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The Algebraic Bivariant Connes-Chern Character

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Abstract

In this thesis we present many properties of bivariant periodic cyclic homology with the purpose of then constructing two bivariant Connes-Chern characters from algebraic versions of Kasparov's KK-theory with values in bivariant periodic cyclic homology. The thesis is naturally divided into three parts.

In the first part, which spans the two first chapters, periodic cyclic theory is presented, starting with the very basic definitions in cyclic theory. The properties of differential homotopy invariance, Morita invariance, and excision, all of which are important for the construction of bivariant Connes-Chern characters, are discussed.

In the second part we discuss algebraic KK-theory based on the reformulations of Kasparov's KK-theory by Cuntz [3], [4], and Zekri [15], [16]. By using the properties of bivariant periodic cyclic theory from the first part, we construct two different bivariant Connes-Chern characters.

In the third part we discuss possible extensions of the theory to topological algebras, in particular a well-behaved class of topological algebras known as m-algebras.

Sammendrag

Denne avhandlingen presenterer en rekke egenskaper ved bivariant periodisk syklisk homologi med formålet å konstruere to bivariante Connes-Chern-karakterer fra algebraiske versjoner av Kasparovs KK-teori med verdier i bivariant periodisk syklisk homologi. Avhandlingen er naturlig delt inn i tre deler.

I den første delen, som omfatter de to første kapitlene, presenterer vi periodisk syklisk teori, bygget opp fra de mest elementære definisjonene i syklisk teori. Egenskaper som homotopiinvarians under glatte homotopier, Morita-invarians, og eksisjonsegenskapen, som alle er av stor interesse i konstruksjonen av bivariante Connes-Chern-karakterer, blir diskutert.

I den andre delen diskuteres algebraisk KK-teori basert på reformuleringer av Kasparovs KK-teori gjort av Cuntz [3], [4], og Zekri [15], [16]. Ved å bruke egenskapene til bivariant periodisk syklisk teori fra den første delen konstruerer vi to forskjellige bivariante Connes-Chern-karakterer.

I den tredje delen diskuterer vi mulige utvidelser av teorien til topologiske algebraer, spesifikt til en spesiell klasse topologiske algebraer kjent som m-algebraer.

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Contents

Introduction and Overview

In 1980 Kasparov introduced what is now known as KK-theory [13] which is a bivariant theory generalizing both K-theory and K-homology for C^*-algebras. The formulation of this theory is typically in terms Kasparov modules, and in general the theory is known to be technically very involved. Therefore, throughout the 1980's an effort was made to reformulate KK-theory algebraically. More specifically, Cuntz and Zekri reformulated the two KK-groups KK_0 and KK_1 in terms of universal C^*-algebras in a series of papers [3], [4], [15], [16]. These reformulations also respected the important product structure of Kasparov KK-theory. The reformulations allow us to define KK-theory for arbitrary algebras, and it is this point of view we shall take in this thesis.

In algebraic topology the Chern character provides a map from topological K-theory to rational cohomology. A major problem in the realm of noncommutative geometry has been to construct a similar map from C^*-algebra K-theory with values in some fitting homology theory. This fitting homology theory is known as periodic cyclic homology. It turns out that we can construct a bivariant theory known as bivariant periodic cyclic homology that generalizes both periodic cyclic homology and its dual theory, periodic cyclic cohomology. Further, we can construct a bivariant map from KK-theory to bivariant periodic cyclic homology generalizing the map from K-theory to periodic cyclic homology. In addition, bivariant periodic cyclic homology is naturally equipped with a product structure, just as KK-theory, and the map can be made to respect the product structures. This is known as the bivariant Connes-Chern character. It should be mentioned that although we get a bivariant Connes-Chern character using bivariant periodic cyclic homology, it is in general not desirable to use this for C^*-algebras. The reason is that (bivariant) periodic cyclic homology give degenerate and pathological results for C^*-algebras. A more modern approach is to use Puschnigg's local theory [14] as a receptacle for a (bivariant) Connes-Chern character for C^*-algebras.

Periodic cyclic homology exists for any algebra, and so with the algebraic reformulation of KK-theory we will in this thesis construct two Connes-Chern characters from algebraic KK-theory to bivariant periodic cyclic homology for arbitrary algebras. This will require several important properties of periodic cyclic homology such as homotopy invariance under differentiable homotopies, Morita invariance, and excision. These properties were established by Cuntz and Quillen in a series of papers in the 1990's [11], [12], [8], where in [12] a suggestion for such a bivariant Connes-Chern character was given. We will use their approach and construct both their Connes-Chern character, as well as a similar Connes-Chern character.

The thesis is divided into three chapters: In Chapter 1 we introduce the very basic definitions of cyclic theory building up to bivariant periodic cyclic theory. In order to introduce this, however, we will need the notions of quasi-free algebras and differentiable homotopies, both of which are also covered in this chapter. Much of the theory presented in this thesis can be developed for a larger category of algebras, known as pro-algebras, and so a short treatment of pro-categories is included.

In Chapter 2 we give an overview of important properties of bivariant periodic cyclic theory, such as homotopy invariance under differentiable homotopies, Morita invariance, and excision. We also show that bivariant periodic cyclic homology indeed generalizes both periodic cyclic homology and periodic cyclic cohomology. Further we calculate the periodic cyclic homology for some simple algebras. We also demonstrate the existence of canonical invertible elements in bivariant periodic cyclic homology for two algebras we will use for our bivariant Connes-Chern characters, namely for an algebra A, the tensor algebra TA and the suspension SA.

Lastly, in Chapter 3 we give an overview of the algebraic reformulation of KK-theory, before at last defining a product in algebraic KK-theory and constructing two bivariant Connes-Chern characters which are compatible with the product structures of algebraic KK-theory and bivariant periodic cyclic theory. We also discuss how we can extend the theory to a certain class of topological algebras known as m-algebras.

The appendix covers some basic material on two universal algebras that will be of significant importance in this thesis.

Conventions

We will work in the category of nonunital algebras. In other words, algebras will not be assumed to be unital, and even if they are, the morphisms will not be assumed to be unit preserving. Also, all algebras will be over the complex numbers, and all tensor products will, unless otherwise stated, be understood to be over this field. It should be noted that most of the theory presented in this thesis would work equally well over any field of characteristic zero, the exception being some material on KK-theory and the Connes-Chern character presented in Chapter 3.

Chapter 1

Basic Cyclic Theory

1.1 Cyclic theory and the algebra of differential forms

Although perhaps a bit unorthodox, we will introduce cyclic homology and Hochschild homology of an algebra A first through the use of the algebra ΩA of differential forms over A. A consequence of this way of presenting the material is that some of the definitions in this section will seem somewhat artificial. However, this version of the theory is perhaps easier to envision. Also, most operators have a very nice form, with the trade-off of being a bit abstract. The connection to what is known as the cyclic complex, the more traditional way to introduce cyclic theory, will be covered in Section 1.3. This viewpoint will make it easier for us to actually perform calculations in cyclic theory, although this is not something we will do much.

Let A be an algebra. The *algebra of differential forms* over A, denoted ΩA, is the universal algebra generated by $x \in A$ with relations of A and symbols dx, $x \in A$, satisfying linearity in x and $d(xy) = xdy + d(x)y$. If A is unital, we will not require $d(1) = 0$. Note that this would be equivalent to introducing $1 \cdot \omega = \omega$ for all ω in ΩA.

We may consider the linear span of all elements of the form $x_0 dx_1 \cdots dx_n$ and $dx_1 \cdots dx_n$, $x_i \in A$ for all i. Denote the span by $\Omega^n A$. Then ΩA may be considered as a vector space in the following way

$$\Omega A = \bigoplus_{n=0}^{\infty} \Omega^n A \tag{1.1}$$

and d can be regarded as a linear map $d\colon \Omega^n A \to \Omega^{n+1} A$ satisfying the following relations

$$\begin{aligned} d\colon x_0 dx_1 \cdots dx_n &\mapsto dx_0 dx_1 \cdots dx_n \\ d\colon dx_1 \cdots dx_n &\mapsto 0 \end{aligned} \tag{1.2}$$

We also see $d^2 = 0$, and so ΩA becomes a differential graded algebra.

Remark 1.1. For dimensions $n \geq 1$ elements of $\Omega^n A$ of the form $x_0 dx_1 \cdots dx_n$ or $dx_1 \cdots dx_n$ will sometimes be written as ωdx_n, with the implicit understanding that ω is the obvious element in $\Omega^{n-1} A$.

Note that for a homogeneous element ω in ΩA, that is, an element of $\Omega^n A$, we say its *degree* is n and denote this by $deg(\omega) = n$. We now present a series of different operators on ΩA which will allow us to define cyclic homology and in turn periodic cyclic homology. Consider first the operator $b\colon \Omega^n A \to \Omega^{n-1} A$ which acts on elements ωdx as

$$b(\omega dx) = (-1)^{deg(\omega)}[\omega, x], \quad b(dx) = 0, \quad \text{and} \quad b(x) = 0 \quad \text{for} \quad x \in A, \tag{1.3}$$

where $[\cdot,\cdot]\colon A \times A \to A$ is the commutator. The operator is extended by linearity to make b an endomorphism on ΩA. A simple, straightforward calculation will show that $b^2 = 0$, and so b defines a differential on ΩA, making $\big(\Omega A, b\big)$ into a chain complex. Furthermore, we introduce the number operator N, defined as as the linear extension of the operator which multiplies a homogeneous element by its degree. In other words, N acts on an element ω in degree n as

$$N(\omega) = deg(\omega)\omega = n\omega. \tag{1.4}$$

Finally we introduce the *Karoubi operator* κ by

$$\kappa(\omega dx) = (-1)^{deg(\omega)} dx \cdot \omega \tag{1.5}$$

for homogeneous elements ω, and we extend by linearity.

