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Neutron Stars

Study of the Mass-Radius Relation and Mean-Field Approaches to the Equation of State

Francesco Pogliano

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Supervisor: Jens Oluf Andersen, IFY

Norwegian University of Science and Technology
Department of Physics

Abstract

The structure and the relation between the total mass, the radius and the central energy density of a neutron star may be found by numerically solving a set of three coupled differential equations. One of these equations is the equation of state, relating the pressure to the energy density. In this thesis, after introducing some important concepts of general relativity, quantum mechanics, quantum field theory and thermal field theory, we will discuss the equation of state using different models. The first is the σ–ω model, later expanded to include leptons and the ρ meson in what in the literature is referred to as $npe\mu$ matter. In the last part we also consider the shift in the vacuum energy due to the presence of matter. Some focus has been given to the first-order phase transition in neutron matter conceived by Chin & Walecka [7]. Although unphysical, the theory behind the phase transition is a first step for understanding more complex phase transitions between hadronic and quark matter in hybrid stars.

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Chapter 1

Introduction

By the word “star” we usually mean enormous celestial bodies made of plasma, held together by gravity and glowing of the light originating from the nuclear fusion in their cores. Although this is indeed the case, it is not the whole truth. The description above is in fact true only for the middle phase of what we may call the “lifetime” of a star, which can be subdivided into three main phases: birth, life and death.
Stars are usually born from the gravitational collapse of parts of a gaseous nebula mainly consisting of hydrogen. When contracting, the density and the temperature increase allowing for the nuclear fusion of hydrogen atom nuclei to happen at its core. This in turn releases energy, increases temperatures even more and eventually results in an enormous, continuous explosion exerting a pressure keeping the sphere of gas from further collapse. When a stable equilibrium between these two pressures is reached, we are into the second phase of the star’s lifetime. This phase may continue for millions or billions of years, until the amount of hydrogen is no longer enough to sustain the nuclear reaction in its core. When this happens, the exerted pressure diminishes and gravity wins, resulting in the collapse of the star. While this implosion may result in higher pressures and temperatures triggering another thermonuclear reaction not previously possible, a point is eventually reached when the attained pressures are either not big enough to sustain the next step of nuclear fusion, or fusion becomes energetically unfavorable leading to the further gravitational collapse of the star’s core. We enter then the third and final phase: the death of the star. In the case when the newly attained pressure do not allow further nuclear fusion to happen, the core becomes an inert collection of atomic nuclei in a sea of electrons, while the outer layers dissipate in a stellar nebula. These core remnants are called white dwarfs: stars held together by gravity and kept from collapse by the Pauli pressure exerted by electrons. This will be the fate of the Sun and of stars that do not exceed approximately three times its mass. The fusion of atomic nuclei heavier than ^{56}Fe is energetically unfavorable, and when a star reaches pressures that allow such a nuclear reaction, it will result in the collapse of its core and the release of enormous amounts of gravitational energy. This in turn provokes the expulsion of the outer layers of the former star: a big explosion, called supernova. While the implosion of the cores of many massive stars results in black holes,

some have just enough mass to win over the Pauli pressure of electrons, but not enough to overcome the one of the nucleons. These remnants are neutron stars.

A neutron star's typical mass lies around 1.5 solar masses, and its radius on the order of 10 km. Since their density is comparable to the one of atomic nuclei, it makes them fully relativistic objects that have to be described in terms of special and general relativity. They are very interesting physical objects as they show conditions that are impossible to emulate on Earth. Our theoretical understanding neutron stars is limited to the extrapolation of the behavior of physical laws we know working for previously observed systems in our laboratories. On the other hand, these star present us with the possibilities of testing these laws and possibly improve them, giving us a tool to observe matter at very high densities.

Although the idea of giant atomic nuclei in space is attributed to Landau in 1931 [27], it was Baade and Zwicky in 1934 who proposed the existence of stars mainly consisting of neutrons as a result of gravitational collapse [1]. The first observation of a variable radio source was by Hewish et al. in 1967 [21], identified later as a pulsar, or rapidly rotating neutron star, for which he was rewarded the Nobel prize in 1975. Since then many other neutron stars have been discovered, increasing the amount of theoretical research done to explain their structure.

This Master's thesis will look into the different models for the structure of neutron stars, and the implied relationship between the central density, radius and total mass. The first chapters will mostly lay the theoretical foundations which will be applied in the rest of the thesis. In Chapter 2 general relativity and the Tolman-Oppenheimer-Volkoff equations are introduced; these, together with the equation of state, are the framework we will use in order to calculate the above mentioned relationship. The equation of state can be derived by what we assume the neutron star is made of, how the constituents interact with each other and the kind of approximation we are using. In Chapter 2 we will also explore the most simple models for a neutron star, idealized as made of cold, relativistic Fermi gas of non-interacting neutrons, based on the 1939 seminal works of Tolman [40], Oppenheimer and Volkoff [29]. In Chapter 3 we will introduce the reader to the path integral formulation of quantum field theory as well as finite-temperature thermal field theory. Chapter 4 improves the model of neutron stars by allowing for interactions between nucleons. This is the σ–ω model (or Walecka model [41]), which the improved $npe\mu$ model of Chapter 5 is based from. Lastly in Chapter 6 the vacuum fluctuations are considered, together with their contribution to the equation of state and the mass-radius relation.

On the front cover are shown the mass-radius relations for each of the models. The magenta line shows the mass-radius relation according to the model of a Fermi gas of noninteracting cold neutrons, the blue line according to the σ–ω model, the green line to the $npe\mu$ model and the cyan one to the renormalized $npe\mu$ model for slightly modified nuclear properties. The dashed lines represent the respective unstable solutions for each model.

Chapter 2

The Tolman-Oppenheimer-Volkoff equation

This chapter derives the Tolman-Oppenheimer-Volkoff (TOV) equation, essential for analyzing the curvature of spacetime inside a neutron star and its structure. The chapter is divided in three parts. The first part introduces important theoretical aspects as classical mechanics, special and general relativity. This will be needed in the second part in order to derive the TOV equation. Finally in the last part we will apply this equation to a naive model of a neutron star only consisting of noninteracting neutrons.

2.1 Theoretical background

2.1.1 Classical mechanics

In classical mechanics we can obtain the equations of motions of a system using the *variational principle* (also known as *Hamilton's principle*). The principle says that when the *action* S of a system is extremized with respect to some generalized coordinates $\boldsymbol{q}(t)$, we obtain the differential equations describing its dynamics. The generalized coordinates are the available degrees of freedom of the system, i.e. the degrees of freedom of a free system minus its constraints [14]. The principle is expressed as

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = 0. \tag{2.1}$$

The action S must be expressed in the same coordinates, and it is defined as the integral of the *Lagrangian* function L in the time interval between t_1 and t_2,

$$S \equiv \int_{t_1}^{t_2} L\big[\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\big]\mathrm{d}t. \tag{2.2}$$

The Lagrangian is defined as a function describing all the physical properties and forces acting on the system taken into consideration. It often takes the form

$$L = T - V, \tag{2.3}$$

where T and V are respectively the expressions for the kinetic and the potential energy of the system. The equations of motion can be found by directly applying the variational principle to the action. Most of the times it is convenient to use the Euler-Lagrange equations, which are the conditions the Lagrangian must satisfy in order to extremize the action:

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \delta L\left(\boldsymbol{q}(t), \dot{\boldsymbol{q}}, t\right) \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \boldsymbol{q}(t)} \delta \boldsymbol{q}(t) + \frac{\partial L}{\partial \dot{\boldsymbol{q}}(t)} \delta \dot{\boldsymbol{q}}(t)\right] \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \boldsymbol{q}(t)} \delta \boldsymbol{q}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{q}(t)} \delta \boldsymbol{q}(t)\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}(t)}\right) \delta \boldsymbol{q}(t)\right] \mathrm{d}t \\
&= \frac{\partial L}{\partial \boldsymbol{q}(t)} \delta \boldsymbol{q}(t)\bigg|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \boldsymbol{q}(t)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}(t)}\right)\right] \delta \boldsymbol{q}(t) \mathrm{d}t = 0.
\end{aligned}$$

In the last step we can evaluate the first term at the boundary points, where $\delta \boldsymbol{q}(t)$ is zero by assumption. This tells us that the variational principle is satisfied when the integrand in the second term is set to zero, yielding the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}(t)} = \frac{\partial L}{\partial \boldsymbol{q}(t)}. \tag{2.4}$$

2.1.2 Scalar fields

A *scalar field* $\phi(\boldsymbol{q}, t)$ is a function that associates a scalar value to every point in space and time it is defined in; these are physical systems and can be described using the Lagrangian formalism [38]. Since these maps are defined in a volume V rather than a single point, we introduce the Lagrangian density $\mathcal{L}$:

$$L \equiv \int_V \mathcal{L} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{2.5}$$

We now define $\mathrm{d}v = \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z$ as the infinitesimal three dimensional volume times the infinitesimal time, and $\Omega = V \times [t_1 : t_2]$ a the three dimensional volume times the time interval where the integration of action is taken. By eliminating every explicit time dependence in the Lagrangian density, (everything should be described by the fields and their derivatives) we can express the action specifically for scalar fields in this way:

$$S = \int_\Omega \mathcal{L}\big[\phi(q, t), \nabla\phi(q, t), \partial_t \phi(q, t)\big] \mathrm{d}v. \tag{2.6}$$

2.1.3 Geometry of spacetime

In the following sections we will introduce the geometry of spacetime and general relativity. This theoretical part is largely inspired by the description of these subjects in the classical textbooks of general relativity from James B. Hartle [19] and Steven Weinberg [42]. In special and general relativity we consider time to be a dimension like the spatial ones. Points in spacetime need four coordinates in order to be specified and are called *events*, where the time coordinate is inserted at the zeroth index. Consequently normal three-dimensional position vectors will be changed to four-vectors. While time intervals and spatial lengths may be relative to the frame of reference according to the special theory of relativity, we can find quantities that are *invariant*, meaning that they are the same in every such frame. One of the most useful invariants is the length of a four dimensional distance vector between two events. A *line element* $\mathrm{d}s$ is the infinitesimal version of such four dimensional distance, and can be used to describe the geometry of spacetime. For flat spacetime this is defined as:

$$\mathrm{d}s^2 = (c\mathrm{d}t)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{2.7}$$

In special and general relativity it is common to operate with $c = 1$ units, where the first term in equation (2.7) becomes $\mathrm{d}t^2$. With this notation we introduce the flat spacetime metric $\eta_{\mu\nu}$, describing the coefficients in front of the terms in $\mathrm{d}s^2$. The metric is a second degree tensor defined as:

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.8}$$

for which the line element described in equation (2.7) can be rewritten as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{2.9}$$

We are using the Einstein convention of repeated indices, so a sum is implied over both μ and ν. In special and general relativity is also usual to indicate a *covariant* tensor with low indices (like $g_{\mu\nu}$) and a *contravatiant* tensor with high indices (like both $\mathrm{d}x^\mu$ and $\mathrm{d}x^\nu$ in 2.9). For curved spacetimes the line element $\mathrm{d}s^2$ is still invariant, but the metric may be different. In curved spacetimes it will be expressed as $g_{\mu\nu}(x)$: a symmetric, position and time dependent, second rank tensor[1]. The line element in curved spacetime will be then given by

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{2.10}$$

By assuming a diagonal $g_{\mu\nu}$[2], we are able to find volume elements in curved spacetimes. The metric in equation (2.10) tells us that for lengths along the x-direction, we will have $\mathrm{d}l^1 = \sqrt{g_{11}(x)}\mathrm{d}x^1$. This is of course valid for every direction with the special case of

[1] When dealing with functions of the coordinates, $f(x)$ will be shorthand for $f(x^0, x^1.x^2, x^3)$, as there is no danger of confusion.

[2] This leads to no loss of generality: every symmetric matrix can be diagonalised.

the one along time, where $\mathrm{d}l^0 = \sqrt{-g_{00}(x)}dx^0$. We can now define what will be the infinitesimal, position and time dependent volume element in curved spacetime:

$$\mathrm{d}v = \sqrt{-g_{00}(x)g_{11}(x)g_{22}(x)g_{33}(x)}\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3. \tag{2.11}$$

If we define $g(x)$ to be the determinant of $g_{\mu\nu}(x)$ considered as a matrix, we can express the four-volume as

$$\mathrm{d}v(x) = \sqrt{-g(x)}\mathrm{d}^4x. \tag{2.12}$$

Equation (2.12) is valid even for non-diagonal matrices. From now on we will drop the x dependence in the metric tensor $g_{\mu\nu}(x)$ and its determinant $g(x)$.

2.1.4 Symbols and definitions

Special and general relativity make use of many symbols and shorthands that are useful for describing its laws in a compact and clear way. The *proper time* τ is the time measured in the frame of reference of a system and is one of the most fundamental concepts in special relativity. It is related to the line element $\mathrm{d}s$ by the identity

$$\mathrm{d}\tau^2 \equiv \mathrm{d}s^2. \tag{2.13}$$

Since time is relative to the frame of reference, most of the laws in relativity are expressed in terms of the proper time. Particles moving freely in spacetime follow paths called *geodesics*. These are described by the *geodesic equation* which makes use of one of the most useful tools in general relativity: the Christoffel symbols $\Gamma^\delta_{\beta\gamma}$. These are defined as

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right). \tag{2.14}$$

With this definition we can express the geodesic equation as

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}. \tag{2.15}$$

In relativity, operations between vectors are defined only at one event. This means we need a new definition of derivative for four-vectors in curved spacetime. This is called *covariant derivative* and for a generic four-vector v^β is defined as

$$\nabla_\alpha v^\beta = \frac{\partial v^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma}v^\gamma. \tag{2.16}$$

With this new tool, we are able to write the geodesic equation more elegantly. By introducing the tangent vector $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ we have

$$\nabla_\alpha u^\alpha = 0. \tag{2.17}$$

A way to describe spacetime curvature is the *Riemann curvature tensor*, a fourth rank tensor defined as

$$R^\alpha_{\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\epsilon}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon}\Gamma^\epsilon_{\beta\gamma}. \tag{2.18}$$

The *Ricci curvature* is a second rank tensor defined as

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}, \tag{2.19}$$

from which we can also define the *Ricci salar* R

$$R \equiv g_{\alpha\beta} R^{\alpha\beta}. \tag{2.20}$$

The *stress-energy tensor* $T_{\alpha\beta}$ is a 4×4 second rank tensor describing the distribution of energy and momentum in spacetime. It can be represented like this:

$$T_{\alpha\beta} = \left(\begin{array}{c|c} \begin{matrix} energy \\ density \end{matrix} & \begin{matrix} energy \\ flux \end{matrix} \\ \hline \begin{matrix} mom. \\ density \end{matrix} & \begin{matrix} stress \\ tensor \end{matrix} \end{array} \right). \tag{2.21}$$

Finally we introduce the *cosmological constant* Λ, which accounts for the energy density in the vacuum.

The Einstein field equations

With the Ricci curvature in (2.19), the Ricci scalar in (2.20) the stress-energy tensor, the speed of light c, the cosmological constant Λ and the gravitational constant G, we are able to write the Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{2.22}$$

We can do a final simplification defining the *Einstein curvature tensor* as

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}, \tag{2.23}$$

with which we can rewrite the field equations in the more compact (and known) form

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{2.24}$$

2.1.5 The Einstein-Hilbert action

We can obtain the equations of motion of a system by knowing its Lagrangian, and by then using the principle of least action in (2.1). This suggests that a carefully chosen Lagrangian could lead us the Einstein field equations in (2.24), the equations of motion of general relativity. The action built from this Lagrangian is called the *Einstein-Hilbert action*, and we will label it as S_{EH}. The Einstein field equations are to be considered as fundamental laws of nature, this because to this day we do not have more fundamental laws to derive it from. It is nevertheless possible to express the same laws in the Lagrangian formalism. According to Hamilton's principle, a variation of the Einstein-Hilbert action in the metric will be zero:

$$\frac{\delta S_{\text{EH}}}{\delta g_{\mu\nu}} = 0. \tag{2.25}$$

The Einstein equations in (2.24) can be seen as consisting of three terms: the distribution of mass and energy $T_{\mu\nu}$, the curvature of spacetime $G_{\mu\nu}$ and the cosmological constant Λ. This means that we can try to split the Einstein-Hilbert action into a *cosmological term* S_Λ, a *matter term* S_{M} and a *curvature term* S_{C}:

$$S_{\text{EH}} = S_\Lambda + S_{\text{M}} + S_{\text{C}}. \tag{2.26}$$

Using the definition of the four volume element in (2.12) we can define S_Λ to be

$$S_\Lambda = -2\int \Lambda\sqrt{-g}d^4x. \tag{2.27}$$

By the definition of Lagrangian density in (2.5), the matter term of S_{EH} can be expressed in a general way introducing a *matter Lagrangian density* $\mathcal{L}_{\text{M}}$ such that

$$S_{\text{M}} = 2\kappa\int \mathcal{L}_{\text{M}}\sqrt{-g}\text{d}^4x. \tag{2.28}$$

When it comes to S_{C}, will see that expressing it as

$$S_{\text{C}} = \int R\sqrt{-g}\text{d}^4x, \tag{2.29}$$

will lead to the curvature term in the Einstein field equations. in (2.28) is $\kappa = 8\pi G$, G the gravitational constant and R the Ricci scalar as defined in (2.20). The variation in the Einstein-Hilbert action can be written as

$$\delta S_{\text{EH}} = \delta S_\Lambda + \delta S_{\text{M}} + \delta S_{\text{C}} = 0, \tag{2.30}$$

and we will now see how the above choices will lead us to the Einstein field equations. We will consider the three terms separately in the following sections. All the derivations here follow the steps described in [42].

The cosmological term

A variation in the metric in the cosmological term of the Einstein Hilbert equation will give us

$$\begin{aligned}\delta S_\Lambda = \frac{\delta S_\Lambda}{\delta g^{\mu\nu}}\delta g^{\mu\nu} &= -\int \frac{\delta(2\Lambda\sqrt{-g})}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\text{d}^4x \\ &= -2\Lambda\int \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\text{d}^4x, \end{aligned}\tag{2.31}$$

where we can take the 2Λ term out of the integration since constant by definition. The $\delta\sqrt{-g}/\delta g^{\mu\nu}$ fraction can be calculated using Jacobi's formula. Considering $\boldsymbol{g}$ as a matrix, Jacobi's formula tells us that

$$\delta \boldsymbol{g} = \mathrm{Tr}\Big(\mathrm{adj}\,(\boldsymbol{g})\,\delta\boldsymbol{g}\Big).$$

The metric $\boldsymbol{g}$ is invertible, thus $\mathrm{adj}(\boldsymbol{g}) = \det(\boldsymbol{g})\boldsymbol{g}^{-1}$, and renaming the determinant of $\boldsymbol{g}$ as $\det(\boldsymbol{g}) = g$, we obtain

$$\delta g = \mathrm{tr}(g\boldsymbol{g}^{-1}\delta\boldsymbol{g}) = g g^{\alpha\beta}\delta g_{\alpha\beta}. \tag{2.32}$$

By also using that $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$, we get

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{-g}{2\sqrt{-g}}\frac{g^{\alpha\beta}\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2}\frac{g^{\alpha\beta}\delta g_{\alpha\beta}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}\frac{g_{\alpha\beta}\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}. \tag{2.33}$$

Plugging in the result from (2.33) into (2.31), we obtain

$$\delta S_\Lambda = \int \Lambda g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x. \tag{2.34}$$

The matter term

We now consider the matter term in S_{EH}, namely equation (2.28). A variation in the metric $g_{\mu\nu}$ will give us:

$$\begin{aligned}\delta S_{\mathrm{M}} = \frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} &= 2\kappa\int\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{M}})}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x\\ &= 2\kappa\int\left(\frac{\mathcal{L}_{\mathrm{M}}}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta\mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}}\right)\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x.\end{aligned}$$

The $\delta\sqrt{-g}/\delta g^{\mu\nu}$ derivative is already evaluated in (2.33), thus

$$\delta S_{\mathrm{M}} = 2\kappa\int\left(-\frac{1}{2}\mathcal{L}_{\mathrm{M}}g_{\mu\nu} + \frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial g^{\mu\nu}}\right)\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x = -\kappa\int T_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\mathrm{d}^4x, \tag{2.35}$$

where we have defined the energy-stress tensor as

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial g^{\mu\nu}} + \mathcal{L}_{\mathrm{M}}g_{\mu\nu}. \tag{2.36}$$

The curvature term

From equation (2.29) we have

$$\delta S_{\mathrm{C}} = \int\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\mathrm{d}^4x = \int\left(\sqrt{-g}\frac{\delta R}{\delta g^{\mu\nu}} + R\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu}\mathrm{d}^4x \tag{2.37}$$

and, by using Jacobi's formula as described in (2.33) we can write

$$\delta S_{\mathrm{C}} = \int \left(\frac{\delta R}{\delta g^{\mu\nu}} - \frac{1}{2} R g_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x. \tag{2.38}$$

We now focus on δR. We know that $R \equiv g^{\mu\nu} R_{\mu\nu}$, and $R_{\mu\nu} \equiv R^{\lambda}_{\mu\lambda\nu}$ so

$$\delta R = g^{\mu\nu} \delta R^{\lambda}_{\mu\lambda\nu} + R_{\mu\nu} \delta g^{\mu\nu} \tag{2.39}$$

Being $R^{\rho}_{\sigma\mu\nu}$ shorthand for $\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu_\lambda} \Gamma^{\lambda}_{\mu\sigma}$, we have

$$\delta R^{\rho}_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} + \delta\Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda} \delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda} \delta\Gamma^{\lambda}_{\mu\sigma}. \tag{2.40}$$

The covariant derivative of the variation of the Christoffel symbol is

$$\nabla_\mu (\delta\Gamma^{\rho}_{\nu\sigma}) = \partial_\mu (\delta\Gamma^{\rho}_{\nu\sigma}) + \Gamma^{\rho}_{\lambda\mu} \delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\mu} \delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\sigma\mu} \delta\Gamma^{\rho}_{\nu\lambda}, \tag{2.41}$$

and using the the identity $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$, equation (2.40) may be rewritten as

$$\begin{aligned} \delta R^{\rho}_{\sigma\mu\nu} &= (\partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda} \delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\sigma} \delta\Gamma^{\rho}_{\nu\lambda}) - (\partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\lambda}_{\nu\sigma} \delta\Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\nu\lambda} \delta\Gamma^{\lambda}_{\mu\sigma}) \\ &= (\partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\lambda\mu} \delta\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\mu} \delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\sigma\mu} \delta\Gamma^{\rho}_{\nu\lambda}) \\ &\qquad - (\partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\lambda\nu} \delta\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\lambda}_{\mu\nu} \delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\sigma\nu} \delta\Gamma^{\rho}_{\mu\lambda}) \\ &= \nabla_\mu (\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_\nu (\delta\Gamma^{\rho}_{\mu\sigma}). \end{aligned} \tag{2.42}$$

Using the result of the calculations in (2.42), $\delta R_{\mu\nu}$ becomes

$$\delta R_{\mu\nu} = \delta R^{\rho}_{\mu\rho\nu} = \nabla_\rho (\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu (\delta\Gamma^{\rho}_{\rho\mu}). \tag{2.43}$$

By remembering the identity $\nabla_\lambda g^{\mu\nu} = 0$, we obtain

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\rho (g^{\mu\nu} \delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu (g^{\mu\nu} \delta\Gamma^{\rho}_{\rho\mu}). \tag{2.44}$$

These are two divergences for the covariant derivative. Using equation (4.7.7) from [42], it becomes

$$g^{\mu\nu} \delta R_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\rho} \left(\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} \right) - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\rho} \right), \tag{2.45}$$

which, in equation (2.39) gives

$$\delta R = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\rho} \left(\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\nu} \right) - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\rho}_{\mu\rho} \right) + R_{\mu\nu} \delta g^{\mu\nu}. \tag{2.46}$$

This will substitute δR in the first term of equation (2.38), which becomes

$$\begin{aligned}
&\int \frac{\delta R}{\delta g^{\mu\nu}} \sqrt{-g}\delta g^{\mu\nu} \mathrm{d}^4 x \\
&= \int \frac{1}{\delta g^{\mu\nu}} \left(\frac{\partial}{\partial x^\rho} \left(\sqrt{-g} g^{\mu\nu} \delta \Gamma^\rho_{\mu\nu} \right) - \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\mu\nu} \delta \Gamma^\rho_{\mu\rho} \right) + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \right) \delta g^{\mu\nu} \mathrm{d}^4 x \\
&= \int \left(\frac{\partial}{\partial x^\rho} \left(\sqrt{-g} g^{\mu\nu} \delta \Gamma^\rho_{\mu\nu} \right) - \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\mu\nu} \delta \Gamma^\rho_{\mu\rho} \right) \right) \mathrm{d}^4 x + \int R_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \mathrm{d}^4 x,
\end{aligned}$$

where the first term is an integral over a cross product that by Stoke's theorem gives the boundary terms. These are fixed by the variational principle, therefore the terms vanishes. We have now shown that $\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}$. We can plug it in equation (2.38) and obtain

$$\delta S_{\mathrm{C}} = \int \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x. \tag{2.47}$$

The Einstein-Hilbert action

For the variational principle, we will obtain

$$\delta S_{\mathrm{EH}} = \delta S_\Lambda + \delta S_{\mathrm{C}} + \delta S_{\mathrm{M}} = 0. \tag{2.48}$$

We now have expressions for δS_Λ, δS_{M} and δS_{C}, namely equations (2.34), (2.35) and (2.47). By substituting into equation (2.48) we obtain

$$\int \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} \mathrm{d}^4 x = 0. \tag{2.49}$$

The integral will be zero if the argument equals zero. We then recognize the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.50}$$

which can be easily rewritten in terms of equation (2.24).

2.2 The Tolman-Oppenheimer-Volkoff equation

The Tolman-Oppenheimer-Volkoff (TOV) equation describes how the rate of change in pressure of a spherically symmetric body of isotropic material changes as a function of the radius r, energy density ϵ, pressure P and the enclosed mass M. The body is in gravitational equilibrium (no explicit time dependence) and the equation is derived from the Einstein field equations in (2.24) [29], [40]. The TOV equation reads:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\epsilon(r)M(r)}{r^2c^2} \left(\frac{P(r)}{\epsilon(r)} + 1 \right) \left(\frac{4\pi r^3 P(r)}{c^2 M(r)} + 1 \right) \left(1 - \frac{2GM(r)}{c^2 r} \right)^{-1}. \tag{2.51}$$

2.2.1 Derivation

Since we are interested in spherically symmetric systems, it makes sense to express the metric in spherical coordinates. This means that the matrix elements g_{22} and g_{33} will be as in flat spacetime. Keeping the g_{00} and g_{11} components generic, we can build a metric tensor in the form:

$$g_{\mu\nu}(r) = \mathrm{diag}\left(e^{\nu(r)}, -e^{\lambda(r)}, -r^2, -r^2 \sin^2\theta\right). \tag{2.52}$$

The fact that we use exponentials to represent the g_{00} and g_{11} functions does not lead to any loss of generality. $\nu(r)$ and $\lambda(r)$ in equation (2.52) are unknown functions to be found given the isotropy and time-independence constraints to the stress-energy tensor and ultimately the Einstein equations. With this metric we can calculate the Einstein tensor $G_{\mu\nu}$ on the left side of equation (2.24). We now define the stress-energy tensor. This can be done by applying isotropy, time-independence and spherical symmetry constraints, obtaining:

$$T_\mu^\nu = \mathrm{diag}\Big(\epsilon(r), -P(r), -P(r), -P(r)\Big). \tag{2.53}$$

With this we have all that is needed to solve the Einstein field equations. The stress-energy tensor is diagonal, so the Einstein tensor must be diagonal as well. This reduces the field equations from ten to four.

We start by evaluating G_{00}. This can be found by hand with the definitions in Section 2.1.4 of the Einstein tensor, Ricci curvature and Riemann tensor, or by using a computer program to evaluate it. The result will be

$$G_{00} = \frac{1}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-\lambda(r)}\right)\right] e^{\nu(r)}. \tag{2.54}$$

The right hand side of the Einstein equations will read $T_{\mu\nu} = g_{\mu\sigma}T_\nu^\sigma$, thus the first one will be

$$\frac{1}{r^2}\left[1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-\lambda(r)}\right)\right] e^{\nu(r)} = \kappa e^{\nu(r)}\epsilon(r), \tag{2.55}$$

which can be simplified to

$$1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-\lambda(r)}\right) = \kappa r^2 \epsilon(r). \tag{2.56}$$

The mass of a shell of thickness $\mathrm{d}r$ at a distance r from the center is given by

$$\mathrm{d}M(r) = 4\pi\epsilon(r) r^2 \mathrm{d}r. \tag{2.57}$$

We can use this to rewrite equation (2.56) as

$$1 - \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-\lambda(r)}\right) = \frac{\kappa}{4\pi}\frac{\mathrm{d}M(r)}{\mathrm{d}r}. \tag{2.58}$$

By integrating both parts from 0 to r, we get the solution for g_{00}:

$$e^{-\lambda(r)} = 1 - \frac{\kappa}{4\pi r} M(r). \tag{2.59}$$

We now consider G_{11} and T_{11}. The Einstein equation will read

$$\frac{1}{r^2} \left(r \frac{\mathrm{d}}{\mathrm{d}r} \nu(r) - e^{\lambda(r)} + 1 \right) = \kappa P(r) e^{\lambda(r)}. \tag{2.60}$$

This can be solved for $\nu'(r)$ and by using the newly obtained expression for $e^{-\lambda(r)}$ in equation (2.59):

$$\nu'(r) = \frac{\mathrm{d}\nu(r)}{\mathrm{d}r} = \left(\kappa r P(r) + \frac{\kappa}{4\pi r^2} M(r) \right) \left(1 - \frac{\kappa}{4\pi r} M(r) \right)^{-1}. \tag{2.61}$$

Another relation for $\nu'(r)$ can be found by using the energy-momentum conservation relation for r.

$$\nabla_\mu T_1^\mu = \frac{\partial T_1^\mu}{\partial x^\mu} - \Gamma^\rho_{1\mu} T^\mu_\rho + \Gamma^\mu_{\rho\mu} T^\rho_1 = 0 \tag{2.62}$$

The only nonzero derivative of the first term of equation (2.62) is $\mathrm{d}T^1_1/\mathrm{d}r$. The equation will then take the form

$$\begin{aligned} \nabla_\mu T_1^\mu = \frac{\mathrm{d}T_1^1}{\mathrm{d}r} - (\Gamma^0_{01} T^0_0 + \Gamma^1_{11} T^1_1 + \Gamma^2_{12} T^2_2 + \Gamma^3_{13} T^3_3) \\ + T^1_1 (\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}). \end{aligned}$$

Noticing that $T^1_1 = T^2_2 = T^3_3$, the expression simplifies to

$$\frac{\mathrm{d}T^1_1}{\mathrm{d}r} + \Gamma^0_{10}(T^1_1 - T^0_0) = 0.$$

We calculate Γ^0_{10} with equation (2.14) and substitute T^1_1 and T^0_0 with the values we defined in (2.53). Solving for $\nu'(r)$ will yield

$$\nu'(r) = -2 \frac{\mathrm{d}P(r)}{\mathrm{d}r} \frac{1}{P(r) + \epsilon(r)} \tag{2.63}$$

which can be plugged into (2.61). Finally, after some math, we obtain

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -2 \Big(P(r) + \epsilon(r) \Big) \left(\kappa r P(r) + \frac{\kappa}{4\pi r} M(r) \right) \left(1 - \frac{\kappa}{4\pi r} M(r) \right)^{-1}. \tag{2.64}$$

By going back to $c \neq 1$ units, writing κ explicitly and taking out some terms from the parenthesis in order to make them dimensionless, we get the TOV equation as described in (2.51):

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\epsilon(r)M(r)}{r^2 c^2} \left(\frac{P(r)}{\epsilon(r)} + 1 \right) \left(\frac{4\pi r^3 P(r)}{c^2 M(r)} + 1 \right) \left(1 - \frac{2GM(r)}{c^2 r} \right)^{-1}. \tag{2.65}$$

This, coupled with the mass equation (2.57) in its differential form

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi\frac{\epsilon(r)}{c^2}r^2, \tag{2.66}$$

and an equation of state relating the pressure to the energy density, form a system of three coupled differential equations describing the structure of spherically symmetrical bodies consisting of isotropic material in gravitational equilibrium. We will idealize neutron stars to have these properties throughout this thesis, and we will thus use these equations in order to derive important relations like their masses and radii. This system of three equations can be solved analytically for some simple cases, numerically otherwise.

2.3 Solutions to the TOV equation

The set of three equations (2.51), (2.66) and the equation of state can be solved either analytically or numerically, depending on the complexity of the latter. In this section we will consider the cases of constant density, non-relativistic and ultra-relativistic Fermi gas, all of which can be solved in analytical form. In the end we will plot the general case, which has to be calculated numerically. Much of the chapter has been inspired by the work of Richard Silbar and Sanjay Reddy *Neutron Stars for Undegraduates* [37].

2.3.1 Constant density

Assuming constant energy density throughout the entire neutron star, we can solve the set of three differential equations analytically. Although unrealistic, the solution gives some insight to some of the properties relativistic stars have. When the energy density is constant, we can integrate equation (2.66) and obtain

$$M(r) = \frac{4\pi}{3}\frac{\epsilon}{c^2}r^3. \tag{2.67}$$

Inserting (2.67) into the TOV equation and setting the energy density as constant, $\epsilon(r) = \epsilon$, we obtain:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{G\epsilon^2 4\pi r}{3c^4}\left(1+\frac{P(r)}{\epsilon}\right)\left(1+\frac{3P(r)}{\epsilon}\right)\left(1-\frac{8\pi G r^2\epsilon}{3c^4}\right)^{-1}. \tag{2.68}$$

Introducing $a^2 = \frac{3\pi c^4}{8\epsilon G}$, we may rewrite this as

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -r\frac{(\epsilon+P(r))\,(\epsilon+3P(r))}{2\epsilon a^2\left(\frac{r^2}{a^2}-1\right)}. \tag{2.69}$$

Equation (2.69) is a separable differential equation and can be rewritten as

$$\int_{P_c}^{P(r')}\frac{2\epsilon\,\mathrm{d}P(r)}{\big(\epsilon+P(r)\big)\big(\epsilon+3P(r)\big)} = -\frac{1}{a^2}\int_0^{r'}\frac{r\,\mathrm{d}r}{1-\frac{r^2}{a^2}}, \tag{2.70}$$

where we have set the limits so that the pressure is $P(r=0) = P_c$ (the central pressure) and $P(r')$ at a radius r'. We first take a look at the integral on the right hand side. By substituting $u = \frac{r^2}{a^2}$ we get $\mathrm{d}u = \frac{2r}{a^2}\mathrm{d}r$, thus

$$-\int_0^{\frac{r'^2}{a^2}} \frac{1}{2}\frac{\mathrm{d}u}{1-u} = \ln\sqrt{1-\tfrac{r'^2}{a^2}}. \tag{2.71}$$

The left hand side integral is instead

$$\int_{P_c}^{P(r')} \frac{2\epsilon\,\mathrm{d}P(r)}{\big(\epsilon+P(r)\big)\big(\epsilon+3P(r)\big)} \tag{2.72}$$

and can be rewritten with partial fraction decomposition as

$$\begin{aligned}\int_{P_c}^{P(r')}\left[\frac{1}{\frac{\epsilon}{3}+P(r)} - \frac{1}{\epsilon+P(r)}\right]\mathrm{d}P(r) &= \ln\left[\frac{\epsilon/3+P(r)}{\epsilon+P(r)}\right]_{P_c}^{P(r')} \\ &= \ln\left[\frac{\epsilon+3P(r')}{\epsilon+P(r')}\right] + \ln\left[\frac{\epsilon+P_c}{\epsilon+3P_c}\right].\end{aligned}$$

Plugging in the integrated left and right hand sides into equation (2.70), we obtain

$$\frac{\epsilon+3P(r')}{\epsilon+P(r')}\frac{\epsilon+P_c}{\epsilon+3P_c} = \sqrt{1-\tfrac{r'^2}{a^2}}. \tag{2.73}$$

All integrations are taken, and we may as well call r' again as r again without fear of confusion. First, we isolate $P(r)$ and obtain an expression for the pressure, now a function of r and with P_c as a parameter:

$$P(r) = \frac{\sqrt{1-\frac{r^2}{a^2}} - \frac{\epsilon+P_c}{\epsilon+3P_c}}{3\frac{\epsilon+P_c}{\epsilon+3P_c} - \sqrt{1-\frac{r^2}{a^2}}}\,\epsilon. \tag{2.74}$$

We may then denote with R the radius at surface. The pressure at the surface is zero, $P(R) = 0$, so (2.73) becomes

$$\frac{\epsilon+P_c}{\epsilon+3P_c} = \sqrt{1-\tfrac{R^2}{a^2}}. \tag{2.75}$$

Isolating R^2, we obtain an analytical solution for the square radius of the star as a function of the central pressure

$$R^2 = a^2\left[1-\left(\frac{\epsilon+P_c}{\epsilon+3P_c}\right)^2\right]. \tag{2.76}$$

It is then possible to take the derivative of it in terms of P_c and look for its roots in order to find for which central pressure we have the largest radius. The derivative will read

$$\frac{\mathrm{d}R^2}{\mathrm{d}P_c} = \frac{a^2\epsilon}{(P_c+\epsilon)^2(3-2\epsilon)^3} = \frac{a^2\epsilon\,(P_c+\epsilon)}{[3(P_c+\epsilon)-2\epsilon]^3} = 0, \tag{2.77}$$

which we notice is only satisfied in the limit where $P_c \to \infty$. The limit can be inserted into equation (2.76), which then yields

$$R_{max}^2 = \frac{8a^2}{9} = \frac{\pi c^4}{3\epsilon G}. \tag{2.78}$$

This is the maximum radius a star with constant density can obtain, showing also that the maximum radius will decrease for increasing densities.

Plotting

By using equations (2.67) and (2.74) we can plot analytically the pressure and the mass in terms of the radius for any choice of P_c and ϵ. For example, by choosing typical values for a neutron star at $\epsilon/c^2 = 4 \times 10^{17}$ kg/m^3 and $P_c = 10^{33}$ Pa we obtain the plots below:

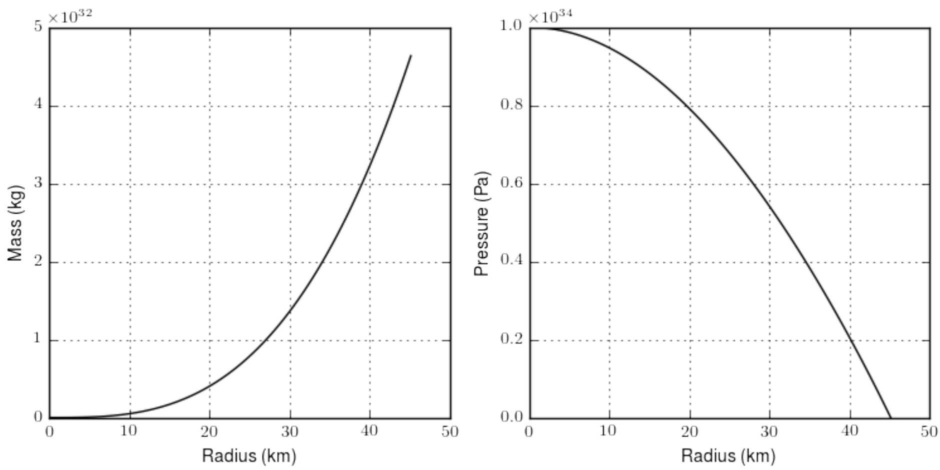

Figure 2.1: Plot of equations (2.67) (left) and (2.74) (right) with $\epsilon/c^2 = 4.0 \times 10^{17}$ kg/m^3 and $P_c = 1.0 \times 10^{34}$ Pa, typical values for a neutron star.

2.3.2 Cold Fermi gas approximation for neutron stars

A neutron star can be approximated to a Fermi gas of neutrons, where the degeneracy pressure from the Pauli exclusion principle holds the star from gravitational collapse. The particle distribution in such gases is described by the Fermi-Dirac distribution [34]

$$n(\epsilon) = \frac{1}{e^{(E-\mu)/k_B T} + 1}, \tag{2.79}$$

where E is the particle energy, μ is the chemical potential, k_B is the Boltzmann constant and T the temperature. A further approximation is to take the zero temperature limit, where $T \to 0$. This is a common and valid approximation, and has its justification in the fact that the star's temperature usually is well below 1 MeV ($\approx 10^{10}$ K). This is small at nuclear scales and the star can be thought as cold, especially when we are only interested in the star's bulk properties such as its mass and radius. Temperature only becomes important when considering the surface of the star. As it treated in more depth in Section 5.1.4, the outer crust consists of fully ionized iron atoms, whose thermal energy becomes more and more important as pressures become low enough not to be able to support the lattice structure. For a common neutron star of 10 km radius and a mass of 1.5 solar masses, this last layer where temperature becomes important is only 0.76 m thick [10]. We choose then to ignore the temperature effects altogether throughout this thesis, and only consider cold, nuclear matter. When only considering a cold Fermi gas, and the distribution becomes

$$\lim_{T\to 0} \frac{1}{e^{(E-\mu)/k_B T}+1} = \theta(\mu - E), \tag{2.80}$$

where θ is the Heaviside step function, defined as:

$$\theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases} \tag{2.81}$$

The Fermi momentum p_F will be then related to E_F, the energy of the most energetic fermion in the cold Fermi gas through

$$E_F = \sqrt{p_F^2 + m^2}. \tag{2.82}$$

The integral for the energy density will be an integral over all momenta of the energy for every particle and their distribution. The limit in (2.80) tells us that we have no particles above p_F and that all states below are occupied by two particles (spin up and down). The integral will read as

$$\epsilon(p_F) = \frac{2}{(2\pi\hbar)^3} \int_0^{p_F} \sqrt{p^2 + m^2}\mathrm{d}^3 p. \tag{2.83}$$

Changing to spherical coordinates, integrating for the angles and introducing the dimensionless variable $u = p/m$ and the constant $x = p_F/m$, we obtain

$$\epsilon(x) = \epsilon_0 \int_0^x \left(u^2+1\right)^{1/2} u^2 \mathrm{d}u, \tag{2.84}$$

where we have defined $\epsilon_0 = \frac{m^4 c^5}{\pi^2 \hbar^3}$ in $c \neq 1$ units. Similarly for the pressure, we have

$$P(x) = \epsilon_0 \int_0^x (\mu - E)\mathrm{d}u = \epsilon_0 \int_0^x \left[\left(x^2+1\right)^{1/2} - \left(u^2+1\right)^{1/2}\right] u^2 \mathrm{d}u. \tag{2.85}$$

Ultrarelativistic (UR) limit for the equation of state

In the UR limit the Fermi energy is much larger than the rest mass, $p_f \gg m$, so $x \gg 1$. Expanding u accordingly, in equation (2.84) the square root $\sqrt{u^2+1}$ becomes a u and we find:

$$\epsilon(x) = \mathcal{E}_0 \int_0^x u^3 \mathrm{d}u = \frac{\epsilon_0}{4} x^4 \tag{2.86}$$

Similarly, for equation (2.85):

$$P(x) = \int_0^x \left(xu^2 - u^3\right) \mathrm{d}u = \frac{\epsilon_0}{12} x^4.$$

These two can be combined, finding the equation of state in the UR limit:

$$P = \frac{1}{3}\epsilon. \tag{2.87}$$

By substituting equation (2.87) into equation (2.51) we will get

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{4GP(r)M(r)}{r^2c^2}\left(\frac{4\pi r^3 P(r)}{c^2 M(r)} + 1\right)\left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}. \tag{2.88}$$

In order to find an analytical solution to equation (2.88), we guess something in the form Kr^n. The terms in the parenthesis are all dimensionless, and by looking at the last factor we understand that the mass must go like $M(r) \propto r$. The pressure must then be $P \propto r^{-2}$ to balance the first parenthesis. We then speculate a solution like $P(r) = \frac{K}{r^2}$, with K a constant to evaluate. By remembering that $\epsilon(r) = 3P(r) = 3K/r^2$, we can immediately compute the mass and the derivative of the pressure:

$$M(r) = \int_0^r 4\pi \frac{\epsilon(r')}{c^2} r'^2 \mathrm{d}r' = \frac{12\pi K}{c^2}\int_0^r \mathrm{d}r' = \frac{12\pi K r}{c^2}, \tag{2.89}$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -2\frac{K}{r^3}. \tag{2.90}$$

Equations (2.89) and (2.90) can be finally inserted in (2.88)

$$1 = \frac{24\pi GK}{c^4}\left(\frac{1}{3} + 1\right)\frac{c^4}{c^4 - 24\pi GK}, \tag{2.91}$$

where we can isolate K, obtaining $K = \frac{c^4}{56\pi G}$, for which the pressure will be

$$P(r) = \frac{c^4}{56\pi G r^2}. \tag{2.92}$$

This solution is unphysical for two reasons: it blows up at $r \to 0$, and has infinite radius (there is no solution for $P(r) = 0$). This has to do with the fact that equation (2.87) used as assumption $P, \epsilon \gg 1$ leading to infinite central mass and densities.

