1$. To estimate the norm of $\mathbf{H}$ on $\mathscr{H}_0^p$ from below, we will choose G in $L^q(\mathbb{T}^\infty)$ with $1/p+1/q = 1$ such that $G(0) = 1$. Then using that $\zeta(s+w)-1$ is the reproducing kernel of $\mathscr{H}_0^2$, we get for $f \in \mathscr{H}_0^p$ that

$$
\langle \mathscr{B}\mathbf{H}f, G\rangle_{L^2(\mathbb{T}^\infty)} = \langle \mathbf{H}f, \mathscr{B}^{-1}P_+G\rangle_{\mathscr{H}^2} = \int_{1/2}^\infty f(w)\overline{(\mathscr{B}^{-1}P_+G(w)-1)}\,dw.
$$

Specifically, we set

$$
G(z) = \prod_{j=1}^\infty\left(1+\frac{2}{q}p_j^{-\alpha}\left(z_j+\frac{(2-q)}{q}\overline{z_j}\right)\right)
$$

where $\alpha > 1/2$. Using Lemma 4 we find that

$$
\begin{aligned}
\|G\|_q^q &= \prod_{j=1}^\infty\left\|1+\frac{2}{q}p_j^{-\alpha}\left(z_j+\frac{(2-q)}{q}\overline{z_j}\right)\right\|_q^q \\
&= \prod_{j=1}^\infty\left(1+\frac{4(q-1)}{q^2}p_j^{-2\alpha}+O(p_j^{-3\alpha})\right).
\end{aligned}
$$

To estimate the Euler products $\prod_{j\geq 1}(1+\lambda p_j^{-s})$ for, say $1 < s < 2$, we use that

$$
\prod_{j=1}^\infty(1+\lambda p_j^{-s}) = \prod_{j=1}^\infty\frac{(1+\lambda p_j^{-s})\big(1-\lambda p_j^{-s}+O(p_j^{-2s})\big)}{(1-p_j^{-s})^\lambda} \asymp \zeta(s)^\lambda \asymp (s-1)^{-\lambda}.
$$

We get that $\|G\|_q \asymp (2\alpha-1)^{-4/(pq^2)}$ as $\alpha \to 1/2$, since $(q-1)/q = 1/p$. If $1/2 < \alpha, w < 1$, then

$$\mathscr{B}^{-1}P_+G(w) = \prod_{j=1}^{\infty}\left(1+(2/q)p_j^{-\alpha-w}\right) \asymp (\alpha+w-1)^{-2/q}.$$

We now choose

$$f(w) = \prod_{j=1}^{\infty}\left(1+(2/p)p_j^{-\alpha-w}\right) - 1 \asymp (\alpha+w-1)^{-2/p}.$$

The norm of f can be computed as in the proof of Lemma 4,

$$\|\mathscr{B}f\|_p = \prod_{j=1}^{\infty}\left\|1+(2/p)p_j^{-\alpha}z_j\right\|_p = \prod_{j=1}^{\infty}\left(1+p_j^{-2\alpha}+O(p_j^{-4\alpha})\right)^{\frac{1}{p}} \asymp (2\alpha-1)^{-1/p}.$$

Combining everything, we get that

$$\frac{\left|\langle \mathscr{B}\mathbf{H}f, G\rangle_{L^2(\mathbb{T}^\infty)}\right|}{\|\mathscr{B}f\|_p\|G\|_q} \gg \int_{1/2}^{1}\frac{(2\alpha-1)^{4/(pq^2)+1/p}}{(\alpha+w-1)^2}\,dw \asymp (2\alpha-1)^{4/(pq^2)+1/p-1}.$$

The exponent is negative if $p \neq 2$ since, in this case, $pq > 4$ so letting $\alpha \to 1/2$ shows that $\mathbf{H}$ is unbounded on $\mathscr{H}_0^p$.

For $p=1$, we make a minor adjustment. We can use the same f (with $p=1$), but we choose

$$G(z) = \prod_{j=1}^{\infty}\left(1+(1/4)p_j^{-\alpha}(z_j-\overline{z_j})\right).$$

The point is that $z_j - \overline{z_j} = 2i\sin(\theta_j)$, if $z_j = e^{i\theta_j}$, so we get that

$$\|G\|_\infty = \prod_{j=1}^{\infty}\sqrt{1+(p_j^{-\alpha}/2)^2} = \prod_{j=1}^{\infty}\left(1+(1/8)p_j^{-2\alpha}+O(p_j^{-4\alpha})\right) \asymp (2\alpha-1)^{-1/8}.$$

The rest of the argument works like above, the conclusion coming from that $1/8 - 1/4 < 0$. □

6. Symbols of the multiplicative Hilbert matrix

To place our discussion in context, we begin with some general considerations concerning Hankel forms, i.e., the bilinear forms associated with (additive or multiplicative) Hankel matrices. We recall that any function ψ in $H^2(\mathbb{T})$ defines a Hankel form H_ψ by the relation

$$H_\psi(f,g) = \langle fg, \psi\rangle_{L^2(\mathbb{T})},$$

which makes sense at least for polynomials f and g. Nehari's theorem [17] says that H_ψ extends to a bounded form on $H^2(\mathbb{T}) \times H^2(\mathbb{T})$ if and only if $\psi = P_+\varphi$

for a bounded function φ in $L^\infty(\mathbb{T})$. Moreover, $\|H_\psi\| = \|\varphi\|_\infty$ if we choose φ to have minimal L^∞ norm. By the Hahn-Banach theorem and the observation that

$$\langle f, \varphi\rangle_{L^2(\mathbb{T})} = \langle f, P_+\varphi\rangle_{L^2(\mathbb{T})},$$

at least for polynomials f, we note an equivalent formulation of the first part of Nehari's theorem: H_ψ defines a bounded form if and only if ψ induces a bounded functional on $H^1(\mathbb{T})$, in the sense that there exist $C > 0$ such that for every polynomial f it holds that $|\langle f, \psi\rangle_{L^2(\mathbb{T})}| \leq C\|f\|_1$. See for example [18, Section 1.4].

In this context let us indicate an alternative proof (in fact, the original approach of Hilbert) of the fact that the usual Hilbert matrix has norm π. Let $\varphi(\theta) = ie^{-i\theta}(\pi - \theta)$, $\theta \in [0, 2\pi)$. Since

$$\sum_{n=0}^{\infty}(n+1)^{-1}e^{in\theta} = P_+\varphi(\theta), \qquad \text{a.e. } \theta,$$

and $\|\varphi\|_\infty = \pi$, it follows that the Hilbert matrix has norm at most π. As noted above, it also follows that

$$\left|\sum_{n=0}^{\infty} c_n(n+1)^{-1}\right| \leq \pi\|f\|_1,$$

where $f(z) = \sum_{n\geq 0} c_n z^n$. In the case of the Hilbert matrix, we have in fact the stronger inequality

$$\sum_{n=0}^{\infty} |c_n|(n+1)^{-1} \leq \pi\|f\|_1, \tag{22}$$

which was proved by Hardy and Littlewood [9].

We turn next to what is known about multiplicative Hankel forms. Every sequence $\varrho = (\varrho_1, \varrho_2, \varrho_3, \dots)$ in ℓ^2 defines in an obvious way a multiplicative Hankel matrix, and we associate with it the corresponding multiplicative Hankel form given by

$$\varrho(a, b) = \sum_{m,n=1}^{\infty} \varrho_{mn} a_m b_n, \tag{23}$$

which initially is defined at least for finitely supported sequences a and b in ℓ^2. We will now explain, using the Bohr lift, that every multiplicative Hankel matrix can be uniquely associated with either a Hankel form on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$ or equivalently a (small) Hankel operator acting on $H^2(\mathbb{T}^\infty)$.

If f, g, and φ are Dirichlet series in $\mathscr{H}^2$ with coefficients a_n, b_n, and $\overline{\varrho_n}$, respectively, a computation shows that

$$\langle fg, \varphi\rangle_{\mathscr{H}^2} = \varrho(a, b).$$

A formal computation gives that

$$\langle \mathscr{B}f\mathscr{B}g, \mathscr{B}\varphi\rangle_{L^2(\mathbb{T}^\infty)} = \langle fg, \varphi\rangle_{\mathscr{H}^2},$$

allowing us to compute the multiplicative Hankel form (23) on $\mathbb{T}^\infty$. This means that we may equivalently study Hankel forms

$$H_\Phi(FG) = \langle FG, \Phi\rangle_{L^2(\mathbb{T}^\infty)}, \qquad F, G \in H^2(\mathbb{T}^\infty). \tag{24}$$

In our previous considerations we required that Φ be in $H^2(\mathbb{T}^\infty)$, but there is nothing to prevent us from considering arbitrary symbols Φ from $L^2(\mathbb{T}^\infty)$. Hence, each Φ in $L^2(\mathbb{T}^\infty)$ induces by (24) a (possibly unbounded) Hankel form H_φ on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$. Of course, this is not a real generalization. Each form H_Φ is also induced by a symbol Ψ in $H^2(\mathbb{T}^\infty)$; setting $\Psi = P_+\Phi$ we have $H_\Phi = H_\Psi$.

On the polydisc, the Hankel form H_Φ is naturally realized as a (small) Hankel operator $\mathbf{H}_\Phi$, which when bounded acts as an operator from $H^2(\mathbb{T}^\infty)$ to the anti-analytic space $\overline{H^2}(\mathbb{T}^\infty)$. Letting $\overline{P_+}$ denote the orthogonal projection of $L^2(\mathbb{T}^\infty)$ onto $\overline{H^2}(\mathbb{T}^\infty)$, we have at least for polynomials F in $H^2(\mathbb{T}^\infty)$ that

$$\mathbf{H}_\Phi F = \overline{P_+}(\overline{\Phi}F).$$

We now come to the question of to which extent Nehari's theorem remains valid in the multiplicative setting. Note first that if Ψ is in $L^\infty(\mathbb{T}^\infty)$, then the corresponding multiplicative Hankel form is bounded, since

$$|H_\Psi(fg)| = |\langle fg, \Psi\rangle| \le \|f\|_2\, \|g\|_2 \|\, \Psi\|_\infty.$$

We say that H_Φ has a bounded symbol if there exists $\Psi \in L^\infty(\mathbb{T}^\infty)$ such that $H_\Phi = H_\Psi$. In [12], Helson proved that every Hankel form in the Hilbert–Schmidt class S_2 has a bounded symbol, but it was shown in [20] that there exist bounded multiplicative Hankel forms without bounded symbols, in sharp contrast to the classical situation. Hence, there are in fact bounded Hankel forms H_Φ for which $f \mapsto H_\Phi(f)$ does not define a bounded functional on $H^1(\mathbb{T}^\infty)$. For when this functional is bounded on $H^1(\mathbb{T}^\infty)$ it has, by Hahn-Banach, a bounded extension to $L^1(\mathbb{T}^\infty)$ and therefore is given by an $L^\infty(\mathbb{T}^\infty)$-function Ψ which must satisfy $H_\Phi = H_\Psi$. The result of [20] was strengthened in [2], where it was shown that there are Hankel forms in Schatten classes S_p without bounded symbols whenever $p > (1 - \log\pi/\log 4)^{-1} = 5.7388...$

In the opposite direction, we have the following positive result about Hankel forms with bounded symbols, reflecting that when $\alpha(n)$ is a multiplicative function, variables separate in a natural way so that the classical Nehari theorem applies to each of the infinitely many copies of the unit circle $\mathbb{T}$.

Theorem 6. *Suppose that $\varphi(s) := \sum_{n\ge1} \alpha(n)n^{-s}$ is in $\mathscr{H}^2$ and that $\alpha(n)$ is a multiplicative function. If $H_{\mathscr{B}\varphi}$ is a bounded Hankel form on $H^2(\mathbb{T}^\infty)\times H^2(\mathbb{T}^\infty)$, then there exist $\Psi \in L^\infty(\mathbb{T}^\infty)$ such that $\mathscr{B}\varphi = P_+\Psi$. Moreover, if the function*

$\alpha(n)$ is completely multiplicative, then the Hankel form $H_{\mathscr{B}\varphi}$ is always bounded on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$.

Proof. We begin by proving the first statement. To this end, by the assumption that $\alpha(n)$ is a multiplicative function, we may factor the symbol $\varphi(s) = \sum_{n\geq 1} \alpha(n)n^{-s}$ into an Euler product,

$$\varphi(s) = \sum_{n=1}^{\infty} \alpha(n)n^{-s} = \prod_{j=1}^{\infty} \left(1 + \sum_{k=1}^{\infty} \alpha\big(p_j^k\big)p_j^{-ks}\right) =: \prod_{j=1}^{\infty} \varphi_j(s),$$

which is absolutely convergent when $\operatorname{Re} s > 1/2$. We observe that $\Phi_j := \mathscr{B}\varphi_j$ depends only on z_j, so that $\Phi(z) := \mathscr{B}\varphi(z) = \prod_{j\geq 1} \Phi_j(z_j)$. Now a version of Lemma 2 in [2] can be used to show that

$$\|H_\Phi\| = \prod_{j=1}^{\infty} \|H_{\Phi_j}\|.$$

Since H_{Φ_j} is a one variable Hankel form, we may appeal to the classical Nehari theorem [17] to infer that there is some Ψ_j in $L^\infty(\mathbb{T})$ so that $H_{\Phi_j} = H_{\Psi_j}$ and moreover that $\|H_{\Phi_j}\| = \|\Psi_j\|_\infty$. Setting $\Psi(z) := \prod_{j\geq 1} \Psi_j(z_j)$, we conclude that $\|H_\Phi\| = \|\Psi\|_\infty$ and that $\Phi = P_+\Psi$.

The second statement of Theorem 6 is just a reformulation of the fact that the set of bounded point evaluations for $H^1(\mathbb{T}^\infty)$ is $\mathbb{D}^\infty \cap \ell^2$ [4]. Following [4, p. 122] or the proof of the first part of the present theorem, we may find corresponding bounded functions explicitly: For every point $z = (z_j)$ on $\mathbb{T}^\infty$, we set

$$\Psi(z) = \prod_{j=1}^{\infty} \frac{1}{1 - |\alpha(p_j)|^2} \frac{1 - \overline{\alpha(p_j)z_j}}{1 - \alpha(p_j)z_j}.$$

This is a bounded function on $\mathbb{T}^\infty$ because $(\alpha(p_j))_{j\geq 1} \in \mathbb{D}^\infty \cap \ell^2$. One may check that $\mathscr{B}^{-1}P_+\Psi(s) = \sum_{n\geq 1} \alpha(n)n^{-s}$ by a direct computation or by checking that Φ represents the functional of point evaluation at $(\alpha(p_j))_{j\geq 1}$. □

Because of the factor $1/\log n$, the analytic symbol (4) of the multiplicative Hilbert matrix does not have multiplicative coefficients, and we know from Theorem 1 that it is not compact. This means that the preceding discussion gives no answer to the following question.

Question. Does the multiplicative Hilbert matrix have a bounded symbol?

Equivalently, we may ask whether we have

$$\left| a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n}\log n} \right| \ll \|f\|_{\mathscr{H}^1} \tag{25}$$

when $f(s)=\sum_{n\geq 1} a_n n^{-s}$ is in $\mathscr{H}^1$. We could even ask if the analogue of the Hardy–Littlewood inequality (22) is valid: Does (25) hold when we put absolute values on a_n, or, in other words, do we have

$$|a_1| + \sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{n}\log n} \ll \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathscr{H}^1} ?$$

To see that we could not hope for a better inequality with $\sqrt{n}\log n$ replaced by a function of slower growth, we look at the function

$$f_N(s) := \left(\sum_{n=1}^{N} n^{-1/2-s} \right)^2 ,$$

which has $\|f_N\|_{\mathscr{H}^1} \sim \log N$. On the other hand, we observe that in this case,

$$\sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{n}\log n} \geq \sum_{n=2}^{N} \frac{d(n)}{n\log n} \sim \log N,$$

where $d(n)$ is the divisor function and the latter estimate follows by Abel's summation formula.

We observe that the left-hand side of (25) can be written as an integral, so that another reformulation of the question is to ask if the linear functional defined by

$$Lf = \int_{1/2}^{\infty} f(w)dw \tag{26}$$

extends to a bounded linear functional on $\mathscr{H}_0^1$. One of the most important open problems in the theory of Hardy spaces of Dirichlet series is to determine whether

$$\int_0^1 |P(1/2+it)|dt \ll \|P\|_{\mathscr{H}^1} \tag{27}$$

holds for all Dirichlet polynomials. If this were the case, then a Carleson measure argument (see [19, Theorem 4]) shows that then we also have

$$\int_{1/2}^{3/2} |f(w)|dw \ll \|f\|_{\mathscr{H}^1}$$

for all f in $\mathscr{H}^1$. The contribution from $\mathrm{Re}(s) \geq 3/2$ can be handled with a point estimate. The easiest way (see also [4]) to deduce a sharp point estimate for $\mathscr{H}_0^1$ is through Helson's inequality [12], which states that $\sum_{n\geq 1} |a_n|^2/d(n) \leq \|f\|^2_{\mathscr{H}^1}$. For $f \in \mathscr{H}_0^1$ and $\mathrm{Re}(s)=\sigma>1/2$ we get that

$$|f(s)| \leq \left(\sum_{n=2}^{\infty} \frac{|a_n|^2}{d(n)} \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} d(n) n^{-2\sigma} \right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{H}_0^1} \left(\zeta(2\sigma)^2 - 1\right)^{\frac{1}{2}} .$$

For instance, if $w \geq 3/2$ then $|f(w)| \ll \|f\|_{\mathscr{H}_0^1} 4^{-w}$. Therefore the validity of the embedding (27) in fact implies that

$$\int_{1/2}^{\infty} |f(w)| dw \ll \|f\|_{\mathscr{H}_0^1}.$$

This inequality is stronger than asking the functional of (26) to be bounded on $\mathscr{H}_0^1$, and hence we have shown that (27) would imply that the multiplicative Hilbert matrix has a bounded symbol. Whether (27) holds is an open problem that has remained unsolved for many years; we refer to [23] for a discussion of it.

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Paper 8

Weak product spaces of Dirichlet series

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WEAK PRODUCT SPACES OF DIRICHLET SERIES

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Abstract. Let $\mathscr{H}^2$ denote the space of ordinary Dirichlet series with square summable coefficients, and let $\mathscr{H}^2_0$ denote its subspace consisting of series vanishing at $+\infty$. We investigate the weak product spaces $\mathscr{H}^2 \odot \mathscr{H}^2$ and $\mathscr{H}^2_0 \odot \mathscr{H}^2_0$, finding that several pertinent problems are more tractable for the latter space. This surprising phenomenon is related to the fact that $\mathscr{H}^2_0 \odot \mathscr{H}^2_0$ does not contain the infinite-dimensional subspace of $\mathscr{H}^2$ of series which lift to linear functions on the infinite polydisc.

The problems considered stem from questions about the dual spaces of these weak product spaces, and are therefore naturally phrased in terms of multiplicative Hankel forms. We show that there are bounded, even Schatten class, multiplicative Hankel forms on $\mathscr{H}^2_0 \times \mathscr{H}^2_0$ whose analytic symbols are not in $\mathscr{H}^2$. Based on this result we examine Nehari's theorem for such Hankel forms. We define also the skew product spaces associated with $\mathscr{H}^2 \odot \mathscr{H}^2$ and $\mathscr{H}^2_0 \odot \mathscr{H}^2_0$, with respect to both half-plane and polydisc differentiation, the latter arising from Bohr's point of view. In the process we supply square function characterizations of the Hardy spaces $\mathscr{H}^p$, for $0 < p < \infty$, from the viewpoints of both types of differentiation. Finally we compare the skew product spaces to the weak product spaces, leading naturally to an interesting Schur multiplier problem.

1. Introduction

In this paper, we investigate certain properties of weak product spaces associated with the Hardy space of Dirichlet series,

$$\mathscr{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \,:\, \|f\|_{\mathscr{H}^2} = \Big(\sum_{n=1}^{\infty} |a_n|^2 \Big)^{\frac{1}{2}} < \infty \right\},$$

and its subspace $\mathscr{H}^2_0$, consisting of those $f \in \mathscr{H}^2$ with $a_1 = f(+\infty) = 0$. The main objects of study are the weak product spaces $\mathscr{H}^2 \odot \mathscr{H}^2$ and $\mathscr{H}^2_0 \odot \mathscr{H}^2_0$. With $\mathscr{X} = \mathscr{H}^2$ or $\mathscr{X} = \mathscr{H}^2_0$, the weak product $\mathscr{X} \odot \mathscr{X}$ is defined as the Banach space completion of the finite sums $F = \sum_k f_k g_k$, where $f_k, g_k \in \mathscr{X}$, under the norm

$$\|F\|_{\mathscr{X} \odot \mathscr{X}} = \inf \sum_k \|f_k\|_{\mathscr{H}^2} \|g_k\|_{\mathscr{H}^2}.$$

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The infimum is taken over all finite representations of F as a sum of products.

While a separate study of $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ may at first be thought unmotivated, we will find that the norm of this space is significantly larger for certain types of Dirichlet series (see Theorem 1 and its corollaries). The presence of such examples is related to the obstructions faced in producing monomials n^{-s} in a product $f_k g_k$, for $f_k, g_k \in \mathscr{H}_0^2$, when n is an integer with a low number of prime factors. In particular, elements of $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ contain no terms of the form p^{-s}, where p is a prime number. Hence there is an easily identifiable infinite-dimensional subspace of $\mathscr{H}^2 \odot \mathscr{H}^2$ which has trivial intersection with $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$.

The weak product space $\mathscr{H}^2 \odot \mathscr{H}^2$ was first investigated by Helson [21, 22] in an attempt to decide whether Nehari's theorem holds for multiplicative Hankel forms (see also Section 2). Helson's work was continued in [25], where it was demonstrated that Nehari's theorem does not hold in full generality. To explain his point of view, note that each sequence $\varrho \in \ell^2$ induces a (not necessarily bounded) multiplicative Hankel form on $\ell^2 \times \ell^2$,

$$\varrho(a,b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \varrho_{mn}, \qquad a, b \in \ell^2. \tag{1}$$

The analytic symbol of (1) is the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \overline{\varrho_n} n^{-s}.$$

Indeed, if f and g are elements of $\mathscr{H}^2$ with coefficients a and b, respectively, we have that

$$H_\varphi(fg) = \langle fg, \varphi \rangle = \varrho(a,b). \tag{2}$$

Here, and throughout the rest of the paper, $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathscr{H}^2$.

Now, from (2) it is clear that the multiplicative Hankel form (1) is bounded on $\ell^2 \times \ell^2$, or equivalently H_φ on $\mathscr{H}^2 \times \mathscr{H}^2$, if and only if φ induces a bounded linear functional on $\mathscr{H}^2 \odot \mathscr{H}^2$ through the $\mathscr{H}^2$-pairing, i.e. if and only if φ is in $\left(\mathscr{H}^2 \odot \mathscr{H}^2\right)^*$.

The first bona fide example of a multiplicative Hankel form was obtained in [11], by the following approach. Note first that the elements of $\mathscr{H}^2$ are analytic functions in the half-plane $\operatorname{Re} s > 1/2$, the reproducing kernel at each such s being given by $\zeta(w + \overline{s})$, where $\zeta(s) = \sum_{n \geq 1} n^{-s}$ is the Riemann zeta function. It is thus natural to consider the Carleman-type operator

$$\mathbf{H} f(s) = \int_{1/2}^{\infty} f(w) \left(\zeta(s+w) - 1\right) \, dw$$

acting on $\mathscr{H}_0^2$, since $(\zeta(s+w)-1)$ is the reproducing kernel of $\mathscr{H}_0^2$ at w, for $w > 1/2$. The matrix of the operator $\mathbf{H}$ is that of the multiplicative Hankel form whose analytic symbol φ is the primitive of $(\zeta(s+1/2)-1)$ in $\mathscr{H}_0^2$. In [11] it was shown that the operator norm of $\mathbf{H}$ on $\mathscr{H}_0^2$ is π, which in terms of its corresponding Hankel form means precisely that $|\langle fg, \varphi\rangle| \le \pi \|f\|_{\mathscr{H}^2}\|g\|_{\mathscr{H}^2}$ for $f, g \in \mathscr{H}_0^2$. More explicitly written,

$$\Big|\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\sqrt{mn}\log(mn)}\Big| \le \pi\Big(\sum_{m=2}^{\infty}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=2}^{\infty}|b_n|^2\Big)^{\frac{1}{2}}. \tag{3}$$

As explained more thoroughly in [11], inequality (3) is a multiplicative analogue of the classical Hilbert inequality

$$\Big|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\Big| \le \pi\Big(\sum_{m=1}^{\infty}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty}|b_n|^2\Big)^{\frac{1}{2}}. \tag{4}$$

There are several other versions of (4) which are also usually referred to as Hilbert's inequality — we direct the interested reader to [19, Ch. IX]. Let us extract a few facts. First, that by discretization of the continuous version of (4) and the Hermite–Hadamard inequality, the following improvement of (4) can be obtained.

$$\Big|\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}\Big| \le \pi\Big(\sum_{m=0}^{\infty}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}|b_n|^2\Big)^{\frac{1}{2}}.$$

We mention without proof that the same procedure (with additional straightforward estimates) yields in the multiplicative setting that

$$\Big|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\sqrt{(m+1/2)(n+1/2)}\log((m+1/2)(n+1/2))}\Big| \le \pi\|a\|_{\ell^2}\|b\|_{\ell^2},$$

which of course no longer represents a multiplicative Hankel form.

The strongest version of Hilbert's inequality (4) is

$$\Big|\sum_{\substack{m,n\ge 0\\ m+n>0}}\frac{a_m b_n}{m+n}\Big| \le \pi\Big(\sum_{m=0}^{\infty}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{\infty}|b_n|^2\Big)^{\frac{1}{2}}. \tag{5}$$

The last variant can also be stated for two-tailed sequences $\{a_m\}_{m\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$. The proof of (5) amounts to a concrete application of Nehari's theorem on $H^2(\mathbb{T})$, since the associated Hankel form has the bounded symbol Φ of supremum norm π,

$$\Phi(z) = -i\,\mathrm{Arg}(z) = i(\pi-\theta), \qquad z = e^{i\theta}. \tag{6}$$

As far as the authors are aware, all proofs of (5) in the literature make use of (a reformulation of) (6).

Whether the multiplicative Hankel form (3) has a bounded symbol is an open problem that is related to a long standing embedding problem of $\mathscr{H}^1$ (see [11, Sec. 6]). It therefore natural to ask if we also have

$$|\langle fg, \varphi \rangle| \leq \pi \|f\|_{\mathscr{H}^2} \|g\|_{\mathscr{H}^2}, \qquad f, g \in \mathscr{H}^2? \tag{7}$$

In light of the discussion above, this question actually turns out to be more subtle than what one might expect at first. We are unable to settle it, seemingly due to the lack of a Nehari theorem for multiplicative Hankel forms.

That (7) is significantly easier to settle for $\mathscr{H}_0^2$ than for $\mathscr{H}^2$ is not a peculiarity, but rather an ongoing theme for all the questions we will ask about product spaces in this paper. Note that inequality (3) is easily recast as a question about (the dual space of) $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$. More generally, elements of $\left(\mathscr{H}_0^2 \odot \mathscr{H}_0^2\right)^*$ correspond to multiplicative Hankel forms of the type (1), but with sums starting at $m, n = 2$. The remainder of this section is an overview of the problems that we will consider.

In Section 2, we investigate the difference between Hankel forms on $\mathscr{H}^2 \times \mathscr{H}^2$ and Hankel forms on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$. After some preliminaries, we obtain the main result of this section, Theorem 1, which allows us to embed any bounded operator $C\colon \ell^2 \to \ell^2$ into a Hankel form on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$. This result is the basis for our observation that $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ is significantly smaller than $\mathscr{H}^2 \odot \mathscr{H}^2$, and it is also an important tool in the proofs of our other main results.

Helson [21] proved that any Hankel form on $\mathscr{H}^2 \times \mathscr{H}^2$ which is of Hilbert-Schmidt class S_2 is induced by a bounded symbol on the infinite polytorus $\mathbb{T}^\infty$. In [10] it was shown that if $p > p_0 \approx 5.74$, then there is a Hankel form on $\mathscr{H}^2 \times \mathscr{H}^2$ of Schatten class S_p that does not have any bounded symbol, leading to the conjecture that the same might be true for all $p > 2$. In Theorem 6 we will prove that $p = 2$ is indeed critical in this sense for multiplicative Hankel forms acting on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$, leading us closer to optimality of Helson's result. In fact, for $p > 2$ we will even demonstrate the existence of forms in S_p that do not have square-integrable symbols on the polytorus.

The penultimate section is devoted to the study of the skew product space $\partial^{-1}\left(\mathscr{H}^2 \odot \partial\mathscr{H}^2\right)$. The motivation to study this space is twofold. Firstly, characterizations of the dual spaces of skew products are often significantly easier to obtain (see [1, 2]). Secondly, for the classical Hardy space H^2, the comparison between $H^2 \odot H^2$ and $\partial^{-1}\left(H^2 \odot \partial H^2\right)$ leads naturally to a Schur multiplier problem for Hankel matrices. Much has been written about this problem, owing to the fact that it was closely related to Pisier's construction of a polynomially bounded operator not similar to a contraction. We refer the reader to [9, 13, 16, 26].

We begin Section 3 by proving a square function characterization of $\mathscr{H}^p$, which is of independent interest for the study of Hardy spaces of Dirichlet series. Due to the notation involved, we defer a precise statement to Theorem 8. We first use this characterization to prove that

$$\mathscr{H}^2 \odot \mathscr{H}^2 \subseteq \partial^{-1}\left(\partial\mathscr{H}^2 \odot \mathscr{H}^2\right) \subsetneq \mathscr{H}^1. \tag{8}$$

We then study whether the first inclusion in (8) is strict. This appears to be a difficult question, but by Schur multiplier methods we are able to demonstrate that this is the case if every appearance of $\mathscr{H}^2$ in (8) is replaced by $\mathscr{H}^2_0$.

Finally, in Section 4, we look at the material of Section 3 again, but with the Hardy spaces of the polydisc in mind. Noting that Dirichlet series differentiation gives rise to a rather unnatural differentiation operator on the polydisc, we prove instead a square function characterization of $H^p(\mathbb{T}^\infty)$ that is adapted to the radial differentiation operator

$$R = \sum_{j=1}^{\infty} z_j \partial_{z_j}. \tag{9}$$

This will allow us to conclude that on finite-dimensional tori, it holds that

$$H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d) = R^{-1}\left(H^2(\mathbb{T}^d) \odot RH^2(\mathbb{T}^d)\right) = H^1(\mathbb{T}^d).$$

It also turns out that radial differentiation has a number theoretic interpretation when considered from the Dirichlet series point of view, something that too will be elaborated upon in Section 4.

Notation. As usual, $\{p_j\}_{j\geq 1}$ denotes the sequence of prime numbers in increasing order, and $\Omega(n)$ will denote the number of prime factors in n, counting multiplicities. We will write $f \ll g$ to indicate that there is some positive constant C so that $|f(x)| \leq C|g(x)|$. If both $f \ll g$ and $g \ll f$, we write $f \asymp g$.

When we speak of a Dirichlet series φ as an element of a dual space $\mathscr{K}^*$, where $\mathscr{K}$ is a Banach space of Dirichlet series in which the space of Dirichlet polynomials $\mathscr{P}$ is dense, we always mean that the functional induced by φ via the $\mathscr{H}^2$-pairing is bounded. That is, $\varphi \in \mathscr{K}^*$ if and only if the functional

$$\upsilon_\varphi(f) = \langle f, \varphi\rangle, \qquad f \in \mathscr{P},$$

extends to a bounded functional on $\mathscr{K}$. Similarly, when we write that $\mathscr{K}^* \subseteq \mathscr{X}$, where $\mathscr{X}$ is a Banach space of Dirichlet series, we mean that for every functional $\upsilon \in \mathscr{K}^*$ there exists a $\varphi \in \mathscr{X}$ such that $\upsilon = \upsilon_\varphi$ (on $\mathscr{P}$) and $\|\varphi\|_{\mathscr{X}} \ll \|\upsilon\|_{\mathscr{K}^*}$.

2. Hankel forms and a matrix embedding

Much of the success in the theory of Hardy spaces of Dirichlet series is due to a simple observation of Bohr [7], which facilitates a link between Dirichlet series and function theory in polydiscs. By identifying each prime number with an independent variable, $z_j = p_j^{-s}$, the Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is lifted to a function in the Hardy space of the countably infinite torus, $H^2(\mathbb{T}^\infty)$. More precisely, the prime factorization

$$n = \prod_{j=1}^{\infty} p_j^{\kappa_j}$$

associates to n the finite non-negative multi-index $\kappa(n) = (\kappa_1, \kappa_2, \kappa_3, \ldots)$. This means that the Bohr lift of f is

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)},$$

where $z = (z_1, z_2, z_3, \ldots)$. The mapping $\mathscr{B}\colon \mathscr{H}^2 \to H^2(\mathbb{T}^\infty)$ is an isometric isomorphism that respects multiplication. $\mathbb{T}^\infty$ is a compact abelian group, and its Haar measure is denoted by m_∞. The measure m_∞ is equal to the product of the normalized Lebesgue measure on $\mathbb{T}$ in each variable. In particular, $H^2(\mathbb{T}^d)$ is a natural subspace of $H^2(\mathbb{T}^\infty)$. We refer to [20, 27] for further properties of $H^2(\mathbb{T}^\infty)$.

In [4], Bayart introduced the spaces $\mathscr{H}^p$, for $1 \leq p < \infty$, as those Dirichlet series f such that $\mathscr{B}f \in H^p(\mathbb{T}^\infty)$, and we define the $\mathscr{H}^p$-norm as

$$\|f\|_{\mathscr{H}^p} = \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f(z)|^p \, dm_\infty(z) \right)^{\frac{1}{p}}.$$

As above, $H^p(\mathbb{T}^d)$ is a natural subspace of $H^p(\mathbb{T}^\infty) \simeq \mathscr{H}^p$.

Returning to the multiplicative Hankel form H_φ defined in (2), the fact that $\mathscr{B}$ respects multiplication implies that

$$H_\varphi(fg) = \langle \mathscr{B}f\mathscr{B}g, \, \mathscr{B}\varphi \rangle_{H^2(\mathbb{T}^\infty)}.$$

From this representation, it is clear that we may replace $\mathscr{B}\varphi$ with any $\psi \in L^2(\mathbb{T}^\infty)$ such that $P\psi = \mathscr{B}\varphi$, where P denotes the Riesz projection from $L^2(\mathbb{T}^\infty)$ to $H^2(\mathbb{T}^\infty)$. In this case, we also denote the Hankel form H_φ by H_ψ. If $\psi \in L^\infty(\mathbb{T}^\infty)$, then $\|H_\varphi\| \leq \|\psi\|_\infty$, where $\|H_\varphi\|$ denotes the norm of H_φ acting on $\mathscr{H}^2 \times \mathscr{H}^2$, and we say that H_φ has bounded symbol ψ. Note that if the functional

$$f \mapsto \langle f, \varphi \rangle, \qquad f \in \mathscr{H}^1,$$

is bounded on $\mathscr{H}^1 \simeq H^1(\mathbb{T}^\infty) \subset L^1(\mathbb{T}^\infty)$, then H_φ has a bounded symbol by the Hahn-Banach theorem. Hence, H_φ has a bounded symbol if and only if $\varphi \in (\mathscr{H}^1)^*$.

The main result of [25] implies that there exist bounded multiplicative Hankel forms that do not have a bounded symbol. It should be pointed out that the proof is non-constructive, and no example of a bounded multiplicative Hankel form without a bounded symbol has been identified. On the other hand, if $d = 1$ then Nehari's theorem [24] states that every bounded Hankel form H_φ

on $H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d)$ has a bounded symbol $\psi \in L^\infty(\mathbb{T}^d)$. Nehari's theorem has been extended to $d < \infty$ by Ferguson–Lacey [17] and Lacey–Terwilleger [23].

The matrix of the multiplicative Hankel form (2) is

$$M_\varrho = \big(\varrho_{mn}\big)_{m,n\geq 1} = \begin{pmatrix} \varrho_1 & \varrho_2 & \varrho_3 & \cdots \\ \varrho_2 & \varrho_4 & \varrho_6 & \cdots \\ \varrho_3 & \varrho_6 & \varrho_9 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{10}$$

By isolating the first row and column in M_ϱ using the inner product representation of H_φ from (2), we obtain

$$H_\varphi(fg) = a_1 b_1 \varrho_1 + a_1 \langle g - b_1, \varphi\rangle + b_1 \langle f - a_1, \varphi\rangle + H_\varphi\big((f-a_1)(g-b_1)\big). \tag{11}$$

The left hand side is a bounded Hankel form if and only if $\varphi \in \big(\mathscr{H}^2 \odot \mathscr{H}^2\big)^*$, while the right hand side is bounded if and only if $\varphi \in \big(\mathscr{H}^2_0\big)^* = \mathscr{H}^2/\mathbb{C}$ and $\varphi \in \big(\mathscr{H}^2_0 \odot \mathscr{H}^2_0\big)^*$. While it is obvious that

$$\big(\mathscr{H}^2 \odot \mathscr{H}^2\big)^* \subseteq \mathscr{H}^2, \tag{12}$$

we shall now see that the corresponding statement for $\mathscr{H}^2_0$ is not true. This will follow immediately from our next result, which also is crucial in establishing the other main results of the paper.

Theorem 1 (Matrix embedding)**.** *Let $C = (c_{j,k})_{j,k\geq 1}$ be an infinite matrix defining an operator on ℓ^2. Consider the Dirichlet series*

$$\varphi(s) = \sum_{j=1}^\infty \sum_{k=1}^\infty c_{j,k}\,(p_{2j-1}p_{2k})^{-s},$$

where $\{p_j\}_{j\geq 1}$ denotes the sequence of primes numbers in increasing order. Then

(a) $\|H_\varphi\|_0 = \|C\|$,

(b) $\|H_\varphi\| \asymp \|C\|_{S_2} = \|\varphi\|_{\mathscr{H}^2}$,

where $\|H_\varphi\|_0$ denotes the norm of H_φ acting on $\mathscr{H}^2_0 \times \mathscr{H}^2_0$, and $\|C\|_{S_2}$ denotes the Hilbert–Schmidt matrix norm of C,

$$\|C\|_{S_2} = \left(\sum_{j=1}^\infty \sum_{k=1}^\infty |c_{j,k}|^2\right)^{\frac{1}{2}}.$$

Proof. Let $f, g \in \mathscr{H}^2_0$ with coefficients $\{a_j\}_{j\geq 1}$ and $\{b_k\}_{k\geq 1}$, respectively. Since there are no constant terms in $\mathscr{H}^2_0$ we have that

$$H_\varphi(fg) = \langle fg, \varphi\rangle = \sum_{j=1}^\infty \sum_{k=1}^\infty \big(a_{p_{2j-1}} b_{p_{2k}} + a_{p_{2k}} b_{p_{2j-1}}\big) c_{j,k}. \tag{13}$$

Note that for every prime p, a_p and b_p each only appear once in this sum. Let

$$\begin{aligned}\mathscr{K}_1 &= \operatorname{span}\{p_{2k-1}^{-s} \,:\, k \geq 1\},\\ \mathscr{K}_2 &= \operatorname{span}\{p_{2k}^{-s} \,:\, k \geq 1\},\\ \mathscr{K}_3 &= \mathscr{H}_0^2 \ominus (\mathscr{K}_1 \oplus \mathscr{K}_2)\,,\end{aligned}$$

and let $P_{\mathscr{K}_j}$ denote the corresponding orthogonal projections. Let $\mathbf{a}_j$ and $\mathbf{b}_j$ denote the coefficient sequences, in the natural basis of $\mathscr{K}_j$, of $P_{\mathscr{K}_j} f$ and $P_{\mathscr{K}_j} g$, respectively. Then we may rewrite (13) as

$$H_\varphi(fg) = \langle C\mathbf{b}_2, \mathbf{a}_1\rangle_{\ell^2} + \langle C\mathbf{a}_2, \mathbf{b}_1\rangle_{\ell^2} = \langle \mathcal{J}(C^T \oplus C)(\mathbf{a}_1, \mathbf{a}_2), (\mathbf{b}_1, \mathbf{b}_2)\rangle_{\ell^2 \oplus \ell^2},$$

where $\mathcal{J}$ is the involution on $\ell^2 \oplus \ell^2$ defined by $\mathcal{J}(\mathbf{a}_1, \mathbf{a}_2) = (\mathbf{a}_2, \mathbf{a}_1)$. We conclude that

$$H_\varphi\big|_{\mathscr{H}_0^2} \simeq \mathcal{J}(C^T \oplus C) \oplus 0,$$

completing the proof of (a). For (b), we first observe that setting $g = 1$ implies $\|H_\varphi\| \geq \|\varphi\|_{(\mathscr{H}^2)^*} = \|\varphi\|_{\mathscr{H}^2} = \|C\|_{S_2}$. Returning to the decomposition (11) we see that $\|H_\varphi\| \leq 4\|C\|_{S_2}$, by using (a). □

As a corollary of Theorem 1, we obtain that a bounded Hankel form on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ does not necessarily have a symbol in $L^2(\mathbb{T}^\infty)$, in stark contrast with the classical situation where bounded Hankel forms have bounded symbols. We also find that (12) does not hold for $\mathscr{H}_0^2$.

Corollary 2. $\left(\mathscr{H}_0^2 \odot \mathscr{H}_0^2\right)^* \not\subseteq \mathscr{H}^2$. *That is, there are bounded multiplicative Hankel forms H_φ on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ with the property that there is no $\psi \in L^2(\mathbb{T}^\infty)$ such that $H_\varphi = H_\psi$.*

Proof. Use Theorem 1 and let C be the matrix of the identity operator on ℓ^2. □

Actually, we have the following stronger version of Corollary 2, which can be proven by considering all diagonal operators C on ℓ^2 and using Theorem 1. It exemplifies concretely that $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ is in some ways significantly smaller than $\mathscr{H}^2 \odot \mathscr{H}^2$.

Corollary 3. *The Dirichlet series*

$$f(s) = \sum_{k=1}^{\infty} a_k (p_{2k-1}p_{2k})^{-s}$$

is in $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ if and only if $a \in \ell^1$, while it is in $\mathscr{H}^2 \odot \mathscr{H}^2$ if and only if $a \in \ell^2$.

Recall that $H^2(\mathbb{T}^d)$ is a natural subspace of $H^2(\mathbb{T}^\infty)$ and that if $f \in \mathscr{H}_0^2$, then $\mathscr{B}f(0) = 0$. We now observe that the inclusions behave as expected for the corresponding finite-dimensional subspaces of the weak product spaces.

Lemma 4. *For* $1 \le d < \infty$, *let* $H_0^2(\mathbb{T}^d)$ *denote the space of functions* $F \in H^2(\mathbb{T}^d)$ *for which* $F(0,0,\ldots,0) = 0$. *Then*

$$\big(H_0^2(\mathbb{T}^d) \odot H_0^2(\mathbb{T}^d)\big)^* \subseteq H_0^2(\mathbb{T}^d) \ominus \operatorname{Lin}(\mathbb{T}^d) \subseteq H_0^2(\mathbb{T}^d),$$

where $\operatorname{Lin}(\mathbb{T}^d)$ *denotes the subspace of* $H_0^2(\mathbb{T}^d)$ *consisting of linear functions,*

$$\operatorname{Lin}(\mathbb{T}^d) = \Big\{L(z) = \sum_{j=1}^d a_j z_j \,:\, a_j \in \mathbb{C}\Big\}.$$

Proof. It is sufficient to show that

$$H_0^2(\mathbb{T}^d) \ominus \operatorname{Lin}(\mathbb{T}^d) \subseteq H_0^2(\mathbb{T}^d) \odot H_0^2(\mathbb{T}^d),$$

since it follows that any functional in $\big(H_0^2(\mathbb{T}^d) \odot H_0^2(\mathbb{T}^d)\big)^*$ must be represented by a unique element of $H_0^2(\mathbb{T}^d) \ominus \operatorname{Lin}(\mathbb{T}^d)$. Every $F \in H_0^2(\mathbb{T}^d) \ominus \operatorname{Lin}(\mathbb{T}^d)$ can be written

$$F(z) = \sum_{j=1}^d z_j F_j(z),$$

where $F_j \in H_0^2(\mathbb{T}^d)$. This representation of F is not unique, but we can always organize it so that $\sum_j \|F_j\|_{H^2(\mathbb{T}^d)}^2 = \|F\|_{H^2(\mathbb{T}^d)}^2$. By the computation

$$\|F\|_{H_0^2 \odot H_0^2} \le \sum_{j=1}^d 1 \cdot \|F_j\|_{H^2(\mathbb{T}^d)} \le \sqrt{d}\Big(\sum_{j=1}^d \|F_j\|_{H^2(\mathbb{T}^d)}^2\Big)^{\frac{1}{2}} = \sqrt{d}\|F\|_{H^2(\mathbb{T}^d)},$$

we see that $F \in H_0^2 \odot H_0^2$. □

It is clear that the final part of this argument breaks down for $d = \infty$; the key point being that the subspace $\operatorname{Lin}(\mathbb{T}^\infty)$ of linear functions in $H_0^2(\mathbb{T}^\infty) \simeq \mathscr{H}_0^2$ is infinite-dimensional, which from the Dirichlet series point of view corresponds to the fact that there are infinitely many prime numbers. Even so, Corollary 2 is surprising. We stress that its conclusion is related to the additional arithmetical obstructions which appear when computing the norm of an element in $\mathscr{H}_0^2 \odot \mathscr{H}_0^2$ rather than in $\mathscr{H}^2 \odot \mathscr{H}^2$. The following result is intended to clarify this statement. In particular, it demonstrates that the subspace of linear functions actually is complemented in $\mathscr{H}^2 \odot \mathscr{H}^2$.

Theorem 5. *For a non-negative integer* m, *let* P_m *denote the projection on* m*-homogeneous Dirichlet series,*

$$P_m \sum_{n=1}^\infty a_n n^{-s} = \sum_{\Omega(n)=m} a_n n^{-s}.$$

Then P_m *is a contraction on* $\mathscr{H}^2 \odot \mathscr{H}^2$.

Proof. The case $m = 0$ is trivial. Let $m \geq 1$ and suppose that

$$F = \sum_k f_k g_k \tag{14}$$

is a finite sum. Then

$$P_m F(s) = \sum_k \sum_{j=0}^{m} P_j f_k(s) P_{m-j} g_k(s).$$

By applying the definition of the norm of $\mathscr{H}^2 \odot \mathscr{H}^2$ and the Cauchy–Schwarz inequality, we find that

$$\begin{aligned} \|P_m F\|_{\mathscr{H}^2 \odot \mathscr{H}^2} &\leq \sum_k \sum_{j=0}^{m} \|P_j f_k\|_{\mathscr{H}^2} \|P_{m-j} g_k\|_{\mathscr{H}^2} \\ &\leq \sum_k \Big(\sum_{j=0}^{m} \|P_j f_k\|_{\mathscr{H}^2}^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=0}^{m} \|P_{m-j} g_k\|_{\mathscr{H}^2}^2 \Big)^{\frac{1}{2}} \\ &\leq \sum_k \|f_k\|_{\mathscr{H}^2} \|g_k\|_{\mathscr{H}^2}, \end{aligned}$$

the final inequality following from the fact that

$$\sum_{j=0}^{\infty} \|P_j f\|_{\mathscr{H}^2}^2 = \|f\|_{\mathscr{H}^2}^2, \qquad f \in \mathscr{H}^2.$$

The proof is completed by taking the infimum over the representations (14). $\square$

We return to the matrix of H_φ acting on $\mathscr{H}^2 \times \mathscr{H}^2$ from (10). The matrix M_ρ^0 corresponding to the action of H_φ on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ is obtained from M_ρ by deleting the first row and column. That is, $M_\rho^0 = (\rho_{mn})_{m,n\geq 2}$ in view of (10).

Now, suppose that H_φ is a compact form, i.e. that its matrix M defines a compact operator on ℓ^2. Let

$$\Lambda = \{\lambda_1, \lambda_2, \ldots\}$$

denote the singular value sequence of M. We say that H_φ is in the Schatten class S_p, $0 < p \leq \infty$, if $\Lambda \in \ell^p$, and we let $\|H_\varphi\|_{S_p} = \|\Lambda\|_{\ell^p}$. When speaking of a Hankel form H_φ we will write $S_p(\mathscr{H}^2)$ or $S_p(\mathscr{H}_0^2)$ to clarify which space is being considered; using Theorem 1 as in Corollary 3, it is easy to construct Hankel forms belonging to the latter Schatten class, but not to the former.

Helson [21] showed that if $H_\varphi \in S_p(\mathscr{H}^2)$ and $p = 2$, then H_φ has a bounded symbol. In [10], the authors showed that this is no longer the case when

$$p > p_0 \approx 5.738817179.$$

We will now investigate symbols for forms $H_\varphi \in S_p(\mathscr{H}_0^2)$. We start by verifying that Helson's result still holds for $S_2(\mathscr{H}_0^2)$.

As in Lemma 4, any bounded Hankel form on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ has a symbol φ in $(\mathscr{H}_0^2 \odot \mathscr{H}_0^2)^*$ of the form

$$\varphi(s) = \sum_{\Omega(n)\geq 2} \varrho_n n^{-s}.$$

From this fact, a computation shows that

$$\begin{aligned}\|H_\varphi\|^2_{S_2(\mathscr{H}_0^2)} &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} |H_\varphi(m^{-s}n^{-s})|^2 \\ &= \sum_{\Omega(n)\geq 2} (d(n)-2)\,|\varrho_n|^2 \asymp \sum_{\Omega(n)\geq 2} d(n)|\varrho_n|^2.\end{aligned}$$

Here $d(n)$ denotes the number of divisors of n, and the final estimate follows from the fact that $d(n)-2 \geq d(n)/3$ for n such that $\Omega(n) \geq 2$, seeing as $d(n) \geq \Omega(n)+1$. Hence we can use Helson's inequality

$$\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)}\right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathscr{H}^1}$$

to conclude that $\varphi \in (\mathscr{H}^1)^*$ whenever $H_\varphi \in S_2(\mathscr{H}_0^2)$. That is, H_φ has a bounded symbol whenever $H_\varphi \in S_2(\mathscr{H}_0^2)$. We now show that Helson's result is optimal for $S_p(\mathscr{H}_0^2)$.

Theorem 6. *For $p > 2$ there exist Hankel forms $H_\varphi \in S_p(\mathscr{H}_0^2)$ such that no ψ in $L^2(\mathbb{T}^\infty)$ satisfies $H_\varphi = H_\psi$. In particular, there exist Hankel forms $H_\varphi \in S_p(\mathscr{H}_0^2)$ for which there are no bounded symbols.*

Proof. Let $C = (c_{j,k})_{j,k\geq 1}$ be a matrix defining an operator on ℓ^2 which belongs to S_p but not to S_2. In accordance with Theorem 1 let

$$\varphi(s) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k}(p_{2j-1}p_{2k})^{-s}.$$

Since, as in the proof of Theorem 1, $H_\varphi\big|_{\mathscr{H}_0^2} \simeq \mathscr{J}\left(C^T \oplus C\right) \oplus 0$, we have that

$$\|H_\varphi\|^p_{S_p(\mathscr{H}_0^2)} = 2\|C\|^p_{S_p} < \infty.$$

On the other hand, we have by assumption that

$$\|\varphi\|_{\mathscr{H}^2} = \|C\|_{S_2} = \infty.$$ □

While Theorem 6 does not concern Hankel forms on $\mathscr{H}^2 \times \mathscr{H}^2$, we do consider it to give us an indication that $p = 2$ might be the critical value also in this case.

Conjecture 1. For every $p > 2$ there exists a multiplicative Hankel form H_φ in $S_p(\mathscr{H}^2)$ without a bounded symbol.

3. A square function characterization of $\mathscr{H}^p$ and skew products

In the context of the classical Hardy spaces, it was Bourgain [9] who recalled the square function characterization of H^p due to Fefferman and Stein [15] and used it to the effect of showing that $H^2 \odot \partial H^2 \subseteq \partial H^1$, where ∂H^p denotes the space consisting of the derivatives of all H^p-functions. In view of the fact that $\partial H^1 = \partial(H^2 \odot H^2) \subseteq H^2 \odot \partial H^2$ this immediately implies that

$$\partial^{-1}(H^2 \odot \partial H^2) = H^2 \odot H^2. \tag{15}$$

In terms of bilinear forms, we can naturally associate a Hankel-type form J_g to every element $g \in \big(\partial^{-1}(H^2 \odot \partial H^2)\big)^*$. If an additive Hankel form H_g on $H^2 \times H^2$ corresponds to the matrix $(\hat{g}(j+k))_{j,k\geq 0}$, then J_g has matrix

$$\left(\frac{j+1}{j+k+1}\hat{g}(j+k)\right)_{j,k\geq 0}.$$

Hence Bourgain's lemma (15) can be equivalently rephrased to say that the map $H_g \mapsto J_g$ is bounded in operator norm. This statement actually carries greater interest than what its face value might suggest. The matrix

$$\left(\frac{j+1}{j+k+1}\right)_{j,k\geq 0}$$

is not a bounded Schur multiplier on all matrices, and hence the map $H_g \mapsto J_g$ is not completely bounded [13]. This observation is at the heart of Pisier's [26] construction of a polynomially bounded operator not similar to a contraction.

We define the skew product space $\partial^{-1}(\mathscr{H}^2 \odot \partial \mathscr{H}^2)$ as the Banach space completion of the space of Dirichlet series F whose derivatives have a finite sum representation $F' = \sum_k f_k g_k'$, where $f_k, g_k \in \mathscr{H}^2$. The completion is taken under the norm

$$\|F\|_{\partial^{-1}(\mathscr{H}^2 \odot \partial \mathscr{H}^2)} = |F(+\infty)| + \inf \sum_k \|f_k\|_{\mathscr{H}^2}\|g_k\|_{\mathscr{H}^2},$$

where the infimum is computed over all finite representations. From the product rule $(fg)' = f'g + fg'$ it is clear that

$$\mathscr{H}^2 \odot \mathscr{H}^2 \subseteq \partial^{-1}(\mathscr{H}^2 \odot \partial \mathscr{H}^2). \tag{16}$$

Our first goal is to establish a square function characterization of $\mathscr{H}^p$, for $0 < p < \infty$, and use it to show that $\partial^{-1}(\mathscr{H}^2 \odot \partial \mathscr{H}^2) \subseteq \mathscr{H}^1$. We begin by recalling that the spaces $\mathscr{H}^p$ are related to the Möbius invariant Hardy spaces in the right half-plane, $\mathbb{C}_0$, defined as

$$H^p_{\mathrm{i}}(\mathbb{C}_0) = \left\{ f \in \mathrm{Hol}(\mathbb{C}_0) \,:\, \|f\|_{H^p_{\mathrm{i}}(\mathbb{C}_0)} = \sup_{\sigma>0}\left(\frac{1}{\pi}\int_{\mathbb{R}} |f(\sigma+it)|^p \frac{dt}{1+t^2}\right)^{\frac{1}{p}} < \infty \right\}.$$

Given a character $\chi \in \mathbb{T}^\infty$, we "twist" the Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ to obtain

$$f_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}, \qquad \chi(n) = \chi^{\kappa(n)}.$$

We will require the following basic result, which can be extracted from Lemma 5 and Theorem 5 in [4].

Lemma 7. *Let $0 < p < \infty$, and suppose that $f \in \mathscr{H}^p$. For almost every $\chi \in \mathbb{T}^\infty$, $f_\chi \in H^p_{\mathrm{i}}(\mathbb{C}_0)$. Moreover,*

$$\|f\|_{\mathscr{H}^p} = \left(\int_{\mathbb{T}^\infty} \|f_\chi\|^p_{H^p_{\mathrm{i}}(\mathbb{C}_0)} \, dm_\infty(\chi) \right)^{\frac{1}{p}}.$$

Remark. The results in [4] are stated only for $p \geq 1$, but the same arguments lead to our statement of Lemma 7.