Lemma 1.2. $\kappa = 1 - (bd + db)$

Proof. This is a simple calculation:

$$\begin{aligned}
&(1 - (bd + db))(\omega dx) = \omega dx - b(d\omega dx) - (-1)^{deg(\omega)} d[\omega, x] \\
&= \omega dx - (-1)^{deg(\omega)+1}[d\omega, x] - (-1)^{deg(\omega)} d[\omega, x] \\
&= \omega dx - [\omega, dx] = \omega dx - \omega dx + (-1)^{deg(\omega)} dx \cdot \omega = \kappa(\omega dx).
\end{aligned}$$

□

An important property of κ is the following

Lemma 1.3. $\Big(\kappa^n - 1\Big)\Big(\kappa^{n+1} - 1\Big) = 0$

Proof. Note first that

$$\kappa^n(a_0 da_1 \cdots da_n) = da_1 \cdots da_n a_0$$

from which we deduce

$$\begin{aligned}
&\kappa^n(a_0 da_1 \cdots da_n) - a_0 da_1 \cdots da_n = [da_1 \cdots da_n, a_0] \\
&= (-1)^n b(da_1 \cdots da_n da_0) = b\kappa^{-1} d(a_0 da_1 \cdots da_n)
\end{aligned}$$

which shows $\kappa^n - 1 = b\kappa^{-1}d$. Then it follows that

$$\kappa^{n+1} = \kappa\Big(1 + b\kappa^{-1}d\Big) = \kappa + \kappa b \kappa^{-1} d$$

Since by Lemma 1.2

$$\begin{aligned}\kappa b &= \Big(1-\big(db+bd\big)\Big)b = b - db^2 - bdb = b - bdb \\ &= b\big(1-db\big) = b\Big(1-\big(bd+db\big)\Big) = b\kappa\end{aligned}$$

b commutes with κ. Hence from the above and Lemma 1.2 we obtain

$$\kappa^{n+1} = \kappa + bd = 1 - bd - db + bd = 1 - db$$

which combined with $d^2 = 0$ now easily gives $\big(\kappa^n - 1\big)\big(\kappa^{n+1} - 1\big) = 0.$ □

As a consequence of Lemma 1.3, we deduce from linear algebra that there is a spectral projection operator, say P, onto the generalized eigenspace for 1 for κ. If we set $L = \big(Nd\big)b + b\big(Nd\big)$ we find through a calculation that on $\Omega^n A$,

$$L = \big(\kappa - 1\big)^2\Big(\kappa^{n-1} + 2\kappa^{n-2} + \cdots + \big(n-1\big)\kappa + n\Big). \tag{1.6}$$

and from this we obtain $\Omega A = \mathit{Ker}(L) \oplus \mathit{Im}(L)$, with P being the projection onto $\mathit{Ker}(L)$ [9].

Remark 1.4. Note that P by construction commutes with N, b and d.

We will not have much interest in the operator P in itself. There is an interpretation of L as being an abstract Laplace operator and elements of $\mathit{Ker}(L) = \mathit{Im}(P)$ being abstract harmonic forms, but we will not explore this. Instead we will be interested in the operator B, defined as $B = NPd$. Then $Bb + bB = 0$ and $B^2 = 0$. These two equalities give us the $\big(B, b\big)$-bicomplex

$$\begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
\Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
\Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\
\downarrow b & & \downarrow b & & & & \\
\Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\
\downarrow b & & & & & & \\
\Omega^0 A & & & & & &
\end{array}$$

Remark 1.5. A sign change from b to $-b$ does not affect the fact that this is a bicomplex. We call the resulting bicomplex with differentials B and $-b$ the *$(B, -b)$-bicomplex.*

Now set

$$D_{2n}^{\Omega A} = \Omega^0 A \oplus \Omega^2 A \oplus \cdots \oplus \Omega^{2n} A \tag{1.7}$$

and

$$D_{2n+1}^{\Omega A} = \Omega^1 A \oplus \Omega^3 A \oplus \cdots \oplus \Omega^{2n+1} A \tag{1.8}$$

Definition 1.6. Let A be an algebra. The *cyclic homology* $HC_n(A)$ of A is the homology of the total complex for the $\big(B,-b\big)$-bicomplex, that is, the homology of the complex

$$\cdots \xrightarrow{B'-b} D_n^{\Omega A} \xrightarrow{B'-b} D_{n-1}^{\Omega A} \xrightarrow{B'-b} \cdots \xrightarrow{B'-b} D_1^{\Omega A} \xrightarrow{B'-b} D_0^{\Omega A} \longrightarrow 0$$

where B' is the *truncated B-operator*, meaning it is equal to B on every component of $D_n^{\Omega A}$ except on $\Omega^n A$, where it is defined to be zero.

Remark 1.7. By [9] the total complexes of the (B,b)-bicomplex and the $(B,-b)$-bicomplex are quasi-isomorphic. Thus for the purposes of homology, the sign change from b to $-b$ does not matter.

Definition 1.8. Let A be an algebra. The *Hochschild homology* $HH_n(A)$ of A is defined as the homology of the complex

$$\cdots \xrightarrow{b} D_n^{\Omega A} \xrightarrow{b} D_{n-1}^{\Omega A} \xrightarrow{b} \cdots \xrightarrow{b} D_1^{\Omega A} \xrightarrow{b} D_0^{\Omega A} \longrightarrow 0$$

We also introduce *Connes' S-operator* and the *SBI-sequence*, an important computational tool in cyclic theory. In our current framework, S acts as simple as "deleting" the top component of $D_n^{\Omega A}$. Equivalently, S is the projection

$$D_n^{\Omega A} = \Omega^n A \oplus \Omega^{n-2} A \oplus \cdots \longrightarrow \Omega^{n-2} A \oplus \Omega^{n-4} A \oplus \cdots = D_{n-2}^{\Omega A} \tag{1.9}$$

Consider $\big(\Omega A, -b\big)$ as the leftmost column in

$$\begin{array}{ccccccc}
\downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} \\
\Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\
\downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \\
\Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\
\downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & & & \\
\Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\
\downarrow{\scriptstyle -b} & & & & & & \\
\Omega^0 A & & & & & &
\end{array}$$

and view it as a chain complex. Denote by $\mathcal{D}(A)$ the total complex with differential $B'-b$ as above. Then we have the following short exact sequence of chain complexes

$$0 \longrightarrow \Omega A \rightarrow \mathcal{D}(A) \longrightarrow \mathcal{D}(A)[2] \rightarrow 0$$

where $\mathcal{D}(A)[2]$ is the double suspension of $\mathcal{D}(A)$. Changing the operator from b to $-b$ does not change the homology of ΩA (under the differential $\pm b$), and so the homology of the leftmost column is still the Hochschild homology of A. We then get a long exact sequence in homology

$$\cdots \longrightarrow HH_{n+2}(A) \xrightarrow{I} HC_{n+2}(A) \xrightarrow{S} HC_n(A) \xrightarrow{\tilde{B}} HH_{n+1}(A) \longrightarrow \cdots$$

where I is induced by the inclusion of the first column into the total complex, S is the map induced by (1.9), and $\tilde{B}$ is the map induced by applying the operator B to the top form in an element of $HC_n(A)$. It is customary to denote both B and $\tilde{B}$ by B.

Lastly we mention that we can apply $Hom_{\mathbb{C}}\Big(-,\mathbb{C}\Big) = Hom(-,\mathbb{C})$ to any diagram in this section to obtain the dual theory, called *cyclic cohomology*.

1.2 ΩA and the Fedosov product

In this section we will construct an isomorphism between a subalgebra of ΩA (with a different product) for an algebra A, and the tensor algebra TA, see Appendix A.2. This isomorphism will provide an interesting point of view when working with the bivariant theory. We first decompose ΩA into even and odd forms,

$$\Omega^{ev}A = \bigoplus_{n\geq 0}^{\infty} \Omega^{2n}A \quad \text{and} \quad \Omega^{odd}A = \bigoplus_{n\geq 0}^{\infty} \Omega^{2n+1}A \tag{1.10}$$

Define now on ΩA the following product $\circ$ given by

$$\omega_1 \circ \omega_2 = \omega_1\omega_2 - d\omega_1 d\omega_2 \tag{1.11}$$

This product is called the *Fedosov product*, and it is straight-forward to verify that this defines an associative and bilinear product on ΩA. In particular, the Fedosov product is compatible with the even-odd-grading given above.