The non-relativistic limit

By taking the non-relativistic approximation of (2.84) and (2.85), i.e. by taking the limit where $x \ll 1$, we obtain

$$\begin{aligned}\epsilon(x) &= \epsilon_0 \int_0^x u^2\sqrt{u^2+1}\mathrm{d}u = \epsilon_0 \int_0^x u^2\left(1 + \frac{u^2}{2} + O(u^4)\right) \\ &= \epsilon_0\left(\frac{1}{3}x^3 + \frac{x^5}{10} + O(x^7)\right) \approx \frac{1}{3}\epsilon_0 x^3 \end{aligned} \tag{2.93}$$

and

$$\begin{aligned} P(x) &= \int_0^x \left[\sqrt{x^2+1} - \sqrt{u^2+1}\right] u^2 \mathrm{d}u \\ &= \epsilon_0 \left(1 + \frac{1}{2}x^2 + O(x^4)\right) \frac{1}{3}x^3 - \epsilon_0 \left(\frac{1}{3}x^3 + \frac{1}{10}x^5 + O(x^7)\right) \\ &\approx \frac{1}{15}\epsilon_0 x^5. \end{aligned} \tag{2.94}$$

We can recast the energy density and the pressure in dimensionless form by using ϵ_0 as scaling constant:

$$\epsilon(x) = \epsilon_0 \overline{\epsilon}(x) \qquad\qquad P(x) = \epsilon_0 \overline{P}(x),$$

and with these write the equation of state for the non-relativistic limit using equations (2.93) and (2.94):

$$\overline{\epsilon} = \frac{15^{\frac{3}{5}}}{3} \overline{P}^{\frac{3}{5}}. \tag{2.95}$$

We can make the TOV equation and the mass equation dimensionless by introducing

$$\overline{M}(r) = \frac{M(r)}{M_\odot} \qquad \beta = \frac{4\pi\epsilon_0}{M_\odot c^2} \qquad R_0 = \frac{GM_\odot}{c^2},$$

where $M_\odot$ is the Sun's mass and R_0 the Schwarzschild radius of the Sun. The new, dimensionless TOV and mass equations will then read

$$\frac{\mathrm{d}\overline{P}}{\mathrm{d}r} = -\frac{R_0 \overline{M}\overline{\epsilon}}{r^2} \left(\frac{\overline{P}}{\overline{\epsilon}} + 1\right) \left(\beta r^3 \frac{\overline{P}}{\overline{M}} + 1\right) \left(1 - \frac{2R_0 \overline{M}}{r}\right)^{-1} \tag{2.96}$$

$$\frac{\mathrm{d}\overline{M}}{\mathrm{d}r} = \beta r^2 \overline{\epsilon}. \tag{2.97}$$

We can then write a script coupling the two differential equations (2.96), (2.97) and the equation of state in (2.95). As boundary term we choose a central normalized pressure $\overline{P}_c$ and we can evaluate the system in a while-loop until $P > 0$, the surface of the star.

2.3.3 The general case

It is of course possible to evaluate the original TOV equation in (2.51) numerically, using the solution of the integrals in (2.84) and (2.85). The solution of the first one can be checked in integral tables [32], and will be

$$\begin{aligned} \epsilon(x) &= \epsilon_0 \int_0^x \left(u^2+1\right)^{1/2} u^2 \mathrm{d}u \\ &= \frac{1}{8}\epsilon_0 \left[\left(2x^3 + x\right)\left(1 + x^2\right)^{1/2} - \sinh^{-1}(x)\right], \end{aligned} \tag{2.98}$$

and we can use this result to evaluate the second integral,

$$\begin{aligned}P(x) &= \epsilon_0 \int_0^x \left(\left(x^2+1\right)^{1/2} - \left(u^2+1\right)^{1/2} \right) u^2 \mathrm{d}u \\ &= \frac{1}{24}\epsilon_0 \left[\left(2x^3 - 3x\right) \left(1+x^2\right)^{1/2} + 3\sinh^{-1}(x) \right]. \end{aligned} \tag{2.99}$$

The numerical evaluation is done as follows. We choose a first central pressure P_c, with which we can find a root for equation (2.99), giving us the associated dimensionless Fermi momentum x. This can be used as input for equation (2.98), which will give us the energy density associated to P_c. This can be done for every pressure P, giving us the possibility to evaluate the energy density ϵ from every pressure P. The equation system for the cold Fermi gas approximation for neutron stars is then closed. We can compare the non-relativistic approximation to the exact relativistic solution by starting from the same central pressure. As an example we can look at the first two plots in Figure 2.4, where we chose $\overline{P}_c = 10^{-6}$. The approximation and the exact relativistic solutions are quite similar, but they diverge more and more as the central pressure increases, as we see in the following plots. The relativistic model ends up having less mass than the non-relativistic model for the same radius. This can be explained by looking at the TOV equation in (2.51). The $P(r)/\epsilon(r)$ and the $4\pi r^3 P(r)/c^2 M(r)$ terms are the relativistic corrections to the Newtonian model, while the last factor adds the correction from the spacetime curvature. All these three terms act as a booster for gravity, and we can think of a "stronger than Newtonian" gravity when taking into account these relativistic effects. This is the reason why the pressure slope will be steeper in the relativistic case, leading to smaller stars with shorter radius than the non-relativistic model when starting with the same central pressure P_c. A way to get a complete picture of how the radii and the masses for the two models diverge would be to plot their end values for the exact numerical solution and the non-relativistic approximation starting from different values of $\overline{P}_c$ (Fig. 2.2). When choosing a realistic criterion for which we can use the non-relativistic approximation, a natural first choice would be when the kinetic energy of the central neutrons, $p_{F,c}c$, is not bigger than its rest mass. Equation (2.99) links the pressure to the ratio between the Fermi energy and the neutron mass: $x = p_F/m_n c$ in $c \neq 1$ units. Analytically, our condition translates to finding for which central pressures P_c this ratio is $x < 1$. By checking for the normalized central pressure $\overline{P}_c(x) = P(x)/\epsilon_0$, we find for $x = 1$ that

$$\overline{P}_c(1) = \frac{1}{24} \left[(2-3)\,(2)^{1/2} + 3\ln\left(1+\sqrt{2}\right) \right] \approx 0.17624 \tag{2.100}$$

Giving us a threshold around $\overline{P}_c = 10^{-1}$, or $P_c \approx 10^{35}$ Pa. This corresponds to the plots in the third row of Figure 2.4, where the difference in total mass is above 20%: too much for the non-relativistic approximation to be valid. We could change approach and require the mass ratio to be less than 10%. With this requirement, we see from the numerical calculations in Figure 2.3) that the mass in the non-relativistic approximation becomes more than 10% bigger for values of P_c around 8.3×10^{-3}, where we can set our limit for the validity of the non-relativistic model. The 10% threshold corresponds to a radius $R_{\mathrm{R}} = 12.73$ km and mass $M_{\mathrm{R}} = 0.62 M_\odot$ for the relativistic case, and radius $R_{\mathrm{NR}} = 13.82$ km and mass $M_{\mathrm{NR}} = 0.68 M_\odot$ for the non-relativistic case.

Maximum mass and unstable solutions

Wheeler et al. [18] and Weinberg in *Gravitation and Cosmology* [42] wrote how a star is stable under density variations until the condition $\partial\mathcal{E}(\rho_c)/\partial\rho_c = 0$ is met, where $\mathcal{E}$ stands for the total energy (mass plus kinetic energy) for a star with central density $\rho(r = 0) = \rho_c$. If all the dissipative forces are absent, dynamical equations will be invariant under time-reversal, and will give real normal modes ω_i, or alternatively positive ω_i^2. This corresponds to oscillating modes or, in case of spherically symmetric bodies, radial (or "breathing") modes. A breathing mode would be described for example by an equation in the form

$$\tilde{r}(t) = Ae^{i\omega_r t} - Be^{-i\omega_r t}, \tag{2.101}$$

where A and B are constants. The oscillation is periodic, as the system returns to the same configuration after every period of $2\pi/\omega_r$. By neglecting the phase and allowing for A and B to be equal, we also see that the equation is invariant under time reversal $t \to -t$. A configuration is instead unstable when the normal mode becomes complex, or $\omega_i^2 < 0$. If $\omega_i \in \mathbb{C}$ and $\mathrm{Im}\{\omega_i\} \neq 0$, the real part of ω_i would give an oscillation, but the imaginary part would contribute with a terms blowing up at either $t \to \infty$ or $t \to -\infty$. These terms are not periodic and make the system not invariant under time-reversal, thus unstable. The transition between stability and instability will occur at $\omega_i^2 = 0$. In our breathing mode example, this would mean either an implosion or an explosion, depending on the signs of A and B. The oscillating modes are described as variations in the density $\delta\rho(r)$. Since any equilibrium configuration is entirely specified by its value at $\rho(0) = \rho_c$, we can choose one for which a frequency ω_r is nearly zero. With this choice, the variations in density will be so slow that the $\rho(r)\ \delta\rho(r)$ will also be an equilibrium configuration with the same chemical composition and same energy $\mathcal{E}$. At the same time $\delta\rho_c$ can not be zero, otherwise $\delta\rho(r)$ would be zero at all radii and the normal mode would vanish. This means that small oscillation in the radial direction in the vicinity of $\omega_r \approx 0$ lead to new equilibrium solution with a new ρ_c, i.e. these solutions are unstable. When going from stability to instability we will then see that

$$\frac{\partial\mathcal{E}(\rho_c)}{\partial\rho_c} = 0. \tag{2.102}$$

Given Einstein's famous identity $\mathcal{E} = mc^2$ and the correspondence between the energy density and the pressure (equation of state), we can rewrite (2.102) in a more revealing way:

$$\frac{\partial M}{\partial P_c} = 0 \tag{2.103}$$

From Figure 2.2 we see how the star decreases in volume and increases in mass as the central pressure becomes larger. By focusing on the relativistic plot, we see how it reaches a top at radius $R \approx 10$ km and mass $M \approx 0.7M_\odot$. Here the condition in (2.103) is met and the equilibrium points from there will be unstable. From these observations we understand that, according to the cold Fermi gas model, a neutron star will not have stable solutions for radii under $R \approx 10$ km and masses above $M \approx 0.7M_\odot$.

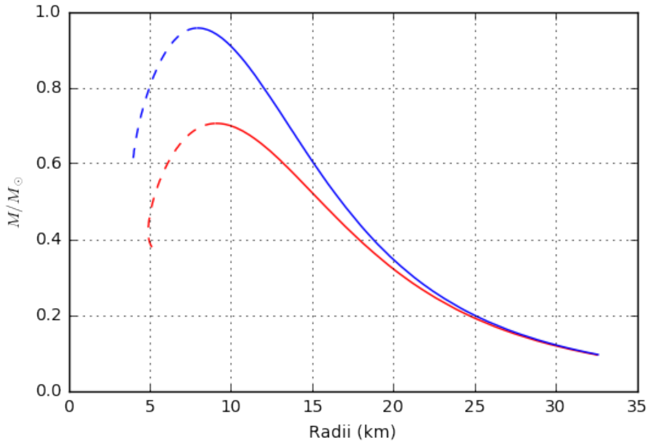

Figure 2.2: Plot of the mass-radius relation for the non-relativistic approximation (blue) and the relativistic, correct equations for $\overline{P}_c$ from 10^{-6} (to the right) growing up to 10^{1} (to the left). We see how they start in a similar way, for then diverge at radii around 20 km, corresponding to central pressures between 10^{-4} and 10^{-3}. The dashed part of the plot corresponds to the unstable solutions.

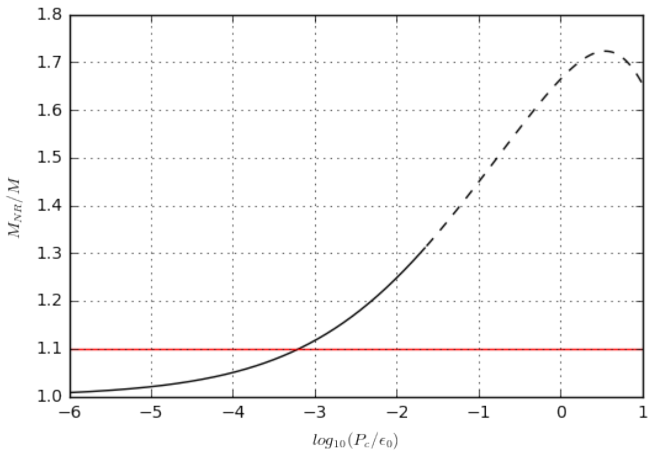

Figure 2.3: Plot of the ratio between the non-relativistic and relativistic mass $M_{\rm NR}/M_{\rm R}$, for the same central pressures $\overline{P}c$. The $M_{\rm NR}/M_{\rm R} = 1.10$ threshold is plotted in red for reference, and corresponds to the radius $R_{\rm R} = 15.6$ km and mass $M_{\rm R} = 0.50 M_\odot$ for the relativistic case, and radius $R_{\rm NR} = 15.97$ km and mass $M_{\rm NR} = 0.55 M_\odot$. The threshold normalized central pressure is $\overline{P}_c = 6.6 \times 10^{-4}$, the crossing point between the plot of the ratio and the threshold (red plot). Again, the dashed plot corresponds to the unstable solutions.

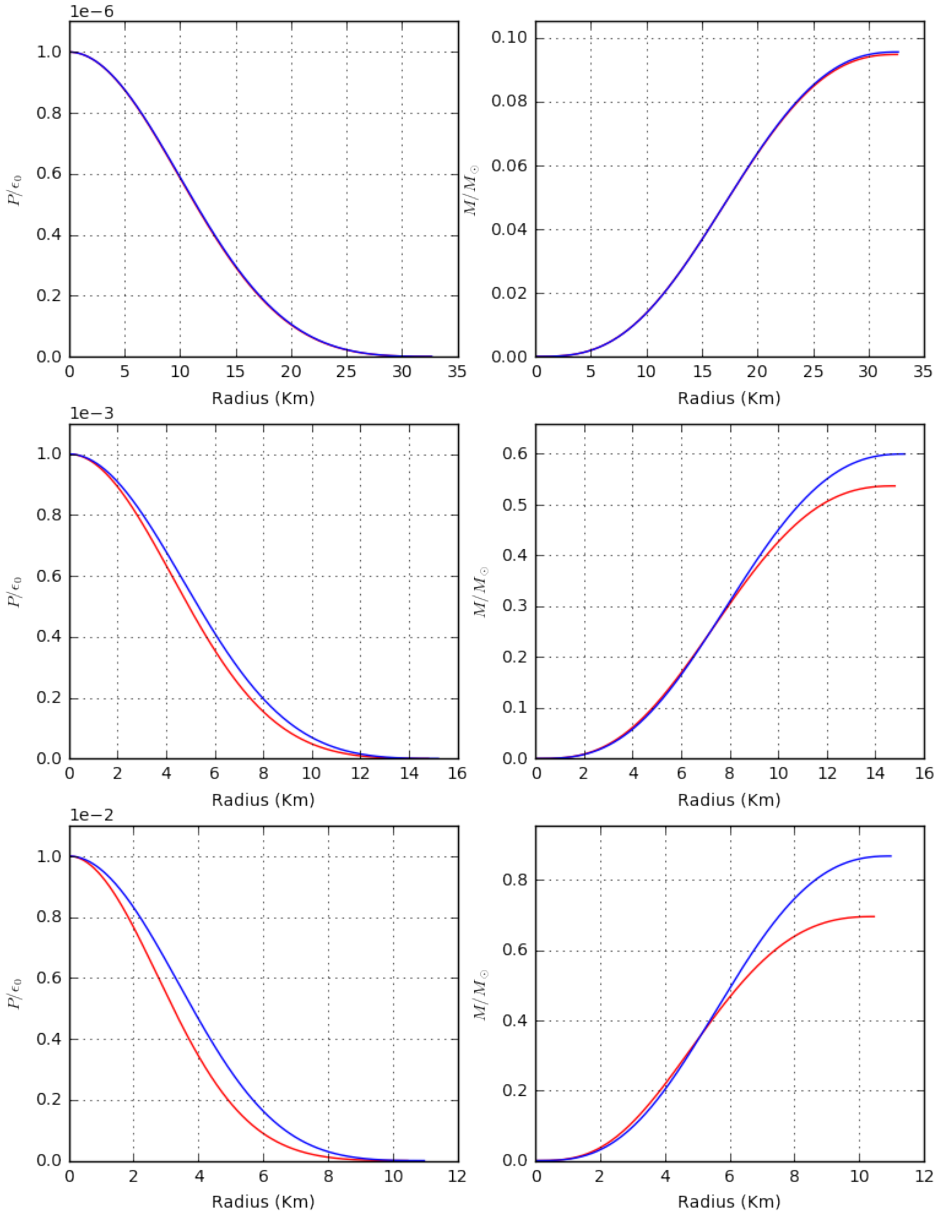

Figure 2.4: Plot of the non-relativistic equations (2.96) in the left panels and (2.97) in the right right panels in blue, and the exact, relativistic equations (2.51) and (2.66) in red in the left and right panels respectively. The central pressure is $\overline{P}_c = 10^{-6}$ in the first row, 10^{-3} in the second and 10^{-2} in the third. We can see how the plots for the radius and the mass are quite similar for low central pressures, and start to diverge as the central pressure increases.

Chapter 3

Thermal Field Theory

In this chapter we will attempt to find an expression for the behavior of fermions starting from concepts of *quantum field theory* (QFT) and thermodynamics. we will start with a brief review of statistical and quantum mechanics, for then using the path integral formalism in order to obtain the grand canonical partition function for a fermion gas. The study of QFT at finite temperatures is often referred to as *thermal field theory* (TFT), and all the derivations in this chapter follow loosely the ones in *Finite-Temperature Field Theory, Principles and Applications* by Gale and Kapusta [26]. When writing Section 3.1, the sources [15], [20] and [25] have been used as reference.

3.1 Review of central concepts for the theory

3.1.1 Classical statistical mechanics

The *grand canonical partition function* Z is an important tool in thermodynamics, as it can be used to find many physical properties of a system. It is defined as

$$Z = \mathrm{tr}\exp\left[-\beta\left(H - \sum_i \mu_i N_i\right)\right], \tag{3.1}$$

where $\beta = T^{-1}$, T is the temperature, H the Hamiltonian of the system, μ_i and N_i respectively the chemical potential and the number of particles of the species i. Here tr indicates the *trace* operation, defined for a $n \times n$ matrix A as

$$\mathrm{tr}A = \sum_{i=1}^{n} a_{ii}, \tag{3.2}$$

where a_{ij} is the element of the matrix A in the i-th row and the j-th column. The exponential in (3.1) is a *matrix exponential*, defined, again for a general square matrix A

as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{3.3}$$

We are again using natural units, $k_B = c = G = \hbar = 1$, where k_B is the Boltzmann constant and $\hbar$ the reduced Planck constant. The physical properties that can be derived from Z are the energy, the pressure, the entropy and the particle numbers in its infinite-volume limit

$$E = - PV + TS + \mu_i N_i \tag{3.4}$$

$$P = \frac{\partial (T \ln Z)}{\partial V} \tag{3.5}$$

$$S = \frac{\partial (T \ln Z)}{\partial T} \tag{3.6}$$

$$N_i = \frac{\partial (T \ln Z)}{\partial \mu_i}. \tag{3.7}$$

In (3.4) and the rest of the thesis repeated indices are assumed to be summed over.

3.1.2 Quantum mechanics and path integrals

Quantum mechanics is generally formulated using the Dirac bra-ket formalism. In this formalism, we describe quantum states as state vectors $|\cdots\rangle$ in a complex, linear vector space called Hilbert space. For any vector $|\alpha\rangle$ there is an associated dual vector $\langle\alpha|$. The two vectors are called *bra* (for $\langle\cdots|$) and *ket* (for $|\cdots\rangle$). The inner product is defined as a complex number

$$\langle a| \cdot |b\rangle \equiv \langle a|b\rangle$$

with the property

$$\langle a|b\rangle = \langle b|a\rangle^*$$

where * means complex conjugation. For an n-dimensional space, we can choose a complete set of orthonormal states $|1\rangle, |2\rangle, |3\rangle \cdots |n\rangle$ as a basis. Orthonormality means that the basis vectors $|n\rangle$ satisfy the relation

$$\langle n_i|n_j\rangle = \delta_{ij} \tag{3.8}$$

where δ_{ij} is the Kronecker delta function, defined as

$$\delta_{ij} = \begin{cases} 1, & \text{for} i = j \\ 0, & \text{for} i \neq j. \end{cases} \tag{3.9}$$

Normal quantum states can be expressed as a linear combination of these basis vectors

$$|\alpha\rangle = \sum_n c_n |n\rangle \tag{3.10}$$

where c_n are normalized amplitudes that satisfy

$$\sum_n |c_n|^2 = 1 \tag{3.11}$$

and that can be found by

$$c_m = \langle m|\alpha\rangle. \tag{3.12}$$

This last identity allows us to rewrite equation (3.10) as

$$|\alpha\rangle = \sum_n \langle n|\alpha\rangle|n\rangle = \sum_n |n\rangle\langle n| \cdot |\alpha\rangle, \tag{3.13}$$

from which follows the completeness relation

$$\sum_n |n\rangle\langle n| = 1. \tag{3.14}$$

An operator $\hat{Q}$ in Hilbert space transforms a vector into another vector

$$\hat{Q}|a\rangle = |b\rangle.$$

When the transformed vector $|b\rangle$ is proportional to $|a\rangle$ by a complex number λ_a we may write

$$\hat{Q}|a\rangle = \lambda_a|a\rangle,$$

then $|a\rangle$ is said to be an *eigenvector* (or *eigenstate* when talking about the respective quantum state) for the operator $\hat{Q}$, and λ_a the associated *eigenvalue* (or *eigenfunction*, when a function). One of the postulates of quantum mechanics is the *Schrödinger equation*

$$i\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = \left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}\boldsymbol{x}^2} + V(\boldsymbol{x},t)\right)\Psi(\boldsymbol{x},t) \tag{3.15}$$

which describes the time evolution for a quantum state Ψ in position space. Although non-relativistic, the Schrödinger equation gives a starting point from which we can develop the path integral formalism, a very important tool in both QFT and TFT.

For free particles the Schrödinger equation will have zero potential

$$i\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t) = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}\boldsymbol{x}^2}\Psi(\boldsymbol{x},t), \tag{3.16}$$

which, using the Dirac formalism, can be rewritten as

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \tag{3.17}$$

where we have identified the operator acting on the right hand side of (3.16) as the *Hamiltonian operator* $\hat{H}$. A powerful property of the Dirac formalism is that the $|\Psi(t)\rangle$ eigenstate does not necessarily have to be in position-space. The *time evolution* operator (or *propagator*) $\hat{U}(t,t_0)$ transforms a state $|\Psi(t_0)\rangle$ into its time evolution at time t:

$$|\Psi(t)\rangle = \hat{U}(t,t_0)|\Psi(t_0)\rangle, \tag{3.18}$$

where the propagator can be expressed as

$$\hat{U}(t, t_0) = e^{-i(t-t_0)\hat{H}}. \tag{3.19}$$

Using the definition of the propagator in (3.18) in (3.19), we can show that it satisfies the Schrödinger equation. We let q be the basis for the position space, and insert our propagator

$$\Psi(q', t') = \langle q'|\Psi(t')\rangle = \langle q'| \exp\left(-i\hat{H}(t'-t)\right)|\Psi(t)\rangle. \tag{3.20}$$

by then inserting the continuous completeness relation $1 = \int \mathrm{d}^3 q |q\rangle\langle q|$, we get

$$\begin{aligned}\Psi(q', t') &= \int \mathrm{d}^3 q \langle q'| \exp\left(-i\hat{H}(t'-t)\right)|q\rangle\langle q|\Psi(t)\rangle \\ &= \int \mathrm{d}^3 q K(q', t'; q, t)\Psi(q, t)\end{aligned} \tag{3.21}$$

where we have defined the position space propagator

$$K(q', t'; q, t) = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle.$$

Being translationally invariant in time, the propagator depends only on the time difference $\tau = t' - t$, so

$$K(q', q; \tau) = \langle q'|e^{-i\hat{H}\tau}|q\rangle. \tag{3.22}$$

The propagator tells us what is the probability amplitude A for a particle originating at a point q to be found in another point q' after a time τ. We can split the time evolution in smaller steps by writing

$$e^{-i\hat{H}\tau} = e^{-i\hat{H}(\tau-\tau_N)}e^{-i\hat{H}(\tau_N-\tau_{N-1})}\cdots e^{-i\hat{H}(\tau_2-\tau_1)}e^{-i\hat{H}\tau_1}$$

where $\tau > \tau_N > \tau_{N-1} > \cdots > \tau_1$. For N equally spaced time intervals of length Δt is $t = N\Delta t$ and by splitting the time evolution in these intervals we can express the probability amplitude for a free particle moving from q to q' as

$$A = \langle q'|e^{-i\hat{H}\Delta t}e^{-i\hat{H}\Delta t}...|q\rangle. \tag{3.23}$$

When talking about the position and momenta of fields, the states will not be discrete but continuous. A Schrödinger-picture field operator $\hat{\phi}(x, 0)$ acting on the field eigenvectors $|\phi\rangle$ at time $t = 0$ gives

$$\hat{\phi}(\boldsymbol{x}, 0)|\phi\rangle = \phi(\boldsymbol{x})|\phi\rangle \tag{3.24}$$

where $\phi(\boldsymbol{x})$ is its eigenfunction, and the completeness (introduced above) and orthonormality relations tell us that

$$\int \mathrm{d}\phi(\boldsymbol{x})|\phi\rangle\langle\phi| = 1, \qquad \langle\phi_a|\phi_b\rangle = \prod_{\boldsymbol{x}} \delta(\phi_a(\boldsymbol{x}) - \phi_b(\boldsymbol{x})). \tag{3.25}$$

For the momentum field operator $\hat{p}(\boldsymbol{x}, 0)$, acting on the field momentum eigenstates $|p\rangle$ we have

$$\hat{p}(\boldsymbol{x}, 0)|p\rangle = p(\boldsymbol{x})|p\rangle \tag{3.26}$$

where $p(\boldsymbol{x})$ is the associated eigenfunction, and from this follow the completeness and orthonormality relations for momentum eigenstates:

$$\int \frac{\mathrm{d}p(\boldsymbol{x})}{2\pi}|p\rangle\langle p| = 1, \qquad \langle p_a|p_b\rangle = \prod_{\boldsymbol{x}} \delta(p_a(\boldsymbol{x}) - p_b(\boldsymbol{x})). \tag{3.27}$$

When talking about quantum fields we can either work in position or momentum space, and we can switch between position and momentum eigenstates by using

$$\langle \phi|p\rangle = e^{i\int \mathrm{d}^3x\phi(\boldsymbol{x})p(\boldsymbol{x})}, \tag{3.28}$$

for which

$$p(\boldsymbol{x}) = \int \mathrm{d}\phi(\boldsymbol{x})\langle\phi|p\rangle\phi(\boldsymbol{x}). \tag{3.29}$$

When dealing with fields, we will use the *Hamiltonian density* $\mathcal{H}$, described in a similar way as the Lagrangian density in (2.5):

$$H = \int d^3x\mathcal{H}(\hat{\phi}, \hat{p}). \tag{3.30}$$

3.1.3 The Dirac equation

The equation describing the motion of a relativistic spin-$^1/_2$ particle is the Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\psi = 0. \tag{3.31}$$

Here ψ is the four-component spinor describing the state of the fermion, m its mass and we have introduced the Feynman slash notation, for which $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ where γ^μ are the four 4×4 matrices called *gamma matrices*. The defining property of the gamma matrices is the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{3.32}$$

where $\eta^{\mu\nu}$ is the flat spacetime metric defined in (2.8). A possible representation of the four matrices is

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{3.33}$$

where *i* runs from 1 to 3, $\mathbb{1}$ stands for the 2×2 identity matrix, and σ_i are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.34}$$

The Lagrangian for a free fermion with mass m is

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{3.35}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$, and $\psi^\dagger$ stand for the hermitian conjugate of ψ. Written in its explicit form, the Lagrangian is

$$\mathcal{L} = \psi^\dagger\gamma^0\left(i\gamma^0\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\cdot\nabla - m\right)\psi. \tag{3.36}$$

From this Lagrangian we can find the Hamiltonian density $\mathcal{H}$ by a Legendre transformation. The canonical momentum is[1]

$$p = \frac{\partial\mathcal{L}}{\partial\left(\partial\psi/\partial t\right)} = i\psi^\dagger, \tag{3.37}$$

so

$$\mathcal{H} = p\frac{\partial\psi}{\partial t} - \mathcal{L} = \overline{\psi}\left(-i\boldsymbol{\gamma}\cdot\nabla + m\right)\psi. \tag{3.38}$$

The Lagrangian in (3.36) has a global U(1) symmetry, and it is invariant under the transformations $\psi \to \psi e^{-i\alpha}$ and $\overline{\psi} \to \overline{\psi}e^{i\alpha}$, where α is a real constant. From this symmetry we are able to find the *conserved Noether current*. This is defined as

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\left(\partial\psi_i\right)}\delta\psi_i, \tag{3.39}$$

where i runs over 1 and 2, being $\psi_1 = \psi$ and $\psi_2 = \overline{\psi}$ the two independent fields. A derivation of the Noether current can be found in the Appendix. By renaming the transformed field as $\psi' = \psi + \delta\psi$, for small $\delta\psi$ we find

$$\delta\psi = \psi' - \psi = \left.\frac{\partial\psi}{\partial\alpha}\right|_{\alpha=0} = -i\psi \tag{3.40}$$

and $\delta\overline{\psi} = i\overline{\psi}$ respectively. The derivative of the Dirac Lagrangian in the gradient of the fields is

$$\frac{\delta\mathcal{L}}{\delta\left(\partial\psi\right)} = i\overline{\psi}\gamma^\mu \tag{3.41}$$

and $\frac{\delta\mathcal{L}}{\delta(\partial\overline{\psi})} = 0$ since the Lagrangian is not dependent on $\partial\overline{\psi}$. This gives the current

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{3.42}$$

which satisfies the relation

$$\partial_\mu j^\mu = \partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0. \tag{3.43}$$

The conserved charge is defined as $Q = \int \mathrm{d}^3x j^0$, so

$$Q = \int \mathrm{d}^3x\overline{\psi}\gamma^0\psi = \int \psi^\dagger\gamma^0\gamma^0\psi = \int \mathrm{d}^3x\psi^\dagger\psi. \tag{3.44}$$

In this case, we can interpret the charge as the fermion number.

[1] We will denote the *canonical momentum* with p and the usual, vector momentum as $\boldsymbol{p}$, in boldface. Later in the section we will take the norm of $\boldsymbol{p}$ and write is as $|\boldsymbol{p}| = p$, the distinction from the canonical momentum should be clear from the context.

3.2 Quantum field formalism for statistical mechanics

Statistical mechanics can help us describe systems in stable equilibrium, i.e. systems that will go back to their original state after some time T_t. Considering a time interval $(0, T_t)$ and dividing it into N steps such that $\Delta t = T_t/N$, we can write the path similarly to (3.23). By alternating between inserting position and momentum completeness relations (from (3.25) and (3.27)) we will obtain

$$\begin{aligned}\langle\phi_a|e^{-iHT_t}|\phi_a\rangle = \lim_{N\to\infty}\int\left(\prod_{i=1}^{N}\mathrm{d}p_i\mathrm{d}\phi_i/(2\pi)\right)&\\ \times\langle\phi_a|p_N\rangle\langle p_N|e^{-iH\Delta t}|\phi_N\rangle\langle\phi_N|p_{N-1}\rangle&\\ \times\langle p_{N-1}|e^{-iH\Delta t}|\phi_{N-1}\rangle\ldots&\\ \times\langle\phi_2|p_1\rangle\langle p_1|e^{-iH\Delta t}|\phi_1\rangle\langle\phi_1|\phi_a\rangle&\end{aligned} \tag{3.45}$$

where from (3.25) we know that $\langle\phi_1|\phi_a\rangle = \delta(\phi_1 - \phi_a)$. Notice that by now we drop the hat for all operators, unless the context makes it necessary to distiguish operators from, say, functions. From (3.28) we have that

$$\langle\phi_{i+1}|p_i\rangle = e^{i\int \mathrm{d}^3x p_i(\boldsymbol{x})\phi_{i+1}(\boldsymbol{x})} \tag{3.46}$$

and by introducing

$$H_i = \int \mathrm{d}^3x\mathcal{H}(p_i(\boldsymbol{x}), \phi_i(\boldsymbol{x})) \tag{3.47}$$

for the limit where $\Delta t \to 0$ we obtain

$$\begin{aligned}\langle\phi_i|e^{-iH_i\Delta t}|p_i\rangle &\approx \langle\phi_i|\left(1 - iH_i\Delta t\right)|p_i\rangle\\ &= \langle\phi_i|p_i\rangle\left(1 - iH_i\Delta t\right)\\ &= e^{-i\int \mathrm{d}^3x p_i\phi_j}e^{-i\Delta t\int \mathrm{d}^3x\mathcal{H}(p_i(\boldsymbol{x}),\phi_i(\boldsymbol{x}))}.\end{aligned} \tag{3.48}$$

By substituting (3.46) and (3.48) into (3.45) we get

$$\begin{aligned}\langle\phi_a|e^{-iHT_t}|\phi_a\rangle = \lim_{N\to\infty}\int\left(\prod_{i=1}^{N}\frac{\mathrm{d}p_i}{2\pi}\mathrm{d}\phi_i\right)\delta(\phi_i - \phi_a)&\\ \times\exp\left[i\Delta t\sum_{j=1}^{N}\int \mathrm{d}^3x\left(p_j(\phi_{j+1} - \phi_j)/\Delta t - \mathcal{H}(p_j, \phi_j)\right)\right].&\end{aligned} \tag{3.49}$$

The final step is to take the $N \to \infty$ limit and introduce the notation of *functional integration*

$$\int\mathcal{D}p = \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{\mathrm{d}p_i}{2\pi} \qquad \int\mathcal{D}\phi = \lim_{N\to\infty}\int\prod_{i=1}^{N}\mathrm{d}\phi_i$$

and then taking the limit $\Delta t \to 0$ transforming the sum in the exponential into an integral. The path integral will be

$$\langle \phi_a | e^{-iHT_t} | \phi_a \rangle = \int \mathcal{D}p\mathcal{D}\phi \exp\left[i \int_0^{T_t} \mathrm{d}t \int \mathrm{d}^3x \left(p\frac{\mathrm{d}\phi}{\mathrm{d}t} - \mathcal{H}(p,\phi) \right) \right]. \tag{3.50}$$

3.2.1 The partition function

By recalling the definition of the partition function in (3.1), we can write

$$Z = \sum_a \int \mathrm{d}\phi_a \langle \phi_a | e^{-\beta\left(H - \sum_i \mu_i N_i\right)} | \phi_a \rangle, \tag{3.51}$$

where the trace is taken by summing over the complete set of states ϕ_a spanning the Hilbert space. Equation (3.51) is very similar to (3.50), and we can express Z as an integral of the fields. In order to make the exponent real we integrate over imaginary time, making the substitutions $T_t = \beta/i$ and $t = \tau/i$, giving

$$Z = \int \mathcal{D}p\mathcal{D}\phi \exp\left[\int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^3x \left(ip\frac{\mathrm{d}\phi}{\mathrm{d}\tau} - \mathcal{H}(p,\phi) \right) \right]. \tag{3.52}$$

3.3 The partition function for relativistic fermions

Equation (3.52) works for bosons. In order to make the fermion version, we will have to use the Dirac Hamiltonian density, together with the transformation for the conserved charge density j^0:

$$\mathcal{H} \to \mathcal{H} + \mu j^0.$$

The Dirac Hamiltonian has two independent fields $\overline{\psi}$ and ψ, so we will have to make the functional integration run over these two fields instead of the field and the momentum as we did for bosons. One peculiarity of the Dirac four-spinors is that, by defining

$$\hat{\psi}(x,0)|\psi\rangle = \psi(x)|\psi\rangle \tag{3.53}$$

and the anticommutation relations for fermionic spinors

$$\{\hat{\psi}_\alpha(\boldsymbol{x},t), \hat{\psi}^\dagger_\beta(\boldsymbol{y},t)\} = \hbar\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{3.54}$$

$$\{\hat{\psi}_\alpha(\boldsymbol{x},t), \hat{\psi}_\beta(\boldsymbol{y},t)\} = \{\hat{\psi}^\dagger_\alpha(\boldsymbol{x},t), \hat{\psi}^\dagger_\beta(\boldsymbol{y},t)\} = 0 \tag{3.55}$$

the eigenvalues must anticommute. This can be tricky since we would like to integrate over these eigenvalues in order to get to the fermion equivalent of (3.52)[2]:

$$Z = \int \mathcal{D}i\psi^\dagger \mathcal{D}\psi \exp\left[\int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^3x \psi^\dagger \left(-\frac{\partial}{\partial\tau} + i\gamma^0\boldsymbol{\gamma}\cdot\nabla - \gamma^0 m + \mu \right) \psi \right]. \tag{3.56}$$

[2]The reason why we are integrating through $i\psi^\dagger$ instead of $\overline{\psi}$ will be clear later.

We can do this by using the Grassmann algebra, made exactly fo the purpose of handling numbers that anticommute.
Grassmann variables $\{\eta_i\}$ are defined by their anticommutation relation:

$$\{\eta_i, \eta_j\} = \{\eta_i^\dagger, \eta_j\} = \{\eta_i^\dagger, \eta_j^\dagger\} = 0, \tag{3.57}$$

and the integration is defined almost as a differentiation,

$$\int \mathrm{d}\eta = 0 \qquad \int \eta \mathrm{d}\eta = 1. \tag{3.58}$$

A general function of Grassmann variables can be expressed as

$$\begin{aligned} f = a + \sum_i a_i \eta_i + \sum_i b_i \eta_i^\dagger + \sum_{i,j} a_{ij} \eta_i \eta_j + \sum_{i,j} b_{ij} \eta_i \eta_j^\dagger \\ + \cdots + C \eta_1^\dagger \eta_1 \eta_2^\dagger \eta_2 \eta_3^\dagger \ldots \eta_N^\dagger \eta_N, \end{aligned} \tag{3.59}$$

where $a_i, b_i, a_{ij} \cdots$ are constants. Integrating the above function over all variables gives

$$\int \mathrm{d}\eta_1^\dagger \mathrm{d}\eta_1 \mathrm{d}\eta_2^\dagger \cdots \mathrm{d}\eta_N^\dagger \mathrm{d}\eta_N f = C, \tag{3.60}$$

and from this follows the property that

$$\begin{aligned} \int \mathrm{d}\eta_1^\dagger \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_N^\dagger \mathrm{d}\eta_N e^{\eta^\dagger D \eta} = \prod_{i=1}^{N} \left(\mathrm{d}\eta_i^\dagger \mathrm{d}\eta_i \right) e^{\eta^\dagger D \eta} \\ = \int \mathcal{D}\eta^\dagger \mathcal{D}\eta e^{\eta^\dagger D \eta} = \det D, \end{aligned} \tag{3.61}$$

where D is a $N \times N$ matrix. We see the similarities with (3.56), but before going on it is useful to expand $\psi_\alpha(\boldsymbol{x}, \tau)$ in a Fourier series

$$\psi_\alpha(\boldsymbol{x}, \tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\boldsymbol{p}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} + \omega_n \tau)} \tilde{\psi}_{\alpha,n}(\boldsymbol{p}), \tag{3.62}$$

where $\boldsymbol{p}$ is the momentum, ω_n the *Matsubara* (imaginary-time) *frequencies*, and V the volume in order to normalize the sum over the momenta.

