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Linearized general relativity and its quantization

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Abstract

We begin by deriving the Einstein equation from the Einstein-Hilbert action integral using variational methods; Variation with respect to the metric and the Palatini variation. The vacuum solution (Schwarzschild solution) is then presented and derived. Weak field approximation is introduced, and the vacuum solution is derived again in this framework. Higher-order Lagrangians and their effects on the gravitational potential is investigated, but found to not be of particular significance to the quantization procedure that follows. The quantization of the theory is accomplished through the use of the background field method, and therefore applies for any background space-time.

Preface

When I first started to work on this master thesis, I had little experience with general relativity and quantum field theory. Therefore, writing was a great learning experience for me. I have investigated everything in the thesis with critical eyes, often wondering about the lack of detail in the existing literature that I drew inspiration from. I have therefore made an attempt in this work to make everything as clear as possible, so that the reader does not wonder about the details of the derivations. Due to this, I present some of my own derivations which yield the same results, but are easier to follow than the literature that I drew inspiration from, in my opinion.

On many occasions, because of my stubbornness to investigate everything deeply, I have gotten stuck where the sources are not detailed enough to follow easily. These periods of being stuck on one problem have been frustrating, but on the hand it has yielded a deeper understanding of the involved concepts. When such breakthroughs did occur, I have done my best to convey my newly found understanding in the thesis, so that the reader does not have to go through the same process.

I want to thank my parents, who have been supporting me to pursue any academic field of interest I have ever shown interest in.

A special thanks also goes out to Francesco Pogliano and Alfredo Sánches García who always had a Lagrangian in mind.

I also want to thank Prof. Jens Oluf Andersen who has been my supervisor for this project. He has been my guide through this learning experience, pointing me in which direction to go.

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Chapter 1

Introduction

The formulation of general relativity greatly reshaped our view of the physical world. Prior to special relativity we believed in absolute space and time, as an axiom upon which all other physics was understood. The discovery that light moves at the same speed in all inertial frames propelled forward a new understanding. We came to understand that simultaneity is relative, and as well, that the Newtonian laws of physics are approximations. General relativity showed us how space and time are influenced by energy. It is the idea that time and space are dynamical fields that is the central theme in this master thesis. We will build upon the central ideas proposed by Einstein, and attempt to linearize the theory in order to make it more manageable. When the theory has been linearized, we may proceed with a quantization procedure.

In the same way that Newtonian physics is an approximation which works well on our scale (the scale we observe in daily life), there is reason to assume that general relativity is an approximation, confined to a scale of relevance. As we see in the beginning of this thesis, it is possible to derive Einstein's equations from a Lagrangian formulation, where we only assume Lorentz invariance for the metric field. However, aware that Lorentz invariance is the central assumption of the theory, it is clear that there are other forms of the action potential that can produce the same results, leaving Einstein's equation behind as an approximation of a bigger picture. These other forms are called the higher-order Einstein-Hilbert actions. Einstein's equation arises from solving the simplest such Lagrangian system, but is merely a first-order approximation. These concerns are further discussed in Chapter 5.

In the second chapter, we see how the Schwarzschild solution arises naturally from Einsteins' equation. It is one of the most famous solutions in general relativity, and provides us with a useful framework for introducing a linearization procedure, called the weak field approximation. This involves investigating the metric field under small perturbations, so that new predictions can be made of the dynamics of such systems.

Once the theory has been linearized (and thereby specifying the domain of the problem to be in the low-energy limit), as well as ruling out the need for higher order terms of the action integral, we are ready to quantize the theory. The quantization of the theory is treated in Chapter 6.

Chapter 2

Derivation of Einstein's field equation

The famous Einstein field equation has been immensely important for the development of modern physics. It is an ever-present concept throughout this thesis, and lays the foundation for the work presented.

However, there is an even deeper layer of concepts and ideas that seem to be fundamental for all concepts put forward in this thesis, and that is the Lagrangian formalism with the variational principle; The motion of a system leaves the action integral of the system stationary (See Appendix C).

This thesis does in other words presuppose that the variational principle is more general, in a sense, than the Einstein field equation. This leads us to believe that the Einstein field equation should be possible to derive from the more general framework, if the correct axioms are chosen.

Of course, in a historical sense, the Einstein field equation was discovered before the Einstein-Hilbert action integral. However, using the Lagrangian formalism as a starting position provides us a foundational perspective on the subject. It also ensures that we maintain the same formalism from start to finish.

The Lagrangian that describes the system should be Lorentz invariant. It should also involve the metric tensor, whose dynamics we want to derive. The simplest Lagrangian that fulfills those criteria is the Einstein-Hilbert Lagrangian, which we will use in this chapter. More complicated Lagrangians which fulfill the criteria are possible, and are investigated in Chapter 5.

In this chapter we will derive the Einstein field equation from the Einstein-Hilbert action. Initially, by a variation of the metric tensor alone, while assuming that the Christoffel symbols are of the form given by B.2. In the subsequent section, we derive the Einstein equation without assuming the form of the Christoffel symbols, in other words, we will be performing a Palatini variation.

2.1 Variation with respect to the metric tensor

In this section, Einstein's field equation is derived from the Einstein-Hilbert action. The Einstein-Hilbert action is given by

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x \left[\frac{1}{2\kappa}R + \mathscr{L}_M\right]\sqrt{-g}\,, \tag{2.1}$$

where $\kappa = 8\pi G$, and G is the gravitational constant. R is the Ricci curvature scalar, $R = g^{\mu\nu}R_{\mu\nu}$. $g_{\mu\nu}$ is the metric tensor, and $R_{\mu\nu}$ is the Ricci tensor. g is in this notation shorthand for the determinant of the metric tensor. Lastly, $\mathscr{L}_M$ is the matter component of the Lagrangian.

According to section C.1, the equations of motion arise from extremizing the action. Let $S_{\mathrm{EH}} \to S'_{\mathrm{EH}} = S_{\mathrm{EH}} + \delta S_{\mathrm{EH}}$, where δS_{EH} is the variation. To extremize the action, we require that δS_{EH} is equal to zero,

$$\delta S_{\mathrm{EH}} = 0\,. \tag{2.2}$$

The variation of the action is

$$\begin{aligned}
\delta S_{\mathrm{EH}} &= \int \mathrm{d}^4x\,\delta\left[\frac{1}{2\kappa}\sqrt{-g}R + \sqrt{-g}\mathscr{L}_M\right]\\
&= \int \mathrm{d}^4x \left[\frac{1}{2\kappa}\delta\left(\sqrt{-g}R\right) + \delta\left(\sqrt{-g}\mathscr{L}_M\right)\right]\\
&= \int \mathrm{d}^4x \left[\frac{1}{2\kappa}\frac{\delta\left(\sqrt{-g}R\right)}{\delta g^{\mu\nu}} + \frac{\delta\left(\sqrt{-g}\mathscr{L}_M\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\\
&= \int \mathrm{d}^4x \left[\frac{1}{2\kappa}\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \frac{\delta R}{\delta g^{\mu\nu}}\sqrt{-g}\right) + \frac{\delta\left(\sqrt{-g}\mathscr{L}_M\right)}{\delta g^{\mu\nu}}\right]\delta g^{\mu\nu}\\
&= \int \mathrm{d}^4x \left[\frac{1}{2\kappa}\left(\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathscr{L}_M\right)}{\delta g^{\mu\nu}}\right]\sqrt{-g}\delta g^{\mu\nu}.
\end{aligned} \tag{2.3}$$

The part of the integrand which is inside the square brackets must be equal to zero, since the variation of the metric tensor, $\delta g^{\mu\nu}$, is completely arbitrary. Thus we obtain the equation of motion

$$\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta R}{\delta g^{\mu\nu}} = -\frac{2\kappa}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_M)}{\delta g^{\mu\nu}}. \tag{2.4}$$

Now, Jacobi's formula in differential form for a matrix, $\boldsymbol{A}(t)$, is [8, pp.169-171]

$$\mathrm{d}\left(\det\boldsymbol{A}(t)\right) = \mathrm{tr}\left(\mathrm{adj}\left(\boldsymbol{A}(t)\right)\mathrm{d}\boldsymbol{A}(t)\right)\,. \tag{2.5}$$

This relation for a differential of a determinant can be extended to the variation of the determinant of the metric tensor. Doing that, we obtain

$$\begin{aligned}
\delta g &= \delta\left(\det\left(\boldsymbol{g}\right)\right)\\
&= \mathrm{tr}\left[\mathrm{adj}\left(\boldsymbol{g}\right)\delta\boldsymbol{g}\right]\\
&= \mathrm{tr}\left[\det\left(\boldsymbol{g}\right)\boldsymbol{g}^{-1}\delta\boldsymbol{g}\right]\\
&= g g^{\mu\nu}\delta g_{\mu\nu}\,.
\end{aligned} \tag{2.6}$$

From (D.23) we know that

$$g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu} \,, \tag{2.7}$$

which we can use to obtain

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu} \,. \tag{2.8}$$

By using what we have learned about the variation of the determinant of the metric, (2.6), we obtain the following relation

$$\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}, \tag{2.9}$$

and hence we insert (2.9) into (2.4) to obtain

$$-\frac{1}{2}Rg_{\mu\nu} + \frac{\delta R}{\delta g^{\mu\nu}} = \kappa\left[\mathscr{L}_M g_{\mu\nu} - 2\frac{\delta\mathscr{L}_M}{\delta g^{\mu\nu}}\right] \,, \tag{2.10}$$

where, by definition, the part within the square brackets is the energy stress tensor, $T_{\mu\nu}$. Now the equation of motion can be written as

$$\frac{\delta R}{\delta g^{\mu\nu}} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \,. \tag{2.11}$$

The only remaining task in order to obtain the equation of motion is to compute the functional derivative of the Ricci scalar with respect to the metric tensor. The Ricci scalar, (B.4), is defined by

$$R = g^{\mu\nu}R_{\mu\nu} \,. \tag{2.12}$$

Since variation obeys the product rule, the variation of the Ricci scalar can be written as

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{2.13}$$

One can always choose a coordinate system in which $\Gamma^{\lambda}_{\mu\nu} = 0$ at a specific point[1] [11, p. 278]. Then the Ricci tensor, (B.3), at that point is

$$R_{\mu\nu} = \Gamma^{\rho}_{\nu\mu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} \,. \tag{2.14}$$

Since space is locally flat at the point of examination, we should expect that the derivatives of the metric tensor remain equal to zero at the point (There is no curvature). Indeed, by examining the definition of the Christoffel symbols (B.2)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \,. \tag{2.15}$$

The Christoffel symbols are equal to zero whenever $\lambda \neq \alpha$ by definition since we are examining a locally flat spacetime. However, when $\lambda = \alpha$, the terms within the parenthesis have to add up to zero. Since all off-diagonal terms of the metric tensor are equal to zero, then so are their derivatives. After having established that off-diagonal terms and their

[1] Translated from mathematical notation to a more physical perspective, we can always choose a coordinate system which is locally flat.

derivatives of the metric tensor are equal to zero, we look at the diagonal terms. Let $\mu = \nu$, while $\lambda = \alpha$, and observe that the only term within the parenthesis left is $-g_{\mu\nu,\alpha}$ which has to be equal to zero. Thus we obtain the relation

$$g_{\mu\nu,\alpha} = 0\,. \tag{2.16}$$

Notice that this relation holds for all μ, ν, and α. It is thus also implied that

$$g_{,\alpha} = 0\,, \tag{2.17}$$

where g without indices is symbolic notation for the trace of the metric tensor, $g^{\mu\nu}g_{\mu\nu} = g^{\mu}_{\mu} = g$.

Moving on, we establish that the term involving the variation of the Ricci tensor can be written as

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\delta\Gamma^{\rho}_{\nu\mu,\rho} - \delta\Gamma^{\rho}_{\rho\mu,\nu}\right)\,. \tag{2.18}$$

If we define a variational four-vector as

$$\delta\omega^{k} = g^{\mu\nu}\delta\Gamma^{k}_{\mu\nu} - g^{\mu\kappa}\Gamma^{\lambda}_{\mu\lambda}\,, \tag{2.19}$$

we observe that (2.18) may be rewritten as

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\kappa}}\delta\omega^{\kappa} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\kappa}}\left(\sqrt{-g}\delta\omega^{\kappa}\right) \tag{2.20}$$

where in the last step (2.16) was used.

Now, remember that $g^{\mu\nu}\delta R_{\mu\nu}$ is part of the four-dimensional integral in (2.1). Since it is a four-divergence, this integral can be rewritten as a surface integral using Gauss' theorem. Since we do not vary the vector $\delta\omega^{\kappa}$ at infinity, this term does not contribute to the overall variation of the action. Thus, we disregard $g^{\mu\nu}\delta R_{\mu\nu}$, and obtain Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{2.21}$$

2.2 Palatini variation

In the Palatini variation method, the Einstein-Hilbert action, (2.1), is independently varied with respect to both the metric tensor and the Christoffel symbol (also called the *connection* in this context). In other words, the connection and metric tensor are assumed to be independent field variables.

The Palatini variational method offers a way to derive Einstein's field equation from the Lagrangian of gravity, with fewer assumptions than the normal variation with respect to the metric tensor. Following the Palatini variation method, one does not assume any specific form or relation between the metric and the connection. One will therefore get two sets of field equations, instead of one. One of these sets of field equations will be the usual Einstein field equation. The other set of equations that one obtains will define the connection, and confirm its form that was assumed, (B.2), in the previous derivation (variation with respect to the metric tensor).

Without assuming any relation between the connection and the metric tensor, the Einstein-Hilbert action, (2.1), becomes

$$S_{\mathrm{EH}} = \int \mathrm{d}^4 x \, \left[\frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu}(\Gamma) + \mathscr{L}_M\right] \sqrt{-g}. \tag{2.22}$$

Varying (2.22) with respect to the metric tensor yields the same result as in the last section, (2.21), because

$$\frac{\delta R_{\mu\nu}(\Gamma)}{\delta g^{\mu\nu}} = 0 \tag{2.23}$$

with the assumptions we started with. When varying (2.22) with respect to the connection, we get

$$\begin{aligned} \delta S_{\mathrm{EH}} &= \int \mathrm{d}^4 x \, \delta \left[\frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu}(\Gamma) + \mathscr{L}_M\right] \sqrt{-g} \\ &= \int \mathrm{d}^4 x \, \frac{1}{2\kappa} \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \tag{2.24}$$

Now, since

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\rho}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} \tag{2.25}$$

it directly follows that

$$\delta R_{\mu\nu} = \delta\partial_\rho \Gamma^{\rho}_{\mu\nu} - \delta\partial_\nu \Gamma^{\rho}_{\mu\rho} + \Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\sigma\rho}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\rho}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\rho}_{\sigma\nu}\delta\Gamma^{\sigma}_{\mu\rho}. \tag{2.26}$$

Since the partial derivatives and the variation commute, (C.4), we can rearrange the first two terms of $\delta R_{\mu\nu}$ into

$$\delta\partial_\rho \Gamma^{\rho}_{\mu\nu} - \delta\partial_\nu \Gamma^{\rho}_{\mu\rho} = \partial_\rho \delta\Gamma^{\rho}_{\mu\nu} - \partial_\nu \delta\Gamma^{\rho}_{\mu\rho}\,. \tag{2.27}$$

Performing a partial integration, we obtain

$$\begin{aligned} \delta S_{\mathrm{EH}} = \int \mathrm{d}^4 x \, \Big[&- \frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^{\mu\nu}) \delta\Gamma^{\rho}_{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\nu}) \delta\Gamma^{\rho}_{\mu\rho} \\ &+ g^{\mu\nu} \left(\Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\sigma\rho}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\rho}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\rho}_{\sigma\nu}\delta\Gamma^{\sigma}_{\mu\rho}\right) \Big] \sqrt{-g}. \end{aligned} \tag{2.28}$$

To obtain the equations of motion for this variation, it is necessary to rearrange some labels. This rearranging needs to be done in such a way that every term inside the square brackets of (2.28) has the same variation as a common factor. In the following equations, (2.29) to (2.34), it is shown how to do this for each term,

$$-\frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^{\mu\nu}) \delta\Gamma^{\rho}_{\mu\nu} = \left[-\frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} g^{\mu\nu})\right] \delta\Gamma^{\rho}_{\mu\nu}\,, \tag{2.29}$$

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\nu}) \delta\Gamma^{\rho}_{\mu\rho} = \left[\frac{1}{\sqrt{-g}} \partial_\gamma (\sqrt{-g} g^{\mu\gamma}) \delta^{\nu}_{\rho}\right] \delta\Gamma^{\rho}_{\mu\nu}\,, \tag{2.30}$$

$$g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\rho}_{\sigma\rho} = g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\delta\Gamma^{\rho}_{\mu\rho} = \left[g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\delta^{\nu}_{\rho}\right]\delta\Gamma^{\rho}_{\mu\nu}\,, \tag{2.31}$$

$$g^{\mu\nu}\Gamma^{\rho}_{\sigma\rho}\delta\Gamma^{\sigma}_{\mu\nu} = g^{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\delta\Gamma^{\rho}_{\mu\nu} = \left[g^{\mu\nu}\Gamma^{\sigma}_{\rho\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu}\,, \tag{2.32}$$

$$-g^{\mu\nu}\Gamma^{\sigma}_{\mu\rho}\delta\Gamma^{\rho}_{\sigma\nu} = -g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho}\delta\Gamma^{\rho}_{\mu\nu} = \left[-g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho}\right]\delta\Gamma^{\rho}_{\mu\nu}\,, \tag{2.33}$$

$$-g^{\mu\nu}\Gamma^{\rho}_{\sigma\nu}\delta\Gamma^{\sigma}_{\mu\rho} = -g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}\delta\Gamma^{\rho}_{\mu\nu} = \left[-g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}\right]\delta\Gamma^{\rho}_{\mu\nu}. \tag{2.34}$$

Combining all the terms in the brackets of the previous block of equations, we construct a tensor

$$\begin{aligned} A^{\mu\nu}_{\rho} = &- \frac{1}{\sqrt{-g}}\partial_{\rho}(\sqrt{-g}g^{\mu\nu}) + \frac{1}{\sqrt{-g}}\partial_{\gamma}(\sqrt{-g}g^{\mu\gamma})\delta^{\nu}_{\rho} \\ &+ g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\delta^{\nu}_{\rho} + g^{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho} - g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}\,. \end{aligned} \tag{2.35}$$

The variation of the action can then be rewritten in a more compact form as

$$\delta S_{\text{EH}} = \int \mathrm{d}^4x\,\sqrt{-g}A^{\mu\nu}_{\rho}\delta\Gamma^{\rho}_{\mu\nu}\,. \tag{2.36}$$