Proposition 1.9. *For any algebra A, $\Big(\Omega^{ev}A,\circ\Big)$ is isomorphic to the tensor algebra over A, TA. Under the same isomorphism, $JA \cong \Big(\oplus_{n\geq 1}\Omega^{2n}A,\circ\Big)$.*

Proof. The isomorphism is given by the extension of the map

$$x \mapsto x, \qquad dxdy \mapsto xy - x\otimes y \tag{1.12}$$

for $x, y \in \Omega^0 A$. If $\rho\colon A \to TA$ is the natural inclusion of the algebra A into its tensor algebra and $\omega(xy) = \rho(xy) - \rho(x)\rho(y)$, then we may describe this extension as

$$x_0dx_1dx_2\cdots dx_n \mapsto \rho(x_0)\omega(x_1,x_2)\omega(x_3,x_4)\cdots\omega(x_{n-1},x_n) \tag{1.13}$$

It is routine to verify that this is an isomorphism. □

1.3 The cyclic complex

A computationally easier approach to the cyclic theory is through the cyclic complex. To introduce this let $\tilde{A}$ be the unitalization of the algebra A. We first note that as vector spaces we have isomorphisms

$$\Omega^n A \cong \tilde{A} \otimes A^{\otimes n} \cong A^{\otimes n+1} \oplus A^{\otimes n} \tag{1.14}$$

given by the map defined by

$$dx_1 dx_2 \cdots dx_n \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_n \tag{1.15}$$
$$x_0 dx_1 \cdots dx_n \mapsto x_0 \otimes x_1 \otimes \cdots \otimes x_n \tag{1.16}$$

and so by transferring all definitions from Section 1.1 we may get a completely equivalent way of looking at cyclic homology. To do this consider the following bicomplex, called the cyclic complex, denoted $CC(A)$

$$\begin{array}{ccccccc}
\downarrow \tilde{b} & & \downarrow -\tilde{b}' & & \downarrow \tilde{b} & & \\
A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & \cdots \\
\downarrow \tilde{b} & & \downarrow -\tilde{b}' & & \downarrow \tilde{b} & & \\
A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & \cdots \\
\downarrow \tilde{b} & & \downarrow -\tilde{b}' & & \downarrow \tilde{b} & & \\
A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & \cdots
\end{array}$$

where we think of the leftmost column as being centered in degree zero. The operator $\tilde{b}'$ is defined as

$$\tilde{b}'\Big(x_0 \otimes x_1 \otimes \cdots \otimes x_n\Big) = \sum_{j=0}^{n-1} (-1)^j x_0 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_n \tag{1.17}$$

while $\tilde{b}$ is can be obtained as extending $\tilde{b}'$ to the last factor in the tensor product as well, that is

$$\tilde{b}\Big(x_0 \otimes x_1 \otimes \cdots \otimes x_n\Big) = \tilde{b}'\Big(x_0 \otimes x_1 \otimes \cdots \otimes x_n\Big) + (-1)^n x_n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}. \tag{1.18}$$

Furthermore, λ is defined as

$$\lambda\Big(x_0 \otimes x_1 \otimes \cdots \otimes x_n\Big) = (-1)^n x_n \otimes x_1 \otimes \cdots \otimes x_{n-1} \tag{1.19}$$

and Q is

$$Q\Big(x_0 \otimes x_1 \otimes \cdots \otimes x_n\Big) = \sum_{j=0}^{n} (-1)^{jn} x_j \otimes \cdots \otimes x_n \otimes x_0 \otimes \cdots \otimes x_{j-1} \tag{1.20}$$

We see that the rows of $CC(A)$ are exact in positive degrees. Lengthy calculations will also show that

$$\tilde{b}^2 = 0 = \tilde{b}'^2, \quad Q\tilde{b} = \tilde{b}'Q, \quad \text{and} \quad \tilde{b}\Big(1-\lambda\Big) = \Big(1-\lambda\Big)\tilde{b}' \tag{1.21}$$

which shows that $CC(A)$ indeed is a bicomplex.

It turns out the cyclic homology of A, $HC_n(A)$, can be realized as the homology of the total complex of $CC(A)$. This is done by via the isomorphism (1.14), under which we find that b corresponds to $\begin{pmatrix} \tilde{b} & 1-\lambda \\ 0 & -\tilde{b}' \end{pmatrix}$, while B corresponds to $\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$, from which we may deduce that the $\left(B, b\right)$-bicomplex and the cyclic bicomplex $CC(A)$ have isomorphic total complexes, meaning they will give the same notion of cyclic homology [9].

As a last version of cyclic homology we also present the following complex, which is also taken from [9]. Let A be an algebra. Set

$$C_n^\lambda(A) \colon = A^{\otimes n+1}/(1-\lambda)A^{\otimes n+1} \tag{1.22}$$

Then $HC_n(A)$ is equal to the homology of the complex

$$\cdots \xrightarrow{\tilde{b}} C_n^\lambda(A) \xrightarrow{\tilde{b}} C_{n-1}^\lambda(A) \xrightarrow{\tilde{b}} \cdots \xrightarrow{\tilde{b}} C_0^\lambda(A) \to 0$$

Using this last picture of cyclic homology we may describe $HC_n(A)$ as homology classes of tensors $x_0 \otimes x_1 \otimes \cdots \otimes x_n$ which satisfies

$$x_0 \otimes x_1 \otimes \cdots \otimes x_n = (-1)^n x_n \otimes x_0 \otimes \cdots \otimes x_{n-1} \tag{1.23}$$

By applying $Hom(-, \mathbb{C})$ we also obtain the cochain complex $C_\lambda^n(A)$, and we can describe the nth cyclic cohomology of A as homology classes of $(n+1)$-linear functionals $\phi \colon A^{\otimes n+1} \to \mathbb{C}$ satisfying

$$\phi(x_0, x_1, ..., x_n) = (-1)^n \phi(x_n, x_1, ..., x_{n-1}) \tag{1.24}$$

with differential $Hom(\tilde{b}, \mathbb{C})$.

1.4 Periodic cyclic homology

In general, cyclic homology HC_n does not enjoy the good properties of homotopy invariance, Morita invariance, and excision. The sense in which these three properties hold for $HC_n(A)$ for an algebra A is very limited and will require restrictions on A itself [10]. However, the properties hold if we stabilize the diagonals of the $\left(B, -b\right)$-bicomplex with respect to Connes' S-operator and then apply the homology functor. This is the motivation for the definition of *periodic cyclic homology.*

Keeping with the periodic aspect of periodic cyclic theory, we define for an algebra A its periodic cyclic homology $HP_i(A)$ as the homology of the periodic total complex of the following extension of the $(B, -b)$-bicomplex

$$
\begin{array}{ccccccccc}
 & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} \\
\cdots & \longleftarrow & \Omega^3 A & \xleftarrow{B} & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A \\
 & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & \\
\cdots & \longleftarrow & \Omega^2 A & \xleftarrow{B} & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & \\
 & & \downarrow{\scriptstyle -b} & & \downarrow{\scriptstyle -b} & & & & \\
\cdots & \longleftarrow & \Omega^1 A & \xleftarrow{B} & \Omega^0 A & & & & \\
 & & \downarrow{\scriptstyle -b} & & & & & & \\
\cdots & \longleftarrow & \Omega^0 A & & & & & &
\end{array}
$$

This easily seen to be the same as the homology of the $\mathbb{Z}/2$-graded complex, or *supercomplex*,

$$
\widehat{\Omega}^{even} A \underset{B-b}{\overset{B-b}{\rightleftarrows}} \widehat{\Omega}^{odd} A
$$

where $\widehat{\Omega}^{even} A$ are the even forms, and $\widehat{\Omega}^{odd} A$ are the odd forms, in $\widehat{\Omega} A = \prod_{n \geq 0} \Omega^n A$. It is this viewpoint which will be easiest when we start working with bivariant periodic cyclic theory, although we shall introduce it without mention of $\widehat{\Omega} A$. Note that $\widehat{\Omega} A$ is the completion of ΩA with respect to the natural filtration (hence the use of the "hats").

Let $D_n^{\Omega A} = \Omega^n A \oplus \Omega^{n-2} A \oplus \cdots$ as before. Remembering the action of Connes' S-operator, see (1.9), we obtain

$$
\widehat{\Omega} A = \varprojlim_{S} \left(D_{2n}^{\Omega A} \oplus D_{2n+1}^{\Omega A} \right) \tag{1.25}
$$

Once again we may do exactly the same for the cohomology case. For an algebra A we have

$$
\left(\widehat{\Omega} A \right)' = \bigoplus_{n \geq 0} \left(\Omega^n A \right)' \tag{1.26}
$$

for which we may once again consider the even and odd forms, and we define the *periodic cyclic cohomology* $HP^i(A)$, $i \in \mathbb{Z}/2$, of A as the homology of the following supercomplex

$$
\left(\widehat{\Omega}^{even} A \right)' \underset{B-b}{\overset{B-b}{\leftrightarrows}} \left(\widehat{\Omega}^{odd} A \right)'
$$

As with $\widehat{\Omega} A$, we also have the following description of $\left(\widehat{\Omega} A \right)'$

$$
\left(\widehat{\Omega} A \right)' = \varinjlim_{S} \left(D_{2n}^{\Omega A} \oplus D_{2n+1}^{\Omega A} \right)'. \tag{1.27}
$$

1.5 Pro-algebras and pro-vector spaces

To properly introduce bivariant periodic cyclic homology, it will be helpful to use *pro-algebras*. A pro-algebra A is an inverse limit of algebras A_i, $i \in I$, where I is a partially ordered set. We will only need the case where I is countable, which simplifies the theory somewhat as we will see below. The theory on *pro-objects* and in particular pro-algebras and pro-vector spaces is covered extensively in both [1] and [12], and we will only give a brief exposition of the material relevant to the sections and chapters to come.

For any category $\mathfrak{C}$ we may consider its corresponding pro-category $pro-\mathfrak{C}$. The objects of $pro-\mathfrak{C}$ are inverse systems $(A_i)_{i\in I}$ for I a directed partially ordered set and $A_i \in Obj\mathfrak{C}$ for all $i \in I$. The set I is not fixed. The morphisms between two objects $X = (X_i)_{i\in I}$ and $Y = (Y_j)_{j\in J}$ of $pro-\mathfrak{C}$ are defined to be

$$Hom(X,Y) = \varprojlim_j \left(\varinjlim_i Hom(X_i, Y_j) \right) \tag{1.28}$$

Hence by definition of inverse limit, a morphism $f\colon X \to Y$ is really a system $(f_j)_{j\in J}$ compatible with the structure maps where $f_j \in \varinjlim_i Hom(X_i, Y_j)$ for all $j \in J$. But by definition of direct limit, this just means that for all $j \in J$ there is $i \in I$ such that

$$f_{ij}\colon X_i \to Y_j \tag{1.29}$$

represents f_j.