With the new expansion of ψ in (3.62) we can rewrite our partition function in (3.56)

$$\begin{aligned}
Z &= \int \mathcal{D}i\psi^\dagger \mathcal{D}\psi \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \psi^\dagger \left(-\frac{\partial}{\partial\tau} + \gamma^0\gamma\cdot\nabla - \gamma^0 m + \mu\right)\psi\right] \\
&= \int \mathcal{D}i\psi^\dagger \mathcal{D}\psi \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \frac{1}{V}\sum_n\sum_{\boldsymbol{p}}\right. \\
&\quad \left. \times\, \tilde{\psi}_n^\dagger(\boldsymbol{p}) e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}+\omega_n\tau)}\left(-\frac{\partial}{\partial\tau} + \gamma^0\gamma\cdot\nabla - \gamma^0 m + \mu\right) e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+\omega_n\tau)}\tilde{\psi}_n(\boldsymbol{p})\right] \\
&= \int \mathcal{D}i\psi^\dagger \mathcal{D}\psi \exp\left[\frac{1}{V}\sum_n\sum_{\boldsymbol{p}}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\right. \\
&\qquad\qquad \left. \times\, \tilde{\psi}_n^\dagger(\boldsymbol{p})\left(-i\omega_n - \gamma^0\gamma\cdot\boldsymbol{p} - m\gamma^0 + \mu\right)\tilde{\psi}_n(\boldsymbol{p})\right] \\
&= \int \mathcal{D}i\psi^\dagger \mathcal{D}\psi \exp\left[\sum_n\sum_{\boldsymbol{p}} i\tilde{\psi}_n^\dagger(\boldsymbol{p})(-i\beta)\left(-i\omega_n - \gamma^0\gamma\cdot\boldsymbol{p} - m\gamma^0 + \mu\right)\tilde{\psi}_n(\boldsymbol{p})\right].
\end{aligned} \tag{3.63}$$

The partition function in (3.63) may then be written as

$$Z = \prod_{\boldsymbol{p}}\int \mathcal{D}(i\psi^\dagger\psi)\exp\left[\sum_n\sum_{\boldsymbol{p}} i\tilde{\psi}_{\alpha,n}^\dagger(\boldsymbol{p}) D_{\alpha\rho}\tilde{\psi}_{\rho,n}(\boldsymbol{p})\right] \tag{3.64}$$

where

$$D = -i\beta\left[(-i\omega_n + \mu) - \gamma^0\gamma\cdot\boldsymbol{p} - m\gamma^0\right]. \tag{3.65}$$

Finally we can apply (3.61) to our partition function in (3.64) and obtain

$$Z = \det D. \tag{3.66}$$

The determinant is taken through all the indices: both the 4×4 gamma matrices, the frequency and the momentum indices. We can start by finding the determinant of the gamma matrices

$$\begin{aligned}
\det D &= \det -i\beta\left[(-i\omega_n + \mu) - \gamma^0\gamma\cdot\boldsymbol{p} - m\gamma^0\right] \\
&= \det\beta\left[-(\omega_n + i\mu) + i\gamma^0\gamma\cdot\boldsymbol{p} + im\gamma^0\right] \\
&= \det\beta\left\{\begin{pmatrix} -\mathbb{1}((\omega_n+i\mu)+im) & 0 \\ 0 & -\mathbb{1}((\omega_n+i\mu)-im)\end{pmatrix} + i\begin{pmatrix} 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0\end{pmatrix}\right\} \\
&= \det\beta\begin{pmatrix} -\mathbb{1}((\omega_n+i\mu)+im) & i\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ i\boldsymbol{p}\cdot\boldsymbol{\sigma} & -\mathbb{1}((\omega_n+i\mu)-im)\end{pmatrix} \\
&= \det\beta^2\left\{\mathbb{1}\left[(\omega_n+i\mu)^2 + m^2\right] + (\boldsymbol{p}\cdot\boldsymbol{\sigma})^2\right\} \\
&= \det\beta^2\left[(\omega_n+i\mu)^2 + m^2 + p^2\right]\mathbb{1} \\
&= \beta^4\left((\omega_n+i\mu)^2 + \omega^2\right)^2
\end{aligned} \tag{3.67}$$

where we have used $(\boldsymbol{p}\cdot\boldsymbol{\sigma})^2 = p^2$, introduced $\omega = \sqrt{m^2+p^2}$, and indicated by $\mathbb{1}$ the 2×2 identity matrix. Between the fourth and fifth step, we have used the property for square matrices of the same order *A*, *B*, *C* and *D* where

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(AD - BC\right) \qquad \text{for} \qquad CD = DC,$$

and consequently in the fifth and sixth step we are taking the determinant of a 2×2 matrix. We are now left with finding the determinant through all n and $\boldsymbol{p}$ for which D is defined in (3.64). Instead of taking the determinant, we can use Jacobi's formula

$$\ln\det A = \mathrm{tr}\ln A, \tag{3.68}$$

valid for any matrix A. Since we are only interested in $\ln Z$ this turns out to be useful, and we apply the formula to D:

$$\ln Z = \ln\det D = \mathrm{tr}\ln D = \sum_n\sum_{\boldsymbol{p}}\ln\beta^4\left((\omega_n+i\mu)^2+\omega^2\right)^2. \tag{3.69}$$

The summation is over both positive and negative Matsubara frequencies, thus we can rewrite (3.69) as

$$\begin{aligned}
\ln Z &= \sum_n\sum_{\boldsymbol{p}}\ln\beta^4\left((\omega_n+i\mu)^2+\omega^2\right)^2 \\
&= \sum_n\sum_{\boldsymbol{p}}\ln\left[\beta^4\left((\omega_n+i\mu)^2+\omega^2\right)\left((-\omega_n+i\mu)^2+\omega^2\right)\right] \\
&= \sum_n\sum_{\boldsymbol{p}}\ln\left[\beta^4\left(\omega_n^2+\omega^2-\mu^2+2i\omega_n\mu\right)\left(\omega_n^2+\omega^2-\mu^2-2i\omega_n\mu\right)\right] \\
&= \sum_n\sum_{\boldsymbol{p}}\ln\left\{\beta^4[\omega_n-i(\omega-\mu)]\,[\omega_n+i(\omega+\mu)]\,[\omega_n+i(\omega-\mu)]\,[\omega_n-i(\omega+\mu)]\right\} \\
&= \sum_n\sum_{\boldsymbol{p}}\ln\left[\beta^4\left(\omega_n^2+(\omega-\mu)^2\right)\left(\omega_n^2+(\omega+\mu)^2\right)\right] \\
&= \sum_n\sum_{\boldsymbol{p}}\left[\ln\beta^2\left(\omega_n^2+(\omega-\mu)^2\right)+\ln\beta^2\left(\omega_n^2+(\omega+\mu)^2\right)\right].
\end{aligned} \tag{3.70}$$

We can find an expression for ω_n by analyzing the relation between the Green functions $G_F(\boldsymbol{x},\boldsymbol{y};\tau,0)$ and $G_F(\boldsymbol{x},\boldsymbol{y};\tau,\beta)$. The Green function is defined as

$$G(\boldsymbol{x},\boldsymbol{y};\tau_1,\tau_2) = Z^{-1}\mathrm{tr}\left[e^{-\beta K}T_\tau\left[\hat{\psi}(\boldsymbol{x},\tau_1)\hat{\psi}(\boldsymbol{y},\tau_2)\right]\right] \tag{3.71}$$

where we have introduced the time-ordering operator T_τ

$$T_\tau\left[\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\right] = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1-\tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2-\tau_1) \tag{3.72}$$

and shortened the notation with the substitution $K = H - \mu Q$. For fermions we will then have

$$\begin{aligned} G_F(\boldsymbol{x},\boldsymbol{y};\tau,0) &= Z^{-1}\text{tr}\left[e^{-\beta K}\hat{\psi}(\boldsymbol{x},\tau)\hat{\psi}(\boldsymbol{y},0)\right] \\ &= Z^{-1}\text{tr}\left[\hat{\psi}(\boldsymbol{y},0)e^{-\beta K}\hat{\psi}(\boldsymbol{x},\tau)\right] \\ &= Z^{-1}\text{tr}\left[e^{-\beta K}e^{\beta K}\hat{\psi}(\boldsymbol{y},0)e^{-\beta K}\hat{\psi}(\boldsymbol{x},\tau)\right] \\ &= Z^{-1}\text{tr}\left[e^{-\beta K}\hat{\psi}(\boldsymbol{y},\beta)\hat{\psi}(\boldsymbol{x},\tau)\right] \\ &= Z^{-1}\text{tr}\left[e^{-\beta K}T_\tau\left[\hat{\psi}(\boldsymbol{x},\tau)\hat{\psi}(\boldsymbol{y},\beta)\right]\right] \\ &= -G_F(\boldsymbol{x},\boldsymbol{y};\tau,\beta), \end{aligned} \tag{3.73}$$

where we have used the cycling property of the trace and the fact that, in the Heinsenberg picture, $\hat{\psi}(\boldsymbol{x},\tau_0) = e^{\tau_0 K}\hat{\psi}((x),0)e^{-\tau_0 K}$. This means that $\psi(\boldsymbol{x},0) = -\psi(\boldsymbol{x},\beta)$ and that the wavefunction will be antiperiodic in the interval $0<\tau<\beta$ for periods of $2\beta/(2n+1)$ where $n = 0,\pm1,\pm2,\pm3\ldots$. We will then have

$$\omega_n(2n+1)\frac{\pi}{\beta}. \tag{3.74}$$

We can take the summation over n in (3.70) by using the definition of ω_n in (3.74), and rewriting both logarithms in integral form

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\ln\left[(2n+1)^2\pi^2+\beta^2(\omega\pm\mu)^2\right] \\ &\qquad = \sum_{n=-\infty}^{\infty}\left[\ln\left[1+(2n+1)^2\pi^2\right]+\int_1^{\beta^2(\omega\pm\mu)^2}\frac{\mathrm{d}\theta^2}{\theta^2+(2n+1)^2\pi^2}\right], \end{aligned} \tag{3.75}$$

where we can drop the first term since it is independent of β, μ and V and would vanish when taking the differentiations in (3.5), (3.6) and (3.7). We now take the summation inside the integral, an by using the identity

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n-x)(n-y)} = \frac{\pi(\cot\pi x-\cot\pi y)}{y-x}$$

we get

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{1}{[(2n+1)\pi + i\theta]\,[(2n+1)\pi - i\theta]} \\
&= \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \frac{1}{\left[n - \left(-\frac{1}{2} - \frac{i\theta}{2\pi}\right)\right]\left[n - \left(-\frac{1}{2} + \frac{i\theta}{2\pi}\right)\right]} \\
&= \frac{1}{(2\pi)^2} \frac{\pi\left(\cot\left(-\frac{\pi}{2} - \frac{i\theta}{2}\right) - \cot\left(-\frac{\pi}{2} + \frac{i\theta}{2}\right)\right)}{i\theta/(2\pi)} \\
&= \frac{1}{4i\theta}\left(\tan\left(\frac{i\theta}{2}\right) - \tan\left(-\frac{i\theta}{2}\right)\right) \\
&= \frac{1}{2i\theta}\tan\frac{i\theta}{2} = \frac{1}{2\theta}(-i)\tan\left(i\frac{\theta}{2}\right) = \frac{1}{2\theta}\tanh\frac{\theta}{2} \\
&= \frac{1}{2\theta}\frac{e^{\theta/2} - e^{-\theta/2}}{e^{\theta/2} + e^{+\theta/2}} = \frac{1}{2\theta}\frac{e^{\theta} - 1}{e^{\theta} + 1} = \frac{1}{2\theta}\left(\frac{e^{\theta} - 1 + 2}{e^{\theta} + 1} - \frac{2}{e^{\theta} + 1}\right) \\
&= \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{e^{\theta} + 1}\right),
\end{aligned}
\tag{3.76}
$$

where we have used the trigonometric identities $\cot(-x) = -\cot(x)$, $\cot(\pi/2 - x) = \tan x$, $\tan(-x) = -\tan x$ and $-i\tan(ix) = \tanh x$. We can now plug the result in (3.76) into (3.75), obtaining

$$
\int_1^{\beta^2(\omega\pm\mu)^2} \frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{e^{\theta} + 1}\right) \mathrm{d}\theta^2. \tag{3.77}
$$

We would now like to do the substitution $\mathrm{d}\theta^2 = 2\theta\mathrm{d}\theta$, but the limit in the integral could be either positive or negative. We then name $C = \beta(\omega \pm \mu)$ and write

$$
\int_{\pm 1}^{\pm C} \left(1 - 2\frac{2}{e^{\theta} + 1}\right) \mathrm{d}\theta = \pm C \mp 1 - 2\int_{\pm 1}^{\pm C} \frac{\mathrm{d}\theta}{e^{\theta} + 1}.
$$

We can now make the substitution $u = e^{\theta}$, thus

$$
\begin{aligned}
&\pm C \mp 1 - 2\int_{e^{\pm 1}}^{e^{\pm C}} \frac{\mathrm{d}u}{u(u+1)} = \pm C \mp 1 - 2\int_{e^{\pm 1}}^{e^{\pm C}} \left(\frac{1}{u} - \frac{1}{u+1}\right) \mathrm{d}u \\
&= \mp C \pm 1 + 2\ln\left(e^{\pm C} + 1\right) - \ln\left(e^{\pm 1} + 1\right),
\end{aligned}
$$

and by dropping the terms independent of C for the same reason as in (3.75), we get

$$
\mp C + 2\ln\left(e^{\pm C} + 1\right). \tag{3.78}
$$

In (3.78) we see that by choosing the upper sign the logarithm diverges, so we have to pick the lower one,

$$
\beta(\omega \pm \mu) + 2\ln\left[e^{-\beta(\omega\pm\mu)} + 1\right], \tag{3.79}
$$

which is the final result for our sum over n in (3.75). Substituting in (3.70) will give

$$\begin{aligned}\ln Z &= \sum_{\boldsymbol{p}} \left\{ \beta(\omega+\mu) + \beta(\omega-\mu) + 2\ln\left[e^{-\beta(\omega+\mu)}+1\right] + 2\ln\left[e^{-\beta(\omega-\mu)}+1\right] \right\} \\ &= 2V \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\beta\omega + \ln\left(e^{-\beta(\omega+\mu)}+1\right) + \ln\left(e^{-\beta(\omega-\mu)}+1\right)\right], \end{aligned} \tag{3.80}$$

where we have taken the continuum limit of the momentum

$$\frac{1}{V}\sum_{\boldsymbol{p}} \to \int \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

In (3.80) we notice the contribution of particles (μ) and antiparticles ($-\mu$), as well as the zero-point energy ($\beta\omega$). We may integrate this expression, and obtain

$$\begin{aligned}\ln Z &= 2V \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\beta\omega + \frac{V}{3\pi^2}\left[p^3 \ln\left(1+e^{-\beta(\omega+\mu)}\right) + \beta \int p^4 \frac{\omega^{-\frac{1}{2}} e^{-\beta(\omega+\mu)}}{1+e^{-\beta(\omega+\mu)}}\mathrm{d}p \right. \\ &\qquad \left. + p^3 \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \beta \int p^4 \frac{\omega^{-\frac{1}{2}} e^{-\beta(\omega-\mu)}}{1+e^{-\beta(\omega-\mu)}}\mathrm{d}p \right]_0^\infty \\ &= 2V \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\beta\omega + \frac{V\beta}{3\pi^2}\int_0^\infty \frac{p^4 \mathrm{d}p}{\sqrt{p^2+m^2}} \left[\frac{1}{1+e^{\beta(\omega+\mu)}} + \frac{1}{1+e^{\beta(\omega-\mu)}}\right], \end{aligned}$$

where the terms appearing from the partial integration vanish at 0 and ∞. The expression for the pressure is easily found by using (3.5) and remembering that $T = \beta^{-1}$:

$$\begin{aligned}P = \frac{\partial(T\ln Z)}{\partial V} &= 2\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\omega \\ &\quad + \frac{1}{3\pi^2}\int_0^\infty \frac{p^4 \mathrm{d}p}{\sqrt{p^2+m^2}} \left[\frac{1}{1+e^{(\omega+\mu)/T}} + \frac{1}{1+e^{(\omega-\mu)/T}}\right]. \end{aligned} \tag{3.81}$$

In the terms in the square parenthesis we recognize the Fermi-Dirac distribution for particles and anti-particles (2.79). By taking the $T \to 0$ limit, the distribution for particles becomes the Heaviside step function, and the anti-particles term vanish. If we then ignore the divergent term[3], we recover the equation for the pressure in a cold, free Fermi gas:

$$
\begin{aligned}
\lim_{T\to 0} P &= \frac{1}{3\pi^2}\int_0^\infty \frac{p^4}{\sqrt{p^2+m^2}}\theta(p_F-p)\mathrm{d}p \\
&= \frac{1}{3\pi^2}\int_0^{p_F} \frac{p^4}{\sqrt{p^2+m^2}}\mathrm{d}p \\
&= \frac{m^4}{3\pi^2}\int_0^{x} \frac{u}{\sqrt{u^2+1}}u^3\mathrm{d}u \\
&= \frac{m^4}{3\pi^2}\left[u^3\sqrt{u^2+1} - \int 3u^2\sqrt{u^2+1}\right]_0^x \\
&= \frac{m^4}{3\pi^2}\int_0^{x} \left(3u^2\sqrt{x^2+1} - 3u^2\sqrt{u^2+1}\right)\mathrm{d}u \\
&= \frac{m^4}{\pi^2}\int_0^{x} \left(\sqrt{x^2+1} - \sqrt{u^2+1}\right)u^2\mathrm{d}u \\
&= \frac{m^4}{24\pi^2}\left[\left(2x^3-3x\right)\left(1+x^2\right)^{1/2} + 3\sinh^{-1}(x)\right]
\end{aligned} \tag{3.82}
$$

which is the same result as in (2.99) expressed in natural units. Similarly, we can find the energy density for particles from (3.4):

$$
\begin{aligned}
\epsilon = \frac{E}{V} &= -P + \frac{T}{V}S + \frac{\mu}{V}N \\
&= -P + \frac{T}{V}\frac{\partial(T\ln Z)}{\partial T} + \mu\frac{\partial(T\ln Z)}{\partial\mu} \\
&= -P + 2T\frac{\partial}{\partial T}\left[T\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(e^{-(\omega-\mu)/T}+1\right)\right] \\
&\qquad + 2\mu T\frac{\partial}{\partial\mu}\left[\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(e^{-\omega/T+\mu/T}+1\right)\right],
\end{aligned} \tag{3.83}
$$

we have already calculated the pressure in (3.82), and the second term would eventually vanish when the $T\to 0$ limit is taken. We then focus on the third term:

$$
\begin{aligned}
&2\mu T\frac{\partial}{\partial\mu}\left[\int\frac{\mathrm{d}^3p}{(2\pi)^3}\ln\left(e^{-\omega/T+\mu/T}+1\right)\right] \\
=&2\mu T\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\frac{1}{T}e^{-(\omega-\mu)/T}}{e^{-(\omega-\mu)/T}+1} \\
=&2\mu\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{e^{(\omega-\mu)/T}+1},
\end{aligned} \tag{3.84}
$$

where, again, the integrand is the Fermi-Dirac distribution. We plug this where we left in

[3]This will be treated in Chapter 6.

(3.83), our result for the zero-temperature pressure, and take the $T \to 0$ limit:

$$\begin{aligned}
\epsilon =& -\frac{m^4}{\pi^2}\int_0^x \left(\sqrt{x^2+1}-\sqrt{u^2+1}\right)u^2\mathrm{d}u + \lim_{T\to 0} 2\mu\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{e^{(\omega-\mu)/T}+1} \\
=& -\frac{m^4}{\pi^2}\int_0^x \left(\sqrt{x^2+1}-\sqrt{u^2+1}\right)u^2\mathrm{d}u + \sqrt{p_F^2+m^2}\int_0^{p_F}\frac{\mathrm{d}p}{\pi^2}p^2 \\
=& -\frac{m^4}{\pi^2}\int_0^x \left(\sqrt{x^2+1}-\sqrt{u^2+1}\right)u^2\mathrm{d}u + \frac{m^4}{\pi^2}\int_0^x u^2\mathrm{d}u\sqrt{x^2+1} \\
=& \frac{m^4}{\pi^2}\int_0^x u^2\mathrm{d}u\sqrt{u^2+1} \\
=& \frac{m^4}{8\pi^2}\left[\left(2x^3+x\right)\left(1+x^2\right)^{1/2}-\sinh^{-1}(x)\right],
\end{aligned} \tag{3.85}$$

which again is the same result as obtained in (2.98).

3.4 The partition function for relativistic bosons

We already have found the partition function for bosons in (3.52), and in case of a scalar, real, neutral and spin-zero field ϕ, we would have a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{3.86}$$

We can find the partition function by finding the canonical momentum[4],

$$p = \frac{\mathcal{L}}{\partial(\partial\phi/\partial t)} = \frac{\partial\phi}{\partial t}, \tag{3.87}$$

then doing the usual Legendre transformation in order to obtain the Hamiltonian density,

$$\mathcal{H} = p\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{1}{2}p^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{3.88}$$

and then inserting this into (3.52):

$$Z = \int \mathcal{D}p\mathcal{D}\phi \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left(ip\frac{\partial\phi}{\partial\tau} - \frac{1}{2}p^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right)\right]. \tag{3.89}$$

We can get rid of the momentum path integral by discretizing it once again as in (3.49):

$$\begin{aligned}
Z = \lim_{N\to\infty}&\left(\prod_{i=1}^N \int_{-\infty}^\infty \frac{\mathrm{d}p_i}{2\pi}\int \mathrm{d}\phi_i\right) \\
&\times \exp\left\{\sum_{j=1}^N \int \mathrm{d}^3x\left[ip_j(\phi_{j+1}-\phi_j)\right.\right. \\
&\left.\left. - \Delta\tau\left(\frac{1}{2}p_j^2 + \frac{1}{2}(\nabla\phi_j)^2 + \frac{1}{2}m^2\phi_j^2\right)\right]\right\}
\end{aligned} \tag{3.90}$$

[4]See footnote in page 30.

and dividing position space into M^3 small cubes, M being an integer and the cubes of volume $V = L^3$, $L = aM$, where a is an infinitesimal length:

$$\int \mathrm{d}^3x = \int \mathrm{d}x_1 \int \mathrm{d}x_2 \int \mathrm{d}x_3 = \sum_{i=1}^{M} a_i \sum_{j=1}^{M} a_j \sum_{k=1}^{M} a_k = \sum_{k}^{M^3} a_k^3. \tag{3.91}$$

We do this in order to enforce the periodicity of ϕ. The cubes are small and we will be taking the $M \to \infty$ limit after the integration is taken. In order to keep Z dimensionless we integrate over A_j, where $p_j = A_j/(a_j^3 \Delta\tau)^{1/2}$. Taking only the integration over momentum in (3.90) with the above substitutions, we obtain

$$\prod_{i=1}^{N} \int \mathrm{d}\phi_i \int \frac{1}{\sqrt{a_i^3 \Delta t}} \frac{\mathrm{d}A_i}{2\pi} \exp\left[\sum_{k=1}^{M^3} a_k^3 \sum_{j=1}^{N} \left(i \frac{A_j(\phi_{j+1} - \phi_j)}{(a_j^3 \Delta t)^{1/2}} - \frac{1}{2} \frac{A_j^2}{a_j^3} \right)\right]. \tag{3.92}$$

Here is only relevant to take integrals where $i = j = k$, so, for each cube,

$$\frac{1}{\sqrt{a_j^3 \Delta\tau}} \int_{-\infty}^{\infty} \frac{\mathrm{d}A_j}{2\pi} \exp\left[-\frac{1}{2} A_j^2 - 2 \frac{(\phi_{j+1} - \phi_j)\sqrt{a^3}}{2i\sqrt{\Delta\tau}}\right] \tag{3.93}$$

Each $\mathrm{d}A_j$ integration is a Gaussian integral that can be looked up on tables [32], with solution

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-bx^2 - 2cx} = \sqrt{\frac{\pi}{b}} e^{\frac{c^2}{b}}. \tag{3.94}$$

In our case is $b = 1/2$ and $c = i(\phi_{j+1} - \phi_j)/\sqrt{a^3/4\Delta t}$, so the solution of the integral is

$$\frac{1}{\sqrt{a^3 \Delta\tau}} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-a^3(\phi_{j+1} - \phi_1)^2}{2\Delta\tau}\right], \tag{3.95}$$

Recovering the integral over position space using (3.91) and (3.95) in the original discrete path integral, we have

$$\begin{aligned} \lim_{M,N \to \infty} \frac{1}{(2\pi a^3 \Delta\tau)^{M^3 N/2}} \int \left(\prod_{i=1}^{N} \mathcal{D}\phi_i \right) \\ \times \exp\left\{ \Delta\tau \sum_{j=1}^{N} \int \mathrm{d}^3x \left[-\frac{1}{2} \left(\frac{\phi_{j+1} - \phi_j}{\Delta\tau} \right)^2 \right.\right. \\ \left.\left. - \frac{1}{2}(\nabla\phi_j)^2 - \frac{1}{2} m^2 \phi_j^2 \right] \right\}. \end{aligned} \tag{3.96}$$

As before, we take the continuous limit and recover the integral over $\mathrm{d}\tau$:

$$\begin{aligned} Z &= N' \int \mathcal{D}\phi \exp\left\{ \int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^3x \left[-\frac{1}{2} \left(\frac{\partial\phi}{\partial\tau} \right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \right\} \\ &= N' \int \mathcal{D}\phi \exp\left(\int_0^{\beta} \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L} \right) = N' \int \mathcal{D}\phi e^S \end{aligned} \tag{3.97}$$

where again we have used the identity $\mathrm{d}\tau = i\mathrm{d}t$ in the first term of the Lagrangian. From now on, as done with fermions, we might drop the constant N'. In contrast to the antiperiodic fermions, ϕ is periodic, so that $\phi(\boldsymbol{x}, 0) = \phi(\boldsymbol{x}, \beta)$. This allow us to partially integrate both the time and space derivatives in the integral in the exponential of (3.97),

$$\begin{aligned}
S =& -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 - m^2\phi^2\right] \\
=& -\frac{1}{2}\left\{\int \left[\phi\frac{\partial\phi}{\partial\tau}\right]_0^\beta \mathrm{d}^3x + \int_0^\beta \left[\phi\nabla\phi\right]_{-\infty}^{+\infty} \mathrm{d}\tau + \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left(-\phi\frac{\partial^2\phi}{\partial\tau^2} - \phi\nabla^2\phi\right)\right\} \\
=& -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\phi \left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right)\phi. \qquad (3.98)
\end{aligned}$$

Where the $[\phi\nabla\phi]$ term vanishes at infinity. We then follow the same procedure used with the fermion partition function by Fourier transforming the field. The ϕ field is real, so two equivalent transformations are possible:

$$\phi(\boldsymbol{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\boldsymbol{p}} e^{-i(\boldsymbol{p}\cdot\boldsymbol{x} + \omega_n\tau)} \phi_n^*(\boldsymbol{p}), \qquad (3.99)$$

$$\phi(\boldsymbol{x}, \tau) = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\boldsymbol{p}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} + \omega_n\tau)} \phi_n(\boldsymbol{p}), \qquad (3.100)$$

where we have inserted $\omega_n = 2\pi n/\beta$, different from the fermion one because of periodicity. Inserting these in the action S of (3.98) and recalling the notation $\omega = \sqrt{p^2 + m^2}$, we obtain:

$$\begin{aligned}
S =& -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \frac{\beta}{V} \sum_n \sum_{\boldsymbol{p}} \left(\omega_n^2 + \omega^2\right) \phi^*(\boldsymbol{p})\phi(\boldsymbol{p}) \\
=& -\frac{1}{2}\beta^2 \sum_n \sum_{\boldsymbol{p}} \left(\omega_n^2 + \omega^2\right) \phi_n^*(\boldsymbol{p})\phi_n(\boldsymbol{p}). \qquad (3.101)
\end{aligned}$$

The sums become products when taken out of the exponential, and the path integral is only dependent on the magnitude of the field $A_n = (A_n^* A_n)^{1/2}$, so

$$Z = \prod_n \prod_{\boldsymbol{p}} \left\{\int_{-\infty}^{\infty} \mathrm{d}A_n(\boldsymbol{p}) \exp\left[-\frac{1}{2}\beta^2 \left(\omega_n^2 + \omega^2\right) A_n^2(\boldsymbol{p})\right]\right\}. \qquad (3.102)$$

This is again a Gaussian integral in the form of (3.94) with $c = 0$, yielding

$$Z = \prod_n \prod_{\boldsymbol{p}} \sqrt{2\pi} \left[\beta^2 \left(\omega_n^2 + \omega^2\right)\right]^{-\frac{1}{2}}, \qquad (3.103)$$

where we can safely drop the $\sqrt{2\pi}$ term. We now take the logarithm of the above function

$$\ln Z = -\frac{1}{2} \sum_n \sum_{\boldsymbol{p}} \ln\left[\beta^2 \left(\omega_n^2 + \omega^2\right)\right], \qquad (3.104)$$

and rewrite it in the same way as in (3.75)

$$\ln Z = -\frac{1}{2}\sum_n \sum_{\boldsymbol{p}} \ln\left[(2\pi n)^2 + \beta^2\omega^2\right] \tag{3.105}$$

$$= -\frac{1}{2}\sum_n \sum_{\boldsymbol{p}} \int_1^{\beta^2\omega^2} \frac{\mathrm{d}\theta^2}{\theta^2 + (2\pi n)^2} - \frac{1}{2}\ln[1 + (2\pi n)^2]. \tag{3.106}$$

In (3.105) we can drop the last term, make the substitution $\mathrm{d}\theta^2 = 2\theta\mathrm{d}\theta$, changing the limits as with the fermion case, and rewrite it as

$$\ln Z = -\sum_n \sum_{\boldsymbol{p}} \frac{1}{4\pi^2} \int_{\pm 1}^{\pm\beta\omega} \frac{\theta\mathrm{d}\theta}{n^2 + \left(\frac{\theta}{2\pi}\right)^2}, \tag{3.107}$$

for so using the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \left(\frac{\theta}{2\pi}\right)^2} = \frac{2\pi^2}{\theta}\left(1 + \frac{2}{e^\theta - 1}\right) \tag{3.108}$$

in (3.107), and obtaining

$$\ln Z = -\sum_{\boldsymbol{p}} \int_{\pm 1}^{\pm\beta\omega} \mathrm{d}\theta \left(\frac{1}{2} + \frac{1}{e^\theta - 1}\right). \tag{3.109}$$

In order to integrate we mimic the procedure used with fermions, making the substitution $u = e^\theta$, where $\mathrm{d}\theta = \mathrm{d}u/u$ and

$$\begin{aligned}
\ln Z &= \sum_{\boldsymbol{p}} \left[\mp\frac{\beta\omega}{2} \pm \frac{1}{2} + \int_{e^{\pm 1}}^{e^{\pm\beta\omega}} \frac{\mathrm{d}u}{u(u-1)}\right] \\
&= \sum_{\boldsymbol{p}} \left[\mp\frac{\beta\omega}{2} \pm \frac{1}{2} + \int_{e^{\pm 1}}^{e^{\pm\beta\omega}} \left(\frac{1}{u-1} - \frac{1}{u}\right)\mathrm{d}u\right] \\
&= \sum_{\boldsymbol{p}} \left[\mp\frac{\beta\omega}{2} \pm \frac{1}{2} + \ln\left(1 - e^{\mp\beta\omega}\right) - \ln\left(1 - e^{\mp 1}\right)\right].
\end{aligned}$$

Here the reasonable choice is to pick the upper signs so to avoid negative numbers in the logarithms. Scrapping also all the terms not dependent on the temperature, we are left with

$$\ln Z = -\sum_{\boldsymbol{p}} \left[\frac{1}{2} + \ln\left(1 - e^{-\beta\omega}\right)\right] \tag{3.110}$$

which, taking the continuous limit for the momentum, gives

$$\ln Z = V \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[-\frac{1}{2}\beta\omega - \ln(1 - e^{-\beta\omega})\right]. \tag{3.111}$$

Equation (3.111) is our partition function for neutral, scalar fields. From this we can obtain all the thermodynamical quantities of interest, such as the pressure and the energy density using equations (3.4) and (3.5). By defining $\ln Z_0$ to be the divergent part of the partition function,

$$\ln Z_0 = V \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(-\frac{1}{2} \beta\omega \right), \tag{3.112}$$

we obtain for the zero-point pressure and energy density

$$P_{\text{ZP}} = \frac{\partial(T \ln Z_0)}{\partial V} = -\frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{p^2 + m^2} \tag{3.113}$$

and

$$\epsilon_{\text{ZP}} = \frac{E_{\text{ZP}}}{V} = \frac{-P_{\text{ZP}} V}{V} = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{p^2 + m^2}. \tag{3.114}$$

These are divergent and independent of the temperature, and will have to be renormalized.

Chapter 4

The σ–ω model

A neutron star is mainly composed of neutrons. Up to now we have considered this kind of star as a Fermi gas consisting of noninteracting cold neutrons, but we should expect a small fractions of other particles such as protons and electrons emerging from the beta decay of neutrons, and possibly heavier leptons and baryons forming at higher densities. Baryons interact by the strong nuclear force, and this interaction should give a contribution to the equations of energy density and pressure, modifying the equation of state. This motivates us to look for a model which takes into account these phenomena. The strong force can be described using relativistic mean-field (RMF) models, where it is mediated by the exchange of massive spin-zero and spin-one mesons between baryons. These interactions are best described by the *quantum chromodynamics* (QCD) theory, but the RMF models are much easier to deal with when handling big statistical systems such as neutron stars. The simplest RMF model is the *σ–ω model*, developed among others by Chin and Walecka [7] [41]. In this description the strong nuclear force is carried by the two massive mesons: the σ and the ω, the first being a scalar field and the second a vector field. The model exploits the fact that the neutron and the proton masses are very similar and treats them as one single particle, the *nucleon*, where the proton and the neutron can be seen as its two possible states. The steps in this chapter follow the ones used in *Compact Stars* by Glendenning [10].

4.1 The σ–ω model

4.1.1 Spin-zero and spin-one Lagrangian

the spin-zero σ field is a spinless, scalar field described by the free Klein-Gordon equation:

$$\left(\Box + m^2\right)\sigma = 0. \tag{4.1}$$

This can be derived by the scalar Lagrangian

$$\mathcal{L}_\sigma = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - \frac{1}{2}m^2\sigma^2 \tag{4.2}$$

using the Euler-Lagrange equations in (2.4) for scalar fields,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \sigma)} = \frac{\partial \mathcal{L}}{\partial \sigma}. \tag{4.3}$$

If the field is complex, we can rewrite the Lagrangian as

$$\mathcal{L}_\sigma = \frac{1}{2}(\partial^\mu \sigma)^*(\partial_\mu \sigma) - \frac{1}{2}m^2 \sigma^* \sigma. \tag{4.4}$$

It has an internal global symmetry when we make the transformation $\sigma' \to e^{-i\theta}\sigma$ and $\sigma^{*\prime} \to e^{i\theta}\sigma^*$ for a constant, real phase θ, and it is manifestly invariant. Since $\delta\sigma = \frac{\partial}{\partial\theta}(\sigma' - \sigma) = -i\sigma$ and similarly $\delta\sigma^* = i\sigma^*$, we can use the Noether theorem to find the conserved current:

$$J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \sigma)}\delta\sigma + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \sigma^*)}\delta\sigma^* = \frac{i}{2}(\sigma^* \partial^\mu \sigma - \sigma \partial^\mu \sigma^*). \tag{4.5}$$

The spin-one ω field is instead a massive vector field described by the Proca equation

$$(\Box + m^2)\omega_\mu - \partial_\mu \partial^\nu \omega_\nu = 0 \tag{4.6}$$

and is derived by the free Lagrangian

$$\mathcal{L}_\omega = -\frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m^2 \omega_\mu \omega^\mu, \tag{4.7}$$

where $\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$. By taking the derivative of the Proca equation,

$$(\Box + m^2)\partial^\mu \omega_\mu - \Box \partial^\mu \omega_\mu = 0, \tag{4.8}$$

it follows that $m^2 \partial^\mu \omega_\mu = 0$. Since m cannot be zero, the field must be divergenceless and the last term of the Proca equation in (4.6) cancels. The equation of motion for each component can be then described with the Klein-Gordon equation in (4.1).

4.1.2 The nucleon spinor and Lagrangian

By assuming the neutron and the proton to have the same mass m, we can write a free nucleon Lagrangian as the sum of the neutron and proton Dirac Lagrangians,

$$\mathcal{L}_{\text{nucl}} = \overline{\psi}_n(i\not{\partial} - m)\psi_n + \overline{\psi}_p(i\not{\partial} - m)\psi_p, \tag{4.9}$$

where ψ_n and ψ_p are the neutron and proton spinors, and $\overline{\psi}_n$ and $\overline{\psi}_p$ their adjoints. With these definitions we can define a new, eight-components nucleon spinor

$$\psi \equiv \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \tag{4.10}$$

with which we can rewrite the nucleon Lagrangian as

$$\mathcal{L}_{\text{nucl}} = \overline{\psi}(i\not{\partial} - m)\psi. \tag{4.11}$$

This Lagrangian still has a global U(1) symmetry, leading to the current already derived in (3.42),

$$j^\mu = \overline{\psi}\gamma^\mu\psi,$$

which in this case means conservation of nucleon number. In addition the Lagrangian in (4.11) has an *isospin* symmetry, i.e. it is invariant under the $\psi' \to e^{-i\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}/2}\psi$ transformation where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denote the Pauli matrices[1] and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$ a normalized vector in isospin space. We find $\delta\psi = -\frac{1}{2}i\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}\psi$ and since $\frac{\delta\mathcal{L}}{\delta(\partial_\mu\overline{\psi})} = 0$, by the Noether theorem we have

$$I^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta\psi = \frac{1}{2}\overline{\psi}\gamma^\mu\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}\psi. \tag{4.12}$$

By making $\boldsymbol{\Lambda}$ an infinitesimal vector, the transformation represents an infinitesimal rotation in isospin-space, from which we get the vector current

$$\boldsymbol{I}^\mu_\psi = \frac{1}{2}\overline{\psi}\gamma^\mu\boldsymbol{\tau}\psi. \tag{4.13}$$

Its third component would look like

$$J^\mu_3 = \frac{1}{2}\overline{\psi}\gamma^\mu\tau_3\psi \tag{4.14}$$

where, since

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.15}$$

the time component will be

$$J^0_3 = \frac{1}{2}(\psi_p^\dagger\psi_p - \psi_n^\dagger\psi_n). \tag{4.16}$$

Here $\psi_p^\dagger\psi_p$ and $\psi_n^\dagger\psi_n$ are the proton and neutron density respectively. This allows us to rewrite the last equation as

$$J^0_3 = \frac{1}{2}(\rho_p - \rho_n). \tag{4.17}$$

4.1.3 Lagrangian for the σ–ω model

The Lagrangian for this model will be a sum of the Lagrangians for the nucleon spinors, the σ and ω fields and the interaction terms between these particles:

$$\mathcal{L} = \mathcal{L}_{\text{nucl}} + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_{\text{int}}. \tag{4.18}$$

[1] It is convention to denote the Pauli matrices with $\boldsymbol{\tau}$ when talking about isospin, and with $\boldsymbol{\sigma}$ when talking about spin. This is though only convention, as they are the same matrices.

The nucleon Lagrangian is the Lagrangian in (4.11), the σ meson Lagrangian is that of a scalar field as seen in (4.2), and the one for the ω spin-one field the one seen in (4.7). We choose the σ and the ω mesons only to interact with the fermion field and we do not consider self-interactions or interactions between the two mesons. If we want the interaction terms to yield a scalar, the σ field will have to couple to the scalar density $\overline{\psi}\psi$, while the vector ω will interact with the vector quantity $\overline{\psi}\gamma^\mu\psi$,

$$\mathcal{L}_{\text{int}} = g_\sigma \sigma \overline{\psi}\psi - g_\omega \omega_\mu \overline{\psi}\gamma^\mu \psi, \tag{4.19}$$

where g_σ and g_ω are the coupling constants for the $\sigma\overline{\psi}\psi$ and the $\omega_\mu\overline{\psi}\gamma^\mu\psi$ interactions respectively. There is no compelling reason to choose the minus sign for g_ω, but by doing that we would end up with a positive value for both coupling constants, as we will show. It is now possible to plug (4.11), (4.2), (4.7) and (4.19) into (4.18), and get

$$\begin{aligned}\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} \\ + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu + g_\sigma\sigma\overline{\psi}\psi - g_\omega\omega_\mu\overline{\psi}\gamma^\mu\psi.\end{aligned} \tag{4.20}$$

The Lagrangian can also be rewritten in the more revealing form

$$\begin{aligned}\mathcal{L} = \overline{\psi}\Big[i\gamma_\mu(\partial^\mu + ig_\omega\omega^\mu) - (m - g_\sigma\sigma)\Big]\psi \\ + \frac{1}{2}\Big[(\partial_\mu\sigma)(\partial^\mu\sigma) - m_\sigma^2\sigma^2\Big] - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu,\end{aligned} \tag{4.21}$$

where we get an insight in how we can interpret it: instead of the Lagrangian of three interacting particles, it can be thought of three free Lagrangians where the σ interaction now acts as a term reducing the mass of the nucleons, while the ω field shares the same behaviour as for the potential four-vector of the photon in the electromagnetic interaction (with the only difference that ω is a massive field). The equations of motion are found by using the Euler-Lagrange equations for fields, The steps leading to the equations of motion can be found in the Appendix in Section A.4. Here are the derived forms for the σ, ω and ψ fields:

$$\sigma\text{-field} \qquad (\Box + m_\sigma^2)\sigma(x) = g_\sigma\overline{\psi}(x)\psi(x), \tag{4.22}$$

$$\omega\text{-field} \qquad (\Box + m_\omega^2)\omega_\mu(x) = g_\omega\overline{\psi}(x)\gamma_\mu\psi(x), \tag{4.23}$$

$$\psi\text{-field} \quad \big[i\gamma^\mu\big(\partial_\mu + ig_\omega\omega_\mu(x)\big) - \big(m - g_\sigma\sigma(x)\big)\big]\psi(x) = 0. \tag{4.24}$$

4.1.4 The relativistic mean-field approximation

In our current model we idealize the neutron star as consisting of static, uniform matter in its ground state. This is done by applying the RMF *approximation*, where instead of the meson fields we use their mean values in the ground state. Being static and uniform, both the currents and the meson fields will be independent of x^μ, giving for (4.22) and (4.23)

$$m_\sigma^2\langle\sigma\rangle = g_\sigma\langle\overline{\psi}\psi\rangle, \tag{4.25}$$

$$m_\omega^2\langle\omega_\mu\rangle = g_\omega\langle\overline{\psi}\gamma_\mu\psi\rangle, \tag{4.26}$$

where the $\langle - \rangle$ parenthesis mean we are taking the mean value of the enclosed quantity. For (4.24) this yields

$$\Big[i\gamma^\mu\big(\partial_\mu + ig_\omega\langle\omega_\mu\rangle\big) - \big(m - g_\sigma\langle\sigma\rangle\big)\Big]\psi(x) = 0. \tag{4.27}$$

Since there is no x-dependence in (4.27), we can easily perform a Fourier transformation and analyze it in momentum space. We make the substitution $\psi(x) = e^{-ix^\nu p_\nu}\psi(p)$ (p here being the four-moment) and take the derivative in the kinetic term:

$$\Big[\gamma^\mu\big(p_\mu - g_\omega\langle\omega_\mu\rangle\big) - \big(m - g_\sigma\langle\sigma\rangle\big)\Big]\psi(p) = 0. \tag{4.28}$$

We return to the definition of the energy-stress tensor in (2.21). In the rest frame, its expectation values for the energy density ϵ and pressure P are on the diagonal,

$$T_{\mu\nu} = T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{4.29}$$

We can now use the definition of the canonical energy-stress tensor

$$T_{\mu\nu} \equiv \eta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\partial_\nu\phi_i \tag{4.30}$$

with the Lagrangian in (4.20) for nucleons in momentum space:

$$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}p^\nu\psi - \overline{\psi}\frac{\partial\mathcal{L}}{\partial(\partial^\mu\overline{\psi})}p^\nu. \tag{4.31}$$

Since $\frac{\partial\mathcal{L}}{\partial(\partial^\mu\overline{\psi})} = 0$, for $\mu = \nu$ we will obtain

$$\epsilon = -\langle\mathcal{L}\rangle + \langle\overline{\psi}\gamma_0 p_0\psi\rangle \tag{4.32}$$

$$P = \langle\mathcal{L}\rangle + \frac{1}{3}\langle\overline{\psi}\gamma_i p_i\psi\rangle. \tag{4.33}$$

In our model for static and uniform meson fields, the expectation value for the Lagrangian $\langle\mathcal{L}\rangle$ will contain only the σ and ω terms without derivatives. Additionally, we have from (4.27) that the first term in the Lagrangian is zero, leaving

$$\langle\mathcal{L}\rangle = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega^\mu\rangle\langle\omega_\mu\rangle. \tag{4.34}$$

From equations (4.32), (4.33) and our expectation value of the Lagrangian in (4.34) we see that we can obtain an equation of state for our neutron star model by finding the values of $\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2$, $\frac{1}{2}m_\omega^2\langle\omega^\mu\rangle\langle\omega_\mu\rangle$, $\langle\overline{\psi}\gamma_0 p_0\psi\rangle$ and $\langle\overline{\psi}\gamma_i p_i\psi\rangle$.