Since the variation, $\delta\Gamma^{\rho}_{\mu\nu}$ is arbitrary, the variational principle implies that

$$A^{\mu\nu}_{\rho} = 0\,. \tag{2.37}$$

If we divide $A^{\mu\nu}_{\rho}$ into two parts, we can more easily use some symmetry arguments to ease our further calculations. Let

$$B^{\mu\nu}_{\rho} = \left[\frac{1}{\sqrt{-g}}\partial_{\gamma}(\sqrt{-g}g^{\mu\gamma}) + g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\right]\delta^{\nu}_{\rho}\,, \tag{2.38}$$

$$C^{\mu\nu}_{\rho} = -\frac{1}{\sqrt{-g}}\partial_{\rho}(\sqrt{-g}g^{\mu\nu}) + g^{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho} - g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}\,, \tag{2.39}$$

$$A^{\mu\nu}_{\rho} = B^{\mu\nu}_{\rho} + C^{\mu\nu}_{\rho}\,. \tag{2.40}$$

Since $\Gamma^{\rho}_{\mu\nu}$ is symmetric in the lower indices, $A^{\mu\nu}_{\rho}$ must also be symmetric in the upper indices. It is obvious that $C^{\mu\nu}_{\rho}$ is symmetric in the lower indices, so $B^{\mu\nu}_{\rho}$ must by extension also be symmetric in its lower indices. All of this adds together to the conclusion that

$$B^{\mu\nu}_{\rho} = 0\,,\ \forall\,\mu,\nu,\rho \tag{2.41}$$

and we obtain that

$$A^{\mu\nu}_{\rho} = C^{\mu\nu}_{\rho} = 0. \tag{2.42}$$

If we contract the indices, μ,ν, on $C^{\mu\nu}_{\rho}$, as such

$$\begin{aligned} g_{\mu\nu}C^{\mu\nu}_{\rho} &= g_{\mu\nu}g^{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - g_{\mu\nu}g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho} - g_{\mu\nu}g^{\mu\alpha}\Gamma^{\nu}_{\alpha\rho} - \frac{1}{\sqrt{-g}}g_{\mu\nu}\partial_{\rho}(\sqrt{-g}g^{\mu\nu}) \\ &= 4\Gamma^{\sigma}_{\rho\sigma} - \delta^{\alpha}_{\mu}\Gamma^{\mu}_{\alpha\rho} - \delta^{\alpha}_{\nu}\Gamma^{\nu}_{\alpha\rho} - \frac{1}{\sqrt{-g}}g_{\mu\nu}\partial_{\rho}(\sqrt{-g}g^{\mu\nu}) \\ &= 4\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\sigma} - \frac{1}{\sqrt{-g}}g_{\mu\nu}\partial_{\rho}(\sqrt{-g}g^{\mu\nu}) \\ &= 0 \end{aligned} \tag{2.43}$$

we obtain

$$2\Gamma^{\sigma}_{\rho\sigma} = \frac{1}{\sqrt{-g}} g_{\mu\nu}\partial_\rho(\sqrt{-g}g^{\mu\nu}). \tag{2.44}$$

Now, as is already known from (2.6),

$$\frac{1}{\sqrt{-g}}\partial_\rho\sqrt{-g} = -\frac{1}{2}g_{\mu\nu}\partial_\rho g^{\mu\nu}\,. \tag{2.45}$$

By inserting (2.45) into (2.44) we obtain

$$\begin{aligned} 2\Gamma^{\sigma}_{\rho\sigma} &= \frac{1}{\sqrt{-g}} g_{\mu\nu}g^{\mu\nu}\partial_\rho\sqrt{-g} + g_{\mu\nu}\partial_\rho g^{\mu\nu} \\ &= \frac{4}{\sqrt{-g}}\partial\sqrt{-g} - \frac{2}{\sqrt{-g}}\partial_\rho\sqrt{-g} \\ &= \frac{2}{\sqrt{-g}}\partial_\rho\sqrt{-g}\,. \end{aligned} \tag{2.46}$$

Hence,

$$\Gamma^{\sigma}_{\rho\sigma} = \frac{1}{\sqrt{-g}}\partial_\rho\sqrt{-g}\,, \tag{2.47}$$

which we insert back into the equation of motion, $C^{\mu\nu}_{\rho} = 0$, to obtain

$$g^{\mu\nu}_{,\rho} + g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho} + g^{\mu\alpha}\Gamma^{\nu}_{\alpha\rho} = 0 \tag{2.48}$$

which is the requirement that the connection is the Christoffel symbol [9, pp.61-62],

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)\,. \tag{2.49}$$

Hence, we obtain the same result as in the previous section; The Einstein field equations. However, this time we have also derived the form of the Christoffel symbols (the connection), and as mentioned in the introduction of this section, derived the Einstein field equations with fewer assumptions than what could be done with a variation of the metric alone.

Chapter 3

The Schwarzschild solution

A classic solution to the Einstein equation is the Schwarzschild solution which describes a Schwarzschild black hole. It is also a quite simple example that is useful for developing an intuition for the mathematics of general relativity, and the methods that are used. The Schwarzschild is a vacuum solution for a spherically symmetric, static metric to the Einstein field equation. The solution will describe the metric in the vacuum around some point mass M. The solution was first published by Schwarzschild in 1916.

The following derivation was greatly aided by Gary Oas's derivation of the Scwarzschild solution [10], but has expanded on some points that were not covered in his notes.

3.1 The form of the metric

Einstein's field equation, which was derived in Chapter 2, is given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \, . \tag{3.1}$$

We are looking for the vacuum solution. Therefore, the equation simplifies to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \, . \tag{3.2}$$

Due to the fact that we are looking for a static solution, the coordinate transformation $t \to -t$ should leave the metric unchanged. This argument also applies to the transformations $\theta \to -\theta$ and $\phi \to -\phi$, when working in spherical coordinates, because of the requirement that the solution is spherically symmetric (invariant under rotations). The off-diagonal metric components change signs under these coordinate transformations. Since they change signs, and should remain unchanged, the only solution is that off-diagonal elements of the metric are equal to zero. Since the metric is diagonal, we immediately recognize that the Ricci tensor is also diagonal. In other words,

$$R_{\mu\nu} = 0, \qquad \mu \neq \nu \, . \tag{3.3}$$

The definitions for the Christoffel symbols, Riemann tensor, Ricci tensor, and Ricci scalar can be found in Appendix B. We may freely assume that the metric should become the metric for flat spacetime when at large r. The line element for a flat space-time is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \,, \tag{3.4}$$

where

$$g_{00} = -1 \qquad g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta \,. \tag{3.5}$$

Generalizing the flat space-time line element to represent the problem in question yields

$$\mathrm{d}s^2 = -U\mathrm{d}t^2 + V\mathrm{d}r^2 + Wr^2\mathrm{d}\theta^2 + Xr^2 \sin^2\theta \mathrm{d}\phi^2 \,. \tag{3.6}$$

Now, since we require spherical symmetry and that the metric is static, we have to require that all of the functions, U, V, W, and X, are functions of r only. Additionally, spherical symmetry dictates that $W = X$. It turns out that we may even set $W = X = 1$, without any loss of generality. To see why, one can perform a rescaling, $R = \sqrt{W}r$, in (3.6). Calculating the new differential yields $\mathrm{d}R = \mathrm{d}r/(W^{-1/2} - RW^{-3/2}/2)$. By direct insertion and some light algebra, one obtains the rescaled line element,

$$\mathrm{d}s^2 = -\mathcal{U}(R)\mathrm{d}t^2 + \mathcal{V}(R)\mathrm{d}R^2 + R^2\mathrm{d}\theta^2 + R^2 \sin^2\theta \mathrm{d}\phi^2 \,, \tag{3.7}$$

hence, it is shown that $W = X = 1$ is a valid choice.

3.2 Solving the problem

Henceforth, we will continue with the general line element,

$$\mathrm{d}s^2 = -U(r)\mathrm{d}t^2 + V(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,, \tag{3.8}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. The non-zero metric components are

$$g_{00} = -U(r) \qquad g_{11} = V(r) \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta \,. \tag{3.9}$$

Due to $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\nu$, where δ^μ_ν is the identity matrix, it is obvious that

$$g^{00} = -\frac{1}{U(r)} \qquad g^{11} = \frac{1}{V(r)} \qquad g^{22} = \frac{1}{r^2} \qquad g^{33} = \frac{1}{r^2 \sin^2\theta} \,. \tag{3.10}$$

From the metric, we can calculate the Christoffel symbols using (B.2). By direct insertion and some light algebra, the non-zero Christoffel symbols are obtained,

$$\begin{aligned}
\Gamma^0_{01} = \Gamma^0_{10} &= \frac{U'}{2U}\,, \\
\Gamma^1_{00} &= \frac{U'}{2V}\,, \\
\Gamma^1_{11} &= \frac{V'}{2V}\,, \\
\Gamma^1_{22} &= -\frac{r}{V}\,, \\
\Gamma^1_{33} &= -\frac{r}{V}\sin^2\theta\,, \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r}\,, \\
\Gamma^2_{33} &= -\cos\theta\sin\theta\,, \\
\Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{r}\,, \\
\Gamma^3_{23} = \Gamma^3_{32} &= \cot\theta\,,
\end{aligned} \tag{3.11}$$

where the prime symbol indicates a differentiation with respect to r, e.g. $U' = \partial_r U$. As mentioned earlier, we infer from the Einstein vacuum equations that only the diagonal components of the Ricci tensor can be non-zero. These are given by

$$\begin{aligned}
R_{00} &= -\frac{U''}{2V} + \frac{U'V'}{4V^2} + \frac{(U')^2}{4UV} - \frac{1}{r}\frac{U'}{V}\,, \\
R_{11} &= \frac{U''}{2U} - \frac{(U')^2}{4U^2} - \frac{U'V'}{4UV} - \frac{V'}{rV}\,, \\
R_{22} &= \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1\,, \\
R_{33} &= \sin^2\theta R_{22}\,.
\end{aligned} \tag{3.12}$$

Using (B.4) and (3.12), the Ricci scalar is obtained

$$R = \frac{U''}{UV} - \frac{U'V'}{2UV^2} - \frac{(U')^2}{2U^2V} + \frac{2U'}{rUV} - \frac{2V'}{rV^2} - \frac{2}{r^2} + \frac{2}{r^2V}\,. \tag{3.13}$$

By inserting the Ricci scalar and the Ricci tensor into the Einstein equation, we can find the four nontrivial equations in terms of r, U, and V;

$$R_{00} - \frac{1}{2}g_{00}R = \frac{V'}{rV^2} + \frac{1}{r^2} - \frac{1}{r^2V} = 0\,, \tag{3.14}$$

$$R_{11} - \frac{1}{2}g_{11}R = -\frac{U'}{rUV} + \frac{1}{r^2} - \frac{1}{r^2V} = 0\,, \tag{3.15}$$

$$R_{22} - \frac{1}{2}g_{22}R = -\frac{U'}{U} + \frac{V'}{V} - \frac{rU''}{U} + \frac{rU'V'}{2UV} + \frac{r(U')^2}{2U} = 0\,, \tag{3.16}$$

$$R_{33} - \frac{1}{2} g_{33} R = R_{22} + \frac{r^2}{2} R = 0 \tag{3.17}$$

We want to find V and U. The first Einstein equation, (3.14), can be rewritten as

$$V' = \frac{V}{r} - \frac{V^2}{r} \,. \tag{3.18}$$

Solving this differential equation by separation of variables yields

$$\ln \frac{V-1}{V} + C = -\ln r = \ln \frac{1}{r} \,, \tag{3.19}$$

where C is some integration constant. By solving for V, we obtain

$$V = \frac{1}{1 - C/r} \,. \tag{3.20}$$

To find the form of U, insert (3.20) into (3.15), and rearrange to find

$$\frac{U'}{U} = \frac{1}{r - C} - \frac{1}{r} \,. \tag{3.21}$$

Again, solving the differential equation by separation of variables yields

$$\ln U = \ln(1 - C/r) + K \,, \tag{3.22}$$

where K is another integration constant. Solving for U, we obtain

$$U = k(1 - C/r) \,, \tag{3.23}$$

where $k = e^K$. Hence, our line element, (3.8) is

$$\mathrm{d}s^2 = -k(1 - C/r)\mathrm{d}t^2 + (1 - C/r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \,. \tag{3.24}$$

Since the line element has to reduce to the line element in flat space in the $r \to \infty$ limit, k has to be equal to one. Also, we expect that in the absence of any mass, $M \to 0$, the same flat line element should be recovered. We see that $C \to 0$ achieves that, and identify $C \propto M^n$ (where $n > 0$), as it is the only free parameter left which we are free to choose.

3.3 Comparison to Kepler's orbits

To get the correct form of C, we write down the line element in SI units,

$$\mathrm{d}s^2 = -(1 - C/r)c^2\mathrm{d}t^2 + \frac{1}{1 - C/r}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \,, \tag{3.25}$$

and compare the geodesics resulting from this metric to already known physics. To do that, we rewrite the metric in terms of mean time, τ,

$$\begin{aligned} \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 &= -c^2 \\ &= -\left(1 - \frac{C}{r}\right) c^2 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{1 - \frac{C}{r}} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \,, \end{aligned} \tag{3.26}$$

which we use to construct an arc-length integral of the path,

$$
\begin{aligned}
I &= \int_{\tau_1}^{\tau_2} \sqrt{-\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2}\,\mathrm{d}\tau \\
&= \int_{\tau_1}^{\tau_2} \sqrt{c^2\left(1-\frac{C}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{1-\frac{C}{r}}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 - r^2\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2}\,\mathrm{d}\tau\,.
\end{aligned}
\tag{3.27}
$$

Next, we make the choice to orient our coordinate system in such a way that a test particle lies in the plane defined by $\theta = \pi/2$. Hence, our integral reduces to

$$
I = \int_{\tau_1}^{\tau_2} \sqrt{c^2\left(1-\frac{C}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{1}{1-\frac{C}{r}}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2}\,\mathrm{d}\tau\,. \tag{3.28}
$$

Now, the goal of this is to acquire the equation of motion of a test particle through the Euler-Lagrange equations[1]. Since the Euler-Lagrange equations are unchanged when the integrand is multiplied by a constant, we can write the integrand simply as $(\mathrm{d}s/\mathrm{d}\tau)^2$ (Remember that $(\mathrm{d}s/\mathrm{d}\tau)^2 = -c^2$). Now, (3.28) appears as

$$
I = \int_{\tau_1}^{\tau_2} \left[-c^2\left(1-\frac{C}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{1}{1-\frac{C}{r}}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right]\mathrm{d}\tau\,. \tag{3.29}
$$

When we apply the Euler-Langrange equations on (3.29), we obtain the following equations of motion

$$
-c^2\frac{C}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \frac{C}{(C-r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + 2r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = \frac{2C}{(C-r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{2}{1-\frac{C}{r}}\frac{\mathrm{d}^2r}{\mathrm{d}\tau^2}\,, \tag{3.30}
$$

$$
0 = 2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + 2r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}\,, \tag{3.31}
$$

$$
0 = -2c^2\frac{C}{r^2}\frac{\mathrm{d}t}{\mathrm{d}\tau} - 2c^2\left(1-\frac{C}{r}\right)\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2}\,. \tag{3.32}
$$

Let us focus on a case of circular motion. In such a case, $\mathrm{d}r/\mathrm{d}\tau = \mathrm{d}^2r/\mathrm{d}\tau^2 = 0$. Hence, (3.30) is then

$$
-c^2\frac{C}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 0\,, \tag{3.33}
$$

which can be solved for C as

$$
C = \frac{2r^3}{c^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\,. \tag{3.34}
$$

[1]Read more about the Euler-Lagrange equations in Appendix C.1.2

Now, Kepler's third law of motion is

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}\,, \tag{3.35}$$

where T is the period of the orbit, a is the length of the semi-major axis of the orbit, G is the gravitational constant, M is the mass of the object which we derived our metric about, and m is the mass of the test particle. For a circular orbit of a test particle with negligible mass, Kepler's third law of motion becomes

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}\,. \tag{3.36}$$

The period of a circular orbit is

$$T = \frac{2\pi}{\mathrm{d}\phi/\mathrm{d}t}\,, \tag{3.37}$$

which we plug into (3.36) to obtain

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{GM}{r^3}\,. \tag{3.38}$$

Now, we use (3.38) in (3.34) to obtain

$$C = \frac{2r^3}{c^2}\frac{GM}{r^3} = \frac{2GM}{c^2}\,. \tag{3.39}$$

Thus, we have solved for C, and can write down the Schwarzschild solution in SI units,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 + \frac{1}{1-\frac{2GM}{c^2 r}}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\,. \tag{3.40}$$

Chapter 4

Weak field approximation

The Einstein equation can be hard to work with, due to it not being necessarily linear. However, in many applications, some details of a system may be negligible. In such systems, some simplifications may be made. The weak field approximation is such a method. By assuming that the perturbations from a ground systems, around some point, are sufficiently small, we may omit terms of a high enough order. In turn, linearization of the theory may be achieved. In succeeding to linearize the Einstein equation, one may find the value of the constant of the Schwarzschild metric, as we will do later in this chapter. One may also develop theories of gravity in specific regimes by this linearized approach. Since the weak field approximation assumes small approximations, the regime that theories that utilize this method is a low energy regime. Now, let us delve into the method of weak field approximation.

4.1 The weak field approximation method

Consider almost flat space-time. A metric of an almost flat space-time can be descirbed as some weak perturbation term, $\kappa h_{\mu\nu}$, where $\kappa h_{\mu\nu} \ll \eta_{\mu\nu}$, added to the metric of flat space-time, $\eta_{\mu\nu}$. Explicitly, that is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + 4\kappa h_{\mu\nu} \,, \tag{4.1}$$

where $\kappa = 8\pi G c^{-4}$ is a constant. The number 4 is there only for convenience and synergy with the other chapters of this work. The approximation requires that we revisit the Ricci tensor and the Ricci scalar. It is necessary to express these entites to first order in $h_{\mu\nu}$ (since the $h_{\mu\nu}$ term is considered weak). Since we omit second-order terms from the calculation, and due to the fact that $g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}_{\nu}$, it follows that

$$g^{\mu\nu} = \eta^{\mu\nu} - 4\kappa h^{\mu\nu} \,. \tag{4.2}$$

The Ricci tensor is given in (B.3). The last two terms will automatically be of order $\mathscr{O}(h^2)$, and will therefore be omitted from the calculations. Additionally, any differential of the flat space metric at any point is always equal to zero. Hence, the Christoffel symbols, (B.2) in the weak field approximation is

$$\Gamma^{\lambda}_{\mu\nu} = 2\kappa \left(\eta^{\lambda\alpha} - 4\kappa h^{\lambda\alpha}\right) \left(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\right) . \tag{4.3}$$

As before, we are only looking at the first order perturbations of the field, and the $h^{\lambda\alpha}$ term is therefore not going to be included. Hence, the christoffel symbols become

$$\Gamma^{\lambda}_{\mu\nu} = 2\kappa \eta^{\lambda\alpha} \left(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\right) . \tag{4.4}$$

In the weak field approximation, raising and lowering of indices of the perturbation term, $h_{\mu\nu}$, can be done by using the flat space metric. The raising and lowering of indices is done by the metric tensor. However, in the weak field approximation, due to only doing calculations to first order, the raising and lowering of indices can be done by the flat space metric alone. This is easily shown by considering

$$\begin{aligned} g^{\alpha\gamma} h_{\gamma\beta} &= \left(\eta^{\alpha\gamma} - 4\kappa h^{\alpha\gamma}\right) h_{\gamma\beta} \\ &= \eta^{\alpha\gamma} h_{\gamma\beta} - 4\kappa h^{\alpha\gamma} h_{\gamma\beta} . \end{aligned} \tag{4.5}$$

Since we are only looking at first order terms, the second term in (4.5) vanishes, and we obtain

$$g^{\alpha\gamma} h_{\gamma\beta} = \eta^{\alpha\gamma} h_{\gamma\beta} . \tag{4.6}$$

By applying what we now know about the raising and lowering of indices in the weak-field approximation, (4.6), to (4.4), we obtain

$$\Gamma^{\lambda}_{\mu\nu} = 2\kappa \left({h^{\lambda}}_{\mu,\nu} + {h^{\lambda}}_{\nu,\mu} - {h_{\mu\nu,}}^{\lambda}\right) . \tag{4.7}$$

By inserting (4.7) into (B.3), we calculate the Ricci tensor

$$R_{\mu\nu} = 2\kappa \left({h^{\rho}}_{\nu,\mu\rho} + {h^{\rho}}_{\mu,\nu\rho} - \Box h_{\nu\mu} - h_{,\mu\nu}\right) , \tag{4.8}$$

where $\Box = \partial^{\rho}\partial_{\rho}$, and $h = {h^{\rho}}_{\rho}$. By insertion, the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$, is

$$R = \left(\eta^{\mu\nu} - 4\kappa h^{\mu\nu}\right) R_{\mu\nu} . \tag{4.9}$$

Since we are only looking at the first order approximation, the term containing $h^{\mu\nu}$ will only lead to second order terms, and will thus be omitted from our calculations. Hence, the Ricci scalar in the weak field approximation is

$$R = \eta^{\mu\nu} R_{\mu\nu} . \tag{4.10}$$

By directly inserting (4.8) into (4.10), we obtain the Ricci scalar,

$$R = 4\kappa \left[{h^{\mu\nu}}_{,\mu\nu} - \Box h\right] . \tag{4.11}$$

We have thus found the Ricci tensor and the Ricci scalar in the weak field approximation.

