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Convolutions for Localization Operators

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Problem description

In a paper from 1984, the physicist Reinhard Werner studied what he called *quantum harmonic analysis on phase space*. Werner was able to extend several results from classical harmonic analysis of functions to results on bounded operators on Hilbert spaces. As his main tools, Werner defined convolutions between operators and functions along with a corresponding Fourier transform of operators.

Today this theory has been expanded and to some degree applied, but mainly in mathematical physics. The original aim of this thesis was therefore to collect and formulate Werner's theory in a precise mathematical form, that would be accessible to mathematicians with no background in physics. In addition to this my supervisor Franz Luef expected that we would discover applications of Werner's theory to time-frequency analysis while preparing the thesis, and that results from other parts of mathematics could be used to shed light on Werner's theory. The aims of this thesis may therefore be summed up as follows:

- Give a precise exposition of Werner's theory of quantum harmonic analysis.
- Investigate possible applications of Werner's theory to time-frequency analysis.
- Apply results from other areas of mathematics to illuminate and improve Werner's results.

Abstract

The theory of quantum harmonic analysis on phase space introduced by Werner is presented and formulated precisely using the terminology of time-frequency analysis and abstract harmonic analysis. Convolutions of functions with operators and of operators with operators are introduced, along with a corresponding Fourier transform of operators – the Fourier-Wigner transform. Using these concepts we formulate and prove a version of Wiener’s Tauberian theorem for operators due to Werner. The main novel result of the thesis is a formulation of the so-called localization operators using the convolution of a function with an operator, which gives a conceptual framework for localization operators and an extension of results by Bayer and Gröchenig. The connection to quantum harmonic analysis provides new perspectives on results in time-frequency analysis. In particular, Lieb’s uncertainty principle is seen to be a special case of a Hausdorff-Young inequality for operators, which in turn leads to an improvement of this Hausdorff-Young inequality. We also show a generalization of the Berezin-Lieb inequalities, and relate this and the convolutions to results by Klauder and Skagerstam. The theory of Banach modules is used to prove new results on the convolutions, and the Fourier-Wigner transform is shown to be related to the so-called Arveson spectrum. Finally the convolutions are considered in the context of modulation spaces, inspired by the existing literature on localization operators and modulation spaces.

Sammendrag

Teorien om kvante-harmonisk analyse på faserommet som introdusert av Werner blir presentert og presist formulert i terminologi fra tid-frekvensanalyse og abstrakt harmonisk analyse. Vi introduserer konvolusjonen av en funksjon med en operator og konvolusjonen av to operatorer, samt en tilhørende Fouriertransformasjon for operatorer. Ved hjelp av disse konseptene viser vi en versjon Wieners Tauberske teorem for operatorer, som først ble vist av Werner. Det fremste nye resultatet i oppgaven er at såkalte lokaliseringoperatorer formuleres som en konvolusjon av en funksjon med en operator, noe som gir lokaliseringoperatorer et konseptuelt rammeverk og fører til en forbedring av resultater av Bayer og Gröchenig. Koblingen til kvante-harmonisk analyse fører til nye perspektiver på kjente resultater i tid-frekvensanalyse. Spesielt viser vi at Lieb’s uskarpheitsrelasjon er et spesialtilfelle av en Hausdorff-Young-ulikhet for operatorer, noe som også fører til en forbedring av denne Hausdorff-Young-ulikheten. Vi viser også en generalisering av Berezin-Lieb-ulikhetene, og knytter denne samt konvolusjonene til teorier utviklet av Klauder og Skagerstam. Videre benyttes teorien om Banachmoduler til å vise nye resultater for konvolusjonene, og at Fourier-Wigner-transformasjonen kan knyttes til det såkalte

Arvesonspektret. Til slutt vises noen resultater om konvolusjonene og moduleringsrom, inspirert av den eksisterende litteraturen om lokaliseringsoperatorer og moduleringsrom.

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1 Introduction

In a paper from 1984, the physicist Reinhard Werner introduced the study of what he called *quantum harmonic analysis on phase space* [46]. His goal was to construct a theory of harmonic analysis that included both classical mechanics, which deals with functions on \mathbb{R}^{2d} , and quantum mechanics, which deals with operators on the Hilbert space $L^2(\mathbb{R}^d)$. In classical harmonic analysis, the convolution of two functions is a key ingredient. A crucial part in Werner's theory is therefore the definition of a convolution of an operator S with a function f , $f * S$, and a convolution of two operators S and T , $S * T$.

A second key ingredient in classical harmonic analysis is the Fourier transform \mathcal{F} of functions, and in order to include operators in harmonic analysis Werner defined a Fourier transform \mathcal{F}_W of operators. This Fourier transform was defined such that the convolutions and Fourier transforms interacted in the expected way; for instance $\mathcal{F}_W(f * S) = \mathcal{F}(f)\mathcal{F}_W(S)$. Equipped with these two concepts, Werner extended theorems in classical harmonic analysis to operators; in particular he obtained an operator-version of Wiener's celebrated Tauberian theorem [29, 46, 48].

Since Werner's theories seem to have received little attention outside of mathematical physics, the first goal for this thesis is to explain Werner's theory in a coherent and precise way suitable for mathematicians with no background in physics. We will formulate the theory using the terminology of abstract harmonic analysis and time-frequency analysis, providing detailed proofs and some novel results along the way.

We will also consider the *localization operators*, first discussed in the works of Berezin [6] as a quantization rule, and later introduced into the context of time-frequency analysis by Daubechies [14]. Given two functions φ_1, φ_2 on \mathbb{R}^d , called windows, and a function f on \mathbb{R}^{2d} , one obtains a localization operator $A_f^{\varphi_1, \varphi_2}$ on $L^2(\mathbb{R}^d)$. Our second goal is to show the novel result that the localization operators can be described as a special case of Werner's theory of convolutions. As we aim to show, this provides localization operators with a conceptual framework, and some of Werner's general results immediately strengthen the known results for localization operators. Our main example is the question asked by Bayer and Gröchenig in a paper from 2014 [3]: what conditions must be imposed on the windows φ_1, φ_2 to guarantee that the set $\{A_f^{\varphi_1, \varphi_2} : f \in L^1(\mathbb{R}^{2d})\}$ is dense in different spaces of operators? We will see that in some cases this question is answered by Werner's generalization of Wiener's Tauberian theorem, and in particular we will be able to improve some of Bayer and Gröchenig's results from implications to equivalences in theorem 7.5.

The third and final goal of the thesis is to show how different mathematical concepts and techniques can be used to shed light on Werner's theories. For instance, localization operators have been studied in the framework of the so-called

modulation spaces of functions [3], and since the localization operators are special cases of convolutions, we will try to study Werner's convolutions using modulation spaces. Many of Werner's results will be shown to be generalizations of familiar theorems in time-frequency analysis. For instance, the Hausdorff-Young inequality for the Fourier transform of operators includes Lieb's uncertainty principle [23] as a special case, and we will use this connection to improve this Hausdorff-Young inequality. We will also show how Werner's extended Fourier transform is related to the concept of spectrum defined by Arveson [1]. Furthermore, we will see that parts of Werner's theory can be thought of as the construction of Banach modules [22], and the celebrated Cohen-Hewitt theorem is then used to prove new results in Werner's theory.

In addition to this, a preprint is being prepared that relates the convolutions of Werner to the phase space representations due to Klauder and Skagerstam [31, 32]. This connection is included in appendix B, where we in particular prove a generalized Berezin-Lieb inequality.

At the time of completion of this thesis, a preprint with the same title has been written by Franz Luef and the author [35]. The preprint is based on the thesis, and in particular aims to introduce the convolutions as a conceptual framework for localization operators, as is done in sections 5 and 7.1. The preprint also includes the connections to Banach modules in section 7.2 and the Arveson spectrum in section 6.2.

The thesis is structured as follows. Section 2 introduces the necessary background material, including a thorough introduction to the theory of vector-valued integration. Thereafter, sections 3 and 4 introduce Werner's convolutions, and the relation between these convolutions and localization operators is made explicit in section 5. Section 6 then introduces and discusses Werner's Fourier transform for operators, and Werner's generalization of Wiener's Tauberian theorem is discussed and proved in section 7, including its consequences for localization operators and some applications of the theory of Banach modules. The convolution on modulation spaces and corresponding classes of operators is then discussed in section 8. Finally some connections to quantum mechanics are explored in two appendices. The first considers Werner's motivation from quantum mechanics for studying his convolutions. The second shows that Werner's convolutions provide a conceptual framework for the phase space representations of Klauder and Skagerstam.

2 Prerequisites

2.1 Notation and conventions

Before we turn our attention to the necessary background material, we will fix some notation and conventions. Much of the time we will work with functions on phase space, i.e. \mathbb{R}^{2d} , and whenever this notation is used we will tacitly assume that $d \in \mathbb{N}$. Furthermore, we will use Latin letters such as f and g to denote functions on phase space \mathbb{R}^{2d} , and Greek letters such as ψ and ϕ to denote functions on \mathbb{R}^d . Elements of \mathbb{R}^{2d} will often be written in the form $z = (x, \omega)$ for $x, \omega \in \mathbb{R}^d$.

A recurrent theme will be duality and the action of bounded linear functionals on a Banach space. If X is a Banach space we will denote its dual space by X^* , and for $x \in X$ and $x^* \in X^*$ we write $\langle x^*, x \rangle$ to denote $x^*(x)$. In order to agree with inner product notation, we will always take the duality bracket $\langle \cdot, \cdot \rangle$ to be antilinear in the second argument. We are therefore strictly speaking considering antilinear functionals, but the antilinear functionals are exactly the pointwise complex conjugates of the linear functionals, so this is of little consequence.

If ψ and ϕ are functions on \mathbb{R}^d , then we write $\psi \otimes \phi$ for the function on \mathbb{R}^{2d} defined by $\psi \otimes \phi(x, \omega) = \psi(x)\phi(\omega)$. Similarly, for two elements ξ, η in some Hilbert space \mathcal{H} , we define the operator $\xi \otimes \eta$ on \mathcal{H} by $\xi \otimes \eta(\zeta) = \langle \zeta, \eta \rangle \xi$, where $\zeta \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle$ is the inner product on \mathcal{H} . In most cases \mathcal{H} will in fact consist of functions, so that $\xi \otimes \eta$ may be interpreted either as a new function or as an operator, but the correct interpretation will always be clear from the context.

The class of Schwartz functions on \mathbb{R}^d will be denoted by $\mathcal{S}(\mathbb{R}^d)$, and the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$.

Finally we will need to fix some notation for operations on functions.

Definition 2.1. Let $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function. We define the functions ψ^* and $\check{\psi}$ on \mathbb{R}^d as well as the *parity operator* P by

$$\begin{aligned}\psi^*(t) &= \overline{\psi(t)} \\ \check{\psi}(t) &= P\psi(t) = \psi(-t).\end{aligned}$$

We will sometimes refer to $\check{\psi}$ as the *reflection* of ψ .

2.2 Positive operators and polar decomposition

Let \mathcal{H} be a Hilbert space. We say that a bounded operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is *positive* if $\langle A\xi, \xi \rangle \geq 0$ for any $\xi \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle$ is the inner product on \mathcal{H} . To introduce the classes of operators that we will work with, we need the following result [36, Ch. 2.2].

Proposition 2.1. *Let A be a positive operator on \mathcal{H} . There exists a unique positive operator \sqrt{A} on \mathcal{H} such that $\sqrt{A}\sqrt{A} = A$. Furthermore, for any operator B the operator B^*B is positive, and we write $\sqrt{B^*B} = |B|$.*

The relation between an operator A and $|A|$ is given by the so-called *polar decomposition*. Recall that a bounded operator U on \mathcal{H} is a *partial isometry* if $U|_{\ker U^\perp}$ is an isometry.

Proposition 2.2. *Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then there exists a unique partial isometry U such that $A = U|A|$.*

2.3 Schatten p -classes of operators

Many of our results will deal with *quantization procedures*, which mathematically speaking are procedures for assigning an operator A_f on a Hilbert space \mathcal{H} to a function f on \mathbb{R}^{2d} . A recurrent question will be whether functions from a given function space give bounded operators with some specified properties. We will therefore need to discuss different classes of operators, and will restrict ourselves to bounded operators on the Hilbert space $L^2(\mathbb{R}^d)$, denoted by $B(L^2(\mathbb{R}^d))$.

All of the classes of operators that we will discuss, except for $B(L^2(\mathbb{R}^d))$ itself, will be subspaces of the *compact operators* $K(L^2(\mathbb{R}^d))$, i.e. operators that are the limit in the operator norm of operators with finite-dimensional range. Operators with finite-dimensional range are frequently referred to as *finite rank* operators. To introduce the classes that we will study, we need the following theorem, which introduces the so-called *singular value decomposition* of a compact operator. A proof may be found in most texts on functional analysis, for instance [12, 40, 42].

Theorem 2.3. *Let T be a compact operator on $L^2(\mathbb{R}^d)$. There exist two orthonormal sets $\{\psi_n\}_{n \in \mathbb{N}}$ and $\{\phi_n\}_{n \in \mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a sequence $\{s_n(T)\}_{n \in \mathbb{N}}$ of positive numbers such that $s_n(T) \rightarrow 0$, and T may be expressed as*

$$T = \sum_{n \in \mathbb{N}} s_n(T) \psi_n \otimes \phi_n,$$

where the convergence of the sum is in the strong topology on $B(L^2(\mathbb{R}^d))$.

The numbers $\{s_n(T)\}_{n \in \mathbb{N}}$ are called the *singular values* of T , and are the eigenvalues of the operator $|T|$. Thus they are in particular uniquely determined.

Convergence in the strong topology means that the sum $\sum_{n \in \mathbb{N}} s_n(T) \langle \xi, \phi_n \rangle \psi_n$ converges to $T\xi$ in the norm of $L^2(\mathbb{R}^d)$ for any fixed $\xi \in L^2(\mathbb{R}^d)$.

The singular values associate a sequence of positive numbers to any compact operator. This sequence will now be used to introduce the Schatten classes of operators.

Definition 2.2. Let $1 \leq p < \infty$. The *Schatten p -class* of operators on $L^2(\mathbb{R}^d)$ is the set \mathcal{T}^p of compact operators given by

$$\mathcal{T}^p = \{T : (s_n(T))_{n \in \mathbb{N}} \in \ell^p\}$$

Furthermore, we let \mathcal{T}^∞ denote $B(L^2(\mathbb{R}^d))$.

Remark. 1. If T is a compact operator, the sequence of singular values of T converge to zero, so we may consider the supremum $\sup_{n \in \mathbb{N}} |s_n(T)|$. It is not difficult to show by direct calculation that this supremum is the operator norm $\|T\|_{B(L^2)}$.

2. The notation $\mathcal{T}^\infty = B(L^2(\mathbb{R}^d))$ is convenient when dealing with complex interpolation (see section 2.10, but the reader should note that other sources such as [3] use \mathcal{T}^∞ to denote the compact operators on $L^2(\mathbb{R}^d)$).

The next theorem uses the singular value decomposition to introduce a norm on the Schatten p -classes, and asserts that they are Banach spaces under this norm. A proof may be found in [43, Thm. 2.7].

Theorem 2.4. Let $1 \leq p < \infty$ and $T \in \mathcal{T}^1$. The expression $\|T\|_{\mathcal{T}^p} = \left(\sum_{n \in \mathbb{N}} s_n(T)^p \right)^{1/p}$ defines a norm on \mathcal{T}^p , and this norm makes \mathcal{T}^p a Banach space under pointwise addition and scalar multiplication. Furthermore, the spaces \mathcal{T}^p are ideals in $B(L^2(\mathbb{R}^d))$, meaning that $A \in B(L^2(\mathbb{R}^d))$ and $T \in \mathcal{T}^p$ implies that $AT, TA \in \mathcal{T}^p$.

Since the norms have been introduced as the usual ℓ^p -norms of sequences, well known results carry over from the theory of ℓ^p -spaces. For instance, $1 \leq p \leq q < \infty$ implies that $\mathcal{T}^p \subset \mathcal{T}^q$ and $\|\cdot\|_{B(L^2)} \leq \|\cdot\|_{\mathcal{T}^q} \leq \|\cdot\|_{\mathcal{T}^p} \leq \|\cdot\|_{\mathcal{T}^1}$. If an operator lies in \mathcal{T}^p for $p < \infty$, the singular value decomposition will converge in the $\|\cdot\|_{\mathcal{T}^p}$ -norms, and as a consequence the finite rank operators are dense in \mathcal{T}^p for $p < \infty$. It follows trivially that \mathcal{T}^p is a dense subspace of \mathcal{T}^q in the norm $\|\cdot\|_q$ whenever $1 \leq p \leq q < \infty$.

2.3.1 The trace and trace class operators

We will be especially interested in the class \mathcal{T}^1 . This space can also be described as the space of *trace class operators*. The concept of the trace of a matrix may be extended to a general operator T on $L^2(\mathbb{R}^d)$ by picking an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$ and defining $\text{tr}(T) = \sum_{n \in \mathbb{N}} \langle T e_n, e_n \rangle$. The sum in the definition of $\text{tr}(T)$ will not converge for an arbitrary compact operator T , and one therefore defines the trace class operators to be those operators T where $\text{tr}(|T|) < \infty$.

To see that the set of trace class operators coincides with \mathcal{T}^1 , write $|T|$ as $|T| = \sum_{n \in \mathbb{N}} s_n(T) \psi_n \otimes \psi_n$, where $\{\psi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of eigenvectors of $|T|$. This is possible by the spectral decomposition of compact operators, see [9, Thm. 7.30]. Calculating $\text{tr}(|T|)$ using the orthonormal basis $\{\psi_n\}_{n \in \mathbb{N}}$ one then finds that $\text{tr}(|T|) = \sum_{n \in \mathbb{N}} s_n(T)$, hence $\text{tr}(|T|) < \infty$ if and only if $T \in \mathcal{T}^1$. In particular we see that $\|T\|_{\mathcal{T}^1} = \text{tr}(|T|)$.

The next proposition, mainly from VI.18 and VI.25 in [40], collects the different properties of the trace that we are going to need later.

Proposition 2.5. *Let $S, T \in \mathcal{T}^1$, $A \in B(L^2(\mathbb{R}^d))$ and $\lambda \in \mathbb{C}$. The trace of S , given by $\text{tr}(S) = \sum_{n \in \mathbb{N}} \langle S e_n, e_n \rangle$ for some orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$, is independent of the orthonormal basis used to calculate it. Furthermore,*

1. $\text{tr}(\lambda S + T) = \lambda \text{tr}(S) + \text{tr}(T)$.
2. $S^* \in \mathcal{T}^1$, and $\text{tr}(S^*) = \overline{\text{tr}(S)}$.
3. $\text{tr}(AS) = \text{tr}(SA)$.
4. $\sum_{n \in \mathbb{N}} |\langle A S e_n, e_n \rangle| \leq \|A\|_{B(L^2)} \|S\|_{\mathcal{T}^1}$.
5. $|\text{tr}(AS)| \leq \|A\|_{B(L^2)} \|S\|_{\mathcal{T}^1}$.

Remark. By the triangle inequality, part (4) of the previous proposition is stronger than part (5). Most sources do not require part (4), and therefore only include part (5) as a proposition. However, the standard way of proving part (5) is to first prove part (4). Such a proof may be found in theorem 18.11 (e) in [12].

Having established that the trace is linear, we are ready to state a version of Hölder's inequality and the duality relations of the Schatten p -classes, proved in theorem 2.8 and 3.2 in [43].

Theorem 2.6. *Let $1 \leq p < \infty$, and let q be the number determined by $\frac{1}{p} + \frac{1}{q} = 1$.*

1. *If $S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$, $ST \in \mathcal{T}^1$ and $\|ST\|_{\mathcal{T}^1} \leq \|S\|_{\mathcal{T}^p} \|T\|_{\mathcal{T}^q}$.*
2. *The dual space of \mathcal{T}^p is \mathcal{T}^q , and the duality may be given by*

$$\langle T, S \rangle = \text{tr}(TS^*)$$

for $S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$.

Furthermore, the dual space of $K(L^2(\mathbb{R}^d))$ is \mathcal{T}^1 under the same duality action.

2.3.2 Hilbert-Schmidt Operators

The space \mathcal{T}^2 is the space of *Hilbert-Schmidt operators* on $L^2(\mathbb{R}^d)$. The Hilbert-Schmidt operators contain the trace class operators as a subspace, and in fact an operator is trace class if and only if it is the product of two Hilbert-Schmidt operators. Using this fact we may define an inner product on the Hilbert-Schmidt operators by

$$\langle S, T \rangle_{\mathcal{T}^2} = \text{tr}(ST^*),$$

for $S, T \in \mathcal{T}^2$, and this inner product makes \mathcal{T}^2 a Hilbert space [40, Thm. VI.22].

2.4 Vector-valued integration

The theory of integration of functions from some measure space to a Banach space may be approached in different ways. The two main approaches consist roughly of either building the theory from first principles similarly to the construction of the Lebesgue integral, or to exploit the fact that we already know how to integrate functions with values in the complex numbers. The first approach leads to the Bochner integral, and the second to what is often referred to as a weak definition of the integral. We will use the second approach as discussed by Folland in [21], restricting the discussion to the measure space \mathbb{R}^d . The connection between vector-valued functions and complex valued functions is provided by bounded linear functionals, and we give the following definition:

Definition 2.3. Let X be a Banach space and $\Psi : \mathbb{R}^d \rightarrow X$ a function. We say that Ψ is *integrable* if $x^* \circ \Psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is integrable for any bounded linear functional x^* on X .

What we would like to call *the integral* $\int_{\mathbb{R}^d} \Psi \, d\mu$ of Ψ , where μ is Lebesgue measure, would be a vector $v \in X$ such that $x^*(v) = \int_{\mathbb{R}^d} x^* \circ \Psi \, d\mu$ for any bounded linear functional x^* on X . If we think of the integral as a limit of sums, this would just be a generalization of the statement that x^* is linear. The existence of such a vector v is not immediately clear, and one would certainly expect that Ψ must satisfy some conditions in order for v to exist. We will confine ourselves with a sufficient condition for the integral to exist [21, Thm. A.22].

Theorem 2.7. Let X be a Banach space, μ Lebesgue measure on \mathbb{R}^d , $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$ a function in $L^1(\mathbb{R}^d)$ and $\Psi : \mathbb{R}^d \rightarrow X$ a bounded and continuous function. In this case the integral $\int_{\mathbb{R}^d} \phi \cdot \Psi \, d\mu$ exists in the sense discussed above, belongs to the closed linear span of the range of Ψ and satisfies the norm estimate

$$\left\| \int_{\mathbb{R}^d} \phi \cdot \Psi \, d\mu \right\|_X \leq \|\phi\|_{L^1(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} \|\Psi(x)\|_X.$$

By definition the weak integral commutes with bounded linear functionals. In fact, it is not difficult to show that it commutes with any bounded linear operator between Banach spaces. We single out the precise statement in a proposition for easy reference, since this property will be used later in the text.

Proposition 2.8. *Let X, Y be Banach spaces, $A : X \rightarrow Y$ a bounded linear operator and μ Lebesgue measure. If $\Psi : \mathbb{R}^d \rightarrow X$ is an integrable function such that the integral $\int_{\mathbb{R}^d} \Psi \, d\mu$ exists in X , then $T \circ \Psi$ is an integrable function such that the integral $\int_{\mathbb{R}^d} T \circ \Psi \, d\mu$ exists in Y . Furthermore, $\int_{\mathbb{R}^d} T \circ \Psi \, d\mu = T \left(\int_{\mathbb{R}^d} \Psi \, d\mu \right)$.*

In this text we will mainly deal with with functions $\Psi : \mathbb{R}^d \rightarrow X$ that are *not* continuous, so that theorem 2.7 does not apply directly. The most common case will be a bounded and *strongly* continuous function $T : \mathbb{R}^{2d} \rightarrow B(L^2(\mathbb{R}^d))$, i.e. a function T such that $z_n \rightarrow z$ in \mathbb{R}^{2d} implies that $T(z_n)\xi \rightarrow T(z)\xi$ for any fixed $\xi \in L^2(\mathbb{R}^d)$. Following [21] we will now show how we may use theorem 2.7 to define the integral of T pointwise.

Let $f \in L^1(\mathbb{R}^{2d})$, fix $\xi \in L^2(\mathbb{R}^d)$ and let $T : \mathbb{R}^{2d} \rightarrow B(L^2(\mathbb{R}^d))$ be a strongly continuous bounded function. Then theorem 2.7 gives that the integral $\iint_{\mathbb{R}^{2d}} f(z)T(z)\xi \, d\mu$ exists in $L^2(\mathbb{R}^d)$. Let I_T be the operator on $L^2(\mathbb{R}^d)$ defined by $I_T\xi = \iint_{\mathbb{R}^{2d}} f(z)T(z)\xi \, d\mu$. Clearly $\xi \mapsto I_T\xi$ is linear, and theorem 2.7 gives the norms estimate

$$\|I_T\xi\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^1} \sup_{z \in \mathbb{R}^{2d}} \|T(z)\xi\|_{L^2} \leq \|\xi\|_{L^2} \|f\|_{L^1} \sup_{z \in \mathbb{R}^{2d}} (\|T(z)\|_{B(L^2)}).$$

In other words, I_T defines a bounded linear operator in $B(L^2(\mathbb{R}^d))$ with norm $\|I_T\|_{B(L^2)} \leq \|f\|_{L^1} \sup_{z \in \mathbb{R}^{2d}} (\|T(z)\|_{B(L^2)})$. We denote the operator I_T by $\iint_{\mathbb{R}^{2d}} f(z)T(z) \, dz$. Note that by the weak interpretation of the integral, I_T is defined by

$$\left\langle \iint_{\mathbb{R}^{2d}} f(z)T(z)\xi \, d\mu, \eta \right\rangle = \iint_{\mathbb{R}^{2d}} f(z) \langle T(z)\xi, \eta \rangle \, d\mu,$$

for $\eta, \xi \in L^2(\mathbb{R}^d)$, where we have used the Riesz representation theorem to identify the dual space of $L^2(\mathbb{R}^d)$ with $L^2(\mathbb{R}^d)$ itself. As a special case we note the following proposition, which will be needed later.

Proposition 2.9. *Let $U : \mathbb{R}^{2d} \rightarrow \mathcal{U}(L^2(\mathbb{R}^d))$ be a strongly continuous function, T a trace class operator on $L^2(\mathbb{R}^d)$, and $f \in L^1(\mathbb{R}^{2d})$. Here $\mathcal{U}(L^2(\mathbb{R}^d))$ denotes the unitary operators. Define the operator I_T by*

$$I_T = \iint_{\mathbb{R}^{2d}} f(z)U(z)TU(z)^* \, dz.$$

I_T is trace class with $\|I_T\|_{\mathcal{T}^1} = \|f\|_{L^1} \|T\|_{\mathcal{T}^1}$, and if $S \in B(L^2(\mathbb{R}^d))$ then

$$\text{tr}(SI_T) = \iint_{\mathbb{R}^{2d}} f(z) \text{tr}(SU(z)TU(z)^*) \, dz.$$

Proof. The strong continuity of $z \mapsto U(z)TU(z)^*$ follows from the strong continuity of $U(z)$, so the integral defining I_T exists by the preceding discussion.

We start by showing that I_T is trace class with the given norm. A slightly tedious but straightforward calculation using proposition 2.8 confirms that $|I_T| = \iint_{\mathbb{R}^{2d}} |f(z)|U(z)|T|U(z)^* dz$ – the calculation consists of checking that the operator on the right is a positive square root of a $I_T^*I_T$. By picking an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ for $L^2(\mathbb{R}^d)$, the trace class norm $\text{tr}(|I_T|)$ is given by

$$\begin{aligned} \sum_{n \in \mathbb{N}} \langle \iint_{\mathbb{R}^{2d}} |f(z)|U(z)|T|U(z)^* dz e_n, e_n \rangle &= \sum_{n \in \mathbb{N}} \iint_{\mathbb{R}^{2d}} |f(z)| \langle U(z)|T|U(z)^* e_n, e_n \rangle dz \\ &= \iint_{\mathbb{R}^{2d}} |f(z)| \sum_{n \in \mathbb{N}} \langle |T|U(z)^* e_n, U(z)^* e_n \rangle dz \\ &= \|T\|_{\mathcal{T}^1} \|f\|_{L^1}. \end{aligned}$$

We have used that $\{U(z)^*\psi_n\}_{n \in \mathbb{N}}$ is another orthonormal basis since $U(z)$ is unitary, and the trace is independent of the basis used to calculate it. We have also used Tonelli's theorem to switch the order of the sum and integral, since the integrand is positive.

In order to prove the last formula, let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. By definition,

$$\begin{aligned} \text{tr}(SI_T) &= \sum_{n \in \mathbb{N}} \langle SI_T e_n, e_n \rangle \\ &= \sum_{n \in \mathbb{N}} \iint_{\mathbb{R}^{2d}} f(z) \langle SU(z)TU(z)^* e_n, e_n \rangle dz, \end{aligned}$$

where we have used proposition 2.8 to move S inside the integral, and then moved the inner product inside the integral by the weak definition of the integral. The result would clearly follow if we could move the sum inside the integral, and we therefore use Fubini's theorem. This is permissible since

$$\iint_{\mathbb{R}^{2d}} \sum_{n \in \mathbb{N}} |f(z) \langle SU(z)TU(z)^* e_n, e_n \rangle| dz \leq \|T\|_{\mathcal{T}^1} \|S\|_{B(L^2)} \iint_{\mathbb{R}^{2d}} |f(z)| dz < \infty$$

by part (4) of proposition 2.5. This concludes the proof. \square

2.5 Modulation spaces

The modulation spaces are a class of spaces of functions and distributions introduced by Feichtinger in a series of papers starting with the introduction of the so-called Feichtinger algebra in [17]. Since then, the modulation spaces have been found

to have many properties that make them natural in the mathematical area of time-frequency analysis, such as an analogue of the Schwartz kernel theorem and invariance under time-frequency shifts [23]. The usefulness of these results is strengthened by the fact that the modulation spaces are Banach spaces and enjoy the natural duality relations. We start by defining the fundamental operators in time-frequency analysis.