We now analyze the mean-field nucleon Dirac equation in (4.28). We introduce the new variables K_μ and m^* defined as $K_\mu \equiv p_\mu - g_\mu\langle\omega_\mu\rangle$ and $m^* \equiv m - g_\sigma\langle\sigma\rangle$, and rewrite the equation of motion as

$$\left[\not{K} - m^*\right]\psi(p) = 0. \tag{4.35}$$

Multiplying by $[\not{K} + m^*]$ on the left side, we obtain

$$\left[\not{K} + m^*\right]\left[\not{K} - m^*\right]\psi(p) = \left(K^\mu K^\nu \frac{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}{2} - m^{*2}\right)\psi(p) = 0. \tag{4.36}$$

Here we have taken into consideration both cases where the index μ and the index ν come first, since K_μ and K_ν commute with themselves and the gamma matrices, but the gamma matrices do not. Given the properties in (3.32), we conclude that

$$\left[K^\mu K_\mu - m^{*2}\right]\psi(p) = 0. \tag{4.37}$$

From the above equation we notice that the nucleon momentum spinor $\psi(p)$ is multiplied with a number(not an operator), giving zero as result. Excluding the trivial solution where $\psi(p) = 0$, the number must equal zero, and given $K^\mu K_\mu = K_0^2 - \boldsymbol{K}^2$, we obtain

$$K_0 = \sqrt{\boldsymbol{K}^2 + m^{*2}}. \tag{4.38}$$

The energy eigenvalues of the nucleon Dirac equation (the time component of the momentum eigenvalues p^μ) can be now written as

$$e(\boldsymbol{p}) = p_0 = K_0(\boldsymbol{p}) + g_\omega\langle\omega\rangle. \tag{4.39}$$

We still have to find the expectation value for the nucleon currents in (4.25) and (4.26). The system we are studying is in its ground state, so the expectation value of any operator $\hat{Q}$ would be

$$\langle\overline{\psi}\hat{Q}\psi\rangle = \sum_i \int \frac{d^3p}{(2\pi)^3}(\overline{\psi}\hat{Q}\psi)_{\kappa_i}\theta(\mu - e(\boldsymbol{p})). \tag{4.40}$$

The Heaviside step function in (4.40) is the particle distribution for fermions in their ground state (the same limit for $T \to 0$ we already have used in the previous chapters), while the sum runs over the possible combinations of nucleon isospin and spin states κ_i. For every energy eigenvalue we have four different combinations of spin and isospin, as both can either take the value of $-1/2$ or $1/2$. The degeneracy for the nucleon is thus four, and being the matter in its groun state at $T = 0$, they are all occupied. The sum over i will then simplify to a coefficient of 4 in front of the integral. From the mean-field Dirac equation for the nucleons in (4.27), it is easy to obtain a Lagrangian,

$$\mathcal{L}_{\text{nucl,mean}} = \overline{\psi}(x)\left[\gamma_\mu(i\partial^\mu - g_\omega\langle\omega^\mu\rangle) - m^*\right]\psi(x), \tag{4.41}$$

from which it is possible to derive the Hamiltonian via a Legendre transformation in a similar way to (3.38). In momentum space it becomes

$$\mathcal{H} = \overline{\psi}(p)\left[\boldsymbol{\gamma}\cdot\boldsymbol{p} + g_\omega\gamma_\mu\langle\omega^\mu\rangle + m^*\right]\psi(p) = \psi^\dagger(p)H_D\psi(p). \tag{4.42}$$

where we have defined the Hamiltonian operator as

$$H_D = \gamma^0\left[\boldsymbol{\gamma}\cdot\boldsymbol{p} + g_\omega\gamma_\mu\langle\omega^\mu\rangle + m^*\right]. \tag{4.43}$$

By now dropping the spinor's (p) for convenience, the expectation value of the energy $\langle\overline{\psi}H_D\psi\rangle_{\kappa_i}$ for a single nucleon state is our energy eigenstates found in (4.39),

$$(\psi^\dagger H_D\psi)_{\kappa_i} = K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle. \tag{4.44}$$

We can take derivatives of any variable α in the Hamiltonian by using

$$\frac{\partial}{\partial\alpha}\left(\psi^\dagger H_D\psi\right)_{\kappa_i} = \left(\psi^\dagger\frac{\partial H_D}{\partial\alpha}\psi\right)_{\kappa_i} + (K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle)\frac{\partial}{\partial\alpha}(\psi^\dagger\psi)_{\kappa_i}, \tag{4.45}$$

thus the differential of (4.44) with respect to $\langle\omega_0\rangle$ is

$$\frac{\partial}{\partial\langle\omega_0\rangle}\left(K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle\right) = (\psi^\dagger\gamma^0\left(g_\omega\gamma^0\omega_0\right)\psi)_{\kappa_i} + (K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle)\frac{\partial}{\partial\langle\omega_0\rangle}(\psi^\dagger\psi)_{\kappa_i}.$$

Since K_0 is independent of $\langle\omega_0\rangle$, this simplifies to

$$g_\omega\left(\langle\psi^\dagger\psi\rangle_{\kappa_i} - 1\right) + (K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle)\frac{\partial}{\partial\langle\omega_0\rangle}\langle\psi^\dagger\psi\rangle_{\kappa_i} = 0, \tag{4.46}$$

which is satisfied for $(\psi^\dagger\psi)_{\kappa_i} = 1$, and shows that the nucleon field is normalized. This gives us a simplified version of (4.45):

$$\frac{\partial}{\partial\alpha}(\psi^\dagger H_D\psi)_{\kappa_i} = \left(\psi^\dagger\frac{\partial H_D}{\partial\alpha}\psi\right)_{\kappa_i}. \tag{4.47}$$

We use the above equation to calculate the derivative in p_x

$$\frac{\partial}{\partial p_x}K_0(\boldsymbol{p}) = (\psi^\dagger\gamma^0\gamma^x\psi)_{\kappa_i} = (\overline{\psi}\gamma^x\psi)_{\kappa_i}. \tag{4.48}$$

This is useful in calculating the nucleon current density:

$$\begin{aligned}
\langle\overline{\psi}\gamma^j\psi\rangle =& 4\int\frac{\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z}{(2\pi)^3}\left(\frac{\partial K_0(\boldsymbol{p})}{\partial p_x}\right)\theta(\mu - e(\boldsymbol{p}))\\
=& 4\int\frac{\mathrm{d}p_y\mathrm{d}p_z}{(2\pi)^2}\int\frac{\mathrm{d}p_x}{2\pi}\left(\frac{\partial K_0(\boldsymbol{p})}{\partial p_x}\right)\theta(\mu - e(\boldsymbol{p}))\\
=& 4\int\frac{\mathrm{d}p_y\mathrm{d}p_z}{(2\pi)^3}\int\mathrm{d}K_0(\boldsymbol{p})\theta(\mu - e(\boldsymbol{p})) = 0.
\end{aligned} \tag{4.49}$$

The result is zero because the Heaviside step functions sets the limits of the integration at $e(\boldsymbol{p}) = \mu = K_0(\boldsymbol{p}) + g_\omega\langle\omega_0\rangle$, meaning that $K_0(\boldsymbol{p}) = \mu - g_\omega\langle\omega_0\rangle$ everywhere on the boundary. Since the nucleon three-current $\langle\overline{\psi}\gamma^j\psi\rangle$ is zero, the mean-field equation of motion for the ω field in (4.26) yields

$$m_\omega\langle\omega_i\rangle = 0, \tag{4.50}$$

and since the field is not massless, we conclude that $\langle\omega_i\rangle = 0$. The ground state occupies a sphere in momentum-space, and we can rewrite the four-momentum eigenvalues as

$$e(p) = K_0(p) + g_\omega\langle\omega_0\rangle, \quad K_0(p) = \sqrt{p^2 + m^{*2}}. \tag{4.51}$$

From the normalization $\langle\psi^\dagger\psi\rangle_{\kappa_i} = 1$, we can again apply the equation for the expectation values (4.40) to find $\langle\psi^\dagger\psi\rangle$:

$$\langle\psi^\dagger\psi\rangle = 4\int \frac{\mathrm{d}^3p}{(2\pi)^3}\theta(\mu - e(p)) = \frac{16\pi}{(2\pi)^3}\int_0^{p_F} p^2\mathrm{d}p = \frac{2p_F^3}{3\pi^2}, \tag{4.52}$$

where the momentum that satisfies $e(p) = \mu$ is the Fermi momentum p_F. The last term to evaluate is $\langle\overline{\psi}\psi\rangle$. This can be found by taking the mass derivative in (4.44) and (4.47), and get

$$\frac{\partial}{\partial m}(\psi^\dagger H_D\psi)_{\kappa_i} = \frac{\partial K_0(p)}{\partial m} = \left(\psi^\dagger\frac{\partial H_D}{\partial m}\psi\right)_{\kappa_i} = (\overline{\psi}\psi)_{\kappa_i}.$$

We can now finally use (4.40) and get

$$\begin{aligned}\langle\overline{\psi}\psi\rangle =& 4\int \frac{d^3p}{(2\pi)^3}\frac{\partial}{\partial m}\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}\theta(\mu - e(p)) \\ =& \frac{2}{\pi^2}\int_0^{p_F} \mathrm{d}p\frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}.\end{aligned} \tag{4.53}$$

We now have everything we need to evaluate the mean σ and ω fields. We combine (4.25) and (4.26) with the newly found results for $\langle\psi^\dagger\psi\rangle$ in (4.52) and for $\langle\overline{\psi}\psi\rangle$ in (4.53), and get

$$g_\sigma\langle\sigma\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2\frac{2}{\pi^2}\int_0^{p_F} \mathrm{d}p\frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}, \tag{4.54}$$

$$g_\omega\langle\omega_0\rangle = \left(\frac{g_\omega}{m_\omega}\right)^2\frac{2{p_F}^3}{3\pi^2}, \tag{4.55}$$

$$g_\omega\langle\omega_i\rangle = 0. \tag{4.56}$$

The three relations above can be easily arranged to yield a result for both $m_\sigma\langle\sigma\rangle$ and $m_\omega\langle\omega_0\rangle$, needed for the expectation value of the Lagrangian in the expressions for the energy density and pressure in (4.32) and (4.33). In order to calculate $\langle\overline{\psi}\gamma_0 p_0\psi\rangle$ and $\langle\overline{\psi}\gamma_i p_i\psi\rangle$ needed to find the equation of state in (4.32) and (4.33), we use the same method as before. The term $\langle\overline{\psi}\gamma_0 p_0\psi\rangle$ is the contribution to the energy density given by the energy momentum states. For each nucleon isospin state, the energy eigenvalues are given by $e(p)$

in (4.51), so we can use (4.40) and find

$$\begin{aligned}
\langle\overline{\psi}\gamma_0 p_0\psi\rangle =&4\int\frac{\mathrm{d}^3p}{(2\pi)^3}e(p)\theta(\mu-e(p))\\
=&\frac{2}{\pi^2}\int_0^\mu \mathrm{d}pp^2\left(g_\omega\langle\omega_0\rangle+\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}\right)\\
=&g_\omega\langle\omega_0\rangle\frac{2{p_F}^3}{3\pi^2}+\frac{2}{\pi^2}\int_0^{p_F}\mathrm{d}pp^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}\\
=&m_\omega^2\langle\omega_0\rangle^2+\frac{2}{\pi^2}\int_0^{p_F}\mathrm{d}pp^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}.
\end{aligned}\tag{4.57}$$

We then observe from (4.47) that by differentiating with respect to $\boldsymbol{p}$ we obtain

$$(\overline{\psi}\boldsymbol{\gamma}\psi)_{\kappa_i}=\frac{\partial K_0(\boldsymbol{p})}{\partial\boldsymbol{p}}.$$

Taking the dot product with $\boldsymbol{p}$ on both sides we can finally use (4.40) and obtain

$$\begin{aligned}
\langle\overline{\psi}\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi\rangle=\langle\overline{\psi}\gamma_i p_i\psi\rangle =&4\int_0^{p_F}\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial K_0(\boldsymbol{p})}{\partial\boldsymbol{p}}\cdot\boldsymbol{p}\\
=&\frac{2}{\pi^2}\int_0^{p_F}\mathrm{d}p\,\frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}}.
\end{aligned}\tag{4.58}$$

We can now plug in the expectation values in (4.57) and (4.58) and for the Lagrangian in (4.34) (where we set $\langle\omega_i\rangle=0$) into the equation of state, obtaining

$$\epsilon=\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2+\frac{2}{\pi^2}\int_0^{p_F}\mathrm{d}pp^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}\tag{4.59}$$

$$P=-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2+\frac{2}{3\pi^2}\int_0^{p_F}\mathrm{d}p\frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}},\tag{4.60}$$

where $m_\sigma^2\langle\sigma\rangle^2$, $m_\omega^2\langle\omega_0\rangle^2$ and $g_\sigma\langle\sigma\rangle$ can be derived from equations (4.54) and (4.55). All quantities in the energy density and pressure can be derived by a chosen value of the Fermi momentum p_F. We can then find all the values of interest for our equation of state by evaluating the above equations for a chosen range of Fermi momenta.

4.2 Peculiarities and computation

The σ–ω model is the simplest RMF framework to describe nuclear matter. It has two free parameters (g_σ/m_σ and g_ω/m_ω) which can be adjusted so that the theory yields experimental values of nuclear matter properties like the binding energy per nucleon and the Fermi momentum at saturation density, that being the density at which the pressure is zero and the system is in static equilibrium. According to recent evidence [28], the binding energy per nucleon for infinite nuclear matter is $B/A=-16.3$ MeV and the Fermi momentum (linked to the radius parameter and the saturation density) is $p_{F,\mathrm{sat}}=1.31\ \mathrm{fm}^{-1}$.

From these values it is possible calculate algebraically the values of the two free parameters of the theory [10]. Walecka in [7] and [41] sets somewhat wrongly the binding energy at -15.75 MeV and the Fermi momentum at saturation density at $p_{F,\text{sat}} = 1.42\ \text{fm}^{-1}$ to model nuclear matter, and obtains the parameters $(g_\sigma/m_\sigma)^2 = 3.033 \times 10^{-4}\ \text{MeV}^{-2}$ and $(g_\omega/m_\omega)^2 = 2.224 \times 10^{-4}\ \text{MeV}^{-2}$. Although managing to yield the above experimental values, the model fails in reproducing other properties of nuclear matter like the compression modulus K, the effective mass m^* and the symmetry energy a_{sym}, which are in poor agreement with the experimental data. It is nevertheless illustrative to show the results of such a model.

4.2.1 Nuclear matter and neutron matter

The σ–ω model idealizes the proton and neutron as the same, neutral particle with four possible states, given by the possible combinations of spin and isospin states. We may call this as *nuclear matter*. Although useful for atomic nuclei, this model cannot be used for neutron stars, which are systems mainly made of neutrons. We can then speak of *neutron matter* when modeling the structure of the star. Neutron matter will be akin to nuclear matter, but with only occupation 2 for the degenerate states: two for spin, but only one for isospin. Mathematically this means that we would only have to count two degenerate states in (4.54) and (4.55), which will become

$$g_\sigma\langle\sigma\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2 \frac{1}{\pi^2}\int_0^{p_F} \mathrm{d}p \frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}, \tag{4.61}$$

$$g_\omega\langle\omega_0\rangle = \left(\frac{g_\omega}{m_\omega}\right)^2 \frac{{p_F}^3}{3\pi^2}. \tag{4.62}$$

Equations (4.57) and (4.58) will also be modified, and subsequently the equation of state in (4.59) and (4.60), which become

$$\epsilon = \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{\pi^2}\int_0^{p_F} \mathrm{d}p\, p^2\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}, \tag{4.63}$$

$$P = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{3\pi^2}\int_0^{p_F} \mathrm{d}p \frac{p^4}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}. \tag{4.64}$$

Nuclear and neutron matter also differ when it comes to binding energy. While nuclear matter has bound states (as we would expect from the existence of stable atomic nuclei), we should expect none for neutron matter, as these bound states have never been observed in nature. Nevertheless, the σ–ω model for neutron matter predicts a local minimum in the binding energy. This minimum can be negative (in which case we should expect a bound state for neutron matter) or positive (which leads to a phase transition between a Fermi liquid and Fermi gas). This can be understood using the same framework used to qualitatively explain phase transitions between the different states in matter – that be solid, liquid and gaseous state. We will use Section 4.2.2 to explain the reasoning behind phase transitions and their application to neutron matter, for then in Section 4.2.3 using the newly found equation of state together with the TOV equation in order to calculate the mass-radius relation for this model.

4.2.2 Phase transition and Maxwell construction

The case with a positive local minimum in the binding energy per nucleon is taken into consideration in the 1974 paper from Chin and Walecka [7]. While using the same σ–ω model for $T = 0$ as developed in this chapter, the star would undergo a phase transition between a liquid core and a gaseous atmosphere, both purely made of neutrons interacting with the mesonic mean-fields. The equation of state for neutron matter is similar in shape to the Van der Waals one, as they both idealize particles as bodies with a strong, short range repulsion and a longer range attraction - ideal molecules or atoms for the Van der Waals equation, neutrons for our star model. Exactly as the Van Der Waals equation of state,

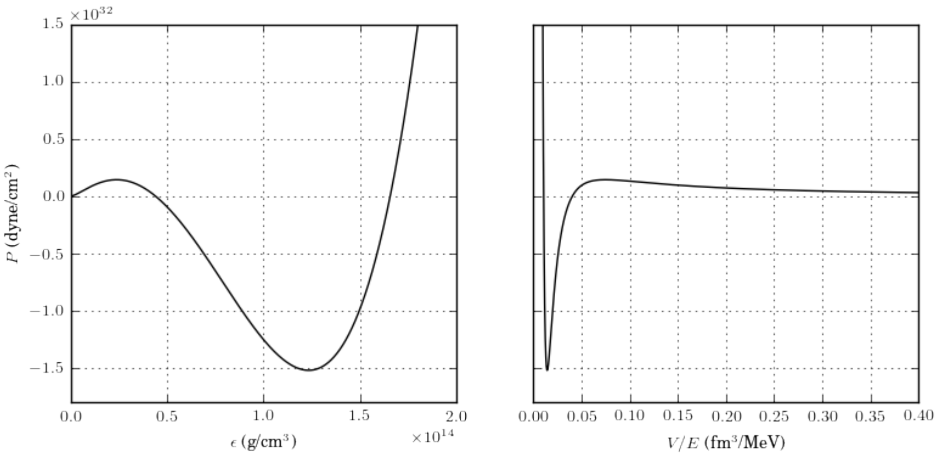

Figure 4.1: Equation of state for neutron matter. Plot of pressure against energy density on the left, and pressure against volume per unit energy on the right. The two plots are equivalent. As we see, they both show a region with negative pressure, and an unstable region with $\mathrm{d}P/\mathrm{d}\epsilon < 0$ (or alternatively $\mathrm{d}P/\mathrm{d}V > 0$). V/E is the relativistic version of the volume occupied by a fixed number of particles, since our equation of state involves the energy density (i.e. energy *per unit volume*) and not the particle density.

in the second panel of Figure 4.1 our shows a region with positive $\mathrm{d}P/\mathrm{d}V$, breaking the Le Chatelier's principle of microscopic stability. The principle simply states that stability in a system is attained when any deviation from equilibrium results in a restoring force towards the original equilibrium state [42]. The quantity we should look to when assessing thermodynamical stability of an equilibrium state is the Gibbs free energy:

$$G = U - TS + PV. \tag{4.65}$$

In (4.65), U is the internal energy of the system, T its temperature, S the entropy, P the pressure and V the volume. The Gibbs free energy takes into consideration the contribution of the temperature and the pressure to the total energy with the $-TS$ and $+PV$ terms respectively. This means that it is the lowest Gibbs free energy, and not the lowest internal energy U, that will characterize the most stable state in thermodynamical equilibrium with its surroundings [34]. In our case the temperature is zero by assumption and the $-TS$ term vanishes. By keeping the volume per energy unit and the internal energy constant, from (4.65) we obtain the relation

$$\mathrm{d}G = V(P)\mathrm{d}P. \tag{4.66}$$

The Gibbs free energy can then be found by numerical integration. As with any system

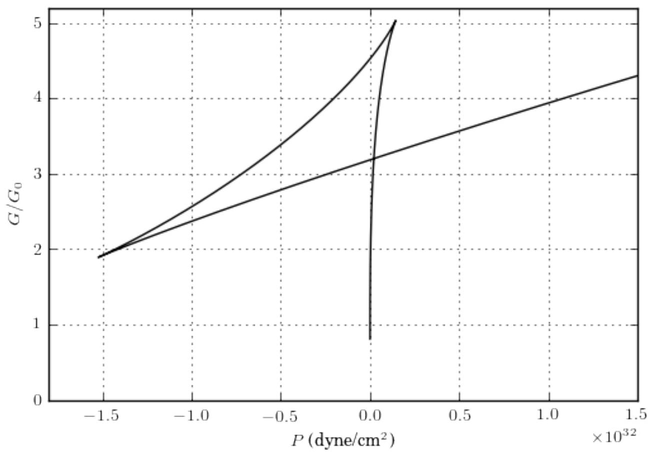

Figure 4.2: The Gibbs free energy calculated by numerical integration of equation (4.66). The states in the loop are unstable. The pressure where the phase transition takes place is the one of the crossing point.

with an equation of state in the form of Figure 4.1, the Gibbs free energy would give up to three equilibrium states for a range of pressures, as we see in Figure 4.2. All states in the triangular loop (except the lower ones in the region with negative pressure) are to be understood as unstable, and thus not present for a neutron star in its thermodynamical ground state. Since the TOV equation do not allow for constant pressure along the radius of the star for nonzero values of pressure or energy density, this phase transition will physically translate in an abrupt jump in energy density at a specific pressure along the radial direction. At the pressure of this jump (the one of the crossing point in Figure 4.2) we will have an interface between a “liquid core” and a gaseous “atmosphere” of neutrons obeying the Fermi-Dirac statistics. Ignoring the different particle compositions, this may be intuitively understood by comparing such interface to the more familiar one between sea and air on Earth. The phase transition pressure for an isotherm equation of state like ours can be either found by plotting the Gibbs free energy as a function of the pressure and check where it crosses itself (Figure 4.2), or by using the Maxwell construction method. The latter is based on the fact that the net change in the Gibbs free energy in the loop is zero:

$$\int_{\text{loop}} \mathrm{d}G = 0. \tag{4.67}$$

By then inserting the relation for the infinitesimal of the Gibbs free energy found in (4.66) we have

$$\int_{\text{loop}} V(P)\mathrm{d}P = 0. \tag{4.68}$$

This integration is shown graphically in Figure 4.3. Since we are now treating P as the independent variable, it will be more clear to plot it along the *x*-axis. We divide our integration in four steps: 1) from the top point where the equation of state crosses the transition pressure, P_{crit}, for the first time. That would be the first time we reach the crossing point in Figure 4.2 starting from below. Here we integrate up to the local maximum of the pressure. In Figure 4.3 this corresponds to the areas shaded in light and dark red. 2) From the local maximum in the pressure to the second occurrence of P_{crit}. This is area shaded in dark red, and since it goes backwards it cancels some of the shaded red area. 3) From the second occurrence of P_{crit} to the minimum in the pressure. This is the area corresponding to the regions shaded in green and blue. Since integrating backwards, this area will have a minus sign. 4) Finally, from the minimum in the pressure to the third and last occurrence of P_{crit}. This is the area shaded in blue. This has positive sign since we integrate forward with respect to P, and cancels some of the area integrated in point 3). After all cancellations,

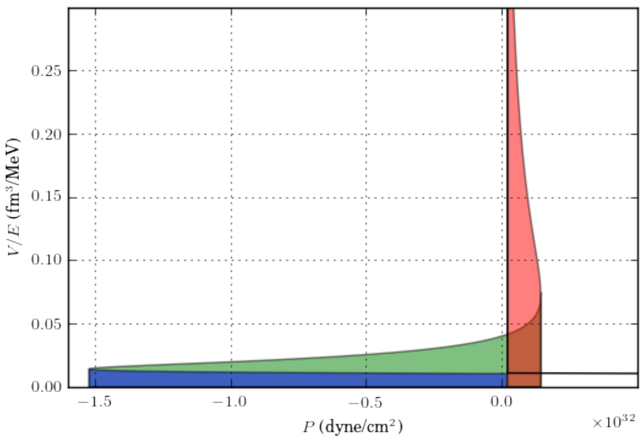

Figure 4.3: Integration in steps of (4.68).

we are only left with the regions shaded in light red, and the one in green. These have opposite signs, and their sum must give zero because of (4.68). By calling the light red area for A and the one in green for B, we write that

$$A - B = 0. \tag{4.69}$$

Doing a Maxwell construction consists in finding the pressure P_{crit} for which these two areas are the same, and cancel each other. When found, we can return to plotting the volume along the *x*-axis, and illustrate the new, stable equation of state in Figure 4.4. Plotted against the energy density, the new, stable equation of state will yield Figure 4.5. Both

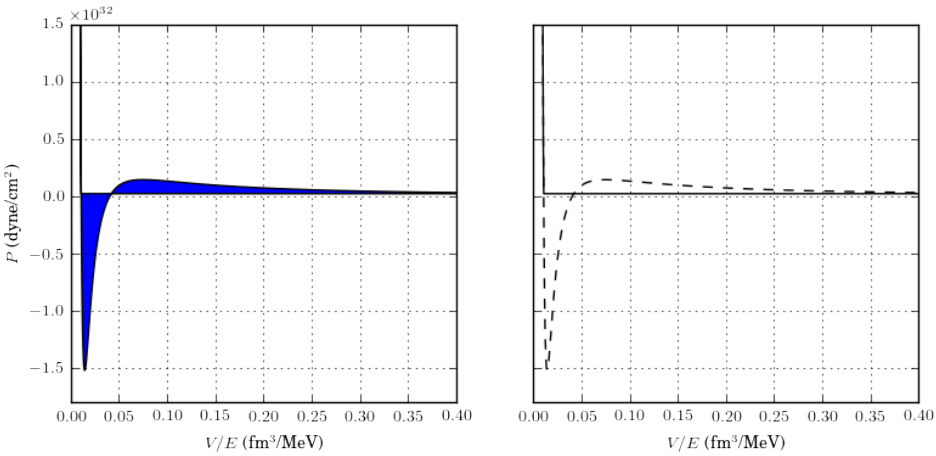

Figure 4.4: Pressure plotted against volume per MeV, with Maxwell construction. On the left panel we see the shaded areas that must be the same when the right P_{crit} is chosen. On the right panel the straight line corresponds to the stable solutions, while the dashed line to the unstable ones.

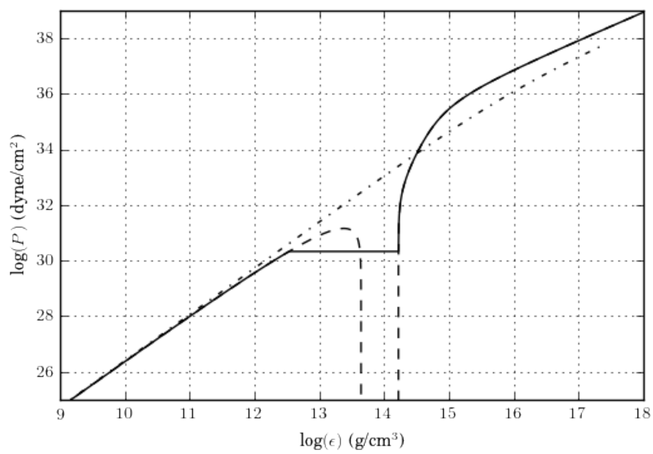

Figure 4.5: Equation of state for neutron matter. The dashed line corresponds to the unstable solution, while the dash-and-point line corresponds to the equation of state of a relativistic Fermi gas in (2.98) and (2.99). We see how this almost overlaps the low density region where we expected the model to yield a Fermi gas, as opposed to the Fermi liquid on the right.

show the critical pressure P_{crit} at 2.13×10^{30} dyne/cm^2. When considering nuclear matter, we would have to use the equation of state derived in chapter 4.1.4, the one occupying all four degenerate states for the nucleons for each energy eigenstate: equations (4.54), (4.55), (4.59) and (4.60). This equation of state does not present any special cases as for neutron matter, but is similar in shape, when not considering the region of phase transition. This is shown in Figure 4.6.

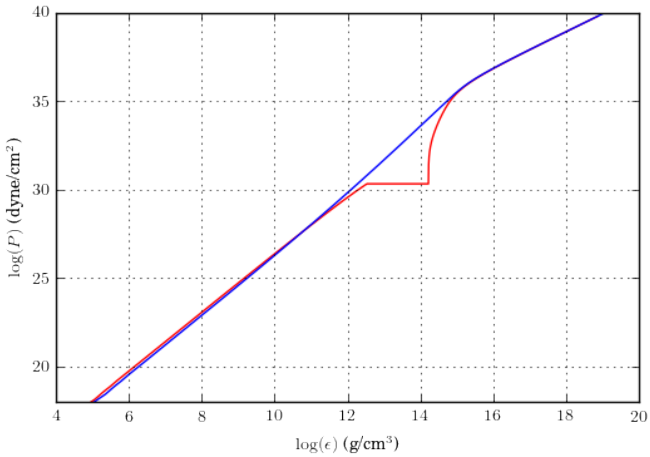

Figure 4.6: Equation of state for neutron matter in red, for nuclear matter in blue.

4.2.3 The TOV equation applied to neutron and nuclear matter

When the equations of state for neutron and nuclear matter are obtained, it is possible to couple them to the TOV and the mass equations in (2.51) and (2.66) respectively, and find solutions for different central pressures in the same way we did for the free, cold, neutron Fermi gas in Chapter 2. For neutron matter, the algorithm gives small stars for low central pressures. For central pressures below the phase transition, stars are much smaller than what obtained with the equation of state in (2.98) and (2.99). It gives though a maximum mass at $M_{max} = 2.60 M_\odot$ at $R = 12.19$ km radius. For nuclear matter the mass and radius are bigger — $M = 3.13 M_\odot$ and $R = 18.57$ km — as we would expect when allowing double as many nucleons per state. The results are shown in picture 4.7. A discussion on the "curls" in the spirals and the corresponding region with $\partial M/\partial \epsilon_0 > 0$ on the right side of plot in the right panel is taken in the Outlook in Chapter 7.

4.2.4 Computation

For this computation, the neutron mass m has been used as normalization constant. The dimensionless fields become

$$g_\sigma \overline{\sigma} = \frac{g_\sigma \langle \sigma \rangle}{m},$$
$$g_\omega \overline{\omega}_0 = \frac{g_\omega \langle \omega \rangle}{m},$$

the momenta and the Fermi momentum are

$$\overline{p} = \frac{p}{m}$$
$$\overline{p}_F = \frac{p_F}{m}$$

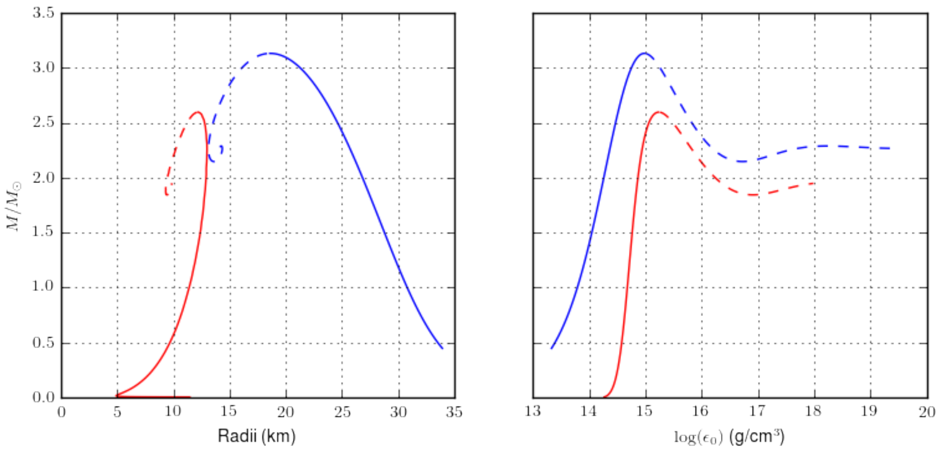

Figure 4.7: Relationship between radii and total masses on the left, and between central densities and total masses on the right. The dashed lines show the unstable solution. Red is for neutron matter, blue for nuclear matter.

and the energy and pressure as

$$\bar{\epsilon} = \frac{\epsilon}{m^4}$$
$$\overline{P} = \frac{P}{m^4}.$$

Expressing then the equation of state in terms of these dimensionless quantities, equations (4.54) and (4.62) become

$$g_\sigma \overline{\sigma} = \left(\frac{g_\sigma}{m_\sigma}\right)^2 \frac{f m^2}{2\pi^2} \int_0^{\overline{p}_F} \mathrm{d}\overline{p} \frac{\overline{p}^2(1 - g_\sigma \overline{\sigma})}{\sqrt{\overline{p}^2 + (1 - g_\sigma\overline{\sigma})^2}},$$
$$g_\omega \overline{\omega}_0 = \left(\frac{g_\omega}{m_\omega}\right)^2 \frac{f m^2 \overline{p}_F^3}{6\pi^2},$$

where f indicates the degeneracy: $f = 4$ for nuclear matter and $f = 2$ for neutron matter. Since $g_\sigma \overline{\sigma}$ is only dependent on $\overline{p_F}$ and not on $\overline{p}$, we can substitute the latter in the $\overline{\sigma}$-field equation with $\overline{u} = \frac{\overline{p}}{(1 - g_\sigma \overline{\sigma})}$ and obtain

$$\begin{aligned} g_\sigma \overline{\sigma} &= \left(\frac{g_\sigma}{m_\sigma}\right)^2 \frac{f}{2\pi^2} m^2 \left(1 - g_\sigma \overline{\sigma}\right)^3 \int_0^{\overline{u}_F} \mathrm{d}\overline{u} \frac{\overline{u}^2}{\sqrt{\overline{u}^2 + 1}} \\ &= \left(\frac{g_\sigma}{m_\sigma}\right)^2 \frac{f}{4\pi^2} m^2 \left(1 - g_\sigma \overline{\sigma}\right)^3 \left(\overline{u}_F \sqrt{\overline{u}_F^2 + 1} - \sinh^{-1}\left(\overline{u}_F\right)\right). \end{aligned} \tag{4.70}$$

The values of $g_\sigma \overline{\sigma}$ can be then found by moving all terms to the left side and using a root finding function for different values of $\overline{p}_F$. From the same values we can also find the other needed quantities like $g_\omega \overline{\omega}_0$ and the integrals in both the energy density and pressure expression. These are also analytically solvable (as seen in (2.98) and (2.99)). When we have evaluated enough data points, we can interpolate the solutions in order to have a continuous equation of state. This can then be coupled to the TOV and the mass equations, obtaining a system of coupled differential equations with the central pressure as the boundary term, uniquely defining the mass, the radius and the structure of the star. The system can finally be evaluated for many different central pressures to obtain Figure 4.7. When making the Maxwell construction, the code loops through all possible straight lines between 0 and the local maximum in the pressure (at around 0.2×10^{32} dyne/cm^2, as we can see in Figures 4.2 and 4.4). For each of these lines, the code finds the crossing points and numerically integrates the enclosed areas. The pressure that yields the least difference between the two enclosed areas corresponds to our closest guess for the phase transition pressure.

Chapter 5

The $npe\mu$ model

5.1 The $npe\mu$ model

The σ–ω model lays the framework for a more general relativistic mean-field (RMF) theory. Even when taking into consideration the strong nuclear force between the nucleons, there are several aspects we have not taken into account when just using the σ–ω model. Some of these aspects are linked to the bulk properties of nuclear matter. The σ–ω model has two parameters (g_σ/m_σ and g_ω/m_ω) and these allow us to fix only two bulk properties, whereas a model with five parameters could help us fixing five properties: the binding energy per nucleon at saturation (B/A), the effective mass at saturation (m^*), the compression modulus (K), the baryon density at saturation (ρ_0) and the symmetry energy density (a_{sym}). In *Compact Stars* by Glendenning [10] and in [41] the parameters are fixed using empirical values of B/A and ρ_0, but fixing only two bulk properties yields values for the others that are far from the ones observed in experiments. Most notably we would get a compression modulus of $K \approx 550$ MeV opposed to the accepted empirical value of $K = 234$ MeV [28]. Other factors that the σ–ω model does not consider are the beta stability of neutrons in neutron matter (there should be a small concentration of protons and electrons in dynamic beta equilibrium with neutrons), and the condition of charge neutrality. As it will be explained in this chapter, we can account for all these five bulk properties by introducing self-interactions in the σ field, and an isospin symmetry restoring force mediated by the ρ meson. The global charge neutrality condition is satisfied by the introduction of leptons (electrons and muons) in the model.

5.1.1 Self-interaction for the σ meson

In order to account for two more bulk properties for high-density nuclear matter we can introduce two self-interaction terms for the scalar field to the Lagrangian, where the coupling constants b and c will be the parameters helping us reproducing the empirical values of the compression modulus K and the reduced mass m^* at saturation density. The first ones to introduce the self interactions in the σ field were Boguta and Bodmer in 1977 [5].

The two self interacting terms will be cubic and quartic in the field, and we can define the potential

$$U(\sigma) = \frac{1}{3} m_n b \left(g_\sigma \sigma\right)^3 + \frac{1}{4} c \left(g_\sigma \sigma\right)^4 . \tag{5.1}$$

In (5.1), b and c are our new parameters and $m_n = 938$ MeV is a constant (originally thought to be like the neutron mass) which makes b dimensionless and the whole term with dimension MeV^4, as we should expect for a Lagrangian term. Adding the self-interaction term to the Lagrangian in (4.21) gives

$$\begin{aligned}\mathcal{L} = \overline{\psi}\Big[i\gamma_\mu(\partial^\mu + ig_\omega\omega^\mu) - (m - g_\sigma\sigma)\Big]\psi + \frac{1}{2}\Big[(\partial_\mu\sigma)(\partial^\mu\sigma) - m_\sigma^2\sigma^2\Big] \\ - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu - \frac{1}{3}m_n b \left(g_\sigma\sigma\right)^3 - \frac{1}{4}c\left(g_\sigma\sigma\right)^4 ,\end{aligned} \tag{5.2}$$

where the choice of sign for the self-interacting terms bear no physical meaning (the minus sign can always be absorbed in the chosen values of b and c). We can obtain the new Euler-Lagrange equation for the σ field following the steps shown in the appendix until (A.13), which with the self-interaction term would instead yield

$$\frac{\partial\left(\mathcal{L}_\sigma + \mathcal{L}_{\text{int}} - U(\sigma)\right)}{\partial\sigma} = -m_\sigma^2\sigma(x) + g_\sigma\left(\overline{\psi}\psi - m_n b(g_\sigma\sigma(x))^2 - c(g_\sigma\sigma(x))^3\right) . \tag{5.3}$$

The mean-field version of this would be

$$m_\sigma^2\langle\sigma\rangle = g_\sigma\left(\langle\overline{\psi}\psi\rangle - m_n b(g_\sigma\langle\sigma\rangle)^2 - c(g_\sigma\langle\sigma\rangle)^3\right) . \tag{5.4}$$

The $\langle\overline{\psi}\psi\rangle$ term was already computed in (4.53). Plugging the result and rewriting in terms of $g_\sigma\langle\sigma\rangle$ we get

$$g_\sigma\langle\sigma\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2\left[-m_n b\left(g_\sigma\langle\sigma\rangle\right)^2 - c\left(g_\sigma\langle\sigma\rangle\right)^3 + \frac{2}{\pi^2}\int_0^{p_F} \mathrm{d}p\frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)}}\right] . \tag{5.5}$$

The expectation value of the Lagrangian in (5.2) is unmodified except for the new terms,

$$\langle\mathcal{L}\rangle = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 - \frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3 - \frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4 . \tag{5.6}$$

The equation of state with the scalar self-interaction is

$$\begin{aligned}\epsilon = \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3 + \frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4 \\ + \frac{2}{\pi^2}\int_0^{p_F}\mathrm{d}p p^2\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2},\end{aligned} \tag{5.7}$$

$$\begin{aligned}P = -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 - \frac{1}{3}bm\left(g_\sigma\langle\sigma\rangle\right)^3 - \frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4 \\ + \frac{2}{3\pi^2}\int_0^{p_F}\mathrm{d}p\frac{p^4}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}.\end{aligned} \tag{5.8}$$

By setting $b = 8.659 \cdot 10^{-3}$ and $c = -2.421 \cdot 10^{-3}$ we fix the value of the compression modulus to $K = 240$ MeV and the reduced mass at saturation density to $m^* = 0.78m$ [10]. Before proceeding, it is important to address the possible problem arising from the negative sign for c. A negative quartic term in the Lagrangian would make the energy density in (5.7) unbounded from below. Although this would be a theoretical problem for a quantum field theory regarding single particles, the mean-field approximation is purely phenomenological and only valid for the statistical limit of many interacting baryons. The model is in addition well behaved in the range of interest, and does not show problems before long beyond the range of validity. This is enough for our effective theory. The value of $g_\sigma\langle\sigma\rangle$ must in fact satisfy equation (5.5), where it grows monotonically from 0 to m, approaching asymptotically the latter value only at high densities [17].