4.2 Weak field approximation of the Einstein equation

The Einstein equation, (2.21), can now be computed in terms of the new Ricci tensor and Ricci scalar. By direct insertion of (4.7), (4.8), and (4.11), into (2.21), we obtain the Einstein equation in the weak field approximation

$$h^{\rho}{}_{\nu,\mu\rho} + h^{\rho}{}_{\mu,\nu\rho} - \Box h_{\nu\mu} - h_{,\mu\nu} - g_{\mu\nu}\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h\right) = \frac{1}{2}T_{\mu\nu}\,. \tag{4.12}$$

Again, the metric of the last term of the left hand side of (4.12) can be replaced with the flat space metric to avoid any second order terms. We thus obtain

$$h^{\rho}{}_{\nu,\mu\rho} + h^{\rho}{}_{\mu,\nu\rho} - \Box h_{\nu\mu} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h^{\alpha\beta}{}_{,\alpha\beta} - \Box h\right) = \frac{1}{2}T_{\mu\nu}\,. \tag{4.13}$$

The trace of $T_{\mu\nu}$ can be determined from (4.13). We find that

$$\frac{1}{4}T^{\mu}{}_{\mu} = -h^{\alpha\beta}{}_{,\alpha\beta} + \Box h\,, \tag{4.14}$$

By inserting (4.14) into (4.13) we obtain

$$h^{\rho}{}_{\nu,\mu\rho} + h^{\rho}{}_{\mu,\nu\rho} - \Box h_{\mu\nu} - h_{,\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\alpha}{}_{\alpha}\right)\,. \tag{4.15}$$

4.2.1 An attempt at inverting the Einstein equation for the weak field

In order to illustrate the need for some gauge choice to be made at this point, we will see what attempting to solve the system as-is would yield. Therefore, we are attempting to find the Green's function to the current equation of motion. However, as we will see, equation (4.15) does not have any Green's function associated with it. To show this, we rearrange (4.15) as

$$H_{\alpha\beta\mu\nu}h^{\alpha\beta} = f_{\mu\nu}\,, \tag{4.16}$$

where $f_{\mu\nu}$ is the right-hand side of (4.15), and $H_{\alpha\beta\mu\nu}$ is

$$H_{\alpha\beta\mu\nu} = -\Box\eta_{\alpha\mu}\eta_{\beta\nu} - \partial_\mu\partial_n u\eta_{\alpha\beta} + \partial_\alpha\partial_\mu\eta_{\beta\nu} + \partial_\beta\partial_\nu\eta_{\alpha\mu}\,. \tag{4.17}$$

By definition, if a Green's function for this operator exists, it will have to obey

$$H_{\alpha\beta\mu\nu}G^{\mu\nu\gamma\delta} = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}\delta(x - x')\,, \tag{4.18}$$

where δ^{γ}_{α} and δ^{δ}_{β} are Kronecker deltas, and $\delta(x - x')$ is the Dirac delta function. We Fourier transform the equation, $\partial_\sigma \to -ik_\sigma$, and we get

$$H_{\alpha\beta\mu\nu} = k^2\eta_{\alpha\mu}\eta_{\beta\nu} + k_\mu k_n u\eta_{\alpha\beta} - k_\alpha k_\mu\eta_{\beta\nu} - k_\beta k_\nu\eta_{\alpha\mu}\,, \tag{4.19}$$

and our condition for the possible inverse function becomes

$$H_{\alpha\beta\mu\nu}G^{\mu\nu\gamma\delta} = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}\,. \tag{4.20}$$

The most general possible inverse function has the form

$$
\begin{aligned}
G^{\mu\nu\gamma\delta} =& A_1\eta^{\mu\nu}\eta^{\gamma\delta} + A_2\eta^{\mu\gamma}\eta^{\delta\nu} + A_3\eta^{\mu\delta}\eta^{\gamma\nu} \\
&+ B_1k^{\mu}k^{\nu}\eta^{\gamma\delta} + B_2k^{\mu}k^{\gamma}\eta^{\nu\delta} + B_3k^{\mu}k^{\delta}\eta^{\gamma\nu} \\
&+ B_4k^{\gamma}k^{\delta}\eta^{\mu\nu} + B_5k^{\nu}k^{\delta}\eta^{\mu\gamma} + B_6k^{\nu}k^{\gamma}\eta^{\mu\delta} \\
&+ Ck^{\mu}k^{\nu}k^{\gamma}k^{\delta}\,, \qquad (4.21)
\end{aligned}
$$

where A_i, B_i and C are undetermined functions depending on k^2. By inserting (4.21) and (4.19) into (4.20), we obtain

$$
\begin{aligned}
H_{\alpha\beta\mu\nu}G^{\mu\nu\gamma\delta} =& A_1\left[2k^2\eta_{\alpha\beta}\eta^{\gamma\delta} - 2k_{\alpha}k_{\beta}\eta^{\gamma\delta}\right] \\
&+ A_2\left[k^2\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} + k^{\gamma}k^{\delta}\eta_{\alpha\beta} - k_{\alpha}k^{\gamma}\delta^{\delta}_{\beta} - k_{\beta}k_{\alpha}\eta^{\gamma\delta}\right] \\
&+ A_3\left[k^2\delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} + k^{\gamma}k^{\delta}\eta_{\alpha\beta} - k_{\alpha}k^{\delta}\delta^{\gamma}_{\beta} - k_{\beta}k^{\gamma}\delta^{\delta}_{\alpha}\right] \\
&+ B_1\left[k^4\eta_{\alpha\beta}\eta^{\gamma\delta} - k^2k_{\alpha}k_{\beta}\eta^{\gamma\delta}\right] \\
&+ D\left[k^2k^{\gamma}k^{\delta}\eta_{\alpha\beta} - k_{\alpha}k_{\beta}k^{\gamma}k^{\delta}\right]\,, \qquad (4.22)
\end{aligned}
$$

where $D = B_2 + B_3 + B_4/2 + B_5 + B_6$. We immediately recognize that the second block containing A_2 does contain a term with the correct solution. But upon closer inspection, we also notice that this block does also contain a unique term not contained in any other block. It is the third term in the A_2 block, $-k_{\alpha}k^{\gamma}\delta^{\delta}_{\beta}$. Since this term is unique, there is no possible way to get the right result, by any choice of the undetermined functions. It is thus shown that there is no $G^{\mu\nu\gamma\delta}$ which obeys our condition, (4.20). In other words, we have shown that (4.15) is not invertible.

4.2.2 The Lorentz gauge

To solve equation (4.15), we must choose a coordinate gauge. As it stands, (4.15) is completely general, and therefore does not impose any coordinate system. To find the form of $h_{\mu\nu}$, we must first choose a coordinate gauge. This coordinate gauge will impose four conditions on $h_{\mu\nu}$. From [7, pp.461-463], we observe that it is possible to choose coordinates where the form of the background spacetime, $\eta_{\mu\nu}$, is conserved, but the (still undetermined) perturbative field is changed. If one considers a coordinate transformation

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x)\,, \qquad (4.23)$$

then that leaves the background spacetime untouched, but transforms the perturbative field as

$$h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\beta,\alpha} - \xi_{\alpha,\beta}\,. \qquad (4.24)$$

Choosing four coordinate conditions, $V_{\alpha}(x) = V'_{\alpha}(x) = 0$, can now be done using the arbitrary, but small, functions, $\xi^{\alpha}(x)$. Since the background spacetime is unperturbed, we know that the change in $h_{\mu\nu} \to h'_{\mu\nu}$ is the same change as the one in $g_{\mu\nu} \to g'_{\mu\nu}$. We may choose what Hartle calls the Lorentz gauge,

$$\partial^{\lambda}h_{\alpha\lambda} = \frac{1}{2}\partial_{\alpha}h\,, \qquad (4.25)$$

provided that $\Box\xi^\alpha = 0$.

As we have seen in this section, a coordinate change will change the metric, but keep the physics unchanged. It is by that reason close to what one calls a gauge choice in electrodynamics. One can easily see the similarity between the Lorentz gauge choice in electrodynamics, and the Lorentz gauge presented here, as a natural four-dimensional generalization of the former.

Choosing to work in the Lorentz gauge, the field equation, (4.15), simplifies to

$$\Box h_{\mu\nu} = -\frac{1}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\alpha{}_\alpha\right) . \tag{4.26}$$

4.3 Weak field approximation for a point mass

It is now useful to look at how we might approximate the space-time around some point mass. When the metric for the point mass is found, we will compare it to the metric that was found for the similar system that was considered in Chapter 3. The point mass we will consider is a static, spherically symmetrical point mass. Due to the symmetries of the system, it is convenient to work in spherical coordinates. The stress-energy tensor for a point mass centered in the origin of the spherical coordinate system we choose to work in, is given by

$$T_{\mu\nu} = \begin{cases} Mc^2\delta^3(\boldsymbol{r}), & \text{if } \mu,\nu = 0 \\ 0, & \text{otherwise} . \end{cases} \tag{4.27}$$

Therefore, we end up with four equations given by

$$\Box h_{00} = -\frac{1}{4}Mc^2\delta^3(\boldsymbol{r}) , \tag{4.28}$$

$$\Box h_{ii} = -\frac{1}{4}Mc^2\delta^3(\boldsymbol{r}) . \tag{4.29}$$

Since we are considering a static point mass, the metric perturbation that describes it, $h_{\mu\nu}$, is constant through time. Therefore, the time-derivative part of the equation vanishes, $h_{\mu\nu,0} = 0$. As a result, the equations of motion simplify to

$$\nabla^2 h_{00} = -\frac{1}{4}Mc^2\delta^3(\boldsymbol{r}) , \tag{4.30}$$

$$\nabla^2 h_{ii} = -\frac{1}{4}Mc^2\delta^3(\boldsymbol{r}) . \tag{4.31}$$

The current form of the equations of motion are convenient, since we know that the dirac delta function can be written as the Laplacian of inverted radius. From (D.5) we know that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\boldsymbol{r}) . \tag{4.32}$$

Thus, we see that the solutions of the perturbations for this point mass are given by

$$h_{00} = \frac{Mc^2}{16\pi r} , \tag{4.33}$$

$$h_{ii} = \frac{Mc^2}{16\pi r}\,. \tag{4.34}$$

The complete metric with the perturbation caused by the point mass is thus given by

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + 4\kappa h_{\mu\nu} \\ &= \mathrm{diag}\left(-1 + \frac{2GM}{c^2 r}, 1 + \frac{2GM}{c^2 r}, 1 + \frac{2GM}{c^2 r}, 1 + \frac{2GM}{c^2 r}\right)\,. \end{aligned} \tag{4.35}$$

Hence, we can write down the line element,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 + \frac{2GM}{c^2 r}\right) \mathrm{d}\sigma^2\,, \tag{4.36}$$

where $\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$, depending on which sets of coordinates one wishes to use. The solution is a metric displayed in *isotropic coordinates*, meaning that all the spatial parts of the metric are the same, $g_{11} = g_{22} = g_{33}$.

This metric is indeed the same as the Schwarzschild metric in isotropic form, if one expands the Schwarzschild metric to first order as shown in [1, pp. 174-177]. The detailed calculations are shown in Appendix F.1. By expanding (F.16) into first order of m/ρ (by assuming that $m \ll \rho$), and writing it out in the same units that we have used throughout this chapter ($G \neq 1, c \neq 1$, and using r instead of ρ), we find that

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 + \left(1 + \frac{2GM}{c^2 r}\right) \mathrm{d}\sigma^2\,. \tag{4.37}$$

We clearly see that in this approximation, the Schwarzschild line element and the weak field approximation of a point mass, M, yield equivalent results.

The weak field approximation lends us a useful tool in the linearization of the Einstein equation. It also provides more insight into specific systems which was shown in this chapter. We were able to use the weak field approximation to determine the constants of the Schwarzschild line element, instead of resorting to a comparison to Kepler's laws. Furthermore, the linearization is immensely helpful when developing a more detailed theory, which is explored in the following chapters of this work. The exercise of finding the metric for a point mass serves as a good introduction to the linearized approach to general relativity.

Chapter 5

Higher-order terms of the action

Our treatment of general relativity has up until this point revolved around the standard Einstein-Hilbert action, (2.1). This action was chosen because it maintains Lorentz invariance, and also, it is a scalar quantity (A necessary condition for an action integral). The only term was thus $\sqrt{-g}R$. It also succeeded in providing a starting point from which to derive Einstein's equation. However, there are other ways to maintain the Lorentz invariance of the action integral constructed from Riemann tensors, and still obtain a scalar quantity. This does require the addition of higher-order terms of the action, such as R^2, $R^{\mu\nu}R_{\mu\nu}$, and so on. In fact, under the arguments so far considered, higher order terms should be included in a more precise action. There is no good argument for why only the first order term should be included, since the first order term and also the higher order terms satisfy the original argument; That the scalar action remains Lorentz invariant. It is therefore necessary to perform an inquiry of what effects the inclusion of higher-order terms in the action integral will have.

5.1 Finding the equation of motion

In light of the introductory discussion, a more complete action integral should be written as

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x \, \left[\frac{c_0}{\kappa}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots + \mathscr{L}_M\right]\sqrt{-g}\,. \tag{5.1}$$

where the c_i are constants.[1]

Before we go further, it is beneficial to make use of the Gauss-Bonnet theorem. One can learn about this theorem from [2, Ch. 12]. The derivation of the theorem is beyond the scope of this thesis. The importance to us, is that by the Gauss-Bonnet theorem, the term that consists of Riemann tensors can be written in terms of Ricci tensors and Ricci scalars,

[1]Obviously, the constants may have dimensions, and should maintain the terms' dimensions in such a way that they are the same dimension as the overall Lagrangian.

i.e.

$$\int \mathrm{d}^4x\, c_3' R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \int \mathrm{d}^4x \left[c_1' R^2 + c_2' R_{\mu\nu} R^{\mu\nu} \right] . \tag{5.2}$$

Hence, we can write equation (5.1) as

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x \left[\frac{c_0}{\kappa} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \cdots + \mathscr{L}_M \right] \sqrt{-g} \, . \tag{5.3}$$

In order to find the effects of including higher order terms in the Lagrangian, we have to find the equations of motion for the system and look at the differences that arise with respect to the Einstein equation (Which should be seen as a first-order approximation).

As usual, the procedure to find the equation of motion is to perform a variation of the action integral and require that the variation is equal to zero. In the following, we will neglect all orders of R higher than two. By applying a variation on the action integral, we obtain

$$\begin{aligned} \delta S_{\mathrm{EH}} = \int \mathrm{d}\omega \Bigg[& \left(\frac{c_0}{\kappa} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} \right) \frac{\delta\sqrt{-g}}{\sqrt{-g}} \\ & + \left(\frac{c_0}{\kappa} \delta R + c_1 \delta(R^2) + c_2 \delta(R_{\mu\nu} R^{\mu\nu}) \right) + \frac{\delta(\mathscr{L}_M \sqrt{-g})}{\sqrt{-g}} \Bigg] , \end{aligned} \tag{5.4}$$

where $\mathrm{d}\omega = \mathrm{d}^4x\, \sqrt{-g}$. To proceed, we have to find the variations of $\sqrt{-g}$, R, R^2, and $R_{\mu\nu}R^{\mu\nu}$. We already know from (2.9) that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \, , \tag{5.5}$$

and we easily see that

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \, . \tag{5.6}$$

Furthermore, we find δR^2 in the same manner as we found δR,

$$\delta R^2 = 2R\delta R = 2R(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}) \, . \tag{5.7}$$

Lastly, we find $\delta(R_{\mu\nu}R^{\mu\nu})$,

$$\begin{aligned} \delta(R_{\mu\nu} R^{\mu\nu}) &= \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta R^{\mu\nu} \\ &= \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \delta(g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta}) \\ &= \delta R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu} \left(\delta g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} + g^{\mu\alpha} \delta g^{\nu\beta} R_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} \delta R_{\alpha\beta} \right) \\ &= 2 \left[\delta R_{\mu\nu} R^{\mu\nu} + g^{\alpha\beta} R_{\alpha\nu} R_{\beta\mu} \delta g^{\mu\nu} \right] \, . \end{aligned} \tag{5.8}$$

By substituting equations (5.5) – (5.8) into (5.4), and identifying the last term with the stress-energy tensor, we obtain

$$\begin{aligned} \delta S_{\mathrm{EH}} = \int \mathrm{d}\omega \Bigg[& -\frac{1}{2} g_{\mu\nu} \left(\frac{c_0}{\kappa} R + c_1 R^2 + c_2 R_{\alpha\beta} R^{\alpha\beta} \right) \\ & + \left(\frac{c_0}{\kappa} R_{\mu\nu} + 2c_1 R R_{\mu\nu} + 2c_2 g^{\alpha\beta} R_{\alpha\nu} R_{\beta\mu} \right) \\ & + \left(\frac{c_0}{\kappa} g^{\mu\nu} + 2c_1 R g^{\mu\nu} + 2c_2 R^{\mu\nu} \right) \frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}} - \frac{1}{2} T_{\mu\nu} \Bigg] \delta g^{\mu\nu} \, , \end{aligned} \tag{5.9}$$

The terms with variation $\delta R_{\mu\nu}$ need to be expressed as a variation of $\delta g^{\mu\nu}$ instead before we can find the equations of motion. Therefore, we will only focus on that term in the following argument. Let us call the term δS_{EH3},

$$\delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \left(\frac{c_0}{\kappa} g^{\mu\nu} + 2c_1 R g^{\mu\nu} + 2c_2 R^{\mu\nu} \right) \delta R_{\mu\nu} \,. \tag{5.10}$$

From our discussion in the last parts of chapter 2.1, we recall that the first term of δS_{EH3} does not contribute to the variation. That term may thus be omitted, and we obtain

$$\delta S_{\mathrm{EH3}} = 2 \int \mathrm{d}\omega \, \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu} \right) \delta R_{\mu\nu} \,. \tag{5.11}$$

The Palatini identity, (E.4), is a useful substitution that may be used in expressing the variation of the Ricci tensor as a variation of the metric tensor. The Palatini identity is

$$\delta R_{\mu\nu} = \nabla_\rho \left(\delta \Gamma^\rho_{\mu\nu} \right) - \nabla_\nu \left(\delta \Gamma^\rho_{\rho\mu} \right) \,. \tag{5.12}$$

The Palatini identity may be recast into a more convenient form for our purposes [16, pp. 290],

$$\delta R_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\nabla_\alpha \nabla_\nu \delta g_{\mu\beta} + \nabla_\alpha \nabla_\mu \delta g_{\nu\beta} - \nabla_\mu \nabla_\nu \delta g_{\alpha\beta} - \nabla_\alpha \nabla_\beta \delta g_{\mu\nu} \right] \,. \tag{5.13}$$

We substitute (5.13) into (5.11) and obtain

$$\begin{aligned} \delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \, \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu} \right) & \left[\nabla^\beta \nabla_\nu \delta g_{\mu\beta} + \nabla^\beta \nabla_\mu \delta g_{\nu\beta} \right. \\ & \left. - g^{\alpha\beta} \nabla_\mu \nabla_\nu \delta g_{\alpha\beta} - \nabla^\beta \nabla_\beta \delta g_{\mu\nu} \right] . \end{aligned} \tag{5.14}$$

It is now necessary to perform a number of partial integrations, in order to isolate the variations of the metric tensor.