For $\tau \in \mathbb{R}$, let Γ_τ be the cone

$$\Gamma_\tau = \{\sigma + it \,:\, |t - \tau| < \sigma\}$$

in the right half-plane $\mathbb{C}_0$, with vertex at $i\tau$. For a holomorphic function f in $\mathbb{C}_0$, let Sf be the square function, or the Lusin area integral,

$$Sf(\tau) = \left(\int_{\Gamma_\tau} |f'(\sigma + it)|^2 \, d\sigma \, dt \right)^{1/2}, \qquad \tau \in \mathbb{R},$$

and let f^* denote the non-tangential maximal function

$$f^*(\tau) = \sup_{s \in \Gamma_\tau} |f(s)|, \qquad \tau \in \mathbb{R}.$$

Since $1/(1+\tau^2)$ is a Muckenhoupt A_q-weight for all $q > 1$, it follows from Gundy and Wheeden [18] that $f \in H^p_{\mathrm{i}}(\mathbb{C}_0)$ if and only if

$$f^* \in L^p_{\mathrm{i}}(\mathbb{R}) = L^p \left((1+\tau^2)^{-1} \, d\tau \right),$$

for $0 < p < \infty$, with comparable norms. Furthermore, if $\lim_{\sigma\to\infty} f(\sigma + it) = 0$, then

$$\|f^*\|_{L^p_{\mathrm{i}}(\mathbb{R})} \asymp \|Sf\|_{L^p_{\mathrm{i}}(\mathbb{R})}. \tag{17}$$

This gives us a norm expression for functions in $\mathscr{H}^p$ in terms of the square function.

Theorem 8. *Let $f(s) = \sum_{n\geq 1} a_n n^{-s}$. Then for any $0 < p < \infty$, we have*

$$\begin{aligned} \|f\|^p_{\mathscr{H}^p} &\asymp |a_1|^p + \int_{\mathbb{T}^\infty} \|S(f_\chi)\|^p_{L^p_{\mathrm{i}}(\mathbb{R})} \, dm_\infty(\chi) \\ &= |a_1|^p + \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} \left(\int_{\Gamma_\tau} |f'_\chi(\sigma + it)|^2 \, d\sigma \, dt \right)^{p/2} \frac{d\tau}{1+\tau^2} \, dm_\infty(\chi). \end{aligned} \tag{18}$$

Proof. In view of (17) and Lemma 7 we obtain (18) for f with constant term $a_1 = 0$, that is, for $f \in \mathscr{H}^p_0$. Note that the linear functional $f \mapsto a_1$ is bounded on $\mathscr{H}^p$, corresponding to the functional $\mathscr{B}f \mapsto \mathscr{B}f(0)$ on $H^p(\mathbb{T}^\infty)$ [12]. Hence, the closed subspace $\mathscr{H}^p_0$ is complemented in $\mathscr{H}^p$ by $\mathbb{C}$, and (18) follows in general for $f \in \mathscr{H}^p$, with one side being finite if and only if the other is. $\square$

Corollary 9. $\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2) \subseteq \mathscr{H}^1$.

Proof. Suppose that $f, g \in \mathscr{H}^2$, and that F is the Dirichlet series such that $F' = fg'$ with $F(+\infty) = 0$. Since $\|g - g(+\infty)\|_{\mathscr{H}^2} \leq \|g\|_{\mathscr{H}^2}$ it is for the purpose of proving the statement justified to assume that $g(+\infty) = 0$. We then have that

$$\begin{aligned}
\|F\|_{\mathscr{H}^1} &\asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\left(\int_{\Gamma_\tau}|f_\chi(\sigma+it)|^2|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\right)^{1/2}\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi)\\
&\leq \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}(f_\chi)^*(\tau)\left(\int_{\Gamma_\tau}|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\right)^{1/2}\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi)\\
&\leq \int_{\mathbb{T}^\infty}\|(f_\chi)^*\|_{L^2_{\mathrm{i}}(\mathbb{R})}\left(\int_{\mathbb{R}}\int_{\Gamma_\tau}|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\,\frac{d\tau}{1+\tau^2}\right)^{1/2}dm_\infty(\chi)\\
&\asymp \int_{\mathbb{T}^\infty}\|f_\chi\|_{H^2_{\mathrm{i}}(\mathbb{C}_0)}\|g_\chi\|_{H^2_{\mathrm{i}}(\mathbb{C}_0)}\,dm_\infty(\chi) \leq \|f\|_{\mathscr{H}^2}\|g\|_{\mathscr{H}^2}.
\end{aligned}$$

This proves that $\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2) \subseteq \mathscr{H}^1$. $\square$

Before proceeding, we give a few remarks on the application of Theorem 8 to the Hardy space $H^p(\mathbb{T}^d)$ of a finite-dimensional polydisc, $d < \infty$. Let D denote the differentiation operator on Dirichlet series,

$$Df(s) = \partial f(s) = f'(s) = -\sum_{n=2}^{\infty} a_n \log(n) n^{-s}.$$

Consider a series f such that $\mathscr{B}f \in H^2(\mathbb{T}^d)$, i.e. such that $a_n = 0$ if $p_j | n$ for some $j > d$. Identifying p_j with the jth complex variable z_j, the differentiation operator D in the usual polydisc notation has the form

$$D\mathscr{B}f(z_1, \ldots, z_d) = -\sum_{j=1}^{d} \log(p_j) z_j \partial_{z_j} \mathscr{B}f(z_1, \ldots, z_d). \tag{19}$$

Hence Theorem 8 gives us a new type of square function characterization of $H^p(\mathbb{T}^d)$, in terms of the differentiation operator D. In analogy with Corollary 9 it can be used to prove that

$$D^{-1}\left(H^2(\mathbb{T}^d) \odot DH^2(\mathbb{T}^d)\right) \subseteq H^1(\mathbb{T}^d)$$

and by the characterization of $H^1(\mathbb{T}^d)$ due to Ferguson–Lacey [17] and Lacey–Terwilleger [23] we conclude that in the finite polydisc we have

$$D^{-1}\left(H^2(\mathbb{T}^d)\odot DH^2(\mathbb{T}^d)\right)=H^2(\mathbb{T}^d)\odot H^2(\mathbb{T}^d)=H^1(\mathbb{T}^d). \tag{20}$$

It should be objected, however, that the weighted differentiation operator D might not be natural in the setting of the polydisc. In Section 4 we shall consider the constructs of the present section for the infinite polydisc, using the radial differentiation operator instead of D, and in the process prove that (20) is valid also for radial differentiation and integration.

We return to the discussion of products of Dirichlet series spaces, and note that Corollary 9 in combination with (16) yields that

$$\mathscr{H}^2\odot\mathscr{H}^2\subseteq\partial^{-1}\left(\partial\mathscr{H}^2\odot\mathscr{H}^2\right)\subseteq\mathscr{H}^1. \tag{21}$$

The remainder of this section is devoted to the investigation of whether these inclusions are strict. We begin with the following observation.

Lemma 10. *Let $\varphi(s)=\sum_{k\geq 1}\overline{\rho_k}k^{-s}$ be a function in $\mathscr{H}^2$. Then φ induces a bounded linear functional υ_φ on $\partial^{-1}(\mathscr{H}^2\odot\partial\mathscr{H}^2)$, via the $\mathscr{H}^2$-pairing, if and only if the form*

$$J_\varphi(a,b)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m b_n\frac{\log n}{\log m+\log n}\rho_{mn} \tag{22}$$

is bounded on $\ell^2\times\ell^2$, where the summand is understood to be 0 if $m=n=1$. The corresponding norms are equivalent,

$$\|\upsilon_\varphi\|\asymp|\rho_1|+\|J_\varphi\|.$$

In particular, if $\rho_k\geq 0$ for all k, then $\varphi\in\left(\partial^{-1}(\mathscr{H}^2\odot\partial\mathscr{H}^2)\right)^*$ if and only if $\varphi\in\left(\mathscr{H}^2\odot\mathscr{H}^2\right)^*$, with equivalent norms.

Proof. Suppose that f and g are Dirichlet series with coefficient sequences a and b, respectively. Let $\partial^{-1}(f'g)$ denote the primitive of $f'g$ with constant term 0. Then

$$\langle\partial^{-1}(f'g),\varphi\rangle=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m b_n\frac{\log n}{\log m+\log n}\rho_{mn},$$

proving the first part of the proposition. For the second part, note as per usual that the action of φ as an element in $\left(\mathscr{H}^2\odot\mathscr{H}^2\right)^*$ corresponds to the multiplicative Hankel form

$$H_\varphi(a,b)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m b_n\rho_{mn}. \tag{23}$$

Hence, if $\rho_k\geq 0$ for all k, then

$$\|\varphi\|_{(\partial^{-1}(\mathscr{H}^2\odot\partial\mathscr{H}^2))^*}\ll\|\varphi\|_{(\mathscr{H}^2\odot\mathscr{H}^2)^*}.$$

The converse inequality is a direct consequence of (16). □

Ortega-Cerdá and Seip [25] showed that $\mathscr{H}^2 \odot \mathscr{H}^2 \subsetneq \mathscr{H}^1$. With Lemma 10, we are able to apply their technique to prove the corresponding statement for $\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2)$.

Theorem 11. $\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2) \subsetneq \mathscr{H}^1$

Proof. Let d be a positive integer and consider the function

$$\varphi_d(s) = \prod_{j=1}^{d} \left(p_{2j-1}^{-s} + p_{2j}^{-s}\right),$$

where $\{p_j\}_{j\geq 1}$ again denotes the prime sequence. The norm of φ_d as an element of the dual of $\mathscr{H}^2 \odot \mathscr{H}^2$ is $2^{d/2}$ [25]. Since the coefficients of φ_d are non-negative, Lemma 10 hence shows that

$$\|\varphi_d\|_{(\partial^{-1}(\mathscr{H}^2\odot\partial\mathscr{H}^2))^*} \asymp 2^{d/2}.$$

On the other hand, consider $f_d = \varphi_d$ as an element of $\mathscr{H}^1$, $\|f_d\|_{\mathscr{H}^1} = (4/\pi)^d$ [25]. Since $\langle f_d, \varphi_d\rangle_2 = 2^d$, the functional induced by φ_d on $\mathscr{H}^1$ has norm at least $(\pi/2)^d$. If it were the case that $\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2) = \mathscr{H}^1$, then the norm of φ_d as a functional on $\mathscr{H}^1$ and the norm as a functional on $\partial^{-1}\left(\mathscr{H}^2 \odot \partial\mathscr{H}^2\right)$ would be equivalent, a contradiction as $d \to \infty$. □

The remaining question of whether

$$\left(\partial^{-1}(\mathscr{H}^2 \odot \partial\mathscr{H}^2)\right)^* = \left(\mathscr{H}^2 \odot \mathscr{H}^2\right)^* \tag{24}$$

or, equivalently, whether the first inclusion in (21) is strict, appears to be subtle. As we just saw in Lemma 10 it can be rephrased as to ask if the forms (22) and (23) are simultaneously bounded, which would mean precisely that

$$\left(\frac{\log n}{\log m + \log n}\right)_{m,n\geq 1}$$

is a Schur multiplier on the class of multiplicative Hankel forms. Specializing to the one-variable case by only considering integers of the form 2^k, we see that the analogue of (24) for the classical Hardy space $H^2(\mathbb{T})$ is equivalent to the statement that $(j+1)/(j+k+1)$ is a Schur multiplier on (additive) Hankel forms, as discussed in the introduction of this section.

However, by applying Theorem 1 in full force together with Schur multiplier techniques, we are able to show that the inclusion is strict when $\mathscr{H}^2$ is replaced by $\mathscr{H}_0^2$. We define $\partial^{-1}\left(\mathscr{H}_0^2 \odot \partial\mathscr{H}_0^2\right)$ in exact analogy with our previous considerations, except that we impose all of its elements f to have constant term $f(+\infty) = 0$.

Theorem 12. $\mathscr{H}_0^2 \odot \mathscr{H}_0^2 \subsetneq \partial^{-1}\left(\mathscr{H}_0^2 \odot \partial\mathscr{H}_0^2\right)$.

Proof. Assume to the contrary that

$$\left(\frac{\log n}{\log m + \log n}\right)_{m,n\geq 2}$$

is a Schur multiplier on bounded multiplicative Hankel forms

$$\rho(a,b) = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_m b_n \rho_{mn}, \qquad a, b \in \ell^2.$$

Applied to every symbol constructed by the procedure of Theorem 1, we conclude that

$$\left(\frac{\log p_{2j-1}}{\log p_{2k} + \log p_{2j-1}}\right)_{j,k\geq 1} \tag{25}$$

is a Schur multiplier on all matrices C defining bounded operators $C : \ell^2 \to \ell^2$. However, (25) cannot be a Schur multiplier, as this would defy Bennett's criterion [5], since

$$\lim_{j\to\infty}\lim_{k\to\infty} \frac{\log p_{2j-1}}{\log p_{2k} + \log p_{2j-1}} = 0,$$

while

$$\lim_{k\to\infty}\lim_{j\to\infty} \frac{\log p_{2j-1}}{\log p_{2k} + \log p_{2j-1}} = 1. \qquad \square$$

It must be stressed that Theorem 12 *does not* imply that the inclusion in (16) is strict. If we attempt to apply the proof to $\mathscr{H}^2 \odot \mathscr{H}^2$, the matrices constructed by Theorem 1 are Hilbert–Schmidt. To be a Schur multiplier on Hilbert–Schmidt matrices means only to have bounded entries, so no contradiction is obtained. However, we do feel that Theorem 12 invokes the natural conjecture.

Conjecture 2. The inclusion between the standard weak product and its skew counterpart is strict, $\mathscr{H}^2 \odot \mathscr{H}^2 \subsetneq \partial^{-1}\left(\mathscr{H}^2 \odot \partial\mathscr{H}^2\right)$.

4. Radial differentiation

From the polydisc point of view, the constructs of the last section all arose from the weighted differentiation operator D of (19), obtained from the Dirichlet series formalism. In the present section we shall consider instead the more natural radial differentiation operator of equation (9). Before commencing, note that as in Theorem 5 every Dirichlet series may be decomposed into m-homogeneous subseries,

$$f(s) = \sum_{n=0}^{\infty} a_n n^{-s} = \sum_{m=0}^{\infty}\Big(\sum_{\Omega(n)=m} a_n n^{-s}\Big) = \sum_{m=0}^{\infty} P_m f(s).$$

Through the Bohr lift, this is equivalent to the corresponding decomposition of a power series in a countably infinite number of variables,

$$F(z) = \sum_{n=0}^{\infty} a_n z^{\kappa(n)} = \sum_{m=0}^{\infty} \Big(\sum_{|\kappa(n)|=m} a_n z^{\kappa(n)} \Big) = \sum_{m=0}^{\infty} P_m F(z), \quad z = (z_1, z_2, \ldots).$$

We recall that $\kappa(n) = (\kappa_1, \kappa_2, \ldots)$ is the finitely supported multi-index associated to every positive integer n through its prime decomposition, so that

$$|\kappa(n)| = \Omega(n) = \sum_j \kappa_j.$$

Consider now, for any $z \in \mathbb{T}^\infty$, the following power series in one variable w.

$$F_z(w) = F(zw) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)} w^{\Omega(n)} = \sum_{m=0}^{\infty} P_m F(z) w^m.$$

Observe in particular that the mth coefficient of $F_z(w)$ is the m-homogeneous subseries of F. From here it is clear that differentiation in the auxiliary variable w allows us to capture the natural radial differentiation of the polydisc, since every monomial of order m is treated equally. This is further justified by the formal computation

$$w \frac{d}{dw} F_z(w) = w \sum_{j=1}^{\infty} z_j \partial_{z_j} F(wz) = (RF)_z(w).$$

We have the following analogue of Lemma 7. We also point out that through the Bohr lift a similar statement can be made for Dirichlet series.

Lemma 13. *Let $F \in H^p(\mathbb{T}^\infty)$, $0 < p < \infty$. Then $F_z \in H^p(\mathbb{T})$ for almost every $z \in \mathbb{T}^\infty$ and*

$$\|F\|_{H^p(\mathbb{T}^\infty)} = \left(\int_{\mathbb{T}^\infty} \|F_z\|^p_{H^p(\mathbb{T})} \, dm_\infty(z) \right)^{1/p}. \tag{26}$$

Proof. This follows from Fubini's theorem and the fact that $z \mapsto F(z)$ and $z \mapsto F(wz)$, for $w \in \mathbb{T}$, have equal $H^p(\mathbb{T}^\infty)$-norm. □

For $\theta \in [0, 2\pi)$, let $\Gamma_\alpha(\theta)$ denote the Stolz angle in $\mathbb{D}$ with vertex at $e^{i\theta}$ and of some fixed aperture $\alpha < \pi/2$. The (slightly non-standard) square function Sg of a function g holomorphic in $\mathbb{D}$ is given by

$$Sg(\theta) = \left(\int_{\Gamma_\alpha(\theta)} |wg'(w)|^2 \, dA(w) \right)^{1/2},$$

where dA denotes the normalized area element. If $g(0) = 0$ we have that $\|g\|_{H^p(\mathbb{T})} \asymp \|Sg\|_{L^p(\mathbb{T})}$. Since $F_z(0) = F(0)$ for every $z \in \mathbb{T}^\infty$, this immediately gives us the analogue of Theorem 8.

Theorem 14. *Let $F \in H^p(\mathbb{T}^\infty)$, $0 < p < \infty$. Then*

$$\|F\|_{H^p(\mathbb{T}^\infty)}^p \asymp |F(0)|^p + \int_{\mathbb{T}^\infty} \int_0^{2\pi} \left(\int_{\Gamma_\alpha(\theta)} |(RF)_z(w)|^2 \, dA(w) \right)^{p/2} \frac{d\theta}{2\pi} \, dm_\infty(z).$$

Now most of the arguments of the previous section can be repeated. We collect the results that follow without providing details. Note in particular the satisfying conclusion obtained for the finite-dimensional polydisc. Indeed, this result partly motivates the existence of this section.

Corollary 15. *We have that*

$$H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty) \subseteq R^{-1} \left(H^2(\mathbb{T}^\infty) \odot RH^2(\mathbb{T}^\infty) \right) \subsetneq H^1(\mathbb{T}^\infty).$$

On the other hand, when $d < \infty$ it holds that

$$H^2(\mathbb{T}^d) \odot H^2(\mathbb{T}^d) = R^{-1} \left(H^2(\mathbb{T}^d) \odot RH^2(\mathbb{T}^d) \right) = H^1(\mathbb{T}^d).$$

We remark that it is not clear how to obtain Corollary 15 directly from the considerations in Section 3, due to the weights $\log p_j$ entering into Dirichlet series differentiation. In fact, suppose that $n = \prod_j p_j^{\kappa_j}$. Then

$$\log n = \sum_j \kappa_j \log p_j \qquad \text{and} \qquad \Omega(n) = \sum_j \kappa_j,$$

illustrating the fact that the R treats every prime equally, while the half-plane differentiation operator D does not. In particular, the proof of Theorem 12 does not yield any information when D is replaced by R, since the Schur multiplier vital to the proof has entries

$$\frac{\Omega(n)}{\Omega(m) + \Omega(n)} = \frac{\Omega(p_{2j-1})}{\Omega(p_{2k}) + \Omega(p_{2j-1})} = \frac{1}{2}.$$

It should also be pointed out that decomposing Dirichlet series (or power series on the infinite polydisc) into homogeneous subseries is not a new idea. It dates back at least to Bohnenblust–Hille [6], and has recently been applied to obtain results for composition operators on spaces of Dirichlet series [3] as well as L^1-estimates for Dirichlet polynomials [8].

We conclude this paper by providing a charming inequality, which follows at once from Lemma 13 and the classical Hardy inequality

$$\sum_{m=0}^{\infty} \frac{|b_m|}{m+1} \leq \pi \left\| \sum_{m=0}^{\infty} b_m w^m \right\|_{H^1(\mathbb{T})}. \tag{27}$$

Corollary 16. *Let* $f(s) = \sum_{n\geq 1} a_n n^{-s} \in \mathscr{H}^1$ *and consider the* m*-homogeneous subseries* $P_m f(s) = \sum_{\Omega(n)=m} a_n n^{-s}$*. Then*

$$\sum_{m=0}^{\infty} \frac{\|P_m f\|_{\mathscr{H}^1}}{m+1} \leq \pi \|f\|_{\mathscr{H}^1}.$$

Corollary 16 can be compared to the estimate $\|P_m f\|_{\mathscr{H}^1} \leq \|f\|_{\mathscr{H}^1}$ appearing in [8, Lem. 3]. Returning to the beginnings of this paper, we mention that Hardy's inequality (27) in turn can be obtained by viewing the bounded symbol for the sharpest version of Hilbert's inequality (6) as an element in the dual of $H^1(\mathbb{T})$ (see [14, pp. 47–49]).

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Paper 9

Contractive inequalities for Bergman spaces and multiplicative Hankel forms

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CONTRACTIVE INEQUALITIES FOR BERGMAN SPACES AND MULTIPLICATIVE HANKEL FORMS

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Abstract. We consider sharp inequalities for Bergman spaces of the unit disc, establishing analogues of the inequality in Carleman's proof of the isoperimetric inequality and of Weissler's inequality for dilations. By contractivity and a standard tensorization procedure, the unit disc inequalities yield corresponding inequalities for the Bergman spaces of Dirichlet series. We use these results to study weighted multiplicative Hankel forms associated with the Bergman spaces of Dirichlet series, reproducing most of the known results on multiplicative Hankel forms associated with the Hardy spaces of Dirichlet series. In addition, we find a direct relationship between the two type of forms which does not exist in lower dimensions. Finally, we produce some counter-examples concerning Carleson measures on the infinite polydisc.

1. Introduction

Hardy spaces of the countably infinite polydisc, $H^p(\mathbb{D}^\infty)$, have in recent years received considerable interest and study, emerging from the foundational papers [16, 23]. Partly, the attraction is motivated by the subject's link with Dirichlet series, realized by identifying each complex variable with a prime Dirichlet monomial, $z_j = p_j^{-s}$ (see [5]). Hardy spaces of Dirichlet series, $\mathscr{H}^p$, are defined by requiring this identification to induce an isometric, multiplicative isomorphism. The connection to Dirichlet series gives rise to a rich interplay between operator theory and analytic number theory — we refer the interested reader to the survey [37] or the monograph [38] as a starting point.

One aspect of the theory is the study of multiplicative Hankel forms on $\ell^2 \times \ell^2$. A sequence $\varrho = (\varrho_1, \varrho_2, \ldots)$ generates a multiplicative Hankel form by the formula

$$\varrho(a, b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \varrho_{mn}, \tag{1}$$

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defined at least for finitely supported sequences a and b. Helson [24] observed that multiplicative Hankel forms are naturally realized as (small) Hankel operators on $H^2(\mathbb{D}^\infty)$, and went on to ask whether every symbol ρ which generates a bounded multiplicative Hankel form on $\ell^2 \times \ell^2$ also induces a bounded linear functional on the Hardy space $H^1(\mathbb{D}^\infty)$. In other words, he asked whether there is an analogue of Nehari's theorem [32] in this context.

Helson's question inspired several papers [9, 11, 25, 26, 35, 36]. Following the program outlined in [26], it was established in [35] that there are bounded Hankel forms that do not extend to bounded functionals on $H^1(\mathbb{D}^\infty)$. In the positive direction, it was proved in [25] that if the Hankel form (1) instead satisfies the stronger property of being Hilbert–Schmidt, then its symbol does extend to a bounded functional on $H^1(\mathbb{D}^\infty)$. Briefly summarizing the most recent development, the result of [35] was generalized in [9], in [11] an analogue of the classical Hilbert matrix was introduced and studied, and in [36] the boundedness of the Hankel form (1) was characterized in terms of Carleson measures in the special case that the form is positive semi-definite.

Very recently, a study of Bergman spaces of Dirichlet series $\mathscr{A}^p$ begun in [3]. In analogy with the Hardy spaces of Dirichlet series, $\mathscr{A}^p$ is constructed from the corresponding Bergman space, $A^p(\mathbb{D}^\infty)$. New difficulties appear in trying to put this theory on equal footing with its Hardy space counterpart. One of them is the lack of contractive inequalities for Bergman spaces in the unit disc. In the Hardy space of the unit disc there is a comparative abundance of such inequalities, each immediately implying a corresponding inequality for $\mathscr{H}^p$. For example, the result of [25] on Hilbert–Schmidt Hankel forms relies essentially on the classical Carleman inequality,

$$\|f\|_{A^2(\mathbb{D})} \leq \|f\|_{H^1(\mathbb{D})}.$$

A second example is furnished by Weissler's inequality: defining for $0 < r \leq 1$ the map $P_r \colon H^p(\mathbb{D}) \to H^q(\mathbb{D})$, by $P_r f(w) = f(rw)$, then P_r is contractive if and only if $r \leq \sqrt{p/q} \leq 1$. Since both of these inequalities are contractive, they carry on to the infinite polydisc by tensorization (see Section 3), thus yielding results for $\mathscr{H}^p$.

We derive analogues of the mentioned inequalities for Bergman spaces of the unit disc in Section 2. Our proofs involve certain variants of the Sobolev inequalities from [4] and [6]. Then, in Section 3, we follow the by now standard tensorization scheme to deduce the corresponding contractive inequalities for the Bergman spaces of Dirichlet series.

Section 4 is devoted to the weighted multiplicative Hankel forms related to the Bergman space, defined by the formula

$$(2) \qquad \varrho_d(a,b) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)}, \qquad a,b \in \ell^2_d.$$

In (2), $d(k)$ denotes the number of divisors of the integer k, and ℓ^2_d denotes the corresponding weighted Hilbert space. Note that the divisor function $d(k)$ counts the number of times ϱ_k appears in (2). In the same way that the forms (1) are realized as Hankel operators on the Hardy space $H^2(\mathbb{D}^\infty)$, the weighted forms (2) are naturally realized as (small) Hankel operators on the Bergman space of the infinite polydisc, $A^2(\mathbb{D}^\infty)$. Equipped with the inequalities from Sections 2 and 3 we successfully obtain the Bergman space counterparts of results from [11, 25, 26, 35].

In Section 4 we will also point out a surprising property of multiplicative Hankel forms. We first observe that $A^2(\mathbb{D}^\infty)$ may be naturally isometrically embedded in the Hardy space $H^2(\mathbb{D}^\infty)$, since the same is true for $A^2(\mathbb{D})$ with respect to $H^2(\mathbb{D}^2)$. Then, we notice that this embedding lifts to the level of Hankel forms, giving us natural map taking weighted Hankel forms (2) to Hankel forms (1). The striking aspect is that this map preserves the singular numbers of the Hankel form, in particular preserving both the uniform and the Hilbert–Schmidt norm.

Finally, in Section 5 we come back to harmonic analysis on the Hardy spaces $H^p(\mathbb{D}^\infty)$. We produce two counter-examples for Carleson measures, again pointing out phenomena that do not exist in finite dimension.

Notation. We will use the notation $f(x) \leq g(x)$ if there is some constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all (appropriate) x. If $f(x) \leq g(x)$ and $g(x) \leq f(x)$, we write $f(x) \asymp g(x)$. As above, $(p_j)_{j\geq 1}$ will denote the increasing sequence of prime numbers.

2. Inequalities of Carleman and Weissler for Bergman spaces

2.1. **Preliminaries.** Let $\alpha > 1$ and $0 < p < \infty$, and define the Bergman space $A^p_\alpha(\mathbb{D})$ as the space of analytic functions f in the unit disc

$$\mathbb{D} = \{z \,:\, |z| < 1\}$$

that are finite with respect to the norm

$$\|f\|_{A^p_\alpha(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(w)|^p \,(\alpha-1)(1-|w|^2)^{\alpha-2}\, dm(w)\right)^{\frac{1}{p}}.$$

Here m denotes the Lebesgue area measure, normalized so that $m(\mathbb{D}) = 1$. It will be convenient to let $dm_\alpha(w) = (\alpha-1)(1-|w|)^{\alpha-2}\, dm(w)$ for $\alpha > 1$, and to let m_1 denote the normalized Lebesgue measure on the torus

$$\mathbb{T} = \{z \,:\, |z| = 1\}.$$

The Hardy space $H^p(\mathbb{D})$ is defined as closure of analytic polynomials with respect to the norm

$$\|f\|_{H^p(\mathbb{D})} = \left(\int_{\mathbb{T}} |f(w)|^p \, dm_1(w)\right)^{\frac{1}{p}}.$$

The Hardy space $H^p(\mathbb{D})$ is the limit of $A^p_\alpha(\mathbb{D})$ as $\alpha \to 1^+$, in the sense that

$$\lim_{\alpha\to 1^+} \|f\|_{A^p_\alpha(\mathbb{D})} = \|f\|_{H^p(\mathbb{D})}$$

for every analytic polynomial f. We therefore let $A^p_1(\mathbb{D}) = H^p(\mathbb{D})$. Our main interest is in the distinguished case $\alpha = 2$, when $m_\alpha = m$ is simply the normalized Lebesgue measure. Therefore we also let $A^p(\mathbb{D}) = A^p_2(\mathbb{D})$. We will only require some basic properties of $A^p_\alpha(\mathbb{D})$ in what follows, and refer generally to the monographs [18, 22].

Let $c_\alpha(j)$ denote the coefficients of the binomial series

$$\frac{1}{(1-w)^\alpha} = \sum_{j=0}^{\infty} c_\alpha(j) w^j, \qquad c_\alpha(j) = \binom{j+\alpha-1}{j}. \tag{3}$$

It is evident from (3) that

$$\sum_{j+k=l} c_\alpha(j) c_\beta(k) = c_{\alpha+\beta}(l). \tag{4}$$

If α is an integer, then $c_\alpha(j)$ denotes the number of ways to write j as a sum of α non-negative integers. Furthermore, if $f(w) = \sum_{j\geq 0} a_j w^j$, then

$$\|f\|_{A^2_\alpha(\mathbb{D})} = \left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{c_\alpha(j)}\right)^{\frac{1}{2}}. \tag{5}$$

Functions f in $A^p_\alpha(\mathbb{D})$ satisfy for $w \in \mathbb{D}$ the sharp pointwise estimate

$$|f(w)| \leq \frac{1}{(1-|w|^2)^{\alpha/p}} \|f\|_{A^p_\alpha(\mathbb{D})}. \tag{6}$$

For the sake of completeness, we will state and prove the results in this section for as general $\alpha > 1$ as we are able, even though we will only make use of the results for $\alpha = 2$ in the following sections.

2.2. **Contractive inclusions of Bergman spaces.** It is well-known that, if $0 < p \leq q$ and $\alpha, \beta \geq 1$, then $A^p_\alpha(\mathbb{D})$ embeds continuously into $A^q_\beta(\mathbb{D})$ if and only if $q/\beta \leq p/\alpha$ (see e.g. [45, Exercise 2.27]). By tensorization, this statement extends to the Bergman spaces on the polydiscs of finite dimension. However, in order for such embeddings to exist on the infinite polydisc, it is necessary that the inclusion map in one variable is contractive.

The first result of the type we are looking for was given by Carleman [13]. For $f \in H^1(\mathbb{D})$ it holds that

$$\|f\|_{A^2(\mathbb{D})} = \|f\|_{A^2_2(\mathbb{D})} \leq \|f\|_{A^1_1(\mathbb{D})} = \|f\|_{H^1(\mathbb{D})}. \tag{7}$$

A modern and natural way to prove (7) can be found in [43]. First, it is easy to verify that

$$\|gh\|_{A^2(\mathbb{D})} \leq \|g\|_{H^2(\mathbb{D})}\|h\|_{H^2(\mathbb{D})},$$

for example by computing by coefficients. If f is a non-vanishing function of $H^1(\mathbb{D})$, writing $f = gh$ with $g = h = f^{1/2}$ now leads to (7). For a general function $f \in H^1(\mathbb{D})$, we first factor out the zeroes through a Blaschke product. This is possible by what seems to be a coincidence: multiplication by a Blaschke product decreases the norm on the left hand side of (7) but preserves the norm on the right hand side.

The ability to factor out zeroes and take roots implies that Carleman's inequality (7) holds for arbitrary $0 < p < \infty$,

$$\|f\|_{A^{2p}(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}.$$

In [12], Burbea generalized Carleman's inequality, showing that for every $0 < p < \infty$ and every non-negative integer n, it holds that

$$\|f\|_{A^{p(1+n)}_{1+n}(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}. \tag{8}$$

Let

$$\alpha_0 = \frac{1+\sqrt{17}}{4} = 1.280776\ldots$$

We offer the following extension of Carleman's inequality.

Theorem 1. *Let $\alpha \geq \alpha_0$ and $0 < p < \infty$. For every $f \in A^p_\alpha(\mathbb{D})$,*

$$\|f\|_{A^{p(\alpha+1)/\alpha}_{\alpha+1}(\mathbb{D})} \leq \|f\|_{A^p_\alpha(\mathbb{D})}.$$

Moreover, if $\alpha > \alpha_0$, we have equality if and only if there exists constants $C \in \mathbb{C}$ and $\xi \in \mathbb{D}$ such that

$$f(w) = \frac{C}{(1-\bar{\xi}w)^{2\alpha/p}}.$$

Let us give two corollaries. The first is mainly decorative, but it illustrates that (8) gets weaker as n increases.

Corollary 2. *Let $f \in H^1(\mathbb{D}) = A^1_1(\mathbb{D})$. Then*

$$\|f\|_{A^1_1(\mathbb{D})} \geq \|f\|_{A^2_2(\mathbb{D})} \geq \|f\|_{A^3_3(\mathbb{D})} \geq \|f\|_{A^4_4(\mathbb{D})} \geq \cdots$$

We also have the following corollary, which will be important in the next section.

Corollary 3. *Let $p = 2/(1+n/2)$ for a non-negative integer n and suppose that $f(w) = \sum_{j\geq 0} a_j w^j$ is in $A^p(\mathbb{D})$. Then*

$$\|f\|_{A^2_{n+2}(\mathbb{D})} = \left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{c_{n+2}(j)}\right)^{\frac{1}{2}} \leq \|f\|_{A^p(\mathbb{D})}.$$

Proof. This follows from n successive applications of Theorem 1, starting from $p = 2/(1+n/2)$ and $\alpha = 2$. □

We now begin the proof of Theorem 1. A version of it was announced in [4][1], following a scheme designed in [7]. Observe also that an analogous result in the Fock space was proved by Carlen [14] using a logarithmic Sobolev inequality. We follow the general strategy of [4, 7], replacing [4, Sec. 5] with a result from [31]. We include many additional details in an attempt to make the scheme used in [4, 7, 14] available to a wider audience.

We shall use two structures on the disk, the Euclidean and the hyperbolic. The usual gradient and Laplacian of u will be denoted by ∇u and Δu, while the hyperbolic gradient and the hyperbolic Laplacian are denoted by $\nabla_{\mathrm{H}}\, u$ and $\Delta_{\mathrm{H}}\, u$. They are connected by the following formulas:

$$\nabla_{\mathrm{H}}\, u(w) = \left(\frac{1-|w|^2}{2}\right) \nabla u(w) \qquad \text{and} \qquad \Delta_{\mathrm{H}}\, u(w) = \left(\frac{1-|w|^2}{2}\right)^2 \Delta u(w).$$

We shall also use the Möbius invariant measure

$$d\mu(w) = \frac{dm(w)}{(1-|w|^2)^2}.$$

We begin with an integral identity (essentially [4, Thm. 3.1]). An analogous result was proven for the Fock space in [14], and a similar result also appears in [7].

Lemma 4. *Let $p > 0$ and $\beta > 1/2$. For an analytic function f in $\overline{\mathbb{D}}$, set $u(w) = |f(w)|^p(1-|w|^2)^\beta$. Then*

$$\int_{\mathbb{D}} |\nabla_{\mathrm{H}}\, u(w)|^2 d\mu(w) = \frac{\beta}{2} \int_{\mathbb{D}} |u(w)|^2 d\mu(w).$$

Proof. Integrating by parts gives

$$\int_{\mathbb{D}} |\nabla_{\mathrm{H}}\, u|^2 d\mu = \frac{1}{4} \int_{\mathbb{D}} |\nabla u|^2 dm = -\frac{1}{4} \int_{\mathbb{D}} u \Delta u dm. \tag{9}$$

[1] Theorem 3.2 in [4] is stated for $kq > 2$, but there seems to be a mistake in the proof of uniqueness on p. 1083. The argument in its entirety seems to apply only when $kq > 3$.

It follows from the assumption $\beta > 1/2$ that boundary terms do not appear here. We compute the Laplacian now. At any point where f does not vanish, we can write

$$\frac{\partial u}{\bar\partial w} = \frac{p}{2}|f|^{p-2} f\overline{f'}(1-|w|^2)^\beta - \beta w|f|^p(1-|w|^2)^{\beta-1},$$

so that

$$\begin{aligned}\frac{\partial^2 u}{\partial w\bar\partial w} &= \frac{p^2}{4}|f'|^2|f|^{p-2}(1-|w|^2)^\beta - \beta\frac{p}{2}|f|^{p-2}f\overline{f'}\bar w(1-|w|^2)^{\beta-1}\\ &\quad - \beta|f|^p(1-|w|^2)^{\beta-1} - \frac{\beta p}{2}|f|^{p-2}f'\bar f(1-|w|^2)^{\beta-1}\\ &\quad + \beta(\beta-1)|w|^2|f|^p(1-|w|^2)^{\beta-2}.\end{aligned}$$

We see that

$$\begin{aligned}-u\Delta u &= -p^2|f'|^2|f|^{2p-2}(1-|w|^2)^{2\beta} + 2\beta p|f|^{2p-2}f\overline{f'}\bar w(1-|w|^2)^{2\beta-1}\\ &\quad + 4\beta|f|^{2p}(1-|w|^2)^{2\beta-2} + 2\beta p|f|^{2p-2}f'\bar f w(1-|w|^2)^{2\beta-1}\\ &\quad - 4\beta^2|w|^2|f|^{2p}(1-|w|^2)^{2\beta-2}.\end{aligned}$$

Coming back to the expression of $\partial u/\bar\partial w$, we find that

$$-\frac{1}{4}u\Delta u = \beta\frac{u^2}{(1-|w|^2)^2} - \left|\frac{\partial u}{\bar\partial w}\right|^2 = \beta\frac{u^2}{(1-|w|^2)^2} - \frac{|\nabla_{\mathrm{H}}\, u|^2}{(1-|w|^2)^2}.$$

Integrating with respect to dm and using (9) gives the result. □

Proof of Theorem 1. We set $q = p(\alpha+1)/\alpha$, $A = (\alpha-2)/(\alpha-1)$ and $B = 1/(\alpha-1)$, so that $A + B = 1$. We want to find the infimum of

$$(\alpha-1)\int_{\mathbb{D}} |f(w)|^p(1-|w|^2)^\alpha d\mu(w)$$

under the constraint

$$\alpha\int_{\mathbb{D}} |f(w)|^q(1-|w|^2)^{\alpha+1}d\mu(w) = 1.$$

Equivalently, using Lemma 4 with

$$u(w) = |f(w)|^{p/2}(1-|w|^2)^{\alpha/2}, \tag{10}$$

we want to find the infimum of

$$A\int_{\mathbb{D}} |u(w)|^2 d\mu(w) + \frac{4B}{\alpha}\int_{\mathbb{D}} |\nabla_{\mathrm{H}}\, u(w)|^2 d\mu(w) \tag{11}$$

under the constraint

$$\alpha\int_{\mathbb{D}} |u(w)|^{2q/p} d\mu(w) = 1. \tag{12}$$

We now solve the latter minimization problem for real-valued u belonging to the Sobolev space $W^{1,2}(\mathbb{D})$, i.e. functions u such that

$$\int_{\mathbb{D}} |\nabla_{\mathrm{H}} u(w)|^2 d\mu(w) < \infty.$$

By the well-known inequality for the bottom of the spectrum of the Laplace–Beltrami operator (see e.g. [31]) we know that for any $u \in W^{1,2}(\mathbb{D})$,

$$\int_{\mathbb{D}} |u(w)|^2 d\mu(w) \leq 4 \int_{\mathbb{D}} |\nabla_{\mathrm{H}} u(w)|^2 d\mu(w).$$

Hence

$$N(u) = \left(A \int_{\mathbb{D}} |u(w)|^2 d\mu(w) + \frac{4B}{\alpha} \int_{\mathbb{D}} |\nabla_{\mathrm{H}} u(w)|^2 d\mu(w) \right)^{1/2}$$

is a norm on $W^{1,2}(\mathbb{D})$ equivalent to the usual norm, since $A > -B/\alpha$. By the Rellich–Kondrakov theorem [30, Ch. 11], which asserts that the inclusion map from $W^{1,2}(\mathbb{D})$ into $L^s(\mathbb{D}, d\mu)$ is compact for any finite s, the problem of finding the infimum of (11) for $u \in W^{1,2}(\mathbb{D})$ satisfying (12) is well-posed. Moreover, this also ensures that minimizers do exist. Indeed, let us take any sequence (u_n) realizing the infimum. This sequence is bounded in the reflexive space $W^{1,2}(\mathbb{D})$, so we may assume that it converges weakly to some $u \in W^{1,2}(\mathbb{D})$. Then (u_n) converges to u in $L^{2q/p}(\mathbb{D}, d\mu)$ so that $\|u\|_{L^{2q/p}}^{2q/p} = 1/\alpha$ whereas $N(u) \leq \liminf_n N(u_n)$.

Next we compute the Euler–Lagrange equation corresponding to the constrained variational problem given by (11) and (12). By standard arguments, we find that any local minimum of the problem is a weak solution of

$$Au - \frac{4B}{\alpha} \Delta_{\mathrm{H}} u = \lambda u^{\frac{2q}{p}-1} \tag{13}$$

for some $\lambda \in \mathbb{R}$. By Lemma 5 below, there are minimizers that are actually $C^2(\mathbb{D})$. Multiplying by u and integrating with respect to μ, we find from (9) that $\lambda > 0$. We now rescale (13) by setting $u = \kappa v$ with

$$\kappa^{2q/p-2} = \frac{4B}{\alpha\lambda}.$$

Then $v \in W^{1,2}(\mathbb{D}) \cap C^2(\mathbb{D})$ satisfies

$$\Delta_{\mathrm{H}} v - \frac{(\alpha-2)\alpha}{4} v + v^{\frac{2q}{p}-1} = 0. \tag{14}$$

We now investigate (13) for our candidate solution $u_0(w) = (1-|w|^2)^{\alpha/2}$. Since

$$\Delta_{\mathrm{H}} u_0(w) = -\frac{\alpha}{2}(1-|w|^2)^{\alpha/2} \left(1 - \frac{\alpha}{2}|w|^2\right)$$

we have that

$$Au_0 - \frac{4B}{\alpha}\Delta_{\mathrm{H}} u_0 = \frac{\alpha}{\alpha-1}(1-|w|^2)^{\frac{\alpha}{2}+1} = \lambda_0 u_0^{\frac{2q}{p}-1},$$

where $\lambda_0 = \alpha/(\alpha-1)$. Hence, if we let $u_0 = \kappa_0 v_0$ with

$$\kappa_0^{2q/p-2} = \frac{4B}{\alpha\lambda_0},$$

then $v_0 \in W^{1,2}(\mathbb{D})$ is a solution of (14). However, by [31, Thm. 1.3] we know that the solution of (14) is unique up to a Möbius transformation, as long as

$$\frac{\alpha(2-\alpha)}{4} < \frac{4q}{p\left(\frac{2q}{p}+2\right)^2}.$$

Replacing q/p by its value, we find that this inequality is satisfied if and only if $\alpha > \alpha_0$. Both the Euler–Lagrange equation and our constraint problem are invariant under Möbius transformations, so we have found all minimizers. Coming back to analytic functions via (10), we have shown that we have equality if and only if there exists $\xi \in \mathbb{D}$ and $\widetilde{C} \in \mathbb{R}$ such that

$$|f(w)|^{p/2} = \widetilde{C}\left|1-\left|\frac{\xi-w}{1-\bar{\xi}w}\right|^2\right|^{\alpha/2}\left|1-|w|^2\right|^{-\alpha/2} = \widetilde{C}\,\frac{\left(1-|\xi|^2\right)^{\alpha/2}}{\left|1-\bar{\xi}w\right|^{\alpha}}.$$

This shows that f has to be a multiple of $(1-\bar{\xi}w)^{-2\alpha/p}$ for some $\xi \in \mathbb{D}$. Finally, the assertion of the theorem for $\alpha = \alpha_0$ is obtained by taking the limit as $\alpha \to \alpha_0^+$. ☐

The following is the regularity result that was used in the proof of the previous theorem.

Lemma 5. *There are minimizers of the variational constrained variational problem given by* (11) *and* (12) *that are* C^2 *smooth in* $\mathbb{D}$.

Proof. Let u be a minimizer. Then it is weak solution of the Euler-Lagrange equation (13). We also know that $u \in L^{2q/p}(\mathbb{D}, d\mu)$. Since the radial rearrangement decreases the Dirichlet norm (by the Polya–Szegö inequality [30, Thm. 16.17]) there is a minimizer u that is positive, radially symmetric and decreasing. Therefore $F(u)$ is bounded in the unit disk, where

$$F(u) := \frac{\alpha}{4B}\left(Au - \lambda u^{\frac{2q}{p}-1}\right)$$

Consider any solution v to the Poisson equation:

$$\Delta v(z) = \frac{F(u(z))}{(1-|z|^2)^2},$$

then $u - v$ satisfies $\Delta(u-v) = 0$ weakly. Therefore $u = v + h$ where h is an harmonic function. One explicit solution to the Poisson equation is given by

$$v(z) = \int_{\mathbb{D}} K(z,w)\frac{F(u(w))}{(1-|w|^2)^2}\,dm(w)$$

where

$$K(z,w) = \frac{1}{2\pi}\left\{\log\left|\frac{w-z}{1-\overline{w}z}\right|^2 + \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2} + |z|^2\left(\frac{1-|w|^2}{|1-\overline{w}z|}\right)^2\right\}.$$

It was shown in [1] that $K(z,w)$ satisfies the estimate

$$|K(z,w)| \leq \frac{(1-|w|^2)^2}{|1-\overline{w}z|^2}\left(1+\log\left|\frac{1-\overline{w}z}{w-z}\right|\right), \qquad z,w\in\mathbb{D}.$$

The difference between u and v is harmonic, thus the regularity of u follows from the regularity of v. □

Remark. The constants A and B, with $A+B=1$, were chosen in the proof so that $u(w) = (1-|w|^2)^{\alpha/2}$ would be a solution of the Euler–Lagrange equation for some $\lambda\in\mathbb{R}$. This is only possible if $\beta=\alpha+1$, and thus explains why this relationship is imposed in the statement of Theorem 1. The condition $\alpha\geq\alpha_0$ comes from [31, Thm. 1.3], but we do not know if it is necessary for the uniqueness of (14).

Question. For any $0<p\leq q$ and $\alpha,\beta\geq 1$ such that $q/\beta\leq p/\alpha$, does the contractive inequality

$$\|f\|_{A^q_\beta(\mathbb{D})} \leq \|f\|_{A^p_\alpha(\mathbb{D})}$$

hold? By Carleman's inequality and Theorem 1, this is true when $\beta=\alpha+n$ for some integer n, and either $\alpha=1$ or $\alpha\geq\alpha_0$. We remark that it is easy to show, for example by computing with coefficients, that

$$\|f\|_{A^4_{2\alpha}(\mathbb{D})} \leq \|f\|_{A^2_\alpha(\mathbb{D})}$$

holds for every $\alpha\geq 1$.

2.3. **Hypercontractivity of the Poisson kernel.** For $r\in[0,1]$, let P_r denote the operator defined on analytic functions in $\mathbb{D}$ by $P_r f(w) = f(rw)$. Clearly, if $r<1$ it follows from (6) that P_r maps any $A^p_\alpha(\mathbb{D})$ into every $A^q_\beta(\mathbb{D})$. We are interested in knowing when this map is contractive.

Theorem 6. *Let $0<p\leq q<\infty$ and let $\alpha=(n+1)/2$ for some $n\in\mathbb{N}$. Then P_r is a contraction from $A^p_\alpha(\mathbb{D})$ to $A^q_\alpha(\mathbb{D})$ if and only if $r\leq\sqrt{p/q}$.*

Weissler [44] proved Theorem 6 when $\alpha=1$. The case $\alpha=3/2$ is also known, see [21, Remark 5.14] or [28], but it appears that these are the only two previously demonstrated cases. To prove Theorem 6 we will use a classical argument of complex analysis to transfer results from Hardy spaces to Bergman spaces in smaller dimensions. This will be accomplished through the following lemma.

Lemma 7 ([40], Sec. 1.4.4)**.** *Let $\mathbb{S}^n$ denote the real unit sphere of dimension $n \geq 1$, and let σ_n denote its normalized surface measure. Extend the function $h\colon \mathbb{D} \to \mathbb{C}$ to $\mathbb{S}^n$ by $\widetilde{h}(x) = h(x_1 + ix_2)$ for $x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{S}^n$. Then*

$$\int_{\mathbb{S}^n} \widetilde{h}(x)\, d\sigma_n(x) = \int_{\mathbb{D}} h(w) dm_{(n+1)/2}(w).$$

We can now demonstrate how Theorem 6 follows from a result of Beckner [6] concerning the unit sphere.

Proof of Theorem 6. Let $\mathcal{P}_r$ denote the Poisson kernel on $\mathbb{S}^n$, defined by

$$\mathcal{P}_r(\xi, \eta) = \frac{1 - r^2}{|r\xi - \eta|^{n+1}}, \qquad \xi, \eta \in \mathbb{S}^n.$$

For a function g on $\mathbb{S}^n$, let

$$(\mathcal{P}_r g)(\xi) = \int_{\mathbb{S}^n} \mathcal{P}_r(\xi, \eta) g(\eta) d\sigma_n(\eta).$$

It is proved in [6] that $\mathcal{P}_r$ defines a contraction from $L^s(\mathbb{S}^n)$ to $L^t(\mathbb{S}^n)$, $1 \leq s \leq t < \infty$, if and only if $r \leq \sqrt{(s-1)/(t-1)}$.

Let us now start with $0 < p \leq q < \infty$ and $r < \sqrt{p/q}$. Let m be a large number such that $mp > 1$ and such that

$$r \leq \sqrt{\frac{mp - 1}{mq - 1}}.$$

Given an analytic polynomial f, we define g on $\mathbb{S}^n$ by

$$g(x_1, x_2, \ldots, x_{n+1}) = |f(x_1 + ix_2)|^{1/m}.$$

Since f is analytic, it follows that g is subharmonic and hence for any

$$(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$$

we get that

$$g(rx_1, \ldots, rx_{n+1}) \leq \mathcal{P}_r g(x_1, \ldots, x_{n+1}).$$

Using Beckner's result with $s = mp$ and $t = mq$ we get that

$$\left(\int_{\mathbb{S}^n} g(rx_1, \ldots, rx_{n+1})^{mq} d\sigma_n(x) \right)^{1/q} \leq \left(\int_{\mathbb{S}^n} g(x_1, \ldots, x_{n+1})^{mp} d\sigma_n(x) \right)^{1/p}.$$

By Lemma 7, this is the same as

$$\left(\int_{\mathbb{D}} |f(rw)|^q dm_{(n+1)/2}(w) \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{D}} |f(w)|^p dm_{(n+1)/2}(w) \right)^{\frac{1}{p}}.$$

It follows that the condition $r \leq \sqrt{p/q}$ is sufficient (by a limiting argument in the endpoint case $r = \sqrt{p/q}$). Conversely, for fixed $r > 0$ and small $\varepsilon > 0$ we have that

$$\left(\int_{\mathbb{D}} |1 + \varepsilon r w|^q \, dm_\alpha(w) \right)^{\frac{1}{q}} = 1 + \frac{qr^2}{4\alpha} \, \varepsilon^2 + O(\varepsilon^4).$$

Letting $\varepsilon \to 0$ shows that $qr^2 \leq p$ is also necessary, for any value of $\alpha \geq 1$. □

Remark. As in the previous subsection, we conjecture that Theorem 6 is true for all values of $\alpha \geq 1$. Several other positive results can be deduced from Theorem 1. For instance, if $\alpha \geq \alpha_0$, then

$$\|P_r f\|_{A^2_\alpha(\mathbb{D})} \leq \|f\|_{A^{2\alpha/(\alpha+1)}_\alpha(\mathbb{D})},$$

for every analytic polynomial f, if and only if $r^2 \leq (\alpha + 1)/\alpha$. In fact, it follows from Theorem 1 that

$$\|f\|_{A^2_{\alpha+1}(\mathbb{D})} \leq \|f\|_{A^{2\alpha/(\alpha+1)}_\alpha(\mathbb{D})}.$$

Computing the norms as in (5), we have that

$$\|P_r f\|_{A^2_\alpha(\mathbb{D})} \leq \|f\|_{A^2_{\alpha+1}(\mathbb{D})}$$

if and only if, for any $k \geq 1$,

$$r^{2k} \leq \frac{c_{\alpha+1}(k)}{c_\alpha(k)} = \frac{\alpha + k}{\alpha}.$$

3. Inequalities on the polydisc and in the half-plane

For $\alpha > 1$, consider the following product measure on $\mathbb{D}^\infty$,

$$\mathbf{m}_\alpha(z) = m_\alpha(z_1) \times m_\alpha(z_2) \times m_\alpha(z_3) \times \cdots,$$

and for $0 < p < \infty$ the corresponding Lebesgue space $L^p_\alpha(\mathbb{D}^\infty)$. We define the Bergman spaces of the infinite polydisc, denoted $A^p_\alpha(\mathbb{D}^\infty)$, as the closure in $L^p_\alpha(\mathbb{D}^\infty)$ of the space of analytic polynomials in an arbitrary number of variables. The Hardy spaces $H^p(\mathbb{D}^\infty)$ are defined as the closure of analytic polynomials with respect to the norm given by the product $m_1 \times m_1 \times \cdots$ on $\mathbb{T}^\infty$, so that

$$\|f\|^p_{H^p(\mathbb{D}^\infty)} = \int_{\mathbb{T}^\infty} |f(z)|^p \, d\mathbf{m}_1(z).$$

As before, $H^p(\mathbb{D}^\infty)$ is the limit as $\alpha \to 1^+$ of $A^p_\alpha(\mathbb{D}^\infty)$, in the sense that

$$\lim_{\alpha \to 1^+} \|f\|_{A^p_\alpha(\mathbb{D}^\infty)} = \|f\|_{H^p(\mathbb{D}^\infty)}$$

for every analytic polynomial f. We distinguish the case $\alpha = 2$ by writing $A^p(\mathbb{D}^\infty) = A^p_2(\mathbb{D}^\infty)$. Applying the point estimate (6) repeatedly we find that if f is a polynomial in $A^p_\alpha(\mathbb{D}^\infty)$, then

$$|f(z)| \le \left(\prod_{j=1}^{\infty} \frac{1}{1-|z_j|^2}\right)^{\alpha/p} \|f\|_{A^p_\alpha(\mathbb{D}^\infty)}, \tag{15}$$

which implies that elements of $A^p_\alpha(\mathbb{D}^\infty)$ are analytic functions on $\mathbb{D}^\infty \cap \ell^2$. Every f in $A^p_\alpha(\mathbb{D}^\infty)$ has a power series expansion convergent in $\mathbb{D}^\infty \cap \ell^2$,

$$f(z) = \sum_{\kappa \in \mathbb{N}_0^\infty} a_\kappa z^\kappa, \tag{16}$$

where $\mathbb{N}_0^\infty$ denotes the set of all finite non-negative multi-indices.

Finally, when $p = 2$ we can compute the norm explicitly. Suppose that f is of the form (16). Then

$$\|f\|_{A^2_\alpha(\mathbb{D}^\infty)} = \left(\sum_{\kappa \in \mathbb{N}_0^\infty} \frac{|a_\kappa|^2}{c_\alpha(\kappa)}\right)^{\frac{1}{2}}, \qquad \text{where} \qquad c_\alpha(\kappa) = \prod_{j=1}^{\infty} c_\alpha(\kappa_j). \tag{17}$$

Note that the final product contains only a finite number of factors not equal to 1, since κ is a finite multi-index.

The contractive inequalities of Section 2 can now be extended to $\mathbb{D}^\infty$ using Minkowski's inequality in the following formulation: if X and Y are measure spaces, g a measurable function on $X \times Y$, and $p \ge 1$, then

$$\left(\int_X \left(\int_Y |g(x,y)|\,dy\right)^p dx\right)^{\frac{1}{p}} \le \int_Y \left(\int_X |g(x,y)|^p\,dx\right)^{\frac{1}{p}} dy.$$

It is sufficient to prove the contractive results on the finite polydiscs $\mathbb{D}^d$, $d < \infty$, as this allows us to conclude by the density of analytic polynomials. This is done by iteratively applying the one dimensional result to each of the variables, and applying Minkowski's inequality in each step. This procedure has been repeated many times (for instance in [5, 8, 25] or in [38, Sec. 6.5.3]) and we do not include the details here.