We see now that for any pro-object $(X_i)_{i\in I}$ we may restrict to another index set I_0 which is cofinal in I [1]. $(X_i)_{i\in I_0}$ is then isomorphic to $(X_i)_{i\in I}$. In particular $(X_i)_{i\in I}$ is isomorphic to $(X_{\alpha(j)})_{j\in J}$ for any order preserving map $\alpha\colon J \to I$ with $\alpha(J)$ cofinal in I.

Restricting to the countable case, we note that any countable directed set I admits an order preserving map $\alpha\colon \mathbb{N} \to I$ with $\alpha(\mathbb{N})$ cofinal in I [12]. Hence the pro-objects in this thesis will be of the form $(X_n)_{n\in\mathbb{N}}$.

Remark 1.10. Note that any object C in a category $\mathfrak{C}$ is also a pro-object in $pro-\mathfrak{C}$ by choosing the index set I to be the one-point partially ordered set.

Remark 1.11. Even if the category $\mathfrak{C}$ does not admit arbitrary inverse limits, the category $pro-\mathfrak{C}$ exists.

It also turns out that given a morphism of pro-objects $f\colon (X_i)_{i\in I} \to (Y_j)_{j\in J}$ we may reindex to one variable [1], [12]. We may represent f as $\{f_{ij}\colon X_i \to Y_j | (i,j) \in F\}$ for F a cofinal subset of $I \times J$ under the product ordering

$$(i,j) \le (i',j') \quad \iff \quad i \le i' \quad \text{and} \quad j \le j'. \tag{1.30}$$

By setting $k = (i,j)$ and $k \le k'$ if and only if $(i,j) \le (i',j')$, we obtain the representation of f as $\{(f_k)_{k\in F} | f_k\colon X_k \to Y_k\}$ with $X_k = X_i$ and $Y_k = Y_j$.

Once again restricting to the countable case, if $f\colon (X_i)_{i\in I} \to (Y_j)_{j\in J}$ is a map of pro-objects with I and J countable we use the above results to obtain $f\colon (X_n)_{n\in\mathbb{N}} \to (Y_n)_{n\in\mathbb{N}}$ represented by $(f_n)_{n\in\mathbb{N}}$ with $f_n\colon X_n \to Y_n$ for all $n \in \mathbb{N}$. Inductively, we may do this for any finite system of maps, reindexing maps one by one.

Pro-algebras and pro-vector spaces are now the obvious pro-categories of the category of algebras and the category of vector spaces, respectively.

For a morphism f of pro-vector spaces we may define the kernel, cokernel, and image of f. They will just be the pro-vector spaces $(Ker f_n)$, $(Coker f_n)$, and $(Im f_n)$. The resulting pro-vector spaces do not depend on the choice of representation of f [12]. The morphism f is said to be injective if $Ker f \cong 0$ and surjective if $Coker f \cong 0$. We note that the category of pro-vector spaces is abelian [1]. We also obtain a notion of *pro-complexes* as a graded pro-vector space with a differential of degree 1. The notion of ideal is extended to pro-algebras by saying that an ideal is a kernel of a morphism of pro-algebras.

We extend the notion of projective and injective objects to pro-objects in the natural way. If (X_n) is projective then $Hom((X_n), -)$ preserves epimorphisms, and if (Y_n) is injective then $Hom(-, (Y_n))$ carries monomorphisms to epimorphisms. For abelian pro-categories we may of course instead say that (X_n) is projective implies $Hom((X_n), -)$ is exact, and (Y_n) being injective implies $Hom(-, (Y_n))$ is exact. The following two results are from [12].

Lemma 1.12. *Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of vector spaces and*

$$W_N = \bigoplus_{n=1}^{N} V_n.$$

Then the pro-vector space (W_N) with natural projection $W_{N+1} \to W_N$ as structure map, is injective.

Lemma 1.13. *Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of vector spaces and*

$$W_N = \bigoplus_{n\geq N} V_n.$$

Then the pro-vector space (W_N) with natural inclusion $W_{N+1} \to W_N$ as structure map, is projective.

In particular the pro-vector space

$$\Big(\bigoplus_{n=0}^{N} \Omega^n A\Big)_{N\in\mathbb{N}} \tag{1.31}$$

is injective for all algebras A, and the pro-vector space

$$\Big(\bigoplus_{n\geq N} \Omega^n A\Big)_{N\in\mathbb{N}} \tag{1.32}$$

is projective for all algebras A.

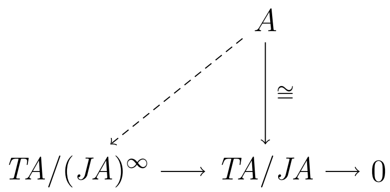
As already mentioned, any object is a pro-object by using the trivial poset with one element. Then, for example, any result which is true for pro-algebras must also be true for algebras. On the other hand, we may often prove theorems for pro-algebras by restricting to the category of algebras. This is true whenever the result is functorial and is natural. In particular, everything introduced until now also holds true for a pro-algebra $(A_k)_k$, replacing $\Omega^n A$ by $(\Omega^n A_k)_k$ and so on. We will not make an effort to make results in this thesis as general as possible with respect to pro-algebras. It is included only because it is a natural way to introduce bivariant periodic cyclic homology.

1.6 Quasi-free algebras

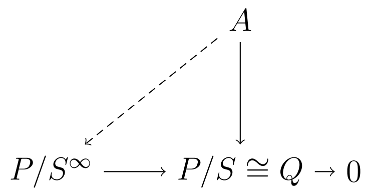
The notion of a *quasi-free* pro-algebra will be of importance when defining bivariant periodic cyclic homology. Given a pro-algebra A and an ideal I in A, denote by A/I^∞ the pro-algebra $(A/I^n)_{n\in\mathbb{N}}$. We list the following theorem from [11], [7]:

Proposition 1.14. *For a pro-algebra A the following conditions are equivalent:*

1. *There exists a morphism of pro-algebras $A \to TA/(JA)^\infty$ lifting the natural quotient map $TA/(JA)^\infty \to TA/JA \cong A$ as in the following diagram*



2. *If $0 \to S \to P \to Q \to 0$ is an extension of pro-algebras which admits a linear splitting and $A \to Q$ a morphism of pro-algebras, then the map $A \to P/S^\infty$ in the following diagram exists and the diagram commutes*



3. *The lifting property in (2) holds for nilpotent extensions, that is, for extensions as in (2) where there exists $k \geq 1$ such that $S^k = 0$.*

Definition 1.15. A quasi-free pro-algebra is a pro-algebra satisfying any of the equivalent conditions in Proposition 1.14.

The list of equivalent conditions in Proposition 1.14 can be extended considerably, and we will come back to one extra equivalent condition when it will be needed in Section 2.3. For now, the existence of lifts is what should be emphasized.

Proposition 1.16. *Let A be a quasi-free pro-algebra and let K be an ideal in A. Then A/K^∞ is also quasi-free.*

Proof. Given a short exact sequence of pro-algebras $0 \to S \to P \to Q \to 0$, a morphism $\alpha\colon A/K^\infty \to Q$ is just a morphism $\beta\colon A \to Q$ vanishing on K^∞. Since A is quasi-free, β lifts to a morphism $\beta'\colon A \to P/S^\infty$ sending K^∞ into S/S^∞, hence $K^\infty \cong (K^\infty)^\infty$ into $(S/S^\infty)^\infty \cong 0$. Hence we have a lift $\alpha'\colon A/K^\infty \to P/S^\infty$ for α. □

We note in particular that for any pro-algebra A, the tensor algebra TA is free, and therefore admits arbitrary lifts. Hence TA satisfies condition (2) in Proposition 1.14, and is therefore a quasi-free pro-algebra. It is the ”canonical” quasi-free pro-algebra associated to a pro-algebra A.

Definition 1.17. An extension of pro-algebras $0 \to Q \to T \to A \to 0$ with T quasi-free will be called a *quasi-free extension.*

1.7 Differentiable homotopies

It turns out (bivariant) periodic cyclic homology can in general not "detect" continuous homotopies, only differentiable homotopies. Thus we have the need for the following definition

Definition 1.18. A *differentiable homotopy* between two pro-algebras $(A_n)_n$ and $(B_m)_m$ is a homomorphism $\phi\colon (A_n) \to (B_m \otimes \mathcal{C}^\infty[0,1])$, where $\mathcal{C}^\infty[0,1]$ here denotes the space of smooth functions on the unit interval $[0,1]$. We will denote by ϕ_t the evaluation of ϕ in the point $t \in [0,1]$. Then ϕ_t is a homomorphism $\phi_t\colon (A_n) \to (B_m)$. Two homomorphisms $\alpha, \beta\colon (A_n) \to (B_m)$ will be called *differentiably homotopic* if there exists a homomorphism $\phi\colon (A_n) \to (B_m \otimes \mathcal{C}^\infty[0,1])$ with $\phi_0 = \alpha$ and $\phi_1 = \beta$.

The following two results from [12] will be important when we define bivariant periodic cyclic homology.

Theorem 1.19. *If $0 \to S \to P \to Q \to 0$ is an extension of pro-algebras admitting a linear splitting and $\alpha\colon T \to Q$ is a morphism of pro-algebras where T is quasi-free, then any two lifts $T \to P/S^\infty$ are differentiably homotopic.*

Then the following is immediate from Proposition 1.16 and Theorem 1.19.

Theorem 1.20. *If $0 \to K_1 \to T_1 \to A \to 0$ and $0 \to K_2 \to T_2 \to A \to 0$ are two quasi-free extensions of A admitting linear splittings, then T_1/K_1^∞ and T_2/K_2^∞ are differentiably homotopy equivalent.*

1.8 The X-complex and bivariant theory

To define the bivariant theory we will need a particular supercomplex.