Definition 2.4. Let ψ be a function $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$, and let $z = (x, \omega) \in \mathbb{R}^{2d}$. The *translation operator* T_x , *modulation operator* M_ω and *time-frequency shifts* $\pi(z)$ are defined by

$$\begin{aligned}(T_x\psi)(t) &= \psi(t - x) \\ (M_\omega\psi)(t) &= e^{2\pi i\omega \cdot t}\psi(t) \\ (\pi(z)\psi)(t) &= (M_\omega T_x\psi)(t) = e^{2\pi i\omega \cdot t}\psi(t - x).\end{aligned}$$

The translation and modulation operators may also be defined for $\psi \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\begin{aligned}\langle T_x\psi, \phi \rangle &= \langle \psi, T_{-x}\phi \rangle \\ \langle M_\omega\psi, \phi \rangle &= \langle \psi, M_{-\omega}\phi \rangle\end{aligned}$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$.

The translation and modulation operators satisfy an important commutation relation, which can be proved by a straightforward calculation.

Lemma 2.10. *Let $z = (x, \omega) \in \mathbb{R}^{2d}$. Then $M_\omega T_x = e^{2\pi i x \cdot \omega} T_x M_\omega$.*

Having defined the time-frequency shifts $\pi(z)$, we now define the short-time Fourier transform of two functions, which is a key part of the most common definition of the modulation spaces.

Definition 2.5. Let $\psi, \phi \in L^2(\mathbb{R}^d)$. The *short-time Fourier transform* (STFT) $V_\phi\psi$ of ψ with window ϕ is the function on \mathbb{R}^{2d} defined by

$$V_\phi\psi(z) = \langle \psi, \pi(z)\phi \rangle$$

for $z \in \mathbb{R}^{2d}$.

We further define the *cross-ambiguity function* $A(\psi, \phi)$ of ψ and ϕ by

$$A(\psi, \phi)(z) = e^{\pi i x \cdot \omega} V_\phi\psi(z).$$

The STFT of two functions in $L^2(\mathbb{R}^d)$ is well-defined, as one may easily check that $\phi \in L^2(\mathbb{R}^d)$ implies that $\pi(z)\phi \in L^2(\mathbb{R}^d)$, and in fact $\|\pi(z)\phi\|_2 = \|\phi\|_2$. However, we will need to define the STFT of a more general function ψ with a

window ϕ . In this text, the window ϕ will generally be a fixed window in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Therefore we may by duality define the STFT of any tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$ with ϕ , by defining that $V_\phi\psi(z) = \langle \psi, \pi(z)\phi \rangle$, where the inner product notation denotes duality. Two of the most important properties of the STFT are given in the following lemma, the proof of which may be found in lemma 3.1.1 and 3.1.3 in [23].

Lemma 2.11. *Let $\psi, \phi \in L^2(\mathbb{R}^d)$, and $(x, \omega) \in \mathbb{R}^{2d}$.*

1. $V_\phi\psi$ is uniformly continuous and vanishes at infinity.
2. For any $(x', \omega') \in \mathbb{R}^{2d}$,

$$V_\phi(\pi(x, \omega)\psi)(x', \omega') = e^{2\pi i x \cdot (\omega' - \omega)} V_\phi\psi(x' - x, \omega' - \omega).$$

We are now in a position to define the modulation spaces.

Definition 2.6. Fix a window $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. For $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ is the set of tempered distributions ψ such that

$$\|\psi\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi\psi(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty.$$

In the special cases where p or q is ∞ , the integral is replaced by an essential supremum:

$$\begin{aligned} \|\psi\|_{M^{\infty,q}} &= \left(\int_{\mathbb{R}^d} \left(\operatorname{ess\,sup}_{x \in \mathbb{R}^d} |V_\phi\psi(x, \omega)| \right)^q d\omega \right)^{1/q} \\ \|\psi\|_{M^{p,\infty}} &= \operatorname{ess\,sup}_{\omega \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi\psi(x, \omega)|^p dx \right)^{1/p}. \end{aligned}$$

Notation. When $p = q$, we will denote the space $M^{p,p}(\mathbb{R}^d)$ by $M^p(\mathbb{R}^d)$.

The main properties of the modulation spaces are now summarized without proof in the following theorem. All of these results may be found in chapters 11.3 and 12.2 in [23].

Theorem 2.12. *Let $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ be a window, and $1 \leq p, q \leq \infty$.*

1. The expressions $\|\psi\|_{M^{p,q}}$ define norms on the modulation spaces $M^{p,q}(\mathbb{R}^d)$, making the spaces into Banach spaces under pointwise addition and scalar multiplication.

2. If $\phi' \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, we obtain the same spaces $M^{p,q}(\mathbb{R}^d)$ by using ϕ' instead of ϕ in definition 2.6. Furthermore, the two norms $\|\psi\|_{M^{p,q}}$ given by using ϕ or ϕ' in definition 2.6 are equivalent.
3. If $p_1 \leq p_2$ and $q_1 \leq q_2$, then $M^{p_1,q_1}(\mathbb{R}^d) \subset M^{p_2,q_2}(\mathbb{R}^d)$.
4. If $1 \leq p, q < \infty$, then we have the dual space relation $(M^{p,q}(\mathbb{R}^d))^* = M^{p',q'}(\mathbb{R}^d)$, where p', q' are the conjugate exponents given by $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$. The duality is given by

$$\langle \psi, \gamma \rangle = \iint_{\mathbb{R}^{2d}} V_\phi \psi(z) \overline{V_\phi \gamma(z)} dz$$

for $\psi \in M^{p,q}(\mathbb{R}^d)$ and $\gamma \in M^{p',q'}(\mathbb{R}^d)$.

The next lemma is sometimes known as Moyal's identity [20, p. 57].

Lemma 2.13. *If $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$, then $V_{\phi_i} \psi_j \in L^2(\mathbb{R}^{2d})$ for $i, j \in \{1, 2\}$, and the relation*

$$\langle V_{\phi_1} \psi_1, V_{\phi_2} \psi_2 \rangle = \langle \psi_1, \psi_2 \rangle \overline{\langle \phi_1, \phi_2 \rangle}$$

holds, where the leftmost inner product is in $L^2(\mathbb{R}^{2d})$ and those on the right are in $L^2(\mathbb{R}^d)$.

2.5.1 Convolutions in modulation spaces

In section 8 we will need a result from Cordero and Gröchenig's paper [13] regarding the convolutions of elements of different modulation spaces. The following is a simplified version of [13, Prop. 2.4].

Proposition 2.14. *Let $1 \leq p, q, r, s, t \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and $\frac{1}{t} + \frac{1}{t'} = 1$.*

$$M^{p,st}(\mathbb{R}^d) * M^{q,st'}(\mathbb{R}^d) \subset M^{r,s}(\mathbb{R}^d),$$

with norm inequality $\|\psi * \gamma\|_{M^{r,s}} \leq C \|\psi\|_{M^{p,st}} \|\gamma\|_{M^{q,st'}}$ for some constant C .

The constant C will depend on the windows used to define the norms on the different modulation spaces.

2.5.2 The Feichtinger algebra $M^1(\mathbb{R}^d)$

Of particular interest is the space $M^1(\mathbb{R}^d)$, sometimes called the *Feichtinger algebra*. As this name suggests, $M^1(\mathbb{R}^d)$ has an algebra structure – in fact it is a Banach algebra under both pointwise multiplication and convolution [23, Prop. 12.1.7].

More precisely, if $\psi, \phi \in M^1(\mathbb{R}^d)$, then $\|\psi \cdot \phi\|_{M^1} \leq \|\psi\|_{M^1} \|\phi\|_{M^1}$ and $\|\psi * \phi\|_{M^1} \leq \|\psi\|_{M^1} \|\phi\|_{M^1}$.

As Jakobsen shows in [27], the Feichtinger algebra is continuously embedded in $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$, and this embedding is dense for $p \neq \infty$. On the other hand, the Schwartz space is densely embedded in $M^1(\mathbb{R}^d)$. By theorem 2.12 the dual of $M^1(\mathbb{R}^d)$ is $M^\infty(\mathbb{R}^d)$. This is a space of tempered distributions, much larger than the function space $L^\infty(\mathbb{R}^d)$. In fact, $L^p(\mathbb{R}^d)$ is continuously embedded in $M^\infty(\mathbb{R}^d)$ for every $1 \leq p \leq \infty$ [27, Lem. 6.1].

The Feichtinger algebra is closed under many natural operations. We collect the results we will need in the following lemma.

Lemma 2.15. *Let $\psi \in M^1(\mathbb{R}^d)$ and $z = (x, \omega) \in \mathbb{R}^{2d}$.*

1. $\pi(z)\psi \in M^1(\mathbb{R}^d)$ with $\|\pi(z)\psi\|_{M^1} = \|\psi\|_{M^1}$.
2. $\check{\psi} \in M^1(\mathbb{R}^d)$, and there is a constant K such that $\frac{1}{K}\|\psi\|_{M^1} \leq \|\check{\psi}\|_{M^1} \leq K\|\psi\|_{M^1}$.
3. $\psi^* \in M^1(\mathbb{R}^d)$, and there is a constant C such that $\frac{1}{C}\|\psi\|_{M^1} \leq \|\psi^*\|_{M^1} \leq C\|\psi\|_{M^1}$.

Proof. 1. Let $z = (x, \omega) \in \mathbb{R}^{2d}$. The result follows from part 2 of lemma 2.11 and a change of variable.

$$\begin{aligned} \|\pi(z)\psi\|_{M^1} &= \iint_{\mathbb{R}^{2d}} |V_\phi(\pi(z)\psi)(x', \omega')| dx' d\omega' \\ &= \iint_{\mathbb{R}^{2d}} |V_\phi\psi(x' - x, \omega' - \omega)| dx' d\omega' = \|\psi\|_{M^1}. \end{aligned}$$

2. Using the change of variable $t \mapsto -t$ we find that

$$\begin{aligned} V_\phi\check{\psi}(x, \omega) &= \int_{\mathbb{R}^d} \psi(-t) \overline{e^{2\pi i \omega \cdot t} \phi(t - x)} dt \\ &= \int_{\mathbb{R}^d} \psi(t) \overline{e^{-2\pi i \omega \cdot t} \phi(-t - x)} dt \\ &= \int_{\mathbb{R}^d} \psi(t) \overline{e^{-2\pi i \omega \cdot t} \check{\phi}(t + x)} dt = V_{\check{\phi}}\psi(-x, -\omega). \end{aligned}$$

Thus

$$\iint_{\mathbb{R}^{2d}} |V_\phi\check{\psi}(x', \omega')| dx' d\omega' = \iint_{\mathbb{R}^{2d}} |V_{\check{\phi}}\psi(x', \omega')| dx' d\omega'.$$

In other words, the $M^1(\mathbb{R}^d)$ -norm of $\check{\psi}$ measured with respect to the window ϕ is the same as the $M^1(\mathbb{R}^d)$ -norm of ψ measured with respect to $\check{\phi}$. Since

the Schwartz space is easily seen to be closed under reflection $\phi \mapsto \check{\phi}$, $\check{\phi}$ is also a valid window. As different windows define equivalent norms on $M^1(\mathbb{R}^d)$, this means that there is some constant K such that

$$\frac{1}{K} \|\psi\|_{M^1} \leq \|\check{\psi}\|_{M^1} \leq K \|\psi\|_{M^1},$$

where all norms are measured with respect to the window ϕ .

3. Similarly to the previous calculation, we calculate that

$$\begin{aligned} V_\phi \psi^*(x, \omega) &= \int_{\mathbb{R}^d} \overline{\psi(t) e^{2\pi i \omega \cdot t} \phi(t-x)} dt \\ &= \int_{\mathbb{R}^d} \psi(t) e^{-2\pi i \omega \cdot t} \overline{\phi^*(t-x)} dt = \overline{V_{\phi^*} \psi(x, -\omega)}. \end{aligned}$$

The proof may now be concluded as we did in the previous part, as the Schwartz space is also closed under complex conjugation. \square

2.5.3 Wilson bases

A very useful property of the modulation spaces $M^p(\mathbb{R})$ is the existence of a so-called *Wilson basis* $\mathcal{W}(\phi) = \{\psi_{k,n}\}_{k \in \mathbb{Z}, n \geq 0}$, where $\phi \in L^2(\mathbb{R})$. We will not discuss the details of this construction, but confine ourselves with knowing that there exists an orthonormal basis $\mathcal{W}(\phi) = \{\psi_{k,n}\}_{k \in \mathbb{Z}, n \geq 0}$ of $L^2(\mathbb{R})$ which also is an unconditional basis (see [23] for background material on unconditional convergence and basis) for $M^p(\mathbb{R})$ for $1 \leq p < \infty$. Furthermore, for every $\psi \in M^1(\mathbb{R})$, the expansion

$$\psi = \sum_{k \in \mathbb{Z}, n \geq 0} \langle \psi, \psi_{k,n} \rangle \psi_{k,n}$$

converges unconditionally in the norm of $M^1(\mathbb{R})$, and the expression $\|\psi\| = \sum_{k,n} |\langle \psi, \psi_{k,n} \rangle|$ is a norm on $M^1(\mathbb{R})$, equivalent to the usual one [18, 25]. A Wilson basis with the same properties for $M^1(\mathbb{R}^d)$ is obtained by taking tensor products. For instance, if $\{\psi_{k,n}\}_{k \in \mathbb{Z}, n \geq 0}$ is a Wilson basis for $M^1(\mathbb{R}^d)$, then $\{\psi_{k,n} \otimes \psi_{i,j}\}_{k,i \in \mathbb{Z}, n,j \geq 0}$ is a Wilson basis for $M^1(\mathbb{R}^{2d})$. Later on we will also need that a Wilson basis $\{w_m\}_{m \in \mathbb{N}}$ for $L^2(\mathbb{R}^d)$ satisfies $\|w_m\|_{M^1} \leq C$ for some constant C [23, Prop. 12.3.8].

2.6 The symplectic Fourier transform

The standard symplectic form σ on \mathbb{R}^{2d} is defined for $(x_1, \omega_1), (x_2, \omega_2) \in \mathbb{R}^{2d}$ by $\sigma(x_1, \omega_1; x_2, \omega_2) = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$. Using the standard symplectic form we

can introduce a version of the Fourier transform that will be suitable for the consideration in this text.

Definition 2.7. Let $f \in L^1(\mathbb{R}^{2d})$. We define the *symplectic Fourier transform* $\mathcal{F}_\sigma f$ of f to be the function

$$\mathcal{F}_\sigma f(z) = \iint_{\mathbb{R}^{2d}} f(z') e^{-2\pi i \sigma(z, z')} dz'$$

for $z \in \mathbb{R}^{2d}$, where σ is the standard symplectic form.

If $\mathcal{F}f$ denotes the regular Fourier transform $\mathcal{F}f(z) = \iint_{\mathbb{R}^{2d}} f(z') e^{-2\pi i z \cdot z'} dz'$, it is easy to see that

$$\mathcal{F}_\sigma f(x, \omega) = \mathcal{F}f(\omega, -x). \quad (1)$$

From this it follows that most properties of the Fourier transform carry over to the symplectic version. One such result that we will need is that the symplectic Fourier transform has no zeros if and only if the same holds for the regular Fourier transform. Furthermore, the symplectic Fourier transform may be extended to $L^2(\mathbb{R}^{2d})$, just as the regular Fourier transform. This extended symplectic Fourier transform is then unitary and its own inverse, a fact that follows easily from the well known equality $\mathcal{F}\mathcal{F}f = \check{f}$ and equation (1). We now collect some simple results that follow from manipulating the definition of the symplectic Fourier transform.

Lemma 2.16. Let $f \in L^1(\mathbb{R}^{2d})$.

1. $\mathcal{F}_\sigma(T_z f) = e^{2\pi i \sigma(z, z')} \mathcal{F}_\sigma f(z')$ for $z, z' \in \mathbb{R}^{2d}$.
2. $\mathcal{F}_\sigma \check{f} = \widetilde{\mathcal{F}_\sigma f}$.
3. $\mathcal{F}_\sigma f^*(z') = \overline{\mathcal{F}_\sigma f(-z')}$.

2.7 Pseudodifferential operators

This section will introduce different procedures for associating a bounded operator on $L^2(\mathbb{R}^d)$ to functions on \mathbb{R}^{2d} , or more generally to distributions in $\mathcal{S}'(\mathbb{R}^d)$. They come with different formalisms and properties that we will take advantage of, but any continuous operator $A : \mathcal{S} \rightarrow \mathcal{S}'$ may be expressed using all of the three procedures that we consider [23, Thm. 14.3.5].

2.7.1 The Weyl calculus

A close relative of the STFT is the *cross-Wigner distribution* of two functions on \mathbb{R}^d . By definition, the cross-Wigner distribution $W(\psi, \phi)$ of two functions ψ and ϕ is given by

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi \left(x + \frac{t}{2} \right) \overline{\phi \left(x - \frac{t}{2} \right)} e^{-2\pi i \omega \cdot t} dt.$$

This expression is similar to the definition of the STFT and the cross-ambiguity function, and in fact $W(\psi, \phi) = \mathcal{F}_\sigma A(\psi, \phi)$ [15, Prop. 175]. We will need a boundedness result for the cross-Wigner distribution on modulation spaces in section 8 [13, Prop. 2.5].

Lemma 2.17. *If $\phi \in M^1(\mathbb{R}^d)$ and $\psi \in M^p(\mathbb{R}^d)$, then $W(\psi, \phi) \in M^{1,p}(\mathbb{R}^{2d})$ and*

$$\|W(\psi, \phi)\|_{M^{1,p}} \leq C \|\psi\|_{M^p} \|\phi\|_{M^1}$$

for some constant C .

Our main motivation for studying the cross-Wigner distribution is its connection with the *Weyl calculus*. The Weyl calculus provides one of the oldest and most common quantization procedures.

Definition 2.8. Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$. The *Weyl transform* L_σ of σ is defined by

$$\langle L_\sigma \psi, \phi \rangle = \langle \sigma, W(\phi, \psi) \rangle.$$

σ is called the *Weyl symbol* of the operator L_σ .

A question that has been discussed extensively in the literature is how the properties of the symbol σ translates into properties of its Weyl transform L_σ . We will pick the results that we need from [13, Thm. 3.1].

Theorem 2.18.

1. *If $1 \leq p \leq 2$ and $\sigma \in M^p(\mathbb{R}^{2d})$, then $L_\sigma \in \mathcal{T}^p$ with $\|L_\sigma\|_{\mathcal{T}^p} \leq C \|\sigma\|_{M^p}$ for some constant C .*
2. *If $2 \leq p \leq \infty$ and $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$ where $\frac{1}{p} + \frac{1}{p'} = 1$, then $L_\sigma \in \mathcal{T}^p$ with $\|L_\sigma\|_{\mathcal{T}^p} \leq C \|\sigma\|_{M^{p,p'}}$ for some constant C .*

2.7.2 The integrated Schrödinger representation and twisted convolution

Another way of associating an operator to a function is to define the operator as a superposition of time-frequency shifts using the theory of vector-valued integration.

Definition 2.9. The *integrated Schrödinger representation* is the map $\rho : L^1(\mathbb{R}^{2d}) \rightarrow B(L^2(\mathbb{R}^d))$ defined by

$$\rho(f) = \iint_{\mathbb{R}^{2d}} f(z) e^{-\pi i x \cdot \omega} \pi(z) dz,$$

where the integral is defined in the weak and pointwise sense discussed in section 2.4. We say that f is the *twisted Weyl symbol* of $\rho(f)$.

Many properties of the integrated Schrödinger representation are proved in [23] and [20]. One such property is the important product formula $\rho(f)\rho(g) = \rho(f \natural g)$, where the product \natural is the *twisted convolution*, defined by

$$f \natural g(z) = \iint_{\mathbb{R}^{2d}} f(z - z') g(z') e^{\pi i \sigma(z, z')} dz'$$

for $f, g \in L^1(\mathbb{R}^{2d})$.

For this text it is essential that ρ may be extended to a unitary operator from $L^2(\mathbb{R}^{2d})$ to \mathcal{T}^2 , and that the twisted convolution $f \natural g$ may be defined for $f, g \in L^2(\mathbb{R}^{2d})$ with norm estimate $\|f \natural g\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^2}$. Both of these facts are proved in [20], in theorem 1.30 and proposition 1.33, respectively.

The relationship between the Weyl calculus and the integrated Schrödinger representation is neatly expressed using the symplectic Fourier transform: for a symbol f we have that $L_f = \rho(\mathcal{F}_\sigma f)$.

2.7.3 Integral operators

Finally one may assign to a function k on \mathbb{R}^{2d} a so-called *integral operator* A_k on $L^2(\mathbb{R}^d)$ by

$$A_k \psi(s) = \int_{\mathbb{R}^d} k(s, t) \psi(t) dy \tag{2}$$

for $\psi \in L^2(\mathbb{R}^d)$. k is called the *kernel* of A_k .

Notation. We will let \mathcal{M} denote the set of integral operators A_k with kernel k in $M^1(\mathbb{R}^{2d})$.

As is shown in [25], \mathcal{M} is also the set of operators with Weyl symbol or twisted Weyl symbol in $M^1(\mathbb{R}^{2d})$. The next theorem (see [25]) shows that operators in \mathcal{M} have a useful decomposition in terms of the Wilson basis.

Theorem 2.19. Let $k \in M^1(\mathbb{R}^{2d})$ and let A_k be the integral operator with kernel k . Let $(w_n)_{n \in \mathbb{N}}$ be a Wilson basis for $L^2(\mathbb{R}^d)$, and denote by W_{mn} the corresponding Wilson basis for $L^2(\mathbb{R}^{2d})$ given by $W_{mn}(x, y) = w_m(x)w_n(y)$.

Then $A_k \in \mathcal{T}^1$, and $A_k = \sum_{m, n \in \mathbb{N}} \langle k, W_{mn} \rangle w_m \otimes w_n$ where the sum converges in the \mathcal{T}^1 norm.

2.8 Localization operators and the Berezin transform

One important class of pseudodifferential operators for this thesis are the so-called *localization operators*, which are also known in the literature as the anti-Wick operators [8, 20]. Closely related to the localization operators is the Berezin transform [3], and so we define both these concepts in the following definition.

Definition 2.10. Let φ_1 and φ_2 be two functions on \mathbb{R}^d , called *windows*. If f is a function on \mathbb{R}^{2d} , then the *localization operator with symbol f* is the operator $A_f^{\varphi_1, \varphi_2}$ on $L^2(\mathbb{R}^d)$ defined by

$$A_f^{\varphi_1, \varphi_2} \psi = \iint_{\mathbb{R}^{2d}} f(z) \cdot V_{\varphi_1} \psi(z) \pi(z) \varphi_2 dz$$

for $\psi \in L^2(\mathbb{R}^d)$. The integral is interpreted in the weak sense discussed in section 2.4.

If $T \in B(L^2(\mathbb{R}^d))$, the *Berezin transform* $\mathcal{B}^{\varphi_1, \varphi_2} T$ is the function on \mathbb{R}^{2d} defined by

$$\mathcal{B}^{\varphi_1, \varphi_2} T(z) = \langle T \pi(z) \varphi_1, \pi(z) \varphi_2 \rangle$$

for $z \in \mathbb{R}^{2d}$. We often write just \mathcal{B} and \mathcal{A} when this does not lead to ambiguity.

We will discuss the relation between the localization operators and Berezin transform in section 5.1.

2.9 Banach modules

The theory of Banach modules includes some very powerful results, in particular the celebrated Cohen-Hewitt theorem [22]. If we are able to phrase our theory using Banach modules, we may apply the Cohen-Hewitt theorem to prove new results with little effort. For this reason we include a short introduction to Banach modules, based on the PhD-thesis of Graven [22].

Definition 2.11. Let A be a Banach algebra. A left *Banach module over A* is a Banach space X together with a module multiplication $A \times X \rightarrow X$ denoted by $(a, x) \mapsto ax$ satisfying the following properties:

1. Module multiplication $(a, x) \mapsto ax$ is bilinear.

2. $a(bx) = (ab)x$ for any $a, b \in A, x \in X$.
3. $\|ax\|_X \leq \|a\|_A \|x\|_X$ for any $a \in A, x \in X$.

Next we introduce the concepts of *order-free* and *essential* Banach modules.

Definition 2.12. Let X be a left Banach module over a Banach algebra A .

1. The *essential submodule* X_e of X is the closed linear span of $\{ax : a \in A, x \in X\}$. We say that X is an *essential* module if $X = X_e$.
2. We say that X is *order-free* if $ax = 0$ for all $a \in A$ implies that $x = 0$.

The previously mentioned Cohen-Hewitt theorem requires the Banach algebra to have a *bounded approximate identity*, which we now define.

Definition 2.13. Let A be a Banach algebra. An *approximate identity* for A is a net $\{e_i\}_{i \in I}$, where I is a directed set, such that

$$\lim_{i \in I} e_i a = a \text{ and } \lim_{i \in I} a e_i = a \text{ for any } a \in A.$$

The approximate identity is *bounded* if $\|e_i\|_A \leq 1$ for any $i \in I$.

It is well known that the Banach algebra $L^1(G)$ has a bounded approximate identity for any locally compact group G [21, Prop. 2.44].

Theorem 2.20 (Cohen-Hewitt factorization theorem). *Let A be a Banach algebra with a bounded approximate identity. If X is a Banach module over A , then $X_e = \{ax : a \in A, x \in X\}$.*

2.9.1 Shifts of Banach spaces

We now turn to the less-known notion of a *shift* in a Banach space, which can be defined for a general locally compact group [22]. However, we restrict ourselves to \mathbb{R}^{2d} . We will consider Banach modules over $L^1(\mathbb{R}^{2d})$, and write the action of $f \in L^1(\mathbb{R}^{2d})$ on $x \in X$ as $f * x$.

Definition 2.14. Let X be a Banach space. A *shift* τ in X is a family of operators $\{\tau_z\}_{z \in \mathbb{R}^{2d}}$ on X with the following properties:

1. For $x \in X$ and $z \in \mathbb{R}^{2d}$, the mapping $x \mapsto \tau_z(x)$ is linear and $\|\tau_z(x)\|_X = \|x\|_X$.
2. If $z, z' \in \mathbb{R}^{2d}$, then $\tau_z \tau_{z'} = \tau_{z+z'}$, where $\tau_z \tau_{z'}$ is the composition of operators.
3. τ_0 is the identity operator on X .

We will be particularly interested in the cases where the map $z \mapsto \tau_z$ is strongly continuous, i.e. $z \mapsto \tau_z(x)$ is continuous from \mathbb{R}^{2d} to X for every fixed $x \in X$. If the module structure agrees with the shift of a Banach space X , we say that that X is a *Banach module with shift*.

Definition 2.15. Let X be a left Banach module over $L^1(\mathbb{R}^{2d})$ with a shift τ . We say that X is a *module with shift* if $\tau_z(f * x) = T_z(f) * x = f * \tau_z(x)$ for any $z \in \mathbb{R}^{2d}$, $f \in L^1(\mathbb{R}^{2d})$ and $x \in X$.

The next theorem will show that any Banach space with a strongly continuous shift can be made into a module with shift in a unique way, and this module will have certain nice properties such as being an essential module. This theorem is [22, Thm. 3.1.5], and we refer the interested reader to the references in that paper.

Theorem 2.21. *Let X be a Banach space with a strongly continuous shift τ . Then there exists a unique module multiplication $*$: $L^1(\mathbb{R}^{2d}) \times X \rightarrow X$ making X into a module with shift. With this module multiplication X is an essential $L^1(\mathbb{R}^{2d})$ -module, and $*$ is given by $f * x = \iint_{\mathbb{R}^{2d}} f(z) \tau_z(x) dz$ for $f \in L^1(\mathbb{R}^{2d})$ and $x \in X$, where the integral is interpreted in the weak sense. Furthermore, if $\{f_i\}_{i \in I}$ is a bounded approximate identity for $L^1(\mathbb{R}^{2d})$, then $\tau_z(x) = \lim_i T_z(f_i) * x$ for any $x \in X$.*

If X is an $L^1(\mathbb{R}^{2d})$ -module with shift τ , we say that τ is strongly continuous at $x \in X$ if $z \mapsto \tau_z(x)$ is continuous from \mathbb{R}^{2d} into X . We can now state a general proposition [22, Thm. 3.1.7] describing such elements for an order-free module X with a shift.

Proposition 2.22. *Let X be an order-free module over $L^1(\mathbb{R}^{2d})$ with shift τ . Then the elements of X where τ is strongly continuous are exactly the elements of the form $f * x$ for $f \in L^1(\mathbb{R}^{2d})$ and $x \in X$.*

2.10 Complex interpolation of Banach spaces

Complex interpolation is a powerful tool for generalizing results on Banach spaces from particular cases to whole classes of Banach spaces. For instance, one may often generalize a result known to be true for $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ to a result for all L^p -spaces for $1 \leq p \leq \infty$. Since we will use this machinery on several occasions, we cite the relevant definitions and theorems from [7], which is the standard reference for this theory.