In particular, Corollary 3 for $n = 2$ yields the next result on the polydisc. Helson [25] proved the corresponding result for the Hardy spaces $H^p(\mathbb{D}^\infty)$, which he used to study Hilbert–Schmidt multiplicative Hankel forms. We shall carry out the analogous study for weighted multiplicative Hankel forms associated with the Bergman space in the next section.

Lemma 8. $\|f\|_{A^2_4(\mathbb{D}^\infty)} \le \|f\|_{A^1(\mathbb{D}^\infty)}$.

Let $\mathbf{r} = (r_1, r_2, \ldots)$ with $r_j \in [0,1]$ and define $P_{\mathbf{r}}f(z) = f(r_1 z_1, r_2 z_2, \ldots)$. Following [5] and using Theorem 6 (with $\alpha = 2$), we get the next result.

Lemma 9. *Let $0 < p \le q < \infty$. The map $P_{\mathbf{r}}$ is a contraction from $A^p(\mathbb{D}^\infty)$ to $A^q(\mathbb{D}^\infty)$ if and only if $r_j \le \sqrt{p/q}$. Moreover, $P_{\mathbf{r}}$ is bounded from $A^p(\mathbb{D}^\infty)$ to $A^q(\mathbb{D}^\infty)$ as soon as $r_j \le \sqrt{p/q}$ for all but a finite set of js.*

When working with multiplicative Hankel forms and Dirichlet series, it is often convenient to recast the expansion (16) in multiplicative notation. Each integer $n \ge 1$ can be written in a unique way as a product of prime numbers,

$$n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

This factorization associates n uniquely to the finite non-negative multi-index $\kappa(n)$. Setting $a_n = a_{\kappa(n)}$, we rewrite (16) as

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}. \tag{18}$$

For $\alpha \ge 1$ we define the general divisor function $d_\alpha(n)$ as the coefficients of the Dirichlet series given by ζ^α, where $\zeta(s) = \sum_{n\ge1} n^{-s}$ is the Riemann zeta function. Using the Euler product of the Riemann zeta function, say for $\mathrm{Re}(s) > 1$, we find that

$$\zeta(s)^\alpha = \left(\prod_{j=1}^{\infty} \frac{1}{1-p_j^{-s}}\right)^{\alpha} = \prod_{j=1}^{\infty}\left(\sum_{k=0}^{\infty} c_\alpha(k) p_j^{-ks}\right) = \sum_{n=1}^{\infty} d_\alpha(n) n^{-s}. \tag{19}$$

It follows that $c_\alpha(\kappa(n)) = d_\alpha(n)$. In multiplicative notation, we restate (17) as

$$\left\| \sum_{n=1}^{\infty} a_n z^{\kappa(n)} \right\|_{A^2_\alpha(\mathbb{D}^\infty)} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d_\alpha(n)} \right)^{\frac{1}{2}}.$$

When $\alpha \ge 1$ is an integer, it is clear that $d_\alpha(n)$ denotes the number of ways to write n as a product of α non-negative integers. In particular, d_2 is the usual divisor function d. It also follows from (19) that

$$\sum_{mn=l} d_\alpha(m) d_\beta(n) = d_{\alpha\beta}(l), \tag{20}$$

in analogy with (4).

The Bohr lift of a Dirichlet series $f(s) = \sum_{n\ge1} a_n n^{-s}$ is the power series defined by

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)},$$

realizing the identification $z_j = p_j^{-s}$. The Bergman space of Dirichlet series $\mathscr{A}^p$ is defined as the completion of Dirichlet polynomials in the norm

$$\|f\|_{\mathscr{A}^p} = \|\mathscr{B}f\|_{A^p(\mathbb{D}^\infty)}.$$

Inequality (15) implies that $\mathscr{A}^p$ is a space of analytic functions in the half-plane $\mathbb{C}_{1/2}$, and that f in $\mathscr{A}^p$ enjoys the sharp pointwise estimate

$$|f(s)| \leq \zeta(2\operatorname{Re} s)^{2/p}\|f\|_{\mathscr{A}^p}. \tag{21}$$

Let $\mathscr{T}$ denote the conformal map of $\mathbb{D}$ to $\mathbb{C}_{1/2}$ given by

$$\mathscr{T}(z) = \frac{1}{2} + \frac{1-z}{1+z}.$$

The conformally invariant Bergman space of $\mathbb{C}_{1/2}$, denoted $A^p_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})$, is the space of analytic functions f in $\mathbb{C}_{1/2}$ with the property that $f \circ \mathscr{T} \in A^p_\alpha(\mathbb{D})$. A computation shows that

$$\|f\|^p_{A^p_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})} = \int_{\mathbb{C}_{1/2}} |f(s)|^p\,(\alpha-1)\left(\operatorname{Re}(s) - \frac{1}{2}\right)^{\alpha-2} \frac{4^{\alpha-1}}{|s+1/2|^{2\alpha}}\,dm(s).$$

By Lemma 8 we have the following version of Carleman's inequality for Dirichlet series in the half-plane.

Theorem 10. *Suppose that $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is in $\mathscr{A}^1$. Then*

$$\left(\sum_{n=1}^\infty \frac{|a_n|^2}{d_4(n)}\right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{A}^1} \tag{22}$$

Moreover, there is a constant $C \geq 1$ such that $\|f\|_{A^2_{4,\mathrm{i}}(\mathbb{C}_{1/2})} \leq C\|f\|_{\mathscr{A}^1}$.

Proof. The inequality (22) is Lemma 8 in multiplicative notation. The second statement follows from the first and Example 2 in [33]. □

For $\varepsilon > 0$, define the translation operator T_ε by $T_\varepsilon f(s) = f(s+\varepsilon)$. Here is a sharp and general version of [3, Prop. 9], which we interpret as Weissler's inequality for Dirichlet series in the half-plane. The corresponding result for $\mathscr{H}^p$ can be found in [5].

Theorem 11. *Let $0 < p \leq q < \infty$. The operator $T_\varepsilon\colon \mathscr{A}^p \to \mathscr{A}^q$ is bounded for every $\varepsilon > 0$, and contractive if and only if $2^{-\varepsilon} \leq \sqrt{p/q}$.*

Proof. This follows from Lemma 9, using the fact that T_ε corresponds to $P_{\mathbf{r}}$ with $r_j = p_j^{-\varepsilon}$. □

We end this section by demonstrating that Lemma 9 also implies a weak generalization of Theorem 10 to more general exponents. In the Hardy space context, it was proven in [8] that if $f(s) = \sum_{n\geq 1} a_n n^{-s}$ and $0 < p \leq 2$, then

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{2/p}(n)}\right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{H}^p}.$$

The Möbius factor $|\mu(n)|$ is 1 if n is square-free and 0 if not. From (8), it follows that this factor may actually be replaced by 1 if $p = 2/(1+n)$ for some non-negative integer n. We have the following extension to Bergman spaces in mind.

Theorem 12. *Let $0 < p \leq 2$ and suppose that $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is in $\mathscr{A}^p$. Then*

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{4/p}(n)}\right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{A}^p}.$$

If $p = 2/(1+n/2)$ for some non-negative integer n, then

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \frac{1}{d_{4/p}(n)}\right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{A}^p}.$$

Proof. Let $\Omega(n)$ denote the number of prime factors of n (counting multiplicity). Using Lemma 9 with $r_j = \sqrt{p/2}$, we have that

$$\begin{aligned} \left\|\sum_{n=1}^{\infty} a_n n^{-s}\right\|_{\mathscr{A}^p} &\geq \left\|\sum_{n=1}^{\infty} a_n \left(\frac{p}{2}\right)^{\Omega(n)/2} n^{-s}\right\|_{\mathscr{A}^2} = \left(\sum_{n=1}^{\infty} |a_n|^2 \frac{1}{(2/p)^{\Omega(n)} d(n)}\right)^{\frac{1}{2}} \\ &\geq \left(\sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{(2/p)^{\Omega(n)} d(n)}\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{4/p}(n)}\right)^{\frac{1}{2}}. \end{aligned}$$

In the final equality we used that $d_\alpha(n) = \alpha^{\Omega(n)}$ when n is square-free. When $p = 2/(1+n/2)$ for a non-negative integer n, tensorizing Corollary 3 (by appealing to Minkowski's inequality) yields that the Möbius factor is actually unnecessary; see Lemma 8 and Theorem 10. □

Remark. Considering the square-free terms only of a Dirichlet series is in many cases sufficient to obtain sharp results, see for example [8]. Often, the reason for this is related to the fact that the square-free zeta function has the same behaviour as the zeta function $\zeta(s)$ near $s = 1$, since

$$\sum_{n=1}^{\infty} |\mu(n)| n^{-s} = \prod_{j=1}^{\infty} (1 + p_j^{-s}) = \prod_{j=1}^{\infty} \frac{1 - p_j^{-2s}}{1 - p_j^{-s}} = \frac{\zeta(s)}{\zeta(2s)}.$$

4. Multiplicative Hankel forms

The multiplicative Hankel form (2) is said to be bounded if there is a constant $C < \infty$ such that

$$|\varrho(a,b)| = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)} \right| \le C \left(\sum_{m=1}^{\infty} \frac{|a_m|^2}{d(m)} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{|b_n|^2}{d(n)} \right)^{\frac{1}{2}}. \tag{23}$$

The smallest such constant is the norm of ϱ. The symbol of the form ϱ is the Dirichlet series $\varphi(s) = \sum_{n\ge1} \overline{\varrho_n} n^{-s}$. If f and g are Dirichlet series with coefficient sequences a and b, respectively, then (23) can be rewritten as $|H_\varphi(fg)| \le C\|f\|_{\mathscr{A}^2}\|g\|_{\mathscr{A}^2}$, where we define

$$H_\varphi(fg) = \langle fg, \varphi\rangle_{\mathscr{A}^2} = \sum_{l=1}^{\infty} \left(\sum_{mn=l} a_m b_n \right) \frac{\varrho_l}{d(l)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)}.$$

Hence, the multiplicative Hankel form is bounded if and only if H_φ is a bounded form on $\mathscr{A}^2 \times \mathscr{A}^2$.

We begin with the following example, giving the Bergman space analogue of the multiplicative Hilbert matrix studied in [11]. Let $\mathscr{A}^2_0$ denote the subspace of $\mathscr{A}^2$ consisting of Dirichlet series $f(s) = \sum_{n\ge1} a_n n^{-s}$ such that $a_1 = f(+\infty) = 0$. As in [11], it is natural to work with Dirichlet series without constant term for convergence reasons. We consider the form

$$H(fg) = \int_{1/2}^{\infty} f(\sigma)g(\sigma)\left(\sigma - \frac{1}{2}\right) d\sigma, \qquad f,g \in \mathscr{A}^2_0. \tag{24}$$

Theorem 13. *The bilinear form* (24) *is a multiplicative Hankel form with symbol*

$$\varphi(s) = \int_{1/2}^{\infty} \left(\zeta(s+\sigma)^2 - 1\right)\left(\sigma - \frac{1}{2}\right) d\sigma = \sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}(\log n)^2} n^{-s}.$$

The form H_φ is bounded, but not compact, on $\mathscr{A}^2_0 \times \mathscr{A}^2_0$.

Proof. To see that φ is the symbol, one can either compute $H(fg)$ at the level of coefficients or use that $\zeta(s+\overline{w})^2 - 1$ is the reproducing kernel of $\mathscr{A}^2_0$. To see that H is bounded, we first use the Cauchy–Schwarz inequality,

$$|H(fg)| \le \left(\int_{1/2}^{\infty} |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma \right)^{\frac{1}{2}} \left(\int_{1/2}^{\infty} |g(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma \right)^{\frac{1}{2}}.$$

By symmetry, we only need to consider one of the factors. We split the integral at $\sigma = 1$.

$$\int_{1/2}^{\infty} |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma = \left(\int_{1/2}^{1} + \int_{1}^{\infty} \right) |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma.$$

The first integral is bounded by a constant multiple of $\|f\|^2_{\mathscr{A}^2}$, as follows from [33, Thm. 3 and Example 4]. For the second integral, we have by the pointwise estimate (21) that

$$|f(\sigma)|^2 \le \|f\|^2_{\mathscr{A}^2} \left(\sum_{n=2}^{\infty} d(n) n^{-2\sigma} \right) \le (2+o(1)) 4^{-\sigma} \|f\|^2_{\mathscr{A}^2},$$

where we in the final inequality used that $\sigma \ge 1$. To show that H_φ is not compact, let $k_\varepsilon(s)$ denote the normalized reproducing kernel of $\mathscr{A}^2_0$ at the point $1/2+\varepsilon/2$,

$$k_\varepsilon(s) = \frac{\zeta^2(s+1/2+\varepsilon/2)-1}{\sqrt{\zeta^2(1+\varepsilon)-1}}.$$

The functions k_ε converge weakly to 0 as $\varepsilon \to 0$, since they converge to 0 on every compact subset of $\mathbb{C}_{1/2}$. By the fact that

$$\zeta(s) = \frac{1}{s-1} + O(1)$$

for $\mathrm{Re}(s) > 1$ close to 1, we get for, say $1/2 < \sigma < 1$, that

$$k_\varepsilon(\sigma) = \frac{(\sigma+1/2+\varepsilon/2-1)^{-2}+O(1)}{(1+\varepsilon-1)^{-1}+O(1)} = \varepsilon \left(\frac{1}{(\sigma-1/2+\varepsilon/2)^2} + O(1) \right).$$

Setting $f = g = k_\varepsilon$, we find that

$$H(fg) = \varepsilon^2 \left(\int_{1/2}^{1} \left(\frac{1}{(\sigma-1/2+\varepsilon/2)^4} + O(1) \right) \left(\sigma - \frac{1}{2} \right) d\sigma + O(1) \right) \gg 1,$$

showing that H is not compact. □

Since the Bohr lift is multiplicative, it holds that

$$\langle fg, \varphi \rangle_{\mathscr{A}^2} = \langle \mathscr{B}f \mathscr{B}g, \mathscr{B}\varphi \rangle_{A^2(\mathbb{D}^\infty)}.$$

For the remainder of this section we will work in the polydisc, and we therefore tacitly identify the Dirichlet series f with its Bohr lift $\mathscr{B}f$. Hence, we consider symbols of the form

$$\varphi(z) = \sum_{n=1}^{\infty} \overline{\varrho_n} z^{\kappa(n)},$$

and define $H_\varphi(fg) = \langle fg, \varphi \rangle_{A^2(\mathbb{D}^\infty)}$, for $f, g \in A^2(\mathbb{D}^\infty)$.

If φ defines a bounded functional on $A^1(\mathbb{D}^\infty)$, then it follows from the Cauchy–Schwarz inequality that

$$|H_\varphi(fg)| = |\langle fg, \varphi \rangle_{A^2}| \le \|\varphi\|_{(A^1)^*} \|fg\|_{A^1} \le \|\varphi\|_{(A^1)^*} \|f\|_{A^2} \|g\|_{A^2},$$

i.e. the Hankel form H_φ is bounded on $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$ in this case. Our first goal is to show that the converse does not hold. We define the weak product

$A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$ as the closure of all finite sums $f = \sum_k g_k h_k$, $g_k, h_k \in A^2(\mathbb{D}^\infty)$, under the norm

$$\|f\|_{A^2(\mathbb{D}^\infty)\odot A^2(\mathbb{D}^\infty)} = \inf \sum_k \|g_k\|_{A^2(\mathbb{D}^\infty)} \|h_k\|_{A^2(\mathbb{D}^\infty)}.$$

Here the infimum is taken over all finite representations $f = \sum_k g_k h_k$. Note that $\|f\|_{A^1(\mathbb{D}^\infty)} \leq \|f\|_{A^2(\mathbb{D}^\infty)\odot A^2(\mathbb{D}^\infty)}$.

Lemma 14. *Suppose that φ generates a Hankel form on $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$. Then*

$$\|H_\varphi\| = \|\varphi\|_{(A^2(\mathbb{D}^\infty)\odot A^2(\mathbb{D}^\infty))^*}.$$

Every bounded Hankel form H_φ extends to a bounded functional on $A^1(\mathbb{D}^\infty)$ if and only if there is a constant $C_\infty < \infty$ such that for any $f \in A^1(\mathbb{D}^\infty)$,

$$\|f\|_{A^2(\mathbb{D}^\infty)\odot A^2(\mathbb{D}^\infty)} \leq C_\infty \|f\|_{A^1(\mathbb{D}^\infty)}.$$

Proof. The first statement is a tautology. The weak product space $A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$ is a Banach space, and therefore the second statement follows from the closed graph theorem and duality (see [9, 25]). □

Factorization and weak factorization of Hardy and Bergman spaces have a long history. Strong factorization for $H^1(\mathbb{D})$ was treated by Nehari [32], and the analogous factorization for $A^1(\mathbb{D})$ was given by Horowitz [27]. Every f in $H^1(\mathbb{D})$ or $A^1(\mathbb{D})$ can be written as a single product $f = gh$, for g, h in $H^2(\mathbb{D})$ or $A^2(\mathbb{D})$, respectively. In Nehari's theorem it is even possible to choose g and h such that $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})}\|h\|_{H^2(\mathbb{D})}$. The same is not possible in the factorization of $A^1(\mathbb{D})$, a simple observation we do not find recorded in the literature.

Factorization on the polydisc $\mathbb{D}^d$ is a much subtler matter, even when $1 < d < \infty$. Strong factorization is certainly not possible, but in [20, 29] it was shown that the corresponding weak factorization holds,

$$H^1(\mathbb{D}) = H^2(\mathbb{D}^d) \odot H^2(\mathbb{D}^d), \qquad d < \infty.$$

The Bergman space analogue was established in [17],

$$A^1(\mathbb{D}^d) = A^2(\mathbb{D}^d) \odot A^2(\mathbb{D}^d), \qquad d < \infty.$$

In [35] it was shown that the best constant C_d in the factorization,

$$\|f\|_{H^2(\mathbb{D}^d)\odot H^2(\mathbb{D}^d)} \leq C_d \|f\|_{H^1(\mathbb{D}^d)},$$

satisfies growth estimate $C_d \geq (\pi^2/8)^{d/4}$ when d is an even integer. This immediately implies that the weak factorization $H^1(\mathbb{D}^\infty) = H^2(\mathbb{D}^\infty) \odot H^2(\mathbb{D}^\infty)$ is impossible. By tensorization, it is explained in [9, Sec. 3] that $C_{kd} \geq C_d^k$ for every positive integer k, a result which effortlessly carries over to the context of Bergman spaces. Hence we have the following.

Theorem 15. *Let C_d denote the best constant in the inequality*

$$\|f\|_{A^2(\mathbb{D}^d)\odot A^2(\mathbb{D}^d)} \le C_d \|f\|_{A^1(\mathbb{D}^d)},$$

for $d = 1, 2, \ldots$. Then

$$C_d \ge \left(\frac{9}{8}\right)^{d/2}.$$

In particular, the factorization in the unit disc is not norm-preserving, and therefore the weak factorization

$$A^1(\mathbb{D}^\infty) = A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$$

does not hold.

Proof. In view of the discussion preceeding the theorem, it is sufficient to prove that $C_1 \ge 3/(2\sqrt{2})$. For every polynomial φ, we get from duality that

$$C_1 \ge \frac{\|\varphi\|_{(A^1(\mathbb{D}))^*}}{\|\varphi\|_{(A^2(\mathbb{D})\odot A^2(\mathbb{D}))^*}} \ge \frac{\|\varphi\|^2_{A^2(\mathbb{D})}}{\|\varphi\|_{A^1(\mathbb{D})}\|\varphi\|_{(A^2(\mathbb{D})\odot A^2(\mathbb{D}))^*}},$$

where we have estimated the $(A^1(\mathbb{D}))^*$-norm by testing φ against itself. As in Lemma 14, we have that

$$\|\varphi\|_{(A^2(\mathbb{D})\odot A^2(\mathbb{D}))^*} = \|H_\varphi\|_{A^2(\mathbb{D})\times A^2(\mathbb{D})}.$$

We choose $\varphi(w) = \sqrt{2}\,w$. Clearly $\|\varphi\|_{A^2(\mathbb{D})} = 1$. The matrix of H_φ with respect to the standard basis of $A^2(\mathbb{D})$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so we find that $\|H_\varphi\|_{A^2(\mathbb{D})\times A^2(\mathbb{D})} = 1$. We are done, since

$$\|\varphi\|_{A^1(\mathbb{D})} = 2\sqrt{2}\int_0^1 r^2\,dr = \frac{2\sqrt{2}}{3}. \qquad \square$$

It would be interesting to decide if the symbol of the Hilbert–type form considered in Theorem 13, which lifts to

$$\varphi(z) = \sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}(\log n)^2}\, z^{\kappa(n)}, \tag{25}$$

defines a bounded linear functional on $A^1(\mathbb{D}^\infty)$. We are unable to settle this problem, but offer the following two observations. First, if f is an analytic polynomial on $\mathbb{D}^\infty$ such that $f(0) = 0$, we may write

$$\langle f, \varphi\rangle_{A^2(\mathbb{D}^\infty)} = \int_{1/2}^{\infty} \left(\mathscr{B}^{-1} f\right)(\sigma + it)\left(\sigma - \frac{1}{2}\right) d\sigma.$$

If we could prove the embedding $\|f\|_{A^1_{\mathrm{i}}(\mathbb{C}_{1/2})} \le \widetilde{C}\|f\|_{\mathscr{A}^1}$, which is a stronger version of the second statement in Theorem 10, then it would follow by simple Carleson measure argument that (25) defines a bounded linear functional on $A^1(\mathbb{D}^\infty)$, through the (inverse) Bohr lift.

Our second observation is contained in the following result.

Theorem 16. *Let φ be as in* (25). *Then φ defines a bounded functional on $A^p(\mathbb{D}^\infty)$ for every $1 < p < \infty$.*

Proof. This is trivial when $p \ge 2$, since $\varphi \in H^2(\mathbb{D}^\infty)$. Let us therefore fix $1 < p < 2$, and suppose that $f(z) = \sum_{n\ge 1} a_n z^{\kappa(n)}$ is in $A^p(\mathbb{D}^\infty)$. Then it follows from the Cauchy–Schwarz inequality and Lemma 9 with $r_j = \sqrt{p/2}$ that

$$\begin{aligned}\left|\langle f,\varphi\rangle_{A^2(\mathbb{D}^\infty)}\right| &= \left|\sum_{n=2}^{\infty} a_n \frac{1}{\sqrt{n}(\log n)^2}\right| \\ &\le \left(\sum_{n=2}^{\infty}\frac{|a_n|^2}{d(n)}\left(\frac{p}{2}\right)^{\Omega(n)}\right)^{\frac12}\left(\sum_{n=2}^{\infty}\left(\frac{2}{p}\right)^{\Omega(n)}\frac{d(n)}{n(\log n)^4}\right)^{\frac12} \\ &\le \|f\|_{A^p(\mathbb{D}^\infty)}\left(\sum_{n=2}^{\infty}\left(\frac{2}{p}\right)^{\Omega(n)}\frac{d(n)}{n(\log n)^4}\right)^{\frac12}\end{aligned}$$

where again $\Omega(n)$ denotes the number of prime factors of n. We may conclude if we can show that

$$\sum_{n=2}^{\infty}\frac{d(n)\alpha^{\Omega(n)}}{n(\log n)^4} < \infty$$

if $1 < \alpha < 2$. This follows at once from Abel summation and the estimate

$$\frac{1}{x}\sum_{n\le x} d(n)\alpha^{\Omega(n)} = C_\alpha(\log x)^{2\alpha-1} + O\left((\log x^{2\alpha-2})\right). \tag{26}$$

To demonstrate (26), we consider the associated Dirichlet series, for say $\mathrm{Re}(s) > 1$, and factor out an appropriate power of the zeta function

$$\begin{aligned} f_\alpha(s) = \sum_{n=1}^{\infty} d(n)\alpha^{\Omega(n)}n^{-s} &= \prod_{j=1}^{\infty}\left(\frac{1}{1-\alpha p_j^{-s}}\right)^2 \\ &= \zeta^{2\alpha}(s)\prod_{j=1}^{\infty}\left(\frac{(1-p_j^{-s})^\alpha}{1-\alpha p_j^{-s}}\right)^2 =: \zeta^{2\alpha}(s)g_\alpha(s).\end{aligned}$$

Note that since

$$\left(\frac{(1-p_j^{-s})^\alpha}{1-\alpha p_j^{-s}}\right)^2 = 1 + (\alpha-1)\alpha\, p_j^{-2s} + O(p_j^{-3s}),$$

the Dirichlet series g_α is absolutely convergent for

$$\operatorname{Re}(s) > \max(1/2, \log_2 \alpha).$$

A standard residue integration argument (see e.g. [42, Ch. II.5]) now gives (26) with $C_\alpha = g_\alpha(1)/\Gamma(2\alpha)$. $\square$

Next, we investigate Hilbert–Schmidt Hankel forms (2), following [25]. Recall that on the finite polydisc $\mathbb{D}^d$, $d < \infty$, a symbol φ generates a Hilbert–Schmidt Hankel form on $H^2(\mathbb{D}^d) \times H^2(\mathbb{D}^d)$ if and only if it generates a Hilbert–Schmidt Hankel form on $A^2(\mathbb{D}^d) \times A^2(\mathbb{D}^d)$. On the infinite polydisc we have the following result. Theorem 10 is its essential ingredient.

Theorem 17. *If the Hankel form generated by φ is Hilbert–Schmidt on $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$, then φ also generates a bounded functional on $A^1(\mathbb{D}^\infty)$. If φ generates a Hilbert–Schmidt form on $H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty)$, then it generates a Hilbert–Schmidt form on $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$, but the converse does not hold.*

Proof. First, we compute the Hilbert–Schmidt norm on $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$ of the form H_φ generated by the symbol $\varphi(s) = \sum_{n \geq 1} \overline{\varrho_n} z^{\kappa(n)}$. An orthonormal basis for $A^2(\mathbb{D}^\infty)$ is given by

$$e_n(z) = z^{\kappa(n)} \sqrt{d(n)}.$$

Hence,

$$\begin{aligned}\|H_\varphi\|^2_{S_2(A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty))} &= \sum_{m=1}^\infty \sum_{n=1}^\infty |H_\varphi(e_m e_n)|^2 = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{|\varrho_{mn}|^2 d(m) d(n)}{[d(mn)]^2} \\ &= \sum_{l=1}^\infty \frac{|\varrho_l|^2}{[d(l)]^2} \sum_{mn=l} d(m) d(n) = \sum_{l=1}^\infty |\varrho_l|^2 \frac{d_4(l)}{[d(l)]^2},\end{aligned}$$

where we have made use of (20) after recalling the convention that $d_2 = d$. The first statement now follows from Theorem 10, since the Cauchy–Schwarz inequality implies that

$$|\langle f, \varphi \rangle_{A^2(\mathbb{D}^\infty)}| = \left| \sum_{n=1}^\infty \frac{a_n \varrho_n}{d(n)} \right| \leq \left(\sum_{n=1}^\infty \frac{|a_n|^2}{d_4(n)} \right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty |\varrho_n|^2 \frac{d_4(n)}{[d(n)]^2} \right)^{\frac{1}{2}}.$$

Similarly we have that

$$\|H_\varphi\|^2_{S_2(H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty))} = \sum_{n=1}^\infty |\varrho_n|^2 d(n).$$

Note that when n is a prime power $n = p_j^k$ we have that

$$d_4(n) = \frac{(k+1)(k+2)(k+3)}{6} \leq (k+1)^3 = [d(n)]^3.$$

Since both $d_4(n)$ and $d(n)$ are multiplicative functions, it follows that $d_4(n) \le [d(n)]^3$ for every n. Hence the second statement is proved.

To see that the converse of the second statement does not hold, consider the set $\mathscr{N} = \{n_1 = 2,\, n_2 = 3 \cdot 5,\, n_3 = 7 \cdot 11 \cdot 13,\, \dots\}$ and define $\varphi(s) = \sum_{n \in \mathscr{N}} \overline{\varrho_n} z^{\kappa(n)}$. Then we have that

$$\|H_\varphi\|^2_{S_2(A^2(\mathbb{D}^\infty)\times A^2(\mathbb{D}^\infty))} = \sum_{j=1}^{\infty} |\varrho_{n_j}|^2,$$

$$\|H_\varphi\|^2_{S_2(H^2(\mathbb{D}^\infty)\times H^2(\mathbb{D}^\infty))} = \sum_{j=1}^{\infty} |\varrho_{n_j}|^2 2^j. \qquad \square$$

The final part of this section is devoted to showing that every Hankel form of the type (2) naturally corresponds to a Hankel form of the type (1) with the same singular numbers. Let D denote the diagonal operator in two variables, $Df(w) = f(w, w)$, for which we have the following observation.

Lemma 18. *The operator D is a contraction from $H^2(\mathbb{D}^2)$ to $A^2(\mathbb{D})$.*

Proof. This is proven in [39], but in an abstract formulation it dates back at least to Aronzajn [2]. The proof of our particular case is very easy and we include it here. Consider

$$f(z_1, z_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j,k} z_1^j z_2^k$$

and use the Cauchy–Schwarz inequality to conclude that

$$\|Df\|^2_{A^2(\mathbb{D})} = \sum_{l=0}^{\infty} \frac{1}{l+1} \left| \sum_{j+k=l} a_{j,k} \right|^2 \le \sum_{l=0}^{\infty} \sum_{j+k=l} |a_{j,k}|^2 = \|f\|^2_{H^2(\mathbb{D}^2)}. \qquad \square$$

The diagonal operator D may be written as an integral operator using the reproducing kernel of $H^2(\mathbb{D}^2)$,

$$Df(w) = \int_{\mathbb{T}^2} f(z_1, z_2)\, \frac{1}{1 - w\overline{z_1}} \frac{1}{1 - w\overline{z_2}}\, dm_1(z_1) dm_1(z_2).$$

Hence its adjoint operator $E\colon A^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is given by

$$Eg(z_1, z_2) = \int_{\mathbb{D}} g(w)\, \frac{1}{1 - z_1\overline{w}} \frac{1}{1 - z_2\overline{w}}\, dA(w).$$

If f and g are in $A^2(\mathbb{D})$, then

$$\langle Ef, Eg \rangle_{H^2(\mathbb{D}^2)} = \langle f, g \rangle_{A^2(\mathbb{D})},$$

that is, E is an isometry. Clearly, the composition DE is the identity operator on $A^2(\mathbb{D})$. Hence we have identified $A^2(\mathbb{D})$ with the subspace $X = EA^2(\mathbb{D})$ of $H^2(\mathbb{D}^2)$ (although perhaps it would be more appropriate to think of it as the factor space induced by the map D). The projection $P\colon H^2(\mathbb{D}^2) \to X$ is given by $P = ED$. Note that P averages the coefficients of monomials of same degree. Precisely, if $f(z) = \sum_{j,k\ge 0} a_{j,k} z_1^j z_2^k$, then

$$Pf(z_1,z_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{j+k} z_1^j z_2^k, \qquad \text{where} \qquad A_l = \frac{1}{l+1}\sum_{j+k=l} a_{j,k}.$$

Clearly, $D(fg) = D(f)D(g)$, but E does not have this property. For example, if $g(w) = w$, then

$$Eg(z_1,z_2) = \frac{z_1+z_2}{2} \qquad \text{and} \qquad E(g^2)(z_1,z_2) = \frac{z_1^2 + z_1z_2 + z_2^2}{3},$$

so that $E(g)E(g) \ne E(g^2)$.

Let us now turn to the relationship between the operator E and Hankel forms. To fix the notation, let Y be a Hilbert space with an orthonormal basis $\{e_j\}_{j\ge 1}$. For a bilinear form $H\colon Y \times Y \to \mathbb{C}$, let $s_n(H)$ denote its nth singular value, i.e.

$$s_n(H) = \inf\{\|H - K\|_{Y\times Y} \;:\; \operatorname{rank} K \le n\},$$

where the rank of a bilinear form $K : Y \times Y \to \mathbb{C}$ is given by

$$\operatorname{rank} K = \operatorname{codim}\ker K = \operatorname{codim}\{f \in Y \;:\; K(f,g) = 0 \text{ for all } g \in Y\}.$$

Of course, $s_n(H)$ is the same as the nth singular value of the operator

$$\{H(e_j,e_k)\}_{j,k\ge 1} : \ell^2 \to \ell^2.$$

The p–Schatten norm of H, $0 < p < \infty$, is given by

$$\|H\|^p_{S_p(Y\times Y)} = \sum_{n=0}^{\infty} |s_n(H)|^p.$$

When $p = 2$ we obtain the Hilbert-Schmidt norm, which can also be computed as the square sum of the coefficients,

$$\|H\|^2_{S_2(Y\times Y)} = \sum_{n=0}^{\infty} |s_n(H)|^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |H(e_j,e_k)|^2.$$

We have the following result.

Lemma 19. *Suppose that $\varphi \in A^2(\mathbb{D})$. Then*

$$s_n(H_\varphi) = s_n(H_{E\varphi}), \qquad n \ge 0.$$

In particular, for $0 < p < \infty$ we have

$$\|H_\varphi\|_{A^2(\mathbb{D})\times A^2(\mathbb{D})} = \|H_{E\varphi}\|_{H^2(\mathbb{D}^2)\times H^2(\mathbb{D}^2)},$$
$$\|H_\varphi\|_{S_p(A^2(\mathbb{D})\times A^2(\mathbb{D}))} = \|H_{E\varphi}\|_{S_p(H^2(\mathbb{D}^2)\times H^2(\mathbb{D}^2))}.$$

Proof. Let $J : X \times X \to \mathbb{C}$ be the restriction of $H_{E\varphi}$ to $X = EA^2(\mathbb{D})$,

$$J(f,g) = \langle fg, E\varphi\rangle_{H^2(\mathbb{D}^2)}, \qquad f,g \in X.$$

For $f, g \in H^2(\mathbb{D}^2)$ we have the identity

$$\langle fg, E\varphi\rangle_{H^2(\mathbb{D}^2)} = \langle D(fg), \varphi\rangle_{A^2(\mathbb{D})} = \langle DfDg, \varphi\rangle_{A^2(\mathbb{D})}. \tag{27}$$

Since $D : X \to A^2(\mathbb{D})$ is unitary, this implies that J is unitarily equivalent to $H_\varphi : A^2(\mathbb{D}) \times A^2(\mathbb{D}) \to \mathbb{C}$. If $K : H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2) \to \mathbb{C}$ is a rank-n form, then its restriction to X, $K' : X \times X \to \mathbb{C}$, has smaller rank, $\operatorname{rank} K' \le n$. Since

$$\|H_{E\varphi} - K\|_{H^2(\mathbb{D}^2)\times H^2(\mathbb{D}^2)} \ge \|J - K'\|_{X\times X}$$

it follows that

$$s_n(H_{E\varphi}) \ge s_n(J) = s_n(H_\varphi), \qquad n \ge 0.$$

Conversely, if the form $K : A^2(\mathbb{D}) \times A^2(\mathbb{D}) \to \mathbb{C}$ has rank n, then clearly $K' : H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2) \to \mathbb{C}$ has smaller rank, where $K'(f,g) = K(Df, Dg)$, for $f, g \in H^2(\mathbb{D}^2)$. However, it follows from (27) and Lemma 18 that

$$\|H_\varphi - K\| = \|H_{E\varphi} - K'\|,$$

proving that also $s_n(H_\varphi) \ge s_n(H_{E\varphi})$. □

Consider $A^2(\mathbb{D}^\infty)$ as a function space over the variables $z = (z_1, z_2, \ldots)$ and $H^2(\mathbb{D}^\infty)$ as a function space over $\xi = (\xi_1, \xi_2, \ldots)$. Define the extension map $\mathscr{E}$ from $A^2(\mathbb{D}^\infty)$ to $H^2(\mathbb{D}^\infty)$ by its integral kernel,

$$K_\xi(z) = \prod_{j=1}^{\infty} \frac{1}{1-\xi_{2j-1}\overline{z_j}}\,\frac{1}{1-\xi_{2j}\overline{z_j}}, \qquad z, \xi \in \mathbb{D}^\infty \cap \ell^2,$$

so that

$$\mathscr{E}f(\xi) = \int_{\mathbb{D}^\infty} f(z)K_\xi(z)\,d\mathbf{m}(z).$$

By tensorization of Lemma 19 (the required technical details may be found in [9, Lem. 2]), we obtain the following.

Theorem 20. *The map $\mathscr{E}$ has the following properties.*

(a) *$\mathscr{E}$ defines an isometric isomorphism from the Bergman space $A^2(\mathbb{D}^\infty)$ to a subspace of the Hardy space $H^2(\mathbb{D}^\infty)$.*

(b) *For $\varphi \in A^2(\mathbb{D}^\infty)$, let $H_\varphi : A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty) \to \mathbb{C}$ be the Hankel form generated by φ, and let $H_{\mathscr{E}\varphi} : H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty) \to \mathbb{C}$ be the Hankel form generated by $\mathscr{E}\varphi$. Then, for every $n \ge 0$, we have that*

$$s_n(H_\varphi) = s_n(H_{\mathscr{E}\varphi}).$$

In particular, H_φ is bounded (p–Schatten, $0 < p < \infty$) if and only if $H_{\mathscr{E}\varphi}$ is bounded (p–Schatten), with equality of the norms.

Remark. In [35], the symbol $\psi(z) = (z_1 + z_2)/2$ is used to show that the weak factorization $H^1(\mathbb{D}^\infty) = H^2(\mathbb{D}^\infty) \odot H^2(\mathbb{D}^\infty)$ cannot hold. In Theorem 15 the symbol $\varphi(w) = w$ is used to demonstrate the corresponding fact for the Bergman spaces. In fact the two examples considered are the same, because $E\varphi = \psi$.

5. Carleson measures on the infinite polydisc

We end this paper by producing two infinite dimensional counter-examples to well-known finite dimensional results for Carleson measures for the Hardy spaces $H^p(\mathbb{D}^d)$. Let μ be a finite positive measure on $\mathbb{D}^d$ (where possibly $d = \infty$), i.e. a finite positive Borel measure on $\overline{\mathbb{D}}^d$ such that $\mu(\overline{\mathbb{D}}^d \setminus \mathbb{D}^d) = 0$. As usual, measures on the compact space $\overline{\mathbb{D}}^d$ correspond to linear functionals on the space of continuous functions $C(\overline{\mathbb{D}}^d)$. We say that μ is a H^p–Carleson measure if there exists a constant $C = C(\mu_d, p) < \infty$ such that

$$\int_{\mathbb{D}^d} |f(z)|^p \, d\mu_d(z) \leq C \|f\|^p_{H^p(\mathbb{D}^d)}$$

for every analytic polynomial f. We say that μ is a L^p–Carleson measure if there exists a constant $C = C(\mu_d, p) < \infty$ such that

$$\int_{\mathbb{D}^d} |\mathscr{P} f(z)|^p \, d\mu_d(z) \leq C \|f\|^p_{L^p(\mathbb{T}^d)}$$

for every trigonometric polynomial f. Here $\mathscr{P} f$ is the Poisson extension of f, defined for $f \in L^p(\mathbb{T}^d)$ by

$$\mathscr{P} f(w) = \int_{\mathbb{T}^d} f(z) \mathscr{P}_w(z) \, d\mathbf{m}_1(z), \qquad \mathscr{P}_w(z) = \prod_{j=1}^{d} \frac{1 - |w_j|^2}{|1 - \overline{z_j} w_j|^2}.$$

This is always well-defined as long as we restrict ourselves to $L^2(\mathbb{T}^d)$-functions f only dependent on a finite number of variables, since we may then suppose that w is finitely supported.

The study of Carleson measures on the infinite polydisc is an important part of the theory of H^p spaces. For instance, the local embedding problem discussed in [41, Sec. 3] can be formulated in terms of Carleson measures. Let $\mathscr{B}^{-1}$ denote the inverse Bohr lift, so that

$$\big(\mathscr{B}^{-1} f\big)(s) = f\big(2^{-s}, \, 3^{-s}, \, 5^{-s}, \dots \, p_j^{-s}, \dots\big).$$

For $0 < p < \infty$, is it true that the measure μ_∞ defined on $\mathbb{D}^\infty$ by

$$\int f(z) d\mu_\infty(z) = \int_0^1 \big(\mathscr{B}^{-1} f\big)(1/2 + it) \, dt, \qquad f \in C(\overline{\mathbb{D}}^d),$$

is a H^p–Carleson measure? A positive answer is only known for even integers. Additionally, the boundedness of positive definite Hankel forms (1) can be formulated in terms of Carleson measures on $\mathbb{D}^\infty$ [36], and the same is true for the Volterra operators studied in [10].

From [15], it is known that a measure μ on $\mathbb{D}^d$, for $d < \infty$, is a H^p–Carleson measure for one $0 < p < \infty$ if and only if it is a Carleson measure for every $0 < p < \infty$. We will now construct a counter-example to this statement when $d = \infty$. We recall that the diagonal restriction operator $Df(w) = f(w,w)$ induces a bounded map from $H^p(\mathbb{D}^2)$ to $A^p(\mathbb{D})$ for every $0 < p < \infty$ (see [19]), and offer the following clarification in the case $0 < p < 2$.

Lemma 21. *The diagonal operator D is not contractive from $H^p(\mathbb{D}^2)$ to $A^p(\mathbb{D})$ when $0 < p < 2$.*

Proof. Let $0 < p < 2$ and consider $f(z_1,z_2) = (z_1+z_2)/2$. Clearly

$$\|Df\|_{A^p(\mathbb{D})}^p = \int_{\mathbb{D}} |f(w,w)|^p \, dm(w) = \frac{2}{2+p},$$

so it is enough to verify that $\|f\|_{H^p(\mathbb{D}^2)}^p < 2/(2+p)$. We factor out z_2 and compute using various identities for the Beta and Gamma functions, obtaining that

$$\begin{aligned}
\|f\|_{H^p(\mathbb{D}^2)}^p &= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1+e^{i\theta}}{2}\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\cos\frac{\theta}{2}\right|^p d\theta \\
&= \frac{1}{\pi}\int_0^{\pi}\left(\cos\frac{\theta}{2}\right)^p d\theta = \frac{2}{\pi}\int_0^1 \frac{t^p}{\sqrt{1-t^2}}\,dt \\
&= \frac{1}{\pi}\int_0^1 t^{(p-1)/2}(1-t)^{-1/2}\,dt = \frac{\mathrm{B}((p+1)/2), 1/2)}{\pi} \\
&= \frac{\Gamma(p/2+1/2)\Gamma(1/2)}{\pi\Gamma(p/2+1)} = \frac{\Gamma(p/2+1/2)}{\Gamma(1/2)(p/2)\Gamma(p/2)} = \frac{2}{p\,\mathrm{B}(p/2,1/2)}.
\end{aligned}$$

To conclude we make use of the identity

$$\mathrm{B}(x,y) = \sum_{n=0}^{\infty}\binom{n-y}{n}\frac{1}{x+n}, \qquad x,y>0.$$

The binomial coefficient is positive for every n when $y = 1/2$, so if $0 < p < 2$ we have that

$$\mathrm{B}(p/2,1/2) > \frac{1}{p/2} + \mathrm{B}(1,1/2) - \frac{1}{1} = \frac{2}{p} + 1. \qquad \square$$

Remark. Lemma 18 implies that D is a contraction from $H^p(\mathbb{D}^2)$ to $A^p(\mathbb{D})$ if p is an even integer. It would be interesting to know if D is a contraction for every $p \geq 2$.

Tensorization of Lemma 18 and Lemma 21 yields the following result.

Theorem 22. *Let μ_∞ be the measure defined for f in $C(\overline{\mathbb{D}}^\infty)$ by*

$$\int_{\mathbb{D}^\infty} f(z_1, z_2, z_3, z_4, \ldots)\, d\mu_\infty(z) = \int_{\mathbb{D}^\infty} f(z_1, z_1, z_3, z_3, \ldots)\, d\mathbf{m}(z), \tag{28}$$

where $\mathbf{m}$ denotes the infinite product of the unweighted normalized Lebesgue measure on $\mathbb{D}$. The measure μ_∞ is a H^p–Carleson measure on $\mathbb{D}^\infty$ if p is an even integer, but not when $0 < p < 2$.

Theorem 22 invites the following question.

Question. If μ defines a H^p–Carleson measure on $\mathbb{D}^\infty$ for some $0 < p < \infty$, does it also define a H^q–Carleson measure for every $p < q < \infty$?

In [15], it is also proven that L^p–Carleson and H^p–Carleson measures coincide on $\mathbb{D}^d$, when $d < \infty$. Again, this is no longer true on $\mathbb{D}^\infty$, as our next two examples will demonstrate.

To obtain the first counter-example, we verify that the measure (28) of Theorem 22 does not define a L^2–Carleson measure on $\mathbb{D}^\infty$ by replacing Lemma 21 with the following result.

Lemma 23. *The operator $D \circ \mathscr{P}$ is not a contraction from $L^2(\mathbb{T}^2)$ to $L^2(\mathbb{D}, m)$.*

Proof. Consider

$$f(e^{i\theta_1}, e^{i\theta_2}) = \frac{1}{\sqrt{3}}\left(e^{i\theta_1} + e^{i\theta_2} + e^{i2\theta_1}e^{-i\theta_2}\right),$$

for which clearly $\|f\|_{L^2(\mathbb{T}^2)} = 1$. Furthermore, we find that

$$\mathscr{P}f(re^{i\theta}, re^{i\theta}) = \frac{e^{i\theta}}{\sqrt{3}}(2r + r^3),$$

so it follows that

$$\int_{\mathbb{D}} |\mathscr{P}f(z,z)|^2\, dm(z) = \frac{2}{3}\int_0^1 \left(2r + r^3\right)^2\, rdr = \frac{43}{36} > 1. \qquad \square$$

Our second counter-example is obtained through the connection with Dirichlet series. In preparation, let us recall a few properties of $L^2(\mathbb{T}^\infty)$. Let $\mathbb{Q}_+$ denote the set of positive rational numbers. Each $q \in \mathbb{Q}_+$ has a finite expansion of the form

$$q = \prod_{j=1}^{\infty} p_j^{\kappa_j},$$

where $\kappa_j \in \mathbb{Z}$. Hence $\mathbb{Q}_+$ can be identified with the set of all finite multi-indices. As in (18), every function $f \in L^2(\mathbb{T}^\infty)$ has an expansion

$$f(z) = \sum_{q \in \mathbb{Q}_+} a_q z^{\kappa(q)}, \qquad \|f\|^2_{L^2(\mathbb{T}^\infty)} = \sum_{q \in \mathbb{Q}_+} |a_q|^2.$$

Note that if $f \in L^2(\mathbb{T}^{d'})$ for some $d' < \infty$ and $s = \sigma + it$, then

$$\big(\mathscr{B}^{-1}\mathscr{P}f\big)(s) = \sum_{q\in\mathbb{Q}_+} a_q (q_+)^{-\sigma} q^{-it}, \qquad \text{where} \qquad q_+ = \prod_{j=1}^{\infty} p_j^{|\kappa_j|}.$$

As our final preliminary, let $\omega(n)$ denote the number of distinct prime factors of n. It is well-known that if $\mathrm{Re}(s) > 1$, then

$$\frac{[\zeta(s)]^2}{\zeta(2s)} = \prod_{j=1}^{\infty} \frac{1+p_j^{-s}}{1-p_j^{-s}} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-s}.$$

Theorem 24. *Let μ_∞ be the measure defined for f in $C(\overline{\mathbb{D}}^\infty)$ by*

$$\int_{\mathbb{D}^\infty} f(z)\, d\mu_\infty(z) = \int_0^1 \big(\mathscr{B}^{-1}f\big)(1/2+\sigma)\, d\sigma.$$

Then μ_∞ is a H^2–Carleson measure but not a L^2–Carleson measure.

Proof. It is well-known that μ_∞ is a H^2-Carleson measure [11, 34]. Let us therefore prove that μ_∞ is not a L^2–Carleson measure. Fix $\varepsilon > 0$ and define $w \in \mathbb{D}^\infty \cap \ell^2$ by $w_j = p_j^{-1/2-\varepsilon}$. We will consider the kernel of the d-dimensional Poisson transform,

$$f_d(z) = \prod_{j=1}^{d} \frac{1-|w_j|^2}{|1-\overline{z_j}w_j|^2}.$$

First observe that

$$\lim_{d\to\infty} \|f_d\|^2_{L^2(\mathbb{T}^\infty)} = \prod_{j=1}^{\infty} \frac{1-p_j^{-2-4\varepsilon}}{(1-p_j^{-1-2\varepsilon})^2} = \frac{[\zeta(1+2\varepsilon)]^2}{\zeta(2+4\varepsilon)} \asymp \varepsilon^{-2}.$$

Next, we have that

$$\lim_{d\to\infty} \big(\mathscr{B}^{-1}\mathscr{P}f_d\big)(1/2+\sigma) = \sum_{q\in\mathbb{Q}_+} q_+^{-1-\varepsilon-\sigma},$$

uniformly convergent in $\sigma \in [0,1]$. Note that

$$\sum_{q\in\mathbb{Q}_+} q_+^{-1-\varepsilon-\sigma} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-1-\varepsilon-\sigma} \asymp (\sigma+\varepsilon)^{-2},$$

since there are $2^{\omega(n)}$ rational numbers $q \in \mathbb{Q}_+$ such that $q_+ = n$. This concludes the argument, since

$$\int_0^1 \frac{d\sigma}{(\sigma+\varepsilon)^4} \asymp \varepsilon^{-3}. \qquad \square$$

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Part 3

Volterra operators and pseudomoments

Paper 10

Volterra operators on Hardy spaces of Dirichlet series

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VOLTERRA OPERATORS ON HARDY SPACES OF DIRICHLET SERIES

OLE FREDRIK BREVIG, KARL-MIKAEL PERFEKT, AND KRISTIAN SEIP

ABSTRACT. For a Dirichlet series symbol $g(s) = \sum_{n\geq 1} b_n n^{-s}$, the associated Volterra operator $\mathbf{T}_g$ acting on a Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is defined by the integral

$$f \mapsto -\int_s^{+\infty} f(w)g'(w)\,dw.$$

We show that $\mathbf{T}_g$ is a bounded operator on the Hardy space $\mathscr{H}^p$ of Dirichlet series with $0 < p < \infty$ if and only if the symbol g satisfies a Carleson measure condition. When appropriately restricted to one complex variable, our condition coincides with the standard Carleson measure characterization of BMOA($\mathbb{D}$). A further analogy with classical BMO is that $\exp(c|g|)$ is integrable (on the infinite polytorus) for some $c > 0$ whenever $\mathbf{T}_g$ is bounded. In particular, such g belong to $\mathscr{H}^p$ for every $p < \infty$. We relate the boundedness of $\mathbf{T}_g$ to several other BMO type spaces: BMOA in half-planes, the dual of $\mathscr{H}^1$, and the space of symbols of bounded Hankel forms. Moreover, we study symbols whose coefficients enjoy a multiplicative structure and obtain coefficient estimates for m-homogeneous symbols as well as for general symbols. Finally, we consider the action of $\mathbf{T}_g$ on reproducing kernels for appropriate sequences of subspaces of $\mathscr{H}^2$. Our proofs employ function and operator theoretic techniques in one and several variables; a variety of number theoretic arguments are used throughout the paper in our study of special classes of symbols g.

1. INTRODUCTION

By a result of Pommerenke [32], the Volterra operator associated with an analytic function g on the unit disc $\mathbb{D}$, defined by the formula

$$T_g f(z) := \int_0^z f(w)g'(w)\,dw, \qquad z \in \mathbb{D}, \tag{1.1}$$

is a bounded operator on the Hardy space $H^2(\mathbb{D})$ if and only if g belongs to the analytic space of bounded mean oscillation BMOA($\mathbb{D}$). In view of the factorization $H^2 \cdot H^2 = H^1$ and C. Fefferman's famous duality theorem, according to

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which BMOA($\mathbb{D}$) is the dual of $H^1(\mathbb{D})$, it follows that T_g is bounded if and only if the corresponding Hankel form H_g is bounded, where

$$H_g(f,h) := \int_{\mathbb{T}} f(z)h(z)\overline{g(z)}\,dm_1(z), \qquad f,h \in H^2(\mathbb{D}).$$

In recent years, it has become known how to give a direct proof of the equivalence of the boundedness of T_g and H_g [3], with no mention of bounded mean oscillation (BMO) or Carleson measures, relying instead on the square function characterization of H^1 to show that $T_g f$ is in $H^1(\mathbb{D})$ whenever f and g are in $H^2(\mathbb{D})$. Although the systematic study of T_g was conducted much later than that of the Hankel form H_g (see [2, 4]), one could now, based on this insight, easily imagine an exposition of the one variable Hardy space theory which considers the boundedness of Volterra operators *before* BMOA and Hankel operators. One advantage would then be that the John–Nirenberg inequality, by Pommerenke's trick [32], has an elementary proof for functions g such that T_g is bounded.

This conception of Volterra operators, as objects of primary interest for understanding BMO, underlies the present investigation of such operators on Hardy spaces of Dirichlet series $\mathscr{H}^p$ with $0 < p < \infty$. The precise definition of these spaces will be given in the next section; suffice it to say at this point that every Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ in $\mathscr{H}^p$ defines an analytic function for $\operatorname{Re} s > 1/2$, and that $\mathscr{H}^p$ can be identified with the Hardy space $H^p(\mathbb{D}^\infty)$ of the countably infinite polydisc $\mathbb{D}^\infty$, through the Bohr lift. For a Dirichlet series symbol $g(s) = \sum_{n\geq 1} b_n n^{-s}$, we consider the Volterra operator $\mathbf{T}_g$ defined by

$$\mathbf{T}_g f(s) := -\int_s^{+\infty} f(w)g'(w)\,dw, \qquad \operatorname{Re} s > 1/2. \tag{1.2}$$

We denote the space of symbols g such that $\mathbf{T}_g : \mathscr{H}^p \to \mathscr{H}^p$ is bounded by $\mathscr{X}_p$. The index $p = 2$ is special, and we frequently write $\mathscr{X}$ instead of $\mathscr{X}_2$.

A general question of interest in the theory of Hardy spaces of Dirichlet series is to reveal how the different roles and interpretations of BMO manifest themselves in this infinite-dimensional setting. The space of symbols generating bounded Hankel forms has been shown to be significantly larger than $(\mathscr{H}^1)^*$ [30], and the space $(\mathscr{H}^1)^*$ itself also lacks many of the familiar features from the finite-dimensional setting. For instance, a function f in $(\mathscr{H}^1)^*$ does not always belong to $\mathscr{H}^p$ for every $p < \infty$ [26]. By Pommerenke's trick, however, it is almost immediate that the corresponding inclusion does hold for the space $\mathscr{X}$, i.e.,

$$\mathscr{X} \subset \bigcap_{0<p<\infty} \mathscr{H}^p.$$

Furthermore, $(\mathscr{H}^1)^*$ is notoriously difficult to deal with, in part owing to the fact that $H^p(\mathbb{D}^\infty)$, viewed as a subspace of $L^p(\mathbb{T}^\infty)$, is not complemented when $p \neq 2$. We shall find that the space $\mathscr{X}$ is significantly easier to manage.

One of our main results is that the spaces $\mathscr{X}_p$ can be characterized by a Carleson measure condition, in analogy with what we have in the classical one variable theory. In our context, the Carleson measure associated with the symbol g will live on the product of $\mathbb{T}^\infty$ and a half-line. Again deferring precise definitions to the next section, we mention that this result takes the following form: The symbol g belongs to $\mathscr{X}_p$ if and only if there exists a constant C (depending on g and p) such that

$$\int_{\mathbb{T}^\infty}\int_0^\infty |f_\chi(\sigma)|^p|g'_\chi(\sigma)|^2\sigma\, d\sigma dm_\infty(\chi) \le C\|f\|^p_{\mathscr{H}^p}$$

holds for every f in $\mathscr{H}^p$. Here m_∞ denotes Haar measure on $\mathbb{T}^\infty$, while χ is a character on $\mathbb{T}^\infty$ and $f_\chi(s) := \sum_{n\ge1} a_n\chi(n)n^{-s}$ for the Dirichlet series $f(s) = \sum_{n\ge1} a_n n^{-s}$. This result, proved in Section 5, is based on an adaption to our setting of an ingenious argument from a recent paper of Pau [31]. Our Carleson measure condition gives us the opportunity to study non-trivial Carleson embeddings on the polydisc $\mathbb{D}^\infty$, see Sections 5.2 and 5.3. Our understanding is incomplete, but some of the questions asked are more tractable than the important embedding problem of $\mathscr{H}^p$ (see [34, Sec. 3]) while still being of a similar character. In the classical setting, the description in terms of Carleson measures shows that T_g is bounded on $H^p(\mathbb{D})$ if and only if it is bounded on $H^2(\mathbb{D})$. We will see that our Carleson measure characterization implies that if g is in $\mathscr{X}_p$, then g is in $\mathscr{X}_{kp}$ for every positive integer k. As is typical in this setting, we have not been able to do better than this for a general symbol g, and the following interesting problem remains open:

Question 1. Is $\mathbf{T}_g$ bounded on $\mathscr{H}^2$ if and only if it is bounded on $\mathscr{H}^p$ for every $p < \infty$?