Definition 1.21. For any pro-algebra A the *X-complex* of A, $X(A)$, is the following supercomplex

$$A \underset{b}{\overset{\natural d}{\rightleftarrows}} \Omega^1 A_\natural$$

where $\Omega^1 A_\natural = \Omega^1 A/([A, \Omega^1 A])$, $b(\natural(xdy)) = [x,y]$, $\natural$ is the natural quotient map and $\natural d(x) = \natural(dx)$.

Given two (pro-)complexes $C = ((C_n), \partial_C)$ and $D = ((D_n), \partial_D)$ (not necessarily supercomplexes) we may define the *Hom-complex* $Hom(C,D)$. The chains of $Hom(C,D)$ are linear maps between the complexes C and D. Indeed $Hom(C,D)$ is graded by the degrees of the linear maps. It becomes a complex with the differential ∂ defined by

$$\partial(\phi) = \partial_D \circ \phi - (-1)^{deg(\phi)} \phi \circ \partial_C \tag{1.33}$$

We are finally ready to define bivariant periodic cyclic homology in a sufficiently general way.

Definition 1.22. Given two pro-algebras A and B we define the bivariant periodic cyclic homology $HP_*(A, B)$, $* = 0, 1$, as $H_*Hom(X(P/Q^\infty), X(T/S^\infty))$, where $0 \to Q \to P \to A \to 0$ is a quasi-free extension of A admitting a linear splitting, and $0 \to S \to T \to B \to 0$ is a quasi-free extension of B admitting a linear splitting.

The fact that this is well-defined, that is, independent of the choices of quasi-free extensions for A and B, will follow from Theorem 1.20 after we discuss homotopy invariance in Section 2.2.

We note that there is now a "canonical" choice of X-complex for calculating the bivariant periodic cyclic homology of two algebras A and B, namely $0 \to JA \to TA \to A \to 0$ and $0 \to JB \to TB \to B \to 0$, giving $HP_*(A, B) = H_*Hom(X(TA/(JA)^\infty), X(TB/(JB)^\infty))$.

If f is a morphism of complexes between $X(P/Q^\infty)$ and $X(T/S^\infty)$ in the setup of Definition 1.22 then f represents an element of $HP_0(A, B)$. Likewise, a linear map between $X(P/Q^\infty)$ and $X(T/S^\infty)$ which is in the kernel of the differential of (1.33) gives an element of $HP_1(A, B)$. We will denote the homology class of f by $ch(f)$ or $[f]$. The use of ch to denote elements will make more sense in Chapter 3, where we construct the Connes-Chern characters.

Remark 1.23. A morphism $A \to B$ of (pro-)algebras induces a map $X(TA/(JA)^\infty) \to X(TB/(JB)^\infty)$ compatible with the differentials, hence induces an element of $HP_0(A, B)$.

Chapter 2

Properties of Periodic Cyclic Theory

2.1 The composition product in bivariant periodic cyclic homology

There is a very natural product in bivariant periodic cyclic homology given by composition of linear maps of X-complexes as in Definition 1.22. To be more specific, given three (pro-)algebras A, B and C we have the following associative product

$$\begin{aligned} HP_i(A,B) \times HP_j(B,C) &\to HP_{i+j}(A,C) \\ (ch(f), ch(g)) &\mapsto ch(f)\cdot ch(g) = ch(g\circ f) \end{aligned} \tag{2.1}$$

The product will just be denoted $\cdot$, that is, we will write $ch(f)\cdot ch(g)$, sometimes even dropping $\cdot$ to ease the notation.

The first thing we notice is that for any (pro-)algebra A, $HP_0(A,A)$ is a unital ring with unit $ch(1_A)$, where 1_A is the identity map on an X-complex defining $HP_*(A,A)$. We may therefore talk about invertible elements. Indeed, we can extend this notion to homology classes of maps between X-complexes defining HP for (pro-)algebras A and B, where A and B are not necessarily the same (pro-)algebra. An element $ch(f) \in HP_*(A,B)$ will be called *invertible* if there is $ch(g) \in HP_*(B,A)$ such that $ch(f)\cdot ch(g) = ch(1_A)$ and $ch(g)\cdot ch(f) = ch(1_B)$. An invertible element of degree zero will be called an *HP-equivalence*. If there is invertible $ch(f) \in HP_0(A,B)$ then the (pro-)algebras A and B will be called *HP-equivalent*. $HP_*(\cdot,\cdot)$ is a bifunctor, and when A and B are HP-equivalent the functor $HP_*(A,\cdot)$ is isomorphic to $HP_*(B,\cdot)$, and similarly in the contravariant case. The isomorphism is implemented by multiplication by the HP-equivalence.

2.2 Homotopy invariance

In order to discuss homotopy invariance of bivariant periodic cyclic homology under differentiable homotopies, we need the notion of *Lie derivative*. We will not treat the Lie derivative in its full generality. Lie derivative will for us mean the induced map on the X-complexes, see [11].

Definition 2.1. Let $f\colon A \to B$ be a morphism of pro-algebras admitting a derivation, that is, there exists a linear map $\dot{f}\colon A \to B$ such that $\dot{f}(xy) = f(x)\dot{f}(y)+\dot{f}(x)f(y)$. The Lie derivative L^f of f is then the map of supercomplexes $X(A) \to X(B)$ given by

$$L^f(x) = \dot{f}(x), \qquad L^f(\natural(xdy)) = \natural(\dot{f}(x)d(f(y)) + f(x)d(\dot{f}(y)). \tag{2.2}$$

Cuntz and Quillen proved the following in [11], adjusting for pro-algebras.

Proposition 2.2. *Let $f\colon A \to B$ be a morphism of pro-algebras admitting a derivation, with A quasifree. Then its associated Lie derivative $L^f\colon X(A) \to X(B)$ is null-homotopic.*

Suppose we now have a *differentiable homotopy* between two pro-algebras A and B, that is, a family $f\colon A \to B \otimes \mathcal{C}^\infty[0,1]$ with $f_t\colon A \to B$ for all $t \in [0,1]$. Suppose further that A is quasifree. From this we get a resulting family of derivatives $\dot{f}_t$, and in turn a family of Lie derivatives $(L_t^f)_{t\in[0,1]} = (L^{f_t})_{t\in[0,1]}$ with $L_t^f\colon X(A) \to X(B)$. For every $t \in [0,1]$ Proposition 2.2 gives a homotopy h_t such that $L_t^f = \partial h_t + h_t\partial$, where ∂ has been used to denote the boundary maps in both $X(A)$ and $X(B)$. By integrating this with respect to t over $[0,1]$ we obtain $L_1^f - L_0^f = \partial H + H\partial$, where

$$H = \int_0^1 h_t dt \tag{2.3}$$

We sum this up in the following proposition

Proposition 2.3. *Let $f\colon A \to B \otimes \mathcal{C}^\infty[0,1]$ be a morphism of pro-algebras, and let A be quasifree. The induced actions of f_0 and f_1 on $HP_*(A,B)$ are the same. In particular, morphisms of quasifree pro-algebras connected by a differentiable homotopy induce the same map in bivariant periodic cyclic homology.*

Now we get immediately from Theorem 1.20 that the definition of bivariant periodic cyclic homology, Definition 1.22, is well defined.

We now recall $\Omega^{ev}A \cong TA$ from Section 1.2. It can be shown [11] that this extends to a natural identification of $X(TA)$ with ΩA compatible with the $\mathbb{Z}/2$-grading, which is also continuous with respect to the filtrations. With this identification one may further show [11], [12], that $X(TA/JA^\infty)$ is homotopy equivalent to the pro-vector space $\xi A = \left(\bigoplus_{k=0}^n \Omega^k A, B - b\right)$ for any pro-algebra A. Hence for algebras A and B, we may rewrite

$$HP_*(A,B) = H_*Hom(\xi A, \xi B) \tag{2.4}$$

by homotopy invariance. In fact, now we are essentially back to looking at morphisms of complexes as in Section 1.4, albeit ΩA is now considered as a pro-vector space. Indeed we may now look at $HP_*(A,B)$ as equivalence classes of *continuous* linear maps between the completions $\widehat{\Omega}A$ and $\widehat{\Omega}B$ with the natural filtration. Let's make the notion of continuity more precise. $\widehat{\Omega}A$ is naturally a complete metric space with the following metric. Let (x_n) and (y_n) be elements of $\widehat{\Omega}A = \prod_{n\geq 0} \Omega^n A$. Their distance is $\leq 2^{-k}$ if the first k entries are equal.

Hence we may regard bivariant periodic cyclic homology as

$$HP_*(A,B) = H_*Hom(\widehat{\Omega}A, \widehat{\Omega}B) \tag{2.5}$$

which is perhaps the most natural version in view of Section 1.4.

Remark 2.4. Two algebras A and B are HP-equivalent if and only if $X(TA/(JA)^{\infty})$ and $X(TB/(JB)^{\infty})$ are homotopy equivalent [12], [9].

Lastly in this section we present an important computational tool, Goodwillie's theorem. In fact, we will present a stronger version of it, known as the generalized Goodwillie's theorem. We need the following lemma whose proof is from [12].