Definition 2.16. A pair (X_0, X_1) of Banach spaces X_0 and X_1 are said to be *compatible* if there exists a topological vector space Z such that X_0 and X_1 are subspaces of Z , and the inclusions $X_0, X_1 \hookrightarrow Z$ are continuous.

In the setting of the definition above, we will in particular have that $X_0 + X_1 \subset Z$, where $X_0 + X_1 = \{x_0 + x_1 : x_0 \in X_0, x_1 \in X_1\}$.

Definition 2.17. If (X_0, X_1) and (Y_0, Y_1) are two compatible pairs of Banach spaces, then a pair of Banach spaces (X, Y) is an *interpolation pair* of (X_0, X_1) and (Y_0, Y_1) if

1. $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and $Y_0 \cap Y_1 \subset Y \subset Y_0 + Y_1$,
2. Whenever T is a linear operator $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ such that both $T|_{X_0} : X_0 \rightarrow Y_0$ and $T|_{X_1} : X_1 \rightarrow Y_1$ are bounded, then $T|_X : X \rightarrow Y$ is also bounded.

The interpolation pair is said to be of interpolation exponent $\theta \in (0, 1)$ if there exists a constant C such that

$$\|T_{X \rightarrow Y}\| \leq C \|T_{X_0 \rightarrow Y_0}\|^{1-\theta} \|T_{X_1 \rightarrow Y_1}\|^\theta,$$

where $\|T_{X_j \rightarrow Y_j}\|$ is the operator norm of T as an operator from X_j to Y_j . If $C = 1$, then (X, Y) is an *exact* interpolation pair.

We now have all we need to state the main theorem, which apart from notation is theorem 4.1.2 in [7].

Theorem 2.23. *If (X_0, X_1) and (Y_0, Y_1) are two pairs of compatible Banach spaces, then for any $\theta \in (0, 1)$ there exist two Banach spaces $(X_0, X_1)_\theta$ and $(Y_0, Y_1)_\theta$ that form an exact interpolation pair with exponent θ .*

There is an explicit description of the spaces $(X_0, X_1)_\theta$ that may be found in [7], but we will only be interested in a few special cases where the interpolation pairs have been identified with well-known spaces. These results have been collected from equation 1.8 in [11] and theorem 5.1.1 in [7].

Theorem 2.24. *Let $1 \leq p_1, q_1 < \infty$ and $0 < \theta < 1$. Then the pairs $(L^{p_1}(\mathbb{R}^d), L^{q_1}(\mathbb{R}^d))$ and $(\mathcal{T}^{p_1}, \mathcal{T}^{q_1})$ are compatible pairs, and*

1. $(\mathcal{T}^{p_1}, \mathcal{T}^{q_1})_\theta = \mathcal{T}^p$, where p is given by $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}$,
2. $(L^{p_1}(\mathbb{R}^d), L^{q_1}(\mathbb{R}^d))_\theta = L^p$, where p is given by $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}$.

2.11 A Banach space result

Recall that if X and Y are Banach spaces and $T : X \rightarrow Y$ is a bounded linear operator, then the *adjoint* $T^* : Y^* \rightarrow X^*$ is defined by $\langle T^*y^*, x \rangle = \langle y^*, Tx \rangle$ for $x \in X$, $y^* \in Y^*$. The bracket denotes duality, and X^* is the dual space of X . We will need the following result relating the properties of T and T^* [42, Thm. 4.12].

Proposition 2.25.

1. *The range of T is dense if and only if T^* is injective.*
2. *The range of T^* is weak* dense if and only if T is injective.*

3 A shift for operators

In [46], Werner defined the convolution of two operators and of a function with an operator. One goal of this thesis is to explain these definitions, and in this section we will study the different components of Werner's definition. First we need to identify these components, and we turn to the well-known convolution of two functions for inspiration. The convolution of two functions f and g on \mathbb{R}^{2d} may be written as $f * g(x) = \iint_{\mathbb{R}^{2d}} f(y)T_x(\check{g})(y) dy$. We see that there are three operations involved: the translation T_x , the reflection \check{g} and the integral. In order to define the convolution of operators, we need to find three corresponding operations on operators. If we start with the integral, it seems reasonable to say that it corresponds to the trace of a trace class operator. After all, the trace is the most natural linear functional associated to trace class operators, just like the integral is for $L^1(\mathbb{R}^{2d})$. As for translation and reflection of operators, the answer is provided in the next definition.

Definition 3.1. Given an operator $A \in B(L^2(\mathbb{R}^d))$ and $z \in \mathbb{R}^{2d}$ we define two operators $\alpha_z A$ and \check{A} by

$$\begin{aligned}\alpha_z A &= \pi(z)A\pi(z)^*, \\ \check{A} &= PAP.\end{aligned}$$

The notation $\alpha_z A$ was introduced by Werner in [46], and will be used in this text as it drastically improves the readability of many statements as well as making it easier to spot when propositions are applicable. When we generalize the convolution to operators, the guiding intuition will be that α_z is the operator-analogue for the translation T_z of functions, and $A \mapsto \check{A}$ is the operator-analogue of $f \mapsto \check{f}$ for functions. Thus we will refer to the operators $\alpha_z A$ as the translates or shifts of A .

The rest of this section will be dedicated to a detailed study of the operations introduced in definition 3.1. In order to understand the operator α_z , we will need to discuss the time-frequency shifts $\pi(z)$.

Lemma 3.1. $\{\pi(z)\}_{z \in \mathbb{R}^{2d}}$ form a projective representation of \mathbb{R}^{2d} corresponding to the cocycle $c(z, z') = e^{-2\pi i \omega' \cdot x}$. In other words, the following hold for $z = (x, \omega)$ and $z' = (x', \omega')$ in \mathbb{R}^{2d} :

1. $\pi(z)\pi(z') = e^{-2\pi i \omega' \cdot x} \pi(z + z')$.
2. $\pi(0)$ is the identity operator on $L^2(\mathbb{R}^d)$.
3. The operators $\pi(z)$ are unitary, and $\pi(z)^* = e^{-2\pi i x \cdot \omega} \pi(-z)$.
4. The map $z \mapsto \pi(z)$ is strongly continuous, which means that if $z_n \rightarrow z$ in \mathbb{R}^{2d} , then $\pi(z_n)\psi \rightarrow \pi(z)\psi$ in $L^2(\mathbb{R}^d)$ for any fixed $\psi \in L^2(\mathbb{R}^d)$.

Proof. 1. By lemma 2.10 we have the commutation relation $T_x M_\omega = e^{-2\pi i \omega' \cdot x} M_{\omega'} T_x$, and since $\pi(z) = M_\omega T_x$ this implies the result:

$$\begin{aligned} \pi(z)\pi(z') &= M_\omega T_x M_{\omega'} T_{x'} \\ &= e^{-2\pi i \omega' \cdot x} M_\omega M_{\omega'} T_x T_{x'} \\ &= e^{-2\pi i \omega' \cdot x} \pi(z + z'). \end{aligned}$$

2. Follows directly from the definition of $\pi(z)$.
3. We know that $\pi(z) = M_\omega T_x$, and therefore the adjoint is $\pi(z)^* = T_x^* M_\omega^*$. Since M_ω is just multiplication with the complex function $e^{2\pi i \omega \cdot t}$, one easily confirms that $M_\omega^* = M_{-\omega}$. For $\psi, \phi \in L^2(\mathbb{R}^d)$ we calculate that

$$\begin{aligned} \langle T_x \psi, \phi \rangle &= \int_{\mathbb{R}^d} \psi(t-x) \overline{\phi(t)} dt \\ &= \int_{\mathbb{R}^d} \psi(t') \overline{\phi(t'+x)} dt \\ &= \langle \psi, T_{-x} \phi \rangle \end{aligned}$$

using the change of variable $t' = t-x$. Thus $T_x^* = T_{-x}$ and $\pi(z)^* = T_{-x} M_{-\omega} = e^{-2\pi i x \cdot \omega} \pi(-z)$, where we have used the commutation relation in lemma 2.10. Using parts (1) and (2) it is then trivial to check that $\pi(z)^* = \pi(z)^{-1}$.

4. We shall divide the proof into three steps. First, we will show that the map $z \mapsto M_\omega \psi$ is continuous from \mathbb{R}^{2d} to $L^2(\mathbb{R}^d)$ for any $\psi \in L^2(\mathbb{R}^d)$. Then we establish that the same is true for the translations $x \mapsto T_x \psi(t)$, and finally prove a theorem on pointwise multiplication of strongly continuous maps to conclude that $z \mapsto \pi(z)$ is strongly continuous.

Consider a sequence ω_n converging to 0 in \mathbb{R}^d . We are interested in $\|M_{\omega_n}\psi - \psi\|_{L^2}^2 = \langle M_{\omega_n}\psi - \psi, M_{\omega_n}\psi - \psi \rangle_{L^2}$, and writing out this inner product we find that

$$\|M_{\omega_n}\psi - \psi\|_{L^2}^2 = \int_{\mathbb{R}^d} |e^{2\pi i\omega_n \cdot t} - 1|^2 |\psi(t)|^2 dt.$$

Since we have that $|e^{2\pi i\omega_n \cdot t} - 1|^2 \leq 4|\psi(t)|^2$, we may appeal to the dominated convergence theorem to conclude that $\lim_{z_n \rightarrow 0} \|M_{\omega_n}\psi - \psi\|_{L^2} = 0$.

Now consider a sequence $x_n \in \mathbb{R}^d$ converging to 0. We want to show that $\|T_{x_n}\psi - \psi\|_{L^2} \rightarrow 0$ as $x_n \rightarrow 0$ for any $\psi \in L^2(\mathbb{R}^d)$. Following the proof in [24, Ex. 10.12], we first assume that ψ is a compactly supported continuous function. Since ψ is continuous, $T_{x_n}\psi \rightarrow \psi$ pointwise as $x_n \rightarrow 0$. This convergence will actually be uniform due to the compact support of ψ and the fact that pointwise continuity on a compact set is uniform. The compactness of the support also allows us to find a compact set $K \subset \mathbb{R}^d$ containing the support of both ψ and $T_{x_n}\psi$ for any n . Combining these results we find that

$$\begin{aligned} \|T_{x_n}\psi - \psi\|_{L^2}^2 &= \int_{\mathbb{R}^d} |T_{x_n}\psi(t) - \psi(t)|^2 dt \\ &= \int_K |T_{x_n}\psi(t) - \psi(t)|^2 dt \\ &\leq \sup_{t \in K} (|T_{x_n}\psi(t) - \psi(t)|^2) \mu(K), \end{aligned}$$

where μ denotes the Lebesgue measure. The final expression will converge to 0 as $x_n \rightarrow 0$ by the aforementioned uniform convergence.

If ψ is a general element of $L^2(\mathbb{R}^d)$, we will use the result that the compactly supported continuous functions are dense in $L^2(\mathbb{R}^d)$ [41, Prop. 3.14]. Given $\epsilon > 0$, we therefore pick a compactly supported and continuous ϕ such that $\|\psi - \phi\|_{L^2} < \epsilon$. By the previous result we may find an $N \in \mathbb{N}$ such that $\|T_{x_n}\phi - \phi\|_{L^2} < \epsilon$ for $n \geq N$. Assuming that $n \geq N$, we find that

$$\begin{aligned} \|T_{x_n}\psi - \psi\|_{L^2} &\leq \|T_{x_n}\psi - T_{x_n}\phi\|_{L^2} + \|T_{x_n}\phi - \phi\|_{L^2} + \|\phi - \psi\|_{L^2} \\ &= \|T_{x_n}\phi - \phi\|_{L^2} + 2\|\phi - \psi\|_{L^2} \\ &< 3\epsilon, \end{aligned}$$

where the equality uses that T_x clearly preserves the norm in $L^2(\mathbb{R}^d)$. Therefore the map $x \mapsto T_x$ is strongly continuous.

Finally, $\pi(z) = M_\omega T_x$, so we would be done if we could show that the pointwise multiplication of strongly continuous maps on \mathbb{R}^{2d} is strongly continuous. In other words, if $z \mapsto A(z)$ and $z \mapsto B(z)$ are strongly continuous from \mathbb{R}^{2d} to $B(L^2(\mathbb{R}^d))$, we want to show that $z \mapsto B(z)A(z)$ is strongly

continuous. Therefore we consider a sequence z_n in \mathbb{R}^{2d} converging to some z . By assumption the sequences $A(z_n)\psi$ and $B(z_n)\psi$ converge for any $\psi \in L^2(\mathbb{R}^d)$. In particular, the sequence $\|B(z_n)\psi\|$ is bounded, and therefore the uniform boundedness principle [9, Thm. 4.10] gives a constant C_B such that $\|B(z_n)\| \leq C_B$ for any $n \in \mathbb{N}$. For a fixed $\psi \in L^2(\mathbb{R}^d)$, we then calculate that

$$\begin{aligned} \|B(z_n)A(z_n)\psi - B(z)A(z)\psi\|_{L^2} &\leq \|B(z_n)A(z_n)\psi - B(z_n)A(z)\psi\|_{L^2} \\ &\quad + \|B(z_n)A(z)\psi - B(z)A(z)\psi\|_{L^2} \\ &\leq C_B \|A(z_n)\psi - A(z)\psi\|_{L^2} + \|(B(z_n) - B(z))(A(z)\psi)\|_{L^2}, \end{aligned}$$

and the final expression converges to zero by the assumptions on $A(z_n)$ and $B(z_n)$. This shows that $z \mapsto B(z)A(z)$ is strongly continuous, as we wanted to prove. \square

Since the operations α_z and $A \mapsto \check{A}$ will generalize translation and reflection on functions to bounded operators, we may expect them to have certain nice properties and to interact with each other in predictable ways. Several results of this nature are collected in the next lemma. For instance, part (2) shows that the composition of two translations α_z and $\alpha_{z'}$ is $\alpha_{z+z'}$ as one would expect for a translation operator.

Lemma 3.2. *Let $A \in B(L^2(\mathbb{R}^{2d}))$, $T \in \mathcal{T}^p$ for $1 \leq p \leq \infty$ and let $z, z' \in \mathbb{R}^{2d}$.*

1. $\|\alpha_z T\|_{\mathcal{T}^p} = \|T\|_{\mathcal{T}^p}$ and $\|\check{T}\|_{\mathcal{T}^p} = \|T\|_{\mathcal{T}^p}$.
2. $\alpha_z(\alpha_{z'} A) = \alpha_{z+z'} A$.
3. $\alpha_z \pi(z') = e^{2\pi i \sigma(z, z')} \pi(z')$, where σ is the standard symplectic form.
4. $(\alpha_z A)^* = \alpha_z A^*$ and $(\check{A})^* = (A^*)^\check{}$.
5. $\pi(z)P = P\pi(-z)$, $\widetilde{\pi(z)} = \pi(-z)$ and $(\alpha_z A)^\check{} = \alpha_{-z} \check{A}$.

Proof. 1. Both statements concern the operation of sending T to UTU^* for some unitary operator U . The Schatten p -class norm of T is defined to be the ℓ^p -norm of the eigenvalues of $|T|$, and hence the Schatten p -class norm of UTU^* is the ℓ^p -norm of the eigenvalues of $|UTU^*|$. It is easy to show that $|UTU^*| = U|T|U^*$ for any unitary operator U . The statement would therefore follow if we could show that $|T|$ and $U|T|U^*$ have the same eigenvalues. The simple exercise of showing this is left to the reader.

2. We know that $\pi(z)\pi(z') = e^{-2\pi ix \cdot \omega'} \pi(z + z')$ and $\pi(z)^* = e^{-2\pi ix \cdot \omega} \pi(-z)$ from the previous lemma. The statement then follows from a calculation:

$$\begin{aligned} \pi(z)\pi(z')A\pi(z')^*\pi(z)^* &= e^{-2\pi ix \cdot \omega'} \pi(z + z')A(\pi(z)\pi(z'))^* \\ &= e^{-2\pi ix \cdot \omega'} \pi(z + z')A(e^{-2\pi ix \cdot \omega'} \pi(z + z'))^* \\ &= \pi(z + z')A\pi(z + z')^*. \end{aligned}$$

3. This is another consequence of $\pi(z)\pi(z') = e^{-2\pi ix \cdot \omega'} \pi(z + z')$, which we apply twice to find that

$$\begin{aligned} \pi(z)\pi(z')\pi(z)^* &= e^{-2\pi ix \cdot \omega'} \pi(z + z')e^{-2\pi ix \cdot \omega} \pi(-z) \\ &= e^{-2\pi ix \cdot (\omega' + \omega)} e^{-2\pi i(x+x') \cdot (-\omega)} \pi(z') \\ &= e^{2\pi i(x' \cdot \omega - x \cdot \omega')} \pi(z') \\ &= e^{2\pi i\sigma(z, z')} \pi(z'). \end{aligned}$$

4. Follows easily from the definitions and well known properties of the adjoint.
5. The result that $P\pi(z)P = \pi(-z)$ is proved by writing the operators out using their definitions – a straightforward calculation left to the reader. From this we immediately get $\pi(z)P = P\pi(-z)$.

Then let $A \in B(L^2(\mathbb{R}^d))$. Using $P\pi(z)P = \pi(-z)$ we find that $(\pi(z)A\pi(z)^*)^\sim = P\pi(z)Ae^{-2\pi ix \cdot \omega} \pi(-z)P = \pi(-z)PAPe^{-2\pi ix \cdot \omega} \pi(z) = \pi(-z)A\pi(-z)^*$, as desired.

□

Remark. • Part (4) shows that we may write \check{S}^* for an operator S without causing confusion about which of the two operations should be performed first.

- Although $z \mapsto \pi(z)$ is merely a projective representation of \mathbb{R}^{2d} over the Hilbert space $L^2(\mathbb{R}^d)$, part 2 shows that $\{\alpha_z\}_{z \in \mathbb{R}^{2d}}$ is a unitary representation of \mathbb{R}^{2d} over the Hilbert Schmidt operators. This was also noted by Feichinger and Kozek in [19].

The fact that $z \mapsto \pi(z)$ is a strongly continuous map from \mathbb{R}^{2d} to $B(L^2(\mathbb{R}^d))$ implies similar results for α_z .

Proposition 3.3. 1. For $p < \infty$, the map $z \mapsto \alpha_z T$ is continuous from \mathbb{R}^{2d} to \mathcal{T}^p for any fixed $T \in \mathcal{T}^p$.

2. The map $z \mapsto \alpha_z T$ is continuous from \mathbb{R}^{2d} to $K(L^2(\mathbb{R}^d))$ for any fixed $T \in K(L^2(\mathbb{R}^d))$.

3. The map $z \mapsto \alpha_z A$ is weak*-continuous from \mathbb{R}^{2d} to $B(L^2(\mathbb{R}^d))$ for any fixed $A \in B(L^2(\mathbb{R}^d))$.

Proof. 1. We refer to Grümmer's convergence theorem [43, Thm. 2.19] for a proof of this fact.

2. Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}^{2d} converging to z , and let $\epsilon > 0$. From the singular value decomposition of $T \in K(L^2(\mathbb{R}^d))$, we find a finite rank operator S such that $\|T - S\|_{B(L^2)} < \frac{\epsilon}{3}$. Since S is a trace class operator and $\|\cdot\|_{B(L^2)} \leq \|\cdot\|_{\mathcal{T}^1}$, we can use part (1) to find $N \in \mathbb{N}$ such that $\|\alpha_{z_n} S - \alpha_z S\|_{B(L^2)} < \frac{\epsilon}{3}$ for $n > N$. For such n we find that

$$\begin{aligned} \|\alpha_z T - \alpha_{z_n} T\|_{B(L^2)} &\leq \|\alpha_z T - \alpha_z S\|_{B(L^2)} + \|\alpha_z S - \alpha_{z_n} S\|_{B(L^2)} + \|\alpha_{z_n} S - \alpha_{z_n} T\|_{B(L^2)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

We have used that α_z is an isometry from part (1) of lemma 3.2, so $\|\alpha_{z_n} S - \alpha_{z_n} T\|_{B(L^2)} = \|\alpha_{z_n}(S - T)\|_{B(L^2)} = \|S - T\|_{B(L^2)}$.

3. Assume that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^{2d} converging to z . We need to show that $\langle \alpha_{z_n} A, T \rangle \rightarrow \langle A, T \rangle$ for any $T \in \mathcal{T}^1$, where the brackets denote duality given by $\langle \alpha_{z_n} A, T \rangle = \text{tr}((\alpha_{z_n} A)T^*)$.

Recall that $\pi(z)$ is strongly continuous and that pointwise multiplication of strongly continuous maps is strongly continuous, as we showed when proving lemma 3.2. Therefore the map $z \mapsto (\alpha_{z_n} A)T^*$ is strongly continuous. Now let $\{\psi_m\}_{m \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$ consisting of eigenvectors of $|T^*|$. By using the definition of the trace and assuming that the sum and limit can be interchanged, we find that

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{tr}((\alpha_{z_n} A)T^*) &= \lim_{n \rightarrow \infty} \sum_{m \in \mathbb{N}} \langle (\alpha_{z_n} A)T^* \psi_m, \psi_m \rangle \\ &= \sum_{n \in \mathbb{N}} \lim_{n \rightarrow \infty} \langle (\alpha_{z_n} A)T^* \psi_m, \psi_m \rangle \\ &= \sum_{n \in \mathbb{N}} \langle (\alpha_z A)T^* \psi_m, \psi_m \rangle = \text{tr}((\alpha_z A)T^*), \end{aligned}$$

where we have also used that the inner product is continuous in both coordinates by the Cauchy-Schwarz inequality. It only remains to show that we are allowed to take the limit inside the sum. To apply the dominated convergence theorem, we need a sequence $\{a_m\}_{m \in \mathbb{N}} \in \ell^1$ such that $|\langle (\alpha_{z_n} A)T^* \psi_m, \psi_m \rangle| \leq a_m$ for every $m \in \mathbb{N}$. Let $T^* = U|T|$ be the polar

decomposition of T^* . We then find that

$$\begin{aligned} |\langle (\alpha_{z_n} A) T^* \psi_m, \psi_m \rangle| &\leq \|(\alpha_{z_n} A) U |T^* \psi_m\|_{L^2} \\ &\leq \|A\|_{B(L^2)} \| |T^* \psi_m \|_{L^2}, \end{aligned}$$

where we have used that $\|U\|_{B(L^2)} \leq 1$ and $\|\alpha_{z_n} A\|_{B(L^2)} = \|A\|_{B(L^2)}$. But $\| |T^* \psi_m \|_{L^2}$ is merely the m 'th singular value of the trace class operator T^* , since we picked our orthonormal basis to be eigenvectors of $|T^*|$. Hence $\{ |T^* \psi_m \}_{m \in \mathbb{N}} \in \ell^1$, and we can safely apply the dominated convergence theorem with $a_m = \|A\|_{B(L^2)} \| |T^* \psi_m \|_{L^2}$. \square

Remark. It is well-known that the analogous results hold for the L^p -spaces; the map $z \mapsto T_z$ is strongly continuous on L^p for $p < \infty$ and weak* continuous on L^∞ [29].

Finally, we will consider how the operations from definition 3.1 affect the Weyl symbol of an operator. This will serve two purposes. One the hand hand it will emphasize that the translations α_z and reflections $S \mapsto \check{S}$ are natural analogues of the translations and reflection of functions. On the other hand it will show that the set \mathcal{M} of operators with Weyl symbol in $M^1(\mathbb{R}^{2d})$ is closed under these operations.

Lemma 3.4. *Let $f \in L^1(\mathbb{R}^{2d})$, and let L_f be the Weyl transform of f .*

- $\alpha_z(L_f) = L_{T_z f}$ for $z \in \mathbb{R}^{2d}$.
- $\widetilde{L}_f = L_{\check{f}}$.
- $L_f^* = L_{f^*}$.

In particular, if $S \in \mathcal{M}$, then $\alpha_z(S), \check{S}, S^ \in \mathcal{M}$.*

Proof. From section 2.7.2 we know that the twisted Weyl symbol of L_f is $\mathcal{F}_\sigma f$, so $L_f = \iint \mathcal{F}_\sigma f(z') e^{-i\pi\omega' \cdot x'} \pi(z') dz'$ where $z' = (x', \omega')$. Using this representation of L_f will allow us to use the results from lemma 3.2.

1. From proposition 2.8 and part (3) of lemma 3.2 we find that

$$\begin{aligned} \pi(z) L_f \pi(z)^* &= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{-i\pi\omega' \cdot x'} \alpha_z(\pi(z')) dz' \\ &= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{2\pi i \sigma(z, z')} e^{-i\pi\omega' \cdot x'} \pi(z') dz' \\ &= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma(T_z f)(z') e^{-i\pi\omega' \cdot x'} \pi(z') dz' = L_{T_z f}. \end{aligned}$$

We have used that $\mathcal{F}_\sigma(T_z f)(z') = \mathcal{F}_\sigma f(z') e^{2\pi i \sigma(z, z')}$ from lemma 2.16.

2. By definition $\widetilde{L}_f = PL_fP$. Using proposition 2.8 and part (5) of lemma 3.2 we compute that

$$\begin{aligned}
PL_fP &= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{-i\pi\omega' \cdot x'} P\pi(z')P \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{-i\pi\omega' \cdot x'} \pi(-z') \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(-z') e^{-i\pi\omega' \cdot x'} \pi(z') \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma \check{f}(z') e^{-i\pi\omega' \cdot x'} \pi(z') \, dz' = L_{\check{f}},
\end{aligned}$$

where the penultimate step uses $\mathcal{F}_\sigma \check{f} = \overline{\mathcal{F}_\sigma f}$ from lemma 2.16.

3. Let $\psi, \phi \in L^2(\mathbb{R}^d)$. By the weak definition of the integral we find that

$$\begin{aligned}
\langle L_f \psi, \phi \rangle &= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{-i\pi\omega' \cdot x'} \langle \pi(z')\psi, \phi \rangle \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(z') e^{i\pi\omega' \cdot x'} \langle \psi, \pi(-z')\phi \rangle \, dz' \\
&= \overline{\iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}_\sigma f(z')} e^{-i\pi\omega' \cdot x'} \langle \pi(-z')\phi, \psi \rangle \, dz'} \\
&= \langle \int_{\mathbb{R}^{2d}} \overline{\mathcal{F}_\sigma f(z')} e^{-i\pi\omega' \cdot x'} \pi(-z') \, dz' \phi, \psi \rangle \\
&= \langle \psi, \int_{\mathbb{R}^{2d}} \mathcal{F}_\sigma f(-z') e^{-i\pi\omega' \cdot x'} \pi(z') \, dz' \phi \rangle = \langle \psi, L_{f^*} \phi \rangle.
\end{aligned}$$

We have used that $\mathcal{F}_\sigma f^*(z') = \overline{\mathcal{F}_\sigma f(-z')}$ from lemma 2.16.

As we have discussed, \mathcal{M} consists of operators with Weyl symbol in $M^1(\mathbb{R}^{2d})$. We have just shown that if S has Weyl symbol f , then $\alpha_z S$, \check{S} and S^* have Weyl symbols $T_z f$, \check{f} and f^* . Since we proved that $f \in M^1(\mathbb{R}^{2d})$ implies $T_z f, \check{f}, f^* \in M^1(\mathbb{R}^{2d})$ in lemma 2.15, we get that $\alpha_z(S), \check{S}, S^* \in \mathcal{M}$. \square

4 Convolutions of operators and functions

Now that we have studied the necessary prerequisites, we will define the convolution of two operators and of an operator with a function. We will follow [46] by first considering functions in $L^1(\mathbb{R}^{2d})$ and operators in \mathcal{T}^1 , before we extend to the other L^p -spaces and Schatten p -classes. In section 8 we then introduce the modulation spaces.

To motivate the definitions, consider the following two alternative expressions for the convolution of two functions:

$$f * g(x) = \iint_{\mathbb{R}^{2d}} f(y)T_x(\check{g})(y) dy \quad (3)$$

$$f * g = \iint_{\mathbb{R}^{2d}} f(y)T_y(g) dy, \quad (4)$$

where the last integral is interpreted in the weak sense to define a function in $L^1(\mathbb{R}^d)$. That this really is an equivalent expression for the convolution is discussed in appendix 4 in [21].

If we use equations (3) and (4) as starting points, and simply replace T_x with α_x , the integral with the trace, and reflection of functions with $S \mapsto \check{S}$ for operators, we are led to the following definition.

Definition 4.1. Let $f, g \in L^1(\mathbb{R}^{2d})$ and $T, S \in \mathcal{T}^1$. We define the following *convolutions*

$$\begin{aligned} f * g(x) &= \iint_{\mathbb{R}^{2d}} f(y)g(x-y) dy, \\ S * T(z) &= \text{tr}(S\alpha_z(\check{T})), \\ f * S := S * f &= \iint_{\mathbb{R}^{2d}} f(y)\alpha_y(S) dy, \end{aligned}$$

where the last integral is interpreted in the weak and pointwise sense as discussed in section 2.4. The first two definitions define a function on \mathbb{R}^{2d} whereas the last definition defines an operator on $L^2(\mathbb{R}^d)$.

Remark. 1. By the linearity of every operation in the definition, we see that the convolutions are linear in both arguments.