A few remarks on partial integrations and the covariant derivative are in order. Since we know that

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g} \,, \tag{5.15}$$

$$\nabla_\mu W^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} W^\mu \right) \,. \tag{5.16}$$

We can thus see that

$$\begin{aligned} \sqrt{-g} \nabla_\mu W^\mu &= \sqrt{-g} \partial_\mu W^\mu + \sqrt{-g} \Gamma^\mu_{\mu\nu} W^\nu \\ &= \sqrt{-g} \partial_\mu W^\mu + \left(\partial_\mu \sqrt{-g} \right) W^\mu \\ &= \partial_\mu \left(\sqrt{-g} W^\mu \right) \,. \end{aligned} \tag{5.17}$$

This last expression is clearly a four-divergence. Additionally, we are concerned with a variation of the metric in the integrals. Using Gauss' Theorem (as in chapter 2.1), one may thus eliminate such terms. In other words,

$$\int \mathrm{d}^4 x \, \sqrt{-g} \nabla_\mu W^\mu = 0 \,. \tag{5.18}$$

Since the covariant derivative obeys the product rule, we can easily verify that partial integrations are possible. However, note that this only holds as there is a factor $\sqrt{-g}$ multiplying the partially integrated part. Put somewhat differently, this only holds when we integrate over $\mathrm{d}\omega = \mathrm{d}^4x\,\sqrt{-g}$.

Now that we know that partial integrations with respect to the covariant derivatives are possible when we integrate over $\mathrm{d}\omega$, we may perform the partial integrations,

$$\begin{aligned}\delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \Big\{ \quad & \nabla_\nu \nabla^\beta \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\mu\beta} \\ & + \nabla_\mu \nabla^\beta \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\nu\beta} \\ & - \nabla_\nu \nabla_\mu \left[\left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) g^{\alpha\beta}\right] \delta g_{\alpha\beta} \\ & - \Box \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\mu\nu} \Big\}\,,\end{aligned} \tag{5.19}$$

where $\Box \equiv \nabla_\alpha \nabla^\alpha$ is the d'Alembertian. From (2.48), we see that

$$\nabla_\alpha g^{\mu\nu} = 0\,. \tag{5.20}$$

Using the fact that the covariant derivative of the metric tensor vanishes, we factorize the third line of (5.19),

$$\begin{aligned}\delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \Big[\quad & \nabla_\nu \nabla^\beta \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\mu\beta} \\ & + \nabla_\mu \nabla^\beta \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\nu\beta} \\ & - g^{\alpha\beta} \nabla_\nu \nabla_\mu \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\alpha\beta} \\ & - \Box \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \delta g_{\mu\nu} \Big]\,.\end{aligned} \tag{5.21}$$

By rearranging the dummy indices, we obtain a more manageable form,

$$\begin{aligned}\delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \Big[\quad & 2 \nabla_\alpha \nabla^\nu \left(c_1 R g^{\mu\alpha} + c_2 R^{\mu\alpha}\right) \\ & - g^{\mu\nu} \nabla_\beta \nabla_\alpha \left(c_1 R g^{\alpha\beta} + c_2 R^{\alpha\beta}\right) \\ & - \Box \left(c_1 R g^{\mu\nu} + c_2 R^{\mu\nu}\right) \Big] \delta g_{\mu\nu}\,.\end{aligned} \tag{5.22}$$

Although the variation of the metric has been isolated, we need the variation of the metric tensor in (5.22) to be in the form with upper indices instead of the lower indices. We can go from lower to upper indices by

$$\delta g_{\sigma\rho} = -g_{\sigma\alpha} g_{\rho\beta} \delta g^{\alpha\beta}\,. \tag{5.23}$$

Inserting (5.23) into (5.22) and additional rearranging of dummy indices gives us

$$\begin{aligned}\delta S_{\mathrm{EH3}} = \int \mathrm{d}\omega \Big[& - 2 \nabla_\alpha \nabla_\nu \left(c_1 R \delta^\alpha_\mu + c_2 R^\alpha_\mu\right) \\ & + g_{\mu\nu} \nabla_\beta \nabla_\alpha \left(c_1 R g^{\alpha\beta} + c_2 R^{\alpha\beta}\right) \\ & + \Box \left(c_1 R g_{\mu\nu} + c_2 R_{\mu\nu}\right) \Big] \delta g^{\mu\nu}\,.\end{aligned} \tag{5.24}$$

By rewriting into factors of c_1 and c_2 inside the square brackets, we arrive at

$$\begin{aligned}\delta S_{\text{EH3}} = \int \mathrm{d}\omega \Big[\quad & 2c_1\big(-\nabla_\mu\nabla_\nu R + g_{\mu\nu}\Box R\big) \\ & + c_2\big(-2\nabla_\alpha\nabla_\nu R^\alpha_\mu + g_{\mu\nu}\nabla_\beta\nabla_\alpha R^{\alpha\beta} + \Box R_{\mu\nu}\big)\Big]\delta g^{\mu\nu}\,.\end{aligned} \tag{5.25}$$

We are now able to insert (5.25) into (5.9) to obtain

$$\begin{aligned}\delta S_{\text{EH}} = \int \mathrm{d}\omega \Big[\quad & \frac{c_0}{\kappa}\Big(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\Big) \\ & + c_1\Big(-2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}\Big) \\ & + c_2\Big(-2\nabla_\alpha\nabla_\nu R^\alpha_\mu + g_{\mu\nu}\nabla_\beta\nabla_\alpha R^{\alpha\beta} + \Box R_{\mu\nu} \\ & \qquad - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2g^{\alpha\beta}R_{\alpha\nu}R_{\beta\mu}\Big) - \frac{1}{2}T_{\mu\nu}\Big]\delta g^{\mu\nu}\,.\end{aligned} \tag{5.26}$$

There is a further simplification that can be made to the variation of the action. Through the contracted Bianchi identity, (E.8), we may rewrite

$$\nabla_\beta\nabla_\alpha R^{\alpha\beta} = \nabla_\beta\nabla_\alpha g^{\beta\gamma}R^\alpha_\gamma = g^{\beta\gamma}\nabla_\beta\nabla_\alpha R^\alpha_\gamma = \nabla^\gamma\left(\frac{1}{2}\nabla_\gamma R\right) = \frac{1}{2}\Box R\,. \tag{5.27}$$

Note that this cannot be done for the first term of the third line of (5.26) due to the fact that the covariant derivatives do not commute amongst themselves; $[\nabla_\alpha, \nabla_\beta] \neq 0$. However, by using (3.2.12) in [15], and (5.27), we obtain

$$\nabla_\alpha\nabla_\nu R^\alpha_\mu = \frac{1}{2}\nabla_\nu\nabla_\mu R + R_{\alpha\nu\mu\beta}R^{\beta\alpha} + R^\beta_\nu R_{\mu\beta} \tag{5.28}$$

Inserting equations (5.27) and (5.28) into the variation of the action, (5.26), yields

$$\begin{aligned}\delta S_{\text{EH}} = \int \mathrm{d}\omega \Big[\quad & \frac{c_0}{\kappa}\Big(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\Big) \\ & + c_1\Big(-2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}\Big) \\ & + c_2\Big(-\nabla_\nu\nabla_\mu R - 2R_{\alpha\nu\mu\beta}R^{\beta\alpha} \\ & \qquad + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\Big) - \frac{1}{2}T_{\mu\nu}\Big]\delta g^{\mu\nu}\,.\end{aligned} \tag{5.29}$$

Since we have to require that the variation is equal to zero, and the variation of the metric tensor is arbitrary, the term within the square brackets must be equal to zero. We have thus found the equation of motion for a Lagrangian to second order in R. The equation of motion is given by

$$\frac{c_0}{\kappa}\bigg(-\frac{1}{2}g_{\mu\nu}R+R_{\mu\nu}\bigg)+c_1\bigg(-2\nabla_\mu\nabla_\nu R+2g_{\mu\nu}\Box R-\frac{1}{2}g_{\mu\nu}R^2+2RR_{\mu\nu}\bigg)$$
$$+c_2\bigg(-\nabla_\nu\nabla_\mu R+\frac{1}{2}g_{\mu\nu}\Box R+\Box R_{\mu\nu}-2R_{\alpha\nu\mu\beta}R^{\beta\alpha}-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\bigg)=\frac{1}{2}T_{\mu\nu}\,. \tag{5.30}$$

Now that we have found the equation of motion, it is useful to find the linearized equation of motion so that a few predictions can be made.

It is worth noting that this equation of motion is not the same result that Stelle obtained and published [13]. It differs in a few signs of second-order terms. Due to the fact that this difference only applies to terms of second order in R, there will be no differences in the following sections of this chapter. That is because the equation of motion will be linearized as in the weak field theory approach, and thus terms of second order or higher will be omitted. However, if one wishes to calculate corrections of an even higher order, it is important to get this result right.

5.2 Linearized solution in Schwarzschild coordinates

In order to investigate the effects of including the second-order terms in the Lagrangian, we turn to the familiar case of the Schwarzschild coordinates, (3.8),

$$\mathrm{d}s^2=-U(r)\mathrm{d}t^2+V(r)\mathrm{d}r^2+r^2\mathrm{d}\Omega^2\,. \tag{5.31}$$

This time, however, we restrict the calculations to first order in κh and its derivatives. We already know from Chapter 3 which forms the Ricci scalar, (3.13), and the Ricci tensor, (3.12), take in the Schwarzschild coordinate system. In addition, we also saw in Chapter 4 how an equation of motion may be linearized. This time there is a deviation from the method used in Chapter 4, since we wish to find the solution in Schwarzschild coordinates directly. We find that

$$\begin{aligned}U&=1+\kappa h_{00}\,,\\ V&=1+\kappa h_{11}\,.\end{aligned} \tag{5.32}$$

An additional detail we have to be wary about when using these old results, is that for the linearized approximation,

$$\begin{aligned}\frac{1}{U}&=1-\kappa h_{00}\,,\\ \frac{1}{V}&=1-\kappa h_{11}\,.\end{aligned} \tag{5.33}$$

The reasoning behind the result in (5.33) is available at (4.2). By directly substituting the linear forms of U and V, and their inverses, into the old results from chapter 3, we find that

$$
\begin{aligned}
R_{tt}/\kappa &= -\frac{h_{00}''}{2} - \frac{h_{00}'}{r}\,, \\
R_{rr}/\kappa &= \frac{h_{00}''}{2} - \frac{h_{11}'}{r}\,, \\
R_{\theta\theta}/\kappa &= \frac{rh_{00}'}{2} - h_{11} - \frac{rh_{11}'}{2}\,, \\
R_{\phi\phi}/\kappa &= \sin^2\theta R_{22}\,.
\end{aligned} \tag{5.34}
$$

$$
R/\kappa = h_{00}'' + \frac{2h_{00}'}{r} - \frac{2h_{11}'}{r} - \frac{2h_{11}}{r^2}\,. \tag{5.35}
$$

When substituting these results into the equation of motion, note that any higher-order terms are automatically omitted because they contain higher order terms of the linearized variables, $h_{\mu\nu}$. Additionally, for the reasons already provided in Chapter 4, all the metric tensors will in this approximation be substituted by the flat space metric tensor, $g_{\mu\nu} \to \eta_{\mu\nu}$. Since we are only working to first order in the perturbation field, the covariant derivatives are replaced with ordinary partial derivatives. This is due to equation (4.7), and the fact that the covariant derivatives only act on the perturbation fields in our equation of motion. In other words, the covariant derivative on any term in the equation of motion will consist of one first-order term (the normal partial derivative), and some higher-order terms (involving the Christoffel symbol). We also pay attention to the fact that the perturbation field, $h_{\mu\nu}$, only depends on the radial coordinate. Combined with the insight that we may utilize the d'Alembertian for spherical coordinates in our investigation, we can replace $\Box h_{\mu\nu} \to \frac{1}{r^2}\partial_r\left(r^2\partial_r h_{\mu\nu}\right)$. The reason why we can utilize the d'Alembertian for spherical coordinates in our Schwarzschild coordinate system is that all the terms that would differ from a spherical coordinate system are second-order terms of the perturbation field. In this investigation, we already limit ourselves to only include the first-order perturbation since we assume that the perturbations are small. Hence, we obtain four equations of motion which describe our system in linearized Schwarzschild coordinates for a point mass. For the time direction we obtain

$$
\begin{aligned}
&\frac{c_0}{\kappa}\left(\frac{1}{2}R + R_{00}\right) + c_1\left(-2\Box R\right) \\
&+c_2\left(-\frac{1}{2}\Box R + \Box R_{00}\right) = \frac{1}{2}T_{00}\,,
\end{aligned} \tag{5.36}
$$

for the radial direction we obtain

$$
\begin{aligned}
&\frac{c_0}{\kappa}\left(-\frac{1}{2}R + R_{11}\right) + c_1\left(-2\partial_r^2 R + 2\Box R\right) \\
&+c_2\left(-\partial_r^2 R + \frac{1}{2}\Box R + \Box R_{11}\right) = \frac{1}{2}T_{11}\,,
\end{aligned} \tag{5.37}
$$

and for the two angular directions we obtain

$$
\begin{aligned}
&\frac{c_0}{\kappa}\left(-r^2\frac{1}{2}R + R_{22}\right) + c_1\left(2r^2\Box R\right) \\
&+c_2\left(\frac{1}{2}r^2\Box R + \Box R_{22}\right) = \frac{1}{2}T_{22}\,,
\end{aligned} \tag{5.38}
$$

and

$$\frac{c_0}{\kappa}\left(-r^2\sin^2\theta\frac{1}{2}R+R_{33}\right)+c_1\left(2r^2\sin^2\theta\Box R\right)$$
$$+c_2\left(\frac{1}{2}r^2\sin^2\theta\Box R+\Box R_{33}\right)=\frac{1}{2}T_{33}\,. \tag{5.39}$$

For the remainder of this derivation, we look at the vacuum solution, i.e., $T_{\mu\nu}=0$. Since $R_{33}=\sin^2\theta R_{22}$, the two angular equations turn out to be the same.[2] However, since the equations are equal to zero, and the Ricci tensor is only radially dependent, we retrieve an additional condition,

$$R_{22}\Box\sin^2\theta=0\implies R_{22}=0\,. \tag{5.40}$$

Therefore, keeping one or the other does not provide any new information to the problem.

The different terms of the equation of motion are easily computed;

$$\Box R_{00}=-\frac{h_{00}^{iv}}{2}-\frac{2h_{00}'''}{r}\,, \tag{5.41}$$

$$\Box R_{11}=\frac{h_{00}^{iv}}{2}+\frac{h_{00}'''}{r}-\frac{h_{11}'''}{r}\,, \tag{5.42}$$

$$\Box R_{22}=\frac{rh_{00}'''}{2}+2h_{00}''+\frac{h_{00}'}{r}-\frac{rh_{11}'''}{2}-3h_{11}''-\frac{3h_{11}'}{r}\,, \tag{5.43}$$

$$\Box R=h_{00}^{iv}+\frac{4h_{00}'''}{r}-\frac{2h_{11}'''}{r}-\frac{2h_{11}''}{r^2}+\frac{4h_{11}'}{r^3}-\frac{4h_{11}}{r^4}\,, \tag{5.44}$$

$$\partial_r^2R=h_{00}^{iv}+\frac{2h_{00}'''}{r}-\frac{4h_{00}''}{r^2}+\frac{4h_{00}'}{r^3}-\frac{2h_{11}'''}{r}+\frac{2h_{11}''}{r^2}+\frac{4h_{11}'}{r^3}-\frac{12h_{11}}{r^4}\,. \tag{5.45}$$

By substituting equations (5.41)–(5.45) into the equations of motion, one obtains a new set of equations represented by h_{00} and h_{11}. The equations were originally found by Stelle in the following form [13]

$$H_{00}=-(c_2+2c_1)h_{00}^{iv}-4(c_2+2c_1)\frac{h_{00}'''}{r}+(c_2+4c_1)\frac{h_{11}'''}{r}+(c_2+4c_1)\frac{h_{11}''}{r^2}$$
$$-2(c_2+4c_1)\frac{h_{11}'}{r^3}+2(c_2+4c_1)\frac{h_{11}}{r^4}-\frac{c_0}{\kappa}\left(\frac{h_{11}'}{r}+\frac{h_{11}}{r^2}\right)\,, \tag{5.46}$$

$$H_{11}=(c_2+4c_1)\frac{h_{00}'''}{r}+2(c_2+4c_1)\frac{h_{00}''}{r^2}-2(c_2+4c_1)\frac{h_{00}'}{r^3}$$
$$-(3c_2+8c_1)\frac{h_{11}''}{r^2}+2(3c_2+8c_1)\frac{h_{11}}{r^4}+\frac{c_0}{\kappa}\left(-\frac{h_{00}'}{r}+\frac{h_{11}}{r^2}\right)\,, \tag{5.47}$$

$$H_{22}=\frac{1}{2}(c_2+4c_1)r^2h_{00}^{iv}+\frac{3}{2}(c_2+4c_1)rh_{00}'''-(c_2+4c_1)h_{00}''+(c_2+4c_1)\frac{h_{00}'}{r}$$
$$-\frac{1}{2}(3c_2+8c_1)rh_{11}'''+(3c_2+8c_1)\frac{h_{11}'}{r}-2(3c_2+8c_1)\frac{h_{11}}{r^2}$$
$$+\frac{1}{2}\frac{c_0}{\kappa}\left(-rh_{00}'-r^2h_{00}''+rh_{11}'\right)\,, \tag{5.48}$$

[2]What is meant here is that the two equations contain exactly the same information.

where $H_{\mu\nu}$ are the left hand sides of the equations of motion, such that $H_{\mu\nu} = \frac{1}{2}T_{\mu\nu}$. In solving this set of equations, it is useful to compute the contracted stress-energy tensor, T^μ_μ, and the relation given by $I^{\mu\nu}T_{\mu\nu}$ where $I^{\mu\nu}$ is the identity matrix. Through solving this system of equations, one arrives at the homogeneous solutions for h_{00} and h_{11}, also found by Stelle,

$$h_{00} = C + \frac{C^{2,0}}{r} + \frac{C^{2+}}{r}e^{m_2 r} + \frac{C^{2-}}{r}e^{-m_2 r} + \frac{C^{0+}}{r}e^{m_0 r} + \frac{C^{0-}}{r}e^{-m_0 r}\,, \tag{5.49}$$

$$\begin{aligned} h_{11} = &-\frac{C^{2,0}}{r} - \frac{C^{2+}}{r}e^{m_2 r} - \frac{C^{2-}}{r}e^{-m_2 r} + \frac{C^{0+}}{r}e^{m_0 r} + \frac{C^{0-}}{r}e^{-m_0 r} \\ &+ \frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}\,, \end{aligned} \tag{5.50}$$

where all the C's are integration constants which are yet to be determined. Furthermore, $m_2 = \sqrt{c_0/c_2\kappa}$, $m_0 = \sqrt{-c_0/2(3c_1 + c_2)\kappa}$. We should also notice that this solution is a Yukawa potential, which is a deviation from the potentials that are found in first-order gravity theories. The masses in this potential are given by m_2 and m_0. At this point, one might be surprised by the appearance of growing exponentials in a gravitational field. However, the rising exponentials are only part of the mathematical solution, and are eliminated when boundary conditions are invoked. For example, we may assess that gravity does not grow stronger with increasing distance. In fact, we should consider the case where $r \to \infty$ and compare it to the Newtonian limit to determine which terms should survive. It is clear from the example that only the decaying exponentials should be a part of the physical solution. We can therefore simplify the solutions. The results of invoking those boundary conditions are thus given by

$$h_{00} = \frac{C^{2,0}}{r} + \frac{C^{2-}}{r}e^{-m_2 r} + \frac{C^{0-}}{r}e^{-m_0 r}\,, \tag{5.51}$$

$$h_{11} = -\frac{C^{2,0}}{r} - \frac{C^{2-}}{r}e^{-m_2 r} + \frac{C^{0-}}{r}e^{-m_0 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} + C^{0-}m_0 e^{-m_0 r}\,. \tag{5.52}$$

We note that we have three unknown integration constants, and three equations to solve them. Solving for the integration constants is thus possible. In the case of the point particle, the solution of the gravitational field is found by Stelle, and is given by

$$h_{00} = -\frac{\kappa M}{8\pi c_0 r} + \frac{\kappa M}{6\pi c_0 r}e^{-m_2 r} - \frac{\kappa M}{24\pi c_0 r}e^{-m_0 r}\,, \tag{5.53}$$

where M is the mass of the point particle at the origin of the coordinate system.