We are able to give an affirmative answer to this question only in the case when g is a linear symbol, i.e., when g has non-zero coefficients only at the primes p_j so that $g(s) = \sum_{j\ge1} a_j p_j^{-s}$.

Before proceeding to give a closer description of our results, we would like to mention another open problem related to Question 1. In Section 6, we will observe that if $\mathbf{T}_g : \mathscr{H}^2 \to \mathscr{H}^2$ is bounded, then the corresponding multiplicative Hankel form is bounded. Furthermore, we will show that if $\mathbf{T}_g : \mathscr{H}^1 \to \mathscr{H}^1$ is bounded, then g is in $(\mathscr{H}^1)^*$. Hence, if the answer to Question 1 is positive, then so is the answer to the following.

Question 2. Do we have $\mathscr{X}_2 \subset (\mathscr{H}^1)^*$?

The reverse inclusion is easily shown to be false. In fact, it is not even true when formulated for the finite-dimensional polydisc $\mathbb{D}^2$ (see Theorem 6.6).

To give appropriate background and motivation for our general result about Carleson measures, we have chosen to begin by exploring in some detail the distinguished space $\mathscr{X}_2$ and its many interesting facets. This will allow us to exhibit the ubiquitous presence of number theoretic arguments in our subject, which is a consequence of our operators $\mathbf{T}_g$ being defined in terms of integrals on the half-plane $\operatorname{Re} s > 1/2$. Roughly speaking, if trying to understand $\mathbf{T}_g$ at the level of the coefficients of $\mathbf{T}_g f$, one has to investigate the interplay between the number of divisors $d(n)$ of an integer n and its logarithm, $\log n$. One may also analyze symbols of number theoretic interest in terms of their function theoretic properties. In fact, our first interesting example of a bounded Volterra operator $\mathbf{T}_g : \mathscr{H}^p \to \mathscr{H}^p$, will be established by the result, shown in Section 2, that the primitive of the Riemann zeta function,

$$g(s) = -\int (\zeta(s+1) - 1)\, ds = \sum_{n=2}^{\infty} \frac{1}{n \log n} n^{-s},$$

is of bounded mean oscillation on the line $\operatorname{Re} s = 0$. Such a BMO condition easily implies that g is in $\mathscr{X}_2$, and also that g is in $\mathscr{X}_p$ for $0 < p < \infty$, once our Carleson measure condition is in place.

To close this introduction, we now describe briefly the contents of the six subsequent sections of this paper. We begin in Section 2 by introducing the Hardy spaces $\mathscr{H}^p$ and start from the preliminary result that $\mathscr{H}^\infty \subset \mathscr{X} \subset \bigcap_{0<p<\infty} \mathscr{H}^p$. In our setting, there is a considerable gap between $\mathscr{H}^\infty$ and $\bigcap_{0<p<\infty} \mathscr{H}^p$, as for instance functions in $\mathscr{H}^\infty$ are bounded analytic functions in the half-plane $\operatorname{Re} s > 0$, while functions in $\bigcap_{0<p<\infty} \mathscr{H}^p$ in general will be analytic in the smaller half-plane $\operatorname{Re} s > 1/2$. In Section 2, the main point is to demonstrate how $\mathscr{X}$ can be thought of as a space of BMO functions in the classical sense. Using the notation $\mathbb{C}_\theta$ for the half-plane $\{s : \operatorname{Re}(s) > \theta\}$ and $\mathscr{D}$ for the class of functions expressible as a Dirichlet series in some half-plane $\mathbb{C}_\theta$, we prove that

$$\mathrm{BMOA}(\mathbb{C}_0) \cap \mathscr{D} \subset \mathscr{X} \subset \mathrm{BMOA}(\mathbb{C}_{1/2}),$$

and we also show that $e^{c|g|}$ is integrable for some positive constant c whenever g is in $\mathscr{X}$.

Section 3 and Section 4 investigate properties of $\mathscr{X}$ with no counterparts in the classical theory. After showing that the primitive of $\zeta(s+\alpha) - 1$ is in $\mathscr{X}$ if and only $\alpha \geq 1$, we make in Section 3 a finer analysis by identifying and studying a scale of symbols associated with the limiting case $\alpha = 1$. More specifically, we find that if we replace p^{-1-s} in the Euler product for $\zeta(s+1)$ by $\lambda(\log p)p^{-1-s}$, then this new symbol is in $\mathscr{X}$ if and only if $\lambda \leq 1$, the point being to nail down the exact edge for a symbol to be in $\mathscr{X}$ when its coefficients enjoy a multiplicative structure. The methods used to prove this result come from two number theoretic papers of respectively Hilberdink [24] and Gál [19].

In Section 4, we deduce conditions on the coefficients b_n of a symbol $g(s) = \sum_{n\geq 1} b_n n^{-s}$ to be in $\mathscr{X}$. We begin by showing that a linear symbol is in $\mathscr{X}$ if and only if g is in $\mathscr{H}^2$. This leads naturally to a consideration of m-homogeneous symbols, i.e., symbols such that b_n is nonzero only if n has m prime factors, counting multiplicity. We obtain optimal weighted ℓ^2-conditions for every $m \geq 2$, showing in particular that the Dirichlet series of g in general converges in $\mathbb{C}_{1/m}$ and in no larger half-plane. Letting m tend to ∞, we find that there exists a positive constant c, not larger than $2\sqrt{2}$, such that

$$\|\mathbf{T}_g\| \leq C\left(|b_2|^2 + \sum_{n=3}^{\infty} |b_n|^2 n e^{-c\sqrt{\log n \log\log n}}\right)^{1/2}$$

holds for every g in $\mathscr{X}$. These results are inspired by and will be compared with analogous results of Queffélec et al. [5, 27] on Bohr's absolute convergence problem for homogeneous Dirichlet series.

Section 5 begins with our general result about Carleson measures and is subsequently concerned with a study of to what extent our results for $\mathscr{X}_2$ carry over to $\mathscr{X}_p$. As already mentioned, our understanding remains incomplete, but we will see that a fair amount of nontrivial conclusions can be drawn from our general condition.

In the last two sections, we return again to the Hilbert space setting. Section 6 explores the relationship between $\mathbf{T}_g$, Hankel operators, and the dual of $\mathscr{H}^1$. In particular, this section gives background for what we have listed as Question 2 above. Finally, Section 7 investigates the compactness of $\mathbf{T}_g$, with particular attention paid to the action of $\mathbf{T}_g$ on reproducing kernels. Here we return to the symbols considered in Section 3 which will allow us to display an example of a non-compact $\mathbf{T}_g$-operator.

Notation. We will use the notation $f(x) \ll g(x)$ if there is some constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all (appropriate) x. If we have both $f(x) \ll g(x)$ and $g(x) \ll f(x)$, we will write $f(x) \asymp g(x)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

we write $f(x) \sim g(x)$. The increasing sequence of prime numbers will be denoted by $\{p_j\}_{j\geq 1}$, and the subscript will sometimes be dropped when there can be no confusion. Given a positive rational number r, we will denote the prime number factorization

$$r = p_1^{\kappa_1} p_2^{\kappa_2} \cdots p_d^{\kappa_d}$$

by $r = (p_j)^\kappa$. This associates uniquely to r the finite multi-index $\kappa(r) = (\kappa_1, \kappa_2, \ldots)$. For χ in $\mathbb{T}^\infty$, we set $\chi(r) := (\chi_j)^\kappa$, when $r = (p_j)^\kappa$. If r is an integer, say n, then the multi-index $\kappa(n)$ will have non-negative entries. We let (m, n) denote the greatest common divisor of two positive integers m and n. The number of prime factors in n will be denoted $\Omega(n)$ (counting multiplicities) and

$\omega(n)$ (not counting multiplicities), and $\pi(x)$ will denote the number of primes less than or equal to x. We will let $\log_k$ denote the k-fold logarithm so that $\log_2 x = \log\log x$, $\log_3 x = \log\log\log x$, and so on. To avoid cumbersome notation, we will use the convention that $\log_k x = 1$ when $x \le x_k$, where $x_2 = e^e$ and $x_{k+1} = e^{x_k}$ for $k \ge 2$.

2. The Hardy spaces $\mathscr{H}^p$, symbols of Volterra operators, and BMO in half-planes

2.1. **Hardy spaces of Dirichlet series.** The Bohr lift of the Dirichlet series $f(s) = \sum_{n\ge1} a_n n^{-s}$ is the power series $\mathscr{B}f(z) = \sum_{n\ge1} a_n z^{\kappa(n)}$. For $0 < p < \infty$, we define $\mathscr{H}^p$ as the space of Dirichlet series f such that $\mathscr{B}f$ is in $H^p(\mathbb{D}^\infty)$, and we set

$$\|f\|_{\mathscr{H}^p} := \|\mathscr{B}f\|_{H^p(\mathbb{D}^\infty)} = \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f(z)|^p \, dm_\infty(z)\right)^{\frac{1}{p}}.$$

Here m_∞ denotes the Haar measure of the infinite polytorus $\mathbb{T}^\infty$, which is simply the product of the normalized Lebesgue measure of the torus $\mathbb{T}$ in each variable. Note that for $p = 2$, we have

$$\|f\|_{\mathscr{H}^2} = \left(\sum_{n=1}^\infty |a_n|^2\right)^{\frac{1}{2}}.$$

We refer to [33] (or to [6, 22]) for a treatment of the properties of $\mathscr{H}^p$, describing briefly the basic results we require below. For a character χ in $\mathbb{T}^\infty$, we define

$$f_\chi(s) := \sum_{n=1}^\infty a_n \chi(n) n^{-s}.$$

For τ in $\mathbb{R}$, the vertical translation of f will be denoted by $f_\tau(s) := f(s + i\tau)$. It is well-known (see [22, Sec. 2]) that if f converges uniformly in some half-plane $\mathbb{C}_\theta$, then f_χ is a normal limit of vertical translations $\{f_{\tau_k}\}_{k\ge1}$ in $\mathbb{C}_\theta$.

The conformally invariant Hardy space $H^p_{\mathrm{i}}(\mathbb{C}_\theta)$ consists of holomorphic functions in $\mathbb{C}_\theta$ that are finite with respect to the norm given by

$$\|f\|_{H^p_{\mathrm{i}}(\mathbb{C}_\theta)} := \sup_{\sigma>\theta} \left(\frac{1}{\pi}\int_{\mathbb{R}} |f(\sigma + it)|^p \frac{dt}{1+t^2}\right)^{\frac{1}{p}}.$$

The following connection between $\mathscr{H}^p$ and $H^p_{\mathrm{i}}(\mathbb{C}_0)$ can be obtained from Fubini's theorem:

$$\|f\|^p_{\mathscr{H}^p} = \int_{\mathbb{T}^\infty} \|f_\chi\|^p_{H^p_{\mathrm{i}}(\mathbb{C}_0)} \, dm_\infty(\chi). \tag{2.1}$$

Based on (2.1), one can deduce Littlewood–Paley type expressions for the norms of $\mathscr{H}^p$. This was first done for $p = 2$ in [7, Prop. 4], and later for $0 < p < \infty$ in

[8, Thm. 5.1], where the formula
(2.2)

$$\|f\|^p_{\mathscr{H}^p} \asymp |f(+\infty)|^p + \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi(\sigma+it)|^{p-2}|f'_\chi(\sigma+it)|^2\sigma d\sigma \frac{dt}{1+t^2}dm_\infty(\chi)$$

was obtained. When $p = 2$, we have equality between the two sides of (2.2). We note in passing that this fact can be used to relate $\mathscr{X}$ to $\mathscr{H}^\infty$, the space of bounded Dirichlet series in $\mathbb{C}_0$ endowed with the norm

$$\|f\|_\infty := \sup_{\sigma>0}|f(s)|, \qquad s = \sigma + it.$$

Indeed, let M_g denote the operator of multiplication by g on $\mathscr{H}^2$, and recall the result that M_g is bounded if and only if g is in $\mathscr{H}^\infty$, with $\|M_g\| = \|g\|_\infty$ (see [22, Thm. 3.1]). Since $(fg)' = f'g+(\mathbf{T}_g f)'$, it then follows from the Littlewood–Paley formula and the triangle inequality that

$$\|\mathbf{T}_g\| \le 2\|g\|_\infty \tag{2.3}$$

and consequently $\mathscr{H}^\infty \subset \mathscr{X}$.

Dirichlet series in $\mathscr{H}^p$ for $0 < p < \infty$ are however generally convergent only in $\mathbb{C}_{1/2}$. In this half-plane, we have the following local embedding from [22, Thm. 4.11]. For every τ in $\mathbb{R}$,

$$\int_\tau^{\tau+1} |f(1/2+it)|^2\,dt \le C\|f\|^2_{\mathscr{H}^2}. \tag{2.4}$$

It is sometimes more convenient to use the equivalent formulation that

$$\|f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2})} \le \widetilde{C}\|f\|^2_{\mathscr{H}^2}. \tag{2.5}$$

It is interesting to compare (2.1) and (2.5). These formulas illustrate why both half-planes $\mathbb{C}_0$ and $\mathbb{C}_{1/2}$ appear in the theory of the Hardy spaces $\mathscr{H}^p$. It will become apparent in what follows that both half-planes show up in a natural way also in the study of Volterra operators.

2.2. BMO **spaces in half-planes.** The space $\mathrm{BMOA}(\mathbb{C}_\theta)$ consists of holomorphic functions in the half-plane $\mathbb{C}_\theta$ that satisfy

$$\|g\|_{\mathrm{BMO}(\mathbb{C}_\theta)} := \sup_{I\subset\mathbb{R}} \frac{1}{|I|}\int_I \left| f(\theta+it) - \frac{1}{|I|}\int_I f(\theta+i\tau)\,d\tau\right|\,dt < \infty.$$

We let as before $\mathscr{D}$ denote the space of functions that can be represented by Dirichlet series in some half-plane. The abscissa of boundedness of a given g in $\mathscr{D}$, denoted by σ_b, is the smallest real number such that $g(s)$ has a bounded analytic continuation to $\mathrm{Re}(s) \ge \sigma_b + \delta$ for every $\delta > 0$. A classical theorem of Bohr [10] states that the Dirichlet series $g(s)$ converges uniformly in $\mathrm{Re}(s) \ge \sigma_b+\delta$ for every $\delta > 0$.

Lemma 2.1. *Assume that g is in $\mathscr{D} \cap \mathrm{BMOA}(\mathbb{C}_0)$. Then*

(i) *g has $\sigma_b \leq 0$;*
(ii) *g_χ is in $\mathrm{BMOA}(\mathbb{C}_0)$ and $\|g_\chi\|_{\mathrm{BMO}} = \|g\|_{\mathrm{BMO}}$ for every character χ;*
(iii) *g is in $\bigcap_{0<p<\infty} \mathscr{H}^p$ and $\exp(c|\mathscr{B}g|)$ is integrable on $\mathbb{T}^\infty$ for some $c > 0$.*

An interesting point is that the space $\mathscr{D} \cap \mathrm{BMOA}(\mathbb{C}_0)$ enjoys a stronger translation invariance, expressed by items (i) and (ii), than what the space $\mathrm{BMOA}(\mathbb{C}_0)$ itself does. Lemma 2.1 can also be interpreted as saying that $\mathscr{D} \cap \mathrm{BMOA}(\mathbb{C}_0)$ is only "slightly larger" than $\mathscr{H}^\infty$. We will later see that part (iii) of Lemma 2.1 holds whenever $\mathbf{T}_g$ is a bounded operator.

Proof of Lemma 2.1. By the definition of σ_b, there exists a positive number M such that $|g(\sigma + it)| \leq M$ whenever $\sigma \geq \sigma_b + 1$. Since g is assumed to be in $\mathrm{BMOA}(\mathbb{C}_0)$, there exists a constant C such that

$$\int_{-\infty}^{\infty} |g(i\tau) - g(\sigma_b + 1 + it)| \frac{\sigma_b + 1}{(\sigma_b + 1)^2 + (\tau - t)^2} \frac{d\tau}{\pi} \leq C.$$

Therefore, by the triangle inequality, we find that

$$\int_{t-\sigma_b-1}^{t+\sigma_b+1} |g(i\tau)| d\tau \leq 2(\sigma_b + 1) \cdot (M + C).$$

Writing g as a Poisson integral, we see that this bound implies (i). Now (ii) follows immediately from the translation invariance of BMOA, the characterization of BMOA in terms of Poisson integrals, and that f_χ is a normal limit of vertical translations of f in $\mathbb{C}_0$ by (i). To prove (iii), we use the John–Nirenberg inequality to conclude that there is $c = c(\|g\|_{\mathrm{BMO}}) > 0$ and $C = C(\|g\|_{\mathrm{BMO}})$ such that

$$\left\|e^{c|g-g(1)|}\right\|_{L^1_{\mathrm{i}}(i\mathbb{R})} := \frac{1}{\pi} \int_{\mathbb{R}} e^{c|g(it)-g(1)|} \frac{dt}{1+t^2} \leq C.$$

Since $\sigma_b(g) \leq 0$, we know that g is absolutely convergent at $s = 1$, so

$$\left\|e^{c|g-g(1)|}\right\|_{L^1_{\mathrm{i}}(i\mathbb{R})} \asymp \left\|e^{c|g|}\right\|_{L^1_{\mathrm{i}}(i\mathbb{R})},$$

where the implied constant depends on g, but only on the absolute value of its coefficients. In particular, we can conclude that

$$\left\|e^{c|g_\chi|}\right\|_{L^1_{\mathrm{i}}(i\mathbb{R})} \leq \widetilde{C},$$

for every $\chi \in \mathbb{T}^\infty$, and $\widetilde{C}$ does not depend on χ, by (ii). Integrating over $\mathbb{T}^\infty$ and using Fubini's theorem as in (2.1) allows us to conclude that $\exp(c|\mathscr{B}g|)$ is in $L^2(\mathbb{T}^\infty)$, which also implies that g is in $\bigcap_{0<p<\infty} \mathscr{H}^p$. □

We require the following standard result, which can be extracted from [20, Sec. VI.1].

Lemma 2.2. *Let g be holomorphic in $\mathbb{C}_\theta$. Then the measure*

$$\mu_g(s) = |g'(\sigma+it)|^2\,(\sigma-\theta)\;d\sigma\,\frac{dt}{1+t^2}$$

is Carleson for $H^p_{\mathrm{i}}(\mathbb{C}_\theta)$ if and only if g is in $\mathrm{BMOA}(\mathbb{C}_\theta)$*, and* $\|\mu_g\|_{\mathrm{CM}(H^p_{\mathrm{i}})} \asymp \|g\|^2_{\mathrm{BMO}(\mathbb{C}_0)}$.

We are now ready for a first result, saying that for the boundedness of $\mathbf{T}_g$ it is sufficient that g is in $\mathrm{BMOA}(\mathbb{C}_0)$ and necessary that it is in $\mathrm{BMOA}(\mathbb{C}_{1/2})$. On the one hand, it is a preliminary result, following rather directly from the available theory of $\mathscr{H}^2$, outlined above. On the other hand, as we shall see in Section 3 and Section 4, $\mathbb{C}_0$ and $\mathbb{C}_{1/2}$ are the extremal half-planes of convergence for symbols g inducing bounded Volterra operators.

Theorem 2.3. *Let $\mathbf{T}_g$ be the operator defined in* (1.2) *for some Dirichlet series g in $\mathscr{D}$.*

(a) *If g is in* $\mathrm{BMOA}(\mathbb{C}_0)$*, then $\mathbf{T}_g$ is bounded on $\mathscr{H}^2$.*

Suppose that $\mathbf{T}_g$ is bounded on $\mathscr{H}^2$. Then,

(b) *g satisfies condition* (iii) *from Lemma 2.1;*
(c) *g is in* $\mathrm{BMOA}(\mathbb{C}_{1/2})$.

Proof. We apply (2.2) to $\mathbf{T}_g f$ and use Lemmas 2.1 and 2.2. Since $(fg')_\chi = f_\chi g'_\chi$ we find that

$$\begin{aligned}\|\mathbf{T}_g\|^2_{\mathscr{H}^2} &\asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |(fg')_\chi(\sigma+it)|^2\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\\ &\ll \int_{\mathbb{T}^\infty} \|f_\chi\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_0)}\|g_\chi\|^2_{\mathrm{BMO}(\mathbb{C}_0)}\,dm_\infty(\chi) = \|f\|^2_{\mathscr{H}^2}\|g\|^2_{\mathrm{BMO}(\mathbb{C}_0)}.\end{aligned}$$

This completes the proof of (a).

For (b), we first observe that $\mathbf{T}_g 1 = g$, so that g is in $\mathscr{H}^2$. Applying $\mathbf{T}_g$ inductively to the powers g^n, for $n = 1, 2, \ldots$, we get that

$$\|g^n\|_{\mathscr{H}^2} \le \|\mathbf{T}_g\|^n n!.$$

Using this and the triangle inequality, we obtain

$$\big\|e^{c|\mathscr{B}g|}\big\|^{1/2}_{L^1(\mathbb{T}^\infty)} = \big\|e^{c|\mathscr{B}g|/2}\big\|_{L^2(\mathbb{T}^\infty)} \le \sum_{n=0}^\infty \Big(\frac{c\|\mathbf{T}_g\|}{2}\Big)^n,$$

which implies that $e^{c|\mathscr{B}g|}$ is integrable whenever $c < 2/\|T_g\|$.

To prove (c), we use the Littlewood–Paley formula for $H^2_{\mathrm{i}}(\mathbb{C}_{1/2})$ and (2.5) to see that

$$\begin{aligned}\int_{\mathbb{R}}\int_{1/2}^{\infty}|f(\sigma+it)|^2\,|g'(\sigma+it)|^2\left(\sigma-\frac{1}{2}\right)\,d\sigma\,\frac{dt}{1+t^2} &\asymp \|\mathbf{T}_g f\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2})}\\ &\ll \|\mathbf{T}_g f\|^2_{\mathscr{H}^2}\\ &\le \|\mathbf{T}_g\|^2\|f\|^2_{\mathscr{H}^2}.\end{aligned}$$

This means that

$$\mu_g(s)=|g'(\sigma+it)|^2\left(\sigma-\frac{1}{2}\right)\,d\sigma\,\frac{dt}{1+t^2}$$

is a Carleson measure for $\mathscr{H}^2$ in $\mathbb{C}_{1/2}$. By [29, Thm. 3], this implies that $\mu_g(s)$ is a Carleson measure for the non-conformal Hardy space $H^2(\mathbb{C}_{1/2})$, which as in Lemma 2.2 means that $h(s):=g(s)/(s+1/2)$ is in $\mathrm{BMO}(\mathbb{C}_{1/2})$. Indeed, we have proved that $\|h\|_{\mathrm{BMO}(\mathbb{C}_{1/2})}\ll\|\mathbf{T}_g\|$.

Let us show that the factor $(s+1/2)^{-1}$ can be removed, so that g is in fact in $\mathrm{BMOA}(\mathbb{C}_{1/2})$. We note first that if $|I|\ge 1$, then it follows from the local embedding (2.4) that

$$\int_I |g(1/2+it)|^2\,dt \ll |I|\cdot\|g\|^2_{\mathscr{H}^2},$$

since g is in $\mathscr{H}^2$ by (b). Hence we only need to consider intervals of length $|I|<1$. For a character χ in $\mathbb{T}^\infty$, we define

$$h_\chi(s):=\frac{g_\chi(s)}{s+1/2}.$$

Clearly, $\|\mathbf{T}_{g_\chi}\|=\|\mathbf{T}_g\|$ for every χ in $\mathbb{T}^\infty$. This means that

$$\sup_{\chi\in\mathbb{T}^\infty}\|h_\chi\|_{\mathrm{BMO}(\mathbb{C}_{1/2})}\ll\|\mathbf{T}_g\|.$$

In particular, the BMO-norm of h is uniformly bounded under vertical translations of g, so that we only need to consider intervals $I=[0,\tau]$ for $\tau<1$. On this interval, $(s+1/2)^{-1}$ and its derivative is bounded from below and above. It follows that g is in $\mathrm{BMO}(\mathbb{C}_{1/2})$. □

Combined with a result from [22], part (b) of Theorem 2.3 yields the following result.

Corollary 2.4. *If $\mathbf{T}_g$ is bounded on $\mathscr{H}^2$, then for almost every character χ on $\mathbb{T}^\infty$, there is a constant C such that*

$$|g_\chi(\sigma+it)|\le C\log\frac{1+|t|}{\sigma} \tag{2.6}$$

holds in the strip $0<\sigma\le 1/2$.

Proof. We assume that $\mathbf{T}_g$ is bounded on $\mathscr{H}^2$. Then by part (b) of Theorem 2.3, there exists a positive number c such that the four functions $e^{\pm cg}$ and $e^{\pm icg}$ are in $\mathscr{H}^2$. Now let f be any of these four functions. Then [22, Thm. 4.2] shows that, for almost every character χ, there exists a constant C (depending on χ) such that

$$|f_\chi(\sigma + it) - f(+\infty)| \leq C\frac{1+\sqrt{|t|}}{\sigma}$$

for every point $\sigma + it$ in $\mathbb{C}_0$. Combining the acquired estimates for the four functions $e^{\pm cg}$ and $e^{\pm icg}$ and taking logarithms, we obtain the desired result. □

Our bound (2.6) shows that almost surely $|g_\chi|$ grows at most as general functions in $\mathrm{BMOA}(\mathbb{C}_0)$ at the boundary of $\mathbb{C}_0$. It would be interesting to know if this result could be strengthened. For instance, is it true that g_χ almost surely satisfies the BMO condition locally, say on finite intervals, whenever $\mathbf{T}_g$ is bounded on $\mathscr{H}^2$? Note that we cannot hope to have the stronger result that g_χ is almost surely in $\mathrm{BMOA}(\mathbb{C}_0)$. Indeed, the proof of part (a) of Theorem 2.3 gives that if g_χ is in $\mathrm{BMOA}(\mathbb{C}_\theta)$ for one character χ, then this holds for all characters χ. In view of this fact and what will be shown in Section 4, g_χ will in general be in $\mathrm{BMOA}(\mathbb{C}_{1/2})$ and in none of the other spaces $\mathrm{BMOA}(\mathbb{C}_\theta)$ for $0 \leq \theta < 1/2$.

2.3. **An unbounded Dirichlet series in** BMO**.** The canonical example of an unbounded function in $\mathrm{BMO}(\mathbb{R})$ is $\log|t|$, the primitive of $1/t$. The Riemann zeta function $\zeta(s)$ is a meromorphic function with one simple pole, at $s = 1$. We now show that the primitive of $-(\zeta(s) - 1)$ has bounded mean oscillation on the line $\sigma = 1$. In view of Theorem 2.3, this supplies us with an example of a bounded $\mathbf{T}_g$-operator.

Theorem 2.5. *The Dirichlet series*

$$g(s) := \sum_{n=2}^{\infty} \frac{1}{n\log n} n^{-s}.$$

is in $\mathrm{BMOA}(\mathbb{C}_0)$.

Proof. We will show that g is in $\mathrm{BMOA}(\mathbb{C}_\varepsilon)$, with BMO-norm uniformly bounded in $\varepsilon > 0$. Since $g(s-1/2)$ is in $\mathscr{H}^2$, we can use the local embedding as in the proof of Theorem 2.3 (c) to conclude that g satisfies the BMO-condition for intervals of length $|I| \geq 1$.

Focusing our attention on short intervals, we fix a real number a and $0 < T < 1$ and set

$$c := \sum_{\log n < 1/T} \frac{1}{n^{1+\varepsilon}\log n} n^{-ia}.$$

To prove the theorem, we will show that

$$\int_a^{a+T} |g(\varepsilon+it)-c|^2\,dt \le CT$$

where C is a universal constant.

Notice first that

$$\int_a^{a+T} |g(\varepsilon+it)-c|^2\,dt = \int_0^{T} |\widetilde{g}(\varepsilon+it)-c|^2\,dt,$$

where

$$\widetilde{g}(s) := \sum_{n=2}^{\infty} \frac{n^{-ia}}{n\log n} n^{-s}.$$

Accordingly, set $b_n := n^{-ia}/(n\log n)$. Then we have that

$$\left(\int_a^{a+T} |g(\varepsilon+it)-c|^2\,dt\right)^{1/2} \le$$

$$\left(\int_0^{T} \left|\sum_{\log n<1/T} b_n n^{-\varepsilon}(n^{-it}-1)\right|^2 dt\right)^{1/2} + \left(\int_0^{T} \left|\sum_{\log n>1/T} b_n n^{-\varepsilon} n^{-it}\right|^2 dt\right)^{1/2}.$$

To deal with the second term, we use the local embedding (2.4) in a similar manner as above, using now that

$$\int_0^T |f(1/2+\varepsilon+it)|^2\,dt \ll \|f\|^2_{\mathscr{H}^2}$$

in this case, since $T<1$. This gives us that

$$\int_0^{T} \left|\sum_{\log n>1/T} b_n n^{-\varepsilon} n^{-it}\right|^2 dt \le \sum_{\log n>1/T} n|b_n|^2 \ll T,$$

as desired.

For the first term, we compute:

$$\int_0^{T} \left|\sum_{\log n<1/T} b_n n^{-\varepsilon}(n^{-it}-1)\right|^2 dt = \sum_{\substack{\log m<1/T \\ \log n<1/T}} b_n\overline{b_m}(mn)^{-\varepsilon} h_{mn}(T), \tag{2.7}$$

where

$$h_{mn}(T) := \frac{(n/m)^{-iT}-1}{i\log\frac{m}{n}} - \frac{n^{-iT}-1}{i\log\frac{1}{n}} - \frac{(1/m)^{-iT}-1}{i\log m} + T.$$

We write h_{mn} as a Taylor series in T, whence

$$h_{mn}(T) = \sum_{k=3}^{\infty} d_{mn}^k T^k,$$

where

$$d_{mn}^k := \frac{(-i)^{k-1}}{k!}\left(\left(\log \frac{n}{m}\right)^{k-1} - (\log n)^{k-1} - \left(\log \frac{1}{m}\right)^{k-1}\right).$$

The point is that in the coefficient d_{mn}^k, the terms of order $(\log m)^{k-1}$ and $(\log n)^{k-1}$ cancel. Estimating the remaining terms in a crude manner, we have that

$$|d_{mn}^k| \ll \frac{2^k}{k!} \sum_{j=1}^{k-2} (\log m)^j (\log n)^{k-j-1}.$$

Note that for $1 \leq j \leq k-2$, we have

$$T^k \sum_{\substack{\log m < 1/T \\ \log n < 1/T}} |b_n||b_m|(\log m)^j (\log n)^{k-j-1} \ll T.$$

We observe that this inequality fails if $j = 0$ or $j = k-1$, corresponding to the terms which disappear from d_{mn}^k.

Combining these estimates with (2.7) we obtain

$$\int_0^T \left| \sum_{\log n < 1/T} b_n n^{-\varepsilon}(n^{-it} - 1)\right|^2 dt \ll T$$

also for the first term, concluding the proof. □

3. Multiplicative symbols

In this section, we study symbols of the form

$$g(s) = \sum_{n=2}^{\infty} \frac{\psi(n)}{\log n} n^{-s}, \tag{3.1}$$

where $\psi(n)$ is a positive multiplicative function. We know from the previous section that if $\psi(n) = n^{-1}$, then g is in $\mathrm{BMOA}(\mathbb{C}_0)$, and therefore g is in $\mathscr{X}$. We begin by considering the distinguished case when the function $\psi(n)$ corresponds to horizontal shifts of the Riemann zeta function. To be more precise, our first task will be to show that g is not in $\mathscr{X}$ when g is the function in $\mathrm{BMOA}(\mathbb{C}_{1-\alpha})$ with coefficients given by $\psi(n) = n^{-\alpha}$ and $1/2 \leq \alpha < 1$. In particular, this means that the Dirichlet series $g(s) = \sum_{n\geq 2} 1/(\sqrt{n}\log n) n^{-s}$, identified in [14] as the symbol of the multiplicative analogue of Hilbert's matrix and shown there to generate a bounded multiplicative Hankel form, is indeed far from belonging to $\mathscr{X}$, as it corresponds to the case $\alpha = 1/2$.

In this section and the next, we will be working at the level of coefficients. Observe that if $f(s) = \sum_{n\geq 1} a_n n^{-s}$ and $g(s) = \sum_{n\geq 2} b_n/(\log n) n^{-s}$, then

$$\mathbf{T}_g f(s) = \sum_{n=2}^{\infty} \frac{1}{\log n} \left(\sum_{\substack{k|n \\ k<n}} a_k b_{n/k} \right) n^{-s}.$$

Since the operator

$$a_1 + \sum_{n=2}^{\infty} a_n n^{-s} \quad \mapsto \quad a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\log n} n^{-s}$$

is trivially bounded and even compact on $\mathscr{H}^2$, we will sometimes tacitly replace $\mathbf{T}_g$ with $\widetilde{\mathbf{T}}_g$,

$$\widetilde{\mathbf{T}}_g f(s) := \sum_{n=2}^{\infty} \frac{1}{\log n} \left(\sum_{k|n} a_k b_{n/k} \right) n^{-s},$$

where it is understood that $b_1 = 1$.

Theorem 3.1. *$\mathbf{T}_g$ is unbounded when g is the primitive of $\zeta(s+\alpha) - 1$ and $\alpha < 1$.*

Proof. If $f(s) = \sum_{n\geq 1} a_n n^{-s}$, then with the convention just described, we have that

$$\mathbf{T}_g f(s) = \sum_{n=2}^{\infty} \frac{1}{n^\alpha \log n} \sum_{k|n} a_k k^\alpha n^{-s}.$$

We now choose $f(s) = \prod_{j=1}^{J} (1 + p_j^{-s})$, which satisfies $\|f\|_{\mathscr{H}^2} = 2^{J/2}$. Let $\mathscr{J}$ be a subset of $\{1, ..., J\}$.

Choosing $n = n_{\mathscr{J}}$, where

$$n_{\mathscr{J}} := \prod_{j\in\mathscr{J}} p_j,$$

we see that

$$\sum_{k|n_{\mathscr{J}}} a_k k^\alpha = n_{\mathscr{J}}^\alpha \prod_{j\in\mathscr{J}} (1 + p_j^{-\alpha}).$$

It follows that

$$\left\|\mathbf{T}_g f\right\|_{\mathscr{H}^2}^2 = \sum_{\mathscr{J}\neq\emptyset} \frac{1}{(\log n_{\mathscr{J}})^2} \prod_{j\in\mathscr{J}} (1 + p_j^{-\alpha})^2,$$

which gives

$$\left\|\mathbf{T}_g f\right\|_{\mathscr{H}^2}^2 \geq 2^{J-1} \min_{|\mathscr{J}|\geq J/2} \frac{1}{(\log n_{\mathscr{J}})^2} \prod_{j\in\mathscr{J}} (1 + p_j^{-\alpha})^2.$$

We conclude that

$$\left\|\mathbf{T}_g f\right\|_{\mathscr{H}^2}^2 \gg e^{cJ^{1-\alpha}(\log J)^{-\alpha}} \|f\|_{\mathscr{H}^2}^2$$

for an absolute constant c. $\square$

The preceding clarification of the case of horizontal shifts of primitives of the Riemann zeta function motivates a more careful examination of what we need to require from the multiplicative function $\psi(n)$ in (3.1) for g to belong to $\mathscr{X}$. We will now see that a surprisingly precise answer can be given if we make a slight modification of the Euler product associated with $\zeta(s)$.

We will need the following simple decomposition of bounded $\mathbf{T}_g$-operators. Let $M_{h,x}$ denote the truncated multiplier associated with $h(s) = \sum_{n\geq 1} c_n n^{-s}$ and $x \geq 1$:

$$M_{h,x}f(s) := \sum_{n\leq x}\left(\sum_{k|n} c_k a_{n/k}\right) n^{-s},$$

where $f(s) = \sum_{n\geq 1} a_n n^{-s}$. We observe that $M_{h,x}$ acts boundedly on $\mathscr{H}^2$ for every Dirichlet series h, but the point of interest is to understand how the norm of $M_{h,x}$ grows with x. Truncated multipliers are linked to $\mathbf{T}_g$ by the following lemma.

Lemma 3.2. *Suppose that $\mathbf{T}_g$ acts boundedly on $\mathscr{H}^2$. Then*

$$\frac{3}{4}\sum_{k=0}^{\infty} 4^{-k}\left\|M_{g',e^{2^k}} f\right\|_{\mathscr{H}^2}^2 \leq \|\mathbf{T}_g f\|_{\mathscr{H}^2}^2 \leq 4\sum_{k=0}^{\infty} 4^{-k}\left\|M_{g',e^{2^k}} f\right\|_{\mathscr{H}^2}^2$$

for every f in $\mathscr{H}^2$.

Proof. We start from the expression

$$\|\mathbf{T}_g f\|_{\mathscr{H}^2}^2 = \sum_{n=2}^{\infty} \frac{1}{(\log n)^2}\left|\sum_{k|n} b_k(\log k) a_{n/k}\right|^2,$$

which we split into blocks in the following way:

$$\sum_{k=0}^{\infty} \frac{1}{4^k} \sum_{e^{2^{k-1}}<n\leq e^{2^k}} \left|\sum_{k|n} b_k(\log k) a_{n/k}\right|^2 \leq \|\mathbf{T}_g f\|_{\mathscr{H}^2}^2,$$

$$\|\mathbf{T}_g f\|_{\mathscr{H}^2}^2 \leq 4\sum_{k=0}^{\infty} \frac{1}{4^k} \sum_{e^{2^{k-1}}<n\leq e^{2^k}} \left|\sum_{k|n} b_k(\log k) a_{n/k}\right|^2.$$

The upper bound is immediate from the right inequality, and the lower bound follows from the left inequality and the fact that

$$\sum_{e^{2^{k-1}}<n\le e^{2^k}} \left|\sum_{k|n} b_k a_{n/k}\right|^2 = \left\|M_{g',e^{2^k}} f\right\|^2_{\mathscr{H}^2} - \left\|M_{g',e^{2^{k-1}}} f\right\|^2_{\mathscr{H}^2}. \qquad \square$$

The preceding lemma, which says that $\mathbf{T}_g$ is bounded whenever the norm of $M_{g',x}$ grows roughly as $\log x$, connects the study of $\mathbf{T}_g$ to the truncated multipliers considered by Hilberdink [24] in a purely number theoretic context. Based on this observation, we shall now present a natural scale of multiplicative symbols g_λ, where $0 < \lambda < \infty$, such that g_λ induces a bounded $\mathbf{T}_g$-operator if and only if $\lambda \le 1$. We shall later see, in Section 7, that $\mathbf{T}_{g_\lambda}$ is non-compact for the pivotal point $\lambda = 1$.

Theorem 3.3. *For $0 < \lambda < \infty$, let g be the Dirichlet series* (3.1)*, where $\psi(n)$ is the completely multiplicative function defined on the primes by $\psi(p) := \lambda p^{-1}(\log p)$. Then $\mathbf{T}_g$ is bounded if and only if $\lambda \le 1$.*

Proof. We begin with the case $\lambda < 1$, for which we adapt the proof of [24, Thm. 2.3]. Hence we let $\varphi(n)$ be an arbitrary positive arithmetic function and note that the Cauchy–Schwarz inequality implies that

$$\|M_{g',x} f\|^2_{\mathscr{H}^2} = \sum_{n\le x}\left|\sum_{d|n}\psi(d)a_{n/d}\right|^2 \le \sum_{n\le x}\sum_{d|n}\frac{\psi(d)}{\varphi(d)}\sum_{k|n}\psi(k)\varphi(k)|a_{n/k}|^2.$$

We therefore find that

$$\|M_{g',x}\|^2_{\mathscr{H}^2} \le \sum_{n\le x}\varphi(n)\psi(n)\max_{m\le x}\sum_{d|m}\frac{\psi(m)}{\varphi(m)}. \tag{3.2}$$

We now require that φ be a multiplicative function satisfying

$$\varphi(p^k) := \begin{cases} 1, & p \le M, \\ K\sum_{r=1}^\infty \psi(p^r), & p > M, \end{cases}$$

where the positive parameters K and M will be determined later. We find that

$$\begin{aligned}\sum_{n\le x}\varphi(n)\psi(n) &\le \prod_p\left(1+\sum_{k=1}^\infty \varphi(p^k)\psi(p^k)\right)\\ &\le \exp\left(\sum_{p\le M}\sum_{k=1}^\infty \psi(p^k) + K\sum_{p>M}\left(\sum_{k=1}^\infty\psi(p^k)\right)^2\right)\\ &= \exp\left(\sum_{p\le M}\frac{\lambda p^{-1}\log p}{1-\lambda p^{-1}\log p} + K\sum_{p>M}\frac{\lambda^2p^{-2}(\log p)^2}{(1-\lambda p^{-1}\log p)^2}\right).\end{aligned}$$

By Abel summation and the prime number theorem in the form

$$\pi(y) = \frac{y}{\log y} + \frac{y}{(\log y)^2} + O\left(\frac{y}{(\log y)^3}\right),$$

we infer that

$$\sum_{n \le N} \varphi(n)\psi(n) \le \exp\left(\lambda \log M + O(1) + O\left(K\frac{\log M}{M}\right)\right). \tag{3.3}$$

We now turn to the second factor on the right-hand side of (3.2). We then use that also

$$\Phi(m) := \sum_{d|m} \frac{\psi(d)}{\varphi(d)}$$

is a multiplicative function. We observe that

$$\Phi(p^k) = \sum_{r=0}^{k} \frac{\psi(p^r)}{\varphi(p^r)} \le \begin{cases} 1 + \sum_{r=1}^{\infty} \psi(p^r), & p \le M, \\ 1 + K^{-1}, & p > M. \end{cases}$$

Consequently

$$\begin{aligned} \Phi(m) &\le \prod_{p \le M} \left(1 + \sum_{r=1}^{\infty} \psi(p^r)\right) \left(1 + K^{-1}\right)^{\omega(m)} \\ &\le \exp\left(\lambda \log M + O(1) + O\left(K^{-1}\frac{\log x}{\log_2 x}\right)\right), \end{aligned} \tag{3.4}$$

where we used that $\omega(m) \ll \log(m)/\log_2(m)$. If we now choose $M = \log x$, $K = (\log x)/\log_2 x$, and insert (3.3) and (3.4) into (3.2), then we find that

$$\|M_{g',x}\|^2 \le C(\log x)^{2\lambda}.$$

Finally, we invoke Lemma 3.2 and conclude that $\mathbf{T}_g$ is bounded whenever $\lambda < 1$.

To show that $\mathbf{T}_g$ is bounded when $\lambda = 1$ we modify the proof. In addition to the function $\varphi(n)$, we use another auxiliary function $h_x(n)$ and use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \|M_{g',x}f\|_{\mathscr{H}^2}^2 &= \sum_{n \le x} \left| \sum_{d|n} \psi(d) a_{n/d} \right|^2 \\ &\le \sum_{n \le x} \sum_{d|n} \frac{\psi(d)}{\varphi(d) h_x(n/d)} \sum_{k|n} \psi(k)\varphi(k)|a_{n/k}|^2 h_x(n/k). \end{aligned}$$

We require from $h_x(n)$ that

$$\sup_m \sum_{e^{2^k} \ge m} h_{e^{2^k}}(m) < \infty.$$

This will ensure boundedness if we can prove that

$$\Phi_h(m) := \sum_{d|m} \frac{\psi(d)}{\varphi(d)h_x(m/d)}$$

enjoys the same uniform bound as that we found for $\Phi(m)$ for a suitable $h_x(n)$. To this end, we choose

$$h_x(n) = \begin{cases} 1, & \sqrt{x} < n \le x, \\ \exp\left(-2\log_2 \frac{\log x}{\log n+1}\right), & 1 \le n \le \sqrt{x}, \end{cases}$$

which implies that

$$\Phi_h(m) \le \Phi(m)e^{2\log_3 x} \le \exp\left(\log_2 m + 2\log_3 x + O(1)\right).$$

This means that in what follows, we may assume that $\log(m) \ge (\log x)/(\log_2 x)^2$. Using again the definition of $h_x(n)$, we also obtain, for $\delta > 0$,

$$\sum_{\substack{d|m \\ m/d \ge x^\delta}} \frac{\psi(d)}{\varphi(d)h_x(m/d)} \le \Phi(m)e^{2\log_2 \frac{1}{\delta}}. \tag{3.5}$$

On the other hand, if $m = x^\beta$ with $0 < \beta < 1$, then arguing as before and choosing the same M and K, we get

$$\Phi(m) \le \exp\left(\log_2 x - \log\frac{1}{\beta} + O(1)\right).$$

Hence, with $\beta = \log m/\log x$ and $\delta = \beta/2$, we find in view of (3.5) that

$$\sum_{\substack{d|m \\ d \le \sqrt{m}}} \frac{\psi(d)}{\varphi(d)h_x(m/d)} \le C\log x.$$

It remains to estimate

$$\sum_{\substack{d|m \\ d \ge \sqrt{m}}} \frac{\psi(d)}{\varphi(d)h_x(m/d)} \le e^{2\log_3 x} \sum_{\substack{d|m \\ d \ge \sqrt{m}}} \frac{\psi(d)}{\varphi(d)}. \tag{3.6}$$

Note first that

$$\sum_{\substack{d|m \\ d \ge \sqrt{m}}} \frac{\psi(d)}{\varphi(d)} \le m^{-\varepsilon/2} \sum_{d|m} \frac{d^\varepsilon \psi(d)}{\varphi(d)} =: m^{-\varepsilon/2}E(m).$$

The definition of $E(m)$ shows that, in particular,

$$E(p^k) = \sum_{r=0}^{k} \frac{p^{\varepsilon r}\psi(p^r)}{\varphi(p^r)} \le \begin{cases} (1 - p^\varepsilon\psi(p))^{-1}, & p \le M, \\ 1 + K^{-1}p^\varepsilon(1-\psi(p))/(1-p^\varepsilon\psi(p)), & p > M. \end{cases}$$

We may assume that ε is so small that the factor $(1-\psi(p))/(1-p^{\varepsilon}\psi(p))$ does not exceed 2. Letting $\mathscr{P}$ denote an arbitrary finite set of primes p, we then get that

$$E(m) \le \prod_{p\le \log m} (1-p^{\varepsilon}\psi(p))^{-1} \max_{\mathscr{P}:\sum_{p\in\mathscr{P}}\log p\le \log m} \prod_{p\in\mathscr{P}} \left(1+2K^{-1}p^{\varepsilon}\right)$$
$$\le \exp\left((\log m)^{\varepsilon}\log_2 m + 2K^{-1} \max_{\frac{\log x}{\log_2 x}\le p\le x} \frac{p^{\varepsilon}}{\log p}\log m + O(1)\right).$$

We now choose

$$\varepsilon := \frac{4\log_3 x}{\log m}.$$

Then the latter estimate becomes

$$E(m) \le \exp\left((\log m)^{\varepsilon}\log_2 m + K^{-1}\frac{(\log x)^{\varepsilon}}{\log_2 x}\log m\right)$$
$$\le \exp\left(\log_2 m + O(1)\right) \le \exp\left(\log_2 x + O(1)\right).$$

We finally observe that the factor $m^{-\varepsilon/2}$ will take care of the term $\log_3 x$ in the exponent on the right-hand side of (3.6).

Following an insight of Gál [19], we argue in the following way in order to show that $\mathbf{T}_g$ is unbounded when $\lambda > 1$. We start from the fact that

$$\prod_{p\le y} p = e^{y(1+o(1))},$$

which is a consequence of the prime number theorem. We let $\varphi(n)$ be the multiplicative function defined by setting

$$\varphi(p^r) := \begin{cases} 1, & p \le \frac{\log x}{\log_2 x} \quad \text{and} \quad r \le \frac{1}{2}\log_2 x, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\varphi(n) = 0$ for $n > x$ if x is large enough. We set $a_n := \varphi(n)/(\sum_n \varphi(n))^{1/2}$ and use the Cauchy–Schwarz inequality to see that

$$\left(\sum_{n\le x}\left|\sum_{d|n} a_d\psi(n/d)\right|^2\right)^{1/2} \ge \frac{\sum_n \varphi(n)\sum_{d|n}\varphi(d)\psi(n/d)}{\sum_n \varphi(n)}.$$

To simplify the writing, we set $y := \log x/\log_2 x$ and $\ell := \lfloor \frac{1}{2}\log_2 x \rfloor$. Then we infer from the preceding estimate that

$$\begin{aligned}\left(\sum_{n\le N}\left|\sum_{d|n} a_d\psi(n/d)\right|^2\right)^{1/2} &\ge \prod_{p\le y}\frac{1+\ell+\ell\psi(p)+(\ell-1)\psi(p^2)+\cdots+\psi(p^\ell)}{1+\ell}\\ &\ge \prod_{p\le y}\left(1+\frac{\ell}{\ell+1}\psi(p)\right) = \exp\left(\frac{\lambda\ell}{\ell+1}\log y + O(1)\right)\\ &\ge (\log x)^{\lambda'}\end{aligned}$$

for some $1 < \lambda' < \lambda$ when x is sufficiently large. We appeal again to Lemma 3.2 to conclude that $\mathbf{T}_g$ is unbounded. □

We notice that, clearly, the symbol g is not in $\mathrm{BMOA}(\mathbb{C}_0)$ for any $\lambda > 0$. In fact, for $\sigma > 0$,

$$\sum_{n=1}^{\infty}\psi(n)n^{-\sigma} = \prod_p \left(1-\psi(p)p^{-\sigma}\right)^{-1} \asymp \exp\left(\lambda\sum_p \frac{\log p}{p^{1+\sigma}}\right) \asymp e^{\lambda/\sigma},$$

which shows that g is not even in the Smirnov class of $\mathbb{C}_0$.

4. Homogeneous symbols and coefficient estimates

The multiplicative symbols of the previous section represent analytic functions in $\mathbb{C}_0$. However, we saw in Theorem 2.3 that for $\mathbf{T}_g$ to be bounded, it is necessary that g be in $\mathrm{BMOA}(\mathbb{C}_{1/2})$. We will begin this section by showing that the latter condition cannot be relaxed by much. Indeed, to begin with, we will prove that linear Dirichlet series give examples of bounded $\mathbf{T}_g$-operators with symbols g converging in $\mathbb{C}_{1/2}$ but in no larger half-plane.

Theorem 4.1. *Let $g(s) = \sum_p b_p p^{-s}$ be any linear symbol in $\mathscr{H}^2$. Then $\|\mathbf{T}_g\| = \|g\|_{\mathscr{H}^2}$.*

Proof. We consider an arbitrary function $f(s) = \sum_{n\ge1} a_n n^{-s}$ in $\mathscr{H}^2$ and compute:

$$\|\mathbf{T}_g f\|^2_{\mathscr{H}^2} = \sum_{n=2}^{\infty}\frac{1}{(\log n)^2}\left|\sum_{p|n} b_p(\log p)a_{n/p}\right|^2.$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}\left|\sum_{p|n} b_p(\log p)a_{n/p}\right|^2 &\le \left(\sum_{p|n}\log p\right)\left(\sum_{p|n}|b_p|^2(\log p)|a_{n/p}|^2\right)\\ &\le (\log n)\sum_{p|n}|b_p|^2(\log p)|a_{n/p}|^2.\end{aligned}$$

This shows that $\|\mathbf{T}_g\| \le \|g\|_{\mathscr{H}^2}$. Since $\mathbf{T}_g 1 = g$, clearly $\|\mathbf{T}_g\| \ge \|g\|_{\mathscr{H}^2}$. □

We note that the space of linear symbols g in $\mathscr{H}^2$ is embedded not only in $\mathrm{BMOA}(\mathbb{C}_{1/2})$ but in fact satisfies the local Dirichlet integral condition

$$\int_0^1 \int_{1/2}^1 |g'(\sigma+it)|^2 d\sigma dt \ll \|g\|_2^2,$$

as shown in [28, Example 4]. We do not know if this stronger embedding can be established for a general symbol in $\mathscr{X}$.

While the norm of a linear function g viewed as an element in the dual of $\mathscr{H}^1$ is also equivalent to $\|g\|_{\mathscr{H}^2}$ (see [23]), there is a striking contrast between the preceding result and the characterization of linear multipliers. Indeed, let again M_g denote the operator of multiplication by g on $\mathscr{H}^2$, and recall that $\|M_g\| = \|g\|_\infty$. (see [22, Thm. 3.1]). Hence, in the special case when g is linear, it follows from Kronecker's theorem that

$$\|M_g\| = \|g\|_\infty = \sup_{\sigma>0} \left| \sum_p b_p p^{-s} \right| = \sum_p |b_p|.$$

The difference between a linear symbol g acting as a multiplier M_g and as a symbol of the Volterra operator $\mathbf{T}_g$ is therefore dramatic: A bounded multiplier has coefficients in ℓ^1, while the boundedness of $\mathbf{T}_g$ means that the coefficients are in ℓ^2. The former implies absolute convergence in $\mathbb{C}_0$ and the latter only in $\mathbb{C}_{1/2}$.

We may understand the phenomenon just observed in the following way. For a general symbol $g(s) = \sum_{n\ge1} b_n n^{-s}$, we have, using also (2.3), the series of inequalities

$$\left(\sum_{n=1}^\infty |b_n|^2\right)^{1/2} \le \|\mathbf{T}_g\| \le 2\|g\|_\infty \le 2\sum_{n=1}^\infty |b_n|. \tag{4.1}$$

The case of linear functions shows that neither the left nor the right inequality can be improved. Loosely speaking, the maximal independence between the terms in a linear symbol serves to make $\|\mathbf{T}_g\|$ minimal and thus equal to $\|g\|_2$ and, at the same time, to make $\|M_g\|$ maximal and hence equal to $\sum_{n\ge1}|b_n|$. This motivates an investigation of what happens when the dependence between the terms in the symbol increases. Such a study, originating in the classical work of Bohnenblust and Hille [9], has already been made in the case of multipliers, in terms of m-homogeneous Dirichlet series. We will now follow the same path for $\mathbf{T}_g$-operators.

Recall that $\Omega(n)$ gives the number of prime factors in n, counting multiplicities. An m-homogeneous Dirichlet series is of the form

$$g(s) := \sum_{\Omega(n)=m} b_n n^{-s}. \tag{4.2}$$

In this terminology, linear symbols are 1-homogeneous Dirichlet series. A precise relationship between boundedness and absolute convergence for m-homogeneous Dirichlet series was found in [5, 27]:

$$\sum_{\Omega(n)=m} |b_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \le C_m \left\| \sum_{\Omega(n)=m} b_n n^{-s} \right\|_\infty .$$

Here the exponent of $\log n$ on the left-hand side cannot be improved. Making the choice $m = \sqrt{\log n / \log_2 n}$ in (4.2), we may obtain the following statement: If for some $c, C > 0$ we have

$$\sum_{n=1}^\infty |b_n| \frac{\exp\left(c\sqrt{\log n \log_2 n}\right)}{\sqrt{n}} \le C \left\| \sum_{n=1}^\infty b_n n^{-s} \right\|_\infty , \tag{4.3}$$

then $c < 1$. It was later shown in [16, 17] that (4.3) holds for $c < 1/\sqrt{2}$, and that this is optimal.