Lemma 2.5. *Let K be an ideal in a pro-algebra B. Then there is a differentiable homotopy equivalence between $T(B/K)/(J(B/K)^{\infty})$ and $T(B/K^{\infty})/(J(B/K^{\infty})^{\infty})$*

Proof. Denote by S the kernel of the composition of natural maps

$$T(B/K^{\infty}) \to T(B/K^{\infty})/J(B/K^{\infty}) \cong B/K^{\infty} \to B/K \tag{2.6}$$

Now note that the following diagram is commutative

$$\begin{array}{ccccc} B/K^{\infty} & \xrightarrow{\rho} & T(B/K^{\infty}) & \xrightarrow{\pi} & T(B/K^{\infty})/(J(B/K^{\infty})) \\ & \searrow & & \swarrow & \\ & & B/K & & \end{array}$$

where ρ is the natural inclusion of an algebra into its tensor algebra, π is the quotient map, the left sloping arrow is the natural quotient map, and the right sloping arrow is the composition $T(B/K^{\infty})/(J(B/K^{\infty})) \cong B/K^{\infty} \to B/K$, where once again the last map is the natural quotient map. The composition of the top arrows yield an isomorphism, from which we get that the kernel of the map $B/K^{\infty} \to B/K$ is isomorphic to the kernel of the map $T(B/K^{\infty})/(J(B/K^{\infty})) \to B/K$. Hence $S/J(B/K^{\infty}) \cong K/K^{\infty}$. This again gives $S^{\infty} \subset J(B/K^{\infty})$, and therefore $T(B/K^{\infty})/J(B/K^{\infty})^{\infty} \cong T(B/K^{\infty})/S^{\infty}$.

Both $0 \to J(B/K) \to T(B/K) \to B/K \to 0$ and $0 \to S \to T(B/K^{\infty}) \to B/K \to 0$ are quasi-free extensions of B/K admitting linear splittings. Thus by invoking Theorem 1.20, we get that $T(B/K^{\infty})/S^{\infty}$ is differentiably homotopy equivalent to $T(B/K)/J(B/K)$, from which the result follows. ☐

Lemma 2.5 immediately yields the following version of the generalized Goodwillie's theorem

Theorem 2.6. *Let K, K' be two ideals in a pro-algebra A such that $K^{\infty} = K'^{\infty}$. Then A/K and A/K' are HP-equivalent. In particular, A/K and A/K^{∞} are HP-equivalent for any ideal K.*

2.3 Restriction to periodic cyclic homology

Now we will verify that bivariant periodic cyclic homology indeed generalizes both periodic cyclic homology and periodic cyclic cohomology. To be more precise, we verify that when using $\mathbb{C}$ as one of the arguments, the resulting groups are isomorphic to the groups obtained from the nonbivariant theory. In particular we will see

$$HP_*(A, \mathbb{C}) = HP^*(A) \quad \text{and} \quad HP_*(\mathbb{C}, A) = HP_*(A) \tag{2.7}$$

This reduction to the nonbivariant theories is a property bivariant periodic cyclic theory has in common with KK-theory.

We finally have need for the following equivalent condition of a pro-algebra being quasi-free [11].

Proposition 2.7. *For a pro-algebra A, the following are equivalent*

1. *A is quasi-free*
2. *There exists a linear map $\phi\colon A \to \Omega^2 A$ satisfying*

$$\phi(xy) = x\phi(y) + \phi(x)y + dxdy. \tag{2.8}$$

By this proposition we obtain

Proposition 2.8. $\mathbb{C}$ *is quasi-free.*

Proof. 1 generates $\mathbb{C}$, and we see that the linear map

$$\phi(1) = d1d1 - 2(1 \cdot d1d1)$$

satisfies the requirement. □

Then the extension $0 \to 0 \to \mathbb{C} \to \mathbb{C} \to 0$ is a quasi-free extension of $\mathbb{C}$. To calculate the bivariant periodic cyclic homology of an algebra A with respect to $\mathbb{C}$ we then need to consider $X(\mathbb{C}/0^\infty) \cong X(\mathbb{C})$. Now consider

$$\mathbb{C} \underset{b}{\overset{\natural d}{\rightleftarrows}} \Omega^1\mathbb{C}_\natural$$

Note that $\Omega^1\mathbb{C}_\natural$ is generated by $\natural(d(1))$ and $\natural(1\cdot d(1))$. Now since $1\cdot d(1) - d(1)\cdot 1 \in [\mathbb{C}, \Omega^1\mathbb{C}_\natural]$,

$$\natural(1 \cdot d(1)) = \natural(1 \cdot d(1 \cdot 1)) = \natural(1 \cdot (1 \cdot d(1) + d(1) \cdot 1)) = \natural(2 \cdot d(1)) = 2\natural(1 \cdot d(1))$$

which shows $\natural(1 \cdot d(1)) = 0$. Then

$$\natural(d(1)) = \natural(d(1 \cdot 1)) = \natural(1 \cdot d(1) + d(1) \cdot 1) = 2\natural(1 \cdot d(1)) = 0$$

as above. Hence $\Omega^1\mathbb{C}_\natural = 0$ and we may instead consider the following complex

$$\mathbb{C} \rightleftarrows 0$$

Combining this with (2.5) in Section 2.2, we see that for any algebra A,

$$HP_*(A, \mathbb{C}) = HP^*(A) \quad \text{and} \quad HP_*(\mathbb{C}, A) = HP_*(A) \tag{2.9}$$

2.4 Morita invariance and the trace map

Cuntz showed in [8] that a generalized version of Morita equivalence holds for bivariant periodic cyclic homology. In fact, this is actually a consequence of the homotopy invariance of HP.

Proposition 2.9. *Let A be an algebra, and let X and Y be linear subspaces. Denote by $A(XY)$ the subalgebra of A generated by products xy, $x \in X$, $y \in Y$. Then $A(XY)$ and $A(YX)$ are HP-equivalent.*

In particular this implies the existence of an invertible element in $HP_0(A, B)$ for any two algebras A and B related by a *Morita context* in a special way. We adopt the following definition. A Morita context

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix} \tag{2.10}$$

for two algebras A and B is an algebra with a splitting into four linear subspaces A, B, X, Y, such that when the elements are written as 2 x 2 matrices the multiplication is consistent with matrix multiplication.

If A and B are connected by a Morita context and in addition satisfy $A = XY$, $B = YX$ in the setting above, then clearly A and B are HP-equivalent by Proposition 2.9. This result will have a very important application for us in the construction of a Connes-Chern character from Kasparov's KK-theory with values in HP in Chapter 3. We take a look at the relevant construction right now.

Denote by $M_n(A)$ the algebra of $n \times n$-matrices over an algebra A, and let $M_\infty(A)$ denote $\varinjlim_n M_n(A)$, where the structure maps are the usual inclusion of $M_n(A)$ into the top left corner of $M_{n+1}(A)$. The elements of $M_\infty(A)$ may be regarded as infinite matrices with only finitely many nonzero terms up to equivalence given by the structure maps. We note that $M_n(A) \cong M_n(\mathbb{C}) \otimes A$ for all $n \in \mathbb{N} \cup \{\infty\}$.

The following is then a Morita context of A and $M_n(A)$ for any finite n: Consider $M_{n+1}(A)$, where we put A and $M_n(A)$ as blocks along the diagonal. Assume without loss of generality that A is in the top left corner. In the same setup we had in the definition of a Morita context with $B = M_n(A)$, we get a linear subspace X which is just the top row in $M_{n+1}(A)$ except for the very first entry where we put A itself. We similarly get a linear subspace Y which is just the leftmost column except for the first entry where we put A. The same construction also works for $M_\infty(A)$. But then we know from the above discussion and Proposition 2.9 that A and $M_n(A)$ are HP-equivalent for all $n \in \mathbb{N} \cup \{\infty\}$.

Indeed, the canonical inclusion in the top left corner $j_k^A \colon A \to M_k(A)$, $k = 1, 2..., \infty$ represents an invertible element. To see this we construct the inverse. Throughout the following discussion, we will write M_k instead of $M_k(\mathbb{C})$ for all $k = 1, 2, ..., \infty$ to ease notation. The map we construct will be from $X(T(M_k \otimes A)/(J(M_k \otimes A)^\infty)$ to $X(TA/(JA)^\infty)$. It is the following composition

$$\begin{aligned} X(T(M_k \otimes A)/(J(M_k \otimes A)^\infty) &\to (M_k)_\natural \otimes X(TA/(JA)^\infty) \\ &\cong X(TA/(JA)^\infty) \end{aligned} \tag{2.11}$$

where the first map is induced by the homomorphism

$$\begin{aligned} T(M_k \otimes A) &\to M_k \otimes TA \\ \rho(m \otimes a) &\mapsto m \otimes \rho(a) \end{aligned} \tag{2.12}$$

and the morphism of complexes

$$\begin{aligned} X(M_k \otimes A) &\to (M_k)_\natural \otimes X(A) \\ \natural((m_1 \otimes a_1)d(m_2 \otimes a_2)) &\mapsto \natural(m_1 m_2) \otimes \natural(d(a_1 a_2)) \end{aligned} \tag{2.13}$$

for $a, a_1, a_2 \in A$, $m, m_1, m_2 \in M_k$, and ρ the canonical inclusion of an algebra into its tensor algebra. It can be seen that the first morphism maps $J(M_k \otimes A)$ into $M_k \otimes JA$. The isomorphism is obtained by realizing that $(M_k)_\natural = \mathbb{C}$. We will denote this composition by Tr^k_A, where Tr is short for trace.

Proposition 2.10. *$ch(j^A_k)$ defines an invertible element in $HP_0(A, M_k(A))$ for $k \in \mathbb{N} \cup \{\infty\}$ with inverse Tr^k_A.*

Proof. Note first that $ch(j^A_k) \cdot ch(Tr^k_A) = ch(Id_A)$ is clear.