5.3 Linearized solution in isotropic coordinates

Stelle has already provided a correction to the gravitational potential in the second-order approximation of the Einstein-Hilbert Lagrangian. However, that does not mean that the derivation of the answer was the most intuitive one. An alternate derivation is shown in this section, by the use of isotropic coordinates. One may find that this derivation is easier to follow.

5.3.1 Linearized equation of motion in isotropic coordinates

By building upon the work in Chapter 4, it is possible to derive the second-order correction of the Einstein-Hilbert action in isotropic coordinates. We will see that working in isotropic coordinates provides us with a rather natural way of obtaining the desired results. From Chapter 4, we already know the linearized form of the Ricci tensor and the Ricci scalar. They are shown in equation (4.8) and equation (4.11), respectively. Also, by choosing to work in the Lorentz gauge, (4.25), we may rewrite the Ricci tensor and the Ricci scalar as

$$R_{\mu\nu} = -2\kappa\Box h_{\mu\nu}\,, \tag{5.54}$$

$$R = -2\kappa\Box h\,, \tag{5.55}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}_{\mu}$. Again, since we are only including first-order terms in the perturbation field, we can replace the covariant derivatives with normal partial derivatives. This is due to equation (4.7), and the fact that the covariant derivatives only act on the perturbation fields in our equation of motion. We also note that any term with more than one Ricci tensor or Ricci scalar, or any combination thereof, will lead to higher-order terms which will be omitted. Additionally, we recall that we can make the replacement $g_{\mu\nu} \to \eta_{\mu\nu}$. The equation of motion, (5.30), thus simplifies to

$$\begin{aligned} &c_0\left(\eta_{\mu\nu}\Box h - 2\Box h_{\mu\nu}\right) + 4c_1\kappa\left(\partial_\mu\partial_\nu\Box h - \eta_{\mu\nu}\Box\Box h\right) \\ &+ c_2\kappa\left(2\partial_\mu\partial_\nu\Box h - \eta_{\mu\nu}\Box\Box h - 2\Box\Box h_{\mu\nu}\right) = \frac{1}{2}T_{\mu\nu}\,. \end{aligned} \tag{5.56}$$

If we desire to find the Green's function, we need to rewrite equation (5.56) into a form such that the left hand side is a differential operator acting on the perturbation field. Thus, we obtain

$$\begin{aligned} &\Big[c_0\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \eta_{\alpha\mu}\eta_{\beta\nu}\Box + 4c_1\kappa\left(\partial_\mu\partial_\nu\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\eta_{\alpha\beta}\Box\Box\right) \\ &+ c_2\kappa\left(2\partial_\mu\partial_\nu\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\eta_{\alpha\beta}\Box\Box - 2\eta_{\mu\alpha}\eta_{\nu\beta}\Box\Box\right)\Big]h^{\alpha\beta} = \frac{1}{2}T_{\mu\nu}\,. \end{aligned} \tag{5.57}$$

The Green's function associated with this equation of motion is derived in Appendix F.2.

5.3.2 The potential for a point mass in isotropic coordinates

Suppose that we want to find the potential for a point mass in the Lagrangian with higher order terms. The Green's function from Appendix F.2 seems to be rather uncomfortable to work with. However, we may circumvent that problem by directly solving the equation of motion, which is demonstrated in this section. Also, in a coordinated effort with Chapter 4, it is natural to choose isotropic coordinates. In this case, we are also working in the weak field limit, and as such, the metric will take the form

$$\mathrm{d}s^2 = -(1+\kappa h_{00})\mathrm{d}t^2 + (1+\kappa h_{11})\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{5.58}$$

The point mass is static and spherically symmetric, hence we obtain $\partial_\alpha h_{\mu\nu} = 0\,, \quad \alpha \neq r$, which in turn leads to $\partial_0 h = 0$. The stress-energy tensor for a point mass is given as

$T_{\mu\nu} = M\delta^0_\mu\delta^0_\nu\delta^3(\vec{r})$. The equation of motion for this system thus becomes

$$c_0\left(\eta_{\mu\nu}\Box h - 2\Box h_{\mu\nu}\right) + 4c_1\kappa\left(\partial_\mu\partial_\nu\Box h - \eta_{\mu\nu}\Box\Box h\right) \\ + c_2\kappa\left(2\partial_\mu\partial_\nu\Box h - \eta_{\mu\nu}\Box\Box h - 2\Box\Box h_{\mu\nu}\right) = \frac{M}{2}\delta^0_\mu\delta^0_\nu\delta^3(\vec{r})\,. \tag{5.59}$$

Since $h_{00} = h_{00}(r)$ and $h_{11} = h_{11}(r)$, the d'Alembertian in (5.59) can be replaced with the Jacobian, ∇^2. The trace of the stress-energy tensor is

$$\frac{1}{2}T^\mu_\mu = \left[2c_0\nabla^2 - 2\kappa(3c_1 + c_2)\nabla^4\right]h\,. \tag{5.60}$$

Since $T_{\mu\nu} = M\delta^0_\mu\delta^0_\nu\delta^3(\vec{r}) = -\frac{M}{4\pi}\nabla^2\frac{1}{r}$, the left hand side of equation (5.60) is $\frac{1}{2}T^\mu_\mu = -\frac{M}{8\pi}\nabla^2\frac{1}{r}$. By inserting this result into equation (5.60) and simplifying, we obtain

$$-\frac{M}{16\pi r} = \left[c_0 - \kappa(3c_1 + c_2)\nabla^2\right]h\,. \tag{5.61}$$

The trace of the perturbation, $h^\mu_\mu = h$, is thus given by

$$h(r) = -\frac{M}{16\pi c_0 r} + \frac{C^- e^{-m_0 r}}{r} + \frac{C^+ e^{m_0 r}}{r}\,, \tag{5.62}$$

where, as before, $m_0 = \sqrt{c_0/2(3c_1 + c_2)\kappa}$, and the big C's are integration constants that have to be determined. We can already infer that C^+ has to be zero for the same reasons given earlier; We cannot allow for growing exponentials to be present because that would make the force of gravity increase over distance. We are thus left with

$$h(r) = -\frac{M}{16\pi c_0 r} + \frac{C^- e^{-m_0 r}}{r}\,. \tag{5.63}$$

From inspection of the metric we are working with, (5.58), we see that

$$h = \eta^{\mu\nu}h_{\mu\nu} = -(-h_{00}) + h_{11} + \frac{1}{r^2}r^2 h_{11} + \frac{1}{r^2\sin^2\theta}r^2\sin^2\theta h_{11} = h_{00} + 3h_{11}\,, \tag{5.64}$$

which in turn leads to

$$3h_{11} = h - h_{00}\,. \tag{5.65}$$

The time component of the equation of motion is

$$-\frac{M}{8\pi r} = \left[-3c_0 + \kappa(4c_1 - c_2)\Box\right]h_{00} + \left[-c_0 + \kappa(4c_1 + c_2)\Box\right]3h_{11}\,. \tag{5.66}$$

By inserting (5.65) into (5.66) and rearranging we obtain

$$-\frac{M}{8\pi r} = -2(c_0 + c_2\kappa\Box)h_{00} + \left[-c_0 + \kappa(4c_1 + c_2)\Box\right]h\,. \tag{5.67}$$

Solving this equation for h_{00} can be complicated. The answer is more easily obtained through a symbolic computation language, such as Mathematica. The solution for h_{00} takes the form[3]

$$h_{00}(r) = \frac{DM}{16\pi c_0 r} + \frac{D_2 e^{-m_2 r}}{r} + \frac{D_0 e^{-m_0 r}}{r}\,, \tag{5.68}$$

where the D's are integration constants to be determined, and $m_2 = \sqrt{-c_0/c_2\kappa}$. The values for the D's were found by Stelle, and the final gravitational potential is given by

$$h_{00}(r) = -\frac{\kappa M}{8\pi c_0 r} + \frac{\kappa M}{6\pi c_0 r} e^{-m_2 r} - \frac{\kappa M}{24\pi c_0 r} e^{-m_0 r}\,. \tag{5.69}$$

We immediately note that we should retrieve the standard Newtonian gravitational potential in the $r \to \infty$ limit. Because of this, we see that $c_0 = 1/2$, which is the familiar value that was used in the Einstein-Hilbert action. We thus obtain

$$h_{00}(r) = -\frac{2GM}{r} + \frac{8GM}{3r} e^{-m_2 r} - \frac{2GM}{3r} e^{-m_0 r}\,. \tag{5.70}$$

5.4 A few final remarks

Stelle notes that one may remove one of the terms in the gravitational potential by having $c_2 = 0$, or $c_2 = -3c_1$, respectively forcing m_2 or m_0 to be infinite. One may also remark that if m_2 or m_0 are large compared to r, then

$$\frac{e^{-m_i r}}{r} \to 4\pi m_i^{-2} \delta^3(\vec{r})\,, \tag{5.71}$$

which in turn leads to a potential

$$h_{00}(r) = -\frac{\kappa M}{8\pi c_0 r} - \frac{\kappa^2 M}{4 c_0^2 r} \left(3c_2 + c_1\right) \delta^3(\vec{r})\,. \tag{5.72}$$

Stelle has set some experimental lower bounds on the masses, m_0, m_2, to be around $\sim 5 \cdot 10^{-13}\,\mathrm{m}^{-1}$. [13] These lower limits are set from hypothetical corrections one may make to the orbit of Mercury. However, short-range laboratory experiments show that the lower bounds of the masses can be set to $m_0, m_2 \geq 10^3 m^{-1}$. [5] This suggests that the corrections to the gravitational potential which arise in the second-order action integral are so small that they are irrelevant for our purposes, and only have relevance at very small scales.

[3] The given form of the solution is the solution after dismissing terms with growing exponentials. This can be achieved through setting the integration constant in front of the terms in question equal to zero.

Chapter 6

Quantization of gravity

For the quantization of the Einstein-Hilbert Lagrangian, we will use the background field method introduced by t'Hooft and Veltman [14]. We will assume a smooth background field (classical field), with a small perturbation field (quantum field),

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{6.1}$$

where $\bar{g}_{\mu\nu}$ is the background field, and $h_{\mu\nu}$ is the perturbation field. Since we want the trace of the metric to be untouched by this perturbation, we also obtain the relation

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\alpha} h^{\alpha\nu} \,, \tag{6.2}$$

which is a valid inverse of the metric up to second order in the perturbation. This is easily verified by computing $g^{\mu\nu} g_{\mu\nu} = 4$.

The perturbation field is, as in the previous chapters, assumed to be very small when compared to the background metric field. As usual, due to the metric tensor being symmetric in its indices, then so is the perturbation metric. This fact will be used extensively throughout the chapter. As in previous chapters on perturbation fields, $\bar{g}$ will be used for lowering and raising of indices (as η was used earlier). That is an effect of the assumption that the perturbation is small.

Due to the results of Chapter 5, we do not concern ourselves with the higher order Lagrangian densities. We therefore return to the first-order Einstein-Hilbert Lagrangian that was first introduced in the beginning of this thesis,

$$\mathscr{L} = \frac{1}{2\kappa} \sqrt{-g} R \,. \tag{6.3}$$

6.1 The expanded Lagrangian density

We want to express the Einstein-Hilbert Lagrangian in terms of $\bar{g}$ and h. This is achieved by direct insertion and some algebraic manipulations. The square root may be expressed in the following manner

$$
\begin{aligned}
\sqrt{-g} &= \sqrt{-\det g} = \sqrt{-\det(\bar{g}+h)} \\
&= \sqrt{-\det(\bar{g})\det(1+\bar{g}^{-1}h)} \\
&= \sqrt{-\det(\bar{g})}\sqrt{\det(1+\bar{g}^{-1}h)} \\
&= \sqrt{-\det(\bar{g})}\exp\left[\frac{1}{2}\ln\left(\det\left(1+\bar{g}^{-1}h\right)\right)\right] .
\end{aligned} \tag{6.4}
$$

Since $\ln\det = \mathrm{Tr}\ln$, we can rewrite the argument of the exponential. Furthermore, we will utilize the fact that the series expansion of the logarithm is

$$
\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \qquad , \quad x < 1 , \tag{6.5}
$$

and the series expansion of the exponential function is

$$
e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots . \tag{6.6}
$$

Thus, by discarding terms of $\mathscr{O}(h^3)$, we obtain

$$
\begin{aligned}
\sqrt{-g} &= \sqrt{-\det\bar{g}}\exp\left[\frac{1}{2}\mathrm{Tr}\ln\left(1+\bar{g}^{-1}h\right)\right] \\
&= \sqrt{-\det\bar{g}}\exp\left[\frac{1}{2}\mathrm{Tr}\left(\bar{g}^{-1}h - \frac{1}{2}(\bar{g}^{-1}h)^2\right)\right] \\
&= \sqrt{-\det\bar{g}}\left[1 + \frac{1}{2}\mathrm{Tr}\left(\bar{g}^{-1}h\right) - \frac{1}{4}\mathrm{Tr}\left(\bar{g}^{-1}h\right)^2 + \frac{1}{8}\mathrm{Tr}^2(\bar{g}^{-1}h)\right] \\
&= \sqrt{-\det\bar{g}}\left[1 + \frac{1}{2}h^{\mu}_{\mu} - \frac{1}{4}\mathrm{Tr}\left(\bar{g}^{-1}h\right)^2 + \frac{1}{8}(h^{\mu}_{\mu})^2\right] .
\end{aligned} \tag{6.7}
$$

Remember that matrix multiplications are expressed in component form as

$$
(AB)_{ij} = A^{ij}B_{jk} . \tag{6.8}
$$

Therefore we may rewrite $\mathrm{Tr}\left(\bar{g}^{-1}h\right)^2$ as

$$
\begin{aligned}
\mathrm{Tr}\left(\bar{g}^{-1}h\right)^2 &= \mathrm{Tr}\left(g^{\alpha\beta}h_{\alpha\beta}g_{\beta\gamma}h^{\beta\gamma}\right) = \mathrm{Tr}\left(g^{\alpha\beta}g_{\beta\gamma}h_{\alpha\beta}h^{\beta\gamma}\right) \\
&= \mathrm{Tr}\left(\delta^{\alpha}_{\gamma}h_{\alpha\beta}h^{\beta\gamma}\right) = \mathrm{Tr}\left(h_{\alpha\beta}h^{\alpha\beta}\right) .
\end{aligned} \tag{6.9}
$$

Since the trace of a scalar is just that scalar itself, we finally obtain

$$
\sqrt{-g} = \sqrt{-\det\bar{g}}\left[1 + \frac{1}{2}h^{\mu}_{\mu} - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{8}(h^{\mu}_{\mu})^2\right] \tag{6.10}
$$

For the expansion of the Ricci scalar, we start with the Ricci tensor. The Ricci tensor is defined by

$$
R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu} . \tag{6.11}
$$

We find the Chrisoffel symbol by using the definitions of the metric tensor expansion and discarding terms of $\mathscr{O}(h^3)$,

$$
\begin{aligned}
\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\alpha}\big(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\big) \\
&= \frac{1}{2}\big(\bar{g}^{\rho\alpha} - h^{\rho\alpha} + h^{\rho}_{\lambda} h^{\lambda\alpha}\big)\big(\bar{g}_{\alpha\mu,\nu} + \bar{g}_{\alpha\nu,\mu} - \bar{g}_{\mu\nu,\alpha} + h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\big) \\
&= \bar{\Gamma}^{\rho}_{\mu\nu} + \bar{g}^{\rho\alpha} H_{\alpha\mu\nu} - h^{\rho\alpha}\left(\bar{\Gamma}_{\alpha\mu\nu} + H_{\alpha\mu\nu}\right) + h^{\rho}_{\lambda} h^{\lambda\alpha}\bar{\Gamma}_{\alpha\mu\nu}\,, \qquad (6.12)
\end{aligned}
$$

where $\bar{\Gamma}^{\rho}_{\mu\nu}$ is the Christoffel symbol of the background field, and

$$
H_{\alpha\mu\nu} = \frac{1}{2}\left(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\right) = \frac{1}{2}\left(h_{\alpha\mu;\nu} + h_{\alpha\nu;\mu} - h_{\mu\nu;\alpha} + 2\bar{\Gamma}^{\lambda}_{\mu\nu} h_{\alpha\lambda}\right)\,. \qquad (6.13)
$$

Thus, when using the fact that the covariant derivative of the background metric tensor is equal to zero, and rewriting all derivatives of the perturbation field into covariant derivatives, we find

$$
\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \hat{\Gamma}^{\rho}_{\mu\nu} - h^{\rho\alpha}\hat{\Gamma}_{\alpha\mu\nu}\,, \qquad (6.14)
$$

$$
\hat{\Gamma}_{\alpha\mu\nu} = \left(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}\right)\,. \qquad (6.15)
$$

Barred symbols are considered to be the corresponding term with respect to the background field, and hatted symbols are considered the corresponding term with respect to the perturbation field. One important detail is that $\hat{\Gamma}^{\rho}_{\mu\nu} = \bar{g}^{\rho\alpha}\hat{\Gamma}_{\alpha\mu\nu}$. Also notice that all the usual rules will therefore apply for the symbols that respect the background fields, while we may not assume so for the hatted symbols.