The series of inequalities (4.1) suggests that we should search for upper ℓ^2-estimates for $\|\mathbf{T}_g\|$ as the appropriate analogues of the lower ℓ^1 estimates (4.2) and (4.3). Therefore, we now aim at finding weights $w_m(n)$ such that

$$\|\mathbf{T}_g\| \le \left(\sum_{\Omega(n)=m} |b_n|^2 w_m(n) \right)^{\frac{1}{2}} \quad \text{for} \quad g(s) = \sum_{\Omega(n)=m} b_n n^{-s}. \tag{4.4}$$

The crucial ingredient in the proof of Theorem 4.1 which covers the case $m = 1$, is the estimate

$$\sum_{p|n} \log p \le \log n.$$

To find a replacement for this estimate, we argue as follows. Observe that if $m \le \Omega(n)$, then

$$\begin{aligned} \sum_{\substack{k|n \\ \Omega(k)=m}} \log k &\le \sum_{p_1|n} \sum_{p_2|n} \cdots \sum_{p_m|n} \log(p_1 p_2 \cdots p_m) \\ &= m \sum_{p_1|n} \cdots \sum_{p_m|n} \log p_m = m\omega(n)^{m-1} \log n. \end{aligned} \tag{4.5}$$

This is sharp, up to a constant depending only on m. Indeed, let n be square-free, so that $\Omega(n) = \omega(n)$. Then

$$\sum_{\substack{k|n \\ \Omega(k)=m}} \log k = \sum_{p|n} \sum_{\substack{p|k|n \\ \omega(k)=m}} \log p = \sum_{p|n} (\log p) \binom{\omega(n)-1}{m-1} = (\log n) \binom{\omega(n)-1}{m-1}.$$

This gives us an example of an admissible weight $w_2(n)$, since $\omega(n)/\log n$ is bounded. It turns out that we can obtain the following optimal result from (4.5).

Theorem 4.2. *The inequality in* (4.4) *holds when* $m = 2$ *with the weight function*

$$w_2(n) = C_2 \frac{\log n}{\log_2 n} \tag{4.6}$$

and C_2 *an absolute constant. This is sharp in the sense that we cannot replace* $\log_2 n$ *in* (4.6) *by* $(\log_2 n)^{1+\varepsilon}$ *for any* $\varepsilon > 0$. *When* $m \geq 3$, *the inequality in* (4.4) *holds with*

$$w_m(n) = C_m \frac{n^{\frac{m-2}{m}}}{(\log n)^{m-2}} \tag{4.7}$$

and C_m *an absolute constant. This is also sharp in the sense that we cannot replace* $(\log n)^{m-2}$ *in* (4.7) *by* $(\log n)^{m+\varepsilon-2}$ *for any* $\varepsilon > 0$.

Proof. To prove that (4.6) is sufficient, we let $\mathbf{T}_g$ act on $f(s) = \sum_{n\geq 1} a_n n^{-s}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \|\mathbf{T}_g f\|^2_{\mathscr{H}^2} &\leq \sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \Bigg(\sum_{\substack{k|n \\ \Omega(k)=2}} (\log_2 k) \log k \Bigg) \Bigg(\sum_{\substack{k|n \\ \Omega(k)=2}} |b_k|^2 \frac{\log k}{\log_2 k} |a_{n/k}|^2 \Bigg) \\ &\leq \sum_{n=2}^{\infty} \frac{\log_2 n}{(\log n)^2} \Bigg(\sum_{\substack{k|n \\ \Omega(k)=2}} \log k \Bigg) \Bigg(\sum_{\substack{k|n \\ \Omega(k)=2}} |b_k|^2 \frac{\log k}{\log_2 k} |a_{n/k}|^2 \Bigg). \end{aligned}$$

We complete the proof by using (4.5) and the well known estimate $\omega(n) \ll \log n/(\log_2 n)$.

To prove that (4.6) is best possible, we assume that there is some $\varepsilon > 0$ such that

$$\|\mathbf{T}_g\| \leq C_2 \Bigg(\sum_{\Omega(n)=2} |b_n|^2 \frac{\log n}{(\log_2 n)^{1+\varepsilon}} \Bigg)^{\frac{1}{2}}$$

for every 2-homogeneous Dirichlet series g. Let x be a large real number and consider the symbol

$$g(s) = \sum_{x/2 < p \leq x} \frac{(\log_2(pq))^{1+\varepsilon/2}}{p} (pq)^{-s},$$

where $q \sim e^x$ is a prime number. The weight condition is then satisfied uniformly in x, since

$$\sum_{\Omega(n)=2} |b_n|^2 \frac{\log n}{(\log_2 n)^{1+\varepsilon}} = \sum_{x/2<p\le x} \frac{\log(pq)\log_2(pq)}{p^2} \asymp \frac{x\log x}{x^2}\pi(x) \asymp 1.$$

We now want to show that $\|\mathbf{T}_g\|$ is unbounded as $x \to \infty$, and choose as a test function

$$f(s) := \prod_{x/2<p\le x} \left(1+p^{-s}\right). \tag{4.8}$$

Let S_x denote the set of square-free numbers generated by the primes $x/2 < p \le x$, so that $\|f\|^2_{\mathscr{H}^2} = |S_x| = 2^{N(x)}$, where $N(x) := \pi(x) - \pi(x/2)$. Note that if n is in S_x, then $\omega(n) \le N(x)$. It follows from the prime number theorem that

$$N(x) \sim \frac{x}{2\log x}.$$

Set $V_x := \{n \in S_x : \omega(n) \ge N(x)/2\}$. By the symmetry of the binomial expansion

$$|S_x| = \sum_{n=0}^{N(x)} \binom{N(x)}{n} = \sum_{n<N(x)/2} \binom{N(x)}{2} + |V_x|,$$

we find that $|V_x| \sim |S_x|/2$. Then

$$\begin{aligned}
\|\mathbf{T}_g\|^2 \ge \frac{\|\mathbf{T}_g f\|^2_{\mathscr{H}^2}}{\|f\|^2_{\mathscr{H}^2}} &\ge \frac{1}{|S_x|}\sum_{n\in V_x} \frac{1}{(\log(nq))^2}\left|\sum_{pq|nq} \frac{(\log_2(pq))^{1+\varepsilon/2}}{p}\log(pq)\right|^2 \\
&\ge \frac{1}{|S_x|}\sum_{n\in V_x}\left|\sum_{p|n}\frac{(\log_2 q)^{1+\varepsilon/2}}{p}\right|^2 \\
&\asymp \frac{1}{|S_x|}\sum_{n\in V_x}\left|\frac{(\log x)^{1+\varepsilon/2}}{x}\,\omega(n)\right|^2 \asymp (\log x)^{\varepsilon},
\end{aligned}$$

giving the desired conclusion.

The proof that (4.7) is sharp is similar. Let $\varepsilon > 0$ be given and consider

$$g(s) = \sum_{\substack{n\in S_x \\ \omega(n)=m}} n^{-1+1/m}(\log n)^{m-1+\varepsilon/2} n^{-s}.$$

We observe that

$$\sum_{\Omega(n)=m} |b_n|^2 n^{1-2/m}(\log n)^{2-m-\varepsilon} = \sum_{\substack{n\in S_x \\ \omega(n)=m}} \frac{(\log n)^m}{n} \asymp \frac{(\log x)^m}{x^m}(\pi(x))^m \asymp 1.$$

Now, if n is in S_x, then it follows from the prime number theorem that $\log n \ll x$. As test function, we use again (4.8). The function

$$t \mapsto t^{-1+1/m}(\log t)^{m-1+\varepsilon/2}$$

is eventually decreasing for every $m \geq 3$ and every $\varepsilon > 0$. We find that

$$\begin{aligned}\|\mathbf{T}_g\|^2 \geq \frac{\|\mathbf{T}_g f\|^2_{\mathscr{H}^2}}{\|f\|^2_{\mathscr{H}^2}} &\geq \frac{1}{|S_x|}\sum_{n\in V_x}\frac{1}{(\log n)^2}\Bigg|\sum_{\substack{k\mid n\\ \Omega(k)=m}} k^{-1+1/m}(\log k)^{m-1+\varepsilon/2}\Bigg|^2 \\ &\gg \frac{1}{|S_x|}\sum_{n\in V_x}\frac{1}{x^2}\left|x^{-m+1}(\log x)^{m+\varepsilon/2}\binom{\omega(n)}{m}\right|^2 \\ &\gg (\log x)^{\varepsilon}\frac{1}{|S_x|}\sum_{n\in V_x} 1 \gg (\log x)^{\varepsilon},\end{aligned}$$

where we used that $k \leq x^m$ in the inner sum.

It remains to establish that (4.4) holds with the weight (4.7). Let $\mathbf{T}_g$ act on $f(s) = \sum_{n\geq 1} a_n n^{-s}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\mathbf{T}_g f\|^2_{\mathscr{H}^2} \leq \sum_{n=2}^{\infty}\frac{1}{(\log n)^2}&\Bigg(\sum_{\substack{k\mid n\\ \Omega(k)=m}} k^{2/m-1}(\log k)^m\Bigg) \\ &\times \Bigg(\sum_{\substack{k\mid n\\ \Omega(k)=m}} |b_k|^2 k^{1-2/m}(\log k)^{2-m}|a_{n/k}|^2\Bigg).\end{aligned}$$

Hence it suffices to show that

$$A_m(n) := \sum_{\substack{k\mid n\\ \Omega(k)=m}} k^{2/m-1}(\log k)^m \ll (\log n)^2.$$

Suppose that n has the prime factorization $n = (p_j)^{\kappa}$. Let $\widetilde{\kappa}$ denote a decreasing rearrangement of κ and let $\widetilde{n} = (p_j)^{\widetilde{\kappa}}$. The function

$$t \mapsto t^{2/m-1}(\log t)^m$$

is eventually decreasing for every $m \geq 3$, so clearly $A_m(n) \ll A_m(\widetilde{n})$. On the other hand $\widetilde{n} \leq n$, so we may without loss of generality assume that $n = \widetilde{n}$. Hence, we have that

$$n = p_1^{\kappa_1}\cdots p_d^{\kappa_d},$$

where $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_d > 0$. By the prime number theorem,

$$p_d \sim \sum_{p\leq p_d}\log p = \log\left(\prod_{j=1}^{d} p_j\right) \leq \log n. \tag{4.9}$$

By summing over the largest prime first, we find that

$$\begin{aligned} A_m(n) &\le \sum_{p\le p_d} (m\log p)^m p^{2/m-1}\left(\sum_{q\le p} q^{2/m-1}\right)^{m-1} \\ &\ll \sum_{p\le p_d} p(\log p) \ll p_d^2 \ll (\log n)^2 \end{aligned}$$

using the prime number theorem twice. $\square$

As promised, Theorem 4.2 exhibits m-homogeneous Dirichlet series g in $\mathscr{X}$ that converge in $\mathbb{C}_{1/m}$, but in no larger half-plane, for every $m \ge 2$. This can be loosely interpreted as saying that the more prime factors we have in each non-zero term, the closer we get to the half-plane $\mathbb{C}_0$. In this sense, the multiplicative symbols of Section 3 correspond to $m = \infty$, and it is therefore not surprising that they converge in $\mathbb{C}_0$.

Setting $m = \sqrt{2\log n/\log_2 n}$, we are led to a family of weights w (cf. (4.3)) that give estimates of the type (4.4) with no reference to homogeneity, allowing arbitrary Dirichlet series g.

Theorem 4.3. *If $c < 2$, then*

$$\|\mathbf{T}_g\| \le C\left(\sum_{n=2}^{\infty} |b_n|^2 n \exp\left(-c\sqrt{\log n \log_2 n}\right)\right)^{\frac{1}{2}}. \tag{4.10}$$

Conversely, if (4.10) *holds for every $\mathbf{T}_g$-operator, then $c \le 2\sqrt{2}$.*

Proof. We observe first that we must have $c \le 2\sqrt{2}$ for (4.10) to hold in view of the sharpness of Theorem 4.2 and the fact that

$$\frac{n^{1-2/m}}{(\log n)^{m-2}} = n(\log n)^2 \exp\left(-2\sqrt{2}\sqrt{\log n \log_2 n}\right),$$

if $m = \sqrt{2\log n/\log_2 n}$.

It remains therefore only to show the positive result that (4.10) holds whenever $0 < c < 2$. To simplify the notation, we set $\varphi_c(k) := \exp\left(c\sqrt{\log k \log_2 k}\right)$. By the Cauchy–Schwarz inequality,

$$\|\mathbf{T}_g f\|_{\mathscr{H}^2}^2 \le \sum_{n=2}^{\infty} \frac{1}{(\log n)^2}\left(\sum_{k|n} \frac{\varphi_c(k)}{k}(\log k)^2\right)\left(\sum_{k|n} |b_k|^2 \frac{k}{\varphi_c(k)}|a_{n/k}|^2\right).$$

Choosing some c', $c < c' < 2$, we find that

$$\sum_{k|n} \frac{\varphi_c(k)}{k}(\log k)^2 \ll \sum_{k|n} \frac{\varphi_{c'}(k)}{k} =: A(n)$$

The rest of the proof is devoted to showing that $A(n) \ll (\log n)^2$, which is precisely what is needed.

Since $x \mapsto \varphi_{c'}(x)/x$ is eventually decreasing on $[1,\infty)$, we may, as in the last part of the proof of Theorem 4.2, assume that $n = \widetilde{n}$. By splitting into homogeneous parts and using (4.9), we find that

$$A(n) = \sum_{m \le \Omega(n)} \sum_{\substack{k\,|\,n \\ \Omega(k)=m}} \frac{\varphi_{c'}(k)}{k} \le \sum_{m \le \Omega(n)} \varphi_{c'}\left((\log n)^m\right) \sum_{\substack{k\,|\,n \\ \Omega(k)=m}} \frac{1}{k}.$$

In each inner sum $\sum k^{-1}$, we divide every prime factor of k by some $a > 0$ and then bound the resulting sum by an Euler product (Rankin's trick), to obtain that

$$\begin{aligned} \sum_{\substack{k\,|\,n \\ \Omega(k)=m}} \frac{1}{k} &\le a^{-m} \prod_{p|n} \left(1 - \frac{a}{p}\right)^{-1} \\ &= a^{-m} \exp\left(a \sum_{p|n} \frac{1}{p} + O(1)\right) \ll a^{-m} \exp\left(a \sum_{p \le p_d} \frac{1}{p}\right) \\ &\asymp a^{-m} \exp\left(a \log_2 p_d\right) \le \exp\left(-m \log a + a \log_3 n\right). \end{aligned}$$

Choosing $a := m/(\log_3 n)$, we obtain in total

$$\begin{aligned} A(n) &\le \sum_{m \le \Omega(n)} \exp\left[c' \sqrt{m(\log_2 n)(\log m + \log_3 n)} - m \log m + m \log_4 n + m\right] \\ &\ll \Omega(n) + \sum_{m \le \log_2 n} \exp\left[c' \sqrt{2m(\log_2 n)(\log_3 n)} - m \log m + m \log_4 n + m\right], \end{aligned}$$

where we first used that the exponential in the sum is bounded when $m \ge \log_2 n$, and then that $\log m \le \log_3 n$ when $m \le \log_2 n$. To estimate the final sum, we use calculus to conclude that the index m of the largest term should satisfy

$$\frac{{c'}^2}{2}(\log_2 n)(\log_3 n) = m\left(\log m - \log_4 n\right)^2,$$

and we see that $m = ({c'}^2/2 + o(1)) \log_2 n / \log_3 n$. Combining this with the standard estimate $\Omega(n) \le \log n / \log 2$, we find that

$$A(n) \ll \log n + (\log_2 n) \exp\left(\left(\frac{{c'}^2}{2} + o(1)\right)(\log_2 n)\right) \ll (\log n)^2,$$

whenever ${c'}^2 < 4$, which is the desired estimate. □

It is not surprising that there is a gap between the necessary and sufficient conditions of Theorem 4.3. When considering the inequality (4.3), the necessary condition obtained from m-homogeneous Dirichlet series misses the sharp condition, also by a factor $\sqrt{2}$. In the latter case, the proof of the sharp necessary condition captures cancellations by L^∞ estimates for random trigonometric polynomials [16]. This suggests that our arguments, which only deal with the absolute values of the coefficients of g, cannot be expected to tell the full story.

5. Boundedness of $\mathbf{T}_g$ on $\mathscr{H}^p$

5.1. **Carleson measure characterization.** We will now consider the action of the Volterra operator $\mathbf{T}_g$ on the Hardy spaces $\mathscr{H}^p$, for $0 < p < \infty$. To this end, we recall that $\mathscr{X}_p$ denotes the space of symbols g in $\mathscr{D}$ such that the Volterra operator $\mathbf{T}_g$ acts boundedly on $\mathscr{H}^p$, and we set

$$\|g\|_{\mathscr{X}_p} := \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}.$$

We will now establish our characterization of the elements of $\mathscr{X}_p$ in terms of Carleson measures.

Applying the Littlewood–Paley formula (2.2) to $\mathbf{T}_g f$, we immediately obtain a characterization of the symbols g that belong to $\mathscr{X}_2$: g is in $\mathscr{X}_2$ if and only if it there is a positive constant $C(g)$ such that

$$\begin{aligned}\|\mathbf{T}_g f\|_{\mathscr{H}^2}^2 &\asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi(\sigma+it)|^2|g'_\chi(\sigma+it)|^2\sigma\, d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\\ &\le C(g)^2\|f\|_{\mathscr{H}^2}^2.\end{aligned}$$

Using Fubini's theorem, we may remove the integral over $\mathbb{R}$, since each t represents a rotation in each variable on $\mathbb{T}^\infty$. From this observation we obtain the characterization

$$\int_{\mathbb{T}^\infty}\int_0^\infty |f_\chi(\sigma)|^2\,|g'_\chi(\sigma)|^2\,\sigma d\sigma\, dm_\infty(\chi) \le C(g)^2\|f\|_{\mathscr{H}^2}^2. \tag{5.1}$$

Clearly, the smallest constant $C(g)$ in (5.1) satisfies $C(g) \asymp \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^2)}$.

Theorem 5.1. $\mathbf{T}_g$ *acts boundedly on* $\mathscr{H}^p$ *for* $0<p<\infty$ *if and only if there is a positive constant* $C(g,p)$ *such that*

$$\int_{\mathbb{T}^\infty}\int_0^\infty |f_\chi(\sigma)|^p|g'_\chi(\sigma)|^2\sigma\, d\sigma dm_\infty(\chi) \le C(g,p)^2\|f\|_{\mathscr{H}^p}^p, \tag{5.2}$$

for all $f\in\mathscr{H}^p$. *Furthermore, if*

$$C(g,p) := \sup_{\|f\|_{\mathscr{H}^p}=1}\left(\int_{\mathbb{T}^\infty}\int_0^\infty |f_\chi(\sigma)|^p|g'_\chi(\sigma)|^2\sigma\, d\sigma dm_\infty(\chi)\right)^{\frac{1}{2}}, \tag{5.3}$$

then $C(g,p) \asymp \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}$.

We observe that if we restrict to only one variable, meaning that we consider only Dirichlet series over powers of a single prime, then the condition of Theorem 5.1 is independent of p and reduces to the familiar one variable description of $\mathrm{BMOA}(\mathbb{D})$.

Our proof of Theorem 5.1 adapts arguments from [31], the main difference being that we will additionally integrate every quantity over $\mathbb{T}^\infty$. Before giving the proof, we collect some preliminary results. By using Fubini's theorem once more, we find that (5.2) is equivalent to

$$\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi(\sigma+it)|^p|g'_\chi(\sigma+it)|^2\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\le C(g,p)^2\|f\|^p_{\mathscr{H}^p}. \tag{5.4}$$

The virtue of introducing an extra parameter in (5.4) is that it allows us to apply techniques adapted to the conformally invariant Hardy space $H^p_{\mathrm{i}}(\mathbb{C}_0)$. In addition to the Littlewood–Paley formula (2.2), we will use the square function formula

$$\|f\|^p_{\mathscr{H}^p}\asymp |a_1|^p+\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\left(\int_{\Gamma_\tau}|f'_\chi(\sigma+it)|^2\,d\sigma\,dt\right)^{p/2}\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi), \tag{5.5}$$

which can be found in [13, Thm. 7]. Here, for τ in $\mathbb{R}$, Γ_τ is the cone

$$\Gamma_\tau=\{\sigma+it\,:\,|t-\tau|<\sigma\}.$$

For a holomorphic function f in $\mathbb{C}_0$, let f^* denote the non-tangential maximal function

$$f^*(\tau):=\sup_{s\in\Gamma_\tau}|f(s)|,\qquad \tau\in\mathbb{R}. \tag{5.6}$$

Since $1/(1+\tau^2)$ is a Muckenhoupt A_q-weight for all $q>1$, it follows from the work of Gundy and Wheeden [21] that f is in $H^p_{\mathrm{i}}(\mathbb{C}_0)$ if and only if f^* is in $L^p_{\mathrm{i}}(\mathbb{R})=L^p\left((1+\tau^2)^{-1}\,d\tau\right)$ for $0<p<\infty$, with comparable norms.

Lemma 5.2. *Let φ be a function and μ a positive measure on $\{\sigma+it\,:\,0<\sigma<1\}$. Then*

$$\int_{\mathbb{R}}\int_0^1|\varphi(\sigma+it)|d\mu(\sigma,t)\asymp\int_{\mathbb{R}}\int_{\Gamma_\tau}|\varphi(\sigma+it)|\frac{1+t^2}{\sigma}d\mu(\sigma,t)\,\frac{d\tau}{1+\tau^2}. \tag{5.7}$$

If μ is a positive measure on all of $\mathbb{C}_0$, then

$$\int_{\mathbb{R}}\int_0^\infty|\varphi(\sigma+it)|d\mu(\sigma,t)\gg\int_{\mathbb{R}}\int_{\Gamma_\tau}|\varphi(\sigma+it)|\frac{1+t^2}{\sigma}d\mu(\sigma,t)\,\frac{d\tau}{1+\tau^2}. \tag{5.8}$$

Proof. For $\sigma+it$ in $\mathbb{C}_0$, we consider the set $I(\sigma+it):=\{\tau\in\mathbb{R}\,:\,\sigma+it\in\Gamma_\tau\}$. A computation shows that

$$\int_{I(\sigma+it)}\frac{d\tau}{1+\tau^2}\asymp\frac{\sigma}{1+t^2},\qquad 0<\sigma\le 1$$

and that

$$\int_{I(\sigma+it)} \frac{d\tau}{1+\tau^2} \ll \frac{\sigma}{1+t^2}, \qquad 0<\sigma<\infty.$$

The estimates (5.7) and (5.8) now follow from Fubini's theorem. $\square$

Proof of Theorem 5.1. We may assume that g is in $\mathscr{H}^p$ since otherwise $\mathbf{T}_g$ is trivially unbounded. Thus, for almost every χ in $\mathbb{T}^\infty$, the measure

$$\mu_{g,\chi}(\sigma,t) = |g_\chi'(\sigma+it)|^2\,\sigma d\sigma\,\frac{dt}{1+t^2}$$

is well-defined on $\mathbb{C}_0$.

Suppose first that $p \geq 2$ and that (5.4) is satisfied. Then by the Littlewood–Paley formula (2.2), Hölder's inequality, and two applications of (5.4), we have that

$$\begin{aligned}
\|\mathbf{T}_g f\|_{\mathscr{H}^p}^p &\asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |(\mathbf{T}_g f)_\chi(\sigma+it)|^{p-2}|f_\chi(\sigma+it)|^2 d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi)\\
&\leq \left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |(\mathbf{T}_g f)_\chi|^p d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi)\right)^{\frac{p-2}{p}}\\
&\qquad\qquad \times \quad \left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi|^p d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi)\right)^{\frac{2}{p}}\\
&\ll C(g,p)^2\|\mathbf{T}_g f\|_{\mathscr{H}^p}^{p-2}\|f\|_{\mathscr{H}^p}^2,
\end{aligned}$$

giving us that $\|\mathbf{T}_g f\|_{\mathscr{H}^p} \ll C(g,p)\|f\|_{\mathscr{H}^p}$.

Suppose now that $\mathbf{T}_g$ acts boundedly on $\mathscr{H}^p$, still considering $p \geq 2$. By (5.7), Hölder's inequality, (5.6), (2.1), and the square function characterization, we have

$$\begin{aligned}
&\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^1 |f_\chi|^p d\mu_{g,\chi}\,dm_\infty\\
&\qquad \ll \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_{\Gamma_\tau} |f_\chi(\sigma+it)|^p|g_\chi'(\sigma+it)|^2 d\sigma dt\frac{d\tau}{1+\tau^2}dm_\infty(\chi)\\
&\qquad \leq \int_{\mathbb{T}^\infty}\int_{\mathbb{R}} \left(f_\chi^*(\tau)\right)^{p-2}\int_{\Gamma_\tau} |(\mathbf{T}_g f)_\chi'|^2 d\sigma dt\frac{d\tau}{1+\tau^2}dm_\infty(\chi)\\
&\qquad \leq \|f\|_{\mathscr{H}^p}^{p-2}\|\mathbf{T}_g f\|_{\mathscr{H}^p}^2 \ll \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}^2\|f\|_{\mathscr{H}^p}^p.
\end{aligned}$$

The remaining integral can be estimated using the uniform pointwise estimates that hold for f and g in $\mathscr{H}^p$ in the half-plane $\mathrm{Re}(s) \geq 1$, yielding that

$$\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_1^\infty |f_\chi|^p d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi) \ll \|f\|_{\mathscr{H}^p}^p\|g\|_{\mathscr{H}^p}^2 \leq \|f\|_{\mathscr{H}^p}^p\|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}^2.$$

Suppose now that $0 < p < 2$ and that (5.4) is satisfied. Using the square function characterization (5.5), (5.6), Hölder's inequality, (2.1), and (5.8), we obtain

$$\|\mathbf{T}_g f\|^p_{\mathscr{H}^p} \asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\left(\int_{\Gamma_\tau}|f_\chi(\sigma+it)|^2|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\right)^{p/2}\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi)$$
$$\leq \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\left(f^*_\chi(\tau)\right)^{\frac{(2-p)p}{2}}\left(\int_{\Gamma_\tau}|f_\chi(\sigma+it)|^p|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\right)^{p/2}\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi)$$
$$\leq \|f\|^{\frac{(2-p)p}{2}}_{\mathscr{H}^p}\left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_{\Gamma_\tau}|f_\chi(\sigma+it)|^p|g'_\chi(\sigma+it)|^2\,d\sigma\,dt\,\frac{d\tau}{1+\tau^2}\,dm_\infty(\chi)\right)^{\frac{p}{2}}$$
$$\ll \|f\|^{\frac{(2-p)p}{2}}_{\mathscr{H}^p}\left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty|f_\chi(\sigma+it)|^p d\mu_{g,\chi}(\sigma,t)\;dm_\infty(\chi)\right)^{\frac{p}{2}}$$
$$\leq \|f\|^{\frac{(2-p)p}{2}}_{\mathscr{H}^p}C(g,p)^p\|f\|^{\frac{p^2}{2}}_{\mathscr{H}^p} = C(g,p)^p\|f\|^p_{\mathscr{H}^p}.$$

Finally we deal with the case when $0 < p < 2$ and $\mathbf{T}_g\colon \mathscr{H}^p \to \mathscr{H}^p$ is bounded. Note first that by the Littlewood–Paley formula (2.2), we have

$$\|\mathbf{T}_g f\|^p_{\mathscr{H}^p} \asymp \int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |(\mathbf{T}_g)_\chi|^{p-2}|f_\chi|^2 d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi).$$

Using Hölder's inequality and this identity, we obtain

$$\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi|^p d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi)$$
$$\ll \|\mathbf{T}_g f\|^{\frac{p^2}{2}}_{\mathscr{H}^p}\left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |(\mathbf{T}_g f)_\chi|^p d\mu_{g,\chi}(\sigma,t)\,dm_\infty(\chi)\right)^{\frac{2-p}{2}}$$
$$\leq \|\mathbf{T}_g f\|^{\frac{p^2}{2}}_{\mathscr{H}^p}C(g,p)^{2-p}\|\mathbf{T}_g f\|^{\frac{p(2-p)}{2}}_{\mathscr{H}^p} \leq C(g,p)^{2-p}\|\mathbf{T}_g\|^p_{\mathscr{L}(\mathscr{H}^p)}\|f\|^p_{\mathscr{H}^p}.$$

By an approximation argument, we can a priori assume that $C(g,p)$ is finite. Then, by taking the supremum over norm-1 Dirichlet series f, we obtain that $C(g,p) \ll \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}$, as desired. □

5.2. **Necessary and sufficient conditions.** Theorem 5.1 can be applied to find necessary and sufficient conditions for membership in $\mathscr{X}_p$, parallel to the result for $\mathscr{X}_2$ proved in Theorem 2.3. However, there is one essential difficulty when passing from $p = 2$ to the general case $0 < p < \infty$, namely that the proof of part (c) of Theorem 2.3 relies on the local embedding property of $\mathscr{H}^2$ expressed by (2.5). The local embedding extends trivially to hold for $p = 2k$, for every positive integer k, since

$$\|f\|^{2k}_{H^{2k}_{\mathrm{i}}(\mathbb{C}_{1/2})} = \left\|f^k\right\|^2_{H^2_{\mathrm{i}}(\mathbb{C}_{1/2})} \leq \widetilde{C}\left\|f^k\right\|^2_{\mathscr{H}^2} = \widetilde{C}\|f\|^{2k}_{\mathscr{H}^{2k}}, \tag{5.9}$$

but it is a well-known open problem whether it holds for any other p. We refer to [34, Sec. 3] for a discussion of the embedding problem.

Arguing similarly for the embedding constant (5.3), we find for every positive integer n that

$$C(g,p) \geq C(g,np). \tag{5.10}$$

We will use this to prove a rather curious incomplete analogue to part (c) of Theorem 2.3. In view of (5.9) and (5.10), we are allowed to apply integral powers before and after using the local embedding property of $\mathscr{H}^2$, leading us to the expected necessary condition for g to belong to $\mathscr{X}_p$, but only for rational p.

Theorem 5.3. *Suppose that g is in $\mathscr{D}$.*

(a) *If g is in* $\mathrm{BMOA}(\mathbb{C}_0)$*, then $\mathbf{T}_g$ is bounded from $\mathscr{H}^p$ to $\mathscr{H}^p$.*
(b) *If g is in $\mathscr{X}_p$, then g satisfies condition* (iii) *from Lemma 2.1.*
(c) *If g is in $\mathscr{X}_p$ and p is in $\mathbb{Q}_+$, then g is in* $\mathrm{BMOA}(\mathbb{C}_{1/2})$*.*

Proof. The proof of (a) is identical to the proof given for $p = 2$ in Theorem 2.3, using Theorem 5.1, (5.4), and that Carleson measures in one variable are independent of p. The proof of (b) is also the same.

For (c) we need two facts which follow from close inspection of the proof of Theorem 5.1. First of all, it is clear from the first part of the proof that for $p \geq 2$ there is a constant C_1, independent of p, such that

$$\|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)} \leq C_1 C(g,p),$$

where $C(g,p)$ is as in Theorem 5.1. Hence, we conclude by (5.10) that there is a constant C_2 such that for every positive integer n we have

$$\|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^{np})} \leq C_2 \|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^p)}. \tag{5.11}$$

Secondly, by mimicking the next part of the proof, also for $p \geq 2$, we see that there is a constant C_3 such that

$$\int_{\mathbb{R}} \int_{1/2}^{1} |f(s)|^p |g'(s)|^2 (\sigma - 1/2) d\sigma \frac{dt}{1+t^2} \leq C_3 \|\mathbf{T}_g f\|^2_{H^p_{\mathrm{i}}(\mathbb{C}_{1/2})} \|f\|^{p-2}_{H^p_{\mathrm{i}}(\mathbb{C}_{1/2})}, \tag{5.12}$$

at least for Dirichlet polynomials f. Here we have implicitly applied the maximal function characterization of $H^p_{\mathrm{i}}(\mathbb{C}_{1/2})$. However, by the inner-outer factorization of H^p_{i}, we see that the constants involved do not blow up as $p \to \infty$. To prove the theorem, let $p = 2k/n > 0$ be a rational number. Hence, by (5.11), $\mathbf{T}_g$ is bounded on $\mathscr{H}^{2k}$, with control of the constant. Combined with (5.12) and the embedding (5.9), we find, setting $C_4 = \widetilde{C}$, that

$$\begin{aligned}\int_{\mathbb{R}} \int_{1/2}^{1} |f(s)|^{2k} |g'(s)|^2 (\sigma - 1/2) d\sigma \frac{dt}{1+t^2} &\leq C_3 \|\mathbf{T}_g f\|^2_{H^{2k}_{\mathrm{i}}(\mathbb{C}_{1/2})} \|f\|^{2(k-1)}_{H^{2k}_{\mathrm{i}}(\mathbb{C}_{1/2})} \\ &\leq C_3 C_4^2 C_2^2 \|\mathbf{T}_g\|^2_{\mathscr{L}(\mathscr{H}^p)} \|f\|^{2k}_{\mathscr{H}^{2k}}.\end{aligned}$$

It follows that $\nu_g(\sigma+it) := |g'(s)|^2(\sigma-1/2)\,d\sigma dt/(1+t^2)$ is a Carleson measure for $\mathscr{H}^{2k}$, with constant uniformly bounded by $\|\mathbf{T}_g\|^2_{\mathscr{L}(\mathscr{H}^p)}$. Clearly, the argument in [29, Thm. 3] produces uniform estimates, so we conclude that ν_g is a Carleson measure on $H^{2k}_{\mathrm{i}}(\mathbb{C}_{1/2})$, with constant uniformly bounded by the same quantity. By appealing to the inner-outer factorization again, we conclude that there is a constant C_5 such that

$$\|\nu_g\|_{\mathrm{CM}(H^2_{\mathrm{i}})} \leq C_5\|\mathbf{T}_g\|^2_{\mathscr{L}(\mathscr{H}^p)} \leq C_6 C(g,p)^2.$$

The proof is now completed by arguing as in the proof of Theorem 2.3. □

Theorem 2.5 now gives us an interesting example of a $\mathbf{T}_g$-operator that is bounded on all $\mathscr{H}^p$-spaces.

Corollary 5.4. *Let g be as in Theorem 2.5, i.e.,*

$$g(s) = \sum_{n=2}^{\infty} \frac{1}{n\log n} n^{-s}.$$

Then $\mathbf{T}_g : \mathscr{H}^p \to \mathscr{H}^p$ is bounded for every $p < \infty$.

5.3. **Linear symbols.** We will now extend Theorem 4.1 by proving that all linear symbols g yield bounded operators $\mathbf{T}_g$ on $\mathscr{H}^p$, for the whole range $0 < p < \infty$. We do this by showing that in this special case, the constant $C(g,p)$ in the Carleson measure condition (5.2) may be chosen independently of p.

Theorem 5.5. *Let*

$$g(s) = \sum_{j=1}^{\infty} b_j p_j^{-s}$$

be given. Then $\mathbf{T}_g$ is bounded on $\mathscr{H}^p$ if and only if g is in $\mathscr{H}^2$. In fact,

$$\sup_{f\in\mathscr{H}^p,\|f\|_{\mathscr{H}^p}\leq 1} \int_{\mathbb{T}^\infty}\int_0^\infty |f_\chi(\sigma)|^p |g'_\chi(\sigma)|^2 \sigma d\sigma dm_\infty(\chi) = \frac{1}{4}\|g\|^2_{\mathscr{H}^2}$$

holds whenever $0 < p < \infty$.

It suffices to consider finitely many, say d, variables. The Poisson kernel on the polydisc is then given by

$$P_z(w) := \prod_{j=1}^{d} \frac{1-|z_j|^2}{|1-\overline{w_j}z_j|^2},$$

where $|z_j| < 1$ and $w = (w_j)$ is a point on $\mathbb{T}^d$. Suppose that $0 < \alpha \leq p$ and that f is in $H^p(\mathbb{D}^d)$. Then $|f|^\alpha$ is separately subharmonic in each variable, which gives us the following.

Lemma 5.6. *If f is in $H^p(\mathbb{D}^d)$, then*
$$|f(z)|^\alpha \le \int_{\mathbb{T}^d} P_z(w)|f(w)|^\alpha dm_d(w)$$
for every point z in $\mathbb{D}^d$ and $0 < \alpha \le p$.

Lemma 5.6 shows that if the Carleson embedding condition (5.2) holds for all harmonic functions f, for one p, then (5.2) holds for all f in $\mathscr{H}^p$, for every p. Hence, to prove Theorem 5.5, we only need to verify that linear functions g in $\mathscr{H}^2$ induce Carleson measures on the harmonic functions for $p = 2$. Obviously this raises the question whether the corresponding statement is true for other symbols g from Sections 3 and 4, or even if it could be true that the Carleson condition for analytic functions implies the same condition for harmonic functions, cf. Question 1 in the introduction. We only have the answer in the simplest case of linear symbols.

To simplify the computations to be given below, we will use the multiplicative notation that comes from identifying the dual of the compact abelian group $\mathbb{T}^\infty$ with the discrete abelian group $\mathbb{Q}_+$ (see [22, 33]). This means that the Fourier series of f on $\mathbb{T}^\infty$ takes the form
$$\sum_{r\in\mathbb{Q}_+} c(r)\chi(r),$$
where $c(r) = \langle f(\chi), \chi(r)\rangle_{L^2(\mathbb{T}^\infty)}$. (The notation $\chi(r)$ is explained at the end of the introduction.)

Proof of Theorem 5.5. To see that the supremum cannot be smaller than 1/4, it suffices to set $g(s) = {p_j}^{-s}$ and $f(s) = 1$. To prove the bound from above, we begin by expanding the function $h_p(\chi) := |f_\chi|^{p/2}$ in a Fourier series on $\mathbb{T}^\infty$,
$$h_p(\chi) = \sum_{r\in\mathbb{Q}^+} c(r)\chi(r).$$
Using Lemma 5.6 with $z_j = p_j^{-\sigma}\chi(p_j)$ and $\alpha = p/2$, we get that
$$|f_\chi(\sigma)|^{p/2} \le \int_{\mathbb{T}^d} h_p(w)P_z(w)dm_d(w) = \sum_{(m,n)=1} c\left(\frac{m}{n}\right)(mn)^{-\sigma}\chi\left(\frac{m}{n}\right),$$
where we in the last step integrated the Fourier series of h_p term by term against the Poisson kernel. It follows that
$$\begin{aligned} I_\sigma &:= \int_{\mathbb{T}^\infty} |f_\chi(\sigma)|^p |g'_\chi(\sigma)|^2 dm_\infty(\chi) \\ &\le \sum_{j,k=1}^{d} \sum_{\frac{m\mu}{n\nu} = \frac{p_j}{p_k}} \left| c\left(\frac{m}{n}\right) c\left(\frac{\mu}{\nu}\right)\right| (mn\mu\nu p_j p_k)^{-\sigma}|b_j b_k| \log p_j \log p_k, \end{aligned}$$

where it is understood that $(m,n)=1$ and $(\mu,\nu)=1$. By symmetry, we get $I_\sigma \le 2I_{\sigma,1}+2I_{\sigma,2}$, where

$$I_{\sigma,1} := \sum_{j,k=1}^{d} \sum_{\substack{\frac{m\mu}{n\nu}=\frac{p_j}{p_k},\\ p_j\,|\,m,\ p_k\,|\,n}} \left|c\left(\frac{m}{n}\right)c\left(\frac{\mu}{\nu}\right)\right|(mn\mu\nu p_jp_k)^{-\sigma}|b_jb_k|\log p_j\log p_k$$

$$I_{\sigma,2} := \sum_{j,k=1}^{d} \sum_{\substack{\frac{m\mu}{n\nu}=\frac{p_j}{p_k},\\ p_j\,|\,m,\ p_k\,|\,\nu}} \left|c\left(\frac{m}{n}\right)c\left(\frac{\mu}{\nu}\right)\right|(mn\mu\nu p_jp_k)^{-\sigma}|b_jb_k|\log p_j\log p_k.$$

We estimate the contribution from these two sums separately. First, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} I_{\sigma,1} &\le \left(\sum_{j,k=1}^{d}\sum_{\substack{(m,n)=1,\\ p_j\,|\,m,\ p_k\,|\,n}}\left|c\left(\frac{m}{n}\right)\right|^2\frac{\log p_j\log p_k}{(mn)^{2\sigma}}\right)^{\frac12} \\ &\qquad\times\left(\sum_{j,k=1}^{d}\sum_{(\mu,\nu)=1}\left|c\left(\frac{\mu}{\nu}\right)\right|^2|b_j|^2|b_k|^2\frac{\log p_j\log p_k}{(p_jp_k)^{2\sigma}}\right)^{\frac12} \\ &\le \left(\sum_{(m,n)=1}\left|c\left(\frac{m}{n}\right)\right|^2\frac{\log m\log n}{(mn)^{2\sigma}}\right)^{\frac12} \\ &\qquad\times\left(\sum_{(\mu,\nu)=1}\left|c\left(\frac{\mu}{\nu}\right)\right|^2\sum_{j,k=1}^{d}|b_j|^2|b_k|^2\frac{\log p_j\log p_k}{(p_jp_k)^{2\sigma}}\right)^{\frac12}, \end{aligned}$$

where we in the final inequality changed the order of summation in the first factor and used that $\sum_{p_j|m}\log p_j \le \log m$. To compute the integrals, we will use the identity

$$\int_0^\infty (\log a)^2 a^{-2\sigma}\sigma d\sigma = \frac14,$$

which is valid for every $a>0$. We use the Cauchy–Schwarz inequality again and take the two integrals into the respective sums, to deduce that

$$\begin{aligned} \int_0^\infty I_{\sigma,1}\,\sigma d\sigma &\le \left(\sum_{(m,n)=1}\left|c\left(\frac{m}{n}\right)\right|^2\frac{\log m\log n}{4(\log mn)^2}\right)^{\frac12} \\ &\qquad\times\left(\sum_{(\mu,\nu)=1}\left|c\left(\frac{\mu}{\nu}\right)\right|^2\sum_{j,k=1}^{d}|b_j|^2|b_k|^2\frac{\log p_j\log p_k}{4(\log p_jp_k)^2}\right)^{\frac12}. \end{aligned}$$

The fractions with logarithms are bounded by 1/16, so in total we get that

$$\int_0^\infty I_{\sigma,1}\,\sigma d\sigma \le \frac{1}{16}\|g\|_{\mathscr{H}^2}^2\|f\|_{\mathscr{H}^p}^p.$$

To estimate $I_{\sigma,2}$, we use the Cauchy–Schwarz inequality and change the order of summation:

$$\begin{aligned}
I_{\sigma,2} &\le \left(\sum_{j,k=1}^{d}\sum_{\substack{(m,n)=1,\\ p_j|m}}\left|c\left(\frac{m}{n}\right)\right|^2|b_k|^2\frac{(\log p_j)^2}{(mp_j)^{2\sigma}}\right)^{\frac12}\\
&\qquad\times\left(\sum_{j,k=1}^{d}\sum_{\substack{(\mu,\nu)=1,\\ p_k|\nu}}\left|c\left(\frac{\mu}{\nu}\right)\right|^2|b_j|^2\frac{(\log p_k)^2}{(\nu p_k)^{2\sigma}}\right)^{\frac12}\\
&=\|g\|_{\mathscr{H}^2}^2\left(\sum_{(m,n)=1}\left|c\left(\frac{m}{n}\right)\right|^2\sum_{p_j|m}\frac{(\log p_j)^2}{(mp_j)^{2\sigma}}\right)^{\frac12}\\
&\qquad\times\left(\sum_{(\mu,\nu)=1}\left|c\left(\frac{\mu}{\nu}\right)\right|^2\sum_{p_k|\nu}\frac{(\log p_k)^2}{(\nu p_k)^{2\sigma}}\right)^{\frac12}.
\end{aligned}$$

The two factors are symmetrical, so by using the Cauchy–Schwarz inequality again we get

$$\begin{aligned}
\int_0^\infty I_{\sigma,2}\,\sigma d\sigma &\le \|g\|_{\mathscr{H}^2}^2\sum_{(m,n)=1}\left|c\left(\frac{m}{n}\right)\right|^2\sum_{p_j|m}\frac{(\log p_j)^2}{4(\log mp_j)^2}\\
&=\frac{\|g\|_{\mathscr{H}^2}^2}{4}\sum_{(m,n)=1}\left|c\left(\frac{m}{n}\right)\right|^2\frac{1}{\log m}\sum_{p_j|m}\frac{\log p_j}{2+\frac{\log m}{\log p_j}+\frac{\log p_j}{\log m}}\\
&\le\frac{\|g\|_{\mathscr{H}^2}^2\|f\|_{\mathscr{H}^p}^p}{16},
\end{aligned}$$

where we used that $\log m/\log p_j + \log p_j/\log m \ge 2$ when $p_j|m$. Combining everything yields

$$\int_0^\infty I_\sigma\,\sigma d\sigma \le 2\int_0^\infty I_{\sigma,1}\,\sigma d\sigma + 2\int_0^\infty I_{\sigma,2}\,\sigma d\sigma \le \frac14\|g\|_{\mathscr{H}^2}^2\|f\|_{\mathscr{H}^p}^p. \qquad \square$$

6. Comparison of $\mathscr{X}$ with other spaces of Dirichlet series of BMO type

6.1. **Hardy spaces $\mathscr{H}^p$ and** $\mathrm{BMOA}(\mathbb{C}_0)$**.** Our initial motivation for studying $\mathbf{T}_g$ was to consider $\mathscr{X} = \mathscr{X}_2$ as a type of BMOA-space for the range of Hardy spaces $\mathscr{H}^p$. From Theorem 2.3, we have the following inclusions, which show that $\mathscr{X}$ is in every $\mathscr{H}^p$, for $0 < p < \infty$.

Corollary 6.1. *We have the following inclusions,*

$$\mathscr{H}^\infty \subsetneq \mathrm{BMOA}(\mathbb{C}_0) \cap \mathscr{D} \subsetneq \mathscr{X} \subsetneq \bigcap_{0<p<\infty} \mathscr{H}^p.$$

Proof. The inclusions are all from Theorem 2.3. That the first inclusion is strict follows from Theorem 2.5. The second inclusion was observed to be strict in the remark at the end of Section 3, but it can also be deduced from any example in Section 4. The strictness of the last inclusion follows from Theorem 4.2 and the fact that

$$\|g\|_{\mathscr{H}^p} \asymp \|g\|_{\mathscr{H}^2} \tag{6.1}$$

when g is an m-homogeneous Dirichlet series, with implied constants depending on m and p. To verify (6.1), we argue as follows. Let $d(n)$ be the number of divisors of the positive integer n. By the extension of Helson's inequality discussed in [12, Sec. 5] and [35, Thm. 3], there exist nonnegative number α and β, depending on p, such that

$$\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^\alpha}\right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathscr{H}^p} \leq \left(\sum_{n=1}^{\infty} |a_n|^2 [d(n)]^\beta\right)^{\frac{1}{2}}. \tag{6.2}$$

The key point is that if $\Omega(n) = m$, then $m+1 \leq d(n) \leq 2^m$, proving (6.1). (In fact, by a suitable application of Hölder's inequality, we can prove (6.1) using only the right inequality in (6.2).) □

In the next three subsections, we will compare $\mathscr{X}$ with two other analogues of BMOA, namely the dual space $(\mathscr{H}^1)^*$ and the space $(\mathscr{H}^2 \odot \mathscr{H}^2)^*$ of symbols generating bounded multiplicative Hankel forms. Let us first recall that neither of these spaces is contained in

$$\bigcap_{0<p<\infty} \mathscr{H}^p.$$

This follows immediately from a result of Marzo and Seip [26], which states that the Riesz projection P on the polytorus is unbounded from $L^\infty(\mathbb{T}^\infty)$ to $H^4(\mathbb{D}^\infty)$. In fact, it is not even known whether $P(L^\infty(\mathbb{T}^\infty))$ is contained in $H^p(\mathbb{D}^\infty)$ for any $p > 2$. Note that $P(L^\infty(\mathbb{T}^\infty))$ is naturally identified with $(\mathscr{H}^1)^*$, and that it is strictly continuously contained in $(\mathscr{H}^2 \odot \mathscr{H}^2)^*$ [30].

6.2. **Hankel forms.** Let us now consider the space of symbols g such that the corresponding Hankel forms $\mathbf{H}_g$ are bounded. The form $\mathbf{H}_g$ is given by

$$\mathbf{H}_g(fh) := \langle fh, g\rangle_{\mathscr{H}^2},$$

from which it is clear, by definition, that $\mathbf{H}_g$ is bounded if and only if g is in $(\mathscr{H}^2 \odot \mathscr{H}^2)^*$. Applying the product rule for derivatives, we find that

$$\mathbf{H}_g(fh) = f(+\infty)h(+\infty)\overline{g(+\infty)} + \langle \partial^{-1}(f'h), g\rangle_{\mathscr{H}^2} + \langle \partial^{-1}(fh'), g\rangle_{\mathscr{H}^2}, \tag{6.3}$$

where

$$\partial^{-1} f(s) := -\int_s^\infty f(w)\,dw.$$

The "half-Hankel" form

$$(f,h) \mapsto \langle \partial^{-1}(f'h), g\rangle_{\mathscr{H}^2} \tag{6.4}$$

is bounded if and only if $g \in (\partial^{-1}(\partial\mathscr{H}^2 \odot \mathscr{H}^2))^*$. It is clear from (6.3) that

$$(\partial^{-1}(\partial\mathscr{H}^2 \odot \mathscr{H}^2))^* \subset (\mathscr{H}^2 \odot \mathscr{H}^2)^*. \tag{6.5}$$

Whether the inclusion in (6.5) is strict, is an open problem. It was observed in [13] that it is equivalent to an interesting Schur multiplier problem.

Corollary 6.2. *Suppose that the Volterra operator $\mathbf{T}_g$ acts boundedly on $\mathscr{H}^2$. Then the Hankel form $\mathbf{H}_g$ is bounded.*

Proof. The Littlewood–Paley formula (2.2) may be polarized, to obtain

$$\begin{aligned}\langle f, g\rangle_{\mathscr{H}^2} &= f(+\infty)\overline{g(+\infty)} \\ &+ \frac{4}{\pi}\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty f'_\chi(\sigma+it)\overline{g'_\chi(\sigma+it)}\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi).\end{aligned} \tag{6.6}$$

We find that

$$\begin{aligned}&\langle \partial^{-1}(f'h), g\rangle_{\mathscr{H}^2} \\ &\qquad = \frac{4}{\pi}\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty f'_\chi(\sigma+it)h_\chi(\sigma+it)\overline{g'_\chi(\sigma+it)}\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi).\end{aligned}$$

Hence, it is clear from Theorem 5.1 that if $\mathbf{T}_g$ is bounded, then so is the form (6.4). Thus we may complete the proof by using the inclusion (6.5). □

On weighted Dirichlet spaces of the disc (including the Hardy space), even in the vector-valued setting, the boundedness of a half-Hankel form also implies the boundedness of the corresponding T_g operator (see [3]). However, by [13, Lem. 10], a half-Hankel form on $\mathscr{H}^2$ generated by a symbol g with positive coefficients is bounded if and only if $\mathbf{H}_g$ is bounded. Since the symbols of Theorem 3.1 generate bounded Hankel forms for $\alpha \geq 1/2$, but not bounded $\mathbf{T}_g$ operators for $\alpha < 1$, this shows that the same relationship between the half-Hankel form and $\mathbf{T}_g$ does not hold in the present context.

6.3. **The dual of $\mathscr{H}^1$.** The most tractable sufficient condition for

$$g(s) = \sum_{n\geq 1} b_n n^{-s}$$

to belong to $(\mathscr{H}^1)^*$ was put forward by Helson [23]: g is in $(\mathscr{H}^1)^*$ if

$$\sum_{n=1}^\infty |b_n|^2 d(n) < \infty, \tag{6.7}$$

where again $d(n)$ denotes the number of divisors of the integer n. In fact, Helson's result is stated in terms of the Hankel form $\mathbf{H}_g$ considered above. If g satisfies (6.7), then $\mathbf{H}_g$ is Hilbert–Schmidt. Note that, by a consideration of zero sets based on [35, Thm. 2], we can show that a Dirichlet series g satisfying (6.7) will not always be in $\mathrm{BMOA}(\mathbb{C}_{1/2})$.

The examples of g in $\mathscr{X}_2$ considered in Sections 3 and 4 are easily seen to satisfy (6.7). Moreover, we see that the symbols in Theorem 3.1, $1/2 < \alpha < 1$, are in $(\mathscr{H}^1)^*$, but not in $\mathscr{X}_2$. Hence $(\mathscr{H}^1)^*$ is not contained in $\mathscr{X}_2$, and it is tempting to conjecture that $\mathscr{X}_2 \subset (\mathscr{H}^1)^*$.

First, let us show how to construct a class of Dirichlet series in $(\mathscr{H}^1)^* \cap \mathscr{X}_2$ that do not satisfy (6.7), showing that Helson's criterion is not well adapted to understanding Volterra operators.

Theorem 6.3. *Suppose that $\mathscr{N} = \{n_1, n_2, \dots\} \subset \mathbb{N}\backslash\{1\}$ is a set with the property that $(n_j, n_k) = 1$ if $j \neq k$. If*

$$g(s) = \sum_{n\in\mathscr{N}} b_n n^{-s}, \tag{6.8}$$

then $\|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^2)} = \|g\|_{\mathscr{H}^2}$. Moreover, for $f(s) = \sum_{n\geq 1} a_n n^{-s}$, we have

$$\left(|a_0|^2 + \sum_{n\in\mathscr{N}} |a_n|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\|f\|_{\mathscr{H}^1}.$$

The second statement in the theorem yields $\|g\|_{(\mathscr{H}^1)^*} \leq \sqrt{2}\|g\|_{\mathscr{H}^2}$, by the Cauchy–Schwarz inequality applied to $\langle f, g\rangle_{\mathscr{H}^2}$. Define the integers $n_1 := 2$, $n_2 := 3\cdot 5$, $n_3 := 7\cdot 11\cdot 13$, and so on. The set $\mathscr{N} := \{n_1, n_2, \dots\}$ satisfies the assumptions of Theorem 6.3, but $d(n_j) = 2^j$, so (6.7) is not always satisfied.

Proof of Theorem 6.3. For the first statement, we simply observe that

$$\sum_{\substack{n|N\\ n\in\mathscr{N}}} \log n \leq \log N,$$

which allows us to follow the proof of Theorem 4.1 to obtain that every Dirichlet series of the form (6.8) satisfies $\|\mathbf{T}_g\| = \|g\|_{\mathscr{H}^2}$.

For the second statement, fix some $n = n_j$, and set $d := \omega(n)$, $m := \Omega(n)$ and $\kappa := \kappa(n)$. By Helson's iterative procedure [23], it is sufficient to demonstrate that for f in $H^1(\mathbb{D}^d)$,

$$\left(|a_0|^2 + \frac{1}{2}|a_\kappa|^2\right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{D}^d)}. \tag{6.9}$$

We begin with Carleman's inequality (see [36]),

$$\left(\sum_{k=0}^{\infty}\frac{|c_k|^2}{k+1}\right)^{\frac{1}{2}} \le \left\|\sum_{k=0}^{\infty} c_k w^k\right\|_{H^1(\mathbb{D})}.$$

Setting $F(w)=\sum_{k\ge 0} c_k w^k$, we use F. Wiener's trick (see [11]) with an mth root of unity, say φ, so that

$$F_m(w^m) := \frac{1}{m}\left(F(w)+F(w\varphi)+F(w\varphi^2)+\cdots+F(w\varphi^{m-1})\right) = \sum_{k=0}^{\infty} c_{mk} w^{mk}.$$

Clearly $\|F_m\|_{H^1(\mathbb{D})} \le \|F\|_{H^1(\mathbb{D})}$, so we find from Carleman's inequality that

$$\left(\sum_{k=0}^{\infty}\frac{|c_{mk}|^2}{k+1}\right)^{\frac{1}{2}} \le \left\|\sum_{k=0}^{\infty} c_k w^k\right\|_{H^1(\mathbb{D})}. \tag{6.10}$$

Returning to our function f in $H^1(\mathbb{D}^d)$, we let f_k denote the k-homogeneous part of f and decompose f accordingly:

$$f(z)=\sum_{k=0}^{\infty} f_k(z).$$

Substituting $z_j \mapsto wz_j$ for $1\le j\le d$, we find, using Fubini's theorem, (6.10), and Minkowski's inequality, that

$$\left(\sum_{k=0}^{\infty}\frac{1}{k+1}\,\|f_{km}\|^2_{H^1(\mathbb{D}^d)}\right)^{\frac{1}{2}} \le \int_{\mathbb{D}^d}\left(\sum_{k=0}^{\infty}\frac{|f_{km}(z)|^2}{k+1}\right)^{\frac{1}{2}} dm_d(z) \le \|f\|_{H^1(\mathbb{D}^d)}.$$

We retain only the two first terms in the sum on the left-hand side. The proof of (6.9) is completed by noting that $\|f_0\|_{H^1(\mathbb{D}^d)}=|a_0|$ and that $|a_\kappa|\le\|f_m\|_{H^1(\mathbb{D}^d)}$, where the latter inequality holds because $|\kappa|=\Omega(n)=m$. $\square$

As for the question of whether $\mathscr{X}_2 \subset (\mathscr{H}^1)^*$, our best result is the following corollary of the characterization given in Theorem 5.1. For its interpretation, one should recall that (5.10) implies that $\mathscr{X}_1 \subset \mathscr{X}_2$. Hence, the corollary also motivates further interest in the question of whether $\mathscr{X}_2=\mathscr{X}_p$ for all p, $0<p<\infty$.