2. For the convolution of two trace class operators S and T , an elementary calculation using lemma 3.2 shows that an equivalent form of $S * T$ is given by $S * T(z) = \text{tr}(\check{S}\alpha_{-z}T)$. We will use this form whenever it is more convenient.
3. If we glance back at the proof of part (3) of proposition 3.3, we see that we actually proved that $S * T$ is a continuous function.

Definition 4.1 raises two natural questions. First, we might ask which spaces of operators and functions the convolutions belong to. Furthermore, if we fix a function or an operator, is convolution with this fixed operator a continuous mapping between these spaces? The next lemma answers these questions except for the convolution of two operators, which is proved separately in lemma 4.2.

Lemma 4.1. *If $f, g \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$, then*

$$\begin{aligned} f * g &\in L^1(\mathbb{R}^{2d}), \\ f * S &\in \mathcal{T}^1, \end{aligned}$$

*with norm estimates $\|f * S\|_{\mathcal{T}^1} \leq \|f\|_{L^1}\|S\|_{\mathcal{T}^1}$ and $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$.*

Proof. The proof for $f * g$ is standard, but included for completeness.

$$\begin{aligned} \|f * g\|_{L^1} &= \iint_{\mathbb{R}^{2d}} \left| \iint_{\mathbb{R}^{2d}} f(y)g(x-y) dy \right| dx \\ &\leq \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} |f(y)g(x-y)| dy dx \\ &\leq \iint_{\mathbb{R}^{2d}} |f(y)| \iint_{\mathbb{R}^{2d}} |g(x-y)| dx dy \\ &= \iint_{\mathbb{R}^{2d}} |f(y)| dy \|g\|_{L^1} \\ &= \|f\|_{L^1}\|g\|_{L^1}, \end{aligned}$$

where we have used Tonelli's theorem to switch the order of integration.

The statement for $f * S$ is exactly the setting for proposition 2.9, so it follows from that proposition. \square

We now proceed to prove the corresponding statement for the convolution of two operators, which is [46, Lem. 3.1]. The reason for making this a separate lemma is that it is associated with a very useful equality that generalizes Moyal's identity, which we will need to refer to on many occasions.

Lemma 4.2. *Let $S, T \in \mathcal{T}^1$. The function $z \mapsto \text{tr}(S\alpha_z T)$ for $z \in \mathbb{R}^{2d}$ is integrable and $\|\text{tr}(S\alpha_z T)\|_{L^1} \leq \|S\|_{\mathcal{T}^1}\|T\|_{\mathcal{T}^1}$.*

Furthermore,

$$\iint_{\mathbb{R}^{2d}} \text{tr}(S\alpha_z T) dz = \text{tr}(S)\text{tr}(T).$$

Proof. We start by showing the norm-inequality. First use the singular value decomposition of the operators S and T to write

$$S = \sum_{m \in \mathbb{N}} s_m \psi_m \otimes \phi_m \qquad T = \sum_{n \in \mathbb{N}} t_n \eta_n \otimes \xi_n,$$

where $\{s_m\}_{m \in \mathbb{N}}$ and $\{t_n\}_{n \in \mathbb{N}}$ are the singular values of S and T , respectively, and the sets $\{\psi_m\}_{m \in \mathbb{N}}$, $\{\phi_m\}_{m \in \mathbb{N}}$, $\{\eta_n\}_{n \in \mathbb{N}}$ and $\{\xi_n\}_{n \in \mathbb{N}}$ are orthonormal in $L^2(\mathbb{R}^d)$. Then

extend the set $\{\psi_m\}_{m \in \mathbb{N}}$ to an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$. Using this basis to calculate the trace, we find that

$$\begin{aligned}
\mathrm{tr}(S\alpha_z T) &= \sum_{i \in \mathbb{N}} \langle S\pi(z)T\pi(z)^* e_i, e_i \rangle \\
&= \sum_{i, n \in \mathbb{N}} t_n \langle \pi(z)^* e_i, \xi_n \rangle \langle S\pi(z)\eta_n, e_i \rangle \\
&= \sum_{i, m, n \in \mathbb{N}} s_m t_n \langle \pi(z)^* e_i, \xi_n \rangle \langle \pi(z)\eta_n, \phi_m \rangle \langle \psi_m, e_i \rangle \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n \langle \pi(z)^* \psi_m, \xi_n \rangle \langle \pi(z)\eta_n, \phi_m \rangle \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n V_{\xi_n} \psi_m(z) \overline{V_{\eta_n} \phi_m(z)}.
\end{aligned}$$

By Moyal's identity, $V_{\xi_n} \psi_m, V_{\eta_n} \phi_m \in L^2(\mathbb{R}^{2d})$, and so $V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m} \in L^1(\mathbb{R}^{2d})$ by Hölder's inequality. The following computation, which exploits both Moyal's identity and Hölder's inequality, then shows that the series above converges absolutely in $L^1(\mathbb{R}^d)$ with the desired norm estimates.

$$\begin{aligned}
\| \sum_{m, n \in \mathbb{N}} s_m t_n V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m} \|_{L^1} &\leq \sum_{m, n \in \mathbb{N}} s_m t_n \|V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m}\|_{L^1} \\
&\leq \sum_{m, n \in \mathbb{N}} s_m t_n \|V_{\xi_n} \psi_m\|_{L^2} \|V_{\eta_n} \phi_m\|_{L^2} \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n \|\xi_n\|_{L^2} \|\psi_m\|_{L^2} \|\eta_n\|_{L^2} \|\phi_m\|_{L^2} \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n = \|S\|_{\mathcal{T}^1} \|T\|_{\mathcal{T}^1}.
\end{aligned}$$

The equality $\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z T) dz = \mathrm{tr}(S)\mathrm{tr}(T)$ now follows by using Moyal's identity. We have already seen in a previous calculation that $\mathrm{tr}(S\alpha_z T) = \sum_{m, n \in \mathbb{N}} s_m t_n V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m}$, and that this expression is integrable. If we integrate the expression, we use Moyal's identity to find that

$$\begin{aligned}
\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z T) dz &= \iint_{\mathbb{R}^{2d}} \sum_{m, n \in \mathbb{N}} s_m t_n V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m} dz \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n \iint_{\mathbb{R}^{2d}} V_{\xi_n} \psi_m \overline{V_{\eta_n} \phi_m} dz \\
&= \sum_{m, n \in \mathbb{N}} s_m t_n \langle \psi_m, \phi_m \rangle \langle \eta_n, \xi_n \rangle \\
&= \mathrm{tr}(S)\mathrm{tr}(T),
\end{aligned}$$

where the last equality follows from a straightforward calculation of the traces of S and T . \square

Remark. If we pick $S = \psi \otimes \phi$ and $T = \eta \otimes \xi$ for $\phi, \psi, \eta, \xi \in L^2(\mathbb{R}^d)$, it is easy to see from the preceding proof that $\text{tr}(S\alpha_z T) = V_\xi \psi \overline{V_\eta \phi}$, and one easily calculates that $\text{tr}(S) = \langle \psi, \phi \rangle$ and $\text{tr}(T) = \langle \eta, \xi \rangle$. In this case the equality in lemma 4.2 is simply Moyal's identity, so the lemma really generalizes this identity.

To sum up this section, we have defined various convolutions between functions in $L^1(\mathbb{R}^{2d})$ and operators in \mathcal{T}^1 , and we have shown that the result of the convolution always is an object belonging to one of those two spaces. Furthermore, the norm estimates provided in the lemmas 4.1 and 4.2 show that taking convolutions with a fixed operator or function always is a continuous operation.

4.1 Extending the domains of the convolutions to the dual spaces

So far we have defined convolutions of objects in $L^1(\mathbb{R}^{2d})$ and \mathcal{T}^1 , but based on the theory of convolutions of functions one could expect that the convolution may be defined for objects in the L^p -spaces and Schatten p -classes. The first step in this direction will be to define the convolution when one factor belongs to $L^1(\mathbb{R}^{2d})$ or \mathcal{T}^1 , and the other factor belongs to one of the dual spaces $L^\infty(\mathbb{R}^{2d})$ or $B(L^2(\mathbb{R}^d))$. Since one of the factors belongs to a dual space, these convolutions may be defined by duality, as we will show shortly. After doing this, the convolutions on L^p -spaces and Schatten p -classes may be defined using the interpolation argument outlined in section 2.10.

To define an object in $B(L^2(\mathbb{R}^d))$ or $L^\infty(\mathbb{R}^{2d})$ by duality means that we consider $B(L^2(\mathbb{R}^d))$ and $L^\infty(\mathbb{R}^{2d})$ as the dual spaces of \mathcal{T}^1 and $L^1(\mathbb{R}^{2d})$, respectively. For instance, to define $h * g \in L^\infty(\mathbb{R}^{2d})$ when $h \in L^\infty(\mathbb{R}^{2d})$ and $g \in L^1(\mathbb{R}^{2d})$, we need to define $\langle h * g, f \rangle$ for any $f \in L^1(\mathbb{R}^{2d})$. To motivate the definition of $\langle h * g, f \rangle$, we will use that definition 4.1 already defines $h * g$ for $h \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$, and we need our definitions to agree in this case. We therefore assume that $h \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ and $f, g \in L^1(\mathbb{R}^{2d})$. We calculate using Fubini's theorem that

$$\begin{aligned} \langle h * g, f \rangle &= \iint_{\mathbb{R}^{2d}} h * g(z) \overline{f(z)} dz \\ &= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} h(y) g(z - y) \overline{f(z)} dy dz \\ &= \iint_{\mathbb{R}^{2d}} h(y) \iint_{\mathbb{R}^{2d}} g(z - y) \overline{f(z)} dz dy \\ &= \langle h, f * \check{g}^* \rangle. \end{aligned}$$

In other words, letting $h * g$ act on f produces the same result as letting h act on $f * \check{g}^*$. But the action of h on $f * \check{g}^*$ is well-defined even when $h \in L^\infty(\mathbb{R}^{2d})$, since $f * \check{g}^* \in L^1(\mathbb{R}^{2d})$ and $L^\infty(\mathbb{R}^{2d})$ is the dual space of $L^1(\mathbb{R}^{2d})$. This inspires the following definition.

Definition 4.2. Let $g \in L^1(\mathbb{R}^{2d})$, $h \in L^\infty(\mathbb{R}^{2d})$, $S \in \mathcal{T}^1$ and $A \in B(L^2(\mathbb{R}^d))$. We then define the *convolutions* $h * g \in L^\infty(\mathbb{R}^{2d})$, $A * S \in L^\infty(\mathbb{R}^{2d})$, $A * g \in B(L^2(\mathbb{R}^d))$ and $h * S \in B(L^2(\mathbb{R}^d))$ by

$$\begin{aligned}\langle h * g, f \rangle &= \langle h, f * \check{g}^* \rangle \quad \forall f \in L^1(\mathbb{R}^{2d}), \\ \langle A * S, f \rangle &= \langle A, f * \check{S}^* \rangle \quad \forall f \in L^1(\mathbb{R}^{2d}), \\ \langle A * g, T \rangle &= \langle A, T * \check{g}^* \rangle \quad \forall T \in \mathcal{T}^1, \\ \langle h * S, T \rangle &= \langle h, T * \check{S}^* \rangle \quad \forall T \in \mathcal{T}^1.\end{aligned}$$

We further define $g * h := h * g$, $S * A := A * S$, $S * h := h * S$ and $g * A := A * g$.

Definition 4.2 was motivated by showing that it agrees with definition 4.1 when all factors are functions. This implies neither that it agrees with definition 4.1 for the other cases, nor that duality actions in definition 4.2 actually define bounded, antilinear functionals. We will need to prove these facts before we proceed, and this is the content of the next two lemmas.

Lemma 4.3. *The definitions 4.2 and 4.1 are compatible.*

Proof. We will need to show that the duality actions of the convolution in definition 4.1 satisfy the relations in definition 4.2 whenever both definitions are applicable. Note that for operators both definitions will be applicable for any trace class operator, as $\mathcal{T}^1 \subset B(L^2(\mathbb{R}^d))$, whereas it is not true that $L^1(\mathbb{R}^{2d}) \subset L^\infty(\mathbb{R}^{2d})$.

1. The case where all factors are functions was shown before definition 4.2.
2. Assume that $f \in L^1(\mathbb{R}^{2d})$ and $A, S \in \mathcal{T}^1$. By applying proposition 2.9 we find that

$$\begin{aligned}\langle A, f * \check{S}^* \rangle &= \text{tr}((f^* * \check{S})A) \\ &= \iint_{\mathbb{R}^{2d}} \overline{f(y)} \text{tr}(A \alpha_y(\check{S})) \, dy \\ &= \langle A * S, f \rangle,\end{aligned}$$

which is what we wanted to show.

3. Omitted as it is straightforward and similar to the other parts.

4. Assume that $T, S \in \mathcal{T}^1$ and $h \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$. Just as in part (2) we use proposition 2.9 to calculate that

$$\begin{aligned}
\langle h * S, T \rangle &= \text{tr}((h * S)T^*) \\
&= \iint_{\mathbb{R}^{2d}} h(y) \text{tr}(T^* \alpha_y S) dy \\
&= \iint_{\mathbb{R}^{2d}} h(y) \overline{\text{tr}(T \alpha_y S^*)} dy \\
&= \langle h, T * \check{S}^* \rangle,
\end{aligned}$$

where we have also used that $(T \alpha_y S^*)^* = \alpha_y(S) T^*$ by lemma 3.2.

□

Lemma 4.4. *Let $g \in L^1(\mathbb{R}^{2d})$, $h \in L^\infty(\mathbb{R}^{2d})$, $S \in \mathcal{T}^1$ and $A \in B(L^2(\mathbb{R}^d))$. Definition 4.2 defines elements of the dual space as claimed, and*

$$\begin{aligned}
\|h * g\|_{L^\infty} &\leq \|h\|_{L^\infty} \|g\|_{L^1}, \\
\|A * S\|_{L^\infty} &\leq \|A\|_{B(L^2)} \|S\|_{\mathcal{T}^1}, \\
\|A * g\|_{B(L^2)} &\leq \|A\|_{B(L^2)} \|g\|_{L^1}, \\
\|h * S\|_{B(L^2)} &\leq \|h\|_{L^\infty} \|S\|_{\mathcal{T}^1}.
\end{aligned}$$

Proof. For instance, let $h \in L^\infty(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$. That $h * S$ acts antilinearly according to definition 4.2 is easy to check directly. To show boundedness, we use the norm estimate in lemma 4.2 to calculate that

$$\begin{aligned}
|\langle h * S, T \rangle| &= |\langle h, T * \check{S}^* \rangle| \\
&\leq \|h\|_{L^\infty} \|T * \check{S}^*\|_{L^1} \\
&\leq \|h\|_{L^\infty} \|S\|_{\mathcal{T}^1} \|T\|_{\mathcal{T}^1}.
\end{aligned}$$

The proofs in the other cases follow the same pattern: first use the estimate associated with the operator norm of elements of the dual space, then use the norm estimate for the convolutions in lemma 4.1 or lemma 4.2.

□

Remark. If $A \in B(L^2(\mathbb{R}^d))$ and $T \in \mathcal{T}^1$, the convolution $A * T$ defined in definition 4.2 is actually given by the function $(A * T)(z) = \text{tr}(A \alpha_z(\check{T}))$, in other words definition 4.1 still holds in this case. To see this, recall that \mathcal{T}^1 is an ideal in $B(L^2(\mathbb{R}^d))$, so the function is well-defined. From proposition 2.5 we get that the bound $|\text{tr}(A \alpha_z(\check{T}))| \leq \|A\|_{B(L^2)} \|T\|_{\mathcal{T}^1}$ holds, so $\text{tr}(A \alpha_z(\check{T})) \in L^\infty(\mathbb{R}^{2d})$. An explicit calculation then reveals that this agrees with $A * T$ defined in definition 4.2, i.e. $\langle \text{tr}(A \alpha_z(\check{T})), f \rangle = \langle A, f * \check{T}^* \rangle \forall f \in L^1(\mathbb{R}^{2d})$.

4.2 Convolutions of L^p -spaces and Schatten p -classes

As promised, we will now prove a result on convolutions between objects in L^p -spaces and Schatten p -classes for different values of p . For the convolution of functions in different L^p -spaces, Young's inequality [39, p. 28] gives a condition for when convolutions are defined. The most common proof of this fact uses complex interpolation, and since theorem 2.24 shows that the interpolation theories of L^p -spaces and Schatten p -classes are similar, we may generalize Young's inequality to the convolutions defined in the previous section. This is the content of the following proposition.

Proposition 4.5. *Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^{2d})$, $g \in L^q(\mathbb{R}^d)$, $S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$, then the following convolutions may be defined and satisfy the norm estimates*

$$\begin{aligned} \|f * g\|_{L^r} &\leq \|f\|_{L^p} \|g\|_{L^q}, \\ \|f * T\|_{\mathcal{T}^r} &\leq \|f\|_{L^p} \|T\|_{\mathcal{T}^q}, \\ \|g * S\|_{\mathcal{T}^r} &\leq \|g\|_{L^q} \|S\|_{\mathcal{T}^p}, \\ \|S * T\|_{L^r} &\leq \|S\|_{\mathcal{T}^p} \|T\|_{\mathcal{T}^q}. \end{aligned}$$

Proof. In the proof we will consider $S * T$, but the same argument works for the other two cases. We divide the proof into two different steps, and will use complex interpolation in both steps.

1. *The statement is true for $p = 1$.* In this case $q = r$, and we consider a fixed $S \in \mathcal{T}^1$. By lemma 4.1 and lemma 4.4 the map $T \mapsto S * T$ is bounded from \mathcal{T}^1 to $L^1(\mathbb{R}^{2d})$ and from $B(L^2(\mathbb{R}^d))$ to $L^\infty(\mathbb{R}^{2d})$, with operator norm no greater than $\|S\|_{\mathcal{T}^1}$ in both cases. Therefore we may use theorem 2.24 with the the map $T \mapsto S * T$ and the compatible pairs of Banach spaces $(\mathcal{T}^1, B(L^2(\mathbb{R}^d)))$ and $(L^1(\mathbb{R}^{2d}), L^\infty(\mathbb{R}^{2d}))$. From that theorem with $p_1 = 1, q_1 = \infty$, we get that $T \mapsto S * T$ is bounded from \mathcal{T}^q to $L^q(\mathbb{R}^{2d})$ with operator norm less than or equal to $\|S\|_{\mathcal{T}^1}$, where $1 < q < \infty$. In other words, $\|S * T\|_{L^q} \leq \|S\|_{\mathcal{T}^1} \|T\|_{\mathcal{T}^q}$.
2. *The statement is true for $1 \leq p, q, r \leq \infty$.* Another way of interpreting the previous statement $\|S * T\|_{L^q} \leq \|S\|_{\mathcal{T}^1} \|T\|_{\mathcal{T}^q}$, is to consider $T \in \mathcal{T}^q$ to be fixed. The statement then says that the map $S \mapsto S * T$ is bounded from \mathcal{T}^1 to $L^q(\mathbb{R}^{2d})$ with operator norm less than or equal to $\|T\|_{\mathcal{T}^q}$. By duality one may also prove, just as we have done in definition 4.2 and lemma 4.4, that the map $S \mapsto S * T$ is bounded from $(\mathcal{T}^q)^* = \mathcal{T}^{q'}$ to $L^\infty(\mathbb{R}^{2d})^1$, where $\frac{1}{q'} + \frac{1}{q} = 1$, and with operator norm less than or equal to $\|T\|_{\mathcal{T}^q}$.

¹An observant reader might note that we need to prove step (1) for $f * T$ to actually write out this duality proof, but this is not an issue since the proof of step (1) is proved exactly like step (1) for $S * T$.

Once again we may apply theorem 2.24, this time with the compatible pairs of Banach spaces $(\mathcal{T}^q, \mathcal{T}^{q'})$ and $(L^q(\mathbb{R}^{2d}), L^\infty(\mathbb{R}^{2d}))$. From theorem 2.24 we get that $S \mapsto S * T$ is bounded with operator norm at most $\|T\|_{\mathcal{T}^q}$ from \mathcal{T}^p to $L^r(\mathbb{R}^{2d})$, where p and r are determined by

$$\begin{aligned}\frac{1}{p} &= 1 - \theta + \frac{\theta}{q'}, \\ \frac{1}{r} &= \frac{1 - \theta}{q}.\end{aligned}$$

By eliminating θ from the equations, we are left with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, as desired. Restating this as a norm estimate, we have shown that $\|S * T\|_{L^r} \leq \|S\|_{\mathcal{T}^p} \|T\|_{\mathcal{T}^q}$.

□

4.2.1 Compact operators and continuous functions vanishing at infinity

Based on the previous proposition, the function space $L^p(\mathbb{R}^{2d})$ seems to correspond to the Schatten p -class in the theory of convolutions. To make this more precise, Werner introduced the notion of *corresponding subspaces* in [46]: a linear subspace X of $B(L^2(\mathbb{R}^d))$ and a linear subspace Y of $L^\infty(\mathbb{R}^{2d})$ are said to be *corresponding* if $S * X \subset Y$ and $S * Y \subset X$ for any $S \in \mathcal{T}^1$. In this terminology we have shown in proposition 4.5 that $L^p(\mathbb{R}^{2d})$ corresponds to \mathcal{T}^p for any $1 \leq p \leq \infty$. We will not study the theory of corresponding subspaces in detail, but we will give one more example. Let $C_0(\mathbb{R}^{2d})$ denote the continuous functions on \mathbb{R}^{2d} vanishing at infinity. The next proposition shows that $C_0(\mathbb{R}^{2d})$ and $K(L^2(\mathbb{R}^d))$ are corresponding subspaces.

Proposition 4.6. *Let $S \in \mathcal{T}^1$. If $f \in C_0(\mathbb{R}^{2d})$ and $T \in K(L^2(\mathbb{R}))$, then $f * S \in K(L^2(\mathbb{R}))$ and $S * T \in C_0(\mathbb{R}^{2d})$.*

Proof. We start by considering $S * T$. Using the singular value decompositions of \check{S} , write

$$\check{S} = \sum_{m \in \mathbb{N}} s_m \psi_m \otimes \phi_m$$

where $\{s_m\}_{m \in \mathbb{N}}$ are the singular values of \check{S} (and also of S), and the sets $\{\psi_m\}_{m \in \mathbb{N}}$, $\{\phi_m\}_{m \in \mathbb{N}}$ are orthonormal in $L^2(\mathbb{R}^d)$. By assumption, the singular values of S are summable. Now assume that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^{2d} such that $\lim_{n \rightarrow \infty} |z_n| = \infty$. We need to show that $\lim_{n \rightarrow \infty} |T * S(z_n)| = 0$. From a straightforward calculation we find that

$$T * S(z_n) = \text{tr}(T \alpha_{z_n} \check{S}) = \sum_{m \in \mathbb{N}} s_m \langle T \pi(z_n) \psi_m, \pi(z_n) \phi_m \rangle.$$

Consider the sequence given by $\pi(z_n)\psi_m$ for some fixed m . For any $\psi \in L^2(\mathbb{R}^d)$, we have that $\langle \psi, \pi(z_n)\psi_m \rangle = V_{\psi_m}\psi(z_n) \rightarrow 0$ as $n \rightarrow \infty$, since lemma 2.11 asserts that the STFT vanishes at infinity. In other words, $\pi(z_n)\psi_m \rightarrow 0$ weakly in $L^2(\mathbb{R}^d)$. Since T is compact we therefore get that $T\pi(z_n)\psi_m \rightarrow 0$ in $L^2(\mathbb{R}^d)$, and by the Cauchy-Schwarz inequality $\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle \rightarrow 0$ as $n \rightarrow \infty$. The reader should also note that $|\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| \leq \|T\|_{B(L^2)}$ by the same inequality.

The idea to conclude the proof is to divide the sum defining $T * S$ into two parts. We will control its tail by the summability of $\{s_m\}_{m \in \mathbb{N}}$, and then control the rest by the fact that each summand vanishes at infinity, as we have just shown. Start by picking any $\epsilon > 0$. Then pick $M \in \mathbb{N}$ such that $\sum_{m=M+1}^{\infty} s_m < \frac{\epsilon}{2\|T\|_{B(L^2)}}$. Then, for any $m \in \{1, 2, \dots, M\}$, pick $N_m \in \mathbb{N}$ such that $|\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| < \frac{\epsilon}{2M\|S\|_{B(L^2)}}$ whenever $n \geq N_m$. If we now let $N = \max\{N_1, \dots, N_M\}$ and pick $n > N$, we find that

$$\begin{aligned} |T * S(z_n)| &\leq \sum_{m \in \mathbb{N}} s_m |\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| \\ &\leq \sum_{m=M+1}^{\infty} s_m \|T\|_{B(L^2)} + \sum_{m=1}^M s_m |\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| \\ &< \frac{\epsilon}{2} + \left(\sup_{m \in \mathbb{N}} s_m \right) \sum_{m=1}^M |\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where we have used that $\|S\|_{B(L^2)} = \sup_{m \in \mathbb{N}} s_m$.

To consider continuity, assume that $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^{2d} where $z_n \rightarrow z$. Observe that if we could take the limit inside the sum we would have

$$\begin{aligned} \lim_{n \rightarrow \infty} T * S(z_n) &= \lim_{n \rightarrow \infty} \sum_{m \in \mathbb{N}} s_m \langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle \\ &= \sum_{m \in \mathbb{N}} s_m \lim_{n \rightarrow \infty} \langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle \\ &= \sum_{m \in \mathbb{N}} s_m \langle T\pi(z)\psi_m, \pi(z)\phi_m \rangle = T * S(z), \end{aligned}$$

by the strong continuity of $\pi(z)$. To justify that we can take the limit inside the sum, note that $s_m |\langle T\pi(z_n)\psi_m, \pi(z_n)\phi_m \rangle| \leq s_m \|T\|$ by the Cauchy-Schwarz inequality, where the right side is summable since $S \in \mathcal{T}^1$. This shows that the dominated convergence theorem applies.

We finally turn to $f * S$. By the Stone-Weierstraß theorem the space $C_c(\mathbb{R}^{2d})$ of continuous functions with compact support is dense in $C_0(\mathbb{R}^{2d})$ with the $\|\cdot\|_{L^\infty}$ norm, so pick a sequence of functions $f_n \in C_c(\mathbb{R}^{2d})$ converging to f . Since f_n is continuous with compact support it is in particular integrable, and thus $f_n * S \in \mathcal{T}^1$ by lemma 4.1. Since $\mathcal{T}^1 \subset K(L^2(\mathbb{R}^d))$, the operator $f_n * S$ is compact. By lemma 4.4 we then find that

$$\begin{aligned} \|f * S - f_n * S\|_{B(L^2)} &= \|(f - f_n) * S\|_{B(L^2)} \\ &\leq \|f - f_n\|_{L^\infty} \|S\|_{\mathcal{T}^1}, \end{aligned}$$

which shows that $f_n * S$ converges to $f * S$ in the operator norm. Since the operators $f_n * S$ are compact, this shows that $f * S$ is compact. □

4.3 Basic properties of the convolutions

Having defined the convolutions and extended the definition to more general classes of functions and operators, we will now prove a few basic properties of the convolutions. The most important of these will be the associativity of convolutions, which is a non-trivial fact that we will need when proving the main results of the text in section 7. First, however, we need to consider how the convolutions interact with the basic operations defined in section 3, such as $\alpha_z S$ and \check{S} for an operator S . The following result shows that these operations combine with our newly defined convolutions in ways that are both natural and similar to the corresponding results for the convolution of functions.

Lemma 4.7. *Let $f, g \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{T}^1$.*

1. $(f * g)^* = f^* * g^*$, $(f * S)^* = f^* * S^*$ and $(S * T)^* = S^* * T^*$.
2. $(f * g)^\check{\cdot} = \check{f} * \check{g}$, $(f * S)^\check{\cdot} = \check{f} * \check{S}$ and $(S * T)^\check{\cdot} = \check{S} * \check{T}$.
3. $T_z(f * g) = (T_z f) * g$, $\alpha_z(f * S) = (T_z f) * S$ and $T_z(S * T) = (\alpha_z S) * T$.
4. *The convolution of positive functions and/or operators is positive.*

Proof. 1. The case with two functions is trivial, so we skip the proof. To consider the adjoint $(f * S)^*$, let $\psi, \phi \in L^2(\mathbb{R}^d)$ and use the weak definition of the

integral to calculate

$$\begin{aligned}
\langle f * S\phi, \psi \rangle &= \iint_{\mathbb{R}^{2d}} \langle f(y)\alpha_y S\phi, \psi \rangle dy \\
&= \iint_{\mathbb{R}^{2d}} \langle \phi, \overline{f(y)}(\alpha_y S)^* \psi \rangle dy \\
&= \iint_{\mathbb{R}^{2d}} \langle \phi, \overline{f(y)}\alpha_y S^* \psi \rangle dy \\
&= \langle \phi, f^* * S^* \psi \rangle,
\end{aligned}$$

where we have used that $(\alpha_y S)^* = \alpha_y S^*$ by lemma 3.2. For $(S * T)^*$, let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}
(S * T)^*(z) &= \overline{\text{tr}(S\alpha_z \check{T})} \\
&= \overline{\sum_{n \in \mathbb{N}} \langle S\alpha_z \check{T}e_n, e_n \rangle} \\
&= \sum_{n \in \mathbb{N}} \langle e_n, S\alpha_z \check{T}e_n \rangle \\
&= \sum_{n \in \mathbb{N}} \langle \alpha_z(\check{T}^*)S^*e_n, e_n \rangle \\
&= \text{tr}(S^*\alpha_z \check{T}^*) \\
&= S^* * T^*(z),
\end{aligned}$$

where we have used lemma 3.2 once more.