Now, $j^{M_k(A)}_k \colon M_k(\mathbb{C}) \otimes A \to M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes A$. Let σ be the automorphism interchanging the two tensor factors $M_k(\mathbb{C})$. This is (differentiably) homotopic to the identity, so $ch(j^{M_k(A)}_k) = ch(\sigma \circ j^{M_k(A)}_k)$, the the following two identities

$$\begin{aligned} ch(j^{M_k(A)}_k) \cdot ch(Tr^k_{M_k(A)}) &= ch(Id_{M_k(A)}) \\ ch(\sigma \circ j^{M_k(A)}_k) \cdot ch(Tr^k_{M_k(A)}) &= ch(Tr^k_A) \cdot ch(j^A_k) \end{aligned}$$

proves the proposition. □

2.5 Excision

The fact that bivariant periodic cyclic homology satisfies excision makes it a lot easier to calculate HP-groups by choosing suitable extensions. We present the Cuntz-Quillen excision theorem [12] and closely follow their exposition.

Theorem 2.11. *For any algebra A and any extension of algebras $0 \to S \to P \to Q \to 0$ there are two natural six-term exact sequences*

$$\begin{array}{ccccc} HP_0(A,S) & \longrightarrow & HP_0(A,P) & \longrightarrow & HP_0(A,Q) \\ \uparrow{\scriptstyle \delta^0_A} & & & & \downarrow{\scriptstyle \delta^1_A} \\ HP_1(A,Q) & \longleftarrow & HP_1(A,P) & \longleftarrow & HP_1(A,S) \end{array}$$

and

$$\begin{array}{ccccc} HP_0(S,A) & \longleftarrow & HP_0(P,A) & \longleftarrow & HP_0(Q,A) \\ \downarrow{\scriptstyle \delta_1^A} & & & & \uparrow{\scriptstyle \delta_0^A} \\ HP_1(Q,A) & \longrightarrow & HP_1(P,A) & \longrightarrow & HP_1(S,A) \end{array}$$

where the horizontal maps are induced by the maps in the extension.

A description of the vertical maps, that is, the connecting morphisms, is of significant interest. In order to do this, we first make note of the following result, which follows from naturality of the connecting morphisms, the definition of the differential in the *Hom*-complex, and the fact that products in HP are composition products.

Proposition 2.12. *Let $0 \to S \to P \to Q \to 0$ be an extension of algebras and denote it by E.*

1. *Let A_1, A_2 be algebras and let δ_1, δ_2 be the connecting morphisms in the six-term exact sequences in $HP_*(A_1, -)$ and $HP_*(A_2, -)$ associated to E. For any $x \in HP_*(A_1, A_2)$ and $y \in HP_*(A_2, Q)$ we have*
$$\delta_1(x \cdot y) = (-1)^{deg(x)}(x \cdot \delta_2(y))$$
2. *Let A_1, A_2 be algebras and let δ_1, δ_2 be the connecting morphisms in the six-term exact sequences in $HP_*(-, A_1)$ and $HP_*(-, A_2)$, associated to E. For any $x \in HP_*(S, A_1)$ and $y \in HP_*(A_1, A_2)$ we have*
$$\delta_2(x \cdot y) = \delta_1(x) \cdot y$$

With this proposition we can start talking about the canonical element $ch(E)$ associated to any extension E of algebras, and in turn get a description of the vertical maps in Theorem 2.11.

Consider an extension $E\colon 0 \to S \to P \to Q \to 0$ as above, and let δ, δ' be the connecting morphisms in the six-term exact sequences obtained from $HP_*(-, S)$ and $HP_*(Q, -)$, respectively. We first note that $\delta(ch(Id_S)), \delta'(ch(Id_Q)) \in HP_1(Q, S)$. In fact, one may through a tedious homological argument show [12]

Proposition 2.13. $\delta'(ch(Id_Q)) = -\delta(ch(Id_S))$

For $E\colon 0 \to S \to P \to Q \to 0$ as above we now define $ch(E)$ to be equal to $\delta'(ch(Id_Q)) = -\delta(ch(Id_S))$. The following description of the vertical maps in Theorem 2.11 is then an immediate consequence of Proposition 2.12.

Theorem 2.14. *Let $E\colon 0 \to S \to P \to Q \to 0$ be an extension of algebras and let A be an algebra. In the first exact sequence of Theorem 2.11, the vertical maps act on $HP_j(A, Q)$ by multiplication on the right by $(-1)^j ch(E)$, while in the second exact sequence both connecting maps are given by multiplication on the left by $-ch(E)$.*

2.6 Contractibility of the cone

When constructing the bivariant Connes-Chern character from algebraic KK-theory to bivariant periodic cyclic homology in Chapter 3, we will have need for the existence of a canonical invertible element in $HP_1(A, SA)$, where SA is the (*smooth*) *suspension* of the algebra A. As bivariant periodic cyclic homology is invariant only under differentiable homotopies, not under continuous homotopies, we will need to use the smooth suspension.

Let A be an algebra. The (*smooth*) *cone over A*, denoted CA, is the algebra $A \otimes \mathcal{C}_0^\infty(0, 1]$. We will formally use $f(0) = 0$ for $f \in CA$. Using the evaluation map in 1, denoted π_1, we obtain the short exact sequence of algebras

$$0 \longrightarrow Ker(\pi_1) \xrightarrow{\iota} CA \xrightarrow{\pi_1} A \longrightarrow 0$$

where ι is the inclusion. We define the the suspension of A to be the kernel of π_1 and we denote it by SA. SA can also be realized as $A \otimes \mathcal{C}_0^\infty(0,1)$. We have the following result.

Proposition 2.15. *For any algebra A, CA is differentiably homotopy equivalent to* 0*.*

Proof. We prove the result by showing that the identity on CA is differentiably homotopic to zero. This can be seen by considering the family $F\colon CA \to CA \otimes \mathcal{C}^\infty[0,1]$, where F_t, the evaluation of F in $t \in [0,1]$, is given by

$$a \otimes f(s) \mapsto a \otimes f(ts) \tag{2.14}$$

that is, given by a "precomposition" with the function $t \mapsto t$. □

By differential homotopy invariance we immediately get that $HP_i(CA, B) = 0$ and $HP_i(B, CA) = 0$ for all algebras B. In particular, once again using the short exact sequence

$$0 \longrightarrow SA \xrightarrow{\iota} CA \xrightarrow{\pi_1} A \longrightarrow 0$$

and the excision theorem, we obtain a long exact sequence of period six of the form

$$\begin{array}{ccccc} HP_0(A,SA) & \longrightarrow & 0 & \longrightarrow & HP_0(A,A) \\ {\scriptstyle \delta_A^0}\uparrow & & & & \downarrow{\scriptstyle \delta_A^1} \\ HP_1(A,A) & \longleftarrow & 0 & \longleftarrow & HP_1(A,SA) \end{array}$$

The vertical maps are then isomorphisms, and it follows that there is a canonical invertible element in $HP_1(A, SA)$, say γ_1^A, namely the image of $ch(Id_A)$ under δ_A^1.

2.7 Contractibility of the tensor algebra

We will also have need for a canonical invertible element in $HP_1(A, JA)$ when constructing the second bivariant Connes-Chern character in Section 3.2.

Let A be an algebra and consider TA, the tensor algebra over A with the natural inclusion ρ.

Proposition 2.16. *TA is differentiably contractible.*

Proof. Consider the family of linear maps $\rho_t\colon A \to TA$ given by $\rho_t = t\rho$ where ρ is the canonical inclusion. This induces a family of algebra homomorphisms $\phi_t\colon TA \to TA$. In particular $\phi_0 = 0$ and $\phi_1 = Id$, showing that TA is differentiably contractible. □

We now consider the short exact sequence $0 \to JA \to TA \to A \to 0$ and use Theorem 2.11 and differential homotopy invariance to obtain the long exact sequence

$$\begin{array}{ccccc} HP_0(A,JA) & \longrightarrow & 0 & \longrightarrow & HP_0(A,A) \\ \delta_A^0 \uparrow & & & & \downarrow \delta_A^1 \\ HP_1(A,A) & \longleftarrow & 0 & \longleftarrow & HP_1(A,JA) \end{array}$$

so as in the previous section, there exists a canonical invertible element in $HP_1(A, JA)$, say $\bar{\gamma}_1^A$, which is equal to $\delta_A^1(Id_A)$.

2.8 The exterior product

With homotopy invariance and excision at our disposal, we include a very brief discussion of the exterior product for the sake of completeness. We shall however have very little need for it in this thesis.

Let A be a unital (pro-)algebra, and let χ_A be the X-complex of the pro-algebra RA/JA^∞ where RA is the unital free tensor algebra over A and JA the kernel of the natural map $RA \to A$.

It was shown in [11] and [12] that for unital (pro-)algebras A and B there is a homotopy equivalence between $\chi_A \otimes \chi_B$ and $\chi_{A\otimes B}$. This shows the existence of an exterior product for unital algebras

$$HP_i(A_1, B_1) \otimes HP_j(A_2, B_2) \to HP_{i+j}(A_1 \otimes A_2, B_1 \otimes B_2) \tag{2.15}$$

Using excision, this has also been extended to the non-unital case, that is, (2.15) holds for arbitrary (pro-)algebras.

2.9 Examples

We will in this section use some of the many results we now have at hand and apply them to specific examples.