Corollary 6.4. *Suppose that the Volterra operator $\mathbf{T}_g$ acts boundedly on $\mathscr{H}^1$. Then g is in $(\mathscr{H}^1)^*$.*

Proof. Let f be a Dirichlet series in $\mathscr{H}^1$ and suppose that $f(+\infty)=0$. Let g be $\mathscr{X}_1$ and apply (6.6) along with the Cauchy–Schwarz inequality,

$$\begin{aligned} |\langle f,g\rangle_{\mathscr{H}^2}| &\asymp \left|\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty f_\chi'(\sigma+it)\overline{g_\chi'(\sigma+it)}\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\right| \\ &\leq \left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty \frac{|f_\chi'(\sigma+it)|^2}{|f_\chi(\sigma+it)|}\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_{\mathbb{T}^\infty}\int_{\mathbb{R}}\int_0^\infty |f_\chi(\sigma+it)|\,|g_\chi'(\sigma+it)|^2\sigma\,d\sigma\,\frac{dt}{1+t^2}\,dm_\infty(\chi)\right)^{\frac{1}{2}}. \end{aligned}$$

We finish the proof by using Theorem 5.1 with $p=1$, since the quantity on the second line is bounded from above and below by $\|f\|_{\mathscr{H}^1}^{1/2}$ in view of the Littlewood–Paley formula (2.2). □

Observe that by part (a) of Theorem 5.3, this shows in particular that if g is in $\mathrm{BMOA}(\mathbb{C}_0)\cap\mathscr{D}$, then g is in $(\mathscr{H}^1)^*$. This inclusion can also be deduced directly from the two Littlewood–Paley formulas (2.2) and (6.6), using the Cauchy–Schwarz inequality and Lemma 2.2.

6.4. **On the finite polydisc $\mathbb{D}^d$.** Let us now confine ourselves to studying Dirichlet series

$$f(s)=\sum_{n=1}^\infty a_n n^{-s}$$

restricted to the first d primes, by demanding that $a_n=0$ if $p_j|n$, for $j>d$. Through the Bohr lift, the restricted Hardy spaces $\mathscr{H}_d^p$ (which are complemented subspaces of $\mathscr{H}^p$) are isometrically identified with $H^p(\mathbb{D}^d)$. We consider now a Dirichlet series g restricted to the first d primes and let $\mathbf{T}_g$ act on $\mathscr{H}_d^p$.

Corollary 6.5. *For $0<p<\infty$, $\mathbf{T}_g$ is bounded on $\mathscr{H}_d^p$ if and only if it is bounded on $\mathscr{H}_d^2$.*

Proof. This follows from Theorem 5.1, since the Carleson measure characterization is now over $\mathbb{D}^d$, and the Carleson measures of $H^p(\mathbb{D}^d)$ are independent of p (see [15]). □

Moreover, using the result that $H^2(\mathbb{D}^d)\odot H^2(\mathbb{D}^d)=H^1(\mathbb{D}^d)$ from [18, 25], we conclude that symbols inducing bounded $\mathbf{T}_g$-operators on the finite polydisc belong to $(H^1(\mathbb{D}^d))^*$. This subsection is devoted to showing that, even in the finite-dimensional setting, the dual of H^1 still does not characterize the bounded $\mathbf{T}_g$-operators.

Let D denote the differentiation operator on Dirichlet series,

$$Df(s) := f'(s) = -\sum_{n=2}^{\infty} a_n (\log n) n^{-s}.$$

Identifying again $\mathscr{H}_d^p$ with $H^p(\mathbb{D}^d)$, we find that we may write

$$Df(z_1, \ldots, z_d) = -\sum_{j=1}^{d} (\log p_j) z_j \partial_{z_j} f(z_1, \ldots, z_d). \tag{6.11}$$

Note the similarity between D and the radial differentiation operator

$$Rf(z_1, \ldots, z_d) := \sum_{j=1}^{d} z_j \partial_{z_j} f(z_1, \ldots, z_d). \tag{6.12}$$

The Volterra operator T_g defined with the radial differentiation operator R and radial integration R^{-1} has previously been investigated on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ by a number of authors. A seminal contribution is that of Pau [31], who proved that T_g is bounded on $H^p(\mathbb{B}_d)$ if and only if g is in $\mathrm{BMOA}(\mathbb{B}_d)$. In particular, for $p = 2$, the T_g operator is bounded if and only if the corresponding Hankel operator is bounded, i.e., if and only if g defines a bounded linear functional on $H^2(\mathbb{B}_d) \odot H^2(\mathbb{B}_d)$.

We shall now see that the corresponding statement is not true on the finite polydisc $\mathbb{D}^2$. The statement and proof are written for the Volterra operator defined in terms of radial differentiation (6.12), but the argument works equally well for the half-plane differentiation (6.11). In the following theorem, we use the notation $g_1 \otimes g_2(z, w) := g_1(z) g_2(w)$.

Theorem 6.6. *There exists functions g_1 in $H^\infty(\mathbb{D})$ and g_2 in* $\mathrm{BMOA}(\mathbb{D})$ *such that $T_{g_1 \otimes g_2}$ is unbounded on $H^2(\mathbb{D}^2)$.*

To obtain the desired conclusion from this theorem, namely that T_g is not bounded simultaneously with the Hankel operator H_g even on the bidisc, it suffices to observe that the symbol $g_1 \otimes g_2$ is in $\mathrm{BMOA}(\mathbb{D}^2)$ and therefore in $\big(H^2(\mathbb{D}^2) \odot H^2(\mathbb{D}^2)\big)^* = \big(H^1(\mathbb{D}^2)\big)^*$.

Proof of Theorem 6.6. Suppose that $f(z, w) = \sum_{m,n \geq 0} a_{m,n} z^m w^n$. Then

$$\begin{aligned} Rf(z, w) &= \sum_{m,n \geq 0} (m + n) a_{m,n} z^m w^n, \\ R^{-1} f(z, w) &= \sum_{\substack{m,n \geq 0 \\ m+n > 0}} \frac{a_{m,n}}{m + n} z^m w^n. \end{aligned}$$

We consider the Volterra operator $T_g f = R^{-1}(fRg)$, choosing $f = f_1 \otimes f_2$, where f_1 and f_2 are both in $H^2(\mathbb{D})$. We compute and find that

$$f(z,w)Rg(z,w) = f_1(z)f_2(w)\left(zg_1'(z)g_2(w) + wg_1(z)g_2'(w)\right). \tag{6.13}$$

We consider first the second term of (6.13), which we write as $h_1(z)h_2(w)$, where

$$h_1(z) := f_1(z)g_1(z) = \sum_{m=0}^{\infty} a_m z^m \quad \text{and} \quad h_2(w) := wf_2(w)g_2'(w) = \sum_{n=1}^{\infty} b_n w^n.$$

Since f_1 is in $H^2(\mathbb{D})$ and g is in $H^\infty(\mathbb{D})$, clearly h_1 is in $H^2(\mathbb{D})$, so $\sum_{m\geq 0}|a_m|^2 < \infty$. In a similar way, we see that h_2 is the derivative of a function in $H^2(\mathbb{D})$ because f_2 is in $H^2(\mathbb{D})$ and g_2 is in $\mathrm{BMOA}(\mathbb{D})$ so that the operator T_{g_2} is bounded on $H^2(\mathbb{D})$. This means that $\sum_{n\geq 1}|b_n|^2/n^2 < \infty$. We conclude therefore that

$$\left\|R^{-1}(h_1h_2)\right\|_{H^2(\mathbb{D}^2)}^2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|^2|b_n|^2}{(m+n)^2} \leq \sum_{m=0}^{\infty}|a_m|^2\sum_{n=1}^{\infty}\frac{|b_n|^2}{n^2} < \infty.$$

Changing our attention to the first term in (6.13), it remains for us to show that we can pick f_1, f_2, g_1, and g_2 satisfying our assumptions, so that the $H^2(\mathbb{D}^2)$-norm of

$$R^{-1}\left(zf_1(z)g_1'(z)f_2(w)g_2(w)\right)$$

is infinite. Replace for the moment $zf_1(z)g_1'(z)$ with an arbitrary function h_1 in $z\partial H^2(\mathbb{D})$, say

$$h_1(z) = \sum_{m=1}^{\infty} a_m z^m.$$

Choose f_2 and g_2 as

$$f_2(w) = \sum_{n=2}^{\infty}\frac{w^n}{\sqrt{n}(\log n)} \qquad \text{and} \qquad g_2(w) = -\log(1-w).$$

The coefficients of $h_2(w) := f_2(w)g_2(w) = \sum_{n\geq 3} b_n w^n$ are given by

$$b_n = \sum_{k=2}^{n-1}\frac{1}{\sqrt{k}(\log k)}\,\frac{1}{(n-k)} \gg \frac{1}{\sqrt{n}(\log n)}\sum_{k=2}^{n-1}\frac{1}{n-k} \gg \frac{1}{\sqrt{n}}.$$

Hence we find that

$$\left\|R^{-1}(h_1h_2)\right\|_{H^2(\mathbb{D}^2)}^2 \gg \sum_{m=1}^{\infty}\sum_{n=3}^{\infty}\frac{|a_m|^2}{(m+n)^2 n} \asymp \sum_{m=1}^{\infty}\frac{|a_m|^2\log(m+2)}{(m+1)^2} = \infty$$

for an appropriate choice of h_1 in $z\partial H^2(\mathbb{D})$. However, by a factorization result of Aleksandrov and Peller [1], there exist f_1^j in $H^2(\mathbb{D})$ and g_1^j in $H^\infty(\mathbb{D})$ for

$1 \leq j \leq 4$, such that

$$h_1(z) = z \sum_{j=1}^{4} f_1^j(z)(g_1^j)'(z).$$

Therefore, at least one of the four pairs (f_1^j, g_1^j), $1 \leq j \leq 4$, will do as the choice of (f_1, g_1). □

7. Compactness of $\mathbf{T}_g$ on $\mathscr{H}^2$

7.1. **Basic results.** We turn to a brief discussion of compactness of $\mathbf{T}_g$. Every polynomial symbol $g(s) = \sum_{n\leq N} b_n n^{-s}$ defines a compact $\mathbf{T}_g$-operator, since in this case $\mathbf{T}_g$ is the sum of N diagonal operators with entries in c_0. This means that all bounded operators from Section 4 actually are compact. To see this, let S_N denote the partial sum operator, acting on a Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ by

$$S_N f(s) = \sum_{n=1}^{N} a_n n^{-s}.$$

Suppose now that we have an estimate of the type $\|\mathbf{T}_g\|^2 \leq \sum_{n\geq 2} |b_n|^2 w(n)$ for some positive weight function $w(n)$. If the right hand side is finite for some Dirichlet series g, then

$$\|\mathbf{T}_g - \mathbf{T}_{S_N g}\|^2 \leq \sum_{n\geq N} |b_n|^2 w(n) \to 0, \qquad N \to \infty,$$

demonstrating that $\mathbf{T}_g$ is compact. In particular, every bounded $\mathbf{T}_g$-operator with a linear symbol is compact, since then $\|\mathbf{T}_g\|_{\mathscr{L}(\mathscr{H}^2)} = \|g\|_{\mathscr{H}^2}$, by Theorem 4.1. Let us also mention that the Volterra operator defined by the primitive of the zeta function considered in Theorem 2.5,

$$g(s) = \sum_{n=2}^{\infty} \frac{1}{n \log n} n^{-s},$$

is compact by this argument and Theorem 4.3. In the next subsection, we will produce a concrete example of a non-compact operator, by testing the Volterra operator of Theorem 3.3, for $\lambda = 1$, against reproducing kernels for suitable subspaces of $\mathscr{H}^2$.

We mention that it is possible to prove versions of Theorems 2.3, 5.1, and 5.3 for compactness, by replacing bounded mean oscillation by vanishing mean oscillation, and embeddings by vanishing embeddings. The details are standard, see for instance [31] for the arguments in a different setting.

We present only two results in this section. The first is that the closure of Dirichlet polynomials in $\mathrm{BMOA}(\mathbb{C}_0)$ is $\mathrm{VMOA}(\mathbb{C}_0) \cap \mathscr{D}$, as it relies on the translation invariance (i) of Lemma 2.1 enjoyed by Dirichlet series in $\mathrm{BMOA}(\mathbb{C}_0)$.

Recall that $\mathrm{VMOA}(\mathbb{C}_0)$ consists of those $g \in \mathrm{BMOA}(\mathbb{C}_0)$ such that

$$\lim_{\delta\to 0^+} \sup_{|I|<\delta} \frac{1}{|I|}\int_I \left| f(it) - \frac{1}{|I|}\int_I f(i\tau)\,d\tau \right| dt = 0.$$

We endow the space $\mathrm{BMO}(\mathbb{C}_\theta) \cap \mathscr{D}$ with the norm $\|f\|_{\mathrm{BMO}(\mathbb{C}_\theta)\cap\mathscr{D}} := |f(+\infty)| + \|f\|_{\mathrm{BMO}(\mathbb{C}_\theta)}$.

Theorem 7.1. *Let g be a symbol in $\mathrm{VMOA}(\mathbb{C}_0)\cap\mathscr{D}$ and ε be a positive number. Then there is a Dirichlet polynomial P such that $\|g-P\|_{\mathrm{BMO}(\mathbb{C}_\theta)\cap\mathscr{D}} < \varepsilon$.*

Proof. Let B_δ denote the horizontal shift operator given by $B_\delta g(s) = g(s+\delta)$, and, as above, let S_N denote the partial sum operator. We choose $P = B_\delta S_N g$, for some $\delta > 0$ and N to be specified later. Clearly $P(+\infty) = b_1 = g(+\infty)$. Since g is in $\mathrm{VMOA}(\mathbb{C}_0)$, we know from [20, Thm. VI.5.1] that

$$\lim_{\delta\to 0} \|g - B_\delta g\|_{\mathrm{BMO}(\mathbb{C}_0)} = 0.$$

Choose $\delta > 0$ so that $\|g - B_\delta g\|_{\mathrm{BMO}(\mathbb{C}_\theta)} < \varepsilon/2$. Then

$$\begin{aligned}\|g-P\|_{\mathrm{BMO}(\mathbb{C}_0)} &\le \|g - B_\delta g\|_{\mathrm{BMO}(\mathbb{C}_0)} + \|B_\delta g - P\|_{\mathrm{BMO}(\mathbb{C}_0)} \\ &< \varepsilon/2 + 2\|B_\delta g - B_\delta S_N g\|_{\mathscr{H}^\infty}.\end{aligned}$$

Now, by (i) of Lemma 2.1, we know that $\sigma_b(g) \le 0$. By a theorem of Bohr [10], this implies that $S_N g(s)$ converges uniformly to $g(s)$ in the closed half-plane $\mathbb{C}_\delta$, for every $\delta > 0$. Hence there is some $N = N(g,\delta)$ such that $\|B_\delta g - B_\delta S_N g\|_{\mathscr{H}^\infty} = \|B_\delta(g - S_N g)\|_{\mathscr{H}^\infty} < \varepsilon/4$. ∎

Our second basic result is that $\mathbf{T}_g$ is never in any Schatten class, unless g is constant. This is in line with [31, Thm. 6.7], showing that a radial Volterra operator $T_g \ne 0$ defined on $H^2(\mathbb{B}_d)$ can be in the Schatten class S_p only for $p > d$.

Theorem 7.2. *Let*

$$g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

be a non-constant Dirichlet series. Then $\mathbf{T}_g \colon \mathscr{H}^2 \to \mathscr{H}^2$ is not in S_p, for any $p < \infty$.

Proof. Since g is not constant, we know there is at least one non-zero term, so set

$$N = \inf\{n \ge 2 \,:\, b_n \ne 0\} < \infty.$$

We will use [37, Thm. 1.33] in the following way: Set $e_n(s) := n^{-s}$ and assume that $2 \le p < \infty$. Then the set $\{e_n\}_{n\ge 1}$ forms an orthonormal basis for $\mathscr{H}^2$, so that:

$$\|\mathbf{T}_g\|_{S_p}^p \ge \sum_{n=N}^{\infty} \|\mathbf{T}_g e_n\|_{\mathscr{H}^2}^p.$$

A simple computation shows that if $n \geq N$, then we have

$$\|\mathbf{T}_g e_n\|_{\mathscr{H}^2}^2 = \sum_{m=2}^{\infty} \frac{|b_m|^2 (\log m)^2}{(\log mn)^2} \geq \frac{|b_N|^2 (\log N)^2}{(\log nN)^2} \geq \frac{|b_N|^2 (\log N)^2}{(2 \log n)^2}.$$

In particular, $\|\mathbf{T}_g e_n\|_{\mathscr{H}^2} \geq (|b_N| \log N)/(2 \log n)$ and hence $\|\mathbf{T}_g\|_{S_p}^p \geq \infty$. The inclusion between Schatten classes allows us to conclude that $\mathbf{T}_g$ cannot be in S_p for any $0 < p < \infty$. □

7.2. **Estimating y-smooth reproducing kernels.** We will now study the action of $\mathbf{T}_g$ on reproducing kernels for suitable subspaces of $\mathscr{H}^2$. The reproducing kernel k_w of $\mathscr{H}^2$ itself at w, where $\mathrm{Re}(w) > 1/2$, is given by

$$k_w(s) := \zeta\left(s + \overline{w}\,\right) = \prod_p \left(1 - p^{-s-\overline{w}}\,\right)^{-1}.$$

Considering these reproducing kernels is insufficient in our analysis of the multiplicative symbol g from Theorem 3.3. Indeed, regardless of the value of λ, the Dirichlet series $g(s)$ converges absolutely all the way down to $\mathrm{Re}(s) = \sigma > 0$. Testing $\mathbf{T}_g$ on the kernels k_w, in $\mathbb{C}_{1/2}$ is therefore not enough to detect that it is unbounded for $\lambda > 1$.

To address this, we consider y-smooth reproducing kernels. Let $P^+(n)$ denote the largest prime factor of n. The integer n is called y-smooth if $P^+(n) \leq y$. The y-smooth reproducing kernels, k_w^y are defined for $\mathrm{Re}(w) > 0$ and $y \geq 1$, by cutting off prime numbers larger than y. This means that we set $k_w^y(s) := \zeta\left(s + \overline{w}, y\right)$, where

$$\zeta\left(s + \overline{w}, y\right) := \prod_{p \leq y} \left(1 - p^{-s-\overline{w}}\right)^{-1}.$$

Notice that we already used another variant of cut-off kernels in the proof of Theorem 3.3. Following Gál's construction, we tested against a finite-dimensional kernel at $\sigma = 0$, cut off to be smooth (in the sense of primes) and retaining only suitable small powers of each prime. Our motivation for turning to the more involved investigation of the reproducing kernels $k_w^y(s)$ is that they provide slightly better estimates than the rougher argument stemming from Gál's work. More specifically, we will see that the multiplicative symbol g from Theorem 3.3 with $\lambda = 1$ provides the only concrete example of a non-compact $\mathbf{T}_g$-operator in this paper. As in Section 3, we consider without loss of generality the operator $\widetilde{\mathbf{T}}_g$ instead of $\mathbf{T}_g$, the difference between the two being compact.

Suppose that $f(s) = \sum_{n\geq 1} \varphi(n)n^{-s}$, where φ is a non-negative completely multiplicative function and that $g(s) = \sum_{n\geq 1} b_n n^{-s}$ has non-negative coefficients. A computation shows that

$$\begin{aligned}&\|\widetilde{\mathbf{T}}_g f\|^2_{\mathscr{H}^2}\\ &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(b_m \log m)(b_n \log n)\varphi\left(\frac{mn}{(m,n)^2}\right)\sum_{k=1}^{\infty}\frac{\varphi(k)^2}{\left(\log k + \log \frac{mn}{(m,n)}\right)^2}.\end{aligned} \tag{7.1}$$

We will now choose f to be a y-smooth reproducing kernel and estimate the innermost sum.

Lemma 7.3. *Let $\varphi(n)$ be the completely multiplicative non-negative function defined by setting*

$$\varphi(n) := \begin{cases} n^{-\sigma}, & \text{if } P^+(n) \leq y,\\ 0, & \text{otherwise.}\end{cases}$$

Fix α, $0 < \alpha < 1$. If $y^\alpha \geq 1/\sigma$, then for sufficiently large y (depending on α), we have

$$\begin{aligned} S_\varphi(m,n) &:= \sum_{k=1}^{\infty} \frac{\varphi(k)^2}{\left(\log k + \log\frac{mn}{(m,n)}\right)^2}\\ &\asymp \frac{\|k_\sigma^y\|^2_{\mathscr{H}^2}}{\left((1+o(1))\,(1-2\sigma)^{-1}y^{1-2\sigma} + \log\frac{mn}{(m,n)}\right)^2},\end{aligned}$$

where $o(1)$ tends to 0 as $y \to \infty$.

Proof. We may assume that $0 < \sigma < 1/2$. Observe first that $\|k_\sigma^y\|^2_{\mathscr{H}^2} = \zeta(2\sigma, y)$. For simplicity of notation, we write $a := \log\frac{mn}{(m,n)}$. By Abel summation, we see that

$$S_\varphi(m,n) \sim 2\sigma \int_1^\infty \frac{\Psi(x,y)\, x^{-2\sigma}}{(\log x + a)^2}\,\frac{dx}{x},$$

where as usual $\Psi(x,y)$ denotes the number of y-smooth integers less than or equal to x. Observe that $\zeta(s,y)$ is the Mellin transform of $\Psi(x,y)$,

$$\zeta(s,y) = s\int_0^\infty x^{s-1}\Psi(x,y)\,dx.$$

Hence by writing $\Psi(x,y)$ as the inverse Mellin transform of $\zeta(s,y)$, integrating over the vertical line $\operatorname{Re} s = \xi$ for some $0 < \xi < 2\sigma$, and then changing the order of integration, we obtain

$$I := \int_1^\infty \frac{\Psi(x,y)\, x^{-2\sigma}}{(\log x + a)^2}\,\frac{dx}{x} = \int_1^\infty \left(\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\zeta(s,y)x^s\,\frac{ds}{s}\right)\frac{x^{-2\sigma}}{(\log x+a)^2}\,\frac{dx}{x}$$
$$= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\zeta(s,y)\underbrace{\left(\int_1^\infty \frac{x^{s-2\sigma}}{(\log x+a)^2}\,\frac{dx}{x}\right)}_{J}\frac{ds}{s}.$$

By substituting $x = e^t$, using the identity

$$\frac{1}{(t+a)^2} = -\frac{d}{da}\int_0^\infty e^{-(t+a)x}dx,$$

and interpreting the resulting integral as a Laplace transform, we find that

$$J = \int_0^\infty \frac{e^{-t(2\sigma-s)}}{(t+a)^2}\,dt = -\frac{d}{da}\left(e^{2\sigma a}\int_{2\sigma}^\infty e^{-at}\mathscr{L}\left\{e^{s\cdot}\right\}(t)dt\right)$$
$$= \int_{2\sigma}^\infty e^{-a(t-2\sigma)}(t-2\sigma)\frac{dt}{s-t}.$$

Therefore, by changing the order of integration again, we obtain that

$$I = \int_{2\sigma}^\infty e^{-a(t-2\sigma)}(t-2\sigma)\left(\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}\zeta(s,y)\frac{ds}{s(s-t)}\right)dt.$$

We evaluate the inner integral by residues, capturing the simple pole in $s = t$, to see that

$$I = \int_{2\sigma}^\infty e^{-a(t-2\sigma)}(t-2\sigma)\frac{\zeta(t,y)}{t}\,dt = \int_0^\infty \frac{\zeta(t+2\sigma,y)}{t+2\sigma}\,te^{-at}\,dt.$$

Hence, to prove the statement of the lemma, we need to estimate

$$\frac{2\sigma}{\zeta(2\sigma,y)}I = \frac{2\sigma}{\zeta(2\sigma,y)}\int_0^\infty \frac{\zeta(t+2\sigma,y)}{t+2\sigma}\,te^{-at}\,dt$$

from below. Observe that

$$\frac{\zeta(t+2\sigma,y)}{\zeta(2\sigma,y)} \geq \exp\left(-Ct\sum_{p\leq y}p^{-2\sigma}\log p\right) \geq \exp\left(-C(1-2\sigma)^{-1}ty^{1-2\sigma}\right)$$

when, say, $t \leq 2y^{-\alpha}$. Here $1 < C = 1 + o(1)$. Assuming that $\sigma \geq y^{-\alpha}$, we have that $2\sigma/(t+2\sigma) \geq 1/2$, and we therefore obtain

$$\frac{2\sigma}{\zeta(2\sigma,y)}\int_0^\infty \frac{\zeta(t+2\sigma,y)}{t+2\sigma}\,te^{-at}dt \gg \int_0^{2y^{-\alpha}} t\exp\left(-\left(a+C(1-2\sigma)^{-1}y^{1-2\sigma}\right)t\right)dt$$
$$\geq \frac{1}{2\left(a+C(1-2\sigma)^{-1}y^{1-2\sigma}\right)^2}$$

for sufficiently large y. On the other hand, the same type of estimates carried out in reverse order shows that

$$\begin{aligned}\frac{2\sigma}{\zeta(2\sigma,y)}\int_0^\infty \frac{\zeta(t+2\sigma,y)}{t+2\sigma}\,te^{-at}dt &\ll \int_0^\infty t\exp\left(-\left(a+C'(1-2\sigma)^{-1}y^{1-2\sigma}\right)t\right)dt\\ &= \frac{1}{(a+C'(1-2\sigma)^{-1}y^{1-2\sigma})^2},\end{aligned}$$

where $1 > C' = 1 + o(1)$. $\square$

Applying (7.1) and Lemma 7.3 to a symbol of multiplicative type (3.1), we find that

$$\begin{aligned}\frac{\|\widetilde{\mathbf{T}}_g k_\sigma^y\|_{\mathscr{H}^2}^2}{\|k_\sigma^y\|_{\mathscr{H}^2}^2} &\asymp \sum_{P^+(m)\le y}\sum_{P^+(n)\le y}\psi(mn)\frac{(m,n)^{2\sigma}}{(mn)^\sigma}\\ &\qquad\times\left((1+o(1))\,(1-2\sigma)^{-1}y^{1-2\sigma}+\log\frac{mn}{(m,n)}\right)^{-2}.\end{aligned} \tag{7.2}$$

under the assumptions on y and σ from Lemma 7.3.

Theorem 7.4. *For $0<\lambda<\infty$, let g be the Dirichlet series* (3.1)*, where $\psi(n)$ is the completely multiplicative function defined on the primes by $\psi(p) := \lambda p^{-1}(\log p)$. Fix α, $0<\alpha<1$. If $\sigma = y^{-\alpha}$, then*

$$\frac{\|\widetilde{\mathbf{T}}_g k_\sigma^y\|_{\mathscr{H}^2}^2}{\|k_\sigma^y\|_{\mathscr{H}^2}^2} \gg y^{2(\lambda-1)}. \tag{7.3}$$

In particular, $\mathbf{T}_g$ is not compact when $\lambda = 1$.

Proof. Let $\mu(n)$ denote the Möbius function, the only property of which we need is that $\mu(n)=0$ unless n is square-free. Restricting the sums in (7.2) to square-free numbers and using that $(m,n)^{2\sigma}\ge 1$, we find that

$$\begin{aligned}\frac{\|\widetilde{\mathbf{T}}_g k_\sigma^y\|_{\mathscr{H}^2}^2}{\|k_\sigma^y\|_{\mathscr{H}^2}^2} &\gg \sum_{\substack{P^+(m)\le y\\ \mu(m)\neq 0}}\sum_{\substack{P^+(n)\le y\\ \mu(n)\neq 0}}\frac{\psi(mn)}{(mn)^\sigma}\\ &\qquad\times\left((1+o(1))\,(1-2\sigma)^{-1}y^{1-2\sigma}+\log\frac{mn}{(m,n)}\right)^{-2}.\end{aligned} \tag{7.4}$$

Now using that m and n are y-smooth and square-free, so that both $\log m$ and $\log n$ are bounded by $\pi(y)\log y \le (1+o(1))y$ by the prime number theorem, we obtain from (7.4) that

$$\frac{\big\|\widetilde{\mathbf{T}}_g k_\sigma^y\big\|^2_{\mathscr{H}^2}}{\big\|k_\sigma^y\big\|^2_{\mathscr{H}^2}} \gg \frac{1}{y^2} \sum_{\substack{P^+(m)\le y\\ \mu(m)\neq 0}} \sum_{\substack{P^+(n)\le y\\ \mu(n)\neq 0}} \frac{\psi(mn)}{(mn)^\sigma}$$
$$= \frac{1}{y^2} \sum_{\substack{P^+(m)\le y\\ \mu(m)\neq 0}} \frac{\psi(m)}{m^\sigma} \sum_{\substack{P^+(n)\le y\\ \mu(n)\neq 0}} \frac{\psi(n)}{n^\sigma} = \Bigg(\frac{1}{y} \sum_{\substack{P^+(m)\le y\\ \mu(m)\neq 0}} \frac{\psi(m)}{m^\sigma}\Bigg)^2.$$

We may now complete the proof of the estimate (7.3) by the following computation:

$$\sum_{\substack{P^+(m)\le y\\ \mu(m)\neq 0}} \frac{\psi(m)}{m^\sigma} = \prod_{p\le y}\left(1+\frac{\psi(p)}{p^\sigma}\right) \asymp \exp\left(\sum_{p\le y}\frac{\psi(p)}{p^\sigma}\right)$$
$$\ge \exp\left(\frac{\lambda}{y^\sigma}\sum_{p\le y}\frac{\log p}{p}\right) \asymp \exp\left(\frac{\lambda}{y^\sigma}\log y\right).$$

In the last step, we used Mertens's first theorem, which asserts that $\sum_{p\le y}\frac{\log p}{p} - \log y$ is bounded in absolute value by 2. Now (7.3) follows because $y^{-\sigma}\log y = \log y + o(1)$ when $y\to\infty$ by our choice of σ.

Finally, let $\{\sigma_j\}_{j\ge1}$ and $\{y_j\}_{j\ge1}$ be sequences such that $\sigma_j\to 0$ and $y_j\to\infty$ as $j\to\infty$. Then for every Dirichlet polynomial P, we have that $\langle P, k_{\sigma_j}^{y_j}\rangle_{\mathscr{H}^2}$ converges as $j\to\infty$. On the other hand, we have that $\|k_{\sigma_j}^{y_j}\|_{\mathscr{H}^2}\to\infty$. Therefore $k_{\sigma_j}^{y_j}/\|k_{\sigma_j}^{y_j}\|_{\mathscr{H}^2}$ converges weakly to 0 in $\mathscr{H}^2$. Hence, the estimate shows, for suitably chosen σ_j and y_j, that $\mathbf{T}_g$ is not compact for $\lambda = 1$. $\square$

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Paper 11

Hardy spaces of Dirichlet series and pseudomoments of the Riemann zeta function

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HARDY SPACES OF DIRICHLET SERIES AND PSEUDOMOMENTS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. We study H^p spaces of Dirichlet series, called $\mathscr{H}^p$, for $0 < p < \infty$. We begin by showing that $\mathscr{H}^p$ may be defined either by taking an appropriate L^p closure of all Dirichlet polynomials or by requiring the sequence of "$m$te Abschnitte" to be uniformly bounded in L^p. After showing that these definitions are equivalent, we proceed to establish upper and lower weighted ℓ^2 estimates (called Hardy–Littlewood inequalities) as well as weighted ℓ^∞ estimates for the coefficients of functions in $\mathscr{H}^p$. We discuss some consequences of these estimates and observe that the Hardy–Littlewood inequalities display what we will call a contractive symmetry between $\mathscr{H}^p$ and $\mathscr{H}^{4/p}$. The relevance of the Hardy–Littlewood inequalities for the study of the dual spaces $(\mathscr{H}^p)^*$ is illustrated by a result about the linear functionals generated by fractional primitives of the Riemann zeta function. We deduce general estimates of the norm of the partial sum operator $\sum_{n=1}^\infty a_n n^{-s} \mapsto \sum_{n=1}^N a_n n^{-s}$ on $\mathscr{H}^p$ with $0 < p \le 1$, supplementing a classical result of Helson for the range $1 < p < \infty$. Finally, we discuss the relevance of our results for the computation of the so-called pseudomoments of the Riemann zeta function $\zeta(s)$ (in the sense of Conrey and Gamburd). We apply our upper Hardy–Littlewood inequality to improve on an earlier asymptotic estimate when $p \to \infty$, but at the same time we show, using ideas from recent work of Harper, Nikeghbali, and Radziwiłł and some probabilistic estimates of Harper, that the Hardy–Littlewood estimate for $p < 1$ fails to give the right asymptotics for the pseudomoments of $\zeta^\alpha(s)$ for $\alpha > 1$.

1. Introduction

H^p spaces of Dirichlet series, to be called $\mathscr{H}^p$ in what follows, have been studied extensively in recent years but mostly in the Banach space case $p \ge 1$, with a view to the operators acting on them. In the present paper, we explore $\mathscr{H}^p$ in the full range $0 < p < \infty$, which in part can be given a number theoretic motivation: The interplay between the additive and multiplicative structure of the integers is displayed in a more transparent way by the results obtained without

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any a priori restriction on the exponent p. As an example, we mention that the multiplicative estimates of Section 3 of this paper exhibit what we will call a contractive symmetry between H^p and $H^{4/p}$, which is particularly significant for the study of $\mathscr{H}^p$. We refer to these estimates as multiplicative because they are obtained by multiplicative iteration via the Bohr lift (see below) of estimates for H^p spaces of the unit disc. We note in passing that, surprisingly, there remain basic problems related to the contractive symmetry that are still open in the case of the unit disc.

By a basic observation of Bohr, the *multiplicative structure* of the integers allows us to view an ordinary Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

as a function of infinitely many variables. Indeed, by the transformation $z_j = p_j^{-s}$ (here p_j is the jth prime number) and the fundamental theorem of arithmetic, we have the Bohr correspondence,

$$f(s) := \sum_{n=1}^{\infty} a_n n^{-s} \quad \longleftrightarrow \quad \mathscr{B}f(z) := \sum_{n=1}^{\infty} a_n z^{\kappa(n)}, \tag{1}$$

where we use multi-index notation and $\kappa(n) = (\kappa_1, \ldots, \kappa_j, 0, 0, \ldots)$ is the multi-index such that $n = p_1^{\kappa_1} \cdots p_j^{\kappa_j}$. This transformation—the so-called Bohr lift—gives an isometric isomorphism between $\mathscr{H}^p$ and the Hardy space $H^p(\mathbb{D}^\infty)$. We will come back to the details of this relation in the next section, where we will show that it ensures an unambiguous definition of $\mathscr{H}^p$ in the full range $0 < p < \infty$. The Bohr lift is of fundamental importance in our subject, and will in particular be what we need in Section 3 and Section 4 to lift coefficient estimates in one complex variable to obtain results for $\mathscr{H}^p$.

The *additive structure* of the integers plays a role whenever we restrict attention to the properties of $f(s)$ viewed as an analytic function in a half-plane or when we consider any problem for which the order of summation matters. A particularly interesting example is that of the partial sum operator

$$S_N f(s) := \sum_{n=1}^{N} a_n n^{-s},$$

viewed as an operator on $\mathscr{H}^p$. By a classical theorem of Helson [30], we know that it is uniformly bounded on $\mathscr{H}^p$ when $1 < p < \infty$. In Section 5, we will give bounds that are essentially best possible in the range $0 < p < 1$ and an improvement by a factor $1/\log\log N$ on the previously known bounds when $p = 1$. We are however still far from knowing the precise asymptotics of the norm of S_N when it acts on either $\mathscr{H}^1$ or $\mathscr{H}^\infty$.

We have found it interesting to relate our discussion and to apply part of our results to a number theoretic problem that deals with the *interplay* between the additive and multiplicative structure of the integers. Thus in the final Section 6 we consider the computation of the so-called pseudomoments of the Riemann zeta function $\zeta(s)$ which were studied by Conrey and Gamburd [15] when p is an even integer. In our terminology, the pseudomoments of $\zeta(s)$ are pth powers of the $\mathscr{H}^p$ norms of the Dirichlet polynomials

$$Z_N(s) := \sum_{n=1}^{N} n^{-1/2-s}.$$

We observe that if we write

$$f_N(s) := \prod_{p_j \le N} \frac{1}{1 - p_j^{-1/2-s}}, \tag{2}$$

then $Z_N = S_N f_N$. Hence Z_N can be obtained by applying the partial sum operator to a Dirichlet series whose coefficients represent a completely multiplicative function. This comes as no surprise of course, but the interesting point is how to estimate the norm of $S_N f_N$. We have essentially two methods, one relying on the multiplicative estimates from Section 3 and another relying on an additive estimate of Helson used in Section 5. We will show that our multiplicative estimates improve on the known estimates from [8] in the range $p > 2$. In general, however, we know the right order of magnitude only when $p > 1$, there being a huge gap between the additive and multiplicative estimates in the range $0 < p < 1$. We are not able to remedy this situation, but we will shed light on it by showing that the Nth partial sum of $[f_N(s)]^\alpha$ for $\alpha > 1$ has $\mathscr{H}^p$ norm of an order of magnitude larger than what is suggested by our multiplicative estimates, provided that p is sufficiently small.

Our study of the pseudomoments of $\zeta(s)$ and more generally $\zeta^\alpha(s)$ highlights another important aspect of the spaces $\mathscr{H}^p$, namely a *probabilistic interpretation* of the Bohr correspondence and the use of probabilistic methods. Our work on pseudomoments in the range $0 < p < 1$ is inspired by the recent paper [26] and relies crucially on some delicate probabilistic estimates due to Harper [25].

To close this introduction, we note that there are many questions about $\mathscr{H}^p$ that are not treated or only briefly mentioned in our paper. Our selection of topics has been governed by what appear to be significant and doable problems for the whole range $0 < p < \infty$. We have chosen to be quite detailed in the groundwork in Section 2, dealing with the definition of $\mathscr{H}^p$, because the infinite-dimensional situation and the non-convexity of the L^p quasi-norms for $0 < p < 1$ require some extra care. In that section, we also summarize briefly some known facts and easy consequences, such as for instance how some results for $\mathscr{H}^2$ can be transferred to $\mathscr{H}^p$ when either $p = 2k$ or $p = 1/(2k)$ for $k = 2, 3, ...$ In Section 3, which deals with upper and lower weighted ℓ^2 estimates for the coefficients of functions in $\mathscr{H}^p$, we will record some functional analytic consequences concerning respectively duality and local embeddings of $\mathscr{H}^p$ into appropriate Bergman spaces when $0 < p < 2$. For further information about known results and open problems, we refer to the monograph [37] and the recent papers [12, 44].

Notation. We will use the notation $f(x) \ll g(x)$ if there is some constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all (appropriate) x. If we have both $f(x) \ll g(x)$ and $g(x) \ll f(x)$, we will write $f(x) \asymp g(x)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

then we write $f(x) \sim g(x)$. As above, the increasing sequence of prime numbers will be denoted by $(p_j)_{j\geq 1}$, and the subscript will sometimes be dropped when there can be no confusion. The number of prime factors in n will be denoted by $\Omega(n)$ (counting multiplicities). We will also use the standard notations $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{N} : n \geq x\}$.

2. Definitions and basic properties of the Hardy spaces $\mathscr{H}^p$ and $H^p(\mathbb{D}^\infty)$

2.1. **Definition of $H^p(\mathbb{D}^\infty)$.** We use the standard notation $\mathbb{T} := \{z : |z| = 1\}$ for the unit circle which is the boundary of the unit disc $\mathbb{D} := \{z : |z| < 1\}$ in the complex plane, and we equip $\mathbb{T}$ with normalized one-dimensional Lebesgue measure μ so that $\mu(\mathbb{T}) = 1$. We write $\mu_d := \mu \times \cdots \times \mu$ for the product of d copies of μ, where d may belong to $\mathbb{N} \cup \{\infty\}$.

We begin by recalling that for every $p > 0$, the classical Hardy space $H^p(\mathbb{D})$ (also denoted by $H^p(\mathbb{T})$) consists of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^p_{H^p(\mathbb{D})} := \sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^p \, d\mu(z) < \infty.$$

This is a Banach space (quasi-Banach in case $0 < p < 1$), and polynomials are dense in $H^p(\mathbb{D})$, so it could as well be defined as the closure of all polynomials in the above norm (or quasi-norm). We refer to [19] or the first chapters of [21] for the definition and basic properties of the Hardy spaces on $\mathbb{D}$.

For the finite dimensional polydisc $\mathbb{D}^d$ with $d \geq 2$, the definition of Hardy spaces can be made in a similar manner: For every $p > 0$, a function $f : \mathbb{D}^d \to \mathbb{C}$ belongs to $H^p(\mathbb{D}^d)$ when it is analytic separately with respect to each of the variables $z_1, \ldots, z_d$ and

$$\|f\|^p_{H^p(\mathbb{D}^d)} := \sup_{r<1} \int_{\mathbb{T}^d} |f(rz)|^p \, d\mu_d(z) < \infty.$$

The standard source for these spaces is Rudin's monograph [41]. As in the one-dimensional case, for almost every z in $\mathbb{T}^d$, the radial boundary limit

$$f^*(z) := \lim_{r\to 1^-} f(rz)$$

exists, and we may write

$$\|f\|^p_{H^p(\mathbb{D}^d)} = \int_{\mathbb{T}^d} |f^*(z)|^p \, d\mu_d(z). \tag{3}$$

This means that $H^p(\mathbb{D}^d)$ is a subspace of $L^p(\mathbb{T}^d, \mu_d)$. Moreover, again as in the one-dimensional case, for every f in $H^p(\mathbb{D}^d)$, we have that

$$\lim_{r\to 1^{-1}} \|f - f_r\|_{H^p(\mathbb{D}^d)} = 0, \tag{4}$$

where $f_r(z) := f(rz)$. This implies that the polynomials are dense in $H^p(\mathbb{D}^d)$, so that the space could equally well be defined as the closure of all polynomials with respect to the norm on the boundary given by (3).

Both (3) and (4) are most conveniently obtained by applying L^p-boundedness of the radial maximal function on $H^p(\mathbb{D}^d)$ for all $p > 0$, a result which can be obtained by considering a dummy variable w in $\mathbb{D}$ and checking first that, given f in $H^p(\mathbb{D}^d)$, the function

$$w \mapsto f(wz_1, \ldots, wz_d)$$

lies in $H^p(\mathbb{D}^d)$ for almost every $(z_1, \ldots z_d) \in \mathbb{T}^d$. By Fubini's theorem, the boundedness of the maximal function then reduces to the classical one-dimensional estimate.

In order to define $H^p(\mathbb{D}^\infty)$, some extra care is needed because functions in $H^p(\mathbb{D}^\infty)$ will in general not be well defined in the whole set $\mathbb{D}^\infty$. To keep things simple, we henceforth consider the set $\mathbb{D}^\infty_{\text{fin}}$ which consists of elements $z = (z_j)_{j\geq 1} \in \mathbb{D}^\infty$ such that $z_j \neq 0$ only for finitely many k. A function $f : \mathbb{D}^\infty_{\text{fin}} \to \mathbb{C}$ is analytic if it is analytic at every point z in $\mathbb{D}^\infty_{\text{fin}}$ separately with respect to each variable. Obviously any analytic $f : \mathbb{D}^\infty_{\text{fin}} \to \mathbb{C}$ can be written by a convergent Taylor series

$$f(z) = \sum_{\kappa \in \mathbb{N}^\infty_{\text{fin}}} c_\kappa z^\kappa, \quad z \in \mathbb{D}^\infty_{\text{fin}},$$

and the coefficients c_κ determine f uniquely. The truncation $A_m f$ of f onto the first m variables $A_m f$ (called "der $m$te Abschnitt" by Bohr) is defined as

$$A_m f(z_1, z_2, \ldots) = f(z_1, \ldots, z_m, 0, 0, \ldots)$$

for every z in $\mathbb{D}^\infty_{\text{fin}}$. By applying the fundamental estimate $|g(0)| \leq \|g\|_{H^p(\mathbb{D}^d)}$, obtained by iterating the case $d = 1$, we deduce that

$$\|A_m f\|_{H^p(\mathbb{D}^m)} \leq \|A_{m'} f\|_{H^p(\mathbb{D}^{m'})} \tag{5}$$

whenever $m' \geq m$.

Definition. Let $p > 0$. The space $H^p(\mathbb{D}^\infty)$ is the space of analytic functions on $\mathbb{D}^\infty_{\text{fin}}$ obtained by taking the closure of all polynomials in the norm (quasi-norm for $0 < p < 1$)

$$\|f\|^p_{H^p(\mathbb{D}^\infty)} := \int_{\mathbb{T}^\infty} |f(z)|^p \, d\mu_\infty(z).$$

Fix a compact set K in $\mathbb{D}^d$ and embed it as the subset $\widetilde{K}$ of $\mathbb{D}^\infty$ so that

$$\widetilde{K} := \{z = (z_1, \ldots, z_d, 0, 0, \ldots) \in \mathbb{D}^\infty \ : \ (z_1, \ldots, z_d) \subset K\}.$$

For all polynomials g we clearly have $\sup_{z \in \widetilde{K}} |g(z)| \leq C_K \|g\|_{H^p(\mathbb{D}^\infty)}$. It follows that any limit of polynomials is analytic on $\mathbb{D}^\infty_{\text{fin}}$, whence $H^p(\mathbb{D}^\infty)$ is well defined. This also implies that every element f in $H^p(\mathbb{D}^\infty)$ has a well-defined Taylor series $f(z) = \sum_\kappa c_\kappa z^\kappa$ and, in turn, this Taylor series determines f uniquely. Namely, by recalling (5), we have that $A_m f$ is in $H^p(\mathbb{D}^m)$ for every $m \geq 1$ and the $A_m f$ are certainly determined by the Taylor series. Finally, by polynomial approximation, it follows that

$$\lim_{m \to \infty} \|f - A_m f\|_{H^p(\mathbb{D}^\infty)} = 0.$$

Obviously, if a function f in $H^p(\mathbb{D}^\infty)$ depends only on the variables $z_1, \ldots z_d$, then we have $\|f\|_{H^p(\mathbb{D}^\infty)} = \|f\|_{H^p(\mathbb{D}^d)}$.

Cole and Gamelin [15] established an optimal estimate for point evaluations on $H^p(\mathbb{D}^\infty)$ by showing that

$$|f(z)| \leq \left(\prod_{j=1}^{\infty} \frac{1}{1 - |z_j|^2} \right)^{1/p} \|f\|_{H^p(\mathbb{D}^\infty)}. \tag{6}$$

Thus the elements in the Hardy spaces continue analytically to the set $\mathbb{D}^\infty \cap \ell^2$.

If f is an integrable function (or a Borel measure) on $\mathbb{T}^\infty$, then we denote its Fourier coefficients by

$$\widehat{f}(\kappa) := \int_{\mathbb{T}^\infty} f(z) \bar{z}^\kappa d\mu_\infty(z)$$

for multi-indices κ in $\mathbb{Z}^\infty_{\text{fin}}$. When $p \geq 1$, it follows directly from the definition of $H^p(\mathbb{D}^\infty)$ that it can be identified as the analytic subspace of $L^p(\mathbb{T}^\infty)$, consisting of the elements in $L^p(\mathbb{T}^\infty)$ whose non-zero Fourier coefficients lie in the positive cone $\mathbb{N}^\infty_{\text{fin}}$ (called the "narrow cone" by Helson [31]).

The following result verifies that, alternatively, $H^p(\mathbb{D}^\infty)$ may be defined in terms of the uniform boundedness of the L^p-norm of the sequence $A_m f$ for $m \geq 1$, and the functions $A_m f$ approximate f in the norm of $H^p(\mathbb{D}^\infty)$.

Theorem 2.1. *Suppose that $0 < p < \infty$ and that f is a formal infinite dimensional Taylor series. Then f is in $H^p(\mathbb{D}^\infty)$ if and only if*

$$\sup_{m\geq 1} \|A_m f\|_{H^p(\mathbb{D}^m)} < \infty. \tag{7}$$

Moreover, for every f in $H^p(\mathbb{D}^\infty)$, it holds that $\|A_m f - f\|_{H^p(\mathbb{D}^\infty)} \to 0$ as $m \to \infty$.

Proof for the case $p \geq 1$. When $p > 1$, the statements follow from the fact that $(A_m f)_{m\geq 1}$ is obviously an L^p-martingale sequence with respect to the natural sigma-algebras. It follows in particular that there is an L^p limit function (still denoted by f) of the sequence $A_m f$ on the distinguished boundary $\mathbb{T}^\infty$, which has the right Fourier series, and the density of polynomials follows immediately from the finite-dimensional approximation. In the case $p = 1$, this fact is stated in [1, Cor. 3], and is derived as consequence of the infinite-dimensional version of the brothers Riesz theorem on the absolute continuity of analytic measures, due to Helson and Lowdenslager [32] (a simpler proof of the result from [32] is also contained in [1]). The approximation property of the $A_m f$ then follows easily. □

The case $0 < p < 1$ requires a new argument and will be presented in the next subsection.

2.2. **Proof of Theorem 2.1 for $0 < p < 1$.** Our aim is to prove Lemma 2.3 below, from which the claim will follow easily. In an effort to make the computations of this section more readable, we temporarily adopt the convention that $\|f\|_{L^p(\mathbb{T}^d)} = \|f\|_p$, where it should be clear from the context what d is. We start with the following basic estimate.

Lemma 2.2. *Let $0 < p < 1$. There is a constant $C_p < \infty$ such that all (analytic) polynomials f on $\mathbb{T}$ satisfy the inequality*

$$\|f - f(0)\|_p^p \leq C_p \left(\|f\|_p^p - |f(0)|^p + |f(0)|^{p-p^2/2} \left(\|f\|_p^p - |f(0)|^p \right)^{p/2} \right). \tag{8}$$

Proof. In this proof, we use repeatedly the elementary inequality $|a+b|^p \leq |a|^p + |b|^p$, which is our replacement for the triangle inequality. We see in particular, by this inequality and the presence of the term $\|f\|_p^p - |f(0)|^p$ inside the brackets on the right-hand side, that (8) is trivial if, say, $\|f\|_p^p \geq (3/2)|f(0)|^2$. We may therefore disregard this case and assume that f satisfies $f(0) = 1$ and $\|f\|_p^p = 1+\varepsilon$ with $\varepsilon < 1/2$. Our aim is to show that, under this assumption,

$$\|f - 1\|_p^p \leq C_p \varepsilon^{p/2}. \tag{9}$$

We begin by writing $f = UI$, where U is an outer function and I is an inner function, such that $U(0) > 0$. By subharmonicity of $|U|^p$, we have $1 \leq |U(0)| \leq (1+\varepsilon)^{1/p} \leq 1 + c_p\varepsilon$. This means that $I(0) \geq (1 + c_p\varepsilon)^{-1} \geq 1 - c_p\varepsilon$. We write $f - 1 = (U - 1)I + I - 1$ and obtain consequently that

$$\|f - 1\|_p^p \leq \|U - 1\|_p^p + \|I - 1\|_p^p. \tag{10}$$

In order to prove (9), it is therefore enough to show that each of the two summands on the right-hand side of (10) is bounded by a constant times $\varepsilon^{p/2}$.

We begin with the second summand on the right-hand side of (10) for which we claim that

$$\|I-1\|_p^p \le C_p' \varepsilon^{p/2} \tag{11}$$

holds for some constant C_p'. We write $I = u + iv$, where u and v are respectively the real and imaginary part of I. Since $1-u \ge 0$, we see that

$$\|1-u\|_1 = \int_{\mathbb{T}} (1-u(z))dm(z) = 1 - I(0) \le c_p \varepsilon. \tag{12}$$

Using Hölder's inequality, we therefore find that

$$\|1-u\|_p^p \le c_p^P \varepsilon^p. \tag{13}$$

In view of (12) and using that $|I| = 1$ and $(1-u^2) \le 2(1-u)$, we also get that

$$\|v\|_p^p \le \|v\|_2^p = \|1-u^2\|_1^{p/2} \le \big(2\|1-u\|_1\big)^{p/2} \le (2c_p)^{p/2}\varepsilon^{p/2}.$$

Combining this inequality with (13), we get the desired bound (11).

We turn next to the first summand on the right-hand side of (10) and the claim that

$$\|U-1\|_p^p \le C_p'' \varepsilon^{p/2} \tag{14}$$

holds for some constant C_p''. By orthogonality, we find that

$$\|U^{p/2} - U(0)^{p/2}\|_2^2 \le \varepsilon$$

and hence

$$\|U^{p/2} - 1\|_2 \le \|U^{p/2} - U(0)^{p/2}\|_2 + (U(0)^{p/2} - 1)^{1/2} \le 2\varepsilon^{1/2}. \tag{15}$$

Since $|U^{p/2} - 1| \ge ||U|^{p/2} - 1| \ge (p/2)\log_+ |U|$ and $U(0) \ge 1$, this implies that

$$\|\log|U|\|_1 = 2\|\log_+|U|\|_1 - \log|U(0)| \le 8p^{-1}\varepsilon^{1/2}. \tag{16}$$

It follows that

$$m\left(\{z: \ |\log|U(z)|| \ge \lambda\}\right) \le 8(p\lambda)^{-1}\varepsilon^{1/2},$$
$$m\left(\{z: \ |\arg U(z)| \ge \lambda\}\right) \le C\lambda^{-1}\varepsilon^{1/2},$$

where the latter inequality is the classical weak-type L^1 estimate for the conjugation operator. We now split $\mathbb{T}$ into three sets

$$E_1 := \{z \,:\, |U(z)| > 3/2\} \cup \{z: \ |U(z)| < 1/2\},$$
$$E_2 := \{z \,:\, 1/2 \le |U(z)| \le 3/2, \ |\arg U(z)| \ge \pi/4\},$$
$$E_3 := \mathbb{T} \setminus (E_1 \cup E_2).$$

It is immediate from (15) that

$$\|\chi_{E_1}(U-1)\|_p^p \ll \varepsilon.$$

Since $m(E_2) \le C\varepsilon^{1/2}$, we have trivially that

$$\|\chi_{E_2}(U-1)\|_p^p \le C(5/2)^p \varepsilon^{1/2}.$$

Finally, on E_3, we have that $|U^{p/2}-1| \simeq |U-1|$, and so it follows from (15) and Hölder's inequality that

$$\|\chi_{E_3}(U-1)\|_p^p \ll \varepsilon^{p/2}.$$

Now the desired inequality (14) follows by combining the latter three estimates. $\square$

One may notice that that in the last step of the proof above we could have used (16) and the fact that the conjugation operator is bounded from L^1 to L^p. It seems that the exponent $p/2$ is the best we can get. It is also curious to note that with $p = 2/k$ and $k \ge 2$ an integer, one could avoid the use of the weak-type estimate for $\arg U$ and get a very slick argument by simply observing that if $g = U^{p/2}$ and $\omega_1, \ldots, \omega_k$ are the kth roots of unity, then by Hölder's inequality,

$$\|U-1\|_p \le \prod_{j=1}^{k} \|g-\omega_j\|_2,$$

and on the right hand side one L^2-norm is estimated by $\varepsilon^{1/2}$ and the others by a constant since we are assuming $\varepsilon \le 1/2$. Again one could raise the question if one can interpolate to get all exponents.