5.1.2 The ρ meson

The ρ meson is a three-component, charged vector meson defined as

$$\boldsymbol{\rho}^\mu = (\rho_1^\mu, \rho_2^\mu, \rho_3^\mu), \tag{5.9}$$

Each $\rho_{i\mu}$ component behaves as the ω meson, so the free Lagrangian can be compactly written as

$$\mathcal{L}_{\rho,\mathrm{f}} = -\frac{1}{4}\boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2}m_\rho^2 \boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu, \tag{5.10}$$

where $\rho_{i\mu\nu} = (\partial_\mu \rho_{i\nu} - \partial_\nu \rho_{i\mu})$. Since the meson is massive, we use the same reasoning as for the ω meson to conclude that every component must obey the Klein-Gordon equation

$$(\Box + m_\rho^2)\rho_i^\mu = 0. \tag{5.11}$$

The Lagrangian in (5.10) is invariant under isospin rotation. We can show this by transforming the field as $\boldsymbol{\rho}_\mu \to \boldsymbol{\rho}_\mu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}_\mu$, and we see that

$$\begin{aligned}
\mathcal{L}'_{\rho,\mathrm{f}} &= -\frac{1}{4}\Bigg[\Big[\partial_\mu(\boldsymbol{\rho}_\nu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}_\nu) - \partial_\nu(\boldsymbol{\rho}_\mu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}_\mu)\Big] \\
&\qquad \cdot \Big[\partial^\mu(\boldsymbol{\rho}^\nu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}^\nu) - \partial^\nu(\boldsymbol{\rho}^\mu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}^\mu)\Big]\Bigg] \\
&\qquad + \frac{1}{2}(\boldsymbol{\rho}_\mu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}_\mu) \cdot (\boldsymbol{\rho}^\mu - \boldsymbol{\Lambda} \times \boldsymbol{\rho}^\mu) \\
&= -\frac{1}{4}\left(\boldsymbol{\rho}_{\mu\nu} - \boldsymbol{\Lambda} \times \boldsymbol{\rho}_{\mu\nu}\right) \cdot \left(\boldsymbol{\rho}^{\mu\nu} - \boldsymbol{\Lambda} \times \boldsymbol{\rho}^{\mu\nu}\right) + \frac{1}{2}\left(\boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu + O(\boldsymbol{\Lambda}^2)\right) \\
&= -\frac{1}{4}\left(\boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + O(\boldsymbol{\Lambda}^2)\right) + \frac{1}{2}\left(\boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu + O(\boldsymbol{\Lambda}^2)\right) \\
&= -\frac{1}{4}\boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2}m_\rho^2 \boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu = \mathcal{L}_{\rho,\mathrm{f}},
\end{aligned} \tag{5.12}$$

where all the dot products such as $\boldsymbol{\rho}_\mu \cdot (\boldsymbol{\Lambda} \times \boldsymbol{\rho}_\mu)$ give zero (since the cross product of a vector is always orthogonal to itself) and if we treat $\boldsymbol{\Lambda}$ an infinitesimal rotation in isospin space we can safely discard the $O(\mathbf{\Lambda}^2)$ terms. This symmetry leads to a conserved current. By rewriting the field transformation in index notation

$$\rho_{i\nu} \to \rho_{i\nu} - \epsilon_{ijk}\Lambda_j\rho_{k\nu}, \tag{5.13}$$

we may describe the variation $\delta\rho$ as

$$\delta\rho_{i\nu} = -\sum_j \epsilon_{ijk}\rho_{k\nu} \tag{5.14}$$

and using the equation for the Noether current in (3.39) we obtain

$$\begin{aligned}\boldsymbol{I}^\mu_\rho &= -\sum_j \frac{\delta\mathcal{L}_\rho}{\delta(\partial_\mu\rho_{i\nu})}\epsilon_{ijk}\rho_{k\nu} \\ &= -\boldsymbol{\rho}_\nu \times \frac{\delta\mathcal{L}_\rho}{\delta(\partial_\mu\boldsymbol{\rho}_\nu)} \\ &= \boldsymbol{\rho}_\nu \times \boldsymbol{\rho}^{\mu\nu},\end{aligned} \tag{5.15}$$

where all the steps are shown in the appendix. The proton and the neutron in this model are treated as the same particle with opposite projection along the third axis in isospin space: the proton with projection $1/2$ and the neutron with projection $-1/2$. If we do not account for external factors such as the electrical charge, the ground state is achieved when all the degenerate energy states are occupied, and there are as many neutrons as protons. We call this *symmetric nuclear matter*. The valley of β stability for ordinary atomic nuclei is an illustration of this reasoning, where the number of protons tends to match the one of neutrons, only to deviate when the electromagnetic repulsion between protons becomes too big to be ignored. The ground state for nuclear matter, as long as only isospin is concerned, is supposed to be symmetric nuclear matter, and all other states would be excited states. We can translate this observation into algebra by introducing a *symmetry restoring force* in the form of a term in the energy density which is quadratic in the deviation from symmetry, such that the least energy is achieved when the proton and the neutron densities are alike. The ρ meson forms a charged isospin triplet and can be chosen to mediate such force between nucleons. This means that we should add a term to the Lagrangian which describes the interaction between the nucleons and the meson. Following the reasoning in [10], the ρ meson should couple to the sum of the nucleon and meson isospin currents in (4.13) and (5.15), so that

$$\mathcal{L}_{\boldsymbol{\rho},\text{int}} = -g_\rho\boldsymbol{\rho}_\mu \cdot \left(\frac{1}{2}\overline{\psi}\gamma^\mu\boldsymbol{\tau}\psi + \boldsymbol{\rho}_\nu \times \boldsymbol{\rho}^{\mu\nu}\right). \tag{5.16}$$

Unfortunately this is not entirely correct, since the ρ current in the second term in (5.16) contains the derivative of the field, which means it would yield another term to the current. This is shown by applying the Noether current expression (3.39) to this current:

$$\begin{aligned}\frac{\delta\mathcal{L}_{\boldsymbol{\rho},\text{int}}}{\delta(\partial_\mu\boldsymbol{\rho}_\nu)}\delta\boldsymbol{\rho}_\nu &= -\boldsymbol{\rho}_\nu \times \frac{\delta\mathcal{L}_{\boldsymbol{\rho},\text{int}}}{\delta(\partial_\mu\boldsymbol{\rho}_\nu)} \\ &= 2g_\rho\left(\boldsymbol{\rho}^\nu \times \boldsymbol{\rho}^\mu\right) \times \boldsymbol{\rho}_\nu\end{aligned} \tag{5.17}$$

(all steps are shown in the appendix). Adding this last term to (5.16) we obtain

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\rho},\text{int}} &= -g_\rho \boldsymbol{\rho}_\mu \cdot \left(\frac{1}{2}\overline{\psi}\gamma^\mu \boldsymbol{\tau}\psi + \boldsymbol{\rho}_\nu \times \boldsymbol{\rho}^{\mu\nu} + 2g_\rho \left(\boldsymbol{\rho}^\nu \times \boldsymbol{\rho}^\mu\right) \times \boldsymbol{\rho}_\nu \right) \\ &= -g_\rho \boldsymbol{\rho}_\mu \cdot \boldsymbol{I}^\mu. \end{aligned} \tag{5.18}$$

With this new interaction term, we can find the new equations of motion for the ψ field:

$$\left[\gamma^\mu \left(i\partial_\mu - g_\omega \omega_\mu(x) - \frac{1}{2} g_\rho \boldsymbol{\tau} \cdot \boldsymbol{\rho}_\mu(x)\right) - \left(m - g_\sigma \sigma(x)\right)\right] \psi(x) = 0. \tag{5.19}$$

When trying to find the mean ρ field, we can simplify the Lagrangian and the equations of motion by considering some facts. First, the ρ meson can be considered as two complex fields and one real,

$$\boldsymbol{\rho}^\mu = \left(\rho^\mu_\pm, \rho^\mu_3\right), \tag{5.20}$$

where the complex fields $\rho^\mu_\pm$ can be defined as the raising and lowering operators in isospin space for the third axis (analogously as in spin space):

$$\rho^\mu_\pm = \frac{1}{\sqrt{2}}\left(\rho^\mu_1 \pm i\rho^\mu_2\right). \tag{5.21}$$

The projection on the third axis in isospin space is the one distinguishing protons from neutrons. In a ground state configuration we would expect the neutron and proton densities to be constant, therefore the expectation value for the raising or lowering operators should be zero. The only nonzero component of ρ would then be ρ^μ_3. Secondly, as for the ω meson, following the same steps leading to (4.50) we should expect the spatial components to disappear, leaving only the ρ^0_3 term (henceforth referred as ρ_{03}). This simplifies the expectation of $\mathcal{L}_\rho$ considerably. The derivatives disappear from both $\mathcal{L}_{\rho,f}$ and $\mathcal{L}_{\rho,\text{int}}$ and the third term in $\mathcal{L}_{\rho,\text{int}}$ also disappears since the only vector surviving from the reasoning above is ρ_0 and the cross product between parallel vectors gives zero. Being left with only the time component of one of the components of ρ vector, the form of the equation of motion for the mean ρ field will be analogous to the one for the ω field:

$$g_\rho\langle\rho_{03}\rangle = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 \langle\overline{\psi}\gamma^0\tau_3\psi\rangle = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 (\rho_p - \rho_n) \tag{5.22}$$

$$g_\rho\langle\rho_{i3}\rangle = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 \langle\overline{\psi}\gamma^i\tau_3\psi\rangle = 0, \tag{5.23}$$

where the value of $\langle\overline{\psi}\gamma^0\tau_3\psi\rangle$ was already found in (4.17). What then survives of the ρ free Lagrangian is

$$\langle\mathcal{L}_{\rho,\text{free}}\rangle = \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2. \tag{5.24}$$

Writing all the fields of the theory, we obtain:

$$
\begin{aligned}
g_\sigma\langle\sigma\rangle &= \left(\frac{g_\sigma}{m_\sigma}\right)^2 \Bigg(- m_n b\,(g_\sigma\langle\sigma\rangle)^2 - c\,(g_\sigma\langle\sigma\rangle)^3 \\
&\qquad + \frac{1}{\pi^2}\sum_B \int_0^{p_B} \frac{p^2(m-g_\sigma\langle\sigma\rangle)\mathrm{d}p}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}}\Bigg),
\end{aligned}
\tag{5.25}
$$

$$
g_\omega\langle\omega_0\rangle = \left(\frac{g_\omega}{m_\omega}\right)^2 \rho, \tag{5.26}
$$

$$
g_\rho\langle\rho_{03}\rangle = \frac{1}{2}\left(\frac{g_\rho}{m_\rho}\right)^2 (\rho_p - \rho_n)\,, \tag{5.27}
$$

$$
\left[\gamma^\mu\big(i\partial_\mu - g_\omega\langle\omega_\mu\rangle - \frac{1}{2}g_\rho\tau_3\langle\rho_{\mu 3}\rangle\big) - \big(m - g_\sigma\langle\sigma\rangle\big)\right]\psi(x) = 0, \tag{5.28}
$$

where $\rho = \sum_B \rho_B$ is the total baryon density, each density is defined at $T = 0$ as

$$
\rho_B = \frac{{p_B}^3}{3\pi^2}. \tag{5.29}
$$

Here the sum over B is over the baryons in the model (here only neutrons and protons) and p_n and p_p are the Fermi momenta of the neutron and the proton respectively. Recalling equations (4.32) and (4.33), we need to find the new values for $\langle\overline{\psi}\gamma_0 p_0\psi\rangle$ and $\langle\overline{\psi}\gamma_i p_i\psi\rangle$. In order to find these, it is necessary to find the new energy eigenvalues for the nucleons. These are found by following the same steps leading to (4.51). First we find the Fourier transform of equation (5.19) and consider the equation of motion in momentum space:

$$
\left[\gamma^\mu\big(p_\mu - g_\omega\langle\omega_0\rangle - \frac{1}{2}g_\rho\tau_3\langle\rho_{03}\rangle\big) - \big(m - g_\sigma\langle\sigma\rangle\big)\right]\psi(p) = 0. \tag{5.30}
$$

Recalling then the procedure leading to the energy eigenvalues in the σ–ω model, we introduce the reduced mass m^*, collect the four-vectors under K_μ and repeat the same steps from (4.35) to (4.39), obtaining this time

$$
e_B(p) = g_\omega\langle\omega_0\rangle + I_B g_\rho\langle\rho_{03}\rangle + \sqrt{p^2 + m^{*2}}, \tag{5.31}
$$

with I_B being the projection of isospin on the third axis. Here we notice how the new term gives a positive value when the baryon taken into consideration is in abundance compared to the other, increasing its energy. Since the ground state is per definition the least energetic state, these states are isospin unfavored, compared to the symmetric ones. As we will see when treating charge neutrality and beta-decay balance, the isospin symmetric state is not necessarily the least energetic one. On the lines of (4.57) and (4.58) we then obtain

$$
\langle\overline{\psi}\gamma_0 p_0\rangle = \frac{1}{\pi^2}\sum_B \int_0^{p_B} p^2\left(g_\omega\langle\omega_0\rangle + I_B g_\rho\langle\rho_{03}\rangle + \sqrt{p^2+m^{*2}}\right)\mathrm{d}p \tag{5.32}
$$

$$
= m_\omega^2\langle\omega_0\rangle^2 + m_\rho^2\langle\rho_{03}\rangle^2 + \frac{1}{\pi^2}\sum_B \int_0^{p_B} p^2\sqrt{p^2+m^{*2}}\mathrm{d}p \tag{5.33}
$$

and

$$\langle\overline{\psi}\gamma_i p_i\psi\rangle = \sum_B \frac{1}{\pi^2}\int_0^{p_B} p^2 \mathrm{d}p \frac{p^2}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}}, \tag{5.34}$$

yielding finally

$$\begin{aligned}\epsilon &= \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 + \frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3 + \frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4 \\ &\quad + \sum_B \frac{1}{\pi^2}\int_0^{p_B} \mathrm{d}p p^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}, \end{aligned} \tag{5.35}$$

$$\begin{aligned}P &= -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3 - \frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4 \\ &\quad + \sum_B \frac{1}{3\pi^2}\int_0^{p_B} \mathrm{d}p \frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}}. \end{aligned} \tag{5.36}$$

5.1.3 Charge neutrality and beta decay

The last parts we would like to incorporate in our equation of state is charge neutrality by introducing leptons such as electrons and muons. These are not affected by the strong nuclear force, and only interact with the weak and electromagnetic force. We will only treat the weak interaction by enforcing beta stability and muon decay statistically, and ignoring the Coulomb contribution of the leptons to the energy density and the pressure. This means that they will contribute to the Lagrangian only with their free Dirac equations:

$$\mathcal{L}_{\text{leptons}} = \sum_\lambda \overline{\psi}_\lambda \left(i\gamma^\mu\partial_\mu - m_\lambda\right)\psi_\lambda. \tag{5.37}$$

Here the sum runs over the leptons in the theory (electrons and muons), while the energy eigenvalues are given by

$$e_e(p) = \sqrt{p^2+m_e^2}, \tag{5.38}$$

$$e_\mu(p) = \sqrt{p^2+m_\mu^2}. \tag{5.39}$$

Both electrons and muons are negatively charged. In order to enforce global charge neutrality we need to find an expression for their charge density, and say that it must be as big as the positive charge density from the protons. Fortunately the absolute value of charge of all these particles is of the same strength ($+1$ or -1), so this is equivalent to saying that the proton density must be the same as the sum of electron and muon densities,

$$\rho_p = \rho_e + \rho_\mu = \frac{1}{3\pi^2}\left(p_e^3 + p_\mu^3\right). \tag{5.40}$$

With the introduction of electrons, we should enforce statistical beta stability, i.e. the fact that in β equilibrium as many neutrons decay into a protons and electrons as there are electron captures from protons, forming new neutrons:

$$n \leftrightarrow p + e^-(+\overline{\nu}_e). \tag{5.41}$$

The term in parenthesis is an electronic antineutrino. In this model we will ignore neutrinos and antineutrinos, since their very small masses make their contribution to the equation of state negligible as long as we assume uniform nuclear matter [17]. Mathematically, this equilibrium translates into the equality of their chemical potentials,

$$\mu_n = \mu_p + \mu_e \tag{5.42}$$

which, for $T = 0$, are the same as their Fermi energy, in other words their energy eigenvalues evaluated at their Fermi momenta:

$$e_n(p_n) = e_p(p_p) + e_e(p_e). \tag{5.43}$$

At high Fermi momenta we would also expect the inverse muon decay to happen, i.e. the capture from an electron of an anti-electronic and a muonic neutrino to give a muon:

$$e^- + \overline{\nu}_e + \nu_\mu \leftrightarrow \mu. \tag{5.44}$$

Neglecting the (nearly) massless neutrinos, we can express this equilibrium by chemical potentials,

$$\mu_e = \mu_\mu, \tag{5.45}$$

or, by the $T = 0$ idealization, their energy eigenvalues evaluated at Fermi momentum

$$\sqrt{p_e^2 + m_e^2} = \sqrt{p_\mu^2 + m_\mu^2}. \tag{5.46}$$

Since the muon mass ($m_\mu = 105.658$ MeV) is much bigger than the electron mass ($m_e = 0.511$ MeV) [31], these particles are highly unstable and decay into electrons for low densities. Muons will not appear before the total energy of the electron is enough to be transformed into the rest mass of a muon at rest. From (5.46) we will then have the following condition for muon presence:

$$p_e^2 > m_\mu^2 - m_e^2. \tag{5.47}$$

Electrons and muons give a contribution with their energies and masses to the equation of state. Their contribution to the pressure and energy density comes from $\langle\overline{\psi}_\lambda\gamma_0 p_0\psi_\lambda\rangle$ and $\langle\overline{\psi}_\lambda\gamma_i p_i\psi_\lambda\rangle$:

$$\langle\overline{\psi}_\lambda\gamma_0 p_0\psi_\lambda\rangle = \sum_\lambda \frac{1}{\pi^2}\int_0^{p_\lambda} p^2 \mathrm{d}p\sqrt{p^2 + m_\lambda^2}, \tag{5.48}$$

$$\frac{1}{3}\langle\overline{\psi}_\lambda\gamma_i p_i\psi_\lambda\rangle = \sum_\lambda \frac{1}{3\pi^2}\int_0^{p_\lambda} \frac{p^4}{\sqrt{p^2 + m_\lambda^2}}\mathrm{d}p, \tag{5.49}$$

giving

$$\begin{aligned}
\epsilon =&\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2+\frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2+\frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3+\frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4\\
&+\sum_\lambda\frac{1}{\pi^2}\int_0^{p_\lambda}p^2\mathrm{d}p\sqrt{p^2+m_\lambda^2}\\
&+\sum_B\frac{1}{\pi^2}\int_0^{p_B}\mathrm{d}pp^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2},
\end{aligned}\tag{5.50}$$

$$\begin{aligned}
P =&-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2+\frac{1}{2}m_\omega^2\langle\omega_0\rangle^2+\frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2-\frac{1}{3}m_n b\left(g_\sigma\langle\sigma\rangle\right)^3-\frac{1}{4}c\left(g_\sigma\langle\sigma\rangle\right)^4\\
&+\sum_\lambda\frac{1}{3\pi^2}\int_0^{p_\lambda}\frac{p^4}{\sqrt{p^2+m_\lambda^2}}\\
&+\sum_B\frac{1}{3\pi^2}\int_0^{p_B}\mathrm{d}p\frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}}.
\end{aligned}\tag{5.51}$$

5.1.4 The crust

Neutron stars were initially thought as densely packed neutrons, held together by the gravitational pull and obeying the laws of statistical quantum physics. This was the idea behind the papers from Tolman [40], Oppenheimer and Volkoff [29] in 1939, reproduced in the results of Section 2.3.3. This is an unrealistic idealization for many reasons already discussed, but also because the energy density grows from zero at the surface, and the neutron matter has to go through a phase with sub-nuclear density. Before reaching nuclear densities we should expect more familiar matter made of atoms, ions and plasma, before becoming proper nuclear matter. We may call this outer layer the crust of the neutron star. A good description of the crust is given in the paper from Baym, Pethick and Sutherland (hereafter referred as BPS) in 1971 [4]. When the matter in a neutron star is in its ground state and at $T = 0$, we should expect a region to be in thermodynamical equilibrium when its Gibbs energy is minimized, i.e. matter prefers to be in its most thermodynamically stable state. This reasoning leads us to think that the outermost layer of a neutron star, when neglecting the possibility for an atmosphere, must consist of a lattice of ^{56}Fe nuclei, the most stable for such densities and pressures. This remains the ground state up to $\epsilon \approx 10^7$ g/cm^3, where the pressure becomes high enough to make an heavier atomic nucleus more stable, ^{62}Ni. The same situation occurs at higher densities: ^{64}Fe becomes the most stable nucleus and appears at 2.71×10^8 g/cm^3, ^{66}Ni at 1.30×10^9 g/cm^3 etc. When a nucleus becomes more energetically favorable than another, we have a phase transition similar to the one considered in the σ–ω model, where we have a jump in energy density for constant pressure. We have 13 such phase transitions according to Haensel & Pichon [16], where nuclei becomes bigger and bigger, and more neutron rich. This phase of crust ends at densities around 4.32×10^{11} g/cm^3, where $\mu_n - m_n$ (the chemical potential of the neutron minus its mass) reaches the value of its lowest continuum state in the lattice. At these densities neutrons start to ”drip out” the nuclei and populate the continuum energy range, in effect being a neutron gas in between the lattice of nuclei. As the energy density and the pressure increase the more we approach the center of the star, we meet atomic nuclei with different shapes than spherical (first elongated structures as rods, then plates). It comes then a point when nuclear matter occupies more than 50% of the volume, and we would have first plates, then rods and finally bubbles of neutron gas inside blocks of nuclear matter [3] [17]. This transition between spherical nuclei to uniform nuclear matter happens between $\epsilon \approx 10^{13.5}$ g/cm^3 and $\epsilon = \epsilon_{cc} \approx 10^{14}$ g/cm^3, where we name the layer where this last density occur as the *crust-core interface*. The core has then densities above ϵ_{cc}, where we can use our RMF theory to describe the nuclear matter in it. Values for the equation of state in the crust region are usually given in tables. Although not a big problem, it is easier to use parametrized versions as the ones shown in [17] for computation. The two equations of state for the crust region are the Skyrme Leon (SLy) developed by Douchin and Haensel [8] and the Friedman Pandharipande and Skyrme (FPS) developed by Pandharipande and Ravenhall [30]. These are quite comparable, the only difference being their modeling of the crust-core interface. While the FPS model takes care of the exotic nuclear shapes (rods, plates etc.) near ϵ_{cc}, the SLy model models the interface as a small phase transition, with a relative jump in energy density of 1%. While the structure of the neutron star may be important for effects such as neutrino emission and elastic properties of the matter, their effect on the final equation of state is small enough to be neglected. For our calculation, we will be using the FPS analytical approximation from [17]:

$$\begin{aligned}\tilde{P} = &\frac{a_1 + a_2\tilde{\epsilon} + a_3\tilde{\epsilon}^3}{1 + a_4\tilde{\epsilon}} f_0(a_5\left(\tilde{\epsilon} - a_6\right)) + \left(a_7 + a_8\tilde{\epsilon}\right)\left(a_9\left(a_{10} - \tilde{\epsilon}\right)\right) \\ &+ \left(a_{11} + a_{12}\tilde{\epsilon}\right) f_0\left(a_{13}\left(a_{14} - \tilde{\epsilon}\right)\right) + \left(a_{15} + a_{16}\tilde{\epsilon}\right) f_0\left(a_{17}\left(a_{18} - \tilde{\epsilon}\right)\right), \end{aligned} \tag{5.52}$$

where $\tilde{P}$ and $\tilde{\epsilon}$ are respectively the logarithms in base 10 of the pressure and the energy density measured in dyne/cm^2 and g/cm^3, and a_{1-18} are fitting constants, with values listed in Table 5.1.

a_1	a_2	a_3	a_4	a_5	a_6
11.4950	-22.775	1.5707	4.3	14.08	27.80
a_7	a_8	a_9	a_{10}	a_{11}	a_{12}
-1.653	1.50	14.67	6.22	6.121	0.005925
a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
0.16326	6.48	11.4971	19.105	0.8938	6.54

Table 5.1: Fitting parameters for the FPS crust equation of state, as retrieved from [17].

The function f_0 is defined as

$$f_0(x) = \frac{1}{e^x + 1}. \tag{5.53}$$

5.2 Mass-radius relation and particle population

The equation of state obtained for the $npe\mu$ model, i.e. the one containing neutrons, protons, electrons and muons given in (5.50) and (5.51) is shown in Figure 5.1. It is softer (i.e.

it grows more slowly) than the one developed in the σ–ω model for both neutron and nuclear matter. When coupled to the TOV and mass equations it yields the mass-radius relation shown in Figure 5.2. This equation of state gives a maximum mass of $M/M_{\odot} = 2.02$ and a corresponding minimum radius of $R = 11.31$ km. At the crust-core interface the

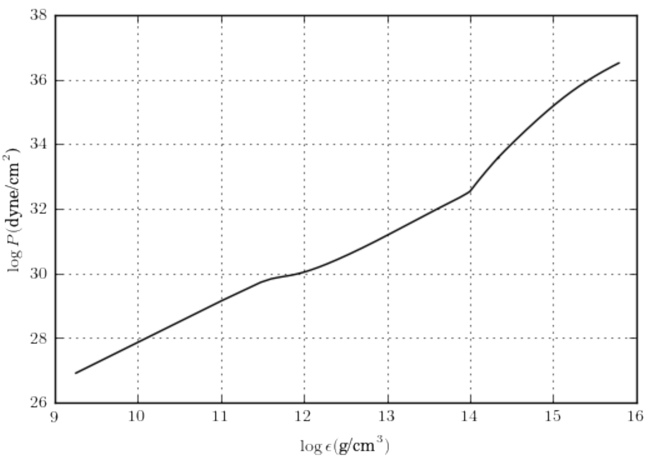

Figure 5.1: Equation of state for $npe\mu$ matter.

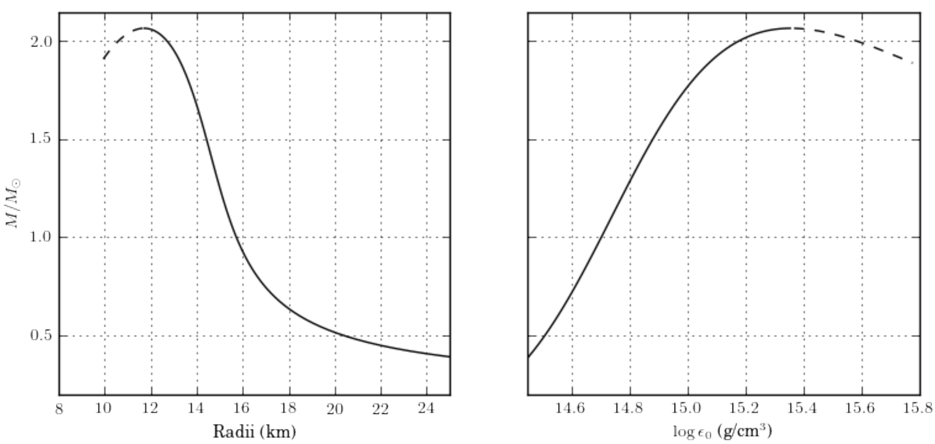

Figure 5.2: Relationship between radii and total masses on the left, and between central densities and total masses on the right, for $npe\mu$ matter. The dashed lines show the unstable solutions.

crust's atoms are very neutron rich, and the star's matter may be approximated as purely consisting of neutrons. Our calculations take then this assumption as a starting point in order to find the particle populations at different densities. As we can see in Figure 5.3, the relative neutron density decreases as a result of the increasing isospin restoring force, while charge neutrality is always preserved by the presence of electrons, and later muons. These appear when the momentum of the electrons is big enough to allow the reaction in (5.44) to happen, at $\rho = 0.128\rho_0$. Here we will idealize the core of neutron stars as pure $npe\mu$ matter. The particle relative population is shown in Figure 5.3.

5.2.1 Computation

The RMF approximation of dense $npe\mu$ matter consists of a system of nonlinear equations:

$$
\begin{aligned}
\rho &= \rho_p + \rho_n, \\
\rho_p &= \frac{p_p^3}{3\pi^2}, \\
\rho_n &= \frac{p_n^3}{3\pi^2}, \\
g_\sigma\langle\sigma\rangle &= \left(\frac{g_\sigma}{m_\sigma}\right)^2 \left[\frac{1}{\pi^2}\left(\int_0^{p_p} \mathrm{d}p \frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)}}\right.\right. \\
&\quad \left.\left. + \int_0^{p_n} \mathrm{d}p \frac{p^2(m - g_\sigma\langle\sigma\rangle)}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)}}\right) - m_n b\left(g_\sigma\langle\sigma\rangle\right)^2 - c\left(g_\sigma\langle\sigma\rangle\right)^3\right], \\
g_\omega\langle\omega_0\rangle &= \left(\frac{g_\omega}{m_\omega}\right)^2 (\rho_n + \rho_p), \\
g_\rho\langle\rho_{03}\rangle &= \left(\frac{g_\rho}{m_\rho}\right)^2 \left(\frac{1}{2}\rho_p - \frac{1}{2}\rho_n\right), \\
\mu_p &= e_p(p_p) = g_\omega\langle\omega_0\rangle + \frac{1}{2}g_\rho\langle\rho_{03}\rangle + \sqrt{p_p^2 + m^{*2}}, \\
\mu_n &= e_n(p_n) = g_\omega\langle\omega_0\rangle - \frac{1}{2}g_\rho\langle\rho_{03}\rangle + \sqrt{p_n^2 + m^{*2}}, \\
\mu_e &= e_e(p_e) = \sqrt{m_e^2 + p_e^2}, \\
\mu_\mu &= e_\mu(p_\mu) = \sqrt{m_\mu^2 + p_\mu^2}, \\
\mu_n &= \mu_p + \mu_e, \\
\mu_e &= \mu_\mu, \\
\rho_p &= \frac{1}{3\pi^2}\left(p_e^3 + p_\mu^3\right).
\end{aligned}
$$

This is just the collection of the equations we have already discussed throughout this chapter. The first three are the baryon, the proton and neutron particle densities (5.29), number 4, 5 and 6 are the σ, ω_0 and ρ_{03} mean-fields encountered in (5.25), (5.26) and (5.27). Number 7, 8, 9 and 10 describe the chemical potential at $T = 0$ for every baryon and lepton species in terms of their energy eigenvalue at Fermi momentum (5.31), 11 and 12 the electron capture/neutron beta decay (5.42) and neutrino capture/muon decay at thermodynamical equilibrium (5.45) respectively, and lastly the charge neutrality condition (5.40). Before muon appearance, p_μ is set to 0. Even if some of these equations

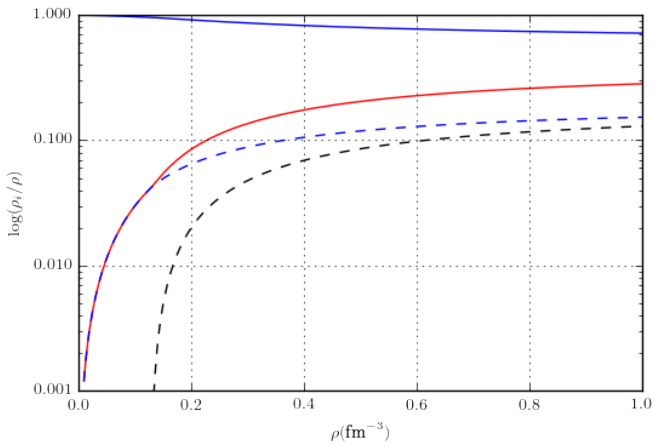

Figure 5.3: Relative particle populations at various densities. Blue indicates the relative population for neutrons, red for protons, dashed blue for electrons and dashed black for muons.

are nonlinear, many other are linear and can be substituted into the others, diminishing the number of equations that eventually will have to be evaluated in a numerical computation. After some tedious algebra, we end up with three nonlinear equations and four unknowns:

$$\begin{aligned} g_\sigma\langle\sigma\rangle - \left(\frac{g_\sigma}{m_\sigma}\right)^2 \Bigg[&- bm_n(g_\sigma\langle\sigma\rangle)^2 - c(g_\sigma\langle\sigma\rangle)^3 \\ &+ m^{*3}\left(\left(\frac{p_p}{m^*}\sqrt{\left(\frac{p_p}{m^*}\right)^2+1} - \sinh^{-1}\left(\frac{p_p}{m^*}\right)\right)\right. \\ &+ \left.\left.\left(\frac{p_n}{m^*}\sqrt{\left(\frac{p_n}{m^*}\right)^2+1} - \sinh^{-1}\left(\frac{p_n}{m^*}\right)\right)\right)\right] = 0, \end{aligned} \tag{5.54}$$

$$g_\rho\langle\rho_{03}\rangle + \sqrt{p_p^2+m^{*2}} - \sqrt{p_n^2+m^{*2}} + \sqrt{p_e^2+m_e^2} = 0, \tag{5.55}$$

$$p_p - p_e = 0, \tag{5.56}$$

where the last one changes to

$$p_p - \left(p_e^3+p_\mu^3\right)^{1/3} = 0 \tag{5.57}$$

after the condition in (5.47) is met. In (5.54) the integrals in (5.25) are written explicitly, as seen in (4.70).
The four unknowns are ρ, ρ_n, $g_\sigma\langle\sigma\rangle$ and p_e, and the other variables in (5.54), (5.55), (5.56)

and (5.57) are uniquely defined in terms of these:

$$
\begin{aligned}
p_p &\equiv \left(3\pi^2(\rho - \rho_n)\right)^{1/3}, \\
p_n &\equiv \left(3\pi^2\rho_n\right)^{1/3}, \\
g_\rho\langle\rho_{03}\rangle &\equiv \left(\frac{g_\rho}{m_\rho}\right)^2 \left(\frac{1}{2}\rho - \rho_n\right), \\
m^* &\equiv m - g_\sigma\langle\sigma\rangle, \\
p_\mu &\equiv \sqrt{m_e^2 + p_e^2 - m_\mu^2}.
\end{aligned}
$$

This system of equations can now be evaluated for a range of values of ρ, i.e. every value of ρ gives unique values for ρ_n, $g_\sigma\langle\sigma\rangle$ and p_e. With these we can obtain all other values of interest in order to find solutions to the equation of state. The computation consists in creating an array of values for ρ, and then building a long equation where all left sides of equations (5.54), (5.55) and (5.56) — the last one substituted by (5.57) after muon appearance — are squared and summed, for then trying to find a root for the equation. The values that the four unknowns have to have in order to give zero in the sum of squares are the values that satisfy the original system of equations. The found values of ρ_n, $g_\sigma\langle\sigma\rangle$ and p_e will be saved in respective arrays, and used as guesses for the next iteration where we will be searching for the values of the same unknowns for slightly increased value of ρ. With every set of four values we will be able to compute the energy density and the pressure in (5.50) and (5.51), giving us a numerical solution for the equation of state, which can be used together with the TOV and mass equations to give the relations in Figure 5.2.

Chapter 6

Renormalization

We have previously shown how we can idealize neutron star matter as a cold Fermi gas consisting of noninteracting neutrons (Chapter 2), how we can derive the same equation of state from QFT and thermodynamical principles (Chapter 3), for then including nucleon interactions in the framework of the σ–ω model using the RMF approximation (Chapter 4). Later we expanded the σ–ω model to include electrons, muons, the ρ meson (Chapter 5) and to account for scalar self-interactions, isospin asymmetries, beta decay, and charge neutrality. However, in order to retrieve the same equations for the relativistic free Fermi gas, in Chapter 3 we have ignored the first, divergent, vacuum term in the partition function in (3.80). If we were to derive the $npe\mu$ equation of state using the more fundamental TFT framework, we would expect to obtain a similar term, as we will show in this chapter.

6.1 Path integral derivation of the RMF $npe\mu$ model

Using the same formalism we developed and used in Chapter 3, we are expected to obtain the same equation of state for $npe\mu$ matter, plus a vacuum term. The derivations in this section follow loosely the ones in [33]. Accounting for the chemical potentials of every fermion, the Lagrangian of the $npe\mu$ model is

$$\begin{aligned}\mathcal{L} = &\sum_B \overline{\psi}_B \Big(i\gamma^\mu\partial_\mu - g_\omega\gamma^\mu\omega_\mu + g_\sigma\sigma - \frac{1}{2}g_\rho\boldsymbol{\rho}_\mu\cdot\gamma^\mu\boldsymbol{\tau} - m_B + \mu_B\gamma^0\Big)\psi_B \\ &-\frac{1}{3}\tilde{b}(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4 - g_\rho\boldsymbol{\rho}\cdot\big(\boldsymbol{\rho}_\nu\times\boldsymbol{\rho}^{\mu\nu} + 2g_\rho(\boldsymbol{\rho}^\nu\times\boldsymbol{\rho}^\mu)\times\boldsymbol{\rho}_\nu\big) \\ &+\sum_\lambda \overline{\psi}_\lambda\left(i\gamma^\mu\partial_\mu - m_\lambda + \mu_\lambda\gamma^0\right)\psi_\lambda \\ &+\frac{1}{2}\Big((\partial^\mu\sigma)(\partial_\mu\sigma) - m_\sigma^2\sigma^2 - \frac{1}{2}\omega^{\mu\nu}\omega_{\mu\nu} + m_\omega^2\omega^\mu\omega_\mu - \frac{1}{2}\boldsymbol{\rho}^{\mu\nu}\cdot\boldsymbol{\rho}_{\mu\nu} + m_\rho^2\boldsymbol{\rho}^\mu\cdot\boldsymbol{\rho}_\mu\Big),\end{aligned} \tag{6.1}$$

where the B and the λ sums run over the baryons (protons and neutrons) and the leptons (electrons and muons) of the model respectively, and $\tilde{b}$ is shorthand for the former coupling constant b times 938 MeV: $\tilde{b} = 938 \times b$ MeV. The first line of (6.1) takes care of the Dirac Lagrangian for both baryons and their interaction terms with the meson fields. The second line includes the self-interactions of the mesons, in the third line is written the Dirac Lagrangian for the two lepton species of the theory, and the fourth line collects the free Lagrangians for the three meson fields. For such a Lagrangian, the partition function would take the form

$$Z = \int \prod_B \left(\mathcal{D}\overline{\psi}_B \mathcal{D}\psi_B\right) \prod_\lambda \left(\mathcal{D}\overline{\psi}_\lambda \mathcal{D}\psi_\lambda\right) \mathcal{D}\sigma \mathcal{D}\omega \mathcal{D}\boldsymbol{\rho} \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L}\right]. \tag{6.2}$$

Allowing the meson fields to have nonzero expectation values, we may rewrite the σ, ω and ρ fields in terms of this and a fluctuation about the mean value:

$$\sigma \to \langle\sigma\rangle + \tilde{\sigma}, \tag{6.3}$$

$$\omega_\mu \to \langle\omega_\mu\rangle + \tilde{\omega}_\mu, \tag{6.4}$$

$$\boldsymbol{\rho}_\mu \to \langle\boldsymbol{\rho}_\mu\rangle + \tilde{\boldsymbol{\rho}}_\mu. \tag{6.5}$$

The RMF approximation then consists in neglecting these fluctuations. We can use the same arguments from Chapters 4 and 5 to find that the only nonzero terms of the ω and ρ fields in static equilibrium are $\langle\omega_0\rangle$ and $\langle\rho_{03}\rangle$. In addition, we can define the *baryonic effective mass* m_B^* and *effective chemical potential* μ_B^* as,

$$m_B^* = m_B - g_\sigma\langle\sigma\rangle, \tag{6.6}$$

$$\mu_B^* = \mu_B - g_\omega\langle\omega_0\rangle - I_B g_\rho\langle\rho_{03}\rangle, \tag{6.7}$$

where I_B is the projection of the isospin along the third axis ($1/2$ for protons and $-1/2$ for neutrons). With these simplifications, the Lagrangian in (6.1) becomes

$$\begin{aligned}\mathcal{L}_{\text{mean}} = &\sum_B \overline{\psi}_B\left(i\gamma^\mu\partial_\mu - m_B^* + \mu_B^*\gamma^0\right)\psi_B + \sum_\lambda \overline{\psi}_\lambda\left(i\gamma^\mu\partial_\mu - m_\lambda + \mu_\lambda\gamma^0\right)\psi_\lambda \\ &-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4,\end{aligned} \tag{6.8}$$

and, subsequently, the partition function in (6.2) becomes

$$Z = \int \prod_B \left(\mathcal{D}\overline{\psi}_B \mathcal{D}\psi_B\right) \prod_\lambda \left(\mathcal{D}\overline{\psi}_\lambda \mathcal{D}\psi_\lambda\right) \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathcal{L}_{\text{mean}}\right]$$

or, in explicit form,

$$\begin{aligned}Z = &\exp\left[V\beta\left(-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4\right)\right] \\ &\times \int \prod_B \left(\mathcal{D}\overline{\psi}_B \mathcal{D}\psi_B\right) \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \sum_B \overline{\psi}_B\left(i\gamma^\mu\partial_\mu - m_B^* + \mu_B^*\gamma^0\right)\psi_B\right] \\ &\times \int \prod_\lambda \left(\mathcal{D}\overline{\psi}_\lambda \mathcal{D}\psi_\lambda\right) \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \sum_\lambda \overline{\psi}_\lambda\left(i\gamma^\mu\partial_\mu - m_\lambda + \mu_\lambda\gamma^0\right)\psi_\lambda\right].\end{aligned} \tag{6.9}$$

Here we have moved the exponentials involving only the meson fields out of the path-integral and we have divided the baryon and lepton path-integrals for more clarity. These are four Dirac path-integrals in the same form as those evaluated in Chapter 3, and the exponentials can be easily translated into (3.56). By following exactly the same steps leading to (3.80), we obtain

$$\begin{aligned}\ln Z = &\left[V\beta\left(-\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4\right)\right] \\ &+ 2V\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\beta\omega^* + \sum_B\left[\ln\left(e^{-\beta(\omega_B^*+\mu_B^*)}+1\right) + \ln\left(e^{-\beta(\omega_B^*-\mu_B^*)}+1\right)\right]\right. \\ &\left.+ \sum_\lambda\left[\ln\left(e^{-\beta(\omega_\lambda+\mu_\lambda)}+1\right) + \ln\left(e^{-\beta(\omega_\lambda-\mu_\lambda)}+1\right)\right]\right\},\end{aligned} \tag{6.10}$$

where $\beta = 1/T$, T is the temperature, and

$$\omega_B^* = \sqrt{p^2 + m_B^{*2}}, \tag{6.11}$$

$$\omega_\lambda = \sqrt{p^2 + m_\lambda^2}, \tag{6.12}$$

$$\omega^* = \sum_B \omega_B^* + \sum_\lambda \omega_\lambda. \tag{6.13}$$

We can then use the thermodynamical identities in (3.4) and (3.5) to obtain the equation of state. Starting from the pressure, we follow the steps leading to (3.81) — this time retaining the divergent term— and obtain

$$\begin{aligned}P =& \frac{\partial(T\ln Z)}{\partial V} \\ =& -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4 \\ &+ \frac{1}{3\pi^2}\sum_B\int_0^\infty\frac{p^4\mathrm{d}p}{\omega_B^*}\left(\frac{1}{e^{(\omega_B^*-\mu_B^*)/T}+1} + \frac{1}{e^{(\omega_B^*+\mu_B^*)/T}+1}\right) \\ &+ \frac{1}{3\pi^2}\sum_\lambda\int_0^\infty\frac{p^4\mathrm{d}p}{\omega_\lambda}\left(\frac{1}{e^{(\omega_\lambda-\mu_\lambda)/T}+1} + \frac{1}{e^{(\omega_\lambda+\mu_\lambda)/T}+1}\right) \\ &+ 2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega^*.\end{aligned} \tag{6.14}$$

The expressions for three meson fields can then be found by maximizing the pressure. For the $\langle\sigma\rangle$ field we have

$$\begin{aligned}0 = \frac{\partial P}{\partial\langle\sigma\rangle} &= -m_\sigma^2\langle\sigma\rangle - \tilde{b}g_\sigma^3\langle\sigma\rangle^2 - cg_\sigma^4\langle\sigma\rangle^3 + 2\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial}{\partial\langle\sigma\rangle}\omega_B^* \\ &+ 2T\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial}{\partial\langle\sigma\rangle}\left[\ln\left(e^{-(\omega_B^*+\mu_B^*)/T}+1\right) + \ln\left(e^{-(\omega_B^*-\mu_B^*)/T}+1\right)\right].\end{aligned} \tag{6.15}$$

We take the partial differentiation in $\langle\sigma\rangle$, and obtain

$$\frac{m_\sigma^2}{g_\sigma}\langle\sigma\rangle = -\tilde{b}(g_\sigma\langle\sigma\rangle)^2 - c(g_\sigma\langle\sigma\rangle)^3 - 2\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m_B^*}{\omega_B^*} + 2\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m_B^*}{\omega_B^*}\left[\frac{e^{-(\omega_B^*+\mu_B^*)/T}}{1+e^{-(\omega_B^*+\mu_B^*)/T}} + \frac{e^{-(\omega_B^*-\mu_B^*)/T}}{1+e^{-(\omega_B^*-\mu_B^*)/T}}\right],$$

where a simple division by the common exponential in both the numerator and the denominator in the two fractions in the square brackets shows us that they are indeed the the Fermi-Dirac distributions for particles and anti-particles. By then taking the $T\to 0$ limit, the first term vanishes and the second becomes a step function, giving

$$g_\sigma\langle\sigma\rangle = \left(\frac{g_\sigma}{m_\sigma}\right)^2\Bigg[-\tilde{b}(g_\sigma\langle\sigma\rangle)^2 - c(g_\sigma\langle\sigma\rangle)^3 + \frac{1}{\pi^2}\sum_B\int_0^{p_B}\frac{m_B-g_\sigma\langle\sigma\rangle}{\sqrt{p^2+(m_B-g_\sigma\langle\sigma\rangle)^2}}p^2\mathrm{d}p - 2\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m_B-g_\sigma\langle\sigma\rangle}{\sqrt{p^2+(m_B-g_\sigma\langle\sigma\rangle)^2}}\Bigg], \quad (6.16)$$

which is the same result as obtained in (5.25), except for the last divergent integral. The other meson fields are obtained in a similar way. For the $\langle\omega_0\rangle$ field we have

$$0 = \frac{\partial P}{\partial\langle\omega_0\rangle} = m_\omega^2\langle\omega_0\rangle + 2T\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial}{\partial\langle\omega_0\rangle}\left[\ln\left(e^{-(\omega_B^*+\mu_B^*)/T}+1\right) + \ln\left(e^{-(\omega_B^*-\mu_B^*)/T}+1\right)\right]. \quad (6.17)$$