The Ricci tensor in terms of the Christoffel symbols is thus

$$
\begin{aligned}
R_{\mu\nu} =& \bar{R}_{\mu\nu} + \partial_{\rho}\left[\hat{\Gamma}^{\rho}_{\mu\nu} - h^{\alpha\rho}\hat{\Gamma}_{\alpha\mu\nu}\right] - \partial_{\nu}\left[\hat{\Gamma}^{\rho}_{\rho\mu} - h^{\alpha\rho}\hat{\Gamma}_{\alpha\rho\mu}\right] \\
&+ \bar{\Gamma}^{\rho}_{\rho\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} + \hat{\Gamma}^{\rho}_{\rho\lambda}\bar{\Gamma}^{\lambda}_{\mu\nu} + \hat{\Gamma}^{\rho}_{\rho\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\rho}_{\nu\lambda}\hat{\Gamma}^{\lambda}_{\mu\rho} - \hat{\Gamma}^{\rho}_{\nu\lambda}\bar{\Gamma}^{\lambda}_{\mu\rho} - \hat{\Gamma}^{\rho}_{\nu\lambda}\hat{\Gamma}^{\lambda}_{\mu\rho} \\
&+ h^{\alpha\rho}\left[\bar{\Gamma}^{\lambda}_{\nu\rho}\hat{\Gamma}_{\alpha\mu\lambda} + \bar{\Gamma}^{\lambda}_{\mu\rho}\hat{\Gamma}_{\alpha\nu\lambda} - \bar{\Gamma}^{\lambda}_{\lambda\rho}\hat{\Gamma}_{\alpha\mu\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}\hat{\Gamma}_{\alpha\rho\lambda}\right]\,, \qquad (6.16)
\end{aligned}
$$

where we are excluding terms of $\mathscr{O}(h^3)$ or higher. Now, we turn our attention to writing the partial derivatives as covariant derivatives, D_α, with respect to the background field $\bar{g}$. This will be very beneficial since we are working with metric tensors, and the background metric commutes with the covariant derivatives. We obtain

$$
\partial_{\rho}\hat{\Gamma}^{\rho}_{\mu\nu} = D_{\rho}\hat{\Gamma}^{\rho}_{\mu\nu} - \bar{\Gamma}^{\rho}_{\rho\lambda}\hat{\Gamma}^{\lambda}_{\mu\nu} + \bar{\Gamma}^{\lambda}_{\rho\mu}\hat{\Gamma}^{\rho}_{\lambda\nu} + \bar{\Gamma}^{\lambda}_{\rho\nu}\hat{\Gamma}^{\rho}_{\lambda\mu}\,, \qquad (6.17)
$$

$$
\partial_{\nu}\hat{\Gamma}^{\rho}_{\rho\mu} = D_{\rho}\hat{\Gamma}^{\rho}_{\rho\mu} - \bar{\Gamma}^{\lambda}_{\mu\nu}\hat{\Gamma}^{\rho}_{\rho\lambda}\,, \qquad (6.18)
$$

$$
\begin{aligned}
\partial_\rho\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\mu\nu}\Big] =&\partial_\rho h^{\alpha\rho}\hat\Gamma_{\alpha\mu\nu} + h^{\alpha\rho}\partial_\rho\hat\Gamma_{\alpha\mu\nu}\\
=&\Big[D_\rho h^{\alpha\rho} - \bar\Gamma^\alpha_{\rho\lambda}h^{\lambda\rho} - \bar\Gamma^\rho_{\rho\lambda}h^{\lambda\alpha}\Big]\hat\Gamma_{\alpha\mu\nu}\\
&+ h^{\alpha\rho}\Big[D_\rho\hat\Gamma_{\alpha\mu\nu} + \bar\Gamma^\lambda\rho\alpha\hat\Gamma_{\lambda\mu\nu} + \bar\Gamma^\lambda\rho\mu\hat\Gamma_{\alpha\lambda\nu} + \bar\Gamma^\lambda\rho\nu\hat\Gamma_{\alpha\mu\lambda}\Big]\\
=&D_\rho\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\mu\nu}\Big] + h^{\alpha\rho}\Big[\bar\Gamma^\lambda_{\rho\alpha}\hat\Gamma_{\lambda\mu\nu} + \bar\Gamma^\lambda_{\rho\mu}\hat\Gamma_{\alpha\lambda\nu}\\
&\qquad + \bar\Gamma^\lambda_{\rho\nu}\hat\Gamma_{\alpha\mu\lambda} - \bar\Gamma^\lambda_{\rho\alpha}\hat\Gamma_{\lambda\mu\nu} - \bar\Gamma^\lambda_{\lambda\rho}\hat\Gamma_{\alpha\mu\nu}\Big]\\
=&D_\rho\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\mu\nu}\Big] + h^{\alpha\rho}\Big[\bar\Gamma^\lambda_{\rho\mu}\hat\Gamma_{\alpha\lambda\nu} + \bar\Gamma^\lambda_{\rho\nu}\hat\Gamma_{\alpha\mu\lambda} - \bar\Gamma^\lambda_{\lambda\rho}\hat\Gamma_{\alpha\mu\nu}\Big], \qquad (6.19)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\nu\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\rho\mu}\Big] =&\partial_\nu h^{\alpha\rho}\hat\Gamma_{\alpha\rho\mu} + h^{\alpha\rho}\partial_\nu\hat\Gamma_{\alpha\rho\mu}\\
=&\Big[D_\nu h^{\alpha\rho} - \bar\Gamma^\alpha_{\nu\lambda}h^{\lambda\rho} - \bar\Gamma^\rho_{\nu\lambda}h^{\lambda\alpha}\Big]\hat\Gamma_{\alpha\rho\mu}\\
&+ h^{\alpha\rho}\Big[D_\nu\hat\Gamma_{\alpha\rho\mu} + \bar\Gamma^\lambda_{\nu\alpha}\hat\Gamma_{\lambda\rho\mu} + \bar\Gamma^\lambda_{\nu\rho}\hat\Gamma_{\alpha\lambda\mu} + \bar\Gamma^\lambda_{\nu\mu}\hat\Gamma_{\alpha\rho\lambda}\Big]\\
=&D_\nu\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\rho\mu}\Big] + h^{\alpha\rho}\Big[\bar\Gamma^\lambda_{\nu\alpha}\hat\Gamma_{\lambda\rho\mu} + \bar\Gamma^\lambda_{\nu\rho}\hat\Gamma_{\alpha\lambda\mu}\\
&\qquad + \bar\Gamma^\lambda_{\nu\mu}\hat\Gamma_{\alpha\rho\lambda} - \bar\Gamma^\lambda_{\nu\alpha}\hat\Gamma_{\lambda\rho\mu} - \bar\Gamma^\lambda_{\nu\rho}\hat\Gamma_{\alpha\lambda\mu}\Big]\\
=&D_\nu\Big[h^{\alpha\rho}\hat\Gamma_{\alpha\rho\mu}\Big] + h^{\alpha\rho}\bar\Gamma^\lambda_{\nu\mu}\hat\Gamma_{\alpha\rho\lambda}\,. \qquad (6.20)
\end{aligned}
$$

Thus, rewriting $R_{\mu\nu}$ in terms of covariant derivatives (with respect to the background field) yields

$$
\begin{aligned}
R_{\mu\nu} =&\bar R_{\mu\nu} + \hat\Gamma^\rho_{\rho\lambda}\hat\Gamma^\lambda_{\mu\nu} - \hat\Gamma^\rho_{\nu\lambda}\hat\Gamma^\lambda_{\mu\rho}\\
&+ D_\rho\left[\hat\Gamma^\rho_{\mu\nu} - h^{\alpha\rho}\hat\Gamma_{\alpha\mu\nu}\right] - D_\nu\left[\hat\Gamma^\rho_{\rho\mu} - h^{\alpha\rho}\hat\Gamma_{\alpha\rho\mu}\right]\,.
\end{aligned} \tag{6.21}
$$

Let $R^{(n)}_{\mu\nu}$ be the n-th order of the perturbation field in $R_{\mu\nu}$. Then, we have that $R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathscr{O}(h^3)$. We thus obtain

$$R^{(0)}_{\mu\nu} = \bar R_{\mu\nu}\,, \tag{6.22}$$

$$R^{(1)}_{\mu\nu} = D_\rho\hat\Gamma^\rho_{\mu\nu} - D_\nu\hat\Gamma^\rho_{\rho\mu}\,, \tag{6.23}$$

$$R^{(2)}_{\mu\nu} = D_\rho\left(h^{\alpha\rho}\hat\Gamma^\rho_{\mu\nu}\right) - D_\nu\left(h^{\alpha\rho}\hat\Gamma^\rho_{\rho\mu}\right) + \hat\Gamma^\rho_{\rho\lambda}\hat\Gamma^\lambda_{\mu\nu} - \hat\Gamma^\rho_{\nu\lambda}\hat\Gamma^\lambda_{\mu\rho},. \tag{6.24}$$

We are ultimately interested in finding $R = g^{\mu\nu}R_{\mu\nu}$ expressed in terms of h and $\bar g$. Thus, for $R = R^{(0)} + R^{(1)} + R^{(2)}$, we find

$$R^{(0)} = \bar g^{\mu\nu}\bar R_{\mu\nu} = \bar R\,. \tag{6.25}$$

At first order, we have

$$R^{(1)} = -h^{\mu\nu}R^{(0)}_{\mu\nu} + \bar g^{\mu\nu}R^{(1)}_{\mu\nu}\,, \tag{6.26}$$

and at second order we have

$$R^{(2)} = h^\mu_\alpha h^{\alpha\nu}R^{(0)} - h^{\mu\nu}R^{(1)}_{\mu\nu} + \bar g^{\mu\nu}R^{(2)}_{\mu\nu}\,. \tag{6.27}$$

When compiling all this information into finding the form of the Lagrangian, and writing everything in terms of the gravitational fields $\bar{g}$ and h, we find that

$$\mathscr{L} = \sqrt{-\bar{g}}\left[\frac{1}{2\kappa}\bar{R} + \mathscr{L}^{(1)} + \mathscr{L}^{(2)}\right] , \tag{6.28}$$

$$\mathscr{L}^{(1)} = \frac{1}{4\sqrt{\kappa}} h_{\mu\nu}\left(\bar{g}^{\mu\nu}\bar{R} - 2\bar{R}^{\mu\nu}\right) , \tag{6.29}$$

$$\begin{aligned}\mathscr{L}^{(2)} =& \frac{1}{8}D_\alpha h_{\mu\nu}D^\alpha h^{\mu\nu} - \frac{1}{8}D_\alpha h D^\alpha h + \frac{1}{4}D_\alpha h D_\beta h^{\alpha\beta} - \frac{1}{4}D_\alpha h_{\mu\beta}D^\beta h^{\alpha\mu} \\ &+ \frac{1}{8}\bar{R}\left(h^2 - h_{\mu\nu}h^{\mu\nu}\right) + \bar{R}^{\mu\nu}\left(\frac{1}{2}h^\lambda_\mu h_{\nu\alpha} - \frac{1}{4}hh_{\mu\nu}\right) .\end{aligned} \tag{6.30}$$

We see that actually, the bracketed term of $\mathscr{L}^{(1)}$ is the left hand side of the Einstein equation, (2.21). Therefore, if $\bar{T}^{\mu\nu} = 0$, then $\mathscr{L}^{(1)} = 0$. In the Lagrangian we are working with in this chapter, this is indeed the case (There is no matter term in the Lagrangian), so we are left with a quadratic Lagrangian. In fact, the result is even more general, as it will apply to non-vacuum states as well. The reason is that if we were to include a matter term into the Lagrangian, it would also have to be expanded around the background field, and the equation of motion would show up within the brackets again. Therefore, by assuming that the equation of motion holds for the background field, we may discard $\mathscr{L}^{(1)}$ altogether.

6.2 Gauge freedom

The Lagrangian we have found, (6.28), is invariant under the infinitesimal gauge transformation [14]

$$h'_{\mu\nu} = h_{\mu\nu} + (\bar{g}_{\alpha\nu} + h_{\alpha\nu})D_\mu\epsilon^\alpha + (\bar{g}_{\mu\alpha} + h_{\mu\alpha})D_\nu\epsilon^\alpha + \epsilon^\alpha D_\alpha h_{\mu\nu} , \tag{6.31}$$

where ϵ^α are infinitesimal functions. We can show that this transformation leaves the Lagrangian unchanged by direct insertion. It is also necessary to use the fact that a total derivative in the Lagrangian density does not contribute to the physics of the system, as it disappears when solving the integral by Gauss' theorem.

The existence a gauge transformation leaves the Lagrangian invariant forces us to fix the gauge.

6.2.1 Why is it necessary to fix the gauge?

The path integral that results from the Lagrangian will count over all paths that the system may take, i.e., all possible field configurations[12, Ch. 6]. However, if there is a gauge symmetry present, that is to say, an unphysical symmetry, then the path integral will count the results from the gauge symmetry as well (They are different field configurations, but not physically distinct configurations). Since the path integral should only count over the physically distinct possibilities, of which the gauge copies are not, it does not work when a gauge symmetry is present in the Lagrangian. Therefore, in order to avoid counting the same physical field configurations more than once, it is necessary to fix the gauge of the Lagrangian. This can be achieved by a gauge-fixing term which is added to the Lagrangian containing the original gauge symmetry.

Normally, one chooses a gauge condition and imposes it on the Lagrangian with the changes that follow. An example of this procedure can be found in Section 4.2.2. However, we want to change the Lagrangian in such a way that it fixes the gauge, i.e., the gauge is chosen beforehand. This would ensure that the path integral does not count over physically equivalent paths.

As an example, one may wish to impose the Lorenz gauge on some general electromagnetic Lagrangian, $\mathscr{L}_{\mathrm{EM}}$. The Lorenz gauge is defined by $\partial_\mu A^\mu = 0$, where A^μ is the electromagnetic four-potential. The gauge-fixing term one would add to the Lagrangian would the be $\mathscr{L}_{\mathrm{gf}} = \partial_\mu A^\mu$. Now, the Lagrangian is not gauge-invariant anymore, but it is also physically unchanged; The gauge has been chosen beforehand.

However, gauge fixing a Lagrangian is not as straight-forward as pictured here. Adding a gauge-fixing Lagrangian is equivalent to imposing a delta function on the path integral, only picking out physically distinct paths. There are subtleties involved in this procedure, from which the Faddeev-Popov ghost Lagrangian associated with the chosen gauge arises.

6.2.2 Gauge-fixing path integrals

Consider the path integral

$$Z = \int \mathcal{D}A_\mu \, e^{iS} \,, \tag{6.32}$$

where S is the action integral over the arbitrary field A_μ.[1] The path integral is invariant under some gauge transformation,

$$A_\mu^U = U A_\mu U^\dagger - iU\partial_\mu U^\dagger \,, \tag{6.33}$$

where U is the unitary matrix associated with the gauge transformation. We want to avoid the path integral overcounting physically equivalent configurations. The overcounting can be avoided by dividing out the equivalent configurations by using a delta function. To see how this is done, consider the integral

$$\Delta^{-1}(A_\mu) = \int \mathcal{D}U \, \delta\left(g\left(A_\mu^U\right)\right) \,, \tag{6.34}$$

where A_μ^U is the gauge-transformed field, g is a gauge condition, and $\mathcal{D}U$ is an integration measure of the gauge group $\mathscr{U}$. In other words, this is an integral that integrates over all of the gauge space, only when the gauge condition is equal to zero. The integration measure is assumed to be gauge invariant,

$$\mathcal{D}U = \mathcal{D}U'' \,, \qquad U'' = UU' \,. \tag{6.35}$$

[1] The number of indices on the arbitrary field is not important to the discussion, but we use A_μ as the field for the example because it is familiar from QED.

The integral can be shown to be gauge-invariant,

$$\begin{aligned}\Delta^{-1}(A_\mu^{U'}) &= \int \mathcal{D}U \, \delta \left[g \left[A_\mu^{UU'} \right) \right] \\ &= \int \mathcal{D}(UU') \, \delta \left(g \left[A_\mu^{UU'} \right) \right] \\ &= \int \mathcal{D}U'' \, \delta \left(g \left[A_\mu^{U''} \right) \right] \\ &= \Delta^{-1}(A_\mu) \, . \end{aligned} \tag{6.36}$$

Equation (6.34) can be rewritten as

$$1 = \Delta(A_\mu) \int \mathcal{D}U \, \delta(g(A_\mu^U)) \, . \tag{6.37}$$

It may thus be inserted into any path integral, without changing it. The term $\Delta(A_\mu)$ is called the Faddeev-Popov determinant, as it was first described by them [3]. Insertion into the path integral yields

$$\begin{aligned} Z &= \iint \mathcal{D}A_\mu \mathcal{D}U \, \Delta(A_\mu) \delta(g(A_\mu^U)) e^{iS} \\ &= \int \mathcal{D}U \int \mathcal{D}A_\mu \, \Delta(A_\mu) \delta(g(A_\mu^U)) e^{iS} \\ &= \int \mathcal{D}U \int \mathcal{D}A_\mu^U \, \Delta(A_\mu^U) \delta(g(A_\mu^U)) e^{iS^U} \\ &= \int \mathcal{D}U \int \mathcal{D}A_\mu \, \Delta(A_\mu) \delta(g(A_\mu)) e^{iS} \, , \end{aligned} \tag{6.38}$$

where the gauge invariance of the integration measure, the action integral, and the Faddeev-Popov determinant, was used in the last two steps. However, even though we assume that the integration measure is invariant under the gauge transformation, it has been pointed out by Fujikawa that this assumption is not always justified [4]. We will nevertheless make this assumption in this derivation. Thus, the integral $\int \mathcal{D}U$ is independent of the rest of the path integral, and may be evaluated in isolation from the rest of the path integral. It is the integral over the gauge group associated with the gauge that the path integral is invariant of. The correct procedure of not overcounting is thus to discard the integral of the gauge group. Then, we can compute the rest of the path integral, without having to worry about overcounting, since there is a delta function which guarantees that only paths under the same gauge condition are counted. The path integral has become a gauge-fixed path integral.

The Faddeev-Popov determinant is found by a change of variables, from U to g, in equation (6.34). The change of variables in an integral requires the inclusion of the Jacobian associated with the change of variables, $\mathcal{D}U = \mathcal{D}g \det \left|\frac{\delta U}{\delta g}\right|$. We thus obtain

$$\begin{aligned}\Delta^{-1}(A_\mu) &= \int \mathcal{D}U\, \delta(g(A_\mu^U)) \\ &= \int \mathcal{D}g\ \det\left|\frac{\delta U}{\delta g}\right| \delta g \\ &= \det\left|\frac{\delta U}{\delta g}\right|_{g=0}, \end{aligned} \tag{6.39}$$

which gives us the form of the Faddeev-Popov determinant

$$\Delta(A_\mu) = \det\left|\frac{\delta g}{\delta U}\right|_{g=0}. \tag{6.40}$$

Everything that has been done this far is completely general, and applies to any path integral that is invariant under some gauge transformation.