Lemma 2.3. *Suppose that* $0 < p < 1$. *If* g *is a polynomial on* $\mathbb{T}^\infty$, *then*

$$\|A_{m+k}g - A_m g\|_p^p \le C_p\Big(\|A_{m+k}g\|_p^p - \|A_m g\|_p^p + \|A_m g\|_p^{p-p^2/2}\left(\|A_{m+k}g\|_p^p - \|A_m g\|_p^p\right)^{p/2}\Big)$$

holds for arbitrary positive integers m *and* k, *where* C_p *is as in Lemma 8.*

Proof. We set $h := A_{m+k}g$ and view h as a function on $\mathbb{T}^m \times \mathbb{T}^k$ so that $A_m g(w, w') = h(w, 0)$. Now fix arbitrary points w in $\mathbb{T}^m$ and w' in $\mathbb{T}^k$. We apply the preceding lemma to the function

$$f(z) := h(w, zw'),$$

which is an analytic function on $\mathbb{D}$. This yields

$$\int_{\mathbb{T}} |h(w,zw') - h(w,0)|^p \, d\mu(z) \le C_p \Bigg(\int_{\mathbb{T}} |h(w,zw')|^p \, d\mu(z) - |h(w,0)|^p + |h(w,0)|^{p-p^2/2} \left(\int_{\mathbb{T}} |h(w,zw')|^p \, d\mu(z) \; - \; |h(w,0)|^p \right)^{p/2} \Bigg).$$

The claim follows by integrating both sides with respect to (w,w') over $\mathbb{T}^{m+k}$ and applying Hölder's inequality to the last term on the right-hand side. □

Proof of Theorem 2.1 for $0 < p < 1$. If f is in $H^p(\mathbb{D}^\infty)$, then clearly (7) holds. To prove the reverse implication, we start from a formal Taylor series f for which (7) holds. Then by assumption $A_m f$ is in $H^p(\mathbb{D}^\infty)$, and we have that $A_m(A_{m'} f) = A_m f$ whenever $m' \ge m \ge 1$. Therefore the quasi-norms $\|A_m f\|_{H^p(\mathbb{D}^\infty)}$ constitute an increasing sequence, and hence (7) implies that

$$\lim_{m\to\infty} \sup_{k\ge 1} \big(\|A_{m+k} f\|_{H^p(\mathbb{D}^\infty)} - \|A_m f\|_{H^p(\mathbb{D}^\infty)} \big) = 0.$$

By Lemma 2.3, we find that $(A_m f)_{m\ge 1}$ is a Cauchy sequence in $H^p(\mathbb{D}^\infty)$, whence $f = \lim_{m\to\infty} A_m f$ in $H^p(\mathbb{D}^\infty)$ since an element in $H^p(\mathbb{D}^\infty)$ is uniquely determined by the sequence $A_m f$. □

2.3. **Definition of $\mathscr{H}^p$.** The systematic study of the Hilbert space $\mathscr{H}^2$ began with the paper [29] which defined $\mathscr{H}^2$ to be the collection of Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

subject to the condition $\|f\|_{\mathscr{H}}^2 := \big(\sum_{n=1}^{\infty} |a_n|^2 \big)^{1/2} < \infty$. The space $\mathscr{H}^2$ consists of functions analytic in the half-plane $\mathbb{C}_{1/2} := \{s = \sigma + it : \ \sigma > 1/2\}$, since the Cauchy–Schwarz inequality shows that the above Dirichlet series converges absolutely for those values of s. Bayart [5] extended the definition to every $p > 0$ by defining $\mathscr{H}^p$ as the closure of all Dirichlet polynomials $f(s) := \sum_{n=1}^{N} a_n n^{-s}$ under the norm (or quasi-norm when $0 < p < 1$)

$$\|f\|_{\mathscr{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \, dt \right)^{1/p}. \tag{17}$$

Computing the limit when $p = 2$, we see that (17) gives back the original definition of $\mathscr{H}^2$. However, at first sight it is not clear that the above definition of $\mathscr{H}^p$ is the right one or that it even yields spaces of convergent Dirichlet series in any right half-plane.

The clarification of these matters is provided by the Bohr lift (1). By Birkhoff's ergodic theorem (or by an elementary argument found in [43, Sec. 3]), we obtain the identity

$$\|f\|_{\mathscr{H}^p} = \|\mathscr{B}f\|_{H^p(\mathbb{D}^\infty)} := \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f(z)|^p \, d\mu_\infty(z)\right)^{1/p}. \tag{18}$$

Since the Hardy spaces on the infinite dimensional torus $H^p(\mathbb{D}^\infty)$ may be defined as the closure of analytic polynomials in the L^p-norm on $\mathbb{T}^\infty$, it follows that the Bohr correspondence gives an isomorphism between the spaces $H^p(\mathbb{D}^\infty)$ and $\mathscr{H}^p$. This linear isomorphism is both isometric and multiplicative, and this results in a fruitful interplay: Many questions in the theory of the spaces $\mathscr{H}^p$ can be better treated by considering the isomorphic space $H^p(\mathbb{D}^\infty)$, and vice versa. An important example is the Cole-Gamelin estimate (6) which immediately implies that for every $p > 0$ the space $\mathscr{H}^p$ consists of analytic functions in the half-plane $\mathbb{C}_{1/2}$. In fact, we infer from (6) that

$$|f(\sigma + it)|^p \le \zeta(2\sigma)\|f\|^p_{\mathscr{H}^p}$$

holds whenever $\sigma > 1/2$, where $\zeta(s)$ is the Riemann zeta function. Moreover, since the coefficients of a convergent Dirichlet series are unique, functions in $\mathscr{H}^p$ are completely determined by their restrictions to the half-plane $\mathbb{C}_{1/2}$. This means in particular that $\mathscr{H}^p$ can be thought of as a space of analytic functions in this half-plane.

To complete the picture, we mention that $\mathscr{H}^\infty$ is defined as the space of Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ that represent bounded analytic functions in the half-plane $\sigma > 0$. We endow $\mathscr{H}^\infty$ with the norm

$$\|f\|_{\mathscr{H}^\infty} := \sup_{\sigma>0} |f(s)|, \quad s = \sigma + it,$$

and then the Bohr lift allows us to associate $\mathscr{H}^\infty$ with $H^\infty(\mathbb{D}^\infty)$. We refer to [37] for this fact and further details about the interesting and rich function theory of $\mathscr{H}^\infty$.

2.4. **A probabilistic interpretation of the Bohr lift.** It is frequently fruitful to think of the product measure μ_∞ on $\mathbb{T}^\infty$ as a probability measure and the infinitely many variables z_k as independent identically distributed (i.i.d.) random variables. From the viewpoint of the Bohr correspondence, we then associate with the sequence of primes $(p_j)_{j\ge1}$ a sequence of independent Steinhaus variables $z(p_j)$, which are random variables equidistributed on $\mathbb{T}$. This sequence defines a random multiplicative function $z(n)$ on the positive integers $\mathbb{N}$ by the rule

$$z(n) = (z(p_j))^{\kappa(n)},$$

where we again use multi-index notation. Functions in $\mathscr{H}^p$ can then, via the Bohr lift, be thought of as linear combinations of these random multiplicative functions. Indeed, we may write the Bohr lift as

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \longleftrightarrow \quad F\left((z(p_j)\right) = \sum_{n=1}^{\infty} a_n z(n)$$

and hence express $\|f\|^p_{\mathscr{H}^p}$ as the pth moment of $|F|$:

$$\|f\|^p_{\mathscr{H}^p} = \mathbb{E}|F|^p.$$

In the final section of this paper, we will make crucial use of this alternate viewpoint, and we then find it natural to switch to this probabilistic terminology.

2.5. **Summary of known results.** The function theory of the two distinguished spaces $\mathscr{H}^2$ and $\mathscr{H}^\infty$ is by now quite well developed; we refer again to [37, 44] for details. The results for the range $1 \le p < \infty$, $p \ne 2$, are less complete. In this section, we mention briefly some key results that extend to the whole range $0 < p < \infty$, as well as some familiar difficulties that arise in our attempts to make such extensions.

We begin with the theorem on multipliers that was first established in [29] for $p = 2$ and extended to the range $1 \le p < \infty$ in [5]. We recall that a multiplier m for $\mathscr{H}^p$ is a function such that the operator $f \mapsto mf$ is bounded on $\mathscr{H}^p$, and the multiplier norm is the norm of this operator. The theorem on multipliers asserts that the space of multipliers for $\mathscr{H}^p$ is equal to $\mathscr{H}^\infty$, and this remains true for $0 < p < 1$, by exactly the same proof as in [5]. Another result that carries over without any change, is the Littlewood–Paley formula of [7, Sec. 5]. The latter result was already used in [12].

For some results, only a partial extension from the case $p = 2$ is known to hold. A well known example is whether the L^p integral of a Dirichlet polynomial $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ over any segment of fixed length on the vertical line $\operatorname{Re} s = 1/2$ is bounded by a universal constant times $\|f\|^p_{\mathscr{H}^p}$. This is known to hold for $p = 2$ and thus trivially for $p = 2k$ for k a positive integer. As shown in [36], this embedding holds if and only if the following is true: The boundedly supported Carleson measures for $\mathscr{H}^p$ satisfy the classical Carleson condition in $\mathbb{C}_{1/2}$.

There is an interesting counterpart for $p < 2$ to the trivial embedding for $p = 2k$ and k a positive integer > 1. This is the following statement about interpolating sequences. If $S = (s_j)$ is a bounded interpolating sequence in $\mathbb{C}_{1/2}$, then we can solve the interpolation problem $f(s_j) = a_j$ in $\mathscr{H}^p$ when

$$\sum_j |a_j|^p (2\sigma_j - 1) < \infty$$

and $p = 2/k$ for k a positive integer. Indeed, choose any kth root $a_j^{1/k}$ and solve $g(s_j) = a_j^{1/k}$ in $\mathscr{H}^2$. Then $f = g^k$ solves our problem in $\mathscr{H}^p$. We do not know if this result extends to any p which is not of the form $p = 2/k$. Comparing the two trivial cases, we observe that there is an interesting "symmetry" between the embedding problem for $\mathscr{H}^p$ and the interpolation problem for $\mathscr{H}^{4/p}$. A similar phenomenon will be explored in the next section.

Before turning to the next two sections which will deal with respectively weighted ℓ^2 and ℓ^∞ bounds for the coefficients, we would like to point out that there are certainly other interesting problems of a similar kind. An interesting example is whether the ℓ^1 estimate

$$\sum_{n=2}^{\infty} \frac{|a_n|}{\sqrt{n}\log n} \le C\|f\|_{\mathscr{H}^1}$$

holds when $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We refer to [13] for background on this problem and again to [44] for a survey of other open problems.

3. Coefficient estimates: Weighted ℓ^2 bounds

3.1. **Contractive Hardy–Littlewood inequalities in the unit disc.**

We begin with some estimates of the $H^p(\mathbb{D})$ norms (or quasi-norms when $0 < p < 1$) in terms of weighted ℓ^2 norms of the coefficient sequence. Such inequalities were first studied systematically by Hardy and Littlewood.

For $\alpha > 1$, the weighted Bergman space $A^p_\alpha(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ for which

$$\|f\|_{A^p_\alpha(\mathbb{D})} := \left(\int_{\mathbb{D}} |f(z)|^p \ (\alpha-1)\left(1-|z|^2\right)^{\alpha-2} \ \frac{dm(z)}{\pi}\right)^{\frac{1}{p}} < \infty,$$

where m denotes Lebesgue area measure on $\mathbb{C}$. We set

$$dm_\alpha(z) := (\alpha-1)\left(1-|z|^2\right)^{\alpha-2}\frac{dm(z)}{\pi}.$$

The Hardy space $H^p(\mathbb{D})$ is the limit of the weighted Bergman spaces $A^p_\alpha(\mathbb{D})$ as $\alpha \to 1^+$, in the sense that

$$\|f\|_{H^p(\mathbb{D})} = \lim_{\alpha\to 1^+} \|f\|_{A^p_\alpha(\mathbb{D})}.$$

We will therefore find it convenient to write $H^p(\mathbb{D}) = A^p_1(\mathbb{D})$ in some formulas, such as in (22) below. For $\alpha \ge 1$ and a non-negative integer j, we define

$$c_\alpha(j) := \binom{j+\alpha-1}{j} = \prod_{l=1}^{\alpha-1}\frac{(j+l)}{l}. \tag{19}$$

Notice that $c_1(j) = 1$ for every j. Identifying $c_\alpha(j)$ as the coefficients of the binomial series $(1-z)^{-\alpha}$, we find that

$$c_{\alpha k}(j) = \sum_{j_1+j_2+\cdots+j_k=j} c_\alpha(j_1)c_\alpha(j_2)\cdots c_\alpha(j_k). \tag{20}$$

In particular, if α is a positive integer, then $c_\alpha(j)$ is the number of ways to write j as a sum of α non-negative integers. We will also require the simple estimate

$$c_\alpha(j+k) \le c_\alpha(j)c_\alpha(k) \tag{21}$$

which can be deduced by comparing factor by factor in the product (19). A computation gives that if $f(z) = \sum_{j\ge 0} a_j z^j$, then

$$\|f\|_{A^2_\alpha(\mathbb{D})} = \left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{c_\alpha(j)}\right)^{\frac{1}{2}}. \tag{22}$$

The following inequality is due to Burbea [14, Cor. 3.4], but we include a short proof in the special case we require, based on (20).

Lemma 3.1. *Suppose that f is in $H^2(\mathbb{D})$, and let k be an integer ≥ 2. Then*

$$\|f\|_{A^{2k}_k(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^{2k}\, dm_k(z)\right)^{\frac{1}{2k}} \le \|f\|_{H^2(\mathbb{D})}.$$

Proof. Suppose that $f(z) = \sum_{j=0}^{\infty} a_j z^j$. We write $|f|^{2k} = |f^k|^2$ and use (22), followed by the Cauchy–Schwarz inequality with (20), to get

$$\begin{aligned}\|f\|^{2k}_{A^{2k}_k(\mathbb{D})} &= \sum_{j=0}^{\infty} \frac{1}{c_k(j)} \left| \sum_{j_1+\cdots+j_k=j} a_{j_1}\cdots a_{j_k}\right|^2 \\ &\le \sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_k=j} |a_{j_1}|^2 \cdots |a_{j_k}|^2 = \left(\sum_{j=0}^{\infty} |a_j|^2\right)^{2k} = \|f\|^{2k}_{H^2(\mathbb{D})}. \qquad \square\end{aligned}$$

It is possible to use Lemma 3.1 and Riesz–Thorin interpolation to prove that

$$\|f\|_{A^{2k}_k(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^{2k}\, dm_k(z)\right)^{\frac{1}{2}} \le C_k \|f\|_{H^2(\mathbb{D})} \tag{23}$$

holds for every real number $k > 1$, but the interpolation process gives a constant $C_k > 1$ when k is not an integer. Numerical evidence has been supplied elsewhere [11] for the conjecture that in fact $C_k = 1$ for all $k > 1$. So far, we have not been able to prove this extension of Lemma 3.1. As a remedy for this situation, we will establish a weaker version for all k which will be satisfactory for the number theoretic applications to be discussed later. For this, we need the following remarkable contractive estimate of Weissler [53, Cor. 2.1] for the dilations

$$f_r(z) := f(rz), \quad r > 0,$$

of functions f in $H^p(\mathbb{D})$.

Lemma 3.2. *Let $0 < p < q < \infty$. The contractive estimate*

$$\|f_r\|_{H^q(\mathbb{D})} \le \|f\|_{H^p(\mathbb{D})}$$

holds for every f in $H^p(\mathbb{D})$ if and only if $r \le \sqrt{p/q}$.

We are now ready to state and prove the main result of this section. To this end, we set

$$\varphi_\alpha(j) := c_{\lfloor\alpha\rfloor}(j)\left(\frac{\alpha}{\lfloor\alpha\rfloor}\right)^j, \quad \alpha \ge 1. \tag{24}$$

Theorem 3.3. *Suppose that $0 < p < \infty$ and that $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is in $H^p(\mathbb{D})$. Then*

$$\|f\|_{H^p(\mathbb{D})} \le \left(\sum_{j=0}^{\infty} |a_j|^2 \varphi_{p/2}(j)\right)^{\frac{1}{2}}, \qquad p \ge 2, \tag{25}$$

$$\left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{\varphi_{2/p}(j)}\right)^{\frac{1}{2}} \le \|f\|_{H^p(\mathbb{D})}, \qquad p \le 2. \tag{26}$$

Here the respective parameters $\alpha = p/2$ and $\alpha = 2/p$ are optimal for contractivity.

Proof. We begin with (25). We will use Lemma 3.2 in reverse with exponents $\lfloor p/2\rfloor = k$ and $p/2$, so we choose $r^2 = \lfloor p/2\rfloor/(p/2)$ and assume that f is a polynomial. Hence

$$\|f\|_{H^p(\mathbb{D})} = \left\|f^2\right\|_{H^{p/2}(\mathbb{D})}^{1/2} \le \left\|f_{1/r}^2\right\|_{H^k(\mathbb{D})}^{1/2} = \left\|f_{1/r}^k\right\|_{H^2(\mathbb{D})}^{\frac{1}{k}}.$$

The right-hand side can be computed at the level of coefficients. We use the Cauchy–Schwarz inequality with (20), and finally (21), to get

$$\begin{aligned}
\|f_{1/r}^k\|_{H^2(\mathbb{D})}^{\frac{1}{k}} &= \left(\sum_{j=0}^{\infty}\left|\sum_{j_1+\cdots+j_k=j} a_{j_1}r^{-j_1}\cdots a_{j_k}r^{-j_k}\right|^2\right)^{\frac{1}{2k}} \\
&\le \left(\sum_{j=0}^{\infty} c_k(j) \sum_{j_1+\cdots+j_k=j} |a_{j_1}|^2 r^{-2j_1}\cdots |a_{j_k}|^2 r^{-2j_k}\right)^{\frac{1}{2k}} \\
&\le \left(\sum_{j=0}^{\infty} \sum_{j_1+\cdots+j_k=j} |a_{j_1}|^2 c_k(j_1) r^{-2j_1}\cdots |a_{j_k}|^2 c_k(j_k) r^{-2j_k}\right)^{\frac{1}{2k}} \\
&= \left(\sum_{j=0}^{\infty} |a_j|^2 c_k(j) r^{-2j}\right)^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof of (25), since $k = \lfloor p/2 \rfloor$ and $r^2 = \lfloor p/2 \rfloor/(p/2)$, so

$$c_k(j)r^{-2j} = c_{\lfloor p/2 \rfloor}(j)\left(\frac{p/2}{\lfloor p/2 \rfloor}\right)^j = \varphi_{p/2}(j).$$

To prove (26), we first assume that $f(z) \neq 0$ for every z in $\mathbb{D}$ so $f^{1/k}$ is an analytic function for $k = \lfloor 2/p \rfloor$. We then use Lemma 3.2 with exponents kp and 2, followed by Lemma 3.1, to get

$$\|f\|_{H^p(\mathbb{D})} = \left\|f^{1/k}\right\|_{H^{kp}(\mathbb{D})}^k \ge \left\|f_{kp/2}^{1/k}\right\|_{H^2(\mathbb{D})}^k \ge \left\|f_{kp/2}\right\|_{A_k^2(\mathbb{D})}. \tag{27}$$

If $f \not\equiv 0$, we factor $f = Bg$, where B is a Blaschke product and g has no zeros in $\mathbb{D}$. Then

$$\|f\|_{H^p(\mathbb{D})} = \|g\|_{H^p(\mathbb{D})} \quad \text{and} \quad \left\|f_{kp/2}\right\|_{A_k^2(\mathbb{D})} \le \left\|g_{kp/2}\right\|_{A_k^2(\mathbb{D})}.$$

It therefore follows that (27) is valid for every f in $H^p(\mathbb{D})$. The right hand side of (27) can be computed at the level of coefficients, and since $k = \lfloor 2/p \rfloor$ we find that

$$\left\|f_{kp/2}\right\|_{A_k^2(\mathbb{D})} = \left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{c_k(j)}\left(\frac{kp}{2}\right)^j\right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\infty} \frac{|a_j|^2}{\varphi_{2/p}(j)}\right)^{\frac{1}{2}},$$

which completes the proof of (26). To see that $\alpha = p/2$ and $\alpha = 2/p$ are optimal, we recall that $\varphi_\alpha(1) = \alpha$ and, for $0 < \varepsilon < 1$, compute

$$\|1+\varepsilon z\|_{H^p(\mathbb{D})}^2 = \left\|(1+\varepsilon z)^{p/2}\right\|_{H^2(\mathbb{D})}^{4/p} = \left(1 + \frac{p^2}{4}\varepsilon^2 + O(\varepsilon^4)\right)^{\frac{2}{p}} = 1 + \frac{p}{2}\varepsilon^2 + O(\varepsilon^4).$$

We complete the proof by letting ε tend to 0. $\square$

When $1 < p \leq 2$ or $2 \leq p < 4$, the result of Theorem 3.3 is simply Lemma 3.2, which gives the inequalities

$$\|f\|_{H^p(\mathbb{D})} \leq \left(\sum_{j=0}^{\infty} |a_j|^2 \left(\frac{p}{2}\right)^j \right)^{\frac{1}{2}}, \qquad q \geq 2,$$

$$\left(\sum_{j=0}^{\infty} |a_j|^2 \left(\frac{2}{p}\right)^j \right)^{\frac{1}{2}} \leq \|f\|_{H^p(\mathbb{D})}, \qquad p \leq 2.$$

The virtue of Theorem 3.3, compared to what Lemma 3.2 will give for every $0 < p < \infty$, is that the geometric factor $(\alpha/\lfloor\alpha\rfloor)^j$ is always dominated by $(2-\delta)^j$ for some $\delta = \delta(\alpha) > 0$. It will become clear why this is crucial in Subsection 3.3.

3.2. **Hardy–Littlewood inequalities for $\mathscr{H}^p$.** We recall the definition of the Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}}, \quad \sigma > 1.$$

Using the Euler product formula, we may define the general divisor function $d_\alpha(n)$ by the rule

$$\zeta^\alpha(s) = \sum_{n=1}^{\infty} d_\alpha(n) n^{-s}, \quad \sigma > 1. \tag{28}$$

A basic observation is that $d_\alpha(n)$ is a multiplicative function, which means that it is completely determined by its values at powers of the prime numbers. The Euler product formula shows that, in fact,

$$d_\alpha(p^j) = c_j(\alpha) \tag{29}$$

for every prime p and every nonnegative integer j, and in general

$$d_\alpha(n) = (c_j(\alpha))^{\kappa(n)}$$

in multi-index notation. We may thus think of $d_\alpha(n)$ as a multiplicative extension of (29).

We will now make a multiplicative extension of Theorem 3.3, similar to the extension from (29) to $d_\alpha(n)$. This will be done by an iterative procedure introduced by Bayart [5] which relies crucially on the contractivity of the estimates of Theorem 3.3. We begin by noting that the multiplicative extension of Theorem 3.3 is known when either $p/2$ or $2/p$ is an integer. If $p/2$ is an integer, (our version of) the inequality in [45, Lem. 8] is

$$\|f\|_{\mathscr{H}^p} \le \left(\sum_{n=1}^{\infty} |a_n|^2 d_{p/2}(n)\right)^{\frac{1}{2}}. \tag{30}$$

On the other hand, it was observed in [8, pp. 203–204], that (26) can be used to prove the corresponding lower inequality

$$\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d_{2/p}(n)}\right)^{\frac{1}{2}} \le \|f\|_{\mathscr{H}^p}, \tag{31}$$

where it is required that $2/p$ is an integer. The case $p = 1$ in (31) is often called Helson's inequality [31]. For both (30) and (31), it is easy to see that the parameters $\alpha = p/2$ and $\alpha = 2/p$ are best possible — a similar statement will be proved in Theorem 3.4.

Both (30) and (31) rely on Theorem 3.3, and we do not know whether any of them extend to the case when either $p/2 > 1$ or $2/p > 1$. We now turn to what we are able to prove, namely the multiplicative extension of Theorem 3.3 for general p. To this end, in accordance with (24), we introduce the multiplicative function

$$\Phi_\alpha(n) := d_{\lfloor\alpha\rfloor}(n) \left(\frac{\alpha}{\lfloor\alpha\rfloor}\right)^{\Omega(n)}, \tag{32}$$

where $\Omega(n)$ denotes the number of prime factors in n, counting multiplicity. We will see later that $\Phi_\alpha(n)$ has the same average order as $d_\alpha(n)$, a fact that for our purposes makes it a satisfactory substitute.

The multiplicative extension of Theorem 3.3 reads as follows.

Theorem 3.4. *If* $f(s) = \sum_{n=1}^{N} a_n n^{-s}$, *then*

$$\|f\|_{\mathscr{H}^p} \le \left(\sum_{n=1}^{N} |a_n|^2 \Phi_{p/2}(n)\right)^{\frac{1}{2}}, \qquad p \ge 2, \tag{33}$$

$$\left(\sum_{n=1}^{N} \frac{|a_n|^2}{\Phi_{2/p}(n)}\right)^{\frac{1}{2}} \le \|f\|_{\mathscr{H}^p}, \qquad p \le 2. \tag{34}$$

The respective parameters $\alpha = p/2$ *and* $\alpha = 2/p$ *are optimal.*

Observe that $\Phi_\alpha(n) = d_\alpha(n)$ whenever α is an integer, so (33) and (34) encompass (30) and (31), respectively. Note also that $\Phi_\alpha(n) = d_\alpha(n)$ if n is square-free.

Theorem 3.4 is an improvement of results[1] from [8, 45]. Indeed, it is proved in [8] that (31) holds if we only consider square-free integers in the lower bound.

[1] The Hardy–Littlewood inequalities in [8, 45] are stated with a weight of the form $[d(n)]^\beta$, where $d(n) = d_2(n)$ denotes the usual divisor function. The difference between $[d(n)]^\beta$ and $d_\alpha(n)$ is marginal, but we have found it more natural to use $d_\alpha(n)$.

Using the Möbius function $\mu(n)$, which is the multiplicative function that is 0 if n is square-free and -1 at each prime number, the Hardy–Littlewood inequality of [8] can be written as

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{2/p}(n)}\right)^{\frac{1}{2}} \le \|f\|_{\mathscr{H}^p}, \tag{35}$$

for $p \le 2$. We will see in Section 6 that certain estimates obtained from (35) cannot be improved (substantially) by using (34). Nevertheless, the fact that $\Phi_\alpha(n) > 0$ for every n allows us to extend an embedding theorem from [6] from $1 \le p < 2$ to $0 < p < 2$. This cannot be achieved using (35), since the lower bound is supported on square-free integers.

In [45], Riesz–Thorin interpolation between the integers $p/2$ in (30) is used to the prove that

$$\|f\|_{\mathscr{H}^q} \le \left(\sum_{n=1}^{\infty} |a_n|^2 d_\alpha(n)\right)^{\frac{1}{2}},$$

where $\alpha = \alpha(p) > p/2$ (unless $p/2$ is an integer). Thus the average order of $d_\alpha(n)$ is larger than that of $d_{p/2}(n)$ and hence than that of $\Phi_p(n)$ as well, as we will see in the next subsection.

We now turn to the proof of Theorem 3.4. It uses a technique which has become standard by now (see e.g. [5, 6, 8, 31]), and we will therefore be brief. For applications of this result, we refer to the subsequent Subsections 3.4 and 3.5 and Sections 5 and 6.

Proof of Theorem 3.4. By (18), we may replace f by $\mathscr{B}f$, and we may assume that $\mathscr{B}f =: F$ is a polynomial. We wish to apply Theorem 3.3 iteratively to the finitely many variables z_j on which F depends. To carry out the iterative argument, we need the following integral version of Minkowski's inequality. Let X and Y be measure spaces, and let g be a measurable function on $X \times Y$. If $r \ge 1$, then

$$\left(\int_X \left(\int_Y |g(x,y)|\,dy\right)^r dx\right)^{\frac{1}{r}} \le \int_Y \left(\int_X |g(x,y)|^r\,dx\right)^{\frac{1}{r}} dy. \tag{36}$$

We use the Euler product of the Riemann zeta function in (28) and recall that $c_\alpha(j)$ are the coefficients of the binomial series $(1-z)^{-\alpha}$, to conclude that Φ_α is the multiplicative function defined by

$$\Phi_\alpha(p^k) = \varphi_\alpha(k).$$

Fix $d \ge 2$, and define the invertible linear operator T_α by

$$T_\alpha\big(z^{\kappa(n)}\big) := \sqrt{\Phi_\alpha(n)}\, z^{\kappa(n)} = \prod_{j=1}^{d} \sqrt{\varphi_\alpha(\kappa_j)}\, z^{\kappa_j}. \tag{37}$$

Let T_α^{-1} denote the inverse operator. In view of (18), it is sufficient to prove that if F is an analytic polynomial in d variables, then

$$\|F\|_{L^p(\mathbb{T}^d)} \le \left\|T_{p/2}F\right\|_{L^2(\mathbb{T}^d)}, \qquad p \ge 2,$$
$$\left\|T_{2/p}^{-1}F\right\|_{L^2(\mathbb{T}^d)} \le \|F\|_{L^p(\mathbb{T}^d)}, \qquad p \le 2.$$

Note that Theorem 3.3 is simply the case $d = 1$. We find it convenient to argue by induction on d, and we will only consider the case $p \ge 2$. We factor the operator T_α as $T_\alpha = R_\alpha S_\alpha$, such that R_α acts on z_j for $1 \le j \le d-1$ and S_α acts on z_d. This is well-defined in view of (37).

The induction hypothesis becomes $\|g\|_{L^q(\mathbb{T}^{d-1})} \le \|R_{p/2}g\|_{L^2(\mathbb{T}^{d-1})}$. To simplify the notation, set $z_d = w$. We begin by using Theorem 3.3 to the effect that

$$\begin{aligned}\|F\|_{L^p(\mathbb{T}^d)} &= \left(\int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} |F(z_1,\ldots,z_{d-1},w)|^p \, d\mu(w)\, d\mu_{d-1}(z_1,\ldots,z_{d-1})\right)^{\frac{1}{p}} \\ &\le \left(\int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} \left|S_{p/2}F(z_1,\ldots,\, z_{d-1},w)\right|^2 \, d\mu(w)\right)^{\frac{p}{2}} d\mu_{d-1}(z)\right)^{\frac{1}{p}}.\end{aligned}$$

We now apply (36) with $X = \mathbb{T}^{d-1}$, $Y = \mathbb{T}$ and $r = p/2$, and find that

$$\le \left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}^{d-1}} \left|S_{p/2}F(z_1,\ldots,z_{d-1},w)\right|^p \, d\mu_{d-1}(z)\right)^{\frac{2}{p}} d\mu(w)\right)^{\frac{1}{2}}.$$

We complete the proof by using the induction hypothesis on $g = S_{p/2}f$.

That the parameter $\alpha = p/2$ is optimal, follows at once from the example considered in the proof of Theorem 3.3, applied multiplicatively. □

3.3. **The average order of $\Phi_\alpha(n)$.**

From (28) it follows by standard techniques (see e.g. [50, Ch. II.5]) that the average order of $d_\alpha(n)$ is given by

$$\frac{1}{N}\sum_{n=1}^{N} d_\alpha(n) = \frac{1}{\Gamma(\alpha)}(\log N)^{\alpha-1} + O\big((\log N)^{\alpha-2}\big). \tag{38}$$

We will now show that $\Phi_\alpha(n)$ has the same average order, up to a bounded factor. To investigate the average order of $\Phi_\alpha(n)$, we consider the associated Dirichlet series and factor out a suitable power of $\zeta(s)$ from the Euler product, to obtain

$$\mathscr{F}_\alpha(s) := \sum_{n=1}^{\infty} \Phi_\alpha(n) n^{-s} = \zeta^\alpha(s) \prod_p \left(1-p^{-s}\right)^\alpha \left(\sum_{j=0}^{\infty} \varphi_\alpha(j)\, p^{-js}\right).$$

For $|z| < \lfloor\alpha\rfloor/\alpha$, it is now convenient to set

$$G_\alpha(z) := (1-z)^\alpha \sum_{j=0}^{\infty} \varphi_\alpha(j) z^j = (1-z)^\alpha \left(1 - \frac{\alpha}{\lfloor\alpha\rfloor} z\right)^{-\lfloor\alpha\rfloor} \tag{39}$$

so that $\mathscr{F}_\alpha(s) = \zeta^\alpha(s)\mathscr{G}_\alpha(s)$, where

$$\mathscr{G}_\alpha(s) := \prod_p G_\alpha(p^{-s}).$$

From (39) we easily find that the Dirichlet series representing $\mathscr{G}_\alpha(s)$ is absolutely convergent for

$$\operatorname{Re} s > \max\big(1/2, \log_2(\alpha/\lfloor\alpha\rfloor)\big).$$

To prove the desired size estimate for $\Phi_\alpha(n)$, we require the following simple estimates.

Lemma 3.5. *If $\alpha \geq 1$ and $0 \leq x < \lfloor\alpha\rfloor/\alpha$, then*

$$G_{\alpha+1}(x) \leq G_\alpha(x). \tag{40}$$

Moreover, G_α enjoys uniform estimates for $0 \leq x \leq 1/2$,

$$1 \leq G_\alpha(x) \leq 1 + x^2 \begin{cases} 16(\alpha-1)/(2-\alpha)^3, & 1 \leq \alpha < 2, \\ 384, & \alpha \geq 2. \end{cases}$$

Proof. To prove (40), we look at the Taylor expansion of the logarithm

$$\log\big(G_\alpha(x)\big) = \sum_{j=2}^{\infty} \frac{x^j}{j} \left(\lfloor\alpha\rfloor \left(\frac{\alpha}{\lfloor\alpha\rfloor}\right)^j - \alpha \right).$$

It is sufficient to show that $C_j(\alpha+1) \leq C_j(\alpha)$, where

$$C_j(\alpha) := \lfloor\alpha\rfloor \left(\frac{\alpha}{\lfloor\alpha\rfloor}\right)^j - \alpha.$$

Clearly $C_j(\lfloor\alpha\rfloor) = C_j([\alpha+1]) = 1$. We set $\alpha = \lfloor\alpha\rfloor + t$ for $0 \leq t < 1$, and differentiate to find that

$$\frac{d}{dt} C_j(\alpha) = j\left(\frac{\alpha}{\lfloor\alpha\rfloor}\right)^{j-1} - 1 \;\geq\; j\left(\frac{\alpha+1}{[\alpha+1]}\right)^{j-1} - 1 = \frac{d}{dt} C_j(\alpha+1).$$

The lower bound in the second statement is just Bernoulli's inequality,

$$\left(1 - \frac{\alpha}{\lfloor\alpha\rfloor} x\right)^{\lfloor\alpha\rfloor/\alpha} \leq 1 - x.$$

The upper bounds can be computed with Taylor's theorem. By (40), we only need to consider $1 \leq \alpha < 2$ and $\alpha \geq 2$. The precise value of the constants are unimportant; we have obtained ours by rather coarse estimates. $\square$

Using standard techniques (see e.g. [50, Ch. II.5]), we now deduce that the average order of $\Phi_\alpha(n)$ is the same as the average order of $d_\alpha(n)$ given by (38).

Lemma 3.6. *Let $\Phi_\alpha(n)$ denote the weight* (32) *for fixed $\alpha \geq 1$. Then*

$$\frac{1}{x}\sum_{n\leq x}\Phi_\alpha(n) = \frac{\mathscr{G}_\alpha(1)}{\Gamma(\alpha)}(\log x)^{\alpha-1} + O\big((\log x)^{\alpha-2}\big).$$

3.4. **A theorem on local embedding for** $0 < p < 2$**.** We will now use Theorem 3.4 and Lemma 3.6 to prove an embedding theorem for the Hardy spaces of Dirichlet series $\mathscr{H}^p$, when $p < 2$. Let $\mathscr{T}$ denote the following conformal map from $\mathbb{D}$ to $\mathbb{C}_{1/2}$,

$$\mathscr{T}(z) := \frac{1}{2} + \frac{1-z}{1+z}.$$

For $\alpha > 1$, define the conformally invariant Bergman space $A^2_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})$ as the space of analytic functions f in $\mathbb{C}_{1/2}$ such that $f \circ \mathscr{T}$ is in $A^2_\alpha(\mathbb{D})$. In particular, set

$$\begin{aligned}\|f\|_{A^2_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})} &= \big\|f\circ\mathscr{T}\big\|_{A^2_\alpha(\mathbb{D})}\\ &= \left(\int_{\mathbb{C}_{1/2}} |f(s)|^2\,(\alpha-1)\left(\sigma-\frac{1}{2}\right)^{\alpha-2}\frac{4^{\alpha-1}dm(s)}{\pi|s+1/2|^{2\alpha}}\right)^{\frac{1}{2}}.\end{aligned}$$

We are able to extend [6, Thm. 1] from $1 \leq p < 2$ to $p < 2$ using Theorem 3.4. Note that this is a Dirichlet series version of (23) in the half-plane $\mathbb{C}_{1/2}$.

Corollary 3.7. *Let $0 < p < 2$. There is a constant $C_p \geq 1$ such that*

$$\|f\|_{A^2_{2/p,\mathrm{i}}(\mathbb{C}_{1/2})} \leq C_p\|f\|_{\mathscr{H}^p}$$

for every $f \in \mathscr{H}^p$. The parameter $\alpha = 2/p$ is optimal.

Proof. Define $\mathscr{H}_\alpha$ as the Hilbert space of Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ that satisfy

$$\|f\|_{\mathscr{H}_\alpha} := \left(\sum_{n=1}^\infty \frac{|a_n|^2}{\Phi_\alpha(n)}\right)^{\frac{1}{2}} < \infty.$$

Here it is crucial that Φ_α is strictly positive. By Lemma 3.6 and [35, Thm. 1] it follows that there is some C_α such that

$$\|f\|_{A^2_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})} \leq C_\alpha\|f\|_{\mathscr{H}_\alpha},$$

whenever $\alpha > 1$. The proof of the first statement is completed using (34). For the proof that $\alpha = 2/p$ is optimal, we can follow the argument given in the proof of [6, Thm. 1]. We set

$$f_{p,\varepsilon}(s) = \zeta^{2/p}(1/2+\varepsilon+s) = \sum_{n=1}^\infty \frac{d_{2/p}(n)}{n^{1/2+\varepsilon}}\,n^{-s},$$

which satisfies

$$f_{p,\varepsilon}(s) = \left(\frac{1}{1/2+\varepsilon+s-1}\right)^{2/p} + O\left(|1/2+\varepsilon+s-1|^{-(2/p-1)}\right)$$

when $1/2 < \operatorname{Re} s = \sigma < 1$ and $0 < \operatorname{Im} s = t < 1$. Then clearly

$$\|f_{p,\varepsilon}\|^2_{A^2_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})} \gg \int_{1/2}^{1}\int_0^1 \left|\frac{1}{\sigma-1/2+\varepsilon+it}\right|^{\frac{4}{p}} \left(\sigma-\frac{1}{2}\right)^{\alpha-2} dtd\sigma \gg \varepsilon^{\alpha-4/p}.$$

Since $\|f_{p,\varepsilon}\|^2_{\mathscr{H}^p} \asymp \varepsilon^{-2/p}$, we get that $\alpha - 4/p \geq -2/p$ is necessary. □

3.5. **Fractional primitives of $\zeta(s)$ and duality.** It was asked in [13, Sec. 5] whether the primitive of the half-shift of the Riemann zeta function

$$\varphi(s) := 1 + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n} n^{-s}$$

defines a bounded linear functional on $\mathscr{H}^1$, or equivalently: Is there a constant C such that

$$\left| a_1 + \sum_{n=2}^{N} \frac{a_n}{\sqrt{n}\log n} \right| \leq C\|f\|_{\mathscr{H}^p} \tag{41}$$

for every Dirichlet polynomial $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ when $p = 1$? Clearly, (41) is satisfied if $p = 2$, and it was shown in [6] that (41) holds whenever $p > 1$.

It was also demonstrated in [6] that φ is in $\mathscr{H}^p$ if and only if $p < 4$. We are still not able to answer the original question from [13, Sec. 5], but we will prove some complementary results that shed more light on this and related questions about duality.

For $\beta > 0$, consider the following fractional primitives of the half-shift of the Riemann zeta function:

$$\varphi_\beta(s) := 1 + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}(\log n)^\beta} n^{-s}. \tag{42}$$

We are interested in the following questions.

(a) For which $\beta > 0$ is φ_β in $\mathscr{H}^p$, when $p \geq 2$?
(b) For which $\beta > 0$ is φ_β in $(\mathscr{H}^p)^*$, when $p \leq 2$?

Before proceeding, let us clarify question (b). The linear functional generated by φ_β can be expressed as

$$\langle f, \varphi_\beta \rangle_{\mathscr{H}^2} := a_1 + \sum_{n=2}^{\infty} \frac{a_n}{\sqrt{n}(\log n)^\beta},$$

when $f(s) = \sum_{n=1}^\infty a_n n^{-s}$. We say that the linear functional generated by φ_β acts boundedly on $\mathscr{H}^p$, or equivalently that φ_β is in $(\mathscr{H}^p)^*$, if there is a constant $C > 0$ such that

$$|\langle f, \varphi_\beta\rangle_{\mathscr{H}^2}| \le C\|f\|_{\mathscr{H}^p}$$

for every Dirichlet polynomial f. Our result is:

Theorem 3.8. *Suppose that $\beta > 0$.*

(a) *Let $p \ge 2$. Then φ_β is in $\mathscr{H}^p$ if and only if $\beta > p/4$.*
(b) *Let $p \le 2$. If $\beta > 1/p$ then φ_β is in $(\mathscr{H}^p)^*$ and if $\beta < 1/p$ then φ_β is not in $(\mathscr{H}^p)^*$.*

It is well-known that the dual space $(\mathscr{H}^p)^*$ for $1 < p < \infty$ is not equal to $\mathscr{H}^q$ with $1/p + 1/q = 1$ (see [43, Sec. 3]). Theorem 3.8 provides additional examples illustrating this fact.

Before proving Theorem 3.8, we note that only the case $\beta = 1$ in Theorem 3.8 can be proved completely using results from [6, 8, 45], and that (33) or (34) are required for either (a) or (b) when $\beta \ne 1$.

Proof of Theorem 3.8 (a). To begin with, we notice that (33) implies that

$$\|\varphi_\beta\|_{\mathscr{H}^p}^2 \le 1 + \sum_{n=2}^\infty \frac{\Phi_{p/2}(n)}{n(\log n)^{2\beta}}.$$

The series on the right-hand side is convergent when $2\beta > p/2$, by Lemma 3.6 and Abel summation, and we have thus proved that φ_β is in $\mathscr{H}^p$ whenever $\beta > p/4$.

To settle the case $\beta = p/4$, we set $k = [p]$, $q = p/k$, and

$$\log^* n = \begin{cases} \log n, & n > 1 \\ 1, & n = 1, \end{cases}$$

and use (35) to the effect that

$$\|\varphi_\beta\|_{\mathscr{H}^p}^p = \|\varphi_\beta^k\|_{\mathscr{H}^q}^q \ge \left(\sum_{n=1}^\infty \frac{|\mu(n)|}{d_{2/q}(n)}\frac{1}{n}\left|\sum_{n_1\cdots n_k = n}\frac{1}{(\log^* n_1)^\beta\cdots(\log^* n_k)^\beta}\right|^2\right)^{\frac{r}{2}}$$
$$\ge \left(\sum_{n=2}^\infty \frac{|\mu(n)|}{d_{2/q}(n)}\frac{[d_k(n)]^2}{n(\log n)^{2k\beta}}\right)^{\frac{r}{2}} = \left(\sum_{n=2}^\infty \frac{|\mu(n)| d_{p[p]/2}(n)}{n(\log n)^{p[p]/2}}\right)^{\frac{r}{2}},$$

where we used thrice that $|\mu(n)|d_\alpha(n) = |\mu(n)|\alpha^{\Omega(n)}$. To see that the final series is divergent, we use Abel summation and the fact that

$$\frac{1}{x}\sum_{n\le x}|\mu(n)|d_\alpha(n) = C_\alpha(\log x)^{\alpha-1} + O\left((\log x)^{\alpha-2}\right),$$

which follows at once from standard techniques since

$$\sum_{n=1}^{\infty} |\mu(n)| d_\alpha(n) n^{-s} = \prod_{j=1}^{\infty} \left(1 + \alpha p_j^{-s}\right). \qquad \square$$

Proof of Theorem 3.8 (b). The first statement follows from (34). The Cauchy–Schwarz inequality gives that

$$|\langle f, \varphi_\beta \rangle_{\mathscr{H}^2}| \leq \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{\Phi_{2/p}(n)} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{\Phi_{2/p}(n)}{n (\log n)^{2\beta}} \right)^{\frac{1}{2}}.$$

Abel summation again gives that the final sum is convergent if $2\beta > 2/p$.

For the second part, suppose that $\beta < 1/p$ and set

$$f(s) = \left(\prod_{p_j \leq N} \frac{1}{1 - p_j^{-1/2-s}} \right)^{2/p}.$$

Clearly, $\|f\|_{\mathscr{H}^p} \asymp (\log N)^{1/p}$. We use Abel summation and (38) and find that

$$\langle f, \varphi_\beta \rangle_{\mathscr{H}^2} \geq \sum_{n=2}^{N} \frac{d_{2/p}(n)}{n (\log n)^\beta} \asymp (\log N)^{2/p - \beta}.$$

We conclude that

$$\frac{\langle f, \varphi_\beta \rangle_{\mathscr{H}^2}}{\|f\|_{\mathscr{H}^p}} \asymp (\log N)^{1/p - \beta}$$

is unbounded as $N \to \infty$, since by assumption $\beta < 1/p$. $\square$

The proof of Theorem 3.8 (b) does not provide any insight into the critical exponent $\beta = 1/p$, except for the trivial case $p = 2$. The final part of this subsection is devoted to some observations on this interesting problem. We begin by considering the corresponding problem for Hardy spaces on the unit disc. To this end, we introduce

$$\psi_\beta(z) := \sum_{j=0}^{\infty} \frac{z^j}{(j+1)^\beta}, \tag{43}$$

which are fractional primitives of $(1 - z)^{-1}$, a function which plays the same role as $\zeta(s)$ in the theory of Hardy spaces in the unit disc. Equivalently, one could consider the linear functional with weights given by

$$\mathrm{Beta}\left(\beta, \frac{j+1}{2}\right) \asymp_\beta (j+1)^\beta.$$

These weights are sometimes more convenient, due to the fact that the associated functional L_β admits the integral representation

$$L_\beta(f) := \int_0^1 f(r)\, 2(1-r^2)^{\beta-1}\, dr. \tag{44}$$

We compile the following result:

Theorem 3.9. *Let ψ_β be as in* (43)*. Then*

(a) *If $1 < p < \infty$, then ψ_β is in $\big(H^p(\mathbb{D})\big)^* = H^{p/(p-1)}(\mathbb{D})$ if and only if $\beta > 1/p$.*

(b) *If $p \le 1$, then ψ_β is in $\big(H^p(\mathbb{D})\big)^*$ if and only if $\beta \ge 1/p$. Moreover, if $\beta \ge 1$, then ψ_β is in $H^p(\mathbb{D})$ for every $p < \infty$.*

Proof. We begin with (a). That $\big(H^p(\mathbb{D})\big)^* = H^{p/(p-1)}(\mathbb{D})$ for $1 < p < \infty$ is well-known (see [19]). We will investigate when ψ_β is in $H^{p/(p-1)}(\mathbb{D})$. To do this, we use a result of Hardy and Littlewood [23]: If $f(z) = \sum_{j=0}^\infty a_j z^j$ has positive and decreasing coefficients and $1 < q < \infty$, then

$$\|f\|_{H^q(\mathbb{D})} \asymp_q \left(\sum_{j=0}^\infty (j+1)^{q-2} a_j^q\right)^{\frac{1}{q}}.$$

Setting $q = p/(p-1)$ we find that

$$\|\psi_\beta\|_{H^q\mathbb{D})}^q \asymp_q \sum_{j=0}^\infty (j+1)^{\frac{p}{p-1}(1-\beta)-2},$$

which is finite if and only if $\beta > 1/p$.

For (b), we begin with the case $\beta = 1$. A stronger version of our statement can be found in [19, Thm. 4.5]. It is also clear that since ψ_1 is in $\big(H^1(\mathbb{D})\big)^*$, ψ_1 is in $H^q(\mathbb{D})$ for every $p < \infty$.

To investigate the case $p < 1$, we require the main result in [20] for which we refer to [19]. We conclude that $\psi_\beta \in \big(H^p(\mathbb{D})\big)^*$ if and only if $\beta \le 1/p$ by combining [19, Thm. 7.5] with [19, Ex. 1 and Ex. 3 on p. 90]. If $\beta < 1$, then ψ_β is a bounded function, so ψ_β is in $H^p(\mathbb{D})$ for every $p < \infty$. □

In analogy with Theorem 3.9, we offer the following conjecture.

Conjecture. Let $p \le 2$. The Dirichlet series $\varphi_{1/p}$ from (42) defines a bounded linear functional on $\mathscr{H}^p$ if and only if $p \le 1$.

Let us now explain how this conjecture is related to another open problem for Hardy spaces of Dirichlet series. Define the space $H^p_{\mathrm{i}}(\mathbb{C}_{1/2})$ in the same way as

$A^2_{\alpha,\mathrm{i}}(\mathbb{C}_{1/2})$ from Corollary 3.7. A computation gives that

$$\|f\|^p_{H^p_{\mathrm{i}}(\mathbb{C}_{1/2})} = \frac{1}{\pi}\int_{-\infty}^{\infty} |f(1/2+it)|^p \, \frac{dt}{1+t^2}.$$

An equivalent formulation of the embedding problem mentioned in Section 2.5 is the following. Is there is a constant $C_p > 0$ such that

$$\|f\|^p_{H^p_{\mathrm{i}}(\mathbb{C}_{1/2})} \leq C_p \|f\|^p_{\mathscr{H}^p} \tag{45}$$

holds for every Dirichlet polynomial f? As mentioned in Section 2.5, the embedding (45) is known to hold only when p is an even integer. See [10] for a simple proof of this fact, which also gives the optimal constant[2] $C_p = 2$. Note that (45) is a stronger statement than Corollary 3.7, since from (23) we get that

$$\|f\|_{A^2_{2/p,\mathrm{i}}(\mathbb{C}_{1/2})} \leq C_p \|f\|_{H^p_{\mathrm{i}}(\mathbb{C}_{1/2})}$$

for $0 < p < 2$. The linear functional generated by φ_β can also be expressed as

$$\langle f, \varphi_\beta\rangle_{\mathscr{H}^2} = a_1 + \int_{1/2}^{\infty} \big(f(\sigma) - a_1\big)\left(\sigma - \frac{1}{2}\right)^{\beta-1} \frac{d\sigma}{\Gamma(\beta)}. \tag{46}$$

Translating (44) from $\mathbb{D}$ to $\mathbb{C}_{1/2}$ we find that

$$\widetilde{L}_{1/p}(F) := \int_{1/2}^{1} F(\sigma)\left(\sigma - \frac{1}{2}\right)^{1/p-1} d\sigma$$

defines a bounded linear functional on $H^p_{\mathrm{i}}(\mathbb{C}_{1/2})$ if and only if $p \leq 1$. Note that the contribution in (46) for $\sigma \geq 1$ can be handled by trivial estimates. Hence we conclude that if (45) holds for $p \leq 1$, then (46) (and hence $\varphi_{1/p}$) defines a bounded linear functional on $\mathscr{H}^p$.

4. Coefficient estimates: Weighted ℓ^∞ bounds

We now turn to weighted ℓ^∞ estimates of the coefficient sequence $(a_n)_{n\geq 1}$ for elements $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ in $\mathscr{H}^p$. Phrased differently, we are interested in estimating the norm of the linear functional $f \mapsto a_n$ for every $n \geq 1$, i.e., the quantity

$$\mathscr{C}(n,p) := \sup_{\|f\|_p = 1} |a_n|.$$

When $p \geq 1$, a_n can be expressed as a Fourier coefficient, implying that this norm is trivially 1 for all n. We will therefore mainly be concerned with the case $0 < p < 1$.

The first observation we make is that, again, it suffices to deal with the one-dimensional situation because the general estimates will appear by multiplicative

[2]The proof given in [10] that $C_2 = 2$ extends effortlessly to show that $C_p = 2$ when p is an even integer.

extension. Before we prove this claim, we recall what is known about the coefficients of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $H^p(\mathbb{D})$ when $0 < p < 1$. For $0 < p < \infty$ and $k \geq 1$, we set

$$C(k,p) := \sup\left\{ \frac{|f^{(k)}(0)|}{k!} \ : \ \|f\|_{H^p(\mathbb{D})} = 1 \right\}. \tag{47}$$

By a classical result [19, p. 98], $C(k,p) \ll k^{1/p-1}$ and $a_k = o(k^{1/p-1})$ for an individual function in $H^p(\mathbb{D})$ when $0 < p < 1$. By a normal family argument, there are extremal functions f_k in $H^p(\mathbb{D})$ for (47).

Turning to the multiplicative extension, we begin by noting that it suffices to consider an arbitrary polynomial

$$F(z) = \sum_{\kappa} c_\kappa z^\kappa$$

on $\mathbb{T}^\infty$ and to estimate the size of c_κ for an arbitrary multi-index

$$\kappa = (\kappa_1, \ldots, \kappa_m, 0, 0, \ldots).$$

Recall that $A_m F$ denotes the $m$te Abschnitt of F. For $0 < p < 1$ we use (5) to find that

$$\begin{aligned}
|c_\kappa|^p &= \left| \int_{\mathbb{T}^m} A_m F(z) \overline{z_1}^{\kappa_1} \cdots \overline{z_m}^{\kappa_m} \, d\mu_m \right|^p \\
&\leq C(\kappa_m, p)^p \int_{\mathbb{T}} \left| \int_{\mathbb{T}^{m-1}} A_m F(z) \overline{z_1}^{\kappa_1} \cdots \overline{z_{m-1}}^{\kappa_{m-1}} d\mu_{m-1} \right|^p d\mu_1 \\
&\leq C(\kappa_1, p)^p \cdots C(\kappa_m, p)^p \|A_m F\|_p^p \\
&\leq C(\kappa_1, p)^p \cdots C(\kappa_m, p)^p \|F\|_p^p.
\end{aligned}$$

This is a best possible estimate because if f_k in $H^p(\mathbb{D})$ satisfies $|a_k|/\|f_k\|_p = C(k,p)$, then clearly the function

$$\prod_{j=1}^{m} f_{\kappa_j}(z_j)$$

will be extremal with respect to the multi-index $\kappa = (\kappa_1, \ldots, \kappa_m)$. Hence we conclude that $n \mapsto \mathscr{C}(n,p)$ is a multiplicative function that takes the value $C(k,p)$ at $n = p_j^k$ for every prime p_j.

To the best of our knowledge, the exact values of $C(k,p)$ from (47) have not been computed previously for any $k \geq 1$ when $0 < p < 1$, and we have therefore made an effort to improve this situation. We begin with the case $k = 1$ which is settled by the following theorem.

Theorem 4.1. *We have*

$$C(1,p) = 1 \quad \text{if} \quad p \geq 1, \quad \text{and} \quad C(1,p) = \sqrt{\frac{2}{p}}\left(1 - \frac{p}{2}\right)^{\frac{1}{p} - \frac{1}{2}} \quad \text{if} \quad 0 < p < 1. \tag{48}$$

The extremals (modulo the trivial modifications $f(z) \mapsto e^{i\theta_1} f(e^{i\theta_2} z)$*) are*

(a) $f(z) = z$ *for* $p > 1$*;*

(b) *the family* $f_a(z) = (a + \sqrt{1-a^2}z)(\sqrt{1-a^2} + az)$ *with* $0 \le a \le 1$ *for* $p = 1$*;*

(c) $f(z) = \big(\sqrt{1-p/2} + z\sqrt{p/2}\big)^{2/p}$ *for* $0 < p < 1$*.*

Proof. As already pointed out, it is obvious that $C(1,p) = 1$ when $p \ge 1$. The uniqueness of the extremal function for $p > 1$ is immediate by the strict convexity of the unit ball of $L^p(\mathbb{T})$.