Here again we take the differentiation in the mean-field, and obtain

$$m_\omega^2\langle\omega_0\rangle = 2g_\omega\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{e^{-(\omega_B^*-\mu_B^*)/T}}{1+e^{-(\omega_B^*-\mu_B^*)/T}} - \frac{e^{-(\omega_B^*+\mu_B^*)/T}}{1+e^{-(\omega_B^*+\mu_B^*)/T}}\right]. \quad (6.18)$$

As before we have the Fermi-Dirac distribution, which for $T\to 0$ gives the usual step function, and the $\langle\omega_0\rangle$ field becomes

$$g_\omega\langle\omega_0\rangle = \left(\frac{g_\omega}{m_\omega}\right)^2\sum_B\int_0^{p_B}\frac{p^2\mathrm{d}p}{\pi^2} = \left(\frac{g_\omega}{m_\omega}\right)^2\frac{p_p^3+p_n^3}{3\pi^2} = \left(\frac{g_\omega}{m_\omega}\right)^2\rho \quad (6.19)$$

as obtained in (5.26). Finally we consider the $\langle\rho_{03}\rangle$ field,

$$0 = \frac{\partial P}{\partial\langle\rho_{03}\rangle} = m_\rho^2\langle\rho_{03}\rangle + 2T\sum_B\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial}{\partial\langle\rho_{03}\rangle}\left[\ln\left(e^{-(\omega_B^*+\mu_B^*)/T}+1\right) + \ln\left(e^{-(\omega_B^*-\mu_B^*)/T}+1\right)\right]. \quad (6.20)$$

The differentiation gives a result in the same form as the one obtained with the $\langle\omega_0\rangle$ field, just with an extra I_B factor:

$$g_\rho\langle\rho_{03}\rangle = \left(\frac{g_\rho}{m_\rho}\right)^2 \sum_B \int_0^{p_B} \frac{I_B p^2 \mathrm{d}p}{\pi^2} = \left(\frac{g_\rho}{m_\rho}\right)^2 \frac{\rho_p - \rho_n}{2}. \tag{6.21}$$

This is again in accordance with what obtained previously in (5.27). We now look for the expression of the pressure in (6.14) at the zero-temperature limit. For $T \to 0$, the second terms in both integrands vanish. The first term, corresponding to particles, gives the usual step function and yields

$$\begin{aligned} P =& -\frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 - \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4 \\ &+ \sum_\lambda \frac{1}{3\pi^2}\int_0^{p_\lambda} \frac{p^4\mathrm{d}p}{\sqrt{p^2+m_\lambda^2}} \\ &+ \sum_B \frac{1}{3\pi^2}\int_0^{p_B} \frac{p^4\mathrm{d}p}{\sqrt{p^2+m_B^{*2}}} + 2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega^*. \end{aligned} \tag{6.22}$$

This is the same expression as in (5.51), except for the last divergent vacuum term. In order to find the expression for the energy density we then use the identity in (3.4):

$$\epsilon = \frac{E}{V} = -P + \frac{T}{V}\frac{\partial(T\ln Z)}{\partial T} + \frac{\mu_i}{V}\frac{\partial(T\ln Z)}{\partial\mu_i}, \tag{6.23}$$

where μ_i runs over μ_p, μ_n, μ_e and μ_μ. The terms dependent on T in $\ln Z$ have the same kind of dependency as in (3.83), thus we do not need to evaluate the second term, which will eventually vanish when taking the zero-temperature limit. The term of interest is the third, which reads:

$$\begin{aligned} \frac{\mu_i}{V}\frac{\partial(T\ln Z)}{\partial\mu_i} =& 2\mu_i \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\partial}{\partial\mu_i}\Bigg\{ \\ & \sum_B \delta_{iB}\left[\ln\left(e^{-\omega_B^*/T+\mu_B^*/T}+1\right) + \ln\left(e^{-\omega_B^*/T-\mu_B^*/T}+1\right)\right] \\ & + \sum_\lambda \delta_{i\lambda}\left[\ln\left(e^{-\omega_\lambda/T+\mu_\lambda/T}+1\right) + \ln\left(e^{-\omega_\lambda/T-\mu_\lambda/T}+1\right)\right]\Bigg\}, \end{aligned}$$

where all the terms independent of μ have vanished, included the divergent term. The δs are to be intended as Kronecker deltas as defined in (3.9), and have been inserted in order to express the sum over i in a more compact and clear way. The lepton expressions are the same as for any generic fermion, and have already been treated in the steps leading to (3.85). The baryon integrals are slightly different. We start by noticing that

$$\frac{\partial}{\partial\mu_B}\frac{\mu_B^*}{T} = \frac{\partial}{\partial\mu_B^*}\frac{\mu_B^*}{T} = \frac{1}{T}$$

and after taking the differentiation in μ_B we can easily follow the steps leading to (3.84), obtaining for the baryon term

$$2\sum_B \mu_B \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(\frac{1}{e^{(\omega_B^*-\mu_B^*)/T}+1}-\frac{1}{e^{(\omega_B^*+\mu_B^*)/T}+1}\right). \tag{6.24}$$

The second term of the integrand in (6.24) vanishes when taking the $T \to 0$ limit, while the first term becomes the usual step function which puts the upper limit of the integral to p_B. Now the Fermi energy ω_B^* equals the reduced chemical potential μ_B^*, and using the definition in (6.7), we obtain for the baryon term in (6.24)

$$\begin{aligned}
&\sum_B \left(g_\omega\langle\omega_0\rangle + I_B g_\rho\langle\rho_{03}\rangle + \sqrt{p_B^2+m_B^{*2}}\right)\int_0^{p_B}\frac{\mathrm{d}p}{\pi^2}p^2 \\
&= g_\omega\langle\omega_0\rangle\frac{p_p^3+p_n^3}{3\pi^2} + \frac{1}{2}g_\rho\langle\rho_{03}\rangle\frac{p_p^3-p_n^3}{3\pi^2} + \sum_B\frac{1}{\pi^2}\int_0^{p_B}p^2\mathrm{d}p\sqrt{p_B^2+m_B^{*2}} \\
&= m_\omega^2\langle\omega_0\rangle^2 + m_\rho^2\langle\rho_{03}\rangle^2 + \frac{1}{\pi^2}\int_0^{p_p}p^2\mathrm{d}p\sqrt{p_p^2+m_p^{*2}} + \frac{1}{\pi^2}\int_0^{p_n}p^2\mathrm{d}p\sqrt{p_n^2+m_n^{*2}},
\end{aligned}$$

where we have used the equations of motion for the $\langle\omega_0\rangle$ and $\langle\rho_{03}\rangle$ fields in (6.19) and (6.21). Using this last result, the third term of (6.23) becomes

$$\begin{aligned}
\frac{\mu_i}{V}\frac{\partial(T\ln Z)}{\partial\mu_i} &= m_\omega^2\langle\omega_0\rangle^2 + m_\rho^2\langle\rho_{03}\rangle^2 + \sum_B\frac{1}{\pi^2}\int_0^{p_B}p^2\mathrm{d}p\sqrt{p_B^2+m_B^{*2}} \\
&\quad + \sum_\lambda\frac{1}{\pi^2}\int_0^{p_\lambda}p^2\mathrm{d}p\sqrt{p_\lambda^2+m_\lambda^2}.
\end{aligned} \tag{6.25}$$

Equation (6.25) will be the third term in (3.84). We may also rewrite the pressure in (6.22) before plugging it in (3.84) using the following equality, already seen in (3.82):

$$\frac{1}{3\pi^2}\int_0^{p_F}\frac{p^4}{\sqrt{p^2+m^2}}\mathrm{d}p = \frac{1}{\pi^2}\int_0^{p_F}\left(\sqrt{p_F^2+m^2}-\sqrt{p^2+m^2}\right)p^2\mathrm{d}p. \tag{6.26}$$

We then plug (6.25) and (6.22) into the expression for the energy density in (3.84) and obtain

$$\begin{aligned}
\epsilon &= -P + \frac{\mu_i}{V}\frac{\partial(T\ln Z)}{\partial\mu_i} \\
&= \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_{03}\rangle^2 + \frac{1}{3}\tilde{b}(g_\sigma\langle\sigma\rangle)^3 + \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4 \\
&\quad + \sum_B\frac{1}{\pi^2}\int_0^{p_B}\sqrt{p^2+m_B^{*2}}p^2\mathrm{d}p \\
&\quad + \sum_\lambda\frac{1}{\pi^2}\int_0^{p_\lambda}\sqrt{p^2+m_\lambda^2}p^2\mathrm{d}p - 2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\omega^*,
\end{aligned} \tag{6.27}$$

which is the same expression as obtained in (5.50), except for the divergent vacuum term.

6.2 The divergent terms

6.2.1 Fermions

We have seen how the TFT framework developed in Chapter 3 yields the same results we obtained so far, adding some divergent terms in the $\langle\sigma\rangle$ field, the energy density and the pressure. We could name the ones obtained in Chapter 5 as $\langle\sigma\rangle_{\text{RMF}}$, ϵ_{RMF} and P_{RMF}, these being equations (5.26), (5.50) and (5.51) respectively. The results obtained in the previous section, equations (6.16), (6.22) and (6.27), will be denoted by $\langle\sigma\rangle_{\text{RMF,V}}$, $\epsilon_{\text{RMF,V}}$ and $P_{\text{RMF,V}}$. The relation between these is

$$g_\sigma\langle\sigma\rangle_{\text{RMF,V}} = g_\sigma\langle\sigma\rangle_{\text{RMF}} - V_\sigma \tag{6.28}$$

$$\epsilon_{\text{RMF,V}} = \epsilon_{\text{RMF}} + V_D \tag{6.29}$$

$$P_{\text{RMF,V}} = P_{\text{RMF}} - V_D, \tag{6.30}$$

where

$$V_D = -2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sum_B \sqrt{p^2+m_B^{*2}} + \sum_\lambda \sqrt{p^2+m_\lambda^2}\right) \tag{6.31}$$

$$V_\sigma = \frac{g_\sigma}{m_\sigma^2}\frac{\partial}{\partial\langle\sigma\rangle}V_D = 2\left(\frac{g_\sigma}{m_\sigma}\right)^2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_B \frac{m_B^*}{\sqrt{p^2+m_B^{*2}}}. \tag{6.32}$$

The m_B^* dependence in (6.31) tells us how the zero-point energy for the fermions is only affected by the presence of the σ field. This motivates us to find what is the shift in the zero-point energy due to the presence of this scalar field and its interaction with the baryons. In order to do this, we define a function $f_V(a)$,

$$f_V(a) = -2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sum_B \sqrt{p^2+a^2} + \sum_\lambda \sqrt{p^2+m_\lambda^2}\right), \tag{6.33}$$

with a as a parameter. We will then use this function in order to compactly express the shift in the vacuum energy due to the interaction between the σ field and the baryons:

$$\begin{aligned} V_{\text{ZP},B} &= f_V(m_B^*) - f_V(m_B) \\ &= -2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sum_B \sqrt{p^2+m_B^{*2}} + \sum_\lambda \sqrt{p^2+m_\lambda^2}\right) \\ &\quad + 2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sum_B \sqrt{p^2+m_B^{2}} + \sum_\lambda \sqrt{p^2+m_\lambda^2}\right) \\ &= -2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_B\left(\sqrt{p^2+m_B^{*2}} - \sqrt{p^2+m_B^2}\right). \end{aligned} \tag{6.34}$$

Due to the fact that leptons do not couple via the strong force and thus the σ field, their contribution to the shift in the vacuum energy vanishes. The expression in (6.34) is divergent and needs to be regularized. Its dimensional regularization is carried out in the

Appendix (section A.6), and yields:

$$V_{\mathrm{ZP},B} = \\ -\frac{1}{16\pi^2}\left\{\left(m_B^{*4}-m_B^4\right)\left[\Gamma\left(-1+\frac{\epsilon}{2}\right)-\frac{1}{2}+\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right]+m_B^{*4}\ln\left(\frac{m_B^{*2}}{m_B^2}\right)\right\}+O(\epsilon). \quad (6.35)$$

By recalling that $m_B^* = m_B - g_\sigma\sigma$, (where for convenience we indicate by σ the mean value $\langle\sigma\rangle$) we expand m_B^{*4} in the first product in (6.35), and obtain

$$-\frac{1}{16\pi^2}\Big(-4m_B^3(g_\sigma\sigma)+6m_B^2(g_\sigma\sigma)^2-4m_B(g_\sigma\sigma)^3+(g_\sigma\sigma)^4\Big) \\ \times\left[\Gamma\left(-1+\frac{\epsilon}{2}\right)-\frac{1}{2}+2\ln\left(\frac{m_B}{\sqrt{4\pi}\mu}\right)\right]. \quad (6.36)$$

From (6.36) we see how the divergence in the gamma function multiplies four terms in growing powers of σ from 1 to 4. These terms represent the four divergent loops arising from the coupling of the σ field to the baryons. In order to renormalize these terms, we have to add counterterms to the Lagrangian in (6.8) in order to absorb the divergences into finite values. These will be dependent upon renormalization scheme. Any finite remainder in the first power of σ yielded from the regularization is a contribution to the expectation value of σ in the vacuum, which we should expect to be zero in our effective theory. This is moreover an odd power of the scalar field, and it makes the energy density not bounded from below. We decide to choose a counterterm $\delta\sigma$ that cancels not only the divergence, but also every finite remainder in this power of σ. The remainders in σ^2 are in the same order of the mass term, and the counterterms from each baryon will act as contributions to m_σ^2 in the Lagrangian. We may call the sum of these baryon counterterms as δm_σ^2, and the new, renormalized mass as $M_\sigma^2 = m_\sigma^2 + \delta m_\sigma^2$. The scalar meson mass is specified by the bulk properties of nuclear matter and is already “physical”, in the sense that its value is fixed in order to yield certain values for some nuclear matter properties. Whichever renormalization scheme we choose, may it be on-shell, MS or $\overline{\mathrm{MS}}$, it would add a divergent and/or a finite contribution to M_σ^2, which will eventually have to be fitted to the same experimental values of the nuclear matter bulk properties. For this reason we may as well renormalize this term in such a way that the δm_σ^2 counterterm cancels exactly all the contributions in σ^2 arising from the regularization, both divergent and finite [36]. In this way the physical mass keeps its previous name: m_σ. The terms in σ^3 and σ^4 are the vacuum-shift contributions to the three and four-couplings $-\tilde{b}g_\sigma^3/3$ and $-cg_\sigma^4/4$, and we may call these counterterms as δb and δc. As with the mass term, both $\tilde{b}$ and c are parameters that have to be fitted to the nuclear matter bulk properties. These may be renormalized in a similar way as with the mass term, where $\delta\tilde{b}$ and δc are chosen in order to cancel all contributions in σ^3 and σ^4 respectively. Our chosen renormalization scheme then corresponds in practice to choosing the counterterms so that all the terms of the zero-point energy shift in the σ field up to the fourth order vanish. These correspond to all the terms in the first product of (6.35), but also the first four terms in the expansion of the last logarithm:

$$
\begin{aligned}
m_B^{*4} \ln \frac{m_B^*}{m_B} &= -m_B^{*4} \left(\sum_{k=1}^{\infty} \frac{(-1)^k \left(-1 + \frac{m_B^*}{m_B}\right)^k}{k} \right) \\
&= -\left(m_B^4 - 4m_B^3(g_\sigma\sigma) + 6m_B^2(g_\sigma\sigma)^2 - 4m_B(g_\sigma\sigma)^3 + (g_\sigma\sigma)^4\right) \\
&\qquad \times \left(\frac{g_\sigma\sigma}{m_B} + \frac{(g_\sigma\sigma)^2}{2m_B^2} + \frac{(g_\sigma\sigma)^3}{3m_B^3} + \frac{(g_\sigma\sigma)^4}{4m_B^4} + O(\sigma^5)\right) \\
&= -m_B^3(g_\sigma\sigma) + \frac{7}{2}m_B^2(g_\sigma\sigma)^2 - \frac{13}{3}m_B(g_\sigma\sigma)^3 + \frac{25}{12}(g_\sigma\sigma)^4 + O(\sigma^5),
\end{aligned}
\tag{6.37}
$$

so that the zero-point shift due to the σ field interaction with baryons will be on the order of $O(\sigma^5)$. This translates to choosing these counterterms:

$$
\begin{aligned}
\delta\sigma &= \sum_B c_{1,B}\sigma \qquad & \delta m_\sigma^2 &= \sum_B c_{2,B}\sigma^2 \\
\delta b &= \sum_B c_{3,B}\sigma^3 \qquad & \delta c &= \sum_B c_{4,B}\sigma^4,
\end{aligned}
\tag{6.38}
$$

where the $c_{i,B}$ coefficients are

$$
\begin{aligned}
c_{1,B} &= \frac{g_\sigma}{16\pi^2}\left[4m_B^3\Gamma\left(-1+\frac{\epsilon}{2}\right) + 8m_B^3 \ln\left(\frac{m_B}{\sqrt{4\pi}\mu}\right)\right] \\
c_{2,B} &= -\frac{g_\sigma^2}{16\pi^2}\left[6m_B^2\Gamma\left(-1+\frac{\epsilon}{2}\right) + 12m_B^2 \ln\left(\frac{m_B}{\sqrt{4\pi}\mu}\right) + 4m_B^2\right], \\
c_{3,B} &= \frac{g_\sigma^3}{16\pi^2}\left[4m_B\Gamma\left(-1+\frac{\epsilon}{2}\right) + 8m_B \ln\left(\frac{m_B}{\sqrt{4\pi}\mu}\right) + \frac{20}{3}m_B\right], \\
c_{4,B} &= -\frac{g_\sigma^4}{16\pi^2}\left[\Gamma\left(-1+\frac{\epsilon}{2}\right) + 2\ln\left(\frac{m_B}{\sqrt{4\pi}\mu}\right) + \frac{11}{3}\right].
\end{aligned}
\tag{6.39}
$$

Here we can clearly see how the coefficients of the gamma functions and the logarithms match and cancel with those in (6.36), and the finite terms yield the ones that cancel the first four terms in the expansion of the logarithm in (6.37). We may then rewrite the single-baryon zero-point energy shift in (6.35), subtracting the counterterms:

$$
\begin{aligned}
\Delta\epsilon_B = V_{\mathrm{ZP},B} - \sum_{i=1}^{4} c_{i,B}\sigma^i = -\frac{1}{8\pi^2}\Bigg[& m_B^{*4}\ln\frac{m_B^*}{m_B} + m_B^3(g_\sigma\sigma) \\
& - \frac{7}{2}m_B^2(g_\sigma\sigma)^2 + \frac{13}{3}m_B(g_\sigma\sigma)^3 - \frac{25}{12}(g_\sigma\sigma)^4\Bigg].
\end{aligned}
\tag{6.40}
$$

Here we recognize the terms in the expansion of the logarithmic term shown in (6.37), canceling the first four terms of the Taylor expansion of the first term. Equation (6.40) is the remainder of the zero-point energy shift after having renormalized the infinities into the existing physical values of m_σ, $\tilde{b}$ and c, and is the same obtained in [6] and [36]. It is important to remember that the zero-point energy shift in (6.40) is valid for only one baryon species. The $npe\mu$ model has two baryon species: neutrons and protons, and both their contributions have to be summed.

6.2.2 Bosons

The form the Lagrangian in (6.1) is written emphasizes the way we treat the physics of nuclear matter in neutron stars. Instead of considering nucleons, leptons and mesons interacting with each other, we treat the σ, ω and ρ particles as fields and the nucleons as quantum particles moving in the mean value of such fields. Since we are using the mean-field approximation we are allowed to operate in terms of an *effective mass* and *effective chemical potential* and treat the fields as free. The ω and the ρ meson fields are essentially free, except for their interaction with the baryons: the model does not include self-interactions for the ω meson, and the self-interacting terms of the ρ meson disappear when enforcing the mean-field approximation, as explained in Chapter 5. This means that we should not expect any contribution to the shift in the vacuum energy from their presence. The σ field has instead self-interaction terms that do not vanish when taking the RMF approximation. We will see in this section how their presence yields a contribution to the shift in the vacuum energy. In order to show this, we account for the quantum fluctuations in the σ field separately. This section closely follows the procedures in [22] and [36]. Naming the quantized field as σ_q, we can divide the classical (σ_0) from the quantum part ($\tilde{\sigma}$):

$$\sigma_q = \sigma_0 + \tilde{\sigma}. \tag{6.41}$$

Accounting only for the quantum corrections up to $O(\tilde{\sigma}^2)$, and recalling the definition of $U(\sigma)$ in (5.1),

$$U(\sigma) = \frac{1}{3}\tilde{b}(g_\sigma\sigma)^3 + \frac{1}{4}c(g_\sigma\sigma)^4, \tag{6.42}$$

the insertion of σ_q in the Lagrangian for our σ field leads to

$$\begin{aligned}
\mathcal{L}_{\sigma_q} =& \frac{1}{2}(\partial^\mu\sigma_q)(\partial_\mu\sigma_q) - \frac{1}{2}m_\sigma^2\sigma_q^2 + g_\sigma\sigma_q\overline{\psi}\psi - U(\sigma_q) \\
=& \frac{1}{2}\left((\partial^\mu\sigma_0)(\partial_\mu\sigma_0) - m_\sigma^2\sigma_0^2\right) + g_\sigma\sigma_0\overline{\psi}\psi \\
&+ (\partial^\mu\sigma_0)(\partial_\mu\tilde{\sigma}) - m_\sigma^2\sigma_0\tilde{\sigma} + g_\sigma\tilde{\sigma}\overline{\psi}\psi + \frac{1}{2}\left((\partial^\mu\tilde{\sigma})(\partial_\mu\tilde{\sigma}) - m_\sigma^2\tilde{\sigma}^2\right) \\
&- \frac{1}{3}\tilde{b}g_\sigma^3\left(\sigma_0^3 + 3\sigma_0^2\tilde{\sigma} + 3\sigma_0\tilde{\sigma}^2\right) - \frac{1}{4}cg_\sigma^4\left(\sigma_0^4 + 4\sigma_0^3\tilde{\sigma} + 6\sigma_0^2\tilde{\sigma}^2\right) + O(\tilde{\sigma}^3).
\end{aligned}$$

This may be rewritten as

$$
\begin{aligned}
\mathcal{L}_{\sigma_q} = & \left[\frac{1}{2}\left((\partial^\mu\sigma_0)(\partial_\mu\sigma_0) - m_\sigma^2\sigma_0^2\right) + g_\sigma\sigma_0\overline{\psi}\psi - U(\sigma_0)\right] \\
& + \left[(\partial^\mu\sigma_0)(\partial_\mu\tilde{\sigma}) - m_\sigma^2\sigma\tilde{\sigma} + g_\sigma\tilde{\sigma}\overline{\psi}\psi - \tilde{\sigma}U'(\sigma_0)\right] \\
& + \left[\frac{1}{2}\left((\partial^\mu\tilde{\sigma})(\partial_\mu\tilde{\sigma}) - m_\sigma^2\tilde{\sigma}^2\right) - \frac{\tilde{\sigma}^2}{2}U''(\sigma_0)\right] \\
= & \mathcal{L}_{\sigma_0} + \mathcal{L}_{\sigma_0\tilde{\sigma}} + \mathcal{L}_{\tilde{\sigma}},
\end{aligned}
\tag{6.43}
$$

where we have divided the Lagrangian in three parts, each one consisting of the terms in the three square parenthesis in the order they have been written, and where $U'(\sigma_0)$ and $U''(\sigma_0)$ represent the first and second derivative in σ_0 respectively. The partition function for the σ_q field is

$$
Z = \int \mathcal{D}\sigma_q \exp\left[\int \mathrm{d}^4x(\mathcal{L}_{\sigma_0}+\mathcal{L}_{\sigma_0\tilde{\sigma}}+\mathcal{L}_{\tilde{\sigma}})\right] = \int \mathcal{D}\sigma_q \exp[S_{\sigma_0}+S_{\sigma_0\tilde{\sigma}}+S_{\tilde{\sigma}}]. \tag{6.44}
$$

The σ_0 field is classical and the action in $\mathcal{L}_{\sigma_0}$ can be taken out of the path integral. We already know its equation of motion from (A.13) and (5.3):

$$
(\Box + m_\sigma^2)\sigma_0 = g_\sigma\overline{\psi}\psi - U'(\sigma_0). \tag{6.45}
$$

By partial integration, we find that $S_{\sigma_0\tilde{\sigma}}$ vanishes:

$$
\begin{aligned}
S_{\sigma\tilde{\sigma}} &= \int \mathrm{d}^4x \left[(\partial^\mu\sigma_0)(\partial_\mu\tilde{\sigma}) - m_\sigma^2\sigma_0\tilde{\sigma} + g_\sigma\tilde{\sigma}\overline{\psi}\psi - \tilde{\sigma}U'(\sigma_0)\right] \\
&= \tilde{\sigma}\partial^\mu\sigma_0 + \int \mathrm{d}^4x \left[-\left(\Box + m_\sigma^2\right)\sigma_0 + g_\sigma\overline{\psi}\psi - U'(\sigma_0)\right]\tilde{\sigma},
\end{aligned}
\tag{6.46}
$$

where the term outside the integral vanishes at the boundary (as usual we require the field and its derivative to go to zero at infinity) and we recognize the equation of motion (6.45) in the integral. Being the σ_0 field classical the path-integral will only be over $\tilde{\sigma}$, giving for the partition function in (6.44)

$$
Z = e^{iS_{\sigma_0}} \int \mathcal{D}\tilde{\sigma} \exp\left\{\frac{1}{2}\int \mathrm{d}^4x \left[(\partial^\mu\tilde{\sigma})(\partial_\mu\tilde{\sigma}) - m_\sigma^2\tilde{\sigma}^2 - \tilde{\sigma}^2U''(\sigma_0)\right]\right\}. \tag{6.47}
$$

Introducing the mean-field approximation, (and reinstating from now on the notation for which $\langle\sigma_0\rangle \to \sigma$) the e^{S_σ} factor corresponds to the terms already found in (6.8) and will not contribute to the shift in the zero-point energy. From the above partition function, we observe how the mean-field potential $U''(\sigma)$ may be regarded as a modification of the mass term. We can then introduce the *effective σ meson mass* m_σ^*,

$$
m_\sigma^{*2} = m_\sigma^2 + U''(\sigma) = m_\sigma^2 + 2\tilde{b}g_\sigma^3\sigma + 3cg_\sigma^4\sigma^2 \tag{6.48}
$$

for which the partition function becomes

$$
Z = \int \mathcal{D}\tilde{\sigma} \exp\left(\int \mathrm{d}^4x\mathcal{L}_{\tilde{\sigma}}\right) = \int \mathcal{D}\tilde{\sigma} \exp\left[\frac{1}{2}\int \mathrm{d}^4x \left((\partial^\mu\tilde{\sigma})(\partial_\mu\tilde{\sigma}) - m_\sigma^{*2}\tilde{\sigma}^2\right)\right]. \tag{6.49}
$$

By then switching to imaginary time $\tau = it$, making the field periodic in τ in the same way as done in Chapter 3, and defining the canonical momentum

$$p = \frac{\mathcal{L}_{\tilde{\sigma}}}{\partial(\partial\tilde{\sigma}/\partial t)} = \frac{\partial\tilde{\sigma}}{\partial t},$$

we rewrite the partition function as

$$Z = \int \mathcal{D}\tilde{\sigma}\mathcal{D}p \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left(i\boldsymbol{p}\frac{\partial\tilde{\sigma}}{\partial\tau} - \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{2}(\nabla\tilde{\sigma})^2 - m_\sigma^{*2}\tilde{\sigma}^2\right)\right]. \quad (6.50)$$

The partition function in (6.50) has the same form as the one for free bosons used in Chapter 3.4. This allows us to use the same results we obtained at the end of the section. We are specifically interested in the zero-point contribution for bosons for the pressure (3.113) and energy density (3.114):

$$P_{\mathrm{ZP},\sigma} - \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{p^2 + m_\sigma^{*2}} \quad (6.51)$$

$$\epsilon_{\mathrm{ZP},\sigma} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{p^2 + m_\sigma^{*2}}. \quad (6.52)$$

As with the fermion case, we need to find the shift in the vacuum energy due to the presence of the $U''(\sigma)$ potential. The integrals look very similar to $V_{\mathrm{ZP},B}$ in (6.34), only different for a factor of $1/4$ and the different effective masses:

$$V_{\mathrm{ZP},\sigma} = -\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sqrt{p^2 + m_\sigma^{*2}} - \sqrt{p^2 + m_\sigma^2}\right). \quad (6.53)$$

The regularization follows the same steps as with the fermion case in Appendix A.6, giving

$$V_{\mathrm{ZP},\sigma} =$$
$$-\frac{1}{(8\pi)^2}\left\{\left(m_\sigma^{*4} - m_\sigma^4\right)\left[\Gamma\left(-1+\frac{\epsilon}{2}\right) - \frac{1}{2} + \ln\left(\frac{m_\sigma^2}{4\pi\mu^2}\right)\right] + m_\sigma^{*4}\ln\left(\frac{m_\sigma^{*2}}{m_\sigma^2}\right)\right\} + O(\epsilon). \quad (6.54)$$

The renormalization scheme follows the same conclusions we made in the fermion case. Since m_σ, g_σ, $\tilde{b}$ and c already are physical, we would like the counterterms to cancel not only the divergences, but also all the finite terms in σ up to the fourth order, included those in the expansion of the logarithmic term. We then have

$$\begin{aligned}\Delta\epsilon_\sigma &= V_{\mathrm{ZP},\sigma} - \text{Counterterms} \\ &= V_{\mathrm{ZP},\sigma} - \sum_i c_{i,\sigma}\sigma^i = \frac{1}{(8\pi)^2}\left[m_\sigma^{*4}\ln\left(\frac{m_\sigma^{*2}}{m_\sigma^2}\right) - C\right],\end{aligned} \quad (6.55)$$

where i runs over 1, 2, 3 and 4, in $c_{i,\sigma}$ are collected the four coefficients of the counterterms as done with the fermion case, and in C are instead collected the first four terms in σ in the Taylor expansion of the first term of (6.55). We can find these terms by expanding this term. The algebra is tedious and is easier done when making the following substitutions:

$$\sigma_1 = \left(\frac{g_\sigma}{m_\sigma}\right)^2 2\tilde{b} g_\sigma \sigma, \qquad \sigma_2 = \left(\frac{g_\sigma}{m_\sigma}\right)^2 3c(g_\sigma \sigma)^2,$$

where

$$\frac{m^{*2}}{m_\sigma^2} = 1 + \frac{U''(\sigma)}{m_\sigma^2} = 1 + \sigma_1 + \sigma_2.$$

The expansion of the logarithmic term yields

$$\begin{aligned} m_\sigma^{*4} \ln\left(\frac{m_\sigma^{*2}}{m_\sigma^2}\right) &= m_\sigma^4 (1+\sigma_1+\sigma_2)^2 \left(-\sum_{k=1}^{\infty} \frac{(-\sigma_1-\sigma_2)^k}{k}\right) \\ &= m_\sigma^4 \Big(1 + 2\sigma_1 + 2\sigma_2 + \sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2\Big) \\ &\quad \times \left((\sigma_1+\sigma_2) - \frac{(\sigma_1+\sigma_2)^2}{2} + \frac{(\sigma_1+\sigma_2)^3}{3} - \frac{(\sigma_1+\sigma_2)^4}{4}\right) + O(\sigma^5) \\ &= m_\sigma^4 \left((\sigma_1+\sigma_2) + \frac{3}{2}(\sigma_1+\sigma_2)^2 + \frac{1}{3}\sigma_1^2 (\sigma_1+3\sigma_2) - \frac{1}{12}\sigma_1^4\right) + O(\sigma^5). \end{aligned}$$

In C in (6.55) we only keep the terms up to $O(\sigma^4)$ of the above result, and obtain

$$\begin{aligned} \Delta\epsilon_\sigma = \frac{m_\sigma^4}{(8\pi)^4} \bigg(&(1+\sigma_1+\sigma_2)^2 \ln(1+\sigma_1+\sigma_2) - (\sigma_1+\sigma_2) - \frac{3}{2}(\sigma_1+\sigma_2)^2 \\ &- \frac{1}{3}\sigma_1^2 (\sigma_1 + 3\sigma_2) + \frac{1}{12}\sigma_1^4 \bigg). \end{aligned} \tag{6.56}$$

This is the result obtained in both [6] and [36] and reported in [10]. It is worth noting how the vacuum shift for the σ meson is dependent on m_σ alone, and not on the ratio g_σ/m_σ. The mass of the σ meson is taken to be 600 MeV in [12] and 550 MeV in [35].

6.3 Vacuum shift contribution to the equation of state

Now we have derived the contributions for the vacuum shift for the baryons in (6.40) and the σ meson in (6.56). Finally we can use these results to rewrite the energy density in (6.29), the pressure in (6.30) and the σ field as

$$\epsilon_{\text{RMF,V}} = \epsilon_{\text{RMF}} + \sum_B \Delta\epsilon_B + \Delta\epsilon_\sigma, \tag{6.57}$$

$$P_{\text{RMF,V}} = P_{\text{RMF}} - \sum_B \Delta\epsilon_B - \Delta\epsilon_\sigma \tag{6.58}$$

$$g_\sigma \sigma_{\text{RMF,V}} = g_\sigma \sigma_{\text{RMF}} - \frac{g_\sigma}{m_\sigma^2} \frac{\partial}{\partial\sigma} \left(\sum_B \Delta\epsilon_B + \Delta\epsilon_\sigma\right). \tag{6.59}$$

The model accounting for the shift in the vacuum energy is frequently called as the *relativistic Hartree approximation* in the literature [10], which is in essence the mean-field approximation applied to the path-integral formalism. The above equation of state has been derived from QFT and thermodynamical principles, using the RMF approximation, and the accounting of only the terms up to $O(\tilde{\sigma}^2)$ in the calculations of the vacuum energy shift contribution by the σ meson's self-interaction.

6.4 Renormalized versus effective RMF theory

Although necessary from a theoretical point of view, the renormalization of the $npe\mu$ model consists in adding terms in $O(\sigma^5)$ to the σ field equation, the pressure and the energy density, and their contribution is not expected to be big. This has already been noticed by [11] and [12] and is confirmed by the plot of the two solutions in Figure 6.1. The difference in the largest mass and radius is minimal. By using the parameters in Table A.4, fitting the same bulk properties as in Chapter 4 except for the compression modulus now set to $K = 300$ MeV, we obtain a mass of $M/M_\odot = 2.0829$ and a corresponding radius of $R = 11.800$ km for the renormalized model, against $M/M_\odot = 2.0662$ and $R = 11.763$ km for the non-renormalized $npe\mu$ model.

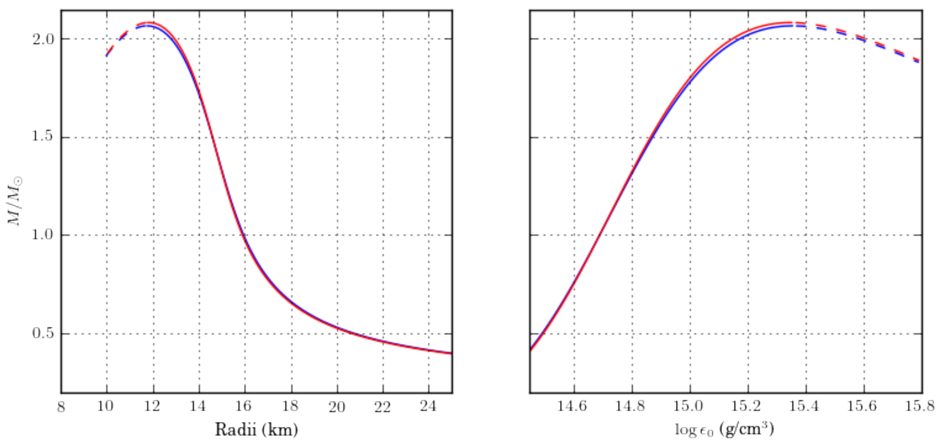

Figure 6.1: Relationship between radii and total masses on the left, and between central densities and total masses on the right. The renormalized solutions are shown in red, while the $npe\mu$ model of Figure 5.2 is reproduced in blue for comparison. We see how the difference is minimal.

Chapter 7

Conclusions and outlook

7.1 Conclusions

In this Master's degree thesis we have shown in Chapter 2 how the mass-radius relation for neutron stars may be found by using a system of three coupled differential equations: the mass equation, the TOV equation and the equation of state. The mass equation gives the mass enclosed by a spherical surface of a certain radius, the TOV equation describes the curvature of spacetime due to the presence of this mass and is a solution to the Einstein equations, and the equation of state links the energy density to the pressure. As a first application of this system of equations we have reproduced the 1939 work of Oppenheimer and Volkoff [29], where we used an equation of state describing non-interacting, relativistic, cold neutrons kept from collapsing by the Pauli exclusion principle. Later we laid the framework for the RMF theory introducing the σ–ω model, otherwise known as Walecka model. The interactions between nucleons are modeled by an attractive, scalar σ meson and a repulsive, vector ω meson. The model has two free parameters allowing us to fit the theory to two of the five bulk properties of neutron and nuclear matter, these being the binding energy per nucleon for saturated nuclear matter, the saturation particle density, the compression modulus, the symmetry energy density and the effective mass ratio at saturation. Although describing neutron stars in a better way, this model yields results for the other three properties that far from the experimental ones [7]. We can though account for all five bulk properties using the $npe\mu$ model, described in Chapter 5. Here the σ–ω model is improved by taking care of beta and isospin balance, global charge neutrality, self-interactions between σ mesons and the appearance of muons. The $npe\mu$ model has five parameters, we are able to fit the theory to all five bulk properties of nuclear matter, and obtain a good description of $npe\mu$ matter. In Chapter 3 we develop the path-integral formalism which, at zero temperature, yields the relativistic Fermi-Dirac and Bose-Einstein particle distributions for fermions and bosons respectively, plus a divergent, temperature independent vacuum term for both. In Chapter 6 we use this formalism to derive the $npe\mu$ model with its respective vacuum term and find out to what degree the shift in the vacuum energy density due to interactions and self-interactions contributes to the equation of state and, eventually, the mass-radius relation. The contribution has been found to be minimal, in accordance with earlier results [11], [12]. A long digression in Chapter 4 has been reserved to the phase transition neutron matter undergoes for the choice of parameters in [7]. For this choice in fact the equation of state shows a region where Le Chatelier's principle of microscopic stability is violated, a situation similar to the equation of state originating from the description of atomic and molecular matter of the Van der Waals model. Although the unphysical nature of pure neutron matter (unstable to beta decay), the framework behind such a system's behavior shows many interesting features. These can be useful in order to obtain a qualitative, approximative understanding of other phenomena involving phase transitions, such as the transition between hadronic and quark matter in hybrid stars [2], the crust-core interface [3], and the transition between stable nuclei at different energies in the neutron star crust [4].

7.2 Outlook

Exotic cores, quark matter and hybrid stars

Current models for neutron star matter include the baryon octet for denser cores, and for the heaviest stars it is thought that the inner core consists of quark-hadron matter [10]. Exotic hadrons start to appear when the momenta of neutrons and protons are large enough to generate new, heavier particles in a similar way as we treated the appearance of muons in Chapter 5. These hadrons are the Λ particle, the Σ particles and the Ξ particles. They consist of at least one strange quark making them *exotic particles*, and their presence softens the equation of state yielding lower values for the largest mass [24]. A more complete discussion of the RMF theory should include these particles [10]. Moreover, quarks will start to be deconfined at baryon densities of $\rho \approx 0.24$ fm^{-3}, coexisting with hadrons at lower densities [39], and then being pure *quark matter* at larger densities. Stars consisting of both hadronic and quark matter are also called *hybrid stars*. The phase transition treated in Chapter 4 is of the first order, and is inadequate for a correct explanation of the transition between $npe\mu$ matter and quark matter. This has to do with the fact that a first-order phase transition is inadequate when handling systems with two conserved charges, here being the baryon number and the electrical charge [10]. More factors that should be taken into account when considering the mass-radius relation are the rotation and the magnetic field of neutron stars.

Third family of compact stars: non-identical twins

Although not yet observed, the current models based on the RMF theory allow for another region stable against gravitational collapse. Bodies in this third family of compact stars (white dwarfs and neutron stars being the first and the second family respectively) are called *non-identical twins of neutron stars* [13]. This stable region for $npe\mu$ matter is shown in Figure 7.1 as an extension of Figure 5.2 for higher densities. Glendenning in [10] shows how a similar region can also be found using a more complete equation of state for hybrid stars, accounting for the presence of hyperions and quarks. This equation of state yields two different stable configurations with the same mass: one is in the family of neutron and hybrid stars, and the other — consisting of smaller and more dense stars — is the new, third family. Two stars belonging to a different family but with the same mass are said to be *non-identical twins* (Figure 7.2).

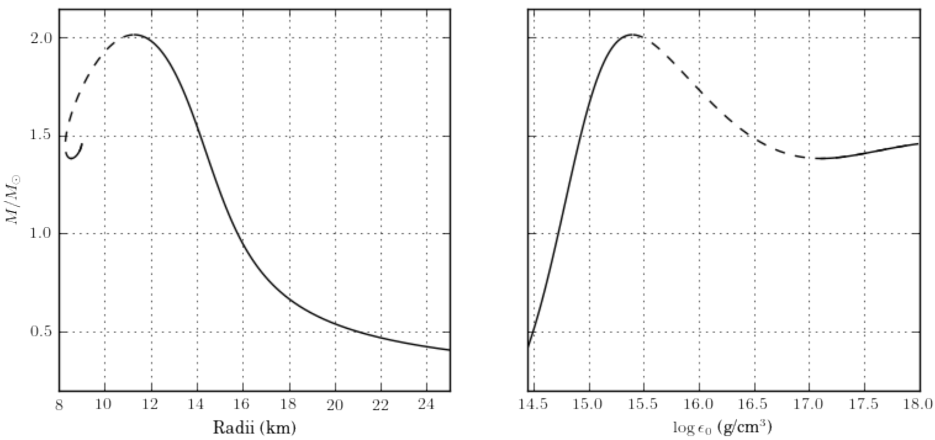

Figure 7.1: Relationship between radii and total masses on the left, and between central densities and total masses on the right, for $npe\mu$ matter. The dashed lines show the unstable solutions. The small, stable region at the "curl of the spiral" in the left panel and its corresponding region on the right of the right panel represent the neutron stars' non-identical twins.

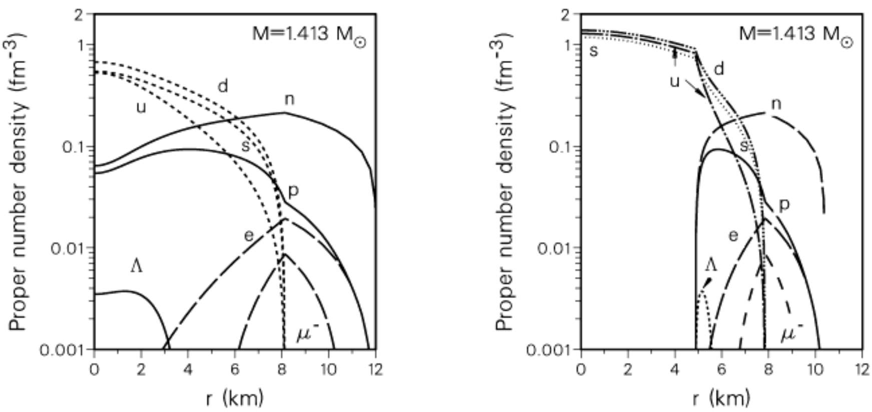

Figure 7.2: The particle distribution of two "twins" of the same mass. The plot on the left panel corresponds to the particle distribution for hybrid stars accounting for the presence of hyperons and free quarks (u stands for *up*, d for *down* and s for *strange quarks*), while the one on the right panel considers the same distribution for a star in the hypothetical third family of compact stars. Courtesy of *Astronomy and Astrophysics* [9].