6.2.3 Choosing a gauge

The path integral of the system is given by

$$Z = \int \mathcal{D}h_{\mu\nu}\, e^{iS}. \tag{6.41}$$

We may impose a gauge on this path integral through the introduction of a delta function. The delta function ensures that only the field configurations that satisfy the gauge condition are counted. Such a path integral takes the form

$$Z = \int \mathcal{D}h_{\mu\nu}\, \delta(g^\alpha) \det\left|\frac{\delta g^\alpha}{\delta \epsilon^\beta}\right| e^{iS}. \tag{6.42}$$

If we assume a more general gauge condition, $g^\alpha = c^\alpha$, where c^α is some arbitrary function not dependent on the gauge variable, ϵ^β. The delta function then takes the form $\delta(g^\alpha - c^\alpha)$. The determinant is unchanged by this arbitrary change of gauge condition since the arbitrary function is not dependent on ϵ^b. Since we may choose arbitrary c^α, we may integrate over a set of possible c^α, and average them around $c^\alpha = 0$. This is referred to in the literature as averaging over c^α with Gaussian weights. Another way to look at this is to notice that since c^α is independent of the path integral, we are free to add any function depending on it to the path integral. Its effect is the same as adding a constant to the overall integral. Then, we may integrate over the delta function containing the gauge condition and obtain the desired form. The function we integrate over is arbitrary. We therefore choose to integrate over an exponential function in such a form that it fits in nicely with our quadratic Lagrangian, hence the use of *Gaussian weights*. To counteract this procedure, we are required to add a normalization constant, $N(\xi)$, as well, so that no real change is made. Completing this procedure yields

$$\begin{aligned} Z_\xi &= N(\xi) \int \mathcal{D}h_{\mu\nu} \mathcal{D}c^\alpha\, e^{-i\int \mathrm{d}^4x\, (c^\alpha)^2/2\xi} \delta(g^\alpha - c^\alpha) \det\left|\frac{\delta g_\alpha}{\delta \epsilon^\beta}\right| e^{iS} \\ &= N(\xi) \int \mathcal{D}h_{\mu\nu}\, e^{-i\int \mathrm{d}^4x\, (g^\alpha)^2/2\xi} \det\left|\frac{\delta g_\alpha}{\delta \epsilon^\beta}\right| e^{iS}. \end{aligned} \tag{6.43}$$

We recognize the integrand of the new exponential as the gauge-fixing Lagrangian, which will be added to the gauge invariant Lagrangian. To complete the procedure we also wish to express the determinant as some exponential so that we may include it in the Lagrangian. This can be achieved by using (D.17), and thus writing the determinant as a fermionic path integral over the fermionic fields η and $\bar{\eta}$,

$$\det M = \int \mathrm{d}\eta \mathrm{d}\bar{\eta}\, e^{i\int \mathrm{d}^4x\, \bar{\eta} M \eta}\,, \tag{6.44}$$

where M is some matrix, and η and $\bar{\eta}$ are fermionic fields. The Lagrangian associated with these fermionic fields is called the Faddeev-Popov ghost Lagrangian. Thus, the ghost Lagrangian is given by

$$\mathscr{L}_{\text{ghost}} = \bar{\eta}^{\alpha} M_{\alpha\beta} \eta^{\beta} \tag{6.45}$$

We will use the following gauge condition, extracted directly from t'Hooft and Veltman's treatment of quantum gravity [14],

$$g^{\alpha} = \sqrt[4]{-\bar{g}}\Big(D^{\nu} h_{\mu\nu} - \frac{1}{2} D_{\mu} h\Big) t^{\mu\alpha}\,, \tag{6.46}$$

where $t^{\mu}_{\beta} t^{\nu\beta} = \bar{g}^{\mu\nu}$. The gauge-fixing Lagrangian, $\mathscr{L}_{\text{gf}} = -\left(g^{\alpha}\right)^2/2\xi$, is thus

$$\mathscr{L}_{\text{gf}} = -\frac{1}{2\xi}\sqrt{-\bar{g}}\Bigg[\Big(D^{\nu} h_{\mu\nu} - \frac{1}{2} D_{\mu} h\Big)\Big(D_{\sigma} h^{\mu\sigma} - \frac{1}{2} D^{\mu} h\Big)\Bigg]. \tag{6.47}$$

The ghost Lagrangian is found through applying the gauge transformation on the gauge-fixing term, and then taking the derivative of it with respect to the gauge variable, ϵ^b. In other words, we need to calculate $g^{\alpha} \to g'^{\alpha}$. Thus, by replacing the perturbation field with its gauge-transformed part, $h_{\mu\nu} \to h'_{\mu\nu}$ and computing the results, we find the determinant. Terms which include the perturbation field are omitted, since the ghost field is non-physical, i.e., it is never external.[2] Performing the gauge transformation on the gauge condition yields

$$\begin{aligned}
C'^{\alpha} &= \sqrt[4]{-\bar{g}}\Big[D^{\nu} h'_{\mu\nu} - \frac{1}{2} D_{\mu} h'\Big] \\
&= \sqrt[4]{-\bar{g}}\Big[D^{\nu}\Big(h_{\mu\nu} + \bar{g}_{\gamma\nu} D_{\mu}\epsilon^{\gamma} + h_{\gamma\nu} D_{\mu}\epsilon^{\gamma} + \bar{g}_{\gamma\mu} D_{\nu}\epsilon^{\gamma} + h_{\gamma\mu} D_{\nu}\epsilon^{\gamma} + \epsilon^{\gamma} D_{\gamma} h\Big) \\
&\qquad - \frac{1}{2} D_{\mu}\Big(h + 2\bar{g}_{\gamma\nu} D^{\nu}\epsilon^{\gamma} + 2h_{\gamma\nu} D^{\nu}\epsilon^{\gamma} + \epsilon^{\gamma} D_{\gamma} h\Big)\Big] t^{\mu\alpha} \\
&= \sqrt[4]{-\bar{g}}\Big[D_{\gamma} D_{\mu}\epsilon^{\gamma} - D_{\mu} D_{\gamma}\epsilon^{\gamma} + \Box\epsilon\mu\Big] t^{\mu\alpha} \\
&= \sqrt[4]{-\bar{g}}\Big[\bar{R}_{\gamma\mu} + \Box\bar{g}_{\gamma\mu}\Big]\epsilon^{\gamma} t^{\mu\alpha}\,,
\end{aligned} \tag{6.48}$$

[2]Another way to look at this is to recognize that in the Feynman diagrams, there will be no vertices of the ghost field with other, physical, fields.

where, in the last step, the commutator relation between covariant derivatives, i.e., $\epsilon^a_{;bc} - \epsilon^a_{;cb} = -R^a_{dbc}\epsilon^d$, was used. Thus, we find that

$$\frac{\partial C'^\alpha}{\partial \epsilon^\gamma} = \sqrt[4]{-\bar{g}}\left[\bar{R}_{\gamma\mu} + \Box \bar{g}_{\gamma\mu}\right] t^{\mu\alpha} \,. \tag{6.49}$$

The procedure may now be completed, so we find the Faddeev-Popov ghost Lagrangian to be

$$\mathscr{L}_{\text{ghost}} = \sqrt{-\bar{g}}\bar{\eta}^\mu \left[\bar{R}_{\mu\nu} + \bar{g}_{\mu\nu}\Box\right]\eta^\nu \,, \tag{6.50}$$

where $\sqrt[4]{-\bar{g}}$ and $t^{\mu\alpha}$ have been transformed into the fermionic fields. The full Lagrangian density, $\mathscr{L}_f = \mathscr{L} + \mathscr{L}_{\text{gf}} + \mathscr{L}_{\text{ghost}}$, may now be written down. Then, the Feynman rules for the theory may be derived.

Chapter 7

Conclusion and outlook

Throughout this master thesis, we have gained a good understanding of general relativity, the weak field approximation and linearization procedure, and finally, the quantisation of gravity. In the investigation, some differences were found with respect to existing literature, e.g., the disagreement with Stelle's results in Chapter 5. This disagreement did not turn out to be of significance to the further discussion that ensued, but it will be important if one wishes to include perturbation terms of second order or higher.

Some conceptual details in the literature have been filled in; Among others, how to perform partial integrations of covariant derivatives comes to mind as an example. Also, in the pursuit to gain a thorough understanding of the procedures, some derivations have been presented which are not inspired by source material, e.g., the use of isotropic coordinates throughout the thesis. It was conceptually easier to solve problems in isotropic coordinates, and it should be employed more in teaching as it seems more pedagogical. The use of Schwarzschild coordinates is not necessary (or most straight-forward) when deriving the Schwarzschild solution in the weak field limit, nor is it the most suitable set of coordinates for illustratory purposes where other coordinate systems are often used (for example the Eddington-Finkelstein coordinates).

By the end of the investigation of Stelle's solution by including higher order terms to the Einstein-Hilbert action integral, we found that the difference to the gravitational potential would be insignificant at the quantum level. The constants in front of the higher-order terms, c_1 and c_2, were not contributing enough to the gravitational potential. We therefore proceeded to quantize only the first-order term of the Einstein-Hilbert action integral.

During the chapter on quantization of gravity, we made the remark that the integration measure, $\mathcal{D}h_{\mu\nu}$ is invariant under gauge transformations. This is assumed in most quantum field theories, but as we are aware, it may not always be assumed. This might be something worth looking into in greater detail. There is also a loose thread, since the matter Lagrangian should also be quantized in the quantum theory of gravity. This should be further expanded on. When that is complete, one may continue to make corrections and predictions of quantum effects of gravity.

One must remember that the quantization of gravity presented here is a low-energy approximation; It is an effective field theory, and not by any means a fundamental theory. This means that the theory has a range of validity, just like the Newtonian mechanics is an approximation which works well on the day to day energy range.

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Appendix A

Notations and conventions

The following notations and conventions are used in this thesis.

Units This thesis makes use of natural units, e.g., $c = h = 1$, unless specified otherwise.

Differentiation The following notation is applied,

$$\frac{\partial}{\partial x^\alpha} g_{\mu\nu} = \partial_\alpha g_{\mu\nu} = g_{\mu\nu,\alpha}. \tag{A.1}$$

A special case for a special notation is the dot notation; $\dot{q} = \partial_t q$. The dot notation will always be a shorthand notation for a differentiation with respect to time. If there is a case where the some entity is only dependent on one variable, the Newton notation may be used,

$$v'(r) = \frac{\mathrm{d}}{\mathrm{d}r} v(r) \tag{A.2}$$

Tensor notation Traces of tensors can be written either as

$$\eta^\alpha_\alpha \qquad \text{or} \qquad \eta\,. \tag{A.3}$$

Whenever a symbol that in the same context has been used as a tensor appears without its indices, the notation represents a trace. This notational convention is used extensively in the thesis for readability.

Metric tensor The sign convention for the metric tensor in this thesis is $(-, +, +, +)$. The metric tensor is assumed to be symmetric; $g_{\mu\nu} = g_{\nu\mu}$.

Additionally, $\det g$ is assumed to be invariant under coordinate transformations.

The Einstein summation convention When a lower and an upper index is repeated, a summation over all the values the indices can take is assumed. More precisely,

$$\sum_{i=1}^{n} a_i b^i = a_i b^i, \qquad i \in [1, n]\,. \tag{A.4}$$

When greek indices are used, the summation is taken from 0 to $n = 3$. When latin indices are used, the summation is taken from 1 to $n = 3$.

Appendix B

Tensor definitions

The Riemann tensor is given by

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{B.1}$$

The Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)\,. \tag{B.2}$$

The Ricci tensor is a special case of the Riemann tensor. It is defined as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}\,. \tag{B.3}$$

Lastly, the Ricci scalar is defined as

$$R = g^{\mu\nu}R_{\mu\nu} \tag{B.4}$$

Appendix C

Variational methods

C.1 Variation of the action

Hamilton's principle states that

> *The motion of the system from time t_1 to time t_2 is such that the line integral (called the* action *or the* action integral*),*
>
> $$I = \int_{t_1}^{t_2} L \, \mathrm{d}t \,, \tag{C.1}$$
>
> *where $L = T - V$, has a stationary value for the actual path of the motion. [6, Chapter 2, pp. 34-35]*

L is called the Lagrangian of the system. T and V are the kinetic and the potential energies of the system, respectively.

In other words, the motion of the system is the path that extremizes the action of the system. To further explore what this means, something has to be said about variational calculus.

C.1.1 Variational calculus

A variation in time, $t \in [a, b]$, of some coordinate variable, $q(t)$, is defined by

$$\delta q = \lim_{\epsilon \to 0} \frac{q(t, \epsilon) - q(t, 0)}{\epsilon} \,, \tag{C.2}$$

where $q(t, \epsilon) = q(t, 0) + \epsilon \eta(t)$. In other words, $\delta q = \eta(t)$, where $\eta(t)$ is completely arbitrary, except for the fact that the endpoints of the variation are fixed, $\eta(a) = \eta(b) = 0$.

Now, if we find the variation of the time derivative of the position variable, we find that it is

$$\delta \partial_t q = \partial_t \eta \,. \tag{C.3}$$

Obviously, when differentiating the variation of the position variable with respect to time, the same result is achieved. Hence,

$$\delta(\partial_t q) = \partial_t(\delta q)\,. \tag{C.4}$$

By extension of this commutation relation, we observe that integration and variation also has to commute, i.e.

$$\delta\left[\int \mathrm{d}t f(t)\right] = \int \mathrm{d}t \delta\left[f(t)\right]\,. \tag{C.5}$$

The variation of the action, (C.1), is hence done by

$$\delta I = \delta \int_{t_1}^{t_2} \mathrm{d}t\, L = \int_{t_1}^{t_2} \mathrm{d}t\, \delta L\,, \tag{C.6}$$

where L can depend on a number of variables. Since our definition of a variation is so similar to the normal definition of differentiation, we can utilize the chain rule in variational calculus as well, yielding

$$\delta L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) = \frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \cdots + \frac{\partial L}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2}\delta \dot{q}_2 + \ldots \tag{C.7}$$

C.1.2 The Euler-Lagrange equations

Consider a variation of some action which is dependent on some Lagrangian, $L(q_i, \dot{q}_i)$, where i ranges from 0 to some integer n. For the variation to yield the equations of motion, it would need to satisfy

$$\delta S = 0\,. \tag{C.8}$$

Applying the variation to the action yields

$$\begin{aligned} \delta S &= \delta \int \mathrm{d}t\, L(q_i, \dot{q}_i)\,, \\ &= \int \mathrm{d}t\, \delta L(q_i, \dot{q}_i)\,, \\ &= \int \mathrm{d}t\, \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right]\,. \end{aligned} \tag{C.9}$$

Utilizing the fact that the variation and the partial differentiation of q_i commutes, we can write $\delta \dot{q}_i = \partial_t \delta q_i$. Combining that and performing a partial integration, we obtain the final form of this integration

$$\begin{aligned} \delta S &= \int \mathrm{d}t\, \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\partial_t \delta q_i\right]\,, \\ &= \int \mathrm{d}t\, \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\bigg|_a^b - \int \mathrm{d}t\, \partial_t \frac{\partial L}{\partial \dot{q}_i}\delta q_i\,, \\ &= \int \mathrm{d}t\, \left[\frac{\partial L}{\partial q_i} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i\,. \end{aligned} \tag{C.10}$$

Since the variation is arbitrary except for at the endpoint (Hence why the integrated part of δS in the second step of (C.10) vanished), the parti inside the square brackets in the last equation has to be equal to zero. We hence have derived the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{C.11}$$

C.1.3 Extension to four-dimensional integrals

When it comes to field theory, an extension to four dimensional integrals, and Lagrangian densities, is necessary. The Lagrangian density is defined by

$$L = \iiint \mathrm{d}^3 q\, \mathcal{L}(q_i, \dot{q}_i)\,, \tag{C.12}$$

where $\mathcal{L}$ is the Lagrangian density. The Euler-Lagrange equations for a system described by a Lagrangian density is given by

$$\frac{\partial \mathcal{L}}{\partial q_i} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,. \tag{C.13}$$

To see this, we generalize the original Euler-Lagrange equations. First, we take a look at the first term

$$\frac{\partial L}{\partial q_i} = \frac{\partial}{\partial q_i}\iiint \mathrm{d}^3 x\, \mathcal{L} = \iiint \mathrm{d}^3 q\, \frac{\partial \mathcal{L}}{\partial q_i}\,, \tag{C.14}$$

and similarly for the second term

$$\partial_t \frac{\partial L}{\partial \dot{q}_i} = \partial_t \frac{\partial}{\partial \dot{q}_i}\iiint \mathrm{d}^3 x\, \mathcal{L} = \iiint \mathrm{d}^3 q\, \partial_t \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,, \tag{C.15}$$

so that the Euler-Lagrange equations for Lagrangian densities are obtained as

$$\frac{\partial \mathcal{L}}{\partial q_i} = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,. \tag{C.16}$$

Appendix D

Integrals and differential identities

D.1 Laplacian and the Dirac delta function

The Laplacian is in spherical coordinates given by

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\theta\frac{\partial}{\partial\phi}\right) . \tag{D.1}$$

The Laplacian of $1/r$ is then easily verified to be zero at every point except for at the origin. The divergence of $1/r$ is however given by

$$\nabla\frac{1}{r} = -\frac{\boldsymbol{r}}{r^3} . \tag{D.2}$$

Integrating the Laplacian of $1/r$ over an arbitrary sphere, and applying Gauss theorem, one obtains

$$\iiint_S \nabla^2\frac{1}{r}\mathrm{d}V = \iint_{\Delta S} -\frac{\boldsymbol{r}}{r^3}\cdot\mathrm{d}\boldsymbol{S} , \tag{D.3}$$

where $\delta\boldsymbol{S} = \hat{\boldsymbol{r}}\mathrm{d}A$, and $\mathrm{d}A$ can be written in normal spherical coordinates as $\mathrm{d}A = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$. We can therefore calculate the integral as

$$\iint_{\Delta S} -\frac{\boldsymbol{r}}{r^3}\cdot\mathrm{d}\boldsymbol{S} = -\int_0^{\pi}\int_0^{2\pi}\sin\theta\mathrm{d}\theta\mathrm{d}\phi = -4\pi . \tag{D.4}$$

Hence, since the Laplacian of $1/r$ is zero everywhere except at the origin, and the integral for any sphere centered around the origin is equal to -4π, the Laplacian must be given by

$$\nabla^2\frac{1}{r} = -4\pi\delta(r) . \tag{D.5}$$

D.2 Grassmann variables

This derivation of the Grassmann variables is taken from Kachelriess's notes on quantum field theory. The derivation towards the end is, however, changed from Kachelriess's notes. This is due to the derivation found there not being very pedagogical. A more pedagogical approach is presented here. One defines some algebra $\mathscr{G}$ where for the variables $a, b \in \mathscr{G}$ one requires the anti-commutation relations

$$\{a, a\} = \{b, b\} = \{a, b\} = 0\,. \tag{D.6}$$

We can thus determine that $a^2 = b^2 = 0$. All higher powers are automatically equal to zero by the same relations, e.g., $a, a^2 = 0$. We may also determine that $ab = -ba$. Any function, f, depending on a and b may thus be expanded into a power series,

$$\begin{aligned} f(a, b) &= f_0 + f_1 a + \tilde{f}_1 b + f_2 ab \\ &= f_0 + f_1 a + \tilde{f}_1 b - f_2 ba\,. \end{aligned} \tag{D.7}$$

The derivatives of the function are given by

$$\frac{\partial f}{\partial a} = f_1 + f_2\,, \qquad \frac{\partial f}{\partial b} = \tilde{f}_1 - f_2\,, \qquad \frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2\,. \tag{D.8}$$

For integration of the Grassmann variables, we require that also their differentiels, $\mathrm{d}a, \mathrm{d}b$ are also Grassmann variables,

$$\{a, \mathrm{d}a\} = \{\mathrm{d}a, \mathrm{d}a\} = \{b, \mathrm{d}b\} = \{\mathrm{d}b, \mathrm{d}b\} = \{a, \mathrm{d}b\} = \{\mathrm{d}a, b\} = \{\mathrm{d}a, \mathrm{d}b\} = 0\,. \tag{D.9}$$

We may now determine some integrals. Grassman variables require that their integrals are linear,

$$\int \mathrm{d}a\, [\alpha f(a) + \beta g(a)] = \alpha \int \mathrm{d}a f(a) + \beta \int \mathrm{d}a g(a)\,, \tag{D.10}$$

where α and β are normal constants. Their integrals are also defined to satisfy the condition that

$$\int \mathrm{d}a \left[\frac{\partial f(a)}{\partial a}\right] = 0\,. \tag{D.11}$$

We find that

$$\left(\int \mathrm{d}a\right)^2 = \left(\int \mathrm{d}a\right)\left(\int \mathrm{d}b\right) = -\left(\int \mathrm{d}b\right)\left(\int \mathrm{d}a\right) = -\left(\int \mathrm{d}a\right)^2\,, \tag{D.12}$$

which implies that $\int \mathrm{d}a = 0$. The second condition also implies that $\int \mathrm{d}a\, a = 1$. Thus, we may note that differentiation and integration are equivalent for Grassman variables.