2. The proof for $f * g$ consists of little more than writing out both sides of the equality. For $S, T \in \mathcal{T}^1$ we need to use part (5) of lemma 3.2 and the property that $\text{tr}(AB) = \text{tr}(BA)$:

$$\begin{aligned}
\check{S} * \check{T} &= \text{tr}(\check{S}\alpha_z T) \\
&= \text{tr}(PSP\alpha_z T) \\
&= \text{tr}(S(\alpha_z T)^\sim) \\
&= \text{tr}(S\alpha_{-z}\check{T}) \\
&= (S * T)^\sim.
\end{aligned}$$

To consider $f * S$ we will need the property of vector-valued integration given in proposition 2.8, with the bounded linear operator P . For $\psi \in L^2(\mathbb{R}^d)$, we

calculate

$$\begin{aligned}
(f * S)^\sim \psi &= P \iint_{\mathbb{R}^{2d}} f(y) (\alpha_y S) P \psi \, dy \\
&= \iint_{\mathbb{R}^{2d}} f(y) P (\alpha_y S) P \psi \, dy \\
&= \iint_{\mathbb{R}^{2d}} f(y) (\alpha_y S)^\sim \psi \, dy \\
&= \iint_{\mathbb{R}^{2d}} f(y) \alpha_{-y} \check{S} \psi \, dy \\
&= (\check{f} * \check{S}) \psi,
\end{aligned}$$

where the last equality follows from the change of variable $y \mapsto -y$. We have also used that $(\alpha_y S)^\sim = \alpha_{-y} \check{S}$ by lemma 3.2.

3. The proof of this part is left to the reader, as it is straightforward and very similar to the preceding proofs.
4. The convolution of two functions is clearly positive, since it is defined by the integral of a positive function. If $f \in L^1(\mathbb{R}^{2d})$ is a positive function and $S \in \mathcal{T}^1$ a positive operator, let $\psi \in L^2(\mathbb{R}^d)$ and use the weak definition of the integral to find that

$$\langle (f * S)\psi, \psi \rangle = \iint_{\mathbb{R}^{2d}} f(z) \langle \alpha_z(S)\psi, \psi \rangle \, dz.$$

This integrand is positive since f and S are positive.

Finally, assume that $S, T \in \mathcal{T}^1$ are positive operators. Since S is positive, it has a positive square root \sqrt{S} . Therefore we can write $S * T(z) = \text{tr}(S \alpha_z \check{T}) = \text{tr}(\sqrt{S} \alpha_z (\check{T}) \sqrt{S})$. A trivial calculation shows that $\sqrt{S} \alpha_z (\check{T}) \sqrt{S}$ is a positive operator, and clearly the trace of a positive operator is a positive number. Hence $S * T(z) = \text{tr}(\sqrt{S} \alpha_z (\check{T}) \sqrt{S})$ is positive. □

Remark. The last part of point (3) in this proposition states that if we fix some function g and define an operator A_g on functions by $A_g(f) = f * g$, then A_g commutes with translations in the sense that $T_z(A_g(f)) = A_g(T_z f)$. There is a well known converse to this, stating that if $A : L^\infty(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ is bounded, linear and commutes with translation, then $A = A_g$ for a unique tempered distribution g [45, Thm.3.16].

The other two parts of point (3) show that a similar property holds for convolutions with a fixed operator S . If we denote by Γ_S the operator that acts on a

function or operator by taking the convolution with S , then $\Gamma_S(T_z f) = \alpha_z \Gamma_S(f)$ and $\Gamma_S(\alpha_z T) = T_z \Gamma_S(T)$. Werner calls these conditions on Γ_S covariance in [46], and in the same paper a converse analogous to the one we just saw for functions is proved. This theory is explained in greater detail and expanded upon in the appendix on informational completeness. We mention these results to give yet another reason why the convolutions we have defined really deserve their name. The interested reader may consult the original paper [46], or the more recent extensions of the theory in [30].

Proposition 4.8. *The convolution operations in definitions 4.1 and 4.2 are commutative and associative.*

Proof. Commutativity For $f, g \in L^1(\mathbb{R}^{2d})$, the proof is standard and consists of introducing the variable $u = x - y$ in the definitions:

$$\begin{aligned} f * g(x) &= \iint_{\mathbb{R}^{2d}} f(y)g(x - y) dy \\ &= \iint_{\mathbb{R}^{2d}} f(x - u)g(u) du \\ &= g * f(x). \end{aligned}$$

Then let $S, T \in \mathcal{T}^1$. We find that

$$\begin{aligned} S * T(z) &= \text{tr}(S\alpha_z \check{T}) \\ &= \text{tr}(S\pi(z)PTP\pi(z)^*) \\ &= \text{tr}(TP\pi(z)^*S\pi(z)P) \\ &= \text{tr}(T(\alpha_{-z}S)^\vee) \\ &= \text{tr}(T\alpha_z \check{S}) \\ &= T * S(z). \end{aligned}$$

We have made extensive use of the property $\text{tr}(AB) = \text{tr}(BA)$, and also used part (5) of lemma 3.2.

Associativity The most interesting case is the convolution of three operators. We will need lemma 4.2 in addition to some more technical calculations. Let $T_1, T_2, T_3 \in \mathcal{T}^1$. To show that $T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3$ it will be helpful to assume an arbitrary operator $T_0 \in \mathcal{T}^1$. If we can show that the dual space actions $\langle T_1 * (T_2 * T_3), T_0 \rangle = \langle (T_1 * T_2) * T_3, T_0 \rangle$ for any T_0 , we will have shown that the two expressions define the same element in the dual space $B(L^2(\mathbb{R}^d))$, and therefore the same operator. Since the duality is given by taking the trace, we need to show that

$$\text{tr}[T_0(T_1 * (T_2 * T_3))] = \text{tr}[T_0((T_1 * T_2) * T_3)].$$

To be pedantic, we should really take the adjoint of the the right argument in $\langle \cdot, \cdot \rangle$, since that is the way we defined the duality. However, two operators are equal if and only if their adjoints are equal, so we may consider the expressions without adjoints.

Writing out the left side of the equation and using proposition 2.9, we find that

$$\begin{aligned} \operatorname{tr} [T_0(T_1 * (T_2 * T_3))] &= \operatorname{tr} \left[T_0 \iint_{\mathbb{R}^{2d}} \operatorname{tr}(T_2 \alpha_x \check{T}_3) \alpha_x T_1 \, dx \right] \\ &= \iint_{\mathbb{R}^{2d}} \operatorname{tr} [T_2 \alpha_x \check{T}_3] \operatorname{tr} [(\alpha_x T_1) T_0] \, dx \\ &= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr} \left[(\alpha_x T_1) T_0 \alpha_y (T_3 \alpha_x \check{T}_2) \right] \, dy \, dx. \end{aligned}$$

The last equality uses lemma 4.2 to introduce the second integral, and also exploits the recently proved commutativity of convolutions to switch the order of T_2 and T_3 . It is a simple exercise to check that $\alpha_y(AB) = (\alpha_y A)(\alpha_y B)$ for operators A and B , in particular $\alpha_y(T_3 \alpha_x \check{T}_2) = (\alpha_y T_3)(\alpha_x \alpha_y \check{T}_2)$. Inserting this into our main calculation we get that

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr} \left[(\alpha_x T_1) T_0 \alpha_y (T_3 \alpha_x \check{T}_2) \right] \, dy \, dx &= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr} \left[(\alpha_x T_1) T_0 (\alpha_y T_3) (\alpha_x \alpha_y \check{T}_2) \right] \, dy \, dx \\ &= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr} \left[(T_0 \alpha_y T_3) (\alpha_x \alpha_y \check{T}_2) (\alpha_x T_1) \right] \, dy \, dx. \end{aligned}$$

We may now use Fubini's theorem to change the order of integration, and then invoke the equality in lemma 4.2 again to reduce the expression to a form that we recognize as the equality we wanted to prove.

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr} \left[(T_0 \alpha_y T_3) \alpha_x ((\alpha_y \check{T}_2) T_1) \right] \, dx \, dy &= \iint_{\mathbb{R}^{2d}} \operatorname{tr} [T_0 \alpha_y T_3] \operatorname{tr} \left[(\alpha_y \check{T}_2) T_1 \right] \, dy \\ &= \operatorname{tr} [T_0 ((T_1 * T_2) * T_3)]. \end{aligned}$$

The other cases are more elementary, using properties of the weak definition of the integral. To give an example, let $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{T}^1$. On the one hand we have by proposition 2.9 that

$$\begin{aligned} S * (T * f)(z) &= \operatorname{tr} \left[S \alpha_z \left(\iint_{\mathbb{R}^{2d}} f(y) \alpha_y T \, dy \right) \right] \\ &= \operatorname{tr} \left[S \iint_{\mathbb{R}^{2d}} f(-y) \alpha_z (\alpha_y \check{T}) \, dy \right] \\ &= \iint_{\mathbb{R}^{2d}} f(-y) \operatorname{tr} [S \alpha_{z+y} \check{T}] \, dy. \end{aligned}$$

On the other hand we find that

$$(S * T) * f(z) = \iint_{\mathbb{R}^{2d}} f(z - y) \operatorname{tr} [S \alpha_y \check{T}] \, dy,$$

and a standard change of variable establishes the desired equality. \square

Propositions 4.5 and 4.8 imply that \mathcal{T}^p is a Banach module over $L^1(\mathbb{R}^{2d})$ for $1 \leq p \leq \infty$, where the module action of $f \in L^1(\mathbb{R}^{2d})$ on $S \in \mathcal{T}^p$ is given by $f * S$. In fact, we could have obtained this module structure by starting with α . Section 3 shows that α is a shift on the Banach spaces \mathcal{T}^p , and even strongly continuous for $p < \infty$ by proposition 3.3. Hence, by theorem 2.21, α induces a unique $L^1(\mathbb{R}^{2d})$ -module structure with shift on \mathcal{T}^p for $p < \infty$, and theorem 2.21 shows that this module structure is given by the convolutions that we have defined in definition 4.1. It should be noted that Werner's convolution of two operators *does not* follow from the theory of Banach modules, yet we will see that the convolution of two operators will be a useful tool to investigate the Banach module structures.

5 The Berezin transform, localization operators and adjoints

The main novel result of this thesis is that the theory of convolutions defined by Werner in [46] may be used to reprove and generalize results on the localization operators and the Berezin transform in [3]. The key fact linking these theories is that both the Berezin transform and localization operators are obtained by picking an operator S of a special form, and then considering convolutions with S .

Theorem 5.1. *Fix $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, and consider the operators $\varphi_2 \otimes \varphi_1$ and $\check{\varphi}_1 \otimes \check{\varphi}_2$. Let $T \in B(L^2(\mathbb{R}^d))$ and let f be a function on \mathbb{R}^{2d} .*

The localization operator $\mathcal{A}_f^{\varphi_1, \varphi_2}$ is given by

$$\mathcal{A}_f^{\varphi_1, \varphi_2} = f * \varphi_2 \otimes \varphi_1.$$

The Berezin transform of T with windows φ_1 and φ_2 is given by

$$\mathcal{B}^{\varphi_1, \varphi_2} T(z) = T * \check{\varphi}_1 \otimes \check{\varphi}_2.$$

Proof. The proof will simply consist of calculating $f * \varphi_2 \otimes \varphi_1$ and $T * \check{\varphi}_1 \otimes \check{\varphi}_2$.

First let $\psi \in L^2(\mathbb{R}^d)$. We find that

$$\begin{aligned}
(f * S)(\psi) &= \iint_{\mathbb{R}^{2d}} f(z)(\alpha_z S)(\psi) dz \\
&= \iint_{\mathbb{R}^{2d}} f(z) \langle \pi(z)^* \psi, \varphi_1 \rangle \pi(z) \varphi_2 dz \\
&= \iint_{\mathbb{R}^{2d}} f(z) V_{\varphi_1} \psi \pi(z) \varphi_2 dz \\
&= A_f^{\varphi_1, \varphi_2} \psi,
\end{aligned}$$

which proves the first statement.

Turning to the Berezin transform, let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Using Parseval's identity, we find that

$$\begin{aligned}
(T * \check{\varphi}_1 \otimes \check{\varphi}_2)(z) &= \text{tr}(\check{T} \alpha_{-z} \check{\varphi}_1 \otimes \check{\varphi}_2) \\
&= \sum_{n \in \mathbb{N}} \langle \check{T} \pi(-z) \check{\varphi}_1 \otimes \check{\varphi}_2 \pi(-z)^* e_n, e_n \rangle \\
&= \sum_{n \in \mathbb{N}} \langle \pi(-z)^* e_n, \check{\varphi}_2 \rangle \langle \check{T} \pi(-z) \check{\varphi}_1, e_n \rangle \\
&= \sum_{n \in \mathbb{N}} \langle e_n, \pi(-z) \check{\varphi}_2 \rangle \langle \check{T} \pi(-z) \check{\varphi}_1, e_n \rangle \\
&= \langle \check{T} \pi(-z) \check{\varphi}_1, \pi(-z) \check{\varphi}_2 \rangle \\
&= \langle PTP \pi(-z) P \varphi_1, \pi(-z) P \varphi_2 \rangle \\
&= \langle T \pi(z) \varphi_1, \pi(z) \varphi_2 \rangle = \mathcal{B}^{\varphi_1, \varphi_2} T(z),
\end{aligned}$$

where we have used lemma 3.2 in the last step. \square

In order to apply the results in section 4 to the Berezin transform and localization operators, we need to discuss some properties of $S = \varphi_2 \otimes \varphi_1$. If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, then $S \in \mathcal{T}^1$ and $\|S\|_{\mathcal{T}^1} = \|S\|_{B(L^2)} = \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}$, as a simple calculation shows. Therefore proposition 4.5 applies, and we immediately obtain the next proposition.

Proposition 5.2. *Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ and $1 \leq p \leq \infty$.*

1. *If $T \in \mathcal{T}^p$, then $\mathcal{B}^{\varphi_1, \varphi_2} T \in L^p(\mathbb{R}^{2d})$ with $\|\mathcal{B}^{\varphi_1, \varphi_2} T\|_{L^p} \leq \|T\|_{\mathcal{T}^p} \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}$.*
2. *If $a \in L^p(\mathbb{R}^{2d})$, then $\mathcal{A}_a^{\varphi_1, \varphi_2} \in \mathcal{T}^p$ with $\|\mathcal{A}_a^{\varphi_1, \varphi_2}\|_{\mathcal{T}^p} \leq \|a\|_{L^p} \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}$.*

The boundedness results in this section for Berezin transforms and localization operators were all shown, along with others, in [3] and [13]. The novel result in this section is therefore that the Berezin transform and localization operators may be considered as special cases of convolutions, which in section 7 will be used to apply theorems from Werner's theory of convolutions to Berezin transforms and localization operators.

5.1 Adjoints

There is one more fundamental connection between the different convolutions that we will need in order to prove the theorems in section 7. In order to explain this, we will extend the notation for the Berezin transforms and localization operators to the case where the windows φ_1, φ_2 are replaced by an operator $S \in \mathcal{T}^1$. From theorem 5.1 the obvious way of doing this is to define \mathcal{A}_S and \mathcal{B}_S by

$$\mathcal{A}_S f = f * S \qquad \mathcal{B}_S T = T * \check{S}^*,$$

for a function f and an operator T . The operators \mathcal{A}_S and \mathcal{B}_S are bounded on different domains, such as L^p -spaces and Schatten p -classes depending on the nature of S , as we have seen in the previous sections. If we choose the domains of \mathcal{A}_S and \mathcal{B}_S in a compatible way, then they are adjoints of each other. This was noted by Werner in [46], but is also contained in [3], although with a very different formalism.

Theorem 5.3. *Fix $S \in \mathcal{T}^1$ and $1 \leq p < \infty$. Let q be the conjugate exponent of p determined by $\frac{1}{p} + \frac{1}{q} = 1$. The Banach space adjoint of $\mathcal{A}_S : L^p(\mathbb{R}^{2d}) \rightarrow \mathcal{T}^p$ is given by*

$$(\mathcal{A}_S)^* = \mathcal{B}_S,$$

where $\mathcal{B}_S : \mathcal{T}^q \rightarrow L^q(\mathbb{R}^{2d})$.

Similarly, the adjoint of $\mathcal{B}_S : \mathcal{T}^p \rightarrow L^p(\mathbb{R}^{2d})$ is given by

$$(\mathcal{B}_S)^* = \mathcal{A}_S,$$

where $\mathcal{A}_S : L^q(\mathbb{R}^{2d}) \rightarrow \mathcal{T}^q$.

Proof. If we let the bracket denote duality, then the adjoint of \mathcal{A}_S is determined by

$$\langle (\mathcal{A}_S)^* T, f \rangle = \langle T, \mathcal{A}_S f \rangle$$

for any $T \in \mathcal{T}^q$ and $f \in L^p(\mathbb{R}^{2d})$. First assume $p < \infty$. Since \mathcal{A}_S and \mathcal{B}_S are convolution operator in disguise, we did in fact check in section 4.1 that

$$\langle \mathcal{B}_S T, f \rangle = \langle T, \mathcal{A}_S f \rangle$$

is true whenever $T \in \mathcal{T}^1$ and $f \in L^1 \cap L^\infty$. The general statement then follows, since \mathcal{T}^1 is a dense subspace of \mathcal{T}^q and $L^1 \cap L^\infty$ is a dense subspace of L^p . The case $p = \infty$ holds by definition 4.2. The proof that $(\mathcal{B}_S)^* = \mathcal{A}_S$ uses exactly the same argument. \square

6 Fourier transforms

We will now introduce a Fourier transform for trace class operators. In the terminology of Werner [28, 29, 46] this is the Fourier-Weyl transform, but we will follow Folland [20] and call it the Fourier-Wigner transform.

Definition 6.1. Let $S \in \mathcal{T}^1$. We define the *Fourier-Wigner transform* $\mathcal{F}_W S$ of S to be the function given by

$$\mathcal{F}_W S(z) = e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S)$$

for $z \in \mathbb{R}^{2d}$.

Various versions of the operator \mathcal{F}_W have been studied in many different settings both in the physics and the mathematics literature – it may in fact be traced all the way back to a fundamental paper in quantum mechanics by Wigner from 1932 [49]. The crucial insight of Werner’s paper [46] is that \mathcal{F}_W might rightly be called a Fourier transform, and that the naturally associated convolution operations are the convolutions discussed in section 4. Much of this section will be used to make these claims precise, by showing that \mathcal{F}_W has several properties analogous to known properties of the Fourier transform of functions. We start by showing that in the simplest case the Fourier-Wigner transform of an operator recovers familiar concepts from time-frequency analysis.

Lemma 6.1. *Let $S = \varphi_2 \otimes \varphi_1$ for two functions $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. The Fourier-Wigner transform of S is given by*

$$\mathcal{F}_W(S)(z) = A(\varphi_2, \varphi_1)(z),$$

where $A(\varphi_2, \varphi_1)(z)$ is the cross-ambiguity function.

Proof. Let $\{\psi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. A calculation using Parseval’s identity shows that

$$\begin{aligned} \mathcal{F}_W(\varphi_2 \otimes \varphi_1)(z) &= e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)\varphi_2 \otimes \varphi_1) \\ &= e^{-\pi i x \cdot \omega} \sum_{n \in \mathbb{N}} \langle (\pi(-z)\varphi_2 \otimes \varphi_1)\psi_n, \psi_n \rangle \\ &= e^{-\pi i x \cdot \omega} \sum_{n \in \mathbb{N}} \langle \pi(-z)\varphi_2, \psi_n \rangle \langle \psi_n, \varphi_1 \rangle \\ &= e^{-\pi i x \cdot \omega} \langle \pi(-z)\varphi_2, \varphi_1 \rangle \\ &= e^{\pi i x \cdot \omega} V_{\varphi_1} \varphi_2(z), \end{aligned}$$

where the last equality follows from the formula for $\pi(z)^*$ given in lemma 3.1 and the definition of the short-time Fourier transform. \square

At some later point we are going to need an example of an operator T such that $\mathcal{F}_W T(z) \neq 0$ for all $z \in \mathbb{R}^{2d}$. We therefore include the following example.

Example 6.1. Consider the Gaussian $\varphi(t) = 2^{d/4} e^{-\pi t \cdot t}$ for $t \in \mathbb{R}^d$ and the operator $S = \varphi \otimes \varphi$. We know that $\mathcal{F}_W S = e^{\pi i x \cdot \omega} V_\varphi \varphi(z)$, and if we write out the definition of $V_\varphi \varphi(z)$, we find that

$$\mathcal{F}_W(\varphi \otimes \varphi)(z) = e^{\pi i x \cdot \omega} \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} \varphi(t+x) \overline{\varphi(t)} dt. \quad (5)$$

Calculating the integral in equation (5) gives that

$$\mathcal{F}_W(\varphi \otimes \varphi)(z) = e^{2\pi i x \cdot \omega} e^{-\frac{1}{2}\pi z \cdot z}.$$

In particular we note that the Fourier-Wigner transform has no zeros.

Next we will show that the Fourier-Wigner transform can be extended to a unitary operator, just as the regular Fourier transform. By doing so we will also find that the inverse of the Fourier-Wigner transform is the integrated Schrödinger representation.

Proposition 6.2. *The Fourier-Wigner transform extends to a unitary operator $\mathcal{F}_W : \mathcal{T}^2 \rightarrow L^2(\mathbb{R}^{2d})$. This extension is the inverse operator of the integrated Schrödinger representation ρ , and satisfies*

$$\mathcal{F}_W(ST) = \mathcal{F}_W(S) \natural \mathcal{F}_W(T)$$

for $S, T \in \mathcal{T}^2$.

Proof. We start by showing that the Fourier-Wigner transform can be extended to the Hilbert-Schmidt operators \mathcal{T}^2 . By the singular value decomposition, elements of the form $S = \sum_{n=1}^N s_n \psi_n \otimes \phi_n$ are dense in \mathcal{T}^2 , where $\{\psi_n\}_{n=1}^N$ and $\{\phi_n\}_{n=1}^N$ are orthonormal sets in $L^2(\mathbb{R}^d)$, $s_n > 0$ are the singular values of S and $N \in \mathbb{N}$. From the previous lemma $\mathcal{F}_W(S) = \sum_{n=1}^N e^{\pi i x \cdot \omega} s_n V_{\phi_n} \psi_n$, and a computation using

Moyal's identity shows that

$$\begin{aligned}
\|\mathcal{F}_W(S)\|_{L^2}^2 &= \left\langle \sum_{n=1}^N s_n e^{\pi i x \cdot \omega} V_{\phi_n} \psi_n, \sum_{m=1}^N s_m e^{\pi i x \cdot \omega} V_{\phi_m} \psi_m \right\rangle_{L^2} \\
&= \sum_{m,n=1}^N s_m s_n \langle V_{\phi_n} \psi_n, V_{\phi_m} \psi_m \rangle_{L^2} \\
&= \sum_{m,n=1}^N s_m s_n \langle \psi_n, \psi_m \rangle_{L^2} \langle \phi_m, \phi_n \rangle_{L^2} \\
&= \sum_{n=1}^N s_n^2 \\
&= \|S\|_{\mathcal{T}^2}^2.
\end{aligned}$$

In words, the Fourier-Wigner transform is an isometry on a dense subspace of \mathcal{T}^2 into $L^2(\mathbb{R}^{2d})$, and therefore extends to an isometry $\mathcal{F}_W : \mathcal{T}^2 \rightarrow L^2(\mathbb{R}^{2d})$ [40, Thm. 1.7]. To show that the extension is the inverse of ρ , we consider $T = \varphi_2 \otimes \varphi_1$ for $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. We have already shown that $\mathcal{F}_W(T)(z) = e^{\pi i x \cdot \omega} V_{\varphi_1} \varphi_2$. If we let $\psi, \phi \in L^2(\mathbb{R}^d)$, we may use the weak definition of the vector-valued integral defining ρ to calculate

$$\begin{aligned}
\langle \rho(e^{\pi i x \cdot \omega} V_{\varphi_1} \varphi_2) \psi, \phi \rangle &= \iint_{\mathbb{R}^{2d}} V_{\varphi_1} \varphi_2 \langle \pi(z) \psi, \phi \rangle dz \\
&= \iint_{\mathbb{R}^{2d}} V_{\varphi_1} \varphi_2(z) \overline{V_{\psi} \phi(z)} dz \\
&= \langle \varphi_2, \phi \rangle \langle \psi, \varphi_1 \rangle,
\end{aligned}$$

where the last equality is Moyal's identity. The last expression clearly equals $\langle Tf, g \rangle$. If we denote the identity operator on \mathcal{T}^2 by $\mathcal{I}_{\mathcal{T}^2}$, we have shown that $\rho \mathcal{F}_W T = \mathcal{I}_{\mathcal{T}^2} T$. By linearity this equality of operators must hold on the dense subspace of \mathcal{T}^2 spanned by such operators T , and therefore $\rho \mathcal{F}_W = \mathcal{I}_{\mathcal{T}^2}$ by continuity. To show that \mathcal{F}_W is a two-sided inverse of ρ , we note that ρ has *some* two-sided inverse ρ^{-1} , since ρ is unitary. If we apply this inverse to the equation $\rho \mathcal{F}_W = \mathcal{I}_{\mathcal{T}^2}$, we find that $\mathcal{F}_W = \rho^{-1}$. Therefore \mathcal{F}_W is the inverse operator of the integrated Schrödinger representation.

The equation $\mathcal{F}_W(ST) = \mathcal{F}_W(S) \natural \mathcal{F}_W(T)$ may now be deduced from two known properties of ρ , namely $\rho(f \natural g) = \rho(f) \rho(g)$ and the fact that it is injective. Since ρ is the inverse of \mathcal{F}_W , $\rho \mathcal{F}_W(ST) = ST$. But we also know that $\rho(\mathcal{F}_W(S) \natural \mathcal{F}_W(T)) = \rho \mathcal{F}_W(S) \rho \mathcal{F}_W(T) = ST$. The equality now follows since ρ is injective. \square

In other words, $\mathcal{F}_W(S)$ is the twisted Weyl symbol of S . As a simple consequence, we obtain a formula for the trace of trace class operators in terms of their Weyl symbol. In section 4 we informally viewed the trace of an operator as an analogue of the integral of a function, and the next formula makes this analogue more precise.

Corollary 6.2.1. *Let $f \in L^1(\mathbb{R}^{2d})$ be a function such that the Weyl transform L_f is a trace class operator. The trace of L_f is given by*

$$\mathrm{tr}(L_f) = \iint_{\mathbb{R}^{2d}} f(z) \, dz.$$

Proof. On the one hand, the previous proposition shows that the $\mathcal{F}_W(L_f)$ is the twisted Weyl symbol of L_f , which we know is $\mathcal{F}_\sigma f$ from section 2.7.2. On the other hand, $\mathcal{F}_W(L_f)(z) = e^{-\pi i x \cdot \omega} \mathrm{tr}(\pi(-z)L_f)$ from the definition of the Fourier-Wigner transform. Therefore $\mathcal{F}_\sigma f(z) = e^{-\pi i x \cdot \omega} \mathrm{tr}(\pi(-z)L_f)$, and evaluating this at $z = 0$ gives the desired equality. \square

Now that we have extended the Fourier-Wigner transform to the Hilbert-Schmidt operators, we may follow Werner [46] and use a simple interpolation argument to prove the analogue of the well-known Hausdorff-Young inequality for Fourier transforms [39, Thm. IX.8].