Appendix A

Theorems and Derivations

A.1 Notation and conventions

Throughout the Appendix and the thesis we use natural units with $\hbar = c = G = k_B = 1$ unless otherwise specified. Here $\hbar$ is the reduced Planck constant, c the speed of light in vacuum, G the Newton gravitational constant and k_B the Boltzmann constant. Energy is measured in MeV$= 10^6$ eV $\approx 1.602 \times 10^{-13}$ J.
Due to the nature of neutron stars, we will operate in two scales: nuclear and stellar. Lengths at nuclear scale are measured in fm $= 10^{-15}$ m, and particle densities in $1/\text{fm}^3$, while at stellar scale lengths are measured in km $= 10^3$ m and masses in $M_\odot$, where $1M_\odot$ correspond to one solar mass, approximately equal to 1.988×10^{30} kg. Otherwise, when switching back to non-natural units, we will follow the conventions set by the literature on neutron stars and use CGS units, except for Chapter 2 where SI units will be used. The use of CGS units means in practice that mass densities will be expressed in g/cm^3 $= 10^3$ kg/m^3 and pressures in dyne/cm^2 $= 10^{-1}$ Pa.
When indexing tensors the Einstein's summation convention is implied, where Greek labels will run over the indices $0, 1, 2, 3$ and Latin labels will run over the indices $1, 2, 3$ if not otherwise specified.
We use the signature -2 of the metric tensor in Minkowski space, where

$$\eta_{\mu\nu} = \eta^{\mu\nu} = (+1, -1, -1, -1),$$

meaning that contravariant and covariant will be

$$x^\mu = (x^0, x^1, x^2, x^3) \qquad x_\mu = (x_0, -x_1, -x_2, -x_3).$$

Boldface will indicate three-vectors $\boldsymbol{v} = (v_x, v_y, v_z)$ or the spacelike part of a four-vector $v^\mu = (v^0, \boldsymbol{v})$. The d'Alembert operator is

$$\Box \equiv \partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu},$$

where ∂_μ is the four-dimensional operator

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \nabla\right).$$

Integrals without set limits are understood to be taken from $-\infty$ to $+\infty$.

A.2 Bulk properties and parameters

In our computations we have used different parameters in order to fit the theory to the experimental data. In Chapter 4 we use the parameters given in [7], fitting the σ–ω model to reproduce a binding energy of $B/A = -15.75$ MeV and and a density at saturation of $\rho_0 = 0.101$ fm^{-3}, and are listed in Table A.1.

$(g_\sigma/m_\sigma)^2$ $\times 10^4$ MeV^{-2}	$(g_\omega/m_\omega)^2$ $\times 10^4$ MeV^{-2}
3.033	2.224

Table A.1: Fitting parameters used in Chapter 4.

In Chapter 5 the parameters are chosen in order to fit the nuclear matter bulk properties values of Table A.2[1]. The fitting parameters are taken from [10] and are listed in Table A.3.

B/A MeV	ρ_0 fm^{-3}	K MeV	a_{sym} MeV	m^*/m
-16.3	0.153	240	32.5	0.78

Table A.2: Values of the bulk properties used in Chapter 5.

$(g_\sigma/m_\sigma)^2$ $\times 10^4$ MeV^{-2}	$(g_\omega/m_\omega)^2$ $\times 10^4$ MeV^{-2}	$(g_\rho/m_\rho)^2$ $\times 10^4$ MeV^{-2}	$\tilde{b}$ $\times 93800$	c $\times$ 100
2.549	1.238	1.230	8.661	-2.421

Table A.3: Fitting parameters used in Chapter 5.

In Chapter 6 we use parameters that fit the same bulk properties as in Table A.2 except for the compression modulus, which now has a value of $K = 300$ MeV. The parameters

[1]The meaning of these bulk properties is explained in the same chapter.

are taken from [12] and are listed in Table A.4. RMF,V indicates the parameters used in the renormalized model, RMF in the effective relativistic mean-field $npe\mu$ model. The σ meson mass is taken to be $m_\sigma = 600$ MeV in accordance with [12].

	$(g_\sigma/m_\sigma)^2$ $\times 10^4$ MeV^{-2}	$(g_\omega/m_\omega)^2$ $\times 10^4$ MeV^{-2}	$(g_\rho/m_\rho)^2$ $\times 10^4$ MeV^{-2}	$\tilde{b}$ $\times 93800$	c $\times$ 100
RMF,V	2.375	1.215	1.238	5.723	0.601
RMF	2.319	1.216	1.239	3.305	1.529

Table A.4: Fitting parameters used in Chapter 6.

A.3 Noether theorem

The Noether theorem observes that global symmetries in a Lagrangian lead to conserved currents [25]. Considering an infinitesimal change in a field $\delta\phi_a$ that keeps the Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu\phi_a)$ invariant, we have

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\partial_\mu\phi_a = 0. \tag{A.1}$$

We now observe that we can use the Lagrange equations $\delta\mathcal{L}/\delta\phi_a = \partial_\mu(\delta\mathcal{L}/\delta\partial_\mu\phi_a)$ in the first term, and the Leibniz rule $\delta\partial_\mu = \partial_\mu\delta$ in the second term, obtaining

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\right)\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a\right). \tag{A.2}$$

The conserved quantity

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a \tag{A.3}$$

is the *Noether current* associated to the symmetry in the field $\phi'_a \to \phi_a + \delta\phi_a$ that leaves the Lagrangian invariant. Moreover we can define the *Noether charge* Q as the integral over space of the zeroth component of the current:

$$Q = \int \mathrm{d}^3x j^0. \tag{A.4}$$

A.4 Derivation of the equations of motion for the σ–ω model

The Lagrangian for the σ–ω model can be seen as the sum of the free Langrangians for the three involved fields and the interaction term,

$$\mathcal{L} = \mathcal{L}_{\text{nucl}} + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_{\text{int}}, \tag{A.5}$$

where

$$\mathcal{L}_{\text{nucl}} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{A.6}$$
$$\mathcal{L}_\sigma = \tfrac{1}{2}\left((\partial_\mu\sigma)(\partial^\mu\sigma) - m_\sigma^2\sigma^2\right) \tag{A.7}$$
$$\mathcal{L}_\omega = -\tfrac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \tfrac{1}{2}m_\omega^2\omega^\mu\omega_\mu \tag{A.8}$$
$$\mathcal{L}_{\text{int}} = g_\sigma\sigma\overline{\psi}\psi - g_\omega\omega_\mu\overline{\psi}\gamma^\mu\psi \tag{A.9}$$

and $\omega^{\mu\nu} = \partial^\mu\omega^\nu - \partial^\nu\omega^\mu$. Explicitly, the Lagrangian can be written as in (4.20):

$$\begin{aligned}\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu}\\ + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu + g_\sigma\sigma\overline{\psi}\psi - g_\omega\omega_\mu\overline{\psi}\gamma^\mu\psi.\end{aligned} \tag{A.10}$$

When deriving the equations of motion, we will have to use the Euler-Lagrange equations for fields

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_i)} = \frac{\partial\mathcal{L}}{\partial\phi_i}, \tag{A.11}$$

where ϕ is a general field, in our case the σ, the ω and the ψ fields.

A.4.1 Equation of motion for the σ field

The only terms of the Lagrangian that are dependent on the σ field are $\mathcal{L}_\sigma$ and $\mathcal{L}_{\text{int}}$, so

$$\partial_\alpha\frac{\partial(\mathcal{L}_\sigma + \mathcal{L}_{\text{int}})}{\partial(\partial_\alpha\sigma)} = \frac{\partial(\mathcal{L}_\sigma + \mathcal{L}_{\text{int}})}{\partial\sigma}. \tag{A.12}$$

The interaction Lagrangian is independent of $\partial_\alpha\sigma$, so $\frac{\partial\mathcal{L}_{\text{int}}}{\partial(\partial_\alpha\sigma)} = 0$. We evaluate first the left hand side (LHS) of (A.12):

$$\begin{aligned}\partial_\alpha\frac{\partial\mathcal{L}_\sigma}{\partial(\partial_\alpha\sigma)} &= \partial_\alpha\frac{\partial}{\partial(\partial_\alpha\sigma)}\frac{1}{2}\left((\partial_\mu\sigma)(\partial^\mu\sigma) - m_\sigma^2\sigma^2\right)\\ &= \frac{1}{2}\partial_\alpha\frac{\partial}{\partial(\partial_\alpha\sigma)}\left[(\partial_\mu\sigma)(\partial_\nu\sigma)\eta^{\nu\mu}\right]\\ &= \frac{1}{2}\eta^{\nu\mu}\partial_\alpha\left[\partial_\nu\sigma\frac{\partial}{\partial(\partial_\alpha\sigma)}(\partial_\mu\sigma) + \partial_\mu\sigma\frac{\partial}{\partial(\partial_\alpha\sigma)}(\partial_\nu\sigma)\right]\\ &= \frac{1}{2}\eta^{\nu\mu}\partial_\alpha\left(\partial_\nu\sigma\delta_\alpha^\mu + \partial_\mu\sigma\delta_\alpha^\nu\right)\\ &= \frac{1}{2}\left(\partial_\alpha\partial^\mu\delta_\alpha^\mu + \partial_\alpha\partial^\nu\delta_\alpha^\nu\right)\\ &= \partial^\mu\partial_\mu\sigma = \Box\sigma,\end{aligned}$$

and then the right hand side (RHS)

$$\frac{\partial(\mathcal{L}_\sigma + \mathcal{L}_{\text{int}})}{\partial\sigma} = -m_\sigma^2\sigma + g_\sigma\overline{\psi}\psi. \tag{A.13}$$

Plugging the results for the LHS and the RHS into (A.12) will give

$$(\Box + m_\sigma^2)\sigma = g_\sigma\overline{\psi}\psi. \tag{A.14}$$

A.4.2 Equation of motion for the ω field

As with the σ field, we only take the terms in the Lagrangian that are dependent on the field taken into consideration. For the ω field we have

$$\partial_\alpha \frac{\partial(\mathcal{L}_\omega + \mathcal{L}_{\text{int}})}{\partial(\partial_\alpha \omega^\beta)} = \frac{\partial(\mathcal{L}_\omega + \mathcal{L}_{\text{int}})}{\partial \omega^\beta}. \tag{A.15}$$

We start by evaluating the LHS. the interaction Lagrangian has again no dependency on $\partial_\alpha \omega^\beta$, so

$$\begin{aligned}
\partial_\alpha \frac{\partial(\mathcal{L}_\omega + \mathcal{L}_{\text{int}})}{\partial(\partial_\alpha \omega^\beta)} &= \partial_\alpha \frac{\partial \mathcal{L}_\omega}{\partial(\partial_\alpha \omega^\beta)} = \partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} \left(-\frac{1}{4}\omega^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu \right) \\
&= -\frac{1}{4} \partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} \left(\omega^{\mu\nu}\omega_{\mu\nu} \right) \\
&= -\frac{1}{4} \partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial^\mu \omega^\nu - \partial^\nu \omega^\mu)(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) \\
&= -\frac{1}{4} \partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial^\mu \omega^\nu \partial_\mu \omega_\nu - \partial^\mu \omega^\nu \partial_\nu \omega_\mu - \partial^\nu \omega^\mu \partial_\mu \omega_\nu + \partial^\nu \omega^\mu \partial_\nu \omega_\mu) \, .
\end{aligned} \tag{A.16}$$

We then evaluate the terms separately:

$$\begin{aligned}
\partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} \partial^\mu \omega^\nu \partial_\mu \omega_\nu &= \partial_\alpha \left[\partial_\mu \omega_\nu \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial_\gamma \omega^\nu \eta^{\gamma\nu}) + \partial^\mu \omega^\nu \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial_\mu \omega^\gamma \eta_{\gamma\nu}) \right] \\
&= \partial_\alpha \left(\partial_\mu \omega_\nu \delta_\alpha^\gamma \delta_\beta^\nu \eta^{\gamma\mu} + \partial^\mu \omega^\nu \delta_\alpha^\mu \delta_\gamma^\beta \eta_{\gamma\nu} \right) \\
&= \partial_\alpha (\partial^\alpha) \omega_\beta + \partial^\alpha \omega_\beta = 2\Box \omega_\beta,
\end{aligned} \tag{A.17}$$

and

$$\begin{aligned}
\partial_\alpha \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} \partial^\mu \omega^\nu \partial_\nu \omega_\mu &= \partial_\alpha \left[\partial_\nu \omega_\mu \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial_\gamma \omega^\nu \eta^{\gamma\nu}) + \partial^\mu \omega^\nu \frac{\partial}{\partial(\partial_\alpha \omega^\beta)} (\partial_\nu \omega^\gamma \eta_{\gamma\mu}) \right] \\
&= \partial_\alpha \left(\partial_\nu \omega_\mu \delta_\alpha^\gamma \delta_\beta^\nu \eta^{\gamma\mu} + \partial^\mu \omega^\nu \delta_\alpha^\nu \delta_\gamma^\beta \eta_{\gamma\mu} \right) \\
&= \partial_\alpha (\partial_\beta \omega^\alpha + \partial_\beta \omega^\alpha) = 2\partial_\beta \omega^\alpha.
\end{aligned} \tag{A.18}$$

The third and fourth term of (A.16) are the same as respectively the first and second term. Plugging these results in, we will get

$$-\frac{1}{4}(4\Box \omega_\beta - 4\partial_\beta \partial^\alpha \omega_\alpha) = -\Box \omega_\beta + \partial_\beta \partial^\alpha \omega_\alpha. \tag{A.19}$$

The RHS of equation (A.15) is a derivation in ω^β:

$$\begin{aligned}
\frac{\partial(\mathcal{L}_\omega + \mathcal{L}_{\text{int}})}{\partial \omega^\beta} &= \frac{1}{2} \frac{\partial}{\partial \omega^\beta} (m_\omega^2 \omega^\mu \omega^\gamma \eta_{\mu\gamma}) - \frac{\partial}{\partial \omega^\beta} g_\omega \omega^\gamma \overline{\psi} \gamma^\mu \psi \eta_{\mu\gamma} \\
&= \frac{m_\omega^2 \eta_{\mu\gamma}}{2} \left(\omega^\gamma \frac{\partial}{\partial \omega^\beta} \omega^\mu + \omega^\mu \frac{\partial}{\partial \omega^\beta} \omega^\gamma \right) - g_\omega \delta_\beta^\gamma \overline{\psi} \gamma^\mu \psi \eta_{\mu\gamma} \\
&= \frac{m_\omega^2}{2} \left(\omega_\mu \delta_\mu^\beta + \omega_\gamma \delta_\gamma^\beta \right) - g_\omega \delta_\beta^\gamma \overline{\psi} \gamma^\mu \psi \eta_{\mu\gamma} \\
&= m_\omega^2 \omega_\beta - g_\omega \overline{\psi} \gamma_\beta \psi.
\end{aligned} \tag{A.20}$$

Finally we can plug the LHS and RHS in (A.19) and (A.20) into (A.15), obtaining

$$-\Box\omega_\beta + \partial_\beta\partial^\alpha\omega_\alpha = m_\omega^2\omega_\beta - g_\omega\overline{\psi}\gamma_\beta\psi,$$

or, rearranged,

$$(\Box + m_\omega^2)\omega_\beta - \partial_\beta\partial^\alpha\omega_\alpha = g_\omega\overline{\psi}\gamma_\beta\psi. \tag{A.21}$$

By taking the divergence of the above equation, we obtain

$$\Box\partial^\beta\omega_\beta + m_\omega^2\partial^\beta\omega_\beta - \Box\partial^\alpha\omega_\alpha = g_\omega\partial^\beta\overline{\psi}\gamma_\beta\psi, \tag{A.22}$$

where the first and third term cancel. $\overline{\psi}\gamma_\beta\psi$ on the right hand side is the Dirac current we can find using the same procedure that leads to (3.42). Its divergence is zero, so we are left with

$$m_\omega^2\partial^\beta\omega_\beta = 0. \tag{A.23}$$

Since the ω field is not massless, we conclude it is divergenceless, giving us the final equation of motion

$$(\Box + m_\omega^2)\omega_\beta = g_\omega\overline{\psi}\gamma_\beta\psi. \tag{A.24}$$

A.4.3 Equation of motion for the ψ field

The terms of the σ–ω Lagrangian dependent on ψ and $\overline{\psi}$ are $\mathcal{L}_{\text{nucl}}$ and $\mathcal{L}_{\text{int}}$. The calculations are easier done in the adjoint field $\overline{\psi}$[2] so

$$\partial_\alpha\frac{\partial(\mathcal{L}_{\text{nucl}} + \mathcal{L}_{\text{int}})}{\partial(\partial_\alpha\overline{\psi})} = \frac{\partial(\mathcal{L}_{\text{nucl}} + \mathcal{L}_{\text{int}})}{\partial\overline{\psi}}. \tag{A.25}$$

The LHS simply becomes 0 (there are no terms dependent on $\partial_\alpha\overline{\psi}$), while the RHS becomes

$$\frac{\partial(\mathcal{L}_{\text{nucl}} + \mathcal{L}_{\text{int}})}{\partial\overline{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi + g_\sigma\sigma\psi - g_\omega\omega_\mu\gamma^\mu\psi, \tag{A.26}$$

giving the equation of motion

$$\left[i\gamma^\mu(\partial_\mu + ig_\omega\omega_\mu) - (m - g_\sigma\sigma)\right]\psi = 0. \tag{A.27}$$

A.5 The ρ meson

A.5.1 Conserved current of the ρ meson

Using the Noether current in (A.3) and the variation in the ρ field in (5.14), using index notation, we get

$$I_j^\mu = \frac{\delta\mathcal{L}_\rho}{\delta(\partial_\mu\rho_{i\nu})}(-\epsilon_{ijk}\rho_{k\nu}), \tag{A.28}$$

[2] By using the ψ field we would reach the adjoint equation of motion, from which is possible to recover the Dirac Equation by conjugate transposing the result.

which in vector notation translates to

$$\boldsymbol{I}^{\mu} = -\boldsymbol{\rho}_{\nu} \times \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})}. \tag{A.29}$$

The only term in the Lagrangian dependent on the derivative of the field is the kinetic term, thus

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})} &= -\frac{1}{4}\frac{\delta}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\left(\partial_{\alpha}\boldsymbol{\rho}_{\beta} - \partial_{\beta}\boldsymbol{\rho}_{\alpha}\right) \cdot \left(\partial^{\alpha}\boldsymbol{\rho}^{\beta} - \partial^{\beta}\boldsymbol{\rho}^{\alpha}\right) \\
&= -\frac{1}{4}\Big[\left(\partial_{\alpha}\boldsymbol{\rho}_{\beta} - \partial_{\beta}\boldsymbol{\rho}_{\alpha}\right)\frac{\delta}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\left(\partial^{\alpha}\boldsymbol{\rho}^{\beta} - \partial^{\beta}\boldsymbol{\rho}^{\alpha}\right) \\
&\quad + \left(\partial^{\alpha}\boldsymbol{\rho}^{\beta} - \partial^{\beta}\boldsymbol{\rho}^{\alpha}\right)\frac{\delta}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\left(\partial_{\alpha}\boldsymbol{\rho}_{\beta} - \partial_{\beta}\boldsymbol{\rho}_{\alpha}\right)\Big] \\
&= -\frac{1}{4}\Big[\left(\partial_{\alpha}\boldsymbol{\rho}_{\beta} - \partial_{\beta}\boldsymbol{\rho}_{\alpha}\right)\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\beta\mu}\eta^{\alpha\nu}\right) \\
&\quad + \left(\partial^{\alpha}\boldsymbol{\rho}^{\beta} - \partial^{\beta}\boldsymbol{\rho}^{\alpha}\right)\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right)\Big] \\
&= -\frac{1}{4}\left(\partial^{\mu}\boldsymbol{\rho}^{\nu} + \partial^{\mu}\boldsymbol{\rho}^{\nu} - \partial^{\nu}\boldsymbol{\rho}^{\mu} - \partial^{\nu}\boldsymbol{\rho}^{\mu}\right) \\
&\quad - \frac{1}{4}\left(\partial^{\mu}\boldsymbol{\rho}^{\nu} + \partial^{\mu}\boldsymbol{\rho}^{\nu} - \partial^{\nu}\boldsymbol{\rho}^{\mu} - \partial^{\nu}\boldsymbol{\rho}^{\mu}\right) \\
&= \frac{1}{2}\boldsymbol{\rho}^{\nu\mu} + \frac{1}{2}\boldsymbol{\rho}^{\nu\mu} = \boldsymbol{\rho}^{\nu\mu}.
\end{aligned}$$

Our current will then be

$$\boldsymbol{\rho}_{\nu} \times \boldsymbol{\rho}^{\mu\nu}$$

A.5.2 Additional current from the $-g_{\rho}\rho_{\mu} \cdot I^{\mu}$ term

$$\begin{aligned}
\boldsymbol{I}^{\mu} &= -\boldsymbol{\rho}_{\nu} \times \frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\boldsymbol{\rho}_{\nu})} = -\boldsymbol{\rho}_{\nu} \times \left[-g_{\rho}\left(\boldsymbol{\rho}_{\alpha} \times \boldsymbol{\rho}_{\beta}\right)\frac{\partial \boldsymbol{\rho}^{\alpha\beta}}{\partial(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\right] \\
&= g_{\rho}\boldsymbol{\rho}_{\nu} \times \left(\boldsymbol{\rho}_{\alpha} \times \boldsymbol{\rho}_{\beta}\right)\left(\frac{\partial}{\partial(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\partial_{\mu}\boldsymbol{\rho}_{\nu}\eta^{\alpha\mu}\eta^{\beta\nu} - \frac{\partial}{\partial(\partial_{\mu}\boldsymbol{\rho}_{\nu})}\partial_{\mu}\boldsymbol{\rho}_{\nu}\eta^{\beta\mu}\eta^{\alpha\nu}\right) \\
&= g_{\rho}\boldsymbol{\rho}_{\nu} \times \left(\boldsymbol{\rho}_{\alpha} \times \boldsymbol{\rho}_{\beta}\right)\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\beta\mu}\eta^{\alpha\nu}\right) \\
&= g_{\rho}\boldsymbol{\rho}_{\nu} \times \left(\boldsymbol{\rho}^{\mu} \times \boldsymbol{\rho}^{\nu} - \boldsymbol{\rho}^{\nu} \times \boldsymbol{\rho}^{\mu}\right) \\
&= 2g_{\rho}\boldsymbol{\rho}_{\nu} \times \left(\boldsymbol{\rho}^{\mu} \times \boldsymbol{\rho}^{\nu}\right) \\
&= 2g_{\rho}\left(\boldsymbol{\rho}^{\nu} \times \boldsymbol{\rho}^{\mu}\right) \times \boldsymbol{\rho}_{\nu}
\end{aligned}$$

A.5.3 Notes on SU(2) symmetry

In Chapter 4 and 5 we use SU(2) symmetry to show the invariance of the nucleon Lagrangian (4.11) and the $\boldsymbol{\rho}$ Lagrangian (5.10) under rotation in isospin space. Although this is the same rotation for both fields, we transform the nucleon spinors as

$$\begin{aligned}\psi' &\rightarrow e^{-\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}}\psi \\ \overline{\psi}' &\rightarrow \overline{\psi}e^{\frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}}\end{aligned} \tag{A.30}$$

where $\boldsymbol{\tau}$ are the Pauli matrices, and the $\boldsymbol{\rho}$ field as

$$\boldsymbol{\rho}' \rightarrow \boldsymbol{\rho} - \boldsymbol{\Lambda} \times \boldsymbol{\rho}. \tag{A.31}$$

The meaning of this section is to explain why these expressions are equivalent.

We may start by considering a three-vector $\boldsymbol{x}$ in SU(2) space. This is spanned by the Pauli matrices, so we can express the rotation as

$$(\boldsymbol{x}' \cdot \boldsymbol{\tau}) = e^{\frac{i\lambda}{2}\boldsymbol{\tau}\cdot\hat{n}}(\boldsymbol{x} \cdot \boldsymbol{\tau})e^{-\frac{i\lambda}{2}\boldsymbol{\tau}\cdot\hat{n}}. \tag{A.32}$$

Here we have made the substitution $\boldsymbol{\Lambda} = \lambda\hat{n}$, where λ is an infinitesimal rotation around the axis along the unit vector $\hat{n}$. The exponentials can be rewritten as

$$\begin{aligned}&\left(\mathbb{1}\cos\frac{\lambda}{2} + i\hat{n}\cdot\boldsymbol{\tau}\sin\frac{\lambda}{2}\right)(\boldsymbol{x}\cdot\boldsymbol{\tau})\left(\mathbb{1}\cos\frac{\lambda}{2} - i\hat{n}\cdot\boldsymbol{\tau}\sin\frac{\lambda}{2}\right) \\ &= (\boldsymbol{x}\cdot\boldsymbol{\tau})\cos^2\frac{\lambda}{2} - i(\boldsymbol{x}\cdot\boldsymbol{\tau})(\hat{n}\cdot\boldsymbol{\tau})\cos\frac{\lambda}{2}\sin\frac{\lambda}{2} + i(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\boldsymbol{\tau})\cos\frac{\lambda}{2}\sin\frac{\lambda}{2} \\ &\qquad + (\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\boldsymbol{\tau})(\hat{n}\cdot\boldsymbol{\tau})\sin^2\frac{\lambda}{2}.\end{aligned} \tag{A.33}$$

We express equation (A.33) in index notation. The first term becomes

$$x_i\tau_i\cos^2\frac{\lambda}{2} = \frac{1}{2}x_i\tau_i\left(1 + \cos\lambda\right), \tag{A.34}$$

using the commutator for Pauli matrices, the second and third terms become

$$\begin{aligned}&i\cos\frac{\lambda}{2}\sin\frac{\lambda}{2}\left(n_i x_j\tau_i\tau_j - x_j n_i\tau_j\tau_i\right) \\ =&\frac{i}{2}n_i x_j\left[\tau_i, \tau_j\right]\sin\lambda \\ =&-n_i x_j\epsilon_{ijk}\tau_k\sin\lambda \\ =&-\left(\hat{n}\times\boldsymbol{x}\right)\cdot\boldsymbol{\tau}\sin\lambda\end{aligned} \tag{A.35}$$

and the fourth

$$
\begin{aligned}
&\left[n_i x_j n_k \tau_i \left(\mathbb{1}\delta_{jk} + i\epsilon_{jkl}\tau_l\right)\right] \frac{1}{2}\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + i n_i x_j n_k \tau_i \epsilon_{jkl}\tau_l\right]\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + i n_i x_j n_k \epsilon_{jkl}\left(\mathbb{1}\delta_{il} + i\epsilon_{ilm}\tau_m\right)\right]\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + i x_j n_l n_k \epsilon_{jkl} - n_i x_j n_k \epsilon_{jkl}\epsilon_{ilm}\tau_m\right]\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + i\boldsymbol{x}\cdot(\hat{n}\times\hat{n}) + n_i x_j n_k \tau_m \epsilon_{jkl}\epsilon_{iml}\right]\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + n_i x_j n_k \tau_m\left(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki}\right)\right]\left(1-\cos\lambda\right)\\
=&\frac{1}{2}\left[(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n}) + (\hat{n}\cdot\boldsymbol{x})(\hat{n}\cdot\boldsymbol{\tau}) - (\boldsymbol{x}\cdot\boldsymbol{\tau})\right]\left(1-\cos\lambda\right)\\
=&(\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n})\left(1-\cos\lambda\right) - \frac{1}{2}(\boldsymbol{x}\cdot\boldsymbol{\tau})\left(1-\cos\lambda\right).
\end{aligned}
\tag{A.36}
$$

We then insert the three terms in (A.34), (A.35) and (A.36) in (A.33), and obtain

$$
\begin{aligned}
&\frac{1}{2}\boldsymbol{x}\cdot\boldsymbol{\tau}(1+\cos\lambda) - (\hat{n}\times\boldsymbol{x})\cdot\boldsymbol{\tau}\sin\lambda + (\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n})\left(1-\cos\lambda\right) - \frac{1}{2}(\boldsymbol{x}\cdot\boldsymbol{\tau})\left(1-\cos\lambda\right)\\
=&(\boldsymbol{x}\cdot\boldsymbol{\tau})\cos\lambda - (\hat{n}\times\boldsymbol{x})\cdot\boldsymbol{\tau}\sin\lambda + (\hat{n}\cdot\boldsymbol{\tau})(\boldsymbol{x}\cdot\hat{n})\left(1-\cos\lambda\right)\\
=&\Big[\boldsymbol{x}\cos\lambda - (\hat{n}\times\boldsymbol{x})\sin\lambda + \hat{n}(\boldsymbol{x}\cdot\hat{n})\left(1-\cos\lambda\right)\Big]\cdot\boldsymbol{\tau},
\end{aligned}
\tag{A.37}
$$

where the term inside the parenthesis in (A.37) is our new, rotated $\boldsymbol{x}'$ vector:

$$
\boldsymbol{x}' = (\boldsymbol{x} - \hat{n}(\boldsymbol{x}\cdot\hat{n}))\cos\lambda - (\hat{n}\times\boldsymbol{x})\sin\lambda + \hat{n}(\boldsymbol{x}\cdot\hat{n}). \tag{A.38}
$$

The rotation of an infinitesimal angle λ around $\hat{n}$ is best understood if we let $\hat{n}$ point in the z direction. We can do this without any loss of generality. The $\hat{n}(\boldsymbol{x}\cdot\hat{n})$ is the projection of $\boldsymbol{x}$ along the z direction, taken in the z direction. This means that the term in $\cos\lambda$ in (A.38) is to be understood as the vector $\boldsymbol{x}$ minus its z-component, i.e., for

$$
\boldsymbol{x} = (x\hat{x} + y\hat{y} + z\hat{z}) \tag{A.39}
$$

is

$$
\boldsymbol{x}' = (x\hat{x} + y\hat{y})\cos\lambda - (\hat{n}\times\boldsymbol{x})\sin\lambda + z\hat{z}. \tag{A.40}
$$

For infinitesimal translations we have $\cos\lambda \to 1$ and $\sin\lambda \to \lambda$, giving

$$
\boldsymbol{x}' = \boldsymbol{x} - \Lambda\times\boldsymbol{x}, \tag{A.41}
$$

which is the expression used for the ρ meson.

A.6 Dimensional regularization of the zero-point energy

In this section we show the steps in order to regularize $V_{\text{ZP},B}$. (6.34). This is reproduced here:

$$V_{\text{ZP},B} = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_B \left[\sqrt{p^2 + m_B^{*2}} - \sqrt{p^2 + m_B^2} \right]. \tag{A.42}$$

Defining $I(m)$ as

$$I(m) = -2 \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{p^2 + m^2}, \tag{A.43}$$

we may recast $V_{\text{ZP},B}$ as

$$V_{\text{ZP}} = I(m_p^*) - I(m_p) + I(m_n^*) - I(m_n). \tag{A.44}$$

This means that we only need to dimensionally regularize (A.43) once in order to obtain the full regularized expression for $V_{\text{ZP},B}$. We begin by rewriting $I(m)$ as

$$I(m) = -2 \frac{\mu^{4-d}}{(2\pi)^{d-1}} \int \mathrm{d}^{d-1} p \sqrt{p^2 + m^2}. \tag{A.45}$$

Here d is the number of dimensions (we will later take limit where $d \to 4$) and μ is a scale factor of dimension m in order to make the integral dimensionless. We then introduce spherical coordinates, obtaining

$$I(m) = -2 \frac{\mu^{4-d}}{(2\pi)^{d-1}} \int \mathrm{d}\Omega_{d-1} \int_0^\infty p^{d-2} \sqrt{p^2 + m^2}\, \mathrm{d}p. \tag{A.46}$$

The angular integral $\mathrm{d}\Omega$ can be evaluated using

$$\int \mathrm{d}\Omega_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}, \tag{A.47}$$

where $\Gamma(x)$ is the gamma function. Both the angular integral and the gamma function are taken as defined in [23]. Plugging this result in (A.46), we obtain

$$I(m) = -4 \frac{\mu^{4-d}}{(4\pi)^{(d-1)/2} \Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty p^{d-2} \sqrt{p^2 + m^2}\, \mathrm{d}p. \tag{A.48}$$

Next we introduce the dimensionless variable $t = p^2/m^2$, for which $\mathrm{d}t = 2p/m^2 \mathrm{d}p$, and obtain

$$I(m) = -2 \frac{m^d \mu^{4-d}}{(4\pi)^{(d-1)/2} \Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty t^{\frac{d-3}{2}} (t+1)^{\frac{1}{2}} \mathrm{d}t. \tag{A.49}$$

In (A.49) we recognize the Beta function in the integral, for which we can use the following identity[23],

$$B(a, b) = \int_0^\infty \mathrm{d}t \frac{t^{a-1}}{(t+1)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{A.50}$$

where for $a = (d-1)/2$ and $b = -d/2$ we recover the integral in (A.49). We get then

$$
\begin{aligned}
I(m) =& \frac{-2\mu^4}{(4\pi)^{(d-1)/2}} \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(-\frac{1}{2}\right)} \left(\frac{m}{\mu}\right)^d \\
=& \frac{-2\mu^4}{(4\pi)^{(d-1)/2}} \frac{\Gamma\left(-d/2\right)}{\Gamma\left(-1/2\right)} \left(\frac{m}{\mu}\right)^d \\
=& \frac{2\mu^4}{(4\pi)^{d/2}} \Gamma\left(-\frac{d}{2}\right) \left(\frac{m}{\mu}\right)^d \\
=& \frac{-4\mu^4}{(4\pi)^{d/2}} \frac{1}{d} \Gamma\left(1-\frac{d}{2}\right) \left(\frac{m}{\mu}\right)^d
\end{aligned}
\tag{A.51}
$$

where we have used that $\Gamma(-1/2) = -2\sqrt{\pi}$ and the property $\Gamma(z+1) = z\Gamma(z)$. We can now make the substitution $d = 4 - \epsilon$,

$$
I(m) = -\frac{4\mu^4}{(4\pi)^{2-\frac{\epsilon}{2}}} \frac{\Gamma\left(-1+\frac{\epsilon}{2}\right)}{4-\epsilon} \left(\frac{m}{\mu}\right)^{4-\epsilon}, \tag{A.52}
$$

and expand all our expressions (except the gamma function) for $\epsilon \to 0$:

$$
(4\pi)^{-2+\frac{\epsilon}{2}} = \frac{1}{(4\pi)^2}\left[1 + \frac{\epsilon}{2}\ln(4\pi) + O(\epsilon^2)\right], \tag{A.53}
$$

$$
\left(\frac{m}{\mu}\right)^{4-\epsilon} = \left(\frac{m}{\mu}\right)^4 \left[1 - \epsilon \ln\left(\frac{m}{\mu}\right) + O(\epsilon^2)\right] \tag{A.54}
$$

$$
\begin{aligned}
\frac{\Gamma\left(-1+\frac{\epsilon}{2}\right)}{4-\epsilon} &= \frac{1}{4}\frac{1}{1-\epsilon/4}\Gamma\left(-1+\frac{\epsilon}{2}\right) \\
&= \frac{1}{4}\left(1+\frac{\epsilon}{4}+O(\epsilon^2)\right)\Gamma\left(-1+\frac{\epsilon}{2}\right).
\end{aligned}
\tag{A.55}
$$

By plugging (A.53), (A.54) and (A.55) into (A.52), and recalling that the ϵ^{-1} divergence is still hiding in the gamma function, we obtain our result for $I(m)$:

$$
I(m) = -\frac{m^4}{2(4\pi)^2}\left[2 + \frac{\epsilon}{2} - \epsilon \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]\Gamma\left(-1+\frac{\epsilon}{2}\right) + O(\epsilon). \tag{A.56}
$$

We are now able to express (A.44). For each baryon we have

$$
\begin{aligned}
V_{\text{ZP},B} &= I(m_B^*) - I(m_B) \\
&= -\frac{1}{32\pi^2}\left\{ m_B^{*4}\left[2+\frac{\epsilon}{2}-\epsilon\ln\left(\frac{m_B^{*2}}{4\pi\mu^2}\right)\right] - m_B^4\left[2+\frac{\epsilon}{2}-\epsilon\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right]\right\} \\
&\qquad\qquad \times\Gamma\left(-1+\frac{\epsilon}{2}\right)+O(\epsilon) \\
&= -\frac{1}{32\pi^2}\left\{ (m_B^{*4}-m_B^4)\left[2+\frac{\epsilon}{2}\right] - \epsilon m_B^{*4}\ln\left(\frac{m_B^{*2}}{4\pi\mu^2}\right) + \epsilon m_B^4\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right. \\
&\qquad\left. + \epsilon m_B^{*4}\ln\left(\frac{m_B^2}{4\pi\mu^2}\right) - \epsilon m_B^{*4}\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right\}\Gamma\left(-1+\frac{\epsilon}{2}\right)+O(\epsilon) \\
&= -\frac{1}{32\pi^2}\left\{ (m_B^{*4}-m_B^4)\left[2+\frac{\epsilon}{2}-\epsilon\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right] - \epsilon m_B^{*4}\ln\left(\frac{m_B^{*2}}{m_B^2}\right)\right\} \\
&\qquad\qquad \times\Gamma\left(-1+\frac{\epsilon}{2}\right)+O(\epsilon).
\end{aligned}
$$

We can then use the fact that the divergent term in $\Gamma(-1+\epsilon/2)$ is $-2/\epsilon$ and multiply it into the expression:

$$
\begin{aligned}
V_{\text{ZP},B} = -\frac{1}{16\pi^2}\Bigg\{ &\left(m_B^{*4}-m_B^4\right)\left[\Gamma\left(-1+\frac{\epsilon}{2}\right)-\frac{1}{2}+\ln\left(\frac{m_B^2}{4\pi\mu^2}\right)\right] \\
&+ m_B^{*4}\ln\left(\frac{m_B^{*2}}{m_B^2}\right)\Bigg\}+O(\epsilon). \qquad \text{(A.57)}
\end{aligned}
$$

Appendix B

Code

All plots and programs were written in Python.