Consider now a complex matrix, $M \in \mathbb{C}$, and complex Grassman variables, $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$, and their complex conjugates $\eta^* = (\eta_1^*, \eta_2^*, \ldots, \eta_n^*)$. We may evaluate the integral

$$\int \mathrm{d}^n\eta \mathrm{d}^n\eta^*\, e^{\bar{\eta} M \eta} = \int \mathrm{d}^n\eta \mathrm{d}^n\eta^*\, e^{\eta_i^* M_{ij} \eta_j} \tag{D.13}$$

When expanding the exponential into a sum, the integrand will contain many different terms. However, as we know from the properties of the Grassman integrals, only one term will surive; That is the term that contains all Grassman variables we are integrating over. We may expand our integrand as

$$\int \mathrm{d}^n\eta \mathrm{d}^n\eta^* \, \frac{1}{n!} \left[\eta^*_{i_1} M_{i_1 j_1} \eta_{j_1}\right] \left[\eta^*_{i_2} M_{i_2 j_2} \eta_{j_2}\right] \cdots \left[\eta^*_{i_n} M_{i_n j_n} \eta_{j_n}\right] . \tag{D.14}$$

The matrix components are normal, complex numbers, and therefore commutes with both the Grassman variables and between themselves. We may thus rewrite

$$\frac{1}{n!} \int \mathrm{d}^n\eta \mathrm{d}^n\eta^* \, \eta^*_{i_1} \eta_{j_1} \eta^*_{i_2} \eta_{j_2} \cdots \eta^*_{i_n} \eta_{j_n} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n} . \tag{D.15}$$

This integral needs to be permuted in such a way that the Grassman variable that is being integrated over is next to its differential operator. Performing that permutation and integrating over the Grassman variables yields

$$\frac{1}{n!} \epsilon_{i_1 \dots i_n} \epsilon_{j_1 \dots i_n} M_{i_1 j_1} \cdots M_{i_n j_n} = \det M , \tag{D.16}$$

where ϵ is the Levi-Civita symbol. We have thus derived that the determinant of some square, complex, matrix may be written as a path integral over Grassman variables (fermionic path integral),

$$\det M = \int \mathrm{d}\bar{\eta} \mathrm{d}\eta \, e^{\bar{\eta} M \eta} . \tag{D.17}$$

It is well known from quantum field theory that these types of integrals are *fermionic path integrals*.

D.3 Derivative of inverse matrix

The inverse of a matrix $\boldsymbol{A}(t)$ is defined such that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \tag{D.18}$$

where I is the identity matrix. In component form, this can be written as

$$A_{ij} A^{jk} = \delta^k_i , \tag{D.19}$$

where δ^k_i is the Kroenecker delta symbol. Applying a differentiation on (D.19) yields

$$\frac{\mathrm{d}A_{ij}}{\mathrm{d}t} A^{jk} = -A_{ij} \frac{\mathrm{d}A^{jk}}{\mathrm{d}t}, \tag{D.20}$$

which in matrix form is

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \boldsymbol{A}^{-1} = -\boldsymbol{A} \frac{\mathrm{d}\boldsymbol{A}^{-1}}{\mathrm{d}t} , \tag{D.21}$$

which leads to the following relation

$$\frac{\mathrm{d}\boldsymbol{A}^{-1}}{\mathrm{d}t} = -\boldsymbol{A}^{-1} \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \boldsymbol{A}^{-1} . \tag{D.22}$$

This result can also be written in component form as

$$\frac{\mathrm{d}A^{ik}}{\mathrm{d}t} = -A^{ij}\,\frac{\mathrm{d}A_{jl}}{\mathrm{d}t}\,A^{lk}\,. \tag{D.23}$$

Appendix E

Important identites

E.1 The Palatini Identity

If one varies the Riemann tensor, (B.1), one gets the expression

$$\delta R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \delta\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\sigma} - \delta\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\delta\Gamma^{\lambda}_{\mu\sigma}\,. \quad \text{(E.1)}$$

We can see from the way in which the variation was defined, (C.2), the variation of a Christoffel symbol is clearly the difference between two tensors, and hence, a tensor itself. We can therefore calculate its covariant derivative

$$\nabla_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) = \partial_{\lambda}(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\sigma\lambda}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\mu\lambda}\delta\Gamma^{\rho}_{\nu\sigma}\,. \quad \text{(E.2)}$$

As a result, (E.1), can be written in terms of covariant derivatives,

$$\delta R^{\rho}_{\sigma\mu\nu} = \nabla_{\mu}\left(\delta\Gamma^{\rho}_{\mu\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\rho}_{\mu\sigma}\right)\,. \quad \text{(E.3)}$$

If we now set the upper index and the second lower index equal, we find the variation of the Ricci tensor in terms of covariant derivatives. This variation is often refered to as the Palatini identity,

$$\delta R_{\mu\nu} = \nabla_{\rho}\left(\delta\Gamma^{\rho}_{\mu\nu}\right) - \nabla_{\nu}\left(\Gamma^{\rho}_{\rho\mu}\right)\,. \quad \text{(E.4)}$$

E.2 The Bianchi Identities

This derivation is taken directly from [16]. The covariant derivative of the Riemann tensor is given by

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\Big(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\Big)\,. \quad \text{(E.5)}$$

By cyclically permuting ν,κ, and η, one obtains the Bianchi identities

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0\,. \quad \text{(E.6)}$$

Contraction of λ and ν yields

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0\,. \tag{E.7}$$

Further contraction yields the contracted Bianchi identities

$$R_{;\eta} - 2R^{\mu}_{\eta;\mu} = 0\,. \tag{E.8}$$

Appendix F

Detailed derivations and calculations

F.1 Schwarzschild metric in isotropic coordinates

The derivation presented here is paraphrased from [1]. The Schawrzschild line element in isotropic form is

$$\mathrm{d}s^2 = -(1 - \frac{2m}{r})\mathrm{d}t^2 + \lambda^2(\rho)\mathrm{d}\sigma^2 \,, \tag{F.1}$$

where $\lambda(\rho)$ is an undetermined function that we will solve for, and $\mathrm{d}\sigma = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + \rho^2 \sin^2\theta\mathrm{d}\phi^2$. The Schwarzschild line element in its normal representation (3.40) is

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)c^2\mathrm{d}t^2 + \frac{1}{1-\frac{2m}{r}}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \,. \tag{F.2}$$

When comparing the two different forms, when comparing the angular part of the line element, we see that we must require that

$$r^2 = \lambda^2\rho^2 \,. \tag{F.3}$$

Additionaly, when comparing the radial parts of the line element, it is clear that

$$\frac{\mathrm{d}r^2}{1 - 2m/r} = \lambda^2\mathrm{d}\rho^2 \,. \tag{F.4}$$

Substituting λ^2 from (F.3) into (F.4) yields

$$\frac{\mathrm{d}r^2}{r^2 - 2mr} = \frac{\mathrm{d}\rho^2}{\rho^2} \,. \tag{F.5}$$

$$\frac{\pm\mathrm{d}r}{\sqrt{r^2 - 2mr}} = \frac{\mathrm{d}\rho}{\rho} \,. \tag{F.6}$$

Integration yields

$$\pm \ln \left[(r^2 - 2m)^{\frac{1}{2}} + r - m\right] = \ln \rho + C\,, \tag{F.7}$$

where C is some integration constant. Consider the limit where $r \ll m$. In that limit, we can approximate our relation between r and ρ as

$$\pm \ln(2r) = \ln \rho + C\,. \tag{F.8}$$

For large r, we want r and ρ to be roughly equal. In other words, we require that as

$$\lim_{r,\rho\to\infty} \frac{\pm \ln(2r)}{\ln \rho + C} = 1\,, \tag{F.9}$$

so we must choose the plus sign, and choose $C = \ln 2$. We then have

$$\sqrt{r^2 - 2mr} + r - m = 2\rho\,. \tag{F.10}$$

Utilizing the fact that

$$\left[r - m + \sqrt{r^2 - 2mr}\right]\left[r - m - \sqrt{r^2 - 2mr}\right] = m^2\,, \tag{F.11}$$

we are able to find, by multiplying (F.11) into (F.9),

$$r - m - \sqrt{r^2 - 2mr} = \frac{m^2}{2\rho}\,. \tag{F.12}$$

By adding (F.12) into (F.9), we find that

$$r = \rho\left(1 + \frac{m}{2\rho}\right)^2\,. \tag{F.13}$$

By applying (F.13) in (F.3), we find that $\lambda(\rho)$ is

$$\lambda(\rho) = \left(1 + \frac{m}{2\rho}\right)^2\,. \tag{F.14}$$

Since we now know the form of $r(\rho)$, we can use this to find the form of the coefficient of the time differential of the Schwarzschild line element in isotropic coordinates, (F.1). Thus, by applying (F.13), we find that

$$\left(1 - \frac{2m}{r}\right) = \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}\,. \tag{F.15}$$

Now, we can use all that we have found, to find the full form of the Schwarzschild element in isotropic coordinates. By applying (F.14) and (F.15) to (F.3), we find that

$$\mathrm{d}s^2 = -\frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}\mathrm{d}t^2 + \left(1 + \frac{m}{2\rho}\right)^4 \mathrm{d}\sigma^2\,. \tag{F.16}$$

F.2 Finding the Green's function to the second-order Einstein equation in isotropic coordinates

The equation of motion, (5.30), can be written in the form

$$Y_{\mu\nu\alpha\beta} h^{\alpha\beta} = T_{\mu\nu} \,, \tag{F.17}$$

where we have defined

$$\begin{aligned} Y = \Big[\quad & \frac{c_0}{\kappa} \left(\eta_{\mu\nu}\eta_{\alpha\beta}\Box - 2\eta_{\alpha\mu}\eta_{\beta\nu}\Box \right) + c_1 \left(\partial_\mu \partial_\nu \eta_{\alpha\beta}\Box - \eta_{\mu\nu}\eta_{\alpha\beta}\Box\Box \right) \\ & + c_2 \left(2\partial_\mu\partial_\nu\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\eta_{\alpha\beta}\Box\Box - 2\eta_{\mu\alpha}\eta_{\nu\beta}\Box\Box \right) \Big] \,. \end{aligned} \tag{F.18}$$

As in chapter 4.2.1, if there is a Green's function, $Z^{\mu\nu\gamma\delta}$ for this differential equation, then it has to obey

$$Y_{\mu\nu\alpha\beta} Z^{\mu\nu\gamma\delta} = \delta^\gamma_\alpha \delta^\delta_\beta \delta(x - x') \,. \tag{F.19}$$

By Fourier-transforming (F.19), we obtain

$$\begin{aligned} Y_{\mu\nu\alpha\beta} = & - \frac{c_0}{\kappa} \left(\eta_{\mu\nu}\eta_{\alpha\beta}k^2 - 2\eta_{\mu\alpha}\eta_{\nu\beta}k^2 \right) - c_1 \eta_{\alpha\beta}k_\mu k_\nu k^2 - c_1 \eta_{\mu\nu}\eta_{\alpha\beta}k^4 \\ & + 2c_2 k_\mu k_\nu \eta_{\alpha\beta} k^2 - c_2 \eta_{\mu\nu}\eta_{\alpha\beta}k^4 - 2c_2 \eta_{\mu\alpha}\eta_{\nu\beta}k^4 \,, \end{aligned} \tag{F.20}$$

$$\begin{aligned} Z^{\mu\nu\gamma\delta} = \quad & A(k^2)\eta^{\mu\nu}\eta^{\gamma\delta} + B(k^2)\eta^{\mu\gamma}\eta^{\nu\delta} + C(k^2)\eta^{\mu\delta}\eta^{\nu\gamma} \\ & + D(k^2)k^\mu k^\nu \eta^{\gamma\delta} + E(k^2)k^\mu k^\gamma \eta^{\nu\delta} + F(k^2)k^\mu k^\delta \eta^{\nu\gamma} \\ & + G(k^2)k^\nu k^\gamma \eta^{\mu\delta} + H(k^2)k^\nu k^\delta \eta^{\mu\gamma} + I(k^2)k^\gamma k^\delta \eta^{\mu\nu} \,, \end{aligned} \tag{F.21}$$

$$Y_{\mu\nu\alpha\beta} Z^{\mu\nu\gamma\delta} = \delta^\gamma_\alpha \delta^\delta_\beta \,. \tag{F.22}$$

We calculate the left hand side of equation (F.22), and look for unique terms which cannot be cancelled out in any other way than to set the coefficient in front of it to zero.

$$\begin{aligned} YZ = \quad & A\left[-2\frac{c_0}{\kappa}\eta_{\alpha\beta}\eta^{\gamma\delta}k^2 + c_1\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 - 5c_1\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 - 4c_2\eta_{\alpha\beta}\eta^{\gamma\delta}k^4\right] \\ & +B\left[\begin{array}{l} -\frac{c_0}{\kappa}\eta_{\alpha\beta}\eta^{\gamma\delta}k^2 + 2\frac{c_0}{\kappa}\delta^\gamma_\alpha\delta^\delta_\beta k^2 - c_1\eta_{\alpha\beta}k^\gamma k^\delta k^2 - c_1\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 \\ +2c_2k^\gamma k^\delta\eta_{\alpha\beta}k^2 - c_2\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 - 2c_2\delta^\gamma_\alpha\delta^\delta_\beta k^4 \end{array}\right] \\ & +C\left[\begin{array}{l} -\frac{c_0}{\kappa}\eta_{\alpha\beta}\eta^{\gamma\delta}k^2 + 2\frac{c_0}{\kappa}\delta^\delta_\alpha\delta^\gamma_\beta k^2 - c_1\eta_{\alpha\beta}k^\delta k^\gamma k^2 - c_1\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 \\ +2c_2k^\gamma k^\delta\eta_{\alpha\beta}k^2 - c_2\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 - 2c_2\delta^\delta_\alpha\delta^\gamma_\beta k^4 \end{array}\right] \\ & +D\left[-\frac{c_0}{\kappa}\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 + 2\frac{c_0}{\kappa}\eta^{\gamma\delta}k_\alpha k_\beta k^2 + c_2\eta_{\alpha\beta}\eta^{\gamma\delta}k^6 - 2c_2k_\alpha k_\beta\eta^{\gamma\delta}k^4\right] \\ & +E\left[-\frac{c_0}{\kappa}\eta_{\alpha\beta}k^\gamma k^\delta k^2 + 2\frac{c_0}{\kappa}k_\alpha k^\gamma\delta^\delta_\beta k^2 + c_2k^\gamma k^\delta\eta_{\alpha\beta}k^4 - c_22k_\alpha k^\gamma\delta^\delta_\beta k^4\right] \\ & +F\left[-\frac{c_0}{\kappa}\eta_{\alpha\beta}k^\gamma k^\delta k^2 + 2\frac{c_0}{\kappa}k_\alpha k^\delta\delta^\gamma_\beta k^2 + c_2k^\gamma k^\delta\eta_{\alpha\beta}k^4 - 2c_2k_\alpha k^\delta\delta^\gamma_\beta k^4\right] \\ & +G\left[-\frac{c_0}{\kappa}\eta_{\alpha\beta}k^\gamma k^\delta k^2 + 2\frac{c_0}{\kappa}k_\beta k^\gamma\delta^\delta_\alpha k^2 + c_2k^\gamma k^\delta\eta_{\alpha\beta}k^4 - 2c_2k_\beta k^\gamma\delta^\delta_\alpha\right] \\ & +H\left[-\frac{c_0}{\kappa}\eta_{\alpha\beta}k^\gamma k^\delta k^2 + 2\frac{c_0}{\kappa}k_\beta k^\delta\delta^\gamma_\alpha k^2 + c_2k^\gamma k^\delta\eta_{\alpha\beta}k^4 - 2c_2k_\beta k^\delta\delta^\gamma_\alpha\right] \\ & +I\left[-2\frac{c_0}{\kappa}\eta_{\alpha\beta}k^\gamma k^\delta k^2 - 5c_1\eta_{\alpha\beta}k^\gamma k^\delta k^4 - 4c_2k^\gamma k^\delta\eta_{\alpha\beta}k^4\right] . \end{aligned} \tag{F.23}$$

Due to unique terms, we see that $C = D = E = F = G = H = 0$. We thus obtain

$$\begin{aligned} YZ = \quad & A\left(-2\frac{c_0}{\kappa} - 5c_1k^2 - 4c_2k^2\right)\eta_{\alpha\beta}\eta^{\gamma\delta}k^2 \\ & +B\left[\begin{array}{l} -\frac{c_0}{\kappa}\eta_{\alpha\beta}\eta^{\gamma\delta}k^2 + 2\frac{c_0}{\kappa}\delta^\gamma_\alpha\delta^\delta_\beta k^2 - c_1\eta_{\alpha\beta}k^\gamma k^\delta k^2 - c_1\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 \\ +c_22k^\gamma k^\delta\eta_{\alpha\beta}k^2 - c_2\eta_{\alpha\beta}\eta^{\gamma\delta}k^4 - 2c_2\delta^\gamma_\alpha\delta^\delta_\beta k^4 \end{array}\right] \\ & +I\left(-2\frac{c_0}{\kappa} - 5c_1k^2 - 4c_2k^2\right)\eta_{\alpha\beta}k^\gamma k^\delta k^2 . \end{aligned} \tag{F.24}$$

For our condition, (F.22), to hold, we need

$$A = -\frac{\frac{c_0}{\kappa} + (c_1 + c_2)k^2}{k^2\left[1 - 2c_2k^2\right]\left[1 + (5c_1 + 4c_2)k^2\right]} , \tag{F.25}$$

$$I = \frac{(2c_2 - c_1)}{k^2\left[1 - 2c_2k^2\right]\left[1 + (5c_1 + 4c_2)k^2\right]} , \tag{F.26}$$

$$B = \frac{2\frac{c_0}{\kappa}}{k^2\left[1 - 2c_2k^2\right]} . \tag{F.27}$$

It is thus clear, after some algebra, that our propagator is given by

$$Z^{\mu\nu\gamma\delta} = \frac{\left[2\frac{c_0}{\kappa} + (5c_1 + 4c_2)k^2\right]\eta^{\mu\gamma}\eta^{\nu\delta} - \left[\frac{c_0}{\kappa} + (c_1 + c_2)k^2\right]\eta^{\mu\nu}\eta^{\gamma\delta} + \left[-c_1 + 2c_2\right]k^\gamma k^\delta\eta^{\mu\nu}}{k^2\left[1 - 2c_2k^2\right]\left[1 + (5c_1 + 4c_2)k^2\right]} . \tag{F.28}$$

When setting $c_1 = c_2 = 0$, we obtain the propagator for the first-order equation of motion,

$$Z^{\mu\nu\gamma\delta} = \frac{c_o}{k^2}\left(2\eta^{\mu\gamma}\eta^{\nu\delta} - \eta^{\mu\nu}\eta^{\gamma\delta}\right) , \tag{F.29}$$

which is easily shown to be the correct propagator when inserting directly into (F.19).