Proposition 6.3 (Hausdorff-Young inequality). *Let $1 \leq p \leq 2$ and let q be the conjugate exponent determined by $\frac{1}{p} + \frac{1}{q} = 1$. If $S \in \mathcal{T}^p$, then $\mathcal{F}_W(S) \in L^q(\mathbb{R}^{2d})$ with norm estimate*

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \|S\|_{\mathcal{T}^p}.$$

Proof. We will use complex interpolation, and therefore start with the endpoints $p = 1$ and $p = 2$. The result for $p = 2$ follows from proposition 6.2, where we even have equality of norms. For $p = 1$, the result follows from part (5) of proposition 2.5, since this proposition gives that

$$\begin{aligned} |F_W S(z)| &= |\mathrm{tr}(\pi(-z)S)| \\ &\leq \|\pi(-z)\|_{B(L^2)} \|S\|_{\mathcal{T}^1} = \|S\|_{\mathcal{T}^1}, \end{aligned}$$

so $\|F_W S\|_{L^\infty} \leq \|S\|_{\mathcal{T}^1}$. For the interpolation argument, theorem 2.24 gives that

$$(\mathcal{T}^1, \mathcal{T}^2)_\theta = \mathcal{T}^p \qquad (L^\infty, L^2)_\theta = L^q,$$

where $\frac{1}{p} = 1 - \frac{\theta}{2}$ and $\frac{1}{q} = \frac{\theta}{2}$ for $\theta \in (0, 1)$. Thus the proposition follows from the machinery of complex interpolation. \square

For $S = \psi \otimes \phi$ with $\psi, \phi \in L^2(\mathbb{R}^d)$, the previous proposition gives that

$$\iint_{\mathbb{R}^{2d}} |V_\phi \psi|^q(z) dz \leq \|\psi\|_{L^2}^q \|\phi\|_{L^2}^q$$

for $2 \leq q < \infty$ and p the conjugate exponent of q , since $\|\psi \otimes \phi\|_{\mathcal{T}^p} = \|\psi\|_{L^2} \|\phi\|_{L^2}$. This is a slightly weaker version of Lieb's uncertainty principle from time-frequency analysis [23, Thm. 3.3.2], which states that

$$\iint_{\mathbb{R}^{2d}} |V_\phi \psi|^q(z) dz \leq \left(\frac{2}{q}\right)^d \|\psi\|_{L^2}^q \|\phi\|_{L^2}^q.$$

On the other hand, if we take Lieb's uncertainty principle as our starting point and expand $S = \sum_{m \in \mathbb{N}} s_m \psi_m \otimes \phi_m$ using the singular value decomposition, we find that

$$\begin{aligned} \|\mathcal{F}_W(S)\|_{L^q} &= \lim_{n \rightarrow \infty} \left\| \mathcal{F}_W \left(\sum_{m=1}^n s_m \psi_m \otimes \phi_m \right) \right\|_{L^q} \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{m=1}^n s_m A(\psi_m, \phi_m) \right\|_{L^q} \\ &\leq \lim_{n \rightarrow \infty} \sum_{m=1}^n s_m \|A(\psi_m, \phi_m)\|_{L^q} \\ &\leq \lim_{n \rightarrow \infty} \sum_{m=1}^n s_m \left(\frac{2}{q}\right)^{d/q} \|\psi_m\|_{L^2} \|\phi_m\|_{L^2} \\ &= \left(\frac{2}{q}\right)^{d/q} \sum_{m=1}^{\infty} s_m = \left(\frac{2}{q}\right)^{d/q} \|S\|_{\mathcal{T}^1}. \end{aligned}$$

The first step in this calculation uses proposition 6.3, which states that the Fourier-Wigner transform is continuous from \mathcal{T}^p to $L^q(\mathbb{R}^{2d})$. We also use Lieb's uncertainty principle to bound $\|A(\psi_m, \phi_m)\|_{L^q}$. By these calculations we obtain an improved version of the Hausdorff-Young inequality

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \left(\frac{2}{q}\right)^{d/q} \|S\|_{\mathcal{T}^p},$$

which includes Lieb's uncertainty principle as a special case. This version is sharp as we have equality in Lieb's uncertainty principle for Gaussians [23, p. 51].

6.1 Convolutions and Fourier transforms

Having introduced both Fourier transforms and convolutions of functions and operators, one would naturally hope that the well-known relationship between these two operations on functions also holds for operators. This is true, but we will need to use the symplectic Fourier transform \mathcal{F}_σ on functions. A first clue to this effect is the appearance of the symplectic form σ in part (3) of lemma 3.2, which will turn out to be the key fact linking the symplectic Fourier transform to the Fourier-Wigner transform and convolutions.

Proposition 6.4. *Let $f, g \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{T}^1$.*

1. $\mathcal{F}_\sigma(f * g) = \mathcal{F}_\sigma(f)\mathcal{F}_\sigma(g)$.
2. $\mathcal{F}_\sigma(S * T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$.
3. $\mathcal{F}_W(f * S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$.

Proof. 1. The case of the Fourier transform of functions is well known, but we include a proof to show how the argument goes for the case of the symplectic Fourier transform. Using Fubini's theorem, which is applicable since the functions are integrable, we find that

$$\begin{aligned}\mathcal{F}_\sigma(f * g)(z) &= \iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,y)} \iint_{\mathbb{R}^{2d}} f(x)g(y-x) dx dy \\ &= \iint_{\mathbb{R}^{2d}} f(x) \iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,y)} g(y-x) dy dx.\end{aligned}$$

Introducing the variable $y' = y - x$, the innermost integral becomes

$$\iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,y'+x)} g(y') dy',$$

and by the bilinearity of σ we may write this as

$$\iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,x)} e^{-2\pi i\sigma(z,y')} g(y') dy'.$$

We then insert this back into our double integral, to find that

$$\begin{aligned}\mathcal{F}_\sigma(f * g)(z) &= \iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,x)} f(x) \iint_{\mathbb{R}^{2d}} e^{-2\pi i\sigma(z,y')} g(y') dy' dx \\ &= \mathcal{F}_\sigma(f)\mathcal{F}_\sigma(g).\end{aligned}$$

2. From the definitions we get that $\mathcal{F}_\sigma(S*T)(z) = \iint_{\mathbb{R}^{2d}} \text{tr} [S\pi(z')\check{T}\pi(z')^*] e^{-2\pi i\sigma(z,z')} dz'$. Using part (3) of lemma 3.2 to get that $e^{-2\pi i\sigma(z,z')}\pi(z') = \alpha_{-z}\pi(z')$, the integrand may be written in an alternative form that will allow us to use lemma 4.2 :

$$\begin{aligned} \text{tr} [S\pi(z')\check{T}\pi(z')^*] e^{-2\pi i\sigma(z,z')} &= \text{tr} \left[S e^{-2\pi i\sigma(z,z')} \pi(z') \check{T} \pi(z')^* \right] \\ &= \text{tr} [S\pi(-z)\pi(z')\pi(-z)^*\check{T}\pi(z')^*]. \end{aligned}$$

Lemma 4.2 then gives that

$$\begin{aligned} \mathcal{F}_\sigma(S * T)(z) &= \iint_{\mathbb{R}^{2d}} \text{tr} [S\pi(-z)\alpha_{z'}(\pi(-z)^*\check{T})] dz' \\ &= \text{tr}(S\pi(-z))\text{tr}(\pi(-z)^*\check{T}) \\ &= \text{tr}(S\pi(-z))\text{tr}(e^{-2\pi i x \cdot \omega} \pi(z)\check{T}) \\ &= \text{tr}(e^{-\pi i x \cdot \omega} S\pi(-z))\text{tr}(e^{-\pi i x \cdot \omega} \pi(-z)T) \\ &= \mathcal{F}_W(S)(z)\mathcal{F}_W(T)(z), \end{aligned}$$

where we have used that $\text{tr}(\pi(z)\check{T}) = \text{tr}(\pi(z)PTP) = \text{tr}(P\pi(z)PT) = \text{tr}(\pi(-z)T)$ from part (5) of lemma 3.2.

3. By proposition 2.9 we may take the trace inside the integral:

$$\begin{aligned} \mathcal{F}_W(f * S)(z) &= e^{-\pi i x \cdot \omega} \text{tr} \left(\pi(-z) \iint_{\mathbb{R}^{2d}} f(z')\pi(z')S\pi(z')^* dz' \right) \\ &= e^{-\pi i x \cdot \omega} \iint_{\mathbb{R}^{2d}} f(z') \text{tr} [\pi(-z)\pi(z')S\pi(z')^*] dz'. \end{aligned}$$

Next we will manipulate the integrand by using part (3) of lemma 3.2 as before. However, we first note that since $\pi(z')^* = \pi(-z')e^{-2\pi i x \cdot \omega}$, a straightforward calculation yields that $\pi(z')^*\pi(-z)\pi(z') = \pi(-z')\pi(-z)\pi(-z')^*$. With this in mind, we find that

$$\begin{aligned} \text{tr} [\pi(-z)\pi(z')S\pi(z')^*] &= \text{tr} [\pi(z')^*\pi(-z)\pi(z')S] \\ &= \text{tr} [\pi(-z')\pi(-z)\pi(-z')^*S] \\ &= e^{-2\pi i\sigma(z,z')}\text{tr}(\pi(-z)S). \end{aligned}$$

Inserting this expression into our calculation concludes the proof, since

$$\begin{aligned} \mathcal{F}_W(f * S)(z) &= e^{-\pi i x \cdot \omega} \iint_{\mathbb{R}^{2d}} f(z')e^{2\pi i\sigma(z,z')}\text{tr}(\pi(-z)S) dz' \\ &= e^{-\pi i x \cdot \omega} \text{tr}(\pi(-z)S) \iint_{\mathbb{R}^{2d}} f(z')e^{-2\pi i\sigma(z,z')} dz' \\ &= \mathcal{F}_\sigma(f)\mathcal{F}_W(S). \end{aligned}$$

□

Remark. One might be tempted to remove the phase factor from the definition of the Fourier-Wigner transform. However, another phase factor would then appear in part (2) of this proposition.

This result is not merely aesthetically pleasing, but will be used in several proofs in the rest of this text. As a first example we obtain the representation of $f * T$ as a pseudodifferential operator.

Corollary 6.4.1. *Let $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$. The twisted Weyl symbol of $f * S$ is the function*

$$\mathcal{F}_\sigma(f)\mathcal{F}_W(S)$$

In particular, if $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, the Weyl symbol of the localization operator $\mathcal{A}_f^{\varphi_1, \varphi_2}$ is the function

$$\sigma = f * W(\varphi_2, \varphi_1).$$

Proof. We know from proposition 6.2 that \mathcal{F}_W is the inverse operator to the integrated Schrödinger representation, and thus returns the twisted Weyl symbol of an operator. From proposition 6.4 we find that $\mathcal{F}_W(f * S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$. In particular, the twisted Weyl symbol of $\mathcal{A}_f^{\varphi_1, \varphi_2}$ is $\mathcal{F}_\sigma(f)A(\varphi_2, \varphi_1)$ by lemma 6.1. The Weyl symbol is the symplectic Fourier transform of the twisted Weyl symbol, thus given by $\mathcal{F}_\sigma[\mathcal{F}_\sigma(f)A(\varphi_2, \varphi_1)] = f * W(\varphi_2, \varphi_1)$. □

Furthermore, this gives a way to calculate the composition of two operators of the form $f * S$ and $g * T$: from section 2.7.2 the twisted Weyl symbol of the composition is the twisted convolution of the symbols of $f * S$ and $g * T$.

We will now prove a proposition by Werner [46] which generalizes the Riemann-Lebesgue lemma to the Fourier-Wigner transform.

Proposition 6.5 (Riemann-Lebesgue lemma). *If $S \in \mathcal{T}^1$, the Fourier-Wigner transform $\mathcal{F}_W(S)$ is continuous and vanishes at infinity, i.e. $\lim_{|z| \rightarrow \infty} |\mathcal{F}_W(z)| = 0$.*

Proof. Vanishes at infinity: Proposition 6.4 gives that $\mathcal{F}_\sigma(S * S) = \mathcal{F}_W(S)^2$. By the usual Riemann-Lebesgue lemma, the left side vanishes at infinity, which clearly implies that $\mathcal{F}_W(S)$ vanishes at infinity.

Continuity: Assume that z_n is a sequence converging to some z in \mathbb{R}^{2d} . We need to show that $\mathcal{F}_W(S)(z_n) \rightarrow \mathcal{F}_W(S)(z)$. Let $\{\psi_m\}_{m \in \mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. Writing out the definition of the trace, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{F}_W(S)(z_n) &= \lim_{n \rightarrow \infty} \sum_{m \in \mathbb{N}} \langle e^{-\pi i x_n \cdot \omega_n} \pi(z_n) S \psi_m, \psi_m \rangle \\ &= \sum_{m \in \mathbb{N}} \lim_{n \rightarrow \infty} \langle e^{-\pi i x_n \cdot \omega_n} \pi(z_n) S \psi_m, \psi_m \rangle, \end{aligned}$$

where we have assumed that the limit can be taken inside the sum for now. By the strong continuity of $z \mapsto e^{-\pi i x \cdot \omega} \pi(z)$, which follows from a trivial addition to the proof of lemma 3.1, we get that $\lim_{n \rightarrow \infty} e^{-\pi i x_n \cdot \omega_n} \pi(z_n) S \psi_m = e^{-\pi i x \cdot \omega} \pi(z) S \psi_m$. Furthermore, the Cauchy-Schwarz inequality gives that the inner product is continuous in both coordinates, so $\lim_{n \rightarrow \infty} \langle e^{-\pi i x_n \cdot \omega_n} \pi(z_n) S \psi_m, \psi_m \rangle = \langle \lim_{n \rightarrow \infty} e^{-\pi i x_n \cdot \omega_n} \pi(z_n) S \psi_m, \psi_m \rangle = \langle e^{-\pi i x \cdot \omega} \pi(z) S \psi_m, \psi_m \rangle$. Inserting this back into the calculation gives the desired result.

The proof that we can take the limit inside the sum follows from the argument we used when proving part (3) of proposition 3.3. □

6.2 The Arveson spectrum

For a function $f \in L^1(\mathbb{R}^{2d})$ one often defines the spectrum of f to be the closure of $\{z \in \mathbb{R}^{2d} : \mathcal{F}f(z) \neq 0\}$. Similarly we could use the Fourier-Wigner transform to define the spectrum of $S \in \mathcal{T}^1$ to be the closure of $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) \neq 0\}$. There is however another notion of spectrum for elements of a von Neumann algebra, such as $B(L^2(\mathbb{R}^d))$, due to Arveson [1]. We wish to show that the Arveson spectrum of $S \in \mathcal{T}^1$ can be described using the set of zeros of the Fourier-Wigner transform of S in a natural way.

We first cite Arveson's definition of the spectrum. Let X be a von Neumann algebra with an automorphism group $\{U_z\}_{z \in \mathbb{R}^{2d}}$ on X , and let $x \in X$. Arveson [2] defined the spectrum $\text{sp}_U(x)$ to be the spectrum of the family of functions $\{z \mapsto \rho(U_z x) : \rho \in X_*\}$, where X_* is the predual of X and we consider X_* as a subspace of the dual space of X .

In our case $U = \alpha$ and $X = B(L^2(\mathbb{R}^d))$, and the predual of X is \mathcal{T}^1 , where $T \in \mathcal{T}^1$ acts on $S \in B(L^2(\mathbb{R}^d))$ by $S \mapsto \text{tr}(TS)$. By the spectrum of a function in $f \in L^1(\mathbb{R}^{2d})$ we will mean the closure of $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) \neq 0\}$, and the spectrum of a family of functions $\{f_i\}_{i \in I}$ in $L^1(\mathbb{R}^{2d})$ is then the closure of $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f_i(z) \neq 0 \text{ for some } i \in I\}$.

Proposition 6.6. *Let $S \in \mathcal{T}^1$. The spectrum $\text{sp}_\alpha(S)$ is the closure of the complement of $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$.*

Proof. By definition, $\text{sp}_\alpha(S)$ is the spectrum of the functions $\text{tr}(T \alpha_z S) = T * \check{S}(z)$ for $T \in \mathcal{T}^1$. By lemma 4.2 these functions belong to $L^1(\mathbb{R}^{2d})$, and hence their spectrum is the closure of the complement of the set $Z := \{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma(T * \check{S}) = 0 \ \forall T \in \mathcal{T}^1\}$. By proposition 6.4, $\mathcal{F}_\sigma(T * \check{S})(z) = \mathcal{F}_W(T)(z) \mathcal{F}_W(\check{S})(z) = \mathcal{F}_W(T)(z) \mathcal{F}_W(S)(-z)$, hence $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$ is a subset of Z . To see that $Z = \{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$, note that we have constructed $T_0 \in \mathcal{T}^1$ with $\mathcal{F}_W(T_0)(z) \neq 0$ for

all $z \in \mathbb{R}^{2d}$ in example 6.1. Hence $\mathcal{F}_W(-z)(S) \neq 0$ implies that $\mathcal{F}_\sigma(T_0 * \check{S})(z) = \mathcal{F}_W(T_0)(z)\mathcal{F}_W(S)(-z) \neq 0$.

□

Note that the minus sign in the previous proposition would disappear if we followed Werner [46] and defined $\mathcal{F}_W S(z) = e^{-\pi i x \cdot \omega} \text{tr}(\pi(z)S)$. The difference between our definition and Werner's is little more than cosmetic; we have chosen our convention to get that $\mathcal{F}_W(S)$ is the twisted Weyl symbol of S when $S \in \mathcal{T}^2$.

7 A generalization of Wiener's Tauberian theorem

The previous section showed that the Fourier-Wigner transform shares several properties with the Fourier transform of functions. The goal of this section is to show that the Wiener's famous Tauberian theorem for functions can be generalized to operators, as was first done by Werner in [46]. We will then apply this result to the rank-one case, which both improves the results of Bayer and Gröchenig [3] and gives these results a conceptual framework using the Fourier-Wigner transform.

Wiener's Tauberian theorem was formulated and proved in a famous 1932 paper by Wiener [48], and characterizes the functions $g \in L^1(\mathbb{R}^{2d})$ whose translates $\{T_z g : z \in \mathbb{R}^{2d}\}$ span a dense subspace of $L^1(\mathbb{R}^d)$ in terms of their Fourier transforms. In order to generalize this theorem to operators, we will first need to fix some terminology.

Definition 7.1. Let $1 \leq p < \infty$. We say that $g \in L^p(\mathbb{R}^{2d})$ is *p-regular* if the translates $\{T_z g : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of $L^p(\mathbb{R}^{2d})$. Similarly, we say that $S \in \mathcal{T}^p$ is *p-regular* if the translates $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of \mathcal{T}^p . We will often refer to 1-regularity as simply *regularity*.

If $g \in L^\infty(\mathbb{R}^{2d})$ we say that g is *∞ -regular* if the translates $\{T_z g : z \in \mathbb{R}^{2d}\}$ span a weak* dense subspace of $L^\infty(\mathbb{R}^{2d})$. We say that $S \in B(L^2(\mathbb{R}^d))$ is *∞ -regular* if the translates $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of $K(L^2(\mathbb{R}^d))$.

Remark. 1. In section 2.3, we saw that $\|\cdot\|_{B(L^2)} \leq \|\cdot\|_{\mathcal{T}^q} \leq \|\cdot\|_{\mathcal{T}^p} \leq \|\cdot\|_{\mathcal{T}^1}$ for $1 \leq p \leq q < \infty$, and also that \mathcal{T}^p is a dense subspace of \mathcal{T}^q . Thus we get that *p-regularity* implies *q-regularity* for an operator S if $p \leq q$. This is also true for $q = \infty$, since any Schatten *p*-class is norm dense in $K(L^2(\mathbb{R}^d))$.

2. An equivalent definition for an operator S to be *∞ -regular* is that the translates of S span a weak* dense subspace of $B(L^2(\mathbb{R}^d))$ [30]. We will use both of these formulations.

We are now ready to state Wiener's theorem using our newly introduced terminology. The first two of these equivalences were proved already in [48] by Wiener, the last one appears for instance as theorem 2.3 in [16].

Theorem 7.1 (Wiener's Tauberian theorem).

1. $f \in L^1(\mathbb{R}^{2d})$ is regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ is empty.
2. $f \in L^2(\mathbb{R}^{2d})$ is 2-regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ has Lebesgue measure zero.
3. $f \in L^\infty(\mathbb{R}^{2d})$ is ∞ -regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ has dense complement.

Remark. The theorem is usually formulated using the regular Fourier transform, but as we have discussed $\mathcal{F}_\sigma f(z) \neq 0$ for all $z \in \mathbb{R}^{2d}$ is equivalent to the same statement for the regular Fourier transform.

For $1 < p < 2$, Lev and Oleviskii [34] have shown the existence of two functions in $L^1(\mathbb{R})$ with the same set of zeros for the Fourier transform, but where one function is p -regular and the other is not. Wiener's Tauberian theorem can therefore not be extended in an obvious way to all values of $1 \leq p \leq \infty$.

The next theorem consists of propositions 1,2 and 3 in [29], and we mainly follow the proof given in that paper. It contains many equivalent statements of p -regularity for an operator, but none concerning the Fourier-Wigner transform as we would have expected in a generalization of theorem 7.1. The reason for this is that such a result is not known to exist for $p \neq 1, 2, \infty$. We therefore state this general result for all $1 \leq p \leq \infty$ first, and then extend it to include the Fourier-Wigner transform for $p = 1, 2, \infty$ in theorem 7.3.

Theorem 7.2. *Let $S \in \mathcal{T}^1$, $1 \leq p \leq \infty$ and let q be the conjugate exponent of p determined by $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:*

1. S is p -regular.
2. If $f \in L^q(\mathbb{R}^{2d})$ and $f * S = 0$, then $f = 0$.
3. $\mathcal{T}^p * S$ is dense in $L^p(\mathbb{R}^{2d})$.
4. If $T \in \mathcal{T}^q$ and $T * S = 0$, then $T = 0$.
5. $L^p(\mathbb{R}^{2d}) * S$ is dense in \mathcal{T}^p .
6. $S * S$ is p -regular.
7. For any regular $T_0 \in \mathcal{T}^1$, $T_0 * S$ is p -regular.

The density in points (3) and (5) is in the p -norm for $p < \infty$, and weak density for $p = \infty$.*

For the case $p = \infty$ we may add two further equivalent statements to the list:

(i) $K(L^2(\mathbb{R}^d)) * S$ is dense in $C_0(\mathbb{R}^{2d})$ in the $\|\cdot\|_{L^\infty}$ norm.

(ii) $C_0(\mathbb{R}^{2d}) * S$ is dense in $K(L^2(\mathbb{R}^d))$ in the operator norm $\|\cdot\|_{B(L^2)}$.

Finally, there exists a p -regular operator S for any $1 \leq p \leq \infty$.

Proof. We begin by considering the last statement, namely the existence of a p -regular operator. From the remark following definition 7.1 it is sufficient to find a regular operator $S \in \mathcal{T}^1$. Now consider the Gaussian φ from example 6.1; we will prove that $S = \varphi \otimes \varphi$ satisfies part (6) of the theorem, so when the proof of the theorem is complete we know that it is in fact regular. The reason for starting with this is that we will need an operator satisfying (6) during the proof. Proposition 6.4 gives that $\mathcal{F}_\sigma(S * S) = \mathcal{F}_W(S)\mathcal{F}_W(S)$. By example 6.1, $\mathcal{F}_W(S)$ has no zeros, thus the same is true for $\mathcal{F}_\sigma(S * S)$. The classical Tauberian theorem of Wiener then states that $S * S$ is regular.

(2) \iff (3): First assume that $p < \infty$. In the notation of section 5.1, statement (3) says that $\mathcal{B}_{\check{S}^*} : \mathcal{T}^p \rightarrow L^p(\mathbb{R}^{2d})$ has dense range. From theorem 5.3 we know that the Banach space adjoint of $\mathcal{B}_{\check{S}^*}$ is $\mathcal{A}_{\check{S}^*}$, and part (1) of proposition 2.25 states that the range of $\mathcal{B}_{\check{S}^*}$ is dense if and only if $\mathcal{A}_{\check{S}^*}$ is injective. Furthermore, the injectivity of $\mathcal{A}_{\check{S}^*}$ is equivalent to the injectivity of \mathcal{A}_S , since $f * \check{S}^* = 0$ implies that $((f * \check{S}^*)^\sim)^* = \check{f}^* * S = 0$ by lemma 4.7. This proves the equivalence.

For $p = \infty$ we need to use part of (2) of proposition 2.25, and that $(\mathcal{A}_S)^* = \mathcal{B}_S$ by theorem 5.3. Otherwise, the proof is the same.

(4) \iff (5): Follows from the same line of reasoning as above, with the roles of \mathcal{A} and \mathcal{B} switched. This is permissible by theorem 5.3, since \mathcal{B} and \mathcal{A} appear in a symmetric way in the theorem.

(2) \implies (4): Assume that we have $T \in \mathcal{T}^q$ such that $T * S = 0$. Taking the convolution with an arbitrary $A \in \mathcal{T}^1$ from the left on both sides of this equality, we find by associativity that $(A * T) * S = 0$. But $A * T \in L^q(\mathbb{R}^{2d})$, so since we are assuming (2) we get that $A * T = 0$ for any $A \in \mathcal{T}^1$. We will now use this to show that $T = 0$.

As we have remarked earlier, the expression $A * T(z) = \text{tr}(\check{A}\alpha_{-z}T)$ is valid, even for $q = \infty$. If we let $z = 0$, we have that $A * T(0) = \text{tr}(\check{A}T) = 0$ for any $A \in \mathcal{T}^1$. If we consider T as an element of the dual space $(\mathcal{T}^1)^* = B(L^2(\mathbb{R}^d))$, which is possible since $\mathcal{T}^q \subset B(L^2(\mathbb{R}^d))$, this says that $\langle T, \check{A}^* \rangle = 0$ for any $A \in \mathcal{T}^1$, where $\langle \cdot, \cdot \rangle$ is a duality bracket. This is clearly equivalent to $\langle T, A \rangle = 0$ for any $A \in \mathcal{T}^1$ since $((\check{A}^*)^\sim)^* = A$, and this proves that $T = 0$.

(1) \iff (4): For $p < \infty$, we will use that \mathcal{T}^q is the dual space of \mathcal{T}^p . A subspace of a Banach space such as \mathcal{T}^p is dense if and only if it separates points of the dual space. In particular, the subspace spanned by $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ is dense in \mathcal{T}^p if and only if $\langle T, \alpha_z S \rangle = 0$ for every $z \in \mathbb{R}^{2d}$ implies that $T = 0$, where the bracket denotes duality and $T \in \mathcal{T}^q$. We have therefore shown that (1) is

equivalent to the statement that $\langle T, \alpha_z S \rangle = 0$ for every $z \in \mathbb{R}^{2d}$ implies that $T = 0$. Since $T = 0$ exactly when $\check{T}^* = 0$, we may equivalently phrase this condition as $\langle \check{T}^*, \alpha_z S \rangle = 0$ for every $z \in \mathbb{R}^{2d}$ implies that $T = 0$.

It remains to show that this last statement is equivalent to (4), which may be achieved by showing that $T * S = 0$ is equivalent to $\langle \check{T}^*, \alpha_z S \rangle = 0$ for any $z \in \mathbb{R}^{2d}$. By definition $T * S(z) = \text{tr}(\check{T} \alpha_{-z} S)$, which we may write in terms of the duality bracket as $T * S(z) = \langle \check{T}^*, \alpha_{-z} S \rangle$. Clearly the left side is zero for all $z \in \mathbb{R}^{2d}$ if and only if the right side is, which is what we wanted to prove.

For $p = \infty$ the same argument works, now using the fact that \mathcal{T}^1 is the dual space of $K(L^2(\mathbb{R}^d))$.

(6) \implies (3): This is a simple consequence of part (3) of lemma 4.7, which states that $T_z(S * S) = \alpha_z(S) * S$. Therefore (6) gives that the set $\{\alpha_z(S) * S : z \in \mathbb{R}^{2d}\}$ is dense in $L^p(\mathbb{R}^{2d})$. However, $\alpha_z(S) \in \mathcal{T}^p$ for any $z \in \mathbb{R}^{2d}$, so the density of $\{\alpha_z(S) * S : z \in \mathbb{R}^{2d}\}$ in particular implies the density of $\mathcal{T}^p * S$.

(7) \implies (3): Follows from an argument similar to the preceding one. Again $T_z(T_0 * S) = \alpha_z(T_0) * S$, and since $T_0 * S$ is assumed to be p -regular, the set $\{\alpha_z(T_0) * S : z \in \mathbb{R}^{2d}\}$ is dense in $L^p(\mathbb{R}^{2d})$. But $\{\alpha_z(T_0) : z \in \mathbb{R}^{2d}\} \subset \mathcal{T}^1 \subset \mathcal{T}^p$, which proves the statement.

(4) \implies (2): We start the proof similarly to the reverse implication. Assume that $f * S = 0$ for $f \in L^q(\mathbb{R}^{2d})$; we want to show that $f = 0$. Taking the convolution with an arbitrary $S' \in \mathcal{T}^1$ from the left and using associativity, $(S' * f) * S = 0$, which by (4) implies that $S' * f = 0$ for any $S' \in \mathcal{T}^1$.

Unfortunately, there is no natural relation between $f * S'$ and duality as there was when proving the reverse inclusion. Instead, we let T be the operator $\varphi \otimes \varphi$ discussed at the very start of the proof, where φ is the Gaussian from example 6.1. We know that T satisfies (6) for $p = 1$, and we have shown that T then satisfies (3), i.e. $T * \mathcal{T}^1$ is a dense subset of $L^1(\mathbb{R}^{2d})$. Since $f * S' = S' * f = 0$ for any $S' \in \mathcal{T}^1$, we must also have $(T * S') * f = 0$ for any $S' \in \mathcal{T}^1$. In other words, $f * g = 0$ for g in the dense subset $T * \mathcal{T}^1 \subset L^1(\mathbb{R}^{2d})$. Since the convolutions are continuous in both arguments, this shows that $f * L^1(\mathbb{R}^{2d}) = 0$ and therefore that $f = 0$.

(3) \implies (6): Assume first that $p < \infty$. Pick an $f \in L^p(\mathbb{R}^{2d})$ and an $\epsilon > 0$. We need to approximate f in the L^p -norm by a finite linear combination of elements of the form $T_z(S * S) = \alpha_z(S) * S$. By investigating the already proved implications, we see that we have proved (3) \implies (2) \implies (4) \implies (1), which lets us assume that the elements $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ span a dense subset of \mathcal{T}^p .

Since (3) holds, we pick an operator $T \in \mathcal{T}^p$ such that $\|T * S - f\|_{L^p} < \frac{\epsilon}{2}$. As we have shown that (1) holds, we then pick $c_i \in \mathbb{C}$ and $z_i \in \mathbb{R}^{2d}$ for $i = 1, 2, \dots, n$

such that $\|T - \sum_{i=1}^n c_i \alpha_{z_i} S\|_p < \frac{\epsilon}{2\|S\|_1}$. An estimate now shows that

$$\begin{aligned} \left\| \sum_{i=1}^n c_i \alpha_{z_i} (S) * S - f \right\|_{L^p} &\leq \left\| \left(\sum_{i=1}^n c_i \alpha_{z_i} S - T \right) * S \right\|_{L^p} + \|T * S - f\|_{L^p} \\ &< \frac{\epsilon}{2\|S\|_{\mathcal{T}^1}} \|S\|_{\mathcal{T}^1} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where we have used proposition 4.5 to estimate the norm of a convolution.