B.1 Chapter 2

```
import math
import numpy as np
import scipy
import scipy.constants as sc
import matplotlib
import matplotlib.pyplot as plt
import scipy.optimize as op

#constants
G = sc.G
c = sc.c
m = sc.m_n

#Normalisation constants
e0 = 1000.0**3*m**4.0*c**5.0/(sc.pi**2.0*sc.hbar**3.0)  # in J/km**3
Mo = 1.989e30  # in kg
R0 = G*Mo/(1000.*c**2.)  # in km
beta = 4.0*sc.pi*e0/(Mo*c**2.)

rstep = 0.01 #in km

#Define functions that will be used in the loops
#TOV-equation
def dPdr(r, M, P, eps):
    return -(R0*eps*M)/r**2.*(P/eps + 1.0)*(beta*r**3.0*P/M + 1.0)
    /(1.0-2.0*R0*M/r)

#Mass equation
def dMdr(r, eps):
    return beta*r**2.*eps

```

```
#Pressure equation, rewritten for rootfinding
def P1(x,Q):
    return ((2*x**3.0 - 3.0*x)*(1+x**2)**(0.5) + 3.0*np.arcsinh(x))/24.0 -
     Q

#EoS function. Takes the pressure as argument, finds the corresponding
#Fermi momentum via rootfinding, and calculates the respective energy
def epsilon(Press):
    xx = op.brentq(P1, -0.01, 5.0,args=(Press,))
    return 1.0/8.0*((2*xx**3.0 + xx)*(1+xx**2)**(0.5) - np.arcsinh(xx))

FirstTimeisbiggerthan10percent=False
P = Pc = PNR = 1.0e-6
M = MNR = 0.0
r = rNR = 0.0

Mlist = []
Rlist = []
MNRlist = []
RNRlist = []
energypercent = []
centralpressures = []
ratios = []

while Pc < 10:
    P = Pc
    PNR = Pc
    M = 0.0
    MNR = 0.0
    r = 0.0
    centralpressures.append(Pc)
    energypercent.append(op.brentq(P1, -0.01, 5.0,args=(Pc,)))
    while (P > 0 or PNR > 0):
        r = r + rstep
        if P > 0:
            eps = epsilon(P)
            M = M + rstep * dMdr(r, eps)
            P = P + rstep * dPdr(r, M, P, eps)
            if P <= 0:
                Rlist.append(r)
                Mlist.append(M)
        if PNR > 0:
            epsNR = 15.**(3./5.)*PNR**(3./5.)/3.
            MNR = MNR + rstep * dMdr(r, epsNR)
            PNR = PNR + rstep * dPdr(r, MNR, PNR, epsNR)
            if PNR <= 0:
                RNRlist.append(r)
                MNRlist.append(MNR)
    ratio=MNRlist[-1]/Mlist[-1]
    ratios.append(ratio)
    if (ratio>1.1 and FirstTimeisbiggerthan10percent==False):
        Pclimit=Pc
        Rlimit=Rlist[-1]
        RNRlimit=RNRlist[-1]
        Mlimit=Mlist[-1]
        MNRlimit=MNRlist[-1]
        FirstTimeisbiggerthan10percent=True
```

```
    Pc = Pc*1.05

print('Limit M is {}'.format(Mlimit))
print('Limit r is {}'.format(Rlimit))
print('Limit MNR is {}'.format(MNRlimit))
print('Limit rNR is {}'.format(RNRlimit))
print('Limit Pc is {}'.format(Pclimit))

indexbiggestmassM=Mlist.index(max(Mlist))
indexbiggestmassMNR=MNRlist.index(max(MNRlist))
StableMlist=Mlist[:(indexbiggestmassM+1)]
UnstableMlist=Mlist[(indexbiggestmassM+1):]
StableRlist=Rlist[:(indexbiggestmassM+1)]
UnstableRlist=Rlist[(indexbiggestmassM+1):]
StableMNRlist=MNRlist[:(indexbiggestmassMNR+1)]
UnstableMNRlist=MNRlist[(indexbiggestmassMNR+1):]
StableRNRlist=RNRlist[:(indexbiggestmassMNR+1)]
UnstableRNRlist=RNRlist[(indexbiggestmassMNR+1):]
#Reproducing the plot in Figure 2.3
plt.ylabel(r'$M/M_\odot$')
plt.xlabel('Radii (km)')
plt.plot(StableRlist, StableMlist, 'r',
         StableRNRlist, StableMNRlist, 'b',
        UnstableRlist,UnstableMlist,'r--',
        UnstableRNRlist,UnstableMNRlist,'b--')
plt.grid()
plt.show()

tenpercentratio=[1.1]*len(centralpressures)
logcentralpressures=[np.log10(item) for item in centralpressures]

StablePc=logcentralpressures[:(indexbiggestmassM+1)]
UnstablePc=logcentralpressures[(indexbiggestmassM+1):]
Stableratios=ratios[:(indexbiggestmassM+1)]
Unstableratios=ratios[(indexbiggestmassM+1):]
#Reproducing the plot in Figure 2.4
plt.ylabel(r'$M_{NR}/M$')
plt.xlabel(r'$log_{10}(P_c/\epsilon_0)$')
plt.plot(StablePc, Stableratios, 'k',
         UnstablePc, Unstableratios, 'k--',
         logcentralpressures, tenpercentratio, 'r')
plt.grid()
plt.show()

#Initialise lists
Mlists = []
MNRlists = []
Rlists = []
RNRlists = []
Plists = []
PNRlists = []

#Reproducing graphs in Figure 2.2
#Make list of central pressures we want to calculate
Pclist=[1e-6,1e-3,1e-2]

#Loop for every central pressure
```

```
for index in range(len(Pclist)):
    P = PNR = Pc = Pclist[index]
    M = 0.0
    MNR = 0.0
    r = 0.0
    Mlist = []
    MNRlist = []
    Rlist = []
    RNRlist = []
    Plist = []
    PNRlist = []
    #For every central pressure, loop until the surface (P=0) is met for both
    #the relativistic and the non-relativistic case. For each, save the pressures,
    #the radii and the masses in lists, and save these when the surface is met.
    while (P > 0 or PNR > 0):
        r = r + rstep
        if P > 0:
            eps = epsilon(P)
            M = M + rstep * dMdr(r, eps)
            P = P + rstep * dPdr(r, M, P, eps)
            Rlist.append(r)
            Mlist.append(M)
            Plist.append(P)
        if PNR > 0:
            epsNR = 15.**(3./5.)*PNR**(3./5.)/3.
            MNR = MNR + rstep * dMdr(r, epsNR)
            PNR = PNR + rstep * dPdr(r, MNR, PNR, epsNR)
            RNRlist.append(r)
            MNRlist.append(MNR)
            PNRlist.append(PNR)
        if r>100000:
            break
    Mlists.append(Mlist)
    MNRlists.append(MNRlist)
    Rlists.append(Rlist)
    RNRlists.append(RNRlist)
    Plists.append(Plist)
    PNRlists.append(PNRlist)

n=len(Pclist)
f, axarr = plt.subplots(n,2,figsize=(9,12))

for i in range(n):
    axarr[i,0].ticklabel_format(style='sci', axis='y', scilimits=(0,0))
    axarr[i,0].grid(True)
    axarr[i,0].plot(Rlists[i], Plists[i], 'r', RNRlists[i], PNRlists[i], 'b')
    axarr[i,0].set_xlabel('Radius (Km)')
    axarr[i,0].set_ylabel(r'$P/\epsilon_0$')
    axarr[i,0].set_ylim([0,Pclist[i]*1.1])

    axarr[i,1].grid(True)
    axarr[i,1].plot(Rlists[i], Mlists[i], 'r',RNRlists[i], MNRlists[i], 'b')
```

```
    axarr[i,1].set_xlabel('Radius (Km)')
    axarr[i,1].set_ylabel(r'$M/M_\odot$')
    axarr[i,1].set_ylim([0,max(MNRlists[i])*1.1])

plt.show()
```

B.2 Chapter 4

```
import math
import numpy as np
import scipy
from scipy import interpolate
import scipy.constants as sc
import scipy.integrate as integrate
import matplotlib
import matplotlib.pyplot as plt
import scipy.optimize as op
from matplotlib import rc
from bokeh.plotting import *
rc('font',**{'family':'sans-serif','sans-serif':['Helvetica']})
rc('text', usetex=True)

#constants
G = sc.G
c = sc.c
hbar=sc.hbar

#constants for translating between units
MeVtoJoules=1.0e6*sc.eV
JoulestoMeV=1.0/MeVtoJoules
MeVtoperm=mtoperMeV=MeVtoJoules/(c*hbar)
permtoMeV=perMeVtom=1.0/MeVtoperm
MeV4togcm3=1000.0*MeVtoJoules/c**2.0*(MeVtoperm/100.0)**3.0
MeV4toJouleskm3=MeVtoJoules*(1000.0*MeVtoperm)**3.0
MeV4tokgkm3=MeV4toJouleskm3/c**2.0
dynetoMeV4=0.1*JoulestoMeV*(permtoMeV)**3.0
gcm3toMeV4=1.0/MeV4togcm3

#other constants
Mo = 1.9891e30  # in kg
R0 = G*Mo/(1000*c**2.0)  # in km
m = (939.5654133 + 938.2720813)/2 #in MeV
gmsigma2=266.9/m**2.0
gmomega2=195.7/m**2.0
e0 = m**4.0
beta = 4.0*sc.pi*e0*MeV4tokgkm3/Mo #in solarmasses per km3
pstep=0.01/m
f=2.0 #degeneracy
C=gmsigma2*f*m**2.0/(2.0*sc.pi**2.0)

#First term in the sigma-omega pressure and energy density expression
def firstterm(fgsigma):
    return fgsigma**2.0/(2.0*m**2.0*gmsigma2)

#Function calculating the omega field from the normalized Fermi momentum
def fgomega(x):
```

```
    return gmomega2*m**2.0*f*x**3.0/(6.0*sc.pi**2.0)

#Second term in the sigma-omega pressure and energy density expression
def secondterm(fgomega):
    return fgomega**2.0/(2.0*m**2.0*gmomega2)

#Function calculating the integral in the pressure expression for the
    sigma-omega model
def pressureintegral(x):
    return ((2.0*x**3.0 - 3.0*x)*(1.0+x**2.0)**(0.5) + 3.0*np.arcsinh(x))
    /8.0

#Function calculating the integral in the energy density expression for
    the sigma-omega model
def energyintegral(x):
    return ((2.0*x**3.0 + x)*(1.0+x**2.0)**(0.5) - np.arcsinh(x))/8.0

#TOV equation
def dPdr(r, M, P, eps):
    return -(R0*eps*M)/r**2.0*(P/eps + 1.0)*((beta*r**3.0*P)/M + 1.0)
    /(1.0-2.0*R0*M/r)

#Mass equation
def dMdr(r, eps):
    return beta*r**2.0*eps

#Function calculating the sigma field for the normalized sigma field and
    Fermi momentum
def gsigmaintegral(y,x):
    xnorm=x/(1.0-y)
    return C*(1.0-y)**3.0*(xnorm*np.sqrt(xnorm**2.0+1.0) - np.arcsinh(
    xnorm))/2.0-y

#Generate an array of n values of the normalized Fermi momentum between 0
    and kfmax, and the corresponding sigma field.
n=50000
kfmax=2500/m
kf=np.linspace(0,kfmax,n).tolist()
gsigmalist=[]
gsigma=0.0
checker=0
for index in range(len(kf)): #looping over all k_F
    currentkf=kf[index]
    gsigma=scipy.optimize.newton(gsigmaintegral,gsigma,None,(currentkf,))
    gsigmalist.append(gsigma)

#Function taking as input the Fermi momentum and sigma field arrays, and
    yielding an array for the pressure values, and one for the energy
    density
def EquationOfStateSigmaOmega(kmax,gsigmalist1):
    pressurelist=[]
    energylist=[]
    for index in range(n):
        k=kf[index]
        gsigma=gsigmalist1[index]
        gomega=fgomega(k)
        pressure = -firstterm(gsigma) + secondterm(gomega) + (1.0-gsigma)
```

```
    **4.0*pressureintegral(k/(1.0-gsigma))*f/(6.0*sc.pi**2.0)
        energy = firstterm(gsigma) + secondterm(gomega) + (1.0-gsigma)
    **4.0*energyintegral(k/(1.0-gsigma))*f/(2.0*sc.pi**2.0)
        energylist.append(energy)
        pressurelist.append(pressure)
    return pressurelist,energylist

#Save the pressure and energy density arrays for neutron matter (
    degeneracy 2) in "pressurelist2" and "energylist2"
[pressurelist2,energylist2]=EquationOfStateSigmaOmega(kfmax,gsigmalist)

#translate the arrays in dyne/cm^2 and g/cm^3
pressureindyne2=[item*e0/dynetoMeV4 for item in pressurelist2]
energyingcm32=[item*e0*MeV4togcm3 for item in energylist2]

#Calculate pressure and energy density arrays for free, cold Fermi gas
freeneutrongaspressure=[]
freeneutrongasenergy=[]
for counter in range(n):
    freeneutrongaspressure.append(pressureintegral(kf[counter])/(3.0*sc.pi
    **2.0))
    freeneutrongasenergy.append(energyintegral(kf[counter])/(sc.pi**2.0))

#translate the arrays in dyne/cm^2 and g/cm^3
freeneutrongaspressuredyne=[item*e0/dynetoMeV4 for item in
    freeneutrongaspressure]
freeneutrongasenergygcm3=[item*e0*MeV4togcm3 for item in
    freeneutrongasenergy]

#interpolate
interpCFG=interpolate.interp1d(freeneutrongasenergy,freeneutrongaspressure
    )

#Make volume and pressure arrays for the pressure-volume EoS
energy2Bconverted=energylist2[:]
afterpressure=pressurelist2[:]
del energy2Bconverted[0]
del afterpressure[0]
volume2=[(1./(item*e0))*(perMeVtom*1.0e15)**3. for item in
    energy2Bconverted]
pressure2=[item*e0/dynetoMeV4 for item in afterpressure]
volume2.reverse()
pressure2.reverse()

#Plot of Figure 4.1
f, axarr = plt.subplots(1,2, sharey=True,figsize=(9,4))

axarr[0].set_ylabel(r'$P$ (dyne/cm$^2$)')
axarr[0].grid(True)
axarr[0].plot(energyingcm32,pressureindyne2,'k')
axarr[0].set_xlabel(r'$\epsilon$ (g/cm$^3$)')
axarr[0].set_xlim(0,2e14)

axarr[1].grid(True)
axarr[1].plot(volume2,pressure2,'k')
axarr[1].set_xlabel(r'$V/E$ (fm$^3$/MeV)')
axarr[1].set_xlim(0,0.4)
```

```

plt.ylim(-1.8e32,1.5e32)
plt.show()

#find local top in pressure graph
leastpressureindex=pressure2.index(min(pressure2))
index=leastpressureindex
found=False
while found==False:
    index=index+1
    if pressure2[index]<pressure2[index-1]:
        found=True
maxpressure=pressure2[index-1]
maxpressureindex=index-1
almostmaxpressure=pressure2[index-2]

#divide pressure into 3 regions: before volume minimum, between volume
    minimum and maximum, after volume maximum. Interpolate
firstpartvolume=volume2[:leastpressureindex]
firstpartpressure=pressure2[:leastpressureindex]
interpvol1=interpolate.interp1d(firstpartpressure,firstpartvolume)

secondpartvolume=volume2[leastpressureindex:maxpressureindex]
secondpartpressure=pressure2[leastpressureindex:maxpressureindex]
interpvol2=interpolate.interp1d(secondpartpressure,secondpartvolume)

thirdpartvolume=volume2[maxpressureindex:]
thirdpartpressure=pressure2[maxpressureindex:]
interpvol3=interpolate.interp1d(thirdpartpressure,thirdpartvolume)

#general interpolation
interpolatedpressure=interpolate.interp1d(volume2,pressure2)

#loop different values for pressure between maxpressure and 0, find the
    two enclosed areas.
n=2000
pressure2loop=np.linspace(pressure2[-1],almostmaxpressure,n).tolist()
Maxwellintegrals1=[]
Maxwellintegrals2=[]
for i in range(n):
    currentpressure=pressure2loop[i]
    vol1=interpvol1(currentpressure)
    vol2=interpvol2(currentpressure)
    vol3=interpvol3(currentpressure)
    firstintegral,err=integrate.quad(interpolatedpressure,vol1,vol2)
    firstintegral=firstintegral - currentpressure*(vol2-vol1)
    secondintegral,err=integrate.quad(interpolatedpressure,vol2,vol3)
    secondintegral=secondintegral - currentpressure*(vol3-vol2)
    Maxwellintegrals1.append(firstintegral)
    Maxwellintegrals2.append(secondintegral)

#find the free Gibbs energy
G=[0]
volume2Gibbs=[item/m*1.0e-45*mtoperMeV**3. for item in volume2]
pressure2Gibbs=[item*dynetoMeV4 for item in pressure2]
currentG=0
for i in range(len(pressure2Gibbs)-1):
```

```
    currentG=currentG+volume2Gibbs[i]*(pressure2Gibbs[i+1]-pressure2Gibbs[
    i])
    G.append(currentG)

#normalize the Gibbs energy
G0=0.009183
Gnorm=[(item/G0 + 1.)*10000 for item in G]
plt.plot(pressure2,Gnorm)

#plot of Figure 4.2
plt.xlabel('$P$ (dyne/cm$^2$)')
plt.ylabel('$G/G_0$')
plt.xlim(-1.8e32,1.5e32)
plt.ylim(0,5.2)
plt.grid(True)
plt.show()

#Calculate the difference between the encloed areas. Find when it is
    closest to 0.
Maxwellintegrals=np.add(Maxwellintegrals1,Maxwellintegrals2).tolist()
AbsMaxwellintegrals=[abs(item) for item in Maxwellintegrals]
minimum=min(AbsMaxwellintegrals)
minimumindex=AbsMaxwellintegrals.index(minimum)

#Find the phase transition pressure
Pcritic=pressure2loop[minimumindex]*dynetoMeV4/e0
print(Pcritic/dynetoMeV4*e0)

#Find the three points where the pressure plot crosses the critical
    pressure
index=0
while pressurelist2[index] < Pcritic:
    index=index+1
firstoccurrence=index-1
while pressurelist2[index] > Pcritic:
    index=index+1
secondoccurrence=index
while pressurelist2[index] < Pcritic:
    index=index +1
thirdoccurrence=index

#Make new pressure arrays accounting for the phase transition
fillin=thirdoccurrence-firstoccurrence
maxwellpressure=pressurelist2[:firstoccurrence] + fillin*[Pcritic] +
    pressurelist2[thirdoccurrence:]
maxwellindyne2=[item*e0/dynetoMeV4 for item in maxwellpressure]

#Make logs for plot, and plot Figure 4.5
def log10neg(x):
    if x>0:
        return np.log10(x)
    else:
        return -np.log10(-x)
logenergyingcm32=[np.log10(item) for item in energyingcm32]
logpressureindyne2=[log10neg(item) for item in pressureindyne2]
logmaxwellindyne2=[np.log10(item) for item in maxwellindyne2]
logfreeneutrongasenergygcm3=[np.log10(item) for item in
```

```
    freeneutrongasenergygcm3]
logfreeneutrongaspressuredyne=[np.log10(item) for item in
    freeneutrongaspressuredyne]

plt.ylabel(r'log$(P)$ (dyne/cm$^2$)')
plt.xlabel(r'log$(\epsilon)$ (g/cm$^3$)')
plt.ylim(25,39)
plt.xlim(9,18)
plt.plot(logenergyingcm32,logpressureindyne2,'k--',
         logenergyingcm32,logmaxwellindyne2,'k',
         logfreeneutrongasenergygcm3,logfreeneutrongaspressuredyne,'k-.')
plt.grid()
plt.show()

#reverse the new pressure array for plotting
revmaxwellindyne=maxwellindyne2[:]
revmaxwellindyne.reverse()
del revmaxwellindyne[0]
PcriticDyne=Pcritic/dynetoMeV4*e0

#Plot Figure 4.4
f, axarr = plt.subplots(1,2, sharey=True,figsize=(9,4))
plt.ylim(-1.8e32,1.5e32)

axarr[0].grid(True)
axarr[0].fill_between(volume2,revmaxwellindyne,pressure2)
axarr[0].set_xlabel(r'$V/E$ (fm$^3$/MeV)')
axarr[0].set_xlim(0,0.4)
axarr[0].set_ylabel(r'$P$ (dyne/cm$^2$)')

axarr[1].grid(True)
axarr[1].plot(volume2,pressure2,'b--',volume2,revmaxwellindyne,'k')
axarr[1].set_xlabel(r'$V/E$ (fm$^3$/MeV)')
axarr[1].set_xlim(0,0.4)
plt.show()

#plot Figure 4.3
firstpartpressuremod=firstpartpressure[:]
firstpartvolumemod=firstpartvolume[:]
for index in range(len(firstpartpressure)):
    if firstpartpressure[index]>PcriticDyne:
        del firstpartpressuremod[0]
        del firstpartvolumemod[0]

thirdpartpressuremod=thirdpartpressure[:]
thirdpartvolumemod=thirdpartvolume[:]
thirdpartpressuremod.reverse()
thirdpartvolumemod.reverse()
for index in range(len(thirdpartpressure)):
    if thirdpartpressure[index]<PcriticDyne:
        del thirdpartpressuremod[0]
        del thirdpartvolumemod[0]

plt.plot(revmaxwellindyne,volume2,'k')
plt.fill_between(secondpartpressure,secondpartvolume,facecolor='green',
    alpha=0.5)
plt.fill_between(thirdpartpressuremod,thirdpartvolumemod,facecolor='red',
```

```
    alpha=0.5)
plt.fill_between(firstpartpressuremod,firstpartvolumemod,facecolor='blue',
    alpha=0.5)

plt.xlim(-1.6e32,0.5e32)
plt.ylim(0,0.3)
plt.ylabel(r'$V/E$ (fm$^3$/MeV)')
plt.xlabel(r'$P$ (dyne/cm$^2$)')
plt.grid()
plt.show()

#interpolate the pressure and the energy density
EoS=interpolate.interp1d(maxwellpressure,energylist2)

#Calculate the mass-radii relation for neutron matter
Pcstart = 2e32*dynetoMeV4/e0
Pcend = 9e38*dynetoMeV4/e0
Pcstep = 1.1
P = Pc = Pcstart
M = 0.0
r = 0.0
rstep = 0.0005 #in km
Mlist2 = []
Rlist2 = []
centralpressures2 = []
while Pc < Pcend:
    P = Pc
    M = 0.0
    r = 0.0
    centralpressures2.append(Pc)
    while P > 0:
        r = r + rstep
        eps = EoS(P)
        M = M + rstep * dMdr(r, eps)
        P = P + rstep * dPdr(r, M, P, eps)
    Rlist2.append(r)
    Mlist2.append(M)
    Pc = Pc*Pcstep

indexbiggestmass2=Mlist2.index(max(Mlist2))
StableMlist2=Mlist2[:(indexbiggestmass2+1)]
StableRlist2=Rlist2[:(indexbiggestmass2+1)]
UnstableMlist2=Mlist2[(indexbiggestmass2+1):]
UnstableRlist2=Rlist2[(indexbiggestmass2+1):]
centraldensities2=[EoS(item)*e0*MeV4togcm3 for item in centralpressures2]
Stablecentraldensities2=centraldensities2[:(indexbiggestmass2+1)]
Unstablecentraldensities2=centraldensities2[(indexbiggestmass2+1):]
logStablecentraldensities2=[np.log10(item) for item in
    Stablecentraldensities2]
logUnstablecentraldensities2=[np.log10(item) for item in
    Unstablecentraldensities2]

#Repeat the procedure, but for nuclear matter (f=4). Here is unnecessary
    with the Maxwell contruction.
f=4.0
n=50000
gsigma4list=[]
```

```
gsigma4=0.0
checker=0
for index in range(len(kf)):
    currentkf=kf[index]
    gsigma4=scipy.optimize.newton(gsigmaintegral,gsigma4,None,(currentkf,))
    gsigma4list.append(gsigma4)

[pressurelist4,energylist4]=EquationOfStateSigmaOmega(kfmax,gsigma4list)

pressureindyne4=[item*e0/dynetoMeV4 for item in pressurelist4]
energyingcm34=[item*e0*MeV4togcm3 for item in energylist4]

#Plot of Figure 4.6
logpressureindyne4=[np.log10(item) for item in pressureindyne4]
logenergyingcm34=[np.log10(item) for item in energyingcm34]
plt.ylabel(r'$\log(P)$ (dyne/cm$^2$)')
plt.xlabel(r'$\log(\epsilon)$ (g/cm$^3$)')
plt.xlim(4,20)
plt.ylim(18,40)
plt.plot(logenergyingcm32,logmaxwellindyne2,'r',logenergyingcm34,logpressureindyne4,'b')
plt.grid(True)
plt.show()

EoS4=interpolate.interp1d(pressurelist4,energylist4)

Pcstart = 2e32*dynetoMeV4/e0
Pcend = pressurelist4[-1]
Pcstep = 1.1
P = Pc = Pcstart
M = 0.0
r = 0.0
rstep = 0.001 #in km
Mlist4 = []
Rlist4 = []
centralpressures4=[]

while Pc < Pcend:
    P = Pc
    centralpressures4.append(Pc)
    M = 0.0
    r = 0.0
    while P > 0:
        r = r + rstep
        eps=EoS4(P)
        M = M + rstep * dMdr(r, eps)
        P = P + rstep * dPdr(r, M, P, eps)
    Rlist4.append(r)
    Mlist4.append(M)
    Pc = Pc*Pcstep

indexbiggestmass4=Mlist4.index(max(Mlist4))
StableMlist4=Mlist4[:(indexbiggestmass4+1)]
StableRlist4=Rlist4[:(indexbiggestmass4+1)]
UnstableMlist4=Mlist4[(indexbiggestmass4+1):]
UnstableRlist4=Rlist4[(indexbiggestmass4+1):]
```

```
centraldensities4=[EoS4(item)*e0*MeV4togcm3 for item in centralpressures4]
Stablecentraldensities4=centraldensities4[:(indexbiggestmass4+1)]
Unstablecentraldensities4=centraldensities4[(indexbiggestmass4+1):]
logStablecentraldensities4=[np.log10(item) for item in
    Stablecentraldensities4]
logUnstablecentraldensities4=[np.log10(item) for item in
    Unstablecentraldensities4]
print(max(Mlist2))
print(Rlist2[indexbiggestmass2])
print(max(Mlist4))
print(Rlist4[indexbiggestmass4])

#Plot of Figure 4.7
f, axarr = plt.subplots(1,2, sharey=True)
axarr[0].set_ylabel(r'$M/M_\odot$')
axarr[0].grid(True)
axarr[0].plot(StableRlist2, StableMlist2, 'r-',
              UnstableRlist2, UnstableMlist2, 'r--',
              StableRlist4, StableMlist4, 'b-',
              UnstableRlist4, UnstableMlist4, 'b--')
axarr[0].set_xlabel('Radii (km)')

axarr[1].grid(True)
axarr[1].plot(logStablecentraldensities2, StableMlist2,'r-',
         logUnstablecentraldensities2, UnstableMlist2,'r--',
         logStablecentraldensities4, StableMlist4,'b-',
         logUnstablecentraldensities4, UnstableMlist4,'b--'
        )
axarr[1].set_xlabel(r'$\log(\epsilon_0)$ (g/cm$^3$)')

plt.ylim(0,3.5)
plt.show()
```

B.3 Chapter 5

```
import numpy as np
import scipy
import scipy.constants as sc
import scipy.integrate as integrate
import matplotlib
import matplotlib.pyplot as plt
import scipy.optimize as op
from scipy import interpolate
from operator import add
from matplotlib import rc
rc('font',**{'family':'sans-serif','sans-serif':['Helvetica']})
rc('text', usetex=True)

#constants and conversion coefficients
G = sc.G
hbar=sc.hbar
Mo = 1.9891e30  # in kg
R0 = G*Mo/(1000*sc.c**2.0)  # in km
MeVtoJoules=1.0e6*sc.eV
JoulestoMeV=1.0/MeVtoJoules
MeVtoperm=mtoperMeV=MeVtoJoules/(sc.c*hbar)
```

```
permtoMeV=perMeVtom=1.0/MeVtoperm
MeV4togcm3=1000.0*MeVtoJoules/sc.c**2.0*(MeVtoperm/100.0)**3.0
MeV4toJouleskm3=MeVtoJoules*(1000.0*MeVtoperm)**3.0
MeV4tokgkm3=MeV4toJouleskm3/sc.c**2.0
dynetoMeV4=0.1*JoulestoMeV*(permtoMeV)**3.0
gcm3toMeV4=1.0/MeV4togcm3
mn = 938. #neutron mass (approximate value for the self-interaction term
    in the sigma field)
m=(939.5654133+938.2720813)/2. #nucleon mass, average of p and n
me = 0.5109989461 #electron mass
mmu = 105.6583745 #muon mass
rho0 = (1.0e15*permtoMeV)**3.0 #normalization constant for baryon density
    in MeV, 1.0*fm^-3
e0=m**4.
beta = 4.0*sc.pi*e0*MeV4tokgkm3/Mo #in solarmasses per km3

#parameters
gmsigma2=9.927*(1.0e-15*mtoperMeV)**2.0
gmomega2=4.820*(1.0e-15*mtoperMeV)**2.0
gmrho2=4.791*(1.0e-15*mtoperMeV)**2.0
b=0.008621
c=-0.002321

def cube(x): #function to find cube of negative numbers
    if x >= 0:
        return x**(1./3.)
    else:
        return -(abs(x)**(1./3.))

def gsigmaintegral(fgsigma,k): #function calculating the integral in
    gsigma. Returns normalized gsigma (gsigma/m)^3
    xnorm=k/(1.0-fgsigma)
    return (1.0-fgsigma)**3.0*(xnorm*np.sqrt(xnorm**2.0+1.0) - np.arcsinh(
    xnorm))/(2.0*sc.pi**2.0)

def squaresbeforemuons(x,rho): #x0=rhon, x1=gsigma, x2=ke. Equation system
     before the appearance of muons
    kp=cube(3.*sc.pi**2.*(rho-x[0]))
    kn=cube(3.*sc.pi**2.*x[0])
    grho=gmrho2*(0.5*rho-x[0])
    mstar=(1.-x[1]/m)*m
    return  (
             (x[1] - gmsigma2*(-b*mn*x[1]**2 - c*x[1]**3 + m**3*(
    gsigmaintegral(x[1]/m,kn/m) + gsigmaintegral(x[1]/m,kp/m))))**2 +
             (kp-x[2])**2.0 +
             (grho + np.sqrt(kp**2. + mstar**2)-np.sqrt(kn**2.+mstar**2.)
    +np.sqrt(x[2]**2 + me**2))**2.
             )

def squaresaftermuons(x,rho): #x0=rhon, x1=gsigma, x2=ke. Equation system
    after the appearance of muons
    kp=cube(3.*sc.pi**2.*(rho-x[0]))
    kn=cube(3.*sc.pi**2.*x[0])
    grho=gmrho2*(0.5*rho-x[0])
    mstar=(1.-x[1]/m)*m
    mue2=me**2. + x[2]**2.
    kmu=np.sqrt(mue2-mmu**2.)
```

```
    return  (
                (x[1] - gmsigma2*(-b*mn*x[1]**2. - c*x[1]**3. + m**3.*(gsigmaintegral(x[1]/m,kn/m) + gsigmaintegral(x[1]/m,kp/m))))**2. +
                (kp-cube(x[2]**3.+kmu**3.))**2. +
                (grho + np.sqrt(kp**2. + mstar**2.)-np.sqrt(kn**2.+mstar**2.) +np.sqrt(mue2))**2.
                )

def dMdr(r, eps):
    return beta*r**2.0*eps #returns sunmasses/km

def dPdr(r, M, P, eps):
    return -(R0*eps*M)/r**2.0*(P/eps + 1.0)*((beta*r**3.0*P)/M + 1.0)/(1.0-2.0*R0*M/r) #returns in J/km**2

bnds=((0,None),(0,None),(0,None)) #rho_n, gsigma and k_e must have positive values
n=2000 #number of steps
startrho=0.01*rho0 #initial baryon density
rhos=np.linspace(startrho,2.0*rho0,n).tolist() #list for all the baryon densities we want to evaluate in the system
guess=np.array([
                startrho,
                gmsigma2*startrho,
                0.12*m*(startrho/rho0)**(2./3.)
                ]) #guess for the initial value of our variables. To be updated in the iteration
#initializing lists
rhons=[]
gsigmas=[]
rhonnorm=[]
rhomuons=[]
rhomuonsnorm=[]
rhoesnorm=[]
kes=[]
kmuons=[]
firsttimemuons=False
for index in range(len(rhos)):
    rho=rhos[index]
    if (guess[2]**2.+me**2.)>=mmu**2. and firsttimemuons==False: #checking condition for muons
        print(rho/rho0)
        firsttimemuons=True
    if firsttimemuons==False:
        roots=op.minimize(squaresbeforemuons,guess,(rho,),bounds=bnds)
    else:
        roots=op.minimize(squaresaftermuons,guess,(rho,),bounds=bnds)
    guess=roots.x
    #save roots in lists
    rhons.append(guess[0])
    rhonnorm.append(guess[0]/rho)
    gsigmas.append(guess[1])
    kes.append(guess[2])
    rhoesnorm.append(guess[2]**3.0/(rho*3*sc.pi**2.))
    if firsttimemuons==True:
        kmuons.append(np.sqrt(guess[2]**2. + me**2. - mmu**2.))
        rhomuons.append(kmuons[-1]**3./(3.*sc.pi**2.))
```

```
        rhomuonsnorm.append(rhomuons[-1]/rho)
    else:
        kmuons.append(0.)
        rhomuons.append(0.)
        rhomuonsnorm.append(0.)

#Plot Figure 5.3
rhopnorm=[1.-item for item in rhonnorm]
rhosnorm=[item/rho0 for item in rhos]
logrhopnorm=[np.log10(item) for item in rhopnorm]
logrhonnorm=[np.log10(item) for item in rhonnorm]
logrhoesnorm=[np.log10(item) for item in rhoesnorm]
logrhomuonsnorm=[np.log10(item) for item in rhomuonsnorm]
plt.rc('text', usetex=True)
plt.rc('font', family='serif')
plt.ylabel(r'$\log (\rho_i/\rho)$')
plt.xlabel(r'$\rho(\textrm{fm}^{-3})$')
plt.ylim(-3,0)
plt.xlim(0,1)
plt.plot(rhosnorm,logrhonnorm,'b',
         rhosnorm,logrhopnorm,'r',
         rhosnorm,logrhoesnorm,'b--',
         rhosnorm,logrhomuonsnorm,'k--')
plt.grid()
plt.show()

#function calculating the integral in the pressure expression for the EoS
def pressureintegral(x):
    return ((2.0*x**3.0 - 3.0*x)*(1.0+x**2.0)**(0.5) + 3.0*np.arcsinh(x))/8.0

#function calculating the integral in the energy density expression for the EoS
def energyintegral(x):
    return ((2.0*x**3.0 + x)*(1.0+x**2.0)**(0.5) - np.arcsinh(x))/8.0

#Function taking as input the values of the fields and the Fermi momenta of the different particles, and yielding the pressure and energy density array
def makeEoS(rholist,rhonlist,gsigmalist,kelist,kmulist):
    energydensitylist=[]
    pressurelist=[]
    for index in range(len(rholist)):
        kp=cube(3.*sc.pi**2.*(rholist[index]-rhonlist[index]))
        kn=cube(3.*sc.pi**2.*rhonlist[index])
        selfinteractions=b*mn*gsigmalist[index]**3./3. + c*gsigmalist[index]**4./4.
        msigmaterm=0.5*gsigmalist[index]**2./gmsigma2
        momegaterm=0.5*gmomega2*rholist[index]**2.
        mrhoterm=0.5*gmrho2*(0.5*rholist[index]-rhonlist[index])**2.
        pressureints = ((m - gsigmalist[index])**4.0 * (pressureintegral(kp / (m - gsigmalist[index]) ) + pressureintegral(kn / (m - gsigmalist[index]))) + me**4. * pressureintegral(kelist[index] / me) + mmu**4. * pressureintegral(kmulist[index] / mmu))/(3. * sc.pi**2.)
        energyints=((m - gsigmalist[index])**4.0 * (energyintegral(kp / (m - gsigmalist[index])) + energyintegral(kn / (m - gsigmalist[index]))) + me**4. * energyintegral(kelist[index] / me) + mmu**4. *
```

```
energyintegral(kmulist[index] / mmu)) /sc.pi**2.
        currentenergydensity = selfinteractions + momegaterm + msigmaterm + mrhoterm + energyints
        currentpressure = -selfinteractions + momegaterm - msigmaterm + mrhoterm + pressureints

        energydensitylist.append(currentenergydensity/e0)
        pressurelist.append(currentpressure/e0)
    return energydensitylist, pressurelist

#Function f0 as defined in Chapter 5
def f0(x):
    return 1.0/(np.exp(x)+1.0)

#EoS for the crust: pressure as function of the energy density
a = [6.22, 6.121, 0.005925, 0.16326, 6.48, 11.4971, 19.105, 0.8938, 6.54, 11.4950, -22.775, 1.5707, 4.3, 14.08, 27.80, -1.653, 1.50, 14.67]
def PfromepsLD(eps):
    return (a[0]+a[1]*eps+a[2]*eps**3.0)*f0(a[4]*a[5]*(eps/a[5]-1.0))/(1.0+a[3]*eps) + (a[6] + a[7]*eps)*f0(a[8]*a[9]*(1.0-eps/a[9])) + (a[10] + a[11]*eps)*f0(a[12]*a[13]*(1.0-eps/a[13])) + (a[14] + a[15]*eps)*f0(a[16]*a[17]*(1.0-eps/a[17]))

#EoS for the crust: energy density as function of the pressure (rootfinding)
def epsLDroot(eps,P):
    return PfromepsLD(eps)-P

def EoSLD(P,guess):
    logdyneP = np.log10(P*e0/dynetoMeV4)
    tempeps = scipy.optimize.newton(epsLDroot,guess,None,(logdyneP,))
    return 10.0**(tempeps)*gcm3toMeV4/e0

#Evaluate the EoS, save the energy densities and the pressures in two arrays
[energy,pressure]=makeEoS(rhos,rhons,gsigmas,kes,kmuons)

#Find crossing point between the npemu EoS and the crust's EoS
energygcm3=[np.log10(item*e0*MeV4togcm3) for item in energy]
pressuredyne=[np.log10(item*e0/dynetoMeV4) for item in pressure]
pressuredyneLD=[PfromepsLD(item) for item in energygcm3]
counter=0
while PfromepsLD(energygcm3[counter])>pressuredyne[counter]:
    counter=counter+1
crossenergy=energy[counter]
crosspressure=pressure[counter]

#Make an array for the low density EoS (crust)
energyLDjoind=np.linspace(10.0e-14,crossenergy,1000).tolist()
energyLDjoindlog=[np.log10(item*e0*MeV4togcm3)for item in energyLDjoind]
pressureLDjoindlog=[PfromepsLD(item) for item in energyLDjoindlog]
pressureLDjoind=[10.0**(item)*dynetoMeV4/e0 for item in pressureLDjoindlog]

#Join the two EoS
joindener=energyLDjoind + energy[counter:]
joindpres=pressureLDjoind + pressure[counter:]
```

```
#Plot of Figure 5.1
joindenerplot=[item*e0*MeV4togcm3 for item in joindener]
joindpresplot=[item*e0/dynetoMeV4 for item in joindpres]
logjoindenerplot=[np.log10(item) for item in joindenerplot]
logjoindpresplot=[np.log10(item) for item in joindpresplot]

plt.rc('text', usetex=True)
plt.rc('font', family='serif')
plt.ylabel(r'$\log P (\textrm{dyne/cm}^2$)')
plt.xlabel(r'$\log \epsilon(\textrm{g/cm}^{3})$')
plt.ylim(26,38)
plt.xlim(9,16)
plt.plot(logjoindenerplot,logjoindpresplot, 'k')
plt.grid()
plt.show()

#Interpolate the EoS
EoS=interpolate.interp1d(joindpres,joindener)

#Calculate the mass-radii relation for npemu matter
Pcstart = 4e33*dynetoMeV4/e0
Pcend = joindpres[-2]
Pcstep = 1.05
P = Pc = Pcstart
M = 0.0
r = 0.0
rstep = 0.001 #in km
Mlist2 = []
Rlist2 = []
centralpressures2 = []
tol=joindpres[0]
while Pc < Pcend:
    P = Pc
    M = 0.0
    r = 0.0
    centralpressures2.append(Pc)
    while P > tol:
        r = r + rstep
        eps=EoS(P)
        M = M + rstep * dMdr(r, eps)
        P = P + rstep * dPdr(r, M, P, eps)
    Rlist2.append(r)
    Mlist2.append(M)
    Pc = Pc*Pcstep
indexbiggestmass2=Mlist2.index(max(Mlist2))
StableMlist2=[]
UnstableMlist2=[]
StableRlist2=[]
UnstableRlist2=[]
StableMlist2=Mlist2[:(indexbiggestmass2+1)]
StableRlist2=Rlist2[:(indexbiggestmass2+1)]
UnstableMlist2=Mlist2[(indexbiggestmass2+1):]
UnstableRlist2=Rlist2[(indexbiggestmass2+1):]

#Print largest mass and smallest radius
print(Rlist2[indexbiggestmass2])
```

```
print(Mlist2[indexbiggestmass2])

#Plot of Figure 5.2
centraldensities2=[EoS(item)*e0*MeV4togcm3 for item in centralpressures2]
Stablecentraldensities2=centraldensities2[:(indexbiggestmass2+1)]
Unstablecentraldensities2=centraldensities2[(indexbiggestmass2+1):]
Stablecentraldensities2log=[np.log10(item) for item in
    Stablecentraldensities2]
Unstablecentraldensities2log=[np.log10(item) for item in
    Unstablecentraldensities2]

f, axarr = plt.subplots(1,2, sharey=True,figsize=(9,4))
axarr[0].grid(True)
axarr[0].plot(StableRlist2, StableMlist2, 'k-',
              UnstableRlist2, UnstableMlist2, 'k--')
axarr[0].set_xlabel('Radii (km)')
axarr[0].set_ylabel(r'$M/M_\odot$')
axarr[0].set_xlim([8,25])
axarr[1].grid(True)
axarr[1].plot(Stablecentraldensities2log, StableMlist2,'k-',
              Unstablecentraldensities2log, UnstableMlist2,'k--')
axarr[1].set_xlabel(r'$\log \epsilon_0$ (g/cm$^3$)')
axarr[1].set_xlim([14.45,15.8])
plt.ylim(0.2,2.15)
plt.show()
```

Bibliography

[1] Baade, W. and Zwicky, F. “On Super-novae”. In: *Proceedings of the National Academy of Science* 20 (1934), pp. 254–259.

[2] Baym, G. and Chin, S. A. “Can a neutron star be a giant MIT bag?” In: *Physics Letters B* 62.2 (1976), pp. 241–244.

[3] Baym, G., Bethe, H. A., and Pethick, C. J. “Neutron star matter”. In: *Nuclear Physics A* 175.2 (1971), pp. 225–271.

[4] Baym, G., Pethick, C. J., and Sutherland, P. “The ground state of matter at high densities: equation of state and stellar models”. In: *The Astrophysical Journal* 170 (1971), p. 299.

[5] Boguta, J. and Bodmer, A. R. “Relativistic calculation of nuclear matter and the nuclear surface”. In: *Nuclear Physics A* 292.3 (1977), pp. 413–428.

[6] Chin, S. A. “A Relativistic Many Body Theory of High Density Matter”. In: *Annals Phys.* 108 (1977), pp. 301–367.

[7] Chin, S. A. and Walecka, J. D. “An equation of state for nuclear and higher-density matter based on relativistic mean-field theory”. In: *Physics Letters B* 52.1 (1974), pp. 24–28.

[8] Douchin, F., Haensel, P., and Meyer, J. “Nuclear surface and curvature properties for SLy Skyrme forces and nuclei in the inner neutron-star crust”. In: *Nuclear Physics A* 665.3–4 (2000), pp. 419–446.

[9] Glendenning, N. K. and Kettner, C. “Possible third family of compact stars more dense than neutron stars”. In: *Astronomy and Astrophysics* 353 (2000), pp. L9–L12.

[10] Glendenning, N. K. *Compact Stars: Nuclear Physics, Particle Physics, and General Relativity*. Astronomy and astrophysics library. Springer, 1997.

[11] Glendenning, N. K. “Vacuum polarization effects on nuclear and neutron star matter”. In: *Phys. Lett.* B208 (1988), pp. 335–338.

[12] Glendenning, N. K. “Vacuum Polarization Effects on Nuclear Matter and Neutron Stars”. In: *Nucl. Phys.* A493 (1989), pp. 521–548.

[13] Glendenning, N. K. and Kettner, C. “Nonidentical neutron star twins”. In: *Astron. Astrophys.* 353 (2000), p. L9.

[14] Goldstein, H., Poole, C., and Safko, J. L. *Classical Mechanics*. Addison-Wesley, 2001.

[15] Griffiths, D. J. *Introduction to Quantum Mechanics*. Pearson/Prentice Hall, 2005.

[16] Haensel, P. and Pichon, B. “Experimental nuclear-masses and the ground-state of cold dense matter”. In: *Astronomy and Astrophysics* 283(1) (1994), pp. 313–318.

[17] Haensel, P., Potekhin, A. Y., and Yakovlev, D. G. *Neutron stars 1: Equation of state and structure*. Vol. 326. Springer Science & Business Media, 2007.

[18] Harrison, B. K. et al. *Gravitation Theory and Gravitational Collapse*. University of Chicago Press, 1965.

[19] Hartle, J. B. *Gravity: An Introduction to Einstein’s General Relativity*. Addison-Wesley, 2003.

[20] Hemmer, P. C. *Kvantemekanikk*. Tapir akademisk forlag, 2005.

[21] Hewish, A. et al. “Observation of a Rapidly Pulsating Radio Source”. In: *Nature* 217 (1968), pp. 709–713.

[22] Itzykson, C. and Zuber, J. B. *Quantum field theory, International series in pure and applied physics*. McGraw-Hill, New York, 1980.

[23] Jeffrey, A. and Zwillinger, D. *Table of Integrals, Series, and Products*. Table of Integrals, Series, and Products Series. Elsevier Science, 2007.

[24] Jia, H.-y. et al. “Describing neutron stars by the relativistic mean field theory”. In: *Chinese Astronomy and Astrophysics* 26.4 (2002), pp. 404–413.

[25] Kachelriess, M. *Quantum Fields notes*.

[26] Kapusta, J. I. and Gale, C. *Finite-temperature field theory: Principles and applications*. Cambridge University Press, 2006.

[27] Landau, L. D. “On the theory of stars”. In: *Phys. Z. Sowjetunion* 1 (1932), pp. 285–288.

[28] Myers, W. D. and Swiatecki, W. J. “Nuclear properties according to the Thomas-Fermi model”. In: *Nuclear Physics A* 601.2 (1996), pp. 141–167.

[29] Oppenheimer, J. R. and Volkoff, G. M. “On Massive Neutron Cores”. In: *Phys. Rev.* 55 (4 1939), pp. 374–381.

[30] Pandharipande, V. R. and Ravenhall, D. G. “Hot nuclear matter”. In: *Nuclear matter and heavy ion collisions*. Springer, 1989, pp. 103–132.

[31] Patrignani, C. et al. “Review of Particle Physics”. In: *Chin. Phys.* C40.10 (2016), p. 100001.

[32] Rottmann, K. *Matematisk formelsamling*. Spektrum forlag, 2001.

[33] Schmitt, A. “Dense matter in compact stars: A pedagogical introduction”. In: *Lect. Notes Phys.* 811 (2010), pp. 1–111.

[34] Schroeder, D. V. *An Introduction to Thermal Physics*. Addison Wesley, 2000.

[35] Serot, B. D. and Uechi, H. "Neutron stars in relativistic hadron-quark models". In: *Annals of Physics* 179.2 (1987), pp. 272–293.

[36] Serot, B. D. and Walecka, J. D. *Advances in nuclear physics. Vol. 16*. Ed. by J. W. Negele and E. Vogt. 1986.

[37] Silbar, R. R. and Reddy, S. "Neutron stars for undergraduates". In: *Am. J. Phys.* 72 (2004), pp. 892–905.

[38] Srednicki, M. *Quantum Field Theory*. Cambridge University Press, 2007.

[39] Staubo, E. F., Heiselberg, H., and Pethick, C. J. "The structure of quark matter in neutron stars". In: *Nuclear Physics A* 566 (1994), pp. 577–580.

[40] Tolman, R. C. "Static Solutions of Einstein's Field Equations for Spheres of Fluid". In: *Phys. Rev.* 55 (4 1939), pp. 364–373.

[41] Walecka, J. D. "A theory of highly condensed matter". In: *Annals of Physics* 83.2 (1974), pp. 491–529.

[42] Weinberg, S. *Gravitation and cosmology : principles and applications of the general theory of relativity*. John Wiley & Sons, Inc., 1972.