To find the extremal functions when $p = 1$, we start from the fact that functions f in the unit ball of $H^1(\mathbb{T})$ can equivalently be written in the form $f = gh$, where h, g are in the unit ball of $H^2(\mathbb{T})$. Writing $g(z) = \sum_{k=0}^\infty g_k z^k$ and $h(z) = \sum_{k=0}^\infty h_k z^k$, our task is to maximize

$$f'(0) = g_0 h_1 + g_1 h_0,$$

under the sole condition that $\sum_{k=0}^\infty |g_k|^2 = 1$ and $\sum_{k=0}^\infty |h_k|^2 = 1$. By the Cauchy–Schwarz inequality, we must have $|g_0|^2 + |g_1|^2 = |h_0|^2 + |h_1|^2 = 1$ and also $(g_0, g_1) = \lambda(h_1, h_0)$ for a unimodular constant λ. We may choose (g_0, g_1) as an arbitrary unit vector, and hence we get the stated extremals.

We turn to the case $0 < p < 1$. By invoking the inner-outer factorization of f, we may write an arbitrary element f with in the unit ball of $H^p(\mathbb{D})$ equivalently as $f = gh^{2/p-1}$, where g, h are in the unit ball of $H^2(\mathbb{T})$ and h has no zeros in $\mathbb{D}$. We denote the coefficients of g and h as before. By applying a suitable transformation $f(z) \mapsto e^{i\theta_1} f(e^{i\theta_2} z)$, we may assume that $h_0, h_1 \ge 0$, and, moreover, that $f(0) = g_0 h_0^{2/p-1}$, where $h_0^{2/p-1} \ge 0$ is chosen to be real and nonnegative. Hence

$$C(1,p) = \sup \left(h_0^{2/p-1} g_1 + \left(\frac{2}{p} - 1\right) h_0^{2/p-2} h_1 g_0 \right),$$

where the supremum is over all pairs (g_0, g_1) with $|g_0|^2 + |g_1|^2 = 1$ and pairs of nonnegative numbers (h_0, h_1) with $h_0^2 + h_1^2 = 1$ and $h_0 \ge h_1$ since h is zero-free.

The maximum occurs when (g_0, g_1) is a multiple of $\big((\frac{2}{p} - 1) h_0^{2/p-2} h_1, h_0^{2/p-1}\big)$ and hence

$$C(1,p)^2 = \max_{h_0^2+h_1^2=1, h_0 \ge h_1 \ge 0} \left(h_0^{4/p-2} + \left(\frac{2}{p} - 1\right)^2 h_0^{4/p-4} h_1^2 \right).$$

Suppressing the condition $h_0 \ge h_1$, we find that

$$\begin{aligned} C(1,p)^2 &\le \max_{x \in [0,1]} \left(x^{4/p-2} + \left(\frac{2}{p} - 1\right)^2 x^{4/p-4} \left(1 - x^2\right) \right) \\ &= \frac{2}{p}\left(1 - \frac{p}{2}\right)^{2/p-1} \end{aligned} \tag{49}$$

by an elementary calculus argument. Since the solution to the extremal problem in (49) corresponds to $h_0 = \sqrt{1-p/2}$, we also have $h_0 \geq h_1$, and the inequality sign in (49) can therefore in fact be replaced by an equality sign. □

For future reference, we notice that the following asymptotic estimates hold:

$$C(1,p) = \begin{cases} 1 + (1-\log 2)(1-p) + O((1-p)^2), & p \nearrow 1 \\ \frac{1}{\sqrt{p}} \cdot \big(\sqrt{2/e} + O(p)\big), & p \searrow 0. \end{cases} \tag{50}$$

For $k \geq 2$, the method used in the preceding proof will lead to a similar finite-dimensional extremal problem. The solution to this problem is plain for all $k \geq 2$ when $p = 1$, but in the range $0 < p < 1$, the complexity increases notably with k, and we have made no attempt to deal with it. Instead, we supply (non-optimal) estimates obtained from the Cauchy integral formula and Lemma 3.2 (Weissler's inequality).

Lemma 4.2. *Suppose that* $0 < p < 1$ *and* $k \geq 1$. *Then*

$$C(k,p) \leq \min_{p \leq x < 1} x^{-k/2}(1-x)^{1/x-1/p}.$$

Proof. Suppose that $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is in $H^p(\mathbb{D})$ with $\|f\|_p = 1$. Then, by Cauchy's formula,

$$|c_k| \leq \frac{1}{2\pi r}\int_{|z|=r} |z^{-k} f(z)||dz|$$

for every r, $0 < r < 1$. Using the pointwise estimate $|f(z)| \leq (1-|z|^2)^{-1/p}\|f\|_p$, we therefore find that

$$|c_k| \leq r^{-k}\big((1-r^2)^{1/p}\big)^{q-1}\|f\|_p^{1-q}\int_{\mathbb{T}} |f(rz)|^q d\mu(z)$$

whenever $0 < r < 1$ and $0 < q < 1$. Choosing $p < q \leq 1$ and $r^2 = p/q$ and invoking Lemma 3.2, we obtain the desired result. □

We notice that (26) of Theorem 3.3 yields the alternate bound[3]

$$C(k,p) \leq \sqrt{c_{\lceil 2/p \rceil}(k)} \tag{51}$$

which is useful when p is close to 0.

We will now use the information gathered above to prove a result about the maximal order of the multiplicative function $n \mapsto \mathscr{C}(n,p)$. To begin with, we notice that, by Theorem 4.1,

$$\mathscr{C}(n,p) = C(1,p)^{\omega(n)}$$

[3]Notice that the bound $C(k,p) \leq \sqrt{\varphi_{2/p}(k)}$ is of no interest in this context because $\varphi_{2/p}(k)$ grows exponentially with k when $2/p$ is not an integer.

when n is a square-free number and hence

$$\limsup_{\mu(n)\neq 0, n\to\infty} \frac{\log \mathscr{C}(n,p)}{\log n/\log\log n} = \log C(1,p) \tag{52}$$

since

$$\limsup_{\mu(n)\neq 0, n\to\infty} \frac{\log \omega(n)}{\log n/\log\log n} = 1.$$

It seems reasonable to expect that the lim sup in (52) is unchanged if we drop the restriction that $\mu(n) \neq 0$. The next theorem is as close as we have been able to get to confirming this conjecture, based on our general bounds for $C(k,p)$.

Theorem 4.3. *Assume that* $0 < p < 1$. *Then*

$$0 < \limsup_{n\to\infty} \frac{\log \mathscr{C}(n,p)}{\log n/\log\log n} < \infty.$$

Moreover,

$$\limsup_{n\to\infty} \frac{\log \mathscr{C}(n,p)}{\log n/\log\log n} = \begin{cases} \frac{1}{2}|\log p|(1+O(p)), & p \searrow 0 \\ c_p(1-p), & p \nearrow 1, \end{cases}$$

where $1 - \log 2 + O(1-p) \le c_p \le 1/2 + O(1-p)$.

Proof. The general lower bound for the lim sup follows from (52), while the lower bounds

$$\limsup_{n\to\infty} \frac{\log \mathscr{C}(n,p)}{\log n/\log\log n} \ge \begin{cases} \frac{1}{2}|\log p|(1+O(p)), & p \searrow 0 \\ (1-\log 2)(1-p) + O((1-p)^2), & p \nearrow 1 \end{cases}$$

follow from (52) along with (50). To get an upper bound for the lim sup when $p \searrow 0$, we use (51) and the fact that

$$\limsup_{n\to\infty} \frac{\log d_\alpha(n)}{\log n/\log\log n} = \log \alpha.$$

Trivially, (51) also gives a general upper bound for the lim sup.

To get an upper bound for the lim sup when $p \nearrow 1$, we argue as follows. Set

$$n = \prod_j p_j^{\kappa_j}.$$

For $\kappa_j \le 1/(1-p)$, we set $x = p$ in Lemma 4.2 and get

$$C(\kappa_j, p) \le p^{-\kappa_j/2}. \tag{53}$$

We note that

$$\sum_{j:\kappa_j\le 1/(1-p)} \kappa_j \le \frac{1}{1-p}\sum_{j\le \log n/(\log\log n)^2} 1 + \frac{(1+o(1))}{\log\log n}\sum_{j:\kappa_j\le 1/(1-p)} \kappa_j \log p_j$$

$$= \frac{\log n}{(1-p)(\log\log n)^2} + \frac{(1+o(1))}{\log\log n}\sum_{j:\kappa_j\le 1/(1-p)} \kappa_j \log p_j. \tag{54}$$

For $\kappa_j > 1/(1-p)$, we set $x = 1-(1-p)/\kappa_j$ in Lemma 4.2 so that

$$C(\kappa_j, p) \le \left(1 - \frac{(1-p)}{\kappa_j}\right)^{-\kappa_j/2} \cdot \left(\frac{1-p}{\kappa_j}\right)^{1-1/p} \le e^{1-p}\kappa_j^{2(1/p-1)}. \tag{55}$$

We observe that, given $\varepsilon > 0$, we will have if p is close enough to 1, then

$$\sum_{j:\kappa_j\ge 1/(1-p)} \log\kappa_j \le \log(\log n/\log 2)\sum_{j\le \log n/(\log\log n)^3} 1 + \frac{\varepsilon}{\log\log n}\sum_{j:\kappa_j > 1/(1-p)} \kappa_j\log p_j$$

$$\le \frac{(1+o(1))\log n}{(\log\log n)^2} + \frac{\varepsilon}{\log\log n}\sum_{j:\kappa_j>1/(1-p)} \kappa_j \log p_j \tag{56}$$

if p is close enough to 1. Putting (54) into (53) and (56) into (55), respectively, we find that

$$\log\mathscr{C}(n,p) = \sum_j \log C(\kappa_j, p) = \sum_{j:\kappa_j\le 1/(1-p)} C(\kappa_j,p) + \sum_{j:\kappa_j>1/(1-p)} C(\kappa_j,p)$$

$$\le o(1)\frac{\log n}{\log\log n} + \frac{|\log p|}{2\log\log n}\sum_{j:\kappa_j\le 1/(1-p)} \kappa_j\log p_j + \frac{2(1/p-1)\varepsilon}{\log\log n}\sum_{j:\kappa_j>1/(1-p)} \kappa_j \log p_j$$

holds for arbitrary $\varepsilon > 0$, if p is close enough to 1. Choosing $\varepsilon < 1/4$, we obtain the desired upper bound for the lim sup. □

We mention finally a consequence of the classical bound $C(k,p) \ll k^{1/p-1}$ that was already used in [9], related to the decomposition of a holomorphic function on $\mathbb{T}^\infty$ into a sum of holomorphic functions with homogeneous power series. Thus we are interested in the orthogonal projection of a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

onto the space of m-homogeneous functions, namely

$$P_m f(s) := \sum_{\Omega(n)=m} a_n n^{-s}.$$

The result in question, to be used in Subsection 6.3, is as follows.

Lemma 4.4. *Suppose that* $0 < p < \infty$. *Then*

$$\|P_m f\|_{\mathscr{H}^p} \leq \begin{cases} \|f\|_{\mathscr{H}^p}, & p \geq 1, \\ \sqrt{e}(m+1)^{1/p-1}\|f\|_{\mathscr{H}^p}, & 0 < p < 1 \end{cases}$$

holds for every f *in* $\mathscr{H}^p$.

Proof. We may assume that f is a Dirichlet polynomial, so that $\mathscr{B}f(z)$ is continuous on $\mathbb{T}^\infty$. We introduce the transformation $wz := (wz_j)$ on $\mathbb{T}^\infty$, where w is a point on the unit circle $\mathbb{T}$. We may then write

$$(\mathscr{B}f)(wz) = \sum_{m=0}^{\infty} (\mathscr{B}P_m f)(z) w^m.$$

It follows that we may consider the functions $(\mathscr{B}P_m f)(z)$ as the coefficients of a function in one complex variable. We set $k = m$ and $x = \max(p, 1 - 1/(m+1))$ in Lemma 4.2 and get

$$|(\mathscr{B}fP_m)(z)|^p \leq \int_{\mathbb{T}} |(\mathscr{B}P_m f)(wz)|^p d\mu_1(w) \qquad 1 \leq p < \infty,$$

and

$$|(\mathscr{B}fP_m)(z)|^p \leq \frac{(m+1)^{1-p}}{(1-1/(m+1))^{pm/2}} \int_{\mathbb{T}} |(\mathscr{B}P_m f)(wz)|^p d\mu_1(w) \qquad 0 < p < 1.$$

Integrating this inequality over $\mathbb{T}^\infty$ with respect to μ_∞ and using Fubini's theorem, we obtain the desired estimate. □

5. Estimates for the partial sum operator

Assume that $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series in $\mathscr{H}^p$ for some $p > 0$. For given $N \geq 1$, the partial sum operator S_N is defined as the map

$$S_N \left(\sum_{n=1}^{\infty} a_n n^{-s} \right) := \sum_{n=1}^{N} a_n n^{-s}.$$

It is of obvious interest to try to determine the norm of S_N when it acts on the Hardy spaces $\mathscr{H}^p$. Helson's version of the M. Riesz theorem [30] shows that S_N is bounded for $1 < p < \infty$, and, moreover, its norm is bounded by the norm of the one-dimensional Riesz projection acting on functions in $H^p(\mathbb{D})$. Furthermore, by the same argument of Helson [30], we have the following.

Lemma 5.1. *Suppose that $0 < p < 1$. We have the estimate*

$$\|S_N f\|_{\mathscr{H}^p} \le \frac{A}{(1-p)} \|f\|_{\mathscr{H}^1}$$

for f in $\mathscr{H}^p$, where A is an absolute constant.

We refer to [2, Sec. 3], where it is explained how the lemma follows from Helson's general result concerning compact Abelian groups whose dual is an ordered group [30]. See also Sections 8.7.2 and 8.7.6 of [42]. In our case, the dual group in question is the multiplicative group of positive rational numbers $\mathbb{Q}_+$ which is ordered by the numerical size of its elements. This means that the bound for $\|S_N\|_{\mathscr{H}^p \to \mathscr{H}^p}$ in the range $1 < p < \infty$ relies on the additive structure of the positive integers.

When $0 < p \le 1$ or $p = \infty$, a natural question is to determine the asymptotic growth of the norm $\|S_N\|_{\mathscr{H}^p \to \mathscr{H}^p}$ when $N \to \infty$. It is known from [4] and [7] that the growth of both $\|S_N\|_{\mathscr{H}^1 \to \mathscr{H}^1}$ and $\|S_N\|_{\mathscr{H}^\infty \to \mathscr{H}^\infty}$ is of an order lying between $\log\log N$ and $\log N$. We will confine our discussion to the range $0 < p \le 1$ and begin with a new result for the case $p = 1$.

Theorem 5.2. *We have*

$$\log\log N \ll \|S_N\|_{\mathscr{H}^1 \to \mathscr{H}^1} \ll \frac{\log N}{\log\log N}.$$

Proof. Using Hölder's inequality with $p = (1+\varepsilon)/\varepsilon$ and $p' = 1+\varepsilon$, we get

$$\|g\|_{\mathscr{H}^1}^{1-\varepsilon} \le \left(\frac{\|g\|_{\mathscr{H}^2}}{\|g\|_{\mathscr{H}^1}} \right)^{2\varepsilon} \|g\|_{\mathscr{H}^{1-\varepsilon}}^{1-\varepsilon}.$$

Setting $g = S_N f$ and applying Lemma 5.1, we get

$$\|S_N f\|_1 \le A \frac{1}{\varepsilon} \left(\frac{\|S_N f\|_2}{\|S_N f\|_1} \right)^{2\varepsilon/(1-\varepsilon)} \|f\|_1.$$

Now we need to understand how large the ratio $\|f\|_2 / \|f\|_1$ can be when f is a Dirichlet polynomial of length N. A precise solution to this problem can be found in the recent paper [18]. For our purpose, the following one-line argument suffices. By Helson's inequality (which is (34) for $p = 1$) and a well-known estimate for the divisor function, we have

$$\|f\|_2 \le \max_{n \le N} \sqrt{d(n)} \|f\|_1 \le e^{c \frac{\log N}{\log\log N}} \|f\|_1$$

for an absolute constant c. This means that we can choose $\varepsilon = (\log\log N)/\log N$ so that we get

$$\|S_N\|_1 \ll (\log N)/\log\log N,$$

as desired.

The lower bound is obvious from the classical one-dimensional result: The Bohr lift maps Dirichlet series in $\mathscr{H}^p$ of the form $\sum_{k=0}^{\infty} c_k 2^{-ks}$ to functions in $H^p(\mathbb{D})$. $\square$

It is interesting to notice that our improved upper bound relies on both an additive argument (Lemma 5.1) and a multiplicative argument (Theorem 3.4). We now turn to the case $0 < p < 1$ which will again require a mixture of additive and multiplicative arguments.

Theorem 5.3. *Suppose that* $0 < p < 1$*. There are positive constants* $\alpha_p \le \beta_p$ *such that*

$$e^{\alpha_p \frac{\log N}{\log\log N}} \ll \|S_N\|_{\mathscr{H}^p \to \mathscr{H}^p} \ll e^{\beta_p \frac{\log N}{\log\log N}}.$$

Moreover, we have

$$\liminf_{N\to\infty} \frac{\log \|S_N\|_{\mathscr{H}^p\to\mathscr{H}^p}}{\log N/\log\log N} \ge \begin{cases} \frac{1}{4}|\log p| + O(1), & p \searrow 0 \\ \frac{1}{2}(1-\log 2)(1-p) + O((1-p)^2), & p \nearrow 1 \end{cases}$$

and

$$\limsup_{N\to\infty} \frac{\log \|S_N\|_{\mathscr{H}^p\to\mathscr{H}^p}}{\log N/\log\log N} \le \begin{cases} \frac{1}{2}|\log p| + O(1), & p \searrow 0 \\ c(1-p) + O((1-p)^2), & p \nearrow 1, \end{cases}$$

where c *is an absolute constant.*

We have made no effort to minimize the constant c, but mention that our proof gives the value $\log 2$ times the norm of the operator $f \mapsto f^*$ from $H^1(\mathbb{D})$ to $L^1(\mathbb{T})$, where f^* is the radial maximal function of f. Comparing with Theorem 4.3, we notice that $\log \|S_N\|_{\mathscr{H}^p\to\mathscr{H}^p}$ has essentially the same maximal order as that of $\log \mathscr{C}(N,p)$.

We will split the proof of Theorem 5.3 into three parts. We begin with the easiest case.

Proof of the upper bound in Theorem 5.3, with asymptotics for $p \searrow 0$. We begin by seting $\alpha := \lceil 2/p \rceil$ and apply the Hardy–Littlewood inequality from Theorem 3.4:

$$\|S_N f\|_{\mathscr{H}^p} \le \|S_N f\|_{\mathscr{H}^2} \le \left(\max_{n\le N} \sqrt{d_\alpha(n)}\right) \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d_\alpha(n)}\right)^{\frac{1}{2}}$$
$$\ll \alpha^{\frac{\log N}{2\log\log N}(1+o(1))} \|f\|_{\mathscr{H}^{2/\alpha}},$$

where we in the last step used that

$$d_\alpha(n) \le \alpha^{\frac{\log n}{\log\log n}(1+o(1))}$$

when $n \to \infty$. We conclude by using that $\|f\|_{\mathscr{H}^{2/\alpha}} \le \|f\|_{\mathscr{H}^p}$, which holds because $2/\alpha \le p$. This argument gives both $\beta_p = \log(1 + 2/p))$, say, and the desired asymptotic estimate when $p \searrow 0$. $\square$

We need a more elaborate argument to get the right asymptotic behavior when $p \nearrow 1$. We prepare for the proof by first establishing an auxiliary result concerning polynomials on $\mathbb{T}$. Here we use again the notation $f_r(z) := f(rz)$, where f is an analytic function on $\mathbb{D}$ and $r > 0$.

Lemma 5.4. *Suppose that $0 < p \le 1$. There exists an absolute constant C, independent of p, such that if $1 - r = C^{-1/p} n^{-1}$, then*

$$\|Q\|^p_{H^p(\mathbb{D})} \le 2\|Q_r\|^p_{H^p(\mathbb{D})} \tag{57}$$

for every polynomial $Q(z) = \sum_{k=0}^{n} c_k z^k$.

Proof. For this proof, we write $\|Q\|_p = \|Q\|_{H^p(\mathbb{D})}$. By the triangle inequality for the L^p quasi-metric, we have

$$\|Q\|^p_p \le \|Q - Q_r\|^p_p + \|Q_r\|^p_p. \tag{58}$$

Since

$$|Q(z) - Q_r(z)| = \left| \int_{rz}^{z} Q'(w)dw \right| \le (1 - r) \max_{0\le\rho\le1} |Q'(\rho z)|,$$

we find that

$$\|Q - Q_r\|^p_p \le A(1 - r)^p \|Q'\|^p_p$$

for an absolute constant A by the H^p boundedness of the radial maximal function. Using Bernstein's inequality for $0 < p \le 1$ [3, 52], we therefore get that

$$\|Q - Q_r\|^p_p \le A(1 - r)^p n^p \|Q\|^p_p.$$

Returning to (58), we see that we get the desired result by setting $C = 2A$. □

Proof of the upper bound in Theorem 5.3 when $p \nearrow 1$. Set

$$m = m(N) := [\log N/(\log\log N)^3]$$

and write $z := (u, v)$ for a point on $\mathbb{T}^\infty$, where $u = (z_1, ..., z_m)$ and $v = (z_{m+1}, z_{m+2}, ...)$, so that u corresponds to the first m primes. Let ξ and η be complex numbers and set $\xi u := (\xi z_1, ...\xi z_m)$ and $\eta v = (\eta z_{m+1}, \eta z_{m+2}, ...)$. Also, if F is a function on $\mathbb{T}^\infty$ and $0 < r, \rho \le 1$, we set $F_{r,\rho}(z) := F(ru, \rho v)$.

We will now apply Lemma 5.4 in two different ways. We begin by applying it to the function $\xi \mapsto (\mathscr{B}S_N f)(\xi u, v)$, which is a polynomial of degree at most $\log N/\log 2$. This gives

$$\int_{\mathbb{T}} |(\mathscr{B}S_N f)(\xi u, v)|^p d\mu(\xi) \le 2 \int_{\mathbb{T}} |(\mathscr{B}S_N f)(r\xi u, v)|^p d\mu(\xi)$$

for every point (u, v) and hence

$$\|\mathscr{B}S_N f\|^p_p \le 2\|\mathscr{B}S_N f_{r,1}\|^p_p$$

by Fubini's theorem, with $1-r = C^{-1/p}(\log N/\log 2)^{-1}$. Next, we apply (57) to the function $\eta \mapsto (\mathscr{B}S_N f_{r,1})(u,\eta v)$, which is a polynomial of degree at most $(1+o(1))\log N/\log\log N$. Hence we find that

$$\|\mathscr{B}S_N f\|_p^p \le 2^2 \|\mathscr{B}S_N f_{r,\rho}\|_p^p$$

with $1-\rho = C^{-1/p}(1+o(1))\log\log N/\log N$. Applying (57) k times in this way, we therefore get that

$$\|\mathscr{B}S_N f\|_p^p \le 2^{k+1}\|\mathscr{B}S_N f_{r,\rho^k}\|_p^p. \tag{59}$$

We choose k such that $\rho^k \le \sqrt{p}$, which is done because our plan is to use Lemma 3.2 (Weissler's inequality). Since $1-\rho = C^{-1/p}(1+o(1))\log\log N/\log N$, we therefore obtain the requirement that

$$k = \frac{\log p}{2\log \rho} = |\log p| \cdot \big(1/2 + o(1)\big) C^{1/p} \log N/\log\log N. \tag{60}$$

We now apply Lemma 5.1 to the right-hand side of (59), which yields

$$\|\mathscr{B}S_N f\|_p \le K(k,p)\|\mathscr{B}f_{r,\rho^k}\|_1,$$

where

$$\begin{aligned} K(k,p) : &= A^p 2^{(k+1)/p}(1-p)^{-1} \\ &= A^p(1-p)^{-1}\exp\left(\Big(\frac{\log 2}{2} + o(1)\Big)|\log p| p^{-1} C^{1/p}\frac{\log N}{\log\log N}\right); \end{aligned} \tag{61}$$

here we took into account (60) to get to the final bound for $K(k,p)$. Note that, in view of (5), we may assume that v is a vector of length $d := \pi(N) - m$. It follows that

$$\begin{aligned} \|\mathscr{B}S_N f\|_p &\le K(k,p)\int_{\mathbb{T}^d}\int_{\mathbb{T}^m} \big|(\mathscr{B}f)(ru,\rho^k v)\big|\, d\mu_m(u) d\mu_d(v) \\ &\le K(k,p)(1-r^2)^{-m(1-p)/p}\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^m}\big|(\mathscr{B}f)(u,\rho^k v)\big|^p\, d\mu_m(u)\right)^{1/p} d\mu_d(v), \end{aligned}$$

where we in the last step used the Cole–Gamelin estimate (6). Using Minkowski's inequality (36) as before, we thus get

$$\|\mathscr{B}S_N f\|_p^p \le K(k,p)^p (1-r^2)^{-m(1-p)}\int_{\mathbb{T}^m}\left(\int_{\mathbb{T}^d}\big|(\mathscr{B}f)(u,\rho^k v)\big|\, d\mu_d(v)\right)^p d\mu_m(u).$$

We now iterate Weissler's inequality along with Minkowski's inequality d times in the same way as in the proof of Theorem 3.4 and get the bound

$$\|\mathscr{B}S_N f\|_p \le K(k,p)(1-r^2)^{-m(1-p)/p}\|f\|_p.$$

Now taking into account our choice of r and m, we find that

$$\limsup_{N\to\infty}\frac{\log\|S_N\|_{\mathscr{H}^p\to\mathscr{H}^p}}{\log N/\log\log N} \le \limsup_{N\to\infty}\frac{\log K(k,p)}{\log N/\log\log N}.$$

Using finally (61), we conclude that

$$\limsup_{N\to\infty} \frac{\log \|S_N\|_{\mathscr{H}^p\to\mathscr{H}^p}}{\log N/\log\log N} \le \frac{C^{1/p}|\log p|\log 2}{2p},$$

and hence we get the desired asymptotics when $p \nearrow 1$ with $c = (C\log 2)/2$. □

Proof of the lower bound in Theorem 5.3. We consider first the special case when M is the product of the first k prime numbers, $M = p_1 \cdots p_k$. By the prime number theorem, we have $k \sim \log M/\log\log M$. We use then the function

$$f_M(s) := \prod_{j=1}^{k} \left(\sqrt{1-p/2} + p_j^{-s}\sqrt{p_j/2}\right)^{2/p}.$$

We recognize each of the factors of this product as the extremal function from Theorem 4.1. Hence $\|f_M\|_p = 1$ and

$$f_M(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with

$$a_M = C(1,p)^k = \left(\sqrt{\frac{2}{p}}\left(1-\frac{p}{2}\right)^{\frac{1}{p}-\frac{1}{2}}\right)^k.$$

Consequently, by the triangle inequality for the L^p quasi-metric,

(62) $$C(1,p)^{pk} \le \|S_{M-1}f_M\|_p^p + \|S_M f_M\|_p^p \le 2\max\left(\|S_{M-1}f_M\|_p^p, \|S_M f_M\|_p^p\right),$$

and therefore at least one of the quasi-norms $\|S_{M-1}f_M\|_p$ or $\|S_M f_M\|_p$ is bounded below by

$$\frac{1}{2}C(1,p)^{(1+o(1))\frac{\log M}{\log\log M}}.$$

Suppose now that an arbitrary N is given. Set $n_j := p_1\cdots p_j$ and

$$J := \max\{j : \ N/n_j \ge n_j + 1\}.$$

It follows that $\log n_J = (1/2 + o(1))\log N$. There are now two cases to consider:

(1) Suppose $\|S_{n_J} f_{n_J}\|_p$ is large. We set $x_N := [N/n_J]$ and define

$$g_N(s) := x_N^{-s} f_{n_J}(s).$$

Then $(S_N g_N)(s) = x_N^{-s}(S_{n_J} f_{n_J})(s)$ because $x_N = N/n_J - \varepsilon$ for some $0 \le \varepsilon < 1$, and so

$$x_N(n_J+1) = (N/n_J - \varepsilon)(n_J+1) = N + N/n_J - \varepsilon(n_J+1) > N,$$

where we in the last step used the definition of J.

(2) Suppose $\|S_{n_J-1}f_{n_J}\|_p$ is large. We set $x_N := \lceil N/n_J \rceil$ and define g_N as in the first case. Then $(S_N g_N)(s) = x_N^{-s}(S_{n_J-1}f_{n_J})(s)$ because $x_N = N/n_J + \varepsilon$ for some $0 \le \varepsilon < 1$, and so

$$x_N(n_J - 1) = (N/n_J + \varepsilon)(n_J - 1) = N - N/n_J + \varepsilon(n_J - 1) < N,$$

where we in the last step again used the definition of J.

In either case, since (62) holds for $M = n_J$ and $\log n_J = (1/2 + o(1)) \log N$, we conclude that

$$\liminf_{N\to\infty} \frac{\log \|S_N\|_{\mathscr{H}^p \to \mathscr{H}^p}}{\log N / \log\log N} \ge \frac{1}{2} \log C(1,p).$$

The proof is finished by invoking the asymptotic estimate (50). □

Up to the precise values of α_p and β_p, the problem of estimating $\|S_N\|_{\mathscr{H}^p \to \mathscr{H}^p}$ for $0 < p < 1$ is solved by Theorem 5.3. This result is, however, somewhat deceptive because it is of no help when we need to estimate $\|S_N f\|_{\mathscr{H}^p}$ for functions f of number theoretic interest, such as (2). In fact, in that case, Lemma 5.1 gives a much better bound. The problem of estimating such norms (or quasi-norms) is the topic of the final section of this paper.

6. Pseudomoments of the Riemann zeta function and related Dirichlet series

6.1. **Generalities about moments and pseudomoments of $\zeta(1/2+it)$.** This section is partially motivated by our desire to understand the distribution of large values of the Riemann zeta function $\zeta(s)$ on the critical line $\sigma = 1/2$. We begin by recalling the classical approximation

$$\zeta(\sigma + it) = \sum_{n \le x} n^{-\sigma - it} - \frac{x^{1-\sigma-it}}{1 - \sigma - it} + O(x^{-\sigma}),$$

which holds uniformly in the range $\sigma \ge \sigma_0 > 0$, $|t| \le x$ (see [51, Thm. 4.11]). This means that

$$\big|\zeta(1/2 + it) - \sum_{n \le 2T} n^{-1/2 - it}\big| = O(T^{-1/2}), \quad T \le t \le 2T,$$

and so our problem is about the size of $\sum_{n \le 2T} n^{-1/2-it}$ on the interval $[T, 2T]$.

We recall briefly some known facts about the distribution of $|\zeta(1/2 + it)|$ on $[T, 2T]$. First, by a celebrated result of Selberg (see [46, 48]), $\log|\zeta(1/2 + it)|$ has an approximate normal distribution with mean zero and variance $\frac{1}{2}\log\log T$ on $[T, 2T]$. This implies that a "typical" value of $|\zeta(1/2 + it)|$ and hence of $|\sum_{n \le 2T} n^{-1/2-it}|$ on $[T, 2T]$ is $e^{\sqrt{(1/2)\log\log T}}$. More precise information about the distribution of $|\zeta(1/2+it)|$ can be acquired from the size of the moments. One expects that

$$M_k(T) := \frac{1}{T}\int_T^{2T} |\zeta(1/2+it)|^{2k} dt \sim A_k (\log T)^{k^2}$$

for some constant A_k for which one even has precise predictions [16]. This asymptotic behavior is known to hold when $k = 1, 2$ by results of respectively Hardy and Littlewood [22] and Ingham [34]. An unconditional lower bound $M_k(T) \gg (\log T)^{k^2}$ is known in the range $k \geq 1$ [38], and this is known to hold conditionally for all $k > 0$ by work of Ramachandra (see [39, 40]) and Heath-Brown [27]. Harper [24], building and improving on

work of Soundararajan [49], showed that the upper bounds of optimal order $M_k \ll (\log T)^{k^2}$ also hold conditionally for all $k > 0$.

By the Bohr correspondence, we may think of the interval $[T, 2T]$ as a subset of $\mathbb{T}^\infty$, and an interesting question is then to understand the distribution of $|\sum_{n \leq 2T} n^{-1/2-it}|$ on the entire torus $\mathbb{T}^\infty$ and, in particular, to compare with what we have on the subset $[T, 2T]$. We use again the notation $Z_N(s) := \sum_{n \leq N} n^{-1/2-s}$ and, following Conrey and Gamburd [16], refer to the corresponding moments

$$\Psi_k(N) := \lim_{T\to\infty} \frac{1}{T}\int_T^{2T} |Z_N(it)|^{2k}\, dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |Z_N(it)|^{2k}\, dt = \|Z_N\|^{2k}_{\mathscr{H}^{2k}}$$

as the pseudomoments of $\zeta(s)$. Conrey and Gamburd found that

(63) $$\Psi_k(N) = C_k (\log N)^{k^2} + O\big((\log N)^{k^2-1}\big)$$

when k is an integer, and a precise value for the constant C_k was given (see the next subsection). For general $k > 0$, one may expect a similar behavior. To this end, it is known from [8] that

(64) $$\Psi_k(N) \asymp_k (\log N)^{k^2}, \quad k > 1/2$$

and that

(65) $$\Psi_k(N) \gg_k (\log N)^{k^2}, \quad k > 0.$$

However, we know only that $\Psi_{1/2}(N) \ll (\log\log N)(\log N)^{1/4}$ and that

$$\Psi_k(N) \ll_k (\log N)^{k/2}, \quad 0 < k < 1/2.$$

Here the upper bounds are established by Helson's theorem on the partial sum operator, and the lower bounds are deduced from Hardy–Littlewood inequalities. We refer to [8] for the details.

There are several remaining problems. The most obvious of these is to get a better upper bound when $0 < k \leq 1/2$. Another problem, to be considered next, is to sharpen the asymptotic bounds in (64); we will obtain fairly precise bounds on the implied constants in this relation.

Unfortunately, we have not been able to improve the estimates in the range $0 < k \leq 1/2$. Instead, we have considered the closely related problem of the pseudomoments of $\zeta^{\alpha}(s)$ for $\alpha > 1$. The somewhat surprising conclusion is that, in this case, the lower bound obtained from the Hardy–Littlewood inequality (the "multiplicative bound") does not give the right asymptotic order for small k.

6.2. **Bounds for the pseudomoments of $\zeta(1/2+it)$ for $k \geq 1$.** For k a positive integer, Conrey and Gamburd [16] computed the constant C_k in (63): They found that $C_k = a_k \gamma_k$, where a_k is an arithmetic factor defined by

$$a_k := \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \frac{c_k^2(j)}{p^j}$$

and γ_k is a geometric factor (the volume of a convex polytope). Bondarenko, Heap, and Seip [8] investigated the asymptotic behavior of $\Psi_k(N)/(\log N)^{k^2}$ and found a lower bound of super-exponential decay using (35) and an upper bound of super-exponential growth using Helson's theorem for the partial sum operator.

From the result in [16] one suspects that super-exponential decay is correct, and this was conjectured in [8, Sec. 5]. We will now verify that for $k \geq 1$, the lower bound is indeed of the correct order. We will do this by replacing the estimates for the partial sum operator with Theorem 3.4. We also include additional details in the computation of the lower estimate from [8] to obtain an explicit lower bound for comparison.

Theorem 6.1. *Suppose that $k \geq 1$. Then*

$$\frac{\Psi_k(N)}{(\log N)^{k^2}} \leq \frac{1}{\Gamma(k+1)^k} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(1 - \frac{k}{[k]}\frac{1}{p}\right)^{-k[k]},$$

$$\frac{\Psi_k(N)}{(\log N)^{k^2}} \geq \frac{1}{\Gamma\big([2k]k+1\big)^{\frac{k}{[2k]}}} \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \left(1 + [2k]k\,\frac{1}{p}\right)^{\frac{k}{[2k]}}.$$

In particular, as $k \to \infty$, we get that

$$\exp\big(\big(-2+o(1)\big)k^2\log k\big) \leq \frac{\Psi_k(N)}{(\log N)^{k^2}} \leq \exp\big(\big(-1+o(1)\big)k^2\log k\big). \tag{66}$$

It is interesting to observe the similarity between the lower bound in (66) and the unconditional bound

$$M_k(T) \geq \exp\big(\big(-2+o(1)\big)k^2\log k\big)\,(\log T)^2$$

obtained by Radziwiłł abd Soundararajan [38]. Likewise, we observe that the upper bound in (66) is in agreement with the expected behavior

$$M_k(T) \sim \exp\big(\big(-1+o(1)\big)k^2\log k\big)\,(\log T)^2,$$

conjectured by Conrey and Gonek [17].

Proof of the upper estimate in Theorem 6.1. Inserting Z_N into (33), we get

$$\Psi_k(N) = \|Z_N\|_{\mathscr{H}^{2k}_{2k}} \le \left(\sum_{n=1}^{N} \frac{d_{[k]}(n)}{n}\left(\frac{k}{[k]}\right)^{\Omega(n)}\right)^{k}.$$

Using Lemma 3.6 and Abel summation, we find that

$$\sum_{n=1}^{N} \frac{d_{[k]}(n)}{n}\left(\frac{k}{[k]}\right)^{\Omega(n)} = \frac{\mathscr{G}_k(1)}{\Gamma(k+1)}(\log N)^k + O\big((\log N)^{k-1}\big).$$

We complete the proof by inspecting the Euler product for $\mathscr{G}_k(1)$ and (39). For the asymptotic estimate, we may safely assume $k \ge 2$, in which case Lemma 3.5 gives $\mathscr{G}_k(1) \asymp 1$. Hence the main contribution to the decay comes from the Gamma function, and the desired result follows from Stirling's formula:

$$\Gamma(k+1)^k = \exp\big((1+o(1))\,k^2\log k\big). \qquad \square$$

The following argument can be extracted from [8, pp. 201–202], but we include some details here for the reader's benefit.

Proof of the lower estimate in Theorem 6.1. We want to use (35), but $k = p/2 \ge 1$. To remedy this, we write $2k = \ell r$ where $\ell \ge [2k]$ is an integer to be chosen later that ensures that $r < 2$. Note that if $n \le N$, then

$$\frac{|\mu(n)|}{d_{2/r}(n)}\left|\sum_{\substack{n_1\cdots n_\ell = n\\ n_1,\ldots,n_\ell \le N}} \frac{1}{\sqrt{n_1}}\cdots\frac{1}{\sqrt{n_k}}\right|^2 = \frac{|\mu(n)|}{d_{2/r}(n)}\,\frac{d_\ell^2(n)}{n} = \frac{|\mu(n)|}{n}\,d_{\ell k}(n).$$

Using (35) and removing all terms in the sum for which $N < n \le N^\ell$, we get the lower bound

$$\|Z_N\|_{2k}^{2k} = \|Z_N^\ell\|_r^r \ge \left(\sum_{n=1}^{N}\frac{|\mu(n)|}{n}d_{\ell k}(n)\right)^{\frac{k}{\ell}}.$$

As above, one checks that

$$\sum_{n=1}^{N}\frac{|\mu(n)|}{n}d_{\ell k}(n) = \widetilde{C}_k(\log N)^{\ell k} + O\big((\log N)^{\ell k-1}\big)$$

with

$$\widetilde{C}_k = \frac{1}{\Gamma(\ell k+1)}\prod_p\left(1-\frac{1}{p}\right)^{\ell k}\left(1+\frac{\ell k}{p}\right). \tag{67}$$

The asymptotic behavior of the Euler product in (67) has been estimated in [8, p. 202], where it was found that

$$\prod_p \left(1-\frac{1}{p}\right)^{\ell k}\left(1+\frac{\ell k}{p}\right) = \exp\big(\big(-1+o(1)\big)\ell k \log\log(\ell k)\big)\,.$$

Therefore the decay is again controlled by $\Gamma(\ell k+1)^{k/\ell}$. Clearly, choosing ℓ as small as possible is optimal, and we therefore set $\ell = [2k]$. The proof is completed by similar considerations as in the preceding argument. □

Theorem 3.4 allows us to improve the more general results of [8] concerning Dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \psi(n) n^{-1/2-s}$$

for a suitable multiplicative function $\psi(n)$, in the same way as done above for the Riemann zeta function, as well as to relax the presumed growth condition on ψ for $k \geq 3$. Since the computations go through as before, we refrain from carrying out the details.

6.3. **Pseudomoments of $\zeta^\alpha(s)$ for $\alpha > 1$ and small k.** We define the pseudomoments of $\zeta^\alpha(s)$ as $\Psi_{k,\alpha}(N) := \|Z_{N,\alpha}\|^{2k}_{\mathscr{H}^{2k}}$, where

$$Z_{N,\alpha}(s) := \sum_{n\leq N} d_\alpha(n) n^{-s-1/2}.$$

Letting F_N be as defined in (2), we see that then $Z_{N,\alpha} = (S_N F_N^\alpha)(s)$. We know from [8] that these pseudomoments satisfy the relation

$$\Psi_{k,\alpha}(N) \asymp (\log N)^{k^2\alpha^2} \tag{68}$$

when $k > 1/2$. We will now show that this result fails for small $k < 1/2$ when $\alpha > 1$. If we agree that the moments of $\zeta^\alpha(s)$ are just the moments of $|\zeta(s)|^\alpha$, then we see that our result implies that, on the Riemann hypothesis, there is a discrepancy between the behavior of the pseudomoments and the moments of $\zeta^\alpha(s)$ for small k when $\alpha > 1$.

Theorem 6.2. *Suppose that $\alpha \geq 1$. For every $k > 0$, there exists a constant $c(k)$ such that*

$$\Psi_{k,\alpha}(N) \gg (\log N)^{k\log\alpha^2} \exp\left(-c(k)\sqrt{\log\log N \log\log\log N}\right)$$

holds for arbitrarily large N.

This is incompatible with (68) when $\alpha > 1$ and $k < (\log \alpha^2)/\alpha^2$. From this we observe that, whenever $k < 1/e$, we can find $\alpha > 1$ such that (68) fails.

We prepare for the proof of Theorem 6.2 by establishing two lemmas.

Lemma 6.3. *Suppose that $\alpha \geq 1$. Then*

$$\mathbb{E}\left|\sum_{M/2<n\leq M} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right| \gg (\log\log M)^{-3+o(1)},$$

where the implicit constant $o(1)$ depends only on M.

Here we applid the probabilistic notation of Subsection 2.4. We defer the proof of Lemma 6.3 until the end of this subsection.

Our second lemma is a result on the distribution of

$$N(x,m) := \sum_{n\leq x,\, \Omega(n)=m} 1,$$

similar in spirit to the Erdős–Kac theorem, saying that $N(x,m)$ is mainly concentrated on the set

$$I_C := \left[\log\log x - C\sqrt{\log\log x \log\log\log x},\, \log\log x + C\sqrt{\log\log x \log\log\log x}\right]$$

when x is large and C is a suitable positive constant. To deduce this result, we rely on an estimate of Sathe (see [47]) saying that

$$N(x,m) \leq C\frac{x}{\log x}\frac{(\log\log x)^{m-1}}{(m-1)!} \tag{69}$$

whenever $x > 10$ and $1 \leq m \leq (3/2)\log\log x$, with C an absolute constant. Choosing C large enough and using Stirling's formula, we therefore find that

$$\sum_{m\leq(3/2)\log\log x,\, m\notin I_C} N(x,m) \leq \frac{x}{2(\log\log x)^8} \tag{70}$$

when x is sufficiently large. Using instead of (69) formula (7) from [33], we deduce that

$$\sum_{m\geq(3/2)\log\log x} N(x,m) \leq \frac{x}{(\log x)^{1/100}} \tag{71}$$

for x large enough. Combining (70) and (71), we obtain the following.

Lemma 6.4. *There exists an absolute constant $C > 0$ such that*

$$\sum_{m\notin I_C} N(x,m) \leq \frac{x}{(\log\log x)^8}$$

for all sufficiently large x.

Proof of Theorem 6.2. We write

$$D_{N,\alpha}(s) := \sum_{N/2<n\leq N} d_\alpha(n)\alpha^{-\Omega(n)}n^{-s-1/2}$$

so that

$$Z_{N,\alpha}(s) - Z_{N/2,\alpha}(s) = \sum_{m\geq 0} \alpha^m P_m D_{N,\alpha}(s).$$

By Lemma 4.4, we have for every m and $0 < q < 1$

$$\|Z_{N,\alpha} - Z_{N/2,\alpha}\|_q \gg \alpha^m m^{1-1/q} \|P_m D_{N,\alpha}\|_q. \tag{72}$$

We will combine (72) with an estimate that we obtain from the two lemmas above.

In what follows, we will use that the L^2 norm of $D_{N,\alpha}$ can be estimated in a trivial way because $d_\alpha(n)\alpha^{-\Omega(n)} \leq 1$. First, applying Hölder's inequality in the form

$$\|f\|_1^{2-q} \leq \|f\|_q^q \|f\|_2^{2-2q}$$

along with Lemma 6.3 and a trivial L^2 estimate, we find that

$$\Big\| \sum_{m\geq 0} P_m D_{N,\alpha} \Big\|_q^q \gg (\log\log N)^{-6+o(1)}$$

whenever $0 < q < 1$. Using the triangle inequality for the L^q quasi-norm and the trivial bound $\|f\|_q \leq \|f\|_2$, we obtain from this that

$$\sum_{m\in I_C} \big\|P_m D_{N,\alpha}\big\|_q^q + \Big\| \sum_{m\notin I_C} P_m D_{N,\alpha} \Big\|_2^q \gg (\log\log N)^{-6+o(1)}.$$

Hence, by a trivial L^2 bound and an application of Lemma 6.4, there exists a constant C such that

$$\sum_{m\in I_C} \big\|P_m D_{N,\alpha}\big\|_q^q \gg (\log\log N)^{-6+o(1)}.$$

Thus, since $|I_C| = O(\sqrt{\log\log N \log\log\log N})$, there exists an m satisfying

$$\log\log N - C\sqrt{\log\log N \log\log\log N} \leq m \leq \log\log N + C\sqrt{\log\log N \log\log\log N}$$

such that

$$\|P_m D_{N,\alpha}\|_q^q \geq (\log\log N)^{-6.5+o(1)}. \tag{73}$$

We now set $q = 2k$. Combining (72) and (73), we find that for some $c(k,\alpha)$

$$\|Z_{N,\alpha} - Z_{N/2,\alpha}\|_{2k}^{2k} \gg (\log N)^{k\log\alpha^2} \exp\Big(-c(k,\alpha)\sqrt{\log\log N \log\log\log N}\Big).$$

Since

$$\|Z_{N,\alpha} - Z_{N/2,\alpha}\|_{2k}^{2k} \leq \|Z_{N,\alpha}\|_{2k}^{2k} + \|Z_{N/2,\alpha}\|_{2k}^{2k},$$

this means that at least one of the pseudomoments $\Psi_{k,\alpha}(N/2)$ or $\Psi_{k,\alpha}(N)$ satisfies the lower bound asserted by the theorem. □

Proof of Lemma 6.3. Let N_x be the set of x-smooth numbers, i.e.,

$$N_x := \{n \in \mathbb{N} : p \text{ a prime such that } p|n \ \Rightarrow p \le x\}.$$

We start with the following identity which holds for every real t:

$$\begin{aligned}\int_1^\infty \frac{\sum_{y/2<n\le y, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}}{y^{1+1/\log x+it}}dy \\ = \left(\frac{1-2^{-1/\log x-it}}{1/\log x+it}\right)\sum_{n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2-1/\log x-it}.\end{aligned} \tag{74}$$

Our first goal is to estimate the supremum of the right hand side in (74) for t from a reasonably short interval. We have

$$\begin{aligned}\Big|\sum_{n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2-1/\log x-it}\Big| \\ = \prod_{p\le x}\Big|1+\sum_{j=1}^\infty c_\alpha(j)\alpha^{-j}z(p)^jp^{-j(1/2+1/\log x+it)}\Big| \\ \asymp \exp\left(\operatorname{Re}\left(\sum_{p\le x} z(p)p^{-1/2-1/\log x-it}\right)\right. \\ \left.+\frac{1}{2\alpha}\operatorname{Re}\left(\sum_{p\le x} z(p)^2p^{-1-2/\log x-2it}\right)\right)\end{aligned} \tag{75}$$

for all points of the configuration space $(z(p))_{p\le x}$. As in [26, Lem. 1], we can modify the proof of [25, Cor. 2] to show that

$$\begin{aligned}\sup_{\substack{1\le t\le 2(\log\log x)^2\\ |1-2^{-it}|\ge 1/4}}\left(\operatorname{Re}\left(\sum_{p\le x} z(p)p^{-1/2-1/\log x-it}\right)\right. \\ \left.+\frac{1}{2\alpha}\operatorname{Re}\left(\sum_{p\le x} z(p)^2p^{-1-2/\log x-2it}\right)\right) \\ \ge \log\log x-\log\log\log x+O((\log\log\log x)^{3/4})\end{aligned}$$

with probability $1-o(1)$ as $x\to\infty$. To achieve this, we add a minor technical detail: In the part of the argument that follows [25, Sec. 6], we only take into account those integers n, $1\le n\le(\log\log x)^2$, such that

$$\min_{2n+1\le t\le 2n+2}|1-2^{-it}|\ge 1/4,$$

noting that the number of such n is bounded below by $C(\log\log x)^2$. Combining the latter inequality with (75), we obtain that with probability $1-o(1)$

$$\sup_{\substack{1\le t\le 2(\log\log x)^2\\ |1-2^{-it}|\ge 1/4}} \left|\sum_{n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2-1/\log x-it}\right| \ge \log x(\log\log x)^{-1+o(1)}.$$

Now taking the supremum of the absolute value of both sides in (74), we find that

$$\int_1^\infty \frac{\left|\sum_{y/2<n\le y, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|}{y^{1+1/\log x}}dy \ge \log x(\log\log x)^{-3+o(1)}$$

with probability $1-o(1)$. Hence taking the expectation over the entire configuration space $(z(p))_{p\le x}$, we finally obtain that, say for all $x>3$,

$$\int_1^\infty \frac{\mathbb{E}\left|\sum_{y/2<n\le y, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|}{y^{1+1/\log x}}dy \ge \log x(\log\log x)^{-3+o(1)}. \tag{76}$$

Now we will show that the assertion of the lemma follows from (76). To this end, we begin by fixing a positive integer M. We will use (76) for x such that $M=x^{10\log\log\log x}$. Applying the Cauchy–Schwarz inequality in the form $(\mathbb{E}|X|)^2\le \mathbb{E}|X|^2$ and recalling that $d_\alpha(n)\alpha^{-\Omega(n)}\le 1$, we find that

$$\int_{\sqrt{M}}^\infty \frac{\mathbb{E}\left|\sum_{y/2<n\le y, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|}{y^{1+1/\log x}}dy \le \sqrt{2}\int_{\sqrt{M}}^\infty \frac{1}{y^{1+1/\log x}}dy = \sqrt{2}\log x(\log\log x)^{-5}.$$

Combining this bound with (76), we find that

$$\int_1^{\sqrt{M}} \frac{\mathbb{E}\left|\sum_{y/2<n\le y, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|}{y^{1+1/\log x}}dy \ge \log x(\log\log x)^{-3+o(1)}, \tag{77}$$

which is the relation to be used below.

Set $S_{M,\alpha}(z):=\sum_{M/2<n\le M} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}$, and let $N_x^\perp$ be the set of integers with prime divisors that are all larger than x. Write

$$S_{M,\alpha}(z)=\sum_{n\in N_x^\perp, 1\le n\le M} c_n d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2},$$

where

$$c_y := \sum_{k\in N_x, M/(2y)\le k\le M/y} d_\alpha(k)\alpha^{-\Omega(k)}z(k)k^{-1/2}.$$

By Helson's inequality (31), we find that

$$\begin{aligned}\mathbb{E}|S_{M,\alpha}| &\ge \mathbb{E}\left(\sum_{n\in N_x^{\perp},1\le n\le M}\frac{|c_n|^2 d_\alpha(n)^2\alpha^{-2\Omega(n)}}{d(n)n}\right)^{1/2}\\ &\ge \mathbb{E}\left(\sum_{x<p\le M}\frac{|c_p|^2}{2p}\right)^{1/2}.\end{aligned}\tag{78}$$

We now want to relate the right-hand side of (78) to the integral

$$\int_x^M\left|\sum_{M/(2y)\le n\le M/y,n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|^2\frac{dy}{y}=\int_x^M|c_y|^2\frac{dy}{y}.$$

To this end, we begin by considering a short interval $[\xi,\xi+\xi^\delta]\subset[x,M]$, where $7/12<\delta<1$ is a fixed parameter. If ξ is sufficiently large, then by [28], this interval contains at least $\xi^\delta/(2\log\xi)$ primes. We partition accordingly the interval into $[\xi^\delta/(2\log\xi)]$ subintervals of equal length $\xi^\delta/[\xi^\delta/(2\log\xi)]$. We make a one-to-one correspondence between these subintervals and the first $[\xi^\delta/(2\log\xi)]$ primes in $[\xi,\xi+\xi^\delta]$, and hence we associate with every y in $[\xi,\xi+\xi^\delta]$ a prime $p=p(y)$ that is also in $[\xi,\xi+\xi^\delta]$. We write $\tilde{c}_y:=c_y-c_{p(y)}$ and notice that

$$|c_y|^2\le 2\big(|c_{p(y)}|^2+|\tilde{c}_y|^2\big),$$

where $\mathbb{E}|\tilde{c}_y|^2\ll\xi^{\delta-1}\ll y^{\delta-1}$. From this we get that

$$\begin{aligned}&\int_\xi^{\xi+\xi^\delta}\left|\sum_{M/(2y)\le n\le M/y,n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|^2\frac{dy}{y}\\ &\qquad\ll(\log\xi)\sum_{\xi\le p\le\xi+\xi^\delta}\frac{|c_p|^2}{p}+\int_\xi^{\xi+\xi^\delta}\frac{|\tilde{c}_y|^2}{y}dy.\end{aligned}$$

Repeating this construction and summing over a suitable collection of intervals $[\xi,\xi+\xi^\delta]$, we then obtain

$$\begin{aligned}&\sum_{x<p\le M}\frac{|c_p|^2}{p}+\int_x^M\frac{|\tilde{c}_y|^2}{y}dy\\ &\qquad\gg\frac{1}{\log M}\int_x^M\left|\sum_{M/(2y)\le n\le M/y,n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)}z(n)n^{-1/2}\right|^2\frac{dy}{y}.\end{aligned}$$

By the change of variables $u = M/y$ in the integral on the right-hand side and using that $\log M = 10 \log x \log\log\log x$, we now deduce that

$$\begin{aligned}
&\sum_{x<p\le M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \\
&\gg \frac{1}{\log x \log\log\log x} \int_1^{M/x} \Bigg| \sum_{u/2\le n\le u, n\in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \Bigg|^2 \frac{du}{u}.
\end{aligned} \tag{79}$$

We are now ready to finish the proof by putting our three basic estimates (77), (78), and (79) together. First, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int_1^{M/x} \Bigg| \sum_{u/2\le n\le u, n\in N_x} d_\alpha(n) \alpha^{-\Omega(n)} z(n) n^{-1/2} \Bigg|^2 \frac{du}{u} \int_1^{M/x} \frac{1}{u^{1+2/\log x}} du \\
&\qquad \ge \left(\int_1^{\sqrt{M}} \frac{\left| \sum_{u/2\le n\le u, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{u^{1+1/\log x}} du. \right)^2
\end{aligned}$$

Therefore, taking expectation in (79) and applying (78) together with (77), we find that

$$\begin{aligned}
\mathbb{E}|S_M| &\gg \mathbb{E}\Big| \Big(\sum_{x<p\le M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \Big)^{1/2} \Big| - \mathbb{E}\Big| \Big(\int_x^M \frac{|\tilde{c}_y|^2}{y} dy \Big)^{1/2} \Big| \\
&\gg \mathbb{E}\Big| \Big(\sum_{x<p\le M} \frac{|c_p|^2}{p} + \int_x^M \frac{|\tilde{c}_y|^2}{y} dy \Big)^{1/2} \Big| - x^{-(1-\delta)/2} \\
&\gg \frac{1}{\log x (\log\log\log x)^{1/2}} \\
&\quad \times \int_1^{\sqrt{M}} \frac{\mathbb{E} \left| \sum_{u/2\le n\le u, n\in N_x} d_\alpha(n)\alpha^{-\Omega(n)} z(n) n^{-1/2} \right|}{u^{1+1/\log x}} du - x^{-(1-\delta)/2} \\
&\ge (\log\log x)^{-3+o(1)} \ge (\log\log M)^{-3+o(1)},
\end{aligned}$$

and hence the desired estimate has been established. ∎

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