If $p = \infty$, the same basic idea applies. First approximate f by $T * S$ in the weak* topology, then approximate T by a finite linear combination of translates of S . We leave to the reader the trivial reformulation of the proof in terms of open sets.

(3) \implies (7): Let $T_0 \in \mathcal{T}^1$ be regular; as noted before, T_0 is also p -regular. The key parts of the previous argument was to first use (3) to approximate f by $T * S$ for some $T \in \mathcal{T}^1$, and then use the p -regularity of S to approximate T by a finite linear combination of translates of S . Exactly the same argument works in this case, except that we need to approximate T with a finite linear combination of translates of T_0 instead of S . We leave the details to the reader.

(ii) \implies (i): Following [29] we start by showing that $K(L^2(\mathbb{R}^d)) * S_0$ is dense in $C_0(\mathbb{R}^{2d})$ for a regular S_0 . Let $f \in C_0(\mathbb{R}^{2d})$ and $\epsilon > 0$. Since $L^1(\mathbb{R}^{2d})$ has an approximate identity [21, Prop. 2.44], there is a $g \in L^1(\mathbb{R}^{2d})$ such that $\|g * f - f\|_\infty < \frac{\epsilon}{2}$. Since we are assuming that S_0 is regular, we know by (3) that there is a $T \in \mathcal{T}^1$ with $\|S_0 * T - g\|_1 < \frac{\epsilon}{2\|f\|_\infty}$. From proposition 4.6 the operator $T * f$ is compact, and an estimate now shows that $S_0 * (T * f)$ approximates f :

$$\begin{aligned} \|S_0 * T * f - f\|_{L^\infty} &\leq \|S_0 * T * f - g * f\|_{L^\infty} + \|g * f - f\|_{L^\infty} \\ &< \|S_0 * T - g\|_{L^1} \|f\|_{L^\infty} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Armed with this knowledge we now prove that (ii) \implies (i) for any $S \in \mathcal{T}^1$, so assume that $C_0(\mathbb{R}^{2d}) * S$ is dense in $K(L^2(\mathbb{R}^d))$. We need to prove that $K(L^2(\mathbb{R}^d)) * S$ is dense in $C_0(\mathbb{R}^{2d})$. If S_0 is some regular operator, then the set $S_0 * C_0(\mathbb{R}^{2d}) * S = \{S_0 * f * S : f \in C_0(\mathbb{R}^{2d})\}$ is a subset of $K(L^2(\mathbb{R}^d)) * S$, and it will be enough to show that this smaller set is dense. Since we assume (ii) we know that $C_0(\mathbb{R}^{2d}) * S$ is dense in $K(L^2(\mathbb{R}^d))$. We also know that $S_0 * K(L^2(\mathbb{R}^d))$ is dense in $C_0(\mathbb{R}^{2d})$ from the first part of the argument, and if we combine these two density results with the continuity of the convolutions, we get that $S_0 * C_0(\mathbb{R}^{2d}) * S$ must be a dense subset of $C_0(\mathbb{R}^{2d})$.

(i) \implies (ii): We will just show that (ii) holds for a regular operator S_0 . The proof is then completed in the same way as (ii) \implies (i). Let $T \in K(L^2(\mathbb{R}^d))$ and $\epsilon > 0$; we will use three density results to find $f \in C_0(\mathbb{R}^{2d})$ with $\|T - f * S_0\|_{B(L^2)} < \epsilon$.

Firstly, we use that T is compact to find a finite rank operator A with $\|T - A\|_{B(L^2)} < \frac{\epsilon}{3}$. Secondly, since A is finite rank it is in particular trace class, so by (3) we may find $g \in L^1(\mathbb{R}^{2d})$ such that $\|A - g * S_0\|_{B(L^2)} < \frac{\epsilon}{3}$. Here we have used that $\|\cdot\|_{B(L^2)} \leq \|\cdot\|_{\mathcal{T}^1}$. Finally the continuous functions with compact support are dense in L^1 , so we can pick $f \in C_0(\mathbb{R}^{2d})$ such that $\|g - f\|_{L^1} \leq \frac{\epsilon}{3\|S_0\|_{B(L^2)}}$. We claim that $\|T - S_0 * f\|_{B(L^2)} < \epsilon$, which would conclude the proof. By the triangle inequality

$$\begin{aligned} \|T - S_0 * f\|_{B(L^2)} &\leq \|T - A\|_{B(L^2)} + \|A - S_0 * g\|_{B(L^2)} + \|S_0 * g - S_0 * f\|_{B(L^2)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|S_0\|_{B(L^2)} \|f - g\|_{L^1} < \epsilon. \end{aligned}$$

(4) \iff (ii) for $p = \infty$: This part follows from the same kind of argument as (2) \iff (3) by using proposition 2.25 with the Banach spaces $K(L^2(\mathbb{R}^d))$ and $C_0(\mathbb{R}^{2d})$. Similar to that argument we get that $C_0(\mathbb{R}^{2d}) * S$ is dense in $K(L^2(\mathbb{R}^d))$ if and only if the map $T \mapsto T * S$ is injective from \mathcal{T}^1 to $C_0(\mathbb{R}^{2d})^*$. The first statement is clearly (ii), and the last statement is almost (4) when $p = \infty$, except that the codomain is $C_0(\mathbb{R}^{2d})^*$ rather than $L^1(\mathbb{R}^{2d})$. However, it should be clear that this is of no importance when determining whether the mapping is injective since $L^1(\mathbb{R}^{2d})$ may be identified with a subset of $C_0(\mathbb{R}^{2d})^*$. We have therefore shown that (4) \iff (ii) for $p = \infty$, which concludes the proof. \square

The next theorem is our generalization of Wiener's Tauberian theorem. The result was proved by Werner for $p = 1$ already in [46] using the theory of convolutions and Fourier-Wigner transforms of operators, whereas the $p = \infty$ result was proved more recently in [29]. We will follow a proof sketched in [29], emphasizing the important role of convolutions in translating results from functions to operators. A more direct proof of the last two parts may be found in the same paper.

Theorem 7.3. *Let $S \in \mathcal{T}^1$.*

1. S is regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ is empty.
2. S is 2-regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ has Lebesgue measure zero.
3. S is ∞ -regular \iff the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ has dense complement.

Proof. Each part of the proof will use the corresponding part of theorem 7.1, and use the equivalences of (1) and (6) in theorem 7.2 along with proposition 6.4. We therefore only prove the first part – the others follow from the same line of reasoning.

By theorem 7.2, S is regular if and only if $S * S$ is regular. The classical Tauberian theorem of Wiener states that $S * S$ is regular if and only if $\mathcal{F}_\sigma(S * S)(z) \neq 0$

for any $z \in \mathbb{R}^{2d}$. However, proposition 6.4 gives that $\mathcal{F}_\sigma(S * S) = (\mathcal{F}_W S)^2$. Thus $\mathcal{F}_\sigma(S * S)(z) \neq 0$ for any $z \in \mathbb{R}^{2d}$ if and only if the same holds for $\mathcal{F}_W S$. If we now follow this chain of equivalences from beginning to end, we observe that we have proved the theorem. \square

When S is a pseudodifferential operator in \mathcal{M} , theorem 7.3 takes a particularly simple form.

Corollary 7.3.1. *Let $S \in \mathcal{M}$ be the operator on $L^2(\mathbb{R}^d)$ with twisted Weyl symbol $f \in M^1(\mathbb{R}^{2d})$.*

1. S is regular \iff the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ is empty.
2. S is 2-regular \iff the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ has Lebesgue measure zero.
3. S is ∞ -regular \iff the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ has dense complement.

Proof. We have shown that the Fourier-Wigner transform of a trace class operator is the twisted Weyl symbol of the operator, so $\mathcal{F}_W(S) = f$. \square

By theorem 7.2, the subset $L^1(\mathbb{R}^{2d}) * S$ is dense in \mathcal{T}^1 for any regular operator S . One might then naturally ask whether there exists some particularly nice operator S such that $L^1(\mathbb{R}^{2d}) * S = \mathcal{T}^1$. Unfortunately, this is not possible. As a counterexample consider S . If there were a function $f \in L^1(\mathbb{R}^{2d})$ such that $S = f * S$, we could apply \mathcal{F}_W to this equation to get that $\mathcal{F}_W(S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$. Since S is regular, $\mathcal{F}_W(S)$ has no zeros, so we can divide by it to get $\mathcal{F}_\sigma f = 1$, which is impossible by the Riemann-Lebesgue lemma.

7.1 Tauberian theorems for localization operators

We now turn to reproving and generalizing the density results for localization operators and Berezin transforms in [3]. Since both the localization operators and Berezin transform may be expressed as a convolution, we can apply theorems 7.2 and 7.3 to these concepts. More precisely, we will pick $S = \varphi_2 \otimes \varphi_1$ for two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, and formulate theorems 7.2 and 7.3 using the terminology of the Berezin transform and localization operators. We start by reformulating theorem 7.2.

Theorem 7.4. *Fix two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ for \mathcal{A} and \mathcal{B} , let $1 \leq p \leq \infty$ and let q be the conjugate exponent of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:*

1. The operator $\varphi_2 \otimes \varphi_1$ is p -regular.
2. \mathcal{A} is injective on $L^q(\mathbb{R}^{2d})$.

3. The set $\{\mathcal{B}T : T \in \mathcal{T}^p\}$ is dense in $L^p(\mathbb{R}^{2d})$.
4. \mathcal{B} is injective on \mathcal{T}^q .
5. The set $\{\mathcal{A}f : f \in L^p(\mathbb{R}^{2d})\}$ is dense in \mathcal{T}^p .
6. $\mathcal{B}(\varphi_2 \otimes \varphi_1)$ is p -regular.
7. For any regular $T_0 \in \mathcal{T}^1$, $\mathcal{B}T$ is p -regular.

The density in points (1), (3) and (5) is in the p norm for $p < \infty$, and weak* density for $p = \infty$.

Remark. When reformulation theorem 7.2, we have substituted $T * S$ with $\mathcal{B}T = T * \check{S}^*$, where $S = \varphi_2 \otimes \varphi_1$. The easiest way of seeing that this is allowed is to observe that part (2) with S is equivalent to part (2) with \check{S}^* , as we did in fact prove when proving (2) \iff (3). Since theorem 7.2 consists of equivalences, we may therefore switch between S and \check{S}^* as we please.

Even more interesting are the results that we obtain by combining part (5) of theorem 7.4 with theorem 7.3.

Theorem 7.5. Fix two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$.

1. The set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^1(\mathbb{R}^{2d})\}$ is dense in $\mathcal{T}^1 \iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty.
2. The set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^2(\mathbb{R}^{2d})\}$ is dense in $\mathcal{T}^2 \iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has zero Lebesgue measure.
3. The set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^\infty(\mathbb{R}^{2d})\}$ is weak* dense in $B(L^2(\mathbb{R}^d)) \iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has dense complement.

Proof. The density statements in this theorem are part (5) of theorem 7.4 for $p = 1, 2, \infty$, and therefore equivalent to S being p -regular. Furthermore, theorem 7.3 relates the condition that S is p -regular for these three values of p to the set of zeros of $\mathcal{F}_W(S) = A(\varphi_2, \varphi_1)$. \square

Remark. Of course, these statements are also equivalent to the other statements in theorem 7.4 for $p = 1$, $p = 2$ and $p = \infty$, respectively.

The equivalence for $p = 2$ was proved by Bayer and Gröchenig in [3] using different methods. They do, however, also sketch an approach similar to ours, where they reduce the statement to Wiener's Tauberian theorem for $p = 2$. This approach is based on Pool's theorem: an operator lies in \mathcal{T}^2 if and only if its Weyl symbol lies in $L^2(\mathbb{R}^{2d})$. There is no corresponding result for \mathcal{T}^p when $p \neq 2$, and the

approach in [3] could therefore not give equivalences for $p = 1, \infty$. Our approach, on the other hand, does not use Pool's theorem and is based on the equivalent statements in Wiener's Tauberian theorem, which is true for $p = 1, 2, \infty$. It was the theory of convolutions and Fourier-Wigner transforms introduced in [46] that allowed us to obtain equivalence statements on the level of operators from Wiener's theorem, and as a consequence obtain theorem 7.5.

For $p \neq 1, 2, \infty$ our approach does not yield equivalences. We may however reprove the results in [3], since p -regularity implies p' -regularity for $p \leq p'$.

Corollary 7.5.1. *Fix two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$.*

1. *Assume $1 \leq p < 2$. If the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty, then the set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^p(\mathbb{R}^{2d})\}$ is norm dense in \mathcal{T}^p .*
2. *Assume $2 \leq p < \infty$. If the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has Lebesgue measure zero, then the set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^p(\mathbb{R}^{2d})\}$ is norm dense in \mathcal{T}^p .*

Proof. If the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty, then the operator $S = \varphi_2 \otimes \varphi_1$ is regular by theorem 7.5 and the following remark. As noted, this implies that S is p -regular for $1 \leq p < \infty$. By theorem 7.4, this implies that $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in L^p(\mathbb{R}^{2d})\}$ is dense in \mathcal{T}^p . Exactly the same argument works for $2 \leq p < \infty$, using part (2) of theorem 7.5. \square

7.2 A Banach module perspective

Recall that the convolutions make \mathcal{T}^p into Banach modules over $L^1(\mathbb{R}^{2d})$. We will show how theorem 7.2 may be used to shed light on these Banach module structures.

Proposition 7.6. 1. *For $1 \leq p < \infty$, the $L^1(\mathbb{R}^{2d})$ -module \mathcal{T}^p is essential.*

2. *For $1 \leq p \leq \infty$, the $L^1(\mathbb{R}^{2d})$ -module \mathcal{T}^p is order-free.*

Proof. 1. Let $T_0 \in \mathcal{T}^1$ be a regular operator, which exists by theorem 7.2. By part (5) of that theorem, $L^1(\mathbb{R}^{2d}) * T_0$ is dense in \mathcal{T}^1 . As we discussed in a remark after definition 7.1, this implies that $L^1(\mathbb{R}^{2d}) * T_0$ is dense in \mathcal{T}^p for any $p < \infty$. In particular this means that the essential submodule of \mathcal{T}^p is all of \mathcal{T}^p , hence \mathcal{T}^p is an essential $L^1(\mathbb{R}^{2d})$ -module.

2. Assume that $S \in \mathcal{T}^p$ is such that $f * S = 0$ for every $f \in L^1(\mathbb{R}^{2d})$. Let $T_0 \in \mathcal{T}^1$ to be a regular operator. Then $L^1(\mathbb{R}^{2d}) * T_0$ is dense in \mathcal{T}^1 by theorem 7.2. By commutativity and associativity of convolutions, $f * T_0 * S = 0$ for any $f \in L^1(\mathbb{R}^{2d})$, and since $L^1(\mathbb{R}^{2d}) * T_0$ is dense in \mathcal{T}^1 this implies by

continuity of the convolution that $T * S = 0$ for any $T \in \mathcal{T}^1$. In particular $T * S(0) = \text{tr}(T\check{S}) = 0$, which shows that $S = 0$ as an element of the dual space $B(L^2(\mathbb{R}^d))$ of \mathcal{T}^1 , and hence $S = 0$.

□

For $p < \infty$ the above result leads to an interesting and immediate application of the Cohen-Hewitt factorization theorem.

Proposition 7.7. *For $1 \leq p < \infty$, any $S \in \mathcal{T}^p$ can be written as $f * T$ for some $f \in L^1(\mathbb{R}^{2d})$ and $T \in \mathcal{T}^p$.*

Since we have not shown that $B(L^2(\mathbb{R}^d)) = \mathcal{T}^\infty$ is an essential module, we can not deduce a similar result for general bounded operators. However, sections 3 and 4 make it clear that $B(L^2(\mathbb{R}^d))$ is an $L^1(\mathbb{R}^{2d})$ -module with shift α , and proposition 7.6 asserts that this Banach module is order-free. Hence proposition 2.22 gives a characterization of the elements of $B(L^2(\mathbb{R}^d))$ where α is strongly continuous.

Proposition 7.8. *The elements of $B(L^2(\mathbb{R}^d))$ where α is strongly continuous are exactly the elements of the form $f * A$ for $f \in L^1(\mathbb{R}^{2d})$ and $A \in B(L^2(\mathbb{R}^d))$.*

The application of Banach module theory to Werner's convolutions, in particular propositions 7.7 and 7.8, appears to be new.

8 Convolutions and Tauberian theorems for modulation spaces

So far we have used the L^p -spaces as our function spaces when defining the convolutions. Now we will seek to understand how the modulation spaces fit into the convolution theory. As mentioned in section 2.5, the modulation space $M^1(\mathbb{R}^{2d})$ is a Banach algebra under convolutions. Furthermore, since $M^1(\mathbb{R}^{2d}) \subset L^1(\mathbb{R}^{2d})$ with $\|f\|_{L^1} \leq \|f\|_{M^1}$ for $f \in M^1(\mathbb{R}^{2d})$, lemma 4.1 shows that we may define $f * T$ for $f \in M^1(\mathbb{R}^{2d})$ and $T \in \mathcal{T}^1$. Unfortunately, the convolution of two trace class operators will not necessarily be an element of $M^1(\mathbb{R}^{2d})$, so we do not get the symmetry that we saw in lemmas 4.1 and 4.2 for $L^1(\mathbb{R}^{2d})$ and \mathcal{T}^1 . This leads to a natural question: which operators will produce functions in $M^1(\mathbb{R}^{2d})$ under convolution with a trace class operator? It turns out that operators in \mathcal{M} have this property, which we now aim to prove. In order to do so we need the following estimate, proved in the proof of corollary 4.3 in [13].

Lemma 8.1. *If $\xi, \eta \in M^1(\mathbb{R}^d)$ and $\psi, \phi \in L^2(\mathbb{R}^d)$, then $V_\xi \psi(z) \overline{V_\eta \phi(z)} \in M^1(\mathbb{R}^{2d})$ with norm estimate*

$$\|V_\xi \psi(z) \overline{V_\eta \phi(z)}\|_{M^1} \leq C_0 \|\xi\|_{M^1} \|\eta\|_{M^1} \|\psi\|_{L^2} \|\phi\|_{L^2}$$

for some constant C_0 .

The main idea for proving theorems 8.2 and 8.3 is to use the proofs in [3] for the localization operators and Berezin transform, and then write $S \in \mathcal{M}$ using the Wilson basis as in theorem 2.19 to get the result for any $S \in \mathcal{M}$.

Theorem 8.2. *Let $S \in \mathcal{M}$ be the trace class operator given on $L^2(\mathbb{R}^d)$ by $S\psi(s) = \int_{\mathbb{R}^d} k(s,t)\psi(t) dt$ with $k \in M^1(\mathbb{R}^{2d})$. For $T \in \mathcal{T}^1$, the function*

$$T * S(z) = \text{tr}(T\pi(z)\check{S}\pi(z)^*)$$

lies in $M^1(\mathbb{R}^{2d})$ and $\|T * S\|_{M^1} \leq C\|k\|_{M^1}\|T\|_{\mathcal{T}^1}$ for some constant C independent of S and T .

Proof. We will prove that $\text{tr}(T\pi(z)S\pi(z)^*)$ lies in $M^1(\mathbb{R}^d)$ with the same norm estimate. The corresponding result with \check{S} then follows since one may easily check that the integral kernel of \check{S} is \check{k} and $\|\check{k}\|_{M^1} \leq K\|k\|_{M^1}$ for some constant K by lemma 2.15.

As in theorem 2.19, let $\{w_i\}_{i \in \mathbb{N}}$ be a Wilson basis for $L^2(\mathbb{R}^d)$, and define the Wilson basis for $L^2(\mathbb{R}^{2d})$ by $W_{ij}(x,y) = w_i(x)\overline{w_j(y)}$. By theorem 2.19 and theorem 2.3 we may expand S and T as

$$S = \sum_{i,j \in \mathbb{N}} \langle k, W_{ij} \rangle w_i \otimes w_j, \quad T = \sum_{n \in \mathbb{N}} t_n \psi_n \otimes \phi_n,$$

where $\{t_n\}_{n \in \mathbb{N}}$ are the singular values of T and $\{\psi_n\}_{n \in \mathbb{N}}$ and $\{\phi_n\}_{n \in \mathbb{N}}$ are orthonormal sets in $L^2(\mathbb{R}^d)$.

We will now calculate the convolution of the operators, and to calculate the trace we will use an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ which is obtained by extending the orthonormal set $\{\psi_n\}_{n \in \mathbb{N}}$. A computation shows that

$$\begin{aligned} T\pi(z)S\pi(z)^*e_k &= T\pi(z) \left[\sum_{i,j \in \mathbb{N}} \langle k, W_{ij} \rangle \langle \pi(z)^*e_k, w_j \rangle w_i \right] \\ &= \sum_{n \in \mathbb{N}} \sum_{i,j \in \mathbb{N}} t_n \langle k, W_{ij} \rangle \langle \pi(z)^*e_k, w_j \rangle \langle \pi(z)w_i, \phi_n \rangle \psi_n \\ &= \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{N}} t_n \langle k, W_{ij} \rangle V_{w_j} e_k(z) \overline{V_{w_i} \phi_n(z)} \psi_n. \end{aligned}$$

Now, since $\{e_k\}_{k \in \mathbb{N}}$ extends $\{\psi_n\}_{n \in \mathbb{N}}$, the trace is given by

$$\begin{aligned} \text{tr}(T\pi(z)S\pi(z)^*) &= \sum_{k \in \mathbb{N}} \langle T\pi(z)S\pi(z)^*e_k, e_k \rangle \\ &= \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{N}} t_n \langle k, W_{ij} \rangle V_{w_j} \psi_n(z) \overline{V_{w_i} \phi_n(z)}, \end{aligned}$$

by the previous computation. Recall from section 2.5.3 that $\|w_i\|_{M^1} \leq C_1$ for some constant C_1 . In addition, the sets $\{\psi_n\}_{n \in \mathbb{N}}$ and $\{\phi_n\}_{n \in \mathbb{N}}$ are orthonormal, so lemma 8.1 implies that the product $V_{w_j} \psi_n(z) \overline{V_{w_i} \phi_n(z)}$ is a function in $M^1(\mathbb{R}^{2d})$ with norm

$$\|V_{w_j} \psi_n(z) \overline{V_{w_i} \phi_n(z)}\|_{M^1} \leq C_2$$

for some constant C_2 .

If we now take the $M^1(\mathbb{R}^{2d})$ norm of our expression for $\text{tr}(T\pi(z)S\pi(z)^*)$, we find that

$$\begin{aligned} \|\text{tr}(T\pi(z)S\pi(z)^*)\|_{M^1} &\leq \sum_{i,j \in \mathbb{N}} \sum_{n \in \mathbb{N}} t_n |\langle k, W_{ij} \rangle| \|V_{w_j} \psi_n(z) \overline{V_{w_i} \phi_n(z)}\|_{M^1} \\ &\leq C_2 \sum_{i,j \in \mathbb{N}} |\langle k, W_{ij} \rangle| \sum_{n \in \mathbb{N}} t_n \\ &\leq C_2 K' \|k\|_{M^1} \|T\|_{\mathcal{T}^1}, \end{aligned}$$

where K' is the positive constant associated with the fact that the ℓ^1 norm on the Wilson basis coefficients is an equivalent norm for $M^1(\mathbb{R}^{2d})$, as discussed in section 2.5.3. \square

Remark. 1. If $A \in B(L^2(\mathbb{R}^d))$ and $S \in \mathcal{M}$, then clearly $A * T \in M^\infty(\mathbb{R}^{2d})$. After all, proposition 4.5 shows that $A * S \in L^\infty(\mathbb{R}^{2d})$, and we know that $L^\infty(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d)$.

2. The theorem is not true for all $S \in \mathcal{T}^1$. If we let $S = T = \psi \otimes \psi$ for $\psi \in L^2(\mathbb{R})$, then the calculations in the proof show that $\text{tr}(T\alpha_z S) = |V_\psi \psi|^2$. Benedetto and Pfander have constructed an example of a $\psi \in L^2(\mathbb{R})$ such that $|V_\psi \psi|^2 \notin M^1(\mathbb{R}^2)$ [4, Remark 4.6].
3. A version of theorem 8.2 for T in other Schatten p -classes was proved in [3]. However, the proof uses that $\|\sigma\|_{M^\infty} \leq \|T\|_{B(L^2)}$, where σ is the Weyl symbol of T as an operator $T : M^1(\mathbb{R}^d) \rightarrow M^\infty(\mathbb{R}^d)$. This inequality is not true in general; in fact the opposite inequality is true by theorem 2.18. We therefore settle for the special cases \mathcal{T}^1 and $B(L^2(\mathbb{R}^d))$.
4. There are other reasons why \mathcal{M} is a natural class of operators. Any class of pseudodifferential operators suitable for the theory of convolutions should be closed under α , and hence by lemma 3.4 the associated class of symbols should be closed under translation. We are therefore led to the class \mathcal{M} , as $M^1(\mathbb{R}^d)$ is the smallest Banach space in $L^1(\mathbb{R}^d)$ isometrically invariant under M_ω and T_x [25].

Next, we would like to prove a similar result for convolutions of functions in different L^p -spaces with $S \in \mathcal{M}$. Because of the connection between localization operators and convolutions discussed in section 5, we will be able to prove this result using an argument from Cordero and Gröchenig's paper [13]. More precisely, we will translate the statement into a statement on Weyl transforms by using the convolution relations in proposition 2.14. The statement then follows from the results on the Weyl transform in theorem 2.18.

Theorem 8.3. *Let $1 \leq p \leq \infty$, $k \in M^1(\mathbb{R}^{2d})$ and let $S \in \mathcal{M}$ be the trace class operator given on $L^2(\mathbb{R}^d)$ by $S\psi(s) = \int_{\mathbb{R}^d} k(s, t)\psi(t) dt$. If $f \in M^{p, \infty}(\mathbb{R}^{2d})$, the function $f * S$ lies in \mathcal{T}^p and $\|f * S\|_{\mathcal{T}^p} \leq C\|k\|_{M^1}\|f\|_{M^{p, \infty}}$ for some constant C independent of S and f .*

Proof. First assume that $S = \varphi_2 \otimes \varphi_1$ for $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^{2d})$ with $\|\varphi_1\|_{M^1} = \|\varphi_2\|_{M^1} = 1$.

1. $1 \leq p \leq 2$. By corollary 6.4.1, $f * S = L_\sigma$ for $\sigma = f * W(\varphi_2, \varphi_1)$. We will estimate the $M^{1, p}(\mathbb{R}^{2d})$ -norm of $W(\varphi_2, \varphi_1)$ using lemma 2.17, so note that $\varphi_2 \in M^p(\mathbb{R}^d)$, since $M^1(\mathbb{R}^d) \subset M^p(\mathbb{R}^d)$. Lemma 2.17 then says that $W(\varphi_2, \varphi_1) \in M^{1, p}(\mathbb{R}^{2d})$, with $\|W(\varphi_2, \varphi_1)\|_{M^{1, p}} \leq C_0\|\varphi_2\|_{M^p}\|\varphi_1\|_{M^1}$.

Now the convolution relations in proposition 2.14 gives that $\sigma \in M^p(\mathbb{R}^{2d})$ with

$$\|\sigma\|_{M^p} \leq C_1\|f\|_{M^{p, \infty}}\|\varphi_2\|_{M^p}\|\varphi_1\|_{M^1}$$

for some constant C_1 .

Now we can apply the relationship between the function space of the Weyl symbol and the Schatten p -class of the Weyl transform given in theorem 2.18. This theorem shows that $f * S \in \mathcal{T}^p$, and $\|f * S\|_p \leq C_2\|f\|_{M^{p, \infty}}\|\varphi_2\|_{M^1}\|\varphi_1\|_{M^1} = C_2\|f\|_{M^{p, \infty}}$ for some constant C_2 , where we have used that $\|\varphi_2\|_{M^p} \leq C_p\|\varphi_2\|_{M^1}$ for some constant C_p .

2. $2 \leq p \leq \infty$. The exactly same argument may be used to reduce the statement to the Weyl symbol of $f * S$. The only difference is that the proof is concluded by part (2) of theorem 2.18 instead of part (1).

In order to extend the result to $S = A_k$ for $k \in M^1(\mathbb{R}^{2d})$, we use the Wilson basis $\{w_m\}_{m \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$ to write $S = \sum_{m, n \in \mathbb{N}} \langle k, W_{mn} \rangle w_m \otimes w_n$, as is allowed by theorem 2.19. Then

$$\begin{aligned} f * S &= f * \sum_{m, n \in \mathbb{N}} \langle k, W_{mn} \rangle w_m \otimes w_n \\ &= \sum_{m, n \in \mathbb{N}} \langle k, W_{mn} \rangle f * (w_m \otimes w_n). \end{aligned}$$

Now take the \mathcal{T}^p norm of this expression to find that

$$\begin{aligned} \|f * S\|_{\mathcal{T}^p} &\leq \sum_{m,n \in \mathbb{N}} |\langle k, W_{mn} \rangle| \cdot \|f * (w_m \otimes w_n)\|_{\mathcal{T}^p} \\ &\leq C_2 \|f\|_{M^{p,\infty}} \sum_{m,n \in \mathbb{N}} |\langle k, W_{mn} \rangle| \\ &\leq C \|f\|_{M^{p,\infty}} \|k\|_{M^1}. \end{aligned}$$

□

If $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then $\varphi_2 \otimes \varphi_1$ is the integral operator A_k with kernel $k(z) = \varphi_2 \otimes \overline{\varphi_1}(z)$, since we calculate when $\psi \in L^2(\mathbb{R}^d)$ that

$$\begin{aligned} A_k(\psi) &= \iint_{\mathbb{R}^{2d}} \varphi_2(x) \overline{\varphi_1(y)} \psi(y) dy \\ &= \varphi_2(x) \iint_{\mathbb{R}^{2d}} \overline{\varphi_1(y)} \psi(y) dy \\ &= \langle \psi, \varphi_1 \rangle \varphi_2 \\ &= \varphi_2 \otimes \varphi_1(\psi). \end{aligned}$$

In particular, $\varphi_2 \otimes \varphi_1 \in \mathcal{M}$, and applying theorems 8.2 and 8.3 to the operator $\varphi_2 \otimes \varphi_1$ allows us to deduce the following boundedness results for localization operators and Berezin transforms from [3].

Proposition 8.4. *Let $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$ and $1 \leq p \leq \infty$.*

1. *If $T \in \mathcal{T}^1$, then $\mathcal{B}T \in M^1(\mathbb{R}^{2d})$ with $\|\mathcal{B}T\|_{M^1} \leq C_1 \|T\|_{\mathcal{T}^1} \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}$ for some constant C_1 .*
2. *If $a \in M^{p,\infty}(\mathbb{R}^{2d})$, then $\mathcal{A}_a^{\varphi_1, \varphi_2} \in \mathcal{T}^p$ with $\|\mathcal{A}_a^{\varphi_1, \varphi_2}\|_{\mathcal{T}^p} \leq C_2 \|\varphi_1\|_{L^2} \|\varphi_2\|_{L^2}$ for some constant C_2 .*

Finally, we include the density results for localization operators with windows in $M^1(\mathbb{R}^{2d})$ from [3]. The proof consists merely of noting that $L^p(\mathbb{R}^{2d}) \subset M^{p,\infty}(\mathbb{R}^{2d})$, so the subsets that we claim are dense are larger than those in corollary 7.5.1. In this sense the following result is weaker than that corollary, and the main point of interest is the previously proved fact that symbols in the large space $M^{p,\infty}(\mathbb{R}^{2d})$ actually give operators in \mathcal{T}^p .

Corollary 8.4.1. *Fix two windows $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$.*

1. *Assume $1 \leq p < 2$. If the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty, then the set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in M^{p,\infty}(\mathbb{R}^{2d})\}$ is norm dense in \mathcal{T}^p .*
2. *Assume $2 \leq p < \infty$. If the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has Lebesgue measure zero, then the set $\{\mathcal{A}_f^{\varphi_1, \varphi_2} : f \in M^{p,\infty}(\mathbb{R}^{2d})\}$ is norm dense in \mathcal{T}^p .*

A Quantum mechanics and informational completeness

In this appendix we will consider the problem that led Kiukas et. al. to proving theorem 7.2 in [29]. This is the problem of informational completeness in quantum mechanics, and may informally be stated as follows: an observable assigns to each state of a system a probability distribution; for which observables can any state be reconstructed from this probability distribution? We shall see that for a special class of observables this question is answered by theorem 7.2.

First, however, we need to make notions such as observable and state more precise.

A.1 Positive operator valued measures

In quantum mechanics, a system is described by a Hilbert space, and we will consider the system described by $L^2(\mathbb{R}^d)$. A common way of introducing observables on a system in quantum mechanics is to define an observable on \mathbb{R}^{2d} to be a bounded, self-adjointed operator on $L^2(\mathbb{R}^d)$. By the spectral theorem [24, Thm. 7.12] there is a one to one correspondence between such operators and the so-called projection valued measures. This suggests a useful generalization of the concept of an observable [10]; we say that a generalized observable is given by a *positive operator valued measure*, which we now define for the special case of the Hilbert space $L^2(\mathbb{R}^d)$.

Definition A.1. Let X be a topological space and let $\mathcal{B}(X)$ denote the σ -algebra of Borel subsets of X . A positive operator valued measure (POVM) on X is a mapping $F : \mathcal{B}(X) \rightarrow B(L^2(\mathbb{R}^d))$ such that

1. $F(M)$ is a positive operator for any $M \in \mathcal{B}(X)$,
2. $F(X)$ is the identity operator on $L^2(\mathbb{R}^d)$,
3. $F(\cup_{i \in \mathbb{N}} M_i) = \sum_{i \in \mathbb{N}} F(M_i)$ for any countable collection of disjoint, measurable subsets $\{M_i\}_{i \in \mathbb{N}}$ of X , where the sum converges in the weak operator topology.

We also need to make precise what we mean by a *state* of the system described by the Hilbert space $L^2(\mathbb{R}^d)$. In quantum mechanics, the state of a system is described by a *density matrix*.

Definition A.2. Let $\rho \in \mathcal{T}^1$. We say that ρ is a *density matrix* if ρ is a positive operator with $\text{tr}(\rho) = 1$.

Since we claim that a POVM F on X describes a generalized observable, we should investigate how we may associate to F some kind of abstract measurement

procedure. Intuitively we would like to ask the following for a subset $M \subset X$: if F is a POVM on X and ρ describes a state, what is the probability that a measurement of the observable F in the state ρ yields a result in M ? An answer to this question can be given by noting that if F is a POVM on X and ρ is a density matrix, then μ_ρ^F defined by

$$\mu_\rho^F(M) = \text{tr}(\rho F(M))$$

is a probability measure on X^2 . In this way a POVM F associates a probability measure μ_ρ^F to any density matrix ρ , and it is this map $\rho \mapsto \mu_\rho^F$ that we refer to as a measurement [10]. Using this we can introduce Prugovečki's concept of informational completeness [38].

Definition A.3. A family $\{A_i\}_{i \in I}$ with $A_i \in B(L^2(\mathbb{R}^d))$ and I some index set is said to be *informationally complete* if, for density matrices ρ and ρ' , we have that

$$\text{tr}(\rho A_i) = \text{tr}(\rho' A_i) \quad \forall i \in I \quad \implies \quad \rho = \rho'.$$

Furthermore, we say that an observable F is informationally complete if $\rho \mapsto \mu_\rho^F$ is injective.

A.2 Covariant observables

We now introduce the class of generalized observables analyzed in [29], namely the so-called *covariant phase space observables*.

Definition A.4. A covariant phase space observable is a POVM F on \mathbb{R}^{2d} such that

$$\alpha_z(F(M)) = F(M + z)$$

for any $z \in \mathbb{R}^{2d}$ and $M \in \mathcal{B}(\mathbb{R}^{2d})$, where $M + z = \{m + z : m \in M\}$.

To relate the covariant observables to the theory of convolutions, we will use the following concept from [46].

Definition A.5. A map $\Gamma : L^\infty(\mathbb{R}^{2d}) \rightarrow B(L^2(\mathbb{R}^d))$ is said to be a *positive correspondence rule* if

1. $\Gamma(T_z f) = \alpha_z \Gamma(f)$ for any $f \in L^\infty(\mathbb{R}^{2d})$ and $z \in \mathbb{R}^{2d}$,
2. Γ sends positive functions to positive operators,

²This is why the generalization from a projection valued measure to a POVM is reasonable; since the operators $F(M)$ are positive, $\mu_\rho^F(M)$ is a positive number and can still be interpreted as a probability.

3. Γ is weak*-weak* continuous,
4. $\Gamma(1) = \mathcal{I}$, where 1 is the constant function $1(z) = 1$ and \mathcal{I} is the identity operator on $L^2(\mathbb{R}^d)$.

Remark. The first property is also called covariance of Γ [46].

The following lemma, originally due to Holevo [26], was mentioned briefly in a remark in the main text. A proof may be found in [46].

Lemma A.1. *If $\Gamma : L^\infty(\mathbb{R}^{2d}) \rightarrow B(L^2(\mathbb{R}^d))$ is a positive correspondence rule, then there exists a positive operator $S \in \mathcal{T}^1$ with $\text{tr}(S) = 1$ such that*

$$\Gamma(f) = f * S$$

for any $f \in L^\infty(\mathbb{R}^{2d})$.

We now sketch how the covariant phase space observables may be identified with the positive correspondence rules. First assume that Γ is a positive correspondence rule. One easily checks that the measure μ_Γ defined by $\mu_\Gamma(M) = \Gamma(\chi_M)$ is a covariant phase space observable, where χ_M is the characteristic function of the Borel subset $M \subset \mathbb{R}^{2d}$. Hence any positive correspondence rule determines a covariant phase space observable.

On the other hand, if F is a covariant phase space observable, then we can construct a positive correspondence rule Γ by defining integration with respect to the POVM F . One can show that for any $f \in L^\infty(\mathbb{R}^{2d})$ there is a unique operator $I_f \in B(L^2(\mathbb{R}^d))$ such that $\langle I_f \psi, \psi \rangle = \int \int_{\mathbb{R}^{2d}} f(z) d\nu_\psi$ for any $\psi \in L^2(\mathbb{R}^d)$, where ν_ψ is the probability measure defined by $\nu_\psi(M) = \langle F(M)\psi, \psi \rangle$ [5]. I_f is called the integral of f with respect to the POVM F . If we define Γ by $\Gamma(f) = I_f$, then Γ is a positive correspondence rule.

One can check that if F is a covariant phase space observable and Γ is the positive correspondence rule obtained above, then $F = \mu_\Gamma$ [47]. In particular, any covariant phase space observable can be obtained by starting with a positive correspondence rule. Using lemma A.1 this leads to the following result.

Lemma A.2. *Let F be a covariant phase space observable. There is a positive operator $S_F \in \mathcal{T}^1$ with $\text{tr}(S) = 1$ such that F is given by*

$$F(M) = \chi_M * S = \iint_{\mathbb{R}^{2d}} \chi_M(z) \alpha_z(S) dz$$

for any $M \in \mathcal{B}(\mathbb{R}^{2d})$.

Proof. As we have discussed above, $F(M) = \Gamma(\chi_M)$ for any $M \in \mathcal{B}(\mathbb{R}^{2d})$, where Γ is some positive correspondence rule. By lemma A.1, $\Gamma(\chi_M) = \chi_M * S$ for some fixed positive operator $S \in \mathcal{T}^1$ with $\text{tr}(S) = 1$. \square

A.3 Informational completeness for covariant observables

We will now investigate informational completeness for a covariant phase space observable F . By lemma A.2, F is given by $F(M) = \chi_M * S$ for a positive operator S with $\text{tr}(S) = 1$. F is informationally complete if $\rho \mapsto \mu_\rho^F$ is injective for density matrices ρ . By definition, μ_ρ^F is given by

$$\begin{aligned} \mu_\rho^F(M) &= \text{tr}(\rho F(M)) \\ &= \text{tr}(\rho \chi_M * S) \\ &= \iint_{\mathbb{R}^{2d}} \chi_M(z) \text{tr}(\rho \alpha_z(S)) \, dz \\ &= \iint_{\mathbb{R}^{2d}} \chi_M(z) \rho * \check{S}(z) \, dz, \end{aligned}$$

where we have used proposition 2.9 to move the trace inside the integral.

Clearly the probability density function of μ_ρ^F with respect to Lebesgue measure is $\rho * \check{S}$. From this it follows that F is informationally complete if and only if the mapping $\rho \mapsto \rho * \check{S}$ is injective. If we recall that $\rho, S \in \mathcal{T}^1$ we see that this statement is covered by theorem 7.4 with $p = \infty$ – the injectivity of $\rho \mapsto \rho * \check{S}$ is then part (4) of the theorem (with \check{S} instead of S , but as discussed before this is not important). In other words F given by $F(M) = \chi_M * S$ is informationally complete if and only if S is ∞ -regular. Finally, by theorem 7.3 this is equivalent to the statement that the set of zeros of $\mathcal{F}_W(S)$ has dense complement, or by proposition 6.6 that the Arveson spectrum of S is all of \mathbb{R}^{2d} .

B The generalized phase space representations of Klauder and Skagerstam

The purpose of this appendix is to show that the convolutions of functions and operators have also been discovered in the physics literature by Klauder and Skagerstam, who introduced what they called *generalized phase space representations* [31, 32]. However, they did not recognize that these representations could be phrased as convolutions and that there is a natural corresponding Fourier transform of operators. By showing that Klauder and Skagerstam's results may be phrased using Werner's convolutions, we will therefore be able to supply a conceptual framework for the generalized phase space representations. In this appendix we will write \mathcal{F} for the symplectic Fourier transform, as we need to reserve the subscript σ for other purposes to agree with the notation in [31, 32].

In physics, one often considers localization operators and the Berezin transform with windows $\varphi_1 = \varphi_2 = \phi$, where $\phi(x) = 2^{d/4} e^{-\pi x \cdot x}$ [6, 33]. In this case the Berezin

transform $\mathcal{B}^{\phi,\phi}S$ is called the *Husimi representation* S_H of $S \in \mathcal{T}^1$, and $f \in L^1(\mathbb{R}^{2d})$ is called the *Glauber-Sudarshan representation* S_{G-S} of the operator $S = \mathcal{A}_f^{\phi,\phi}$ [31]. Furthermore, the mapping $f \mapsto \mathcal{A}_f^{\phi,\phi}$ is the *Berezin quantization* [33].

By theorem 5.1, Werner's convolutions provide a generalization of the Husimi and Glauber-Sudarshan representations by replacing the operator $\phi \otimes \phi$ with a general trace class operator. However, rather than jumping to this conclusion, we will follow the reasoning of [31] to show another way of arriving at the convolutions of Werner.

B.1 Generalizing the Husimi and Glauber-Sudarshan representations

The goal of Klauder and Skagerstam [31] was to generalize the Husimi and Glauber-Sudarshan representations by replacing the Gaussian $\phi(x) = 2^{d/4}e^{-\pi x \cdot x}$ with a trace class operator σ . To find a natural way to do this, they started from the relation

$$\mathrm{tr}(S^*T) = \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}_W(S)(z')} \mathcal{F}_W(T)(z') dz' \quad (6)$$

for $S, T \in \mathcal{T}^1$, which follows from evaluating $\mathcal{F}_W(S^*T)(z) = \mathcal{F}_W(S^*) \sharp \mathcal{F}_W(T)(z)$ at $z = 0$. They then showed that this could be expressed using the Husimi and Glauber-Sudarshan representations as

$$\mathrm{tr}(S^*T) = \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}(S_{G-S})(z')} \mathcal{F}(T_H)(z') dz', \quad (7)$$

which we will prove easily once the connection to Werner's convolutions has been established.

Klauder and Skagerstam then fixed $\sigma \in \mathcal{T}^1$ such that $\mathcal{F}_W(\sigma)$ vanishes nowhere, in order to generalize S_{G-S} to a representation $S_{-\sigma}$, and T_H to a representation T_σ . They required that $S_{-\sigma}$ and T_σ should satisfy the obvious generalization of (7), and observed that this would hold if $S_{-\sigma}$ and T_σ were introduced by the following formal calculation based on equation (6):

$$\begin{aligned} \mathrm{tr}(S^*T) &= \iint_{\mathbb{R}^{2d}} \left[\frac{\mathcal{F}_W(S)(z')}{\mathrm{tr}(\pi(z')^* \sigma^*)} \right]^* \mathrm{tr}(\pi(z') \sigma) \mathcal{F}_W(T)(z') dz' \\ &:= \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}(S_{-\sigma})(z')} \mathcal{F}(T_\sigma)(z') dz'. \end{aligned}$$

One may then derive explicit expressions for $S_{-\sigma}$ and T_σ [31]. The generalized Husimi representation T_σ is given by $T_\sigma(z) = \mathrm{tr}(T \alpha_z(\sigma))$, which clearly equals $T * \check{\sigma}$ from the definition of the convolution of operators. The most relevant

expression for the generalized Glauber-Sudarshan representation $S_{-\sigma}$, is that when $S = \iint_{\mathbb{R}^{2d}} f(z) \alpha_z \sigma^* dz$ for some $f \in L^1(\mathbb{R}^{2d})$, then $f = S_{-\sigma}$. In other words, if $S = f * \sigma^*$, then $f = S_{-\sigma}$. One may easily check that $T_\sigma = T_H$ and $S_{-\sigma} = S_{G-S}$ when $\sigma = \phi \otimes \phi$, where $\phi(x) = 2^{d/4} e^{-\pi x \cdot x}$. We summarize the results in the next theorem.

Theorem B.1. *Fix $\sigma \in \mathcal{T}^1$ and let $S \in \mathcal{T}^1$.*

1. $S_\sigma = S * \check{\sigma}$.
2. *If $S = f * \sigma^*$ for some $f \in L^1(\mathbb{R}^{2d})$, then $S_{-\sigma} = f$.*

Furthermore, if $\sigma = \phi \otimes \phi$ where $\phi(x) = 2^{d/4} e^{-\pi x \cdot x}$ for $x \in \mathbb{R}^d$, then $S_\sigma = S_H$ and $S_{-\sigma} = S_{G-S}$.

From proposition 4.5 we now get precise conditions for when the generalized Husimi and Glauber-Sudarshan representations of an operator S belong to various function spaces, depending on which Schatten p -class S and σ belong to. Also, theorem B.1 shows that S_σ and $S_{-\sigma}$ may be defined using Werner's convolutions even when $\mathcal{F}_W(\sigma)$ has zeros. In fact, we can apply our generalization of Wiener's Tauberian theorem in theorems 7.2 and 7.3 to obtain results on the representations S_σ and $S_{-\sigma}$ in terms of the zero set of $\mathcal{F}_W(\sigma)$.

B.2 Using convolutions to reprove relations between Weyl and Berezin quantization

Since we have related the generalized Husimi and Glauber-Sudarshan representations to Werner's convolutions, we may now use the theory of these convolutions to shed light on relations between the twisted Weyl symbol and the Husimi and Glauber-Sudarshan representations. Known relations between these representations can now be expressed neatly as the associativity of convolutions, or as the relations $\mathcal{F}_W(f * T) = \mathcal{F}(f) \mathcal{F}_W(T)$ and $\mathcal{F}(S * T) = \mathcal{F}_W(S) \mathcal{F}_W(T)$.

Example B.1. We have already used this approach to prove equation (6), and claimed that equation (7) follows from theorem B.1. By that theorem, $S = S_{-\sigma} * \sigma^*$ if $S_{-\sigma} \in L^1(\mathbb{R}^{2d})$ exists. We can use $\mathcal{F}_W(S) = \mathcal{F}_W(S_{-\sigma} * \sigma^*) = \mathcal{F}(S_{-\sigma}) \mathcal{F}_W(\sigma^*)$ to

write equation (6) as

$$\begin{aligned}
\mathrm{tr}(S^*T) &= \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}_W(S)(z')} \mathcal{F}_W(T)(z') \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}(S_{-\sigma})(z')} \overline{\mathcal{F}_W(\sigma^*)(z')} \mathcal{F}_W(T)(z') \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}(S_{-\sigma})(z')} \mathcal{F}_W(\check{\sigma})(z') \mathcal{F}_W(T)(z') \, dz' \\
&= \iint_{\mathbb{R}^{2d}} \overline{\mathcal{F}S_{-\sigma}(z')} \mathcal{F}(T_\sigma)(z') \, dz',
\end{aligned}$$

where the last equality uses $\mathcal{F}_W(\check{\sigma})\mathcal{F}_W(T) = \mathcal{F}(T * \check{\sigma}) = \mathcal{F}T_\sigma$ by theorem B.1. By picking $\sigma = \phi \otimes \phi$ with $\phi(x) = 2^{d/4}e^{-\pi x \cdot x}$ we recover equation (7).

Example B.2. There is also a relation between S_σ and $S_{-\sigma}$ [31], which may be formulated as the associativity of the convolutions. If $S_{-\sigma} \in L^1(\mathbb{R}^{2d})$ exists for $\sigma \in \mathcal{T}^1$, theorem B.1 shows that $S = S_{-\sigma} * \sigma^*$ and $S_\sigma = S * \check{\sigma}$. Hence

$$S_\sigma = (S_{-\sigma} * \sigma^*) * \check{\sigma} = S_{-\sigma} * (\sigma^* * \check{\sigma}) = S_{-\sigma} * \mathrm{tr}(\sigma^* \alpha_z \sigma),$$

where the last equality uses the definition of $\sigma^* * \check{\sigma}$. One should also note that the last convolution is a regular convolution of functions. Again, by picking $\sigma = \phi \otimes \phi$ for $\phi(x) = 2^{d/4}e^{-\pi x \cdot x}$ we obtain a known relation between the Husimi and Glauber-Sudarshan representations [31].

Example B.3. Finally, $\mathcal{F}_W(f * T) = \mathcal{F}(f)\mathcal{F}_W(T)$ provides a known link between Weyl and Berezin quantization. Assume that $S \in \mathcal{T}^1$ may be represented as $S = S_{-\sigma} * \sigma^*$, where $S_{-\sigma} \in L^1(\mathbb{R}^{2d})$ and $\sigma \in \mathcal{T}^1$. By proposition 6.2, $\mathcal{F}_W(S)$ is the twisted Weyl symbol of S , and by proposition 6.4

$$\mathcal{F}_W(S) = \mathcal{F}(S_{-\sigma})\mathcal{F}_W(\sigma^*).$$

If $\sigma = \phi \otimes \phi$ with $\phi(x) = 2^{d/4}e^{-\pi x \cdot x}$, one may calculate that $\mathcal{F}_W(\sigma^*) = e^{2\pi i x \cdot \omega} e^{\frac{\pi}{2}(z \cdot z)}$. In this case $S_{-\sigma} = S_{G-S}$, and we obtain the relation

$$\mathcal{F}(S_{G-S}) = \mathcal{F}_W(S) e^{-2\pi i x \cdot \omega} e^{-\frac{\pi}{2}(z \cdot z)},$$

which relates the symbol of S in Weyl quantization, $\mathcal{F}_W(S)$, to the symbol of S in Berezin quantization, S_{G-S} . Similar formulae for the Husimi representation are obtained in the same way.

B.3 Berezin-Lieb inequalities

In [32], Klauder and Skagerstam proved and applied a Berezin-Lieb type inequality for the extended Husimi and Glauber-Sudarshan representations $S_{-\sigma}$ and S_{σ} . Let $\sigma, \sigma' \in \mathcal{T}^1$ be positive operators with $\text{tr}(\sigma) = \text{tr}(\sigma') = 1$. For an operator $T \in \mathcal{T}^1$ and $\beta \in \mathbb{R}$, Klauder and Skagerstam proved that

$$\iint_{\mathbb{R}^{2d}} e^{-\beta T_{\sigma'}(z)} dz \leq \text{tr}(e^{-\beta T}) \leq \iint_{\mathbb{R}^{2d}} e^{-\beta T_{-\sigma}(z)} dz. \quad (8)$$

By theorem B.1, one might expect that this result can be formulated using Werner's convolutions. In fact, Werner proved this result already in [46] in a more general form. Werner's proof uses the following three properties of the convolutions.

Lemma B.2. *Let S be a positive trace class operator with $\text{tr}(S) = 1$, and consider $f \in L^{\infty}(\mathbb{R}^{2d})$ and $T \in B(L^2(\mathbb{R}^d))$ with $f \geq 0$ and $T \geq 0$. Then*

1. $f * S \geq 0$ and $T * S \geq 0$.
2. $\text{tr}(f * S) = \iint_{\mathbb{R}^{2d}} f(z) dz$ and $\iint_{\mathbb{R}^{2d}} T * S(z) dz = \text{tr}(T)$.
3. If 1 denotes the function $1(z) = 1$ on \mathbb{R}^{2d} and \mathcal{I} is the identity operator on $L^2(\mathbb{R}^d)$, then $1 * S = \mathcal{I}$ and $\mathcal{I} * S = 1$.

Proof. 1. This was proved in lemma 4.7.

2. Consider first $\text{tr}(f * S)$. When $f \in L^1(\mathbb{R}^{2d})$, we can use that $\mathcal{F}_W(f * S)(z) = \mathcal{F}(f)(z)\mathcal{F}_W(S)(z)$. Applying this at $z = 0$ gives the result. To prove the result when $\int_{\mathbb{R}^{2d}} f(z) dz = \infty$, one can approximate f from below by functions in $L^1(\mathbb{R}^{2d})$.

The second part is lemma 4.2 when $T \in \mathcal{T}^1$, and when $\text{tr}(T) = \infty$ one can approximate T by trace class operators to prove the result.

3. The convolution of $f \in L^{\infty}(\mathbb{R}^{2d})$ with $S \in \mathcal{T}^1$ is defined by duality, using the condition $\langle f * S, T \rangle = \langle f, T * \check{S}^* \rangle$ for any $T \in \mathcal{T}^1$. One easily checks using the definitions and lemma 4.2 that $\langle 1, T * \check{S}^* \rangle = \iint_{z \in \mathbb{R}^{2d}} \overline{\text{tr}(T \alpha_z S^*)} dz = \overline{\text{tr}(T)} = \langle \mathcal{I}, T \rangle$, hence $1 * S = \mathcal{I}$. That $\mathcal{I} * S = 1$ is proved similarly. □

In words, convolution with S fixed preserves trace/integral, positivity and identity. This is the key to Werner's proof of the Berezin-Lieb inequality, which now follows in an elaborated version.

Proposition B.3. Fix a positive trace class operator S with $\text{tr}(S) = 1$, and let $T = T^* \in B(L^2(\mathbb{R}^d))$ and $f = f^* \in L^\infty(\mathbb{R}^{2d})$. If Φ is a positive, convex and continuous function on a domain containing the spectrums of T and $S * T$, then

$$\iint_{\mathbb{R}^{2d}} \Phi \circ (S * T)(z) dz \leq \text{tr}(\Phi(T)). \quad (9)$$

Similarly, if Φ is a positive, convex and continuous function on a domain containing the spectrums of f and $f * S$, then

$$\text{tr}(\Phi(f * S)) \leq \iint_{\mathbb{R}^{2d}} \Phi \circ f(z) dz. \quad (10)$$

Proof. First we will explain how the proof may be reduced to the case $\Phi(t) = t_+$, i.e. the function that returns the positive part of t . The reader may confirm that the set of functions Φ where equations (9) and (10) hold is a convex cone and closed under taking the supremum. It is also closed under reflection $\Phi \mapsto \check{\Phi}$ and translations $\Phi \mapsto T_x \Phi$ for $x \in \mathbb{R}$ – these facts follow from the spectral calculus and that convolutions with S preserve identity. Since Φ is assumed to be positive, convex and continuous on a compact set, it can be approximated uniformly by positive piecewise linear convex functions [37, p. 35]. As is shown in [37, Thm. 1.5.7], any positive piecewise linear convex function can be written as a linear combination with positive coefficients of translates and reflections of the function t_+ – hence Φ can be approximated by such functions. It is therefore enough to prove the result for t_+ .

We will restrict the rest of the proof to inequality (9), inequality (10) follows from a similar argument. Observe that by the spectral calculus,

$$\text{tr}(T_+) = \inf\{\text{tr}(A) : A \geq 0, A \geq T\},$$

since $0 \leq T \leq A$ implies $\text{tr}(T) \leq \text{tr}(A)$. Now consider the following calculation:

$$\begin{aligned} \inf\{\text{tr}(A) : A \geq 0, A \geq T\} &= \inf\left\{\iint_{\mathbb{R}^{2d}} A * S(z) dz : A \geq 0, A \geq T\right\} \\ &\geq \inf\left\{\iint_{\mathbb{R}^{2d}} A * S(z) dz : A * S \geq 0, A * S \geq T * S\right\} \\ &\geq \inf\left\{\iint_{\mathbb{R}^{2d}} g(z) dz : g \geq 0, g \geq T * S\right\} \\ &= \iint_{\mathbb{R}^{2d}} (T * S)_+ dz. \end{aligned}$$

The first equality is simply part (2) of lemma B.2. The two inequalities follow since we take the infimum of larger sets: in the first case this is true by part (1) of lemma B.2, and in the second case it is trivially true. The final equality follows by simple integration theory. \square

To obtain Klauber and Skagerstam's result in equation (8), let $\Phi(t) = e^{-\beta x}$. Then let $S = \check{\sigma}$ in (9), and let $S = \sigma^*$ and $f = T_{-\sigma}$ in (10).

Remark. Equation (8) has an interesting interpretation in physics. If $H \in B(L^2(\mathbb{R}^d))$ is the Hamiltonian of a quantum mechanical system and $\beta = \frac{1}{k_B T}$, where k_B is Boltzmann's constant and T is the temperature, then $\text{tr}(e^{-\beta H})$ is the so-called partition function of the system. In classical mechanics, on the other hand, the partition function of an observable $f \in L^\infty(\mathbb{R}^{2d})$ is given by $\iint_{\mathbb{R}^{2d}} e^{-\beta f(z)} dz$. Hence equation (8) bounds the quantum mechanical partition function of H by the classical partition functions of $H_{\sigma'}$ and $H_{-\sigma}$. If the aim is to determine $\text{tr}(e^{-\beta H})$, one can try to optimize the bounds by choosing σ and σ' cleverly. A detailed discussion of the physical consequences of equation (8) is far beyond the scope of this thesis, but the interested reader may consult [32, 44].

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