Example 2.17. Although not explicitly mentioned earlier, we have immediately from Section 2.3 that

$$HP_0(\mathbb{C}) = HP^0(\mathbb{C}) = \mathbb{C}, \quad HP_1(\mathbb{C}) = HP^1(\mathbb{C}) = 0 \tag{2.16}$$

By Morita invariance we have, for any algebra A, $HP_*(A) \cong HP_*(M_n(A))$ and $HP^*(A) \cong HP^*(M_n(A))$ for $* = 0, 1$ and $n \in \mathbb{N} \cup \{\infty\}$. In particular we get

$$HP_0(M_n(\mathbb{C})) = HP^0(M_n(\mathbb{C})) = \mathbb{C}, \quad HP_1(M_n(\mathbb{C})) = HP^1(M_n(\mathbb{C})) = 0 \tag{2.17}$$

Given any algebra A, the inclusion $j\colon A^n \to A$ gives an HP-equivalence between A^n and A. To see this, note that we have a canonical short exact sequence $0 \to A^n \xrightarrow{j} A \to A/A^n \to 0$. Using the excision theorem and the fact that $HP_*(A/A^n, B) = 0$ for all algebras B by Theorem 2.6, we get $HP_*(A^n, B) \cong HP_*(A, B)$ and $HP_*(B, A^n) \cong HP_*(B, A)$ for any algebra B, and the isomorphisms are induced by $ch(j)$. The following is then immediate

Proposition 2.18. *Any nilpotent algebra is HP-equivalent to* 0.

Example 2.19. Consider an infinite-dimensional separable Hilbert space $\mathcal{H}$ with an orthonormal basis $(\xi_n)_{n\in\mathbb{N}}$. Let $\mathbb{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$. Given positive $S \in \mathbb{B}(\mathcal{H})$ we may consider

$$Tr(S) = \sum_{i=1}^{\infty} \langle S\xi_i, \xi_i\rangle \tag{2.18}$$

There is no reason why this should be finite. Indeed for general operators S it is not. We can however always say that it is ≥ 0. It can further be shown that $Tr(S^*S) = Tr(SS^*)$ for all $S \in \mathbb{B}(\mathcal{H})$, and that $Tr(S)$ is independent of the choice of orthonormal basis. We now define

$$\mathcal{S}^p = \{S \in \mathbb{B}(\mathcal{H}) : Tr(|S|^p) < \infty\} \tag{2.19}$$

where $|x| = (x^*x)^{1/2}$ by standard functional calculus. This can be shown to be an ideal in $\mathbb{B}(\mathcal{H})$ for all $p \geq 1$ and in general $\mathcal{S}^p \subset \mathcal{S}^q$ for $p \leq q$. $\mathcal{S}^p$ is known as the *p'th Schatten ideal.* By the above discussion all Schatten ideals are *HP*-equivalent [12]. This is true also if one consider the Schatten ideals as m-algebras [10] (see Section 3.3).

Chapter 3

Bivariant Connes-Chern Characters

In this chapter we will construct bivariant Connes-Chern characters from the algebraic version of Kasparov's KK-theory to bivariant periodic cyclic homology. To do this, we will first present Cuntz's and Zekri's reformulations [3], [4], [15], [16] of Kasparov's KK-theory. We will not present Kasparov's formulation in terms of Hilbert C^*-modules as this does not make sense for arbitrary algebras. We will also no longer specify when results hold for pro-algebras rather than algebras.

3.1 An algebraic reformulation of KK-theory

In the trivially graded, or ungraded, version of Kasparov's KK-theory, it is possible to make a reformulation of KK_0, the first Kasparov KK-theory group, in terms of generalized homomorphisms.

Definition 3.1. Let A and B be C^*-algebras. A *prequasihomomorphism* from A to B is a triple (Φ, E, μ), where $\Phi = (\phi, \bar{\phi})$ is a pair of homomorphisms from A into a C^*-algebra E, and μ is a homomorphism $J \to B$, where J is an ideal in E such that for all $a \in A$, $\phi(a) - \bar{\phi}(a) \in J$.

A *quasihomomorphism* from A to B is a prequasihomomorphism where in addition

1. μ is an inclusion
2. E is the C^*-algebra generated by $\phi(A)$ and $\bar{\phi}(A)$
3. J is the closed, two-sided ideal generated by $\phi(A) - \bar{\phi}(A)$
4. The composition of $\phi\colon A \to E$ and the quotient map $E \to E/J$ is injective.

Every prequasihomomorphism determines a unique quasihomomorphism [4].

Given any pair of homomorphisms (α, β) from A to E we obtain a unique homomorphism $QA \to E$, sending $\iota(x)$ and $\bar{\iota}(x)$ to $\alpha(x)$ and $\beta(x)$, respectively. Here ι and $\bar{\iota}$ are the canonical inclusions $A \to QA$. Conversely, given a map $\Psi\colon QA \to B$, we get a unique pair (α, β) of morphisms $A \to B$ by setting $\alpha(x) = \Psi(\iota(x))$ and $\beta(x) = \Psi(\bar{\iota}(x))$ for all $x \in A$.

Let (Φ, E, μ) be a quasihomomorphism from a C^*-algebra A to a C^*-algebra B. Under the above correspondence we get a homomorphism $qA \to B$ since the ideal J in E is the ideal generated by $\phi(A) - \bar{\phi}(A)$.

In [3] it was shown that Kasparov's KK_0 and the product $KK_0 \times KK_0 \to KK_0$ can be equivalently formulated by looking at homotopy classes of quasihomomorphisms between C^*-algebras, with product being a somewhat technical "composition" of (generalized) homomorphisms. Building on this, it was shown in [4] that for any pair of C^*-algebras A and B, $KK_0(A, B)$ can be realized as homotopy classes of homomorphisms from qA to $\mathbb{K} \otimes B$, where $\mathbb{K}$ is the C^*-algebra of compact operators on a infinite-dimensional, separable Hilbert space. We denote the set of homotopy classes of such homomorphisms from qA to $\mathbb{K} \otimes B$ by $[qA, \mathbb{K} \otimes B]$. The product in KK-theory also becomes easy to write down but builds on homotopy invariance and stability of the functor KK, as well as a lifting property which holds for C^*-algebras.

Denote by $j_0 \colon B \to M_2(B)$ the inclusion in the top left corner, and let $\pi_0 \colon qB \to B$ be the restriction to qB of the homomorphism $QB \to B$ defined by the pair $(Id_B, 0)$. The following theorem from [4] is important to define the product.

Theorem 3.2. *Let A be a separable C^*-algebra. There exists a homomorphism $\phi_A \colon qA \to M_2(q^2A)$ such that $\pi_0\phi_A$ is homotopic to $j_0 \colon qA \to M_2(qA)$, and such that $\phi_A\pi_0$ is homotopic to $j_0 \colon q^2A \to M_2(q^2A)$. Here $q^2A = q(qA)$, and π_0 is also used for the obvious extension to $M_2(q^2A)$.*

Using Theorem 3.2, that is, homotopy invariance of $q^nA = q(q^{n-1}A)$ modulo stabilization by 2×2-matrices for separable A, we can now define the product: Let A, B and C be C^*-algebras, with A separable and B, C stable. Then the pairing $[qA, B] \times [qB, C] \to [qA, C]$ given by

$$([\alpha], [\beta]) \mapsto [\beta' q(\alpha)' \phi_A] \tag{3.1}$$

defines a bilinear, associative product. Here β' and $q(\alpha)'$ are the natural extensions of β and $q(\alpha)$ to 2×2-matrices, and $q(\alpha)$ is the naturally induced map, that is, given $\alpha \colon A \to B$, $q(\alpha) \colon qA \to qB$ is the map

$$a_0 q(a_1) \cdots q(a_n) \mapsto \alpha(a_0) q(\alpha(a_1)) \cdots q(\alpha(a_n)). \tag{3.2}$$

Now, $\mathbb{K} \otimes B$ is stable for any C^*-algebra B, so we thus have that the above pairing gives for C^*-algebras A, B and C, with A separable, a bilinear, associative product

$$[qA, \mathbb{K} \otimes B] \times [qB, \mathbb{K} \otimes C] \to [qA, \mathbb{K} \otimes C] \tag{3.3}$$

A similar reformulation for KK_1 was done by Zekri in [15]. Indeed, there exists another universal algebra construction which describes KK_1 for separable C^*-algebras.

First we need the notion of *KK-equivalence*. Denote by $[x]$ an element of $KK_i(A, B)$, $i = 0, 1$, for arbitrary C^*-algebras A and B. Note that in general, homomorphisms of C^*-algebras induce elements of the KK-groups [2]. Thus it makes sense to talk about $[Id_A]$ in $KK_0(A, A)$. Two C^*-algebras A and B are called *KK-equivalent* if there exists an element $[x] \in KK_0(A, B)$ and an element $[y] \in KK_0(B, A)$ such that $[x][y] = [Id_A]$ and $[y][x] = [Id_B]$, where the product is the Kasparov product (or, if we want we could do this in terms of quasihomomorphisms as in the discussion above). If two C^*-algebras are

KK-equivalent, they share all KK-theoretic properties, which is the only fact about KK-equivalence we will need in the sequel.

Let A be a C^*-algebra. We set EA to be the universal algebra generated by A and a self-adjoint F, where $F^2 = 1$ acts on EA as the identity. This is a C^*-algebra when equipped with the greatest C^*-norm. We define εA to be the ideal in EA generated by $[A, F] = \{aF - Fa | a \in A\}$. Zekri showed in [15] that the group $KK_1(A, B)$ could, for separable A and B, be identified with homotopy classes of homomorphisms $\varepsilon A \to \mathbb{K} \otimes B$, which we once again denote by $[\varepsilon A, \mathbb{K} \otimes B]$. It was also shown that graded products $KK_i \times KK_j \to KK_{i+j}$, addition modulo 2, exist and coincide with the products in Kasparov's KK-theory. Furthermore, it was shown in [16] that εA is KK-equivalent to SA, where $SA = A \otimes \mathcal{C}_0(0, 1)$ is the C^*-algebra suspension of A.

In order to reformulate KK-theory in a purely algebraic manner we now need to "delete" some topological aspects. In particular, tensor products with $\mathbb{K}$ will be replaced by tensor products with $M_\infty(\mathbb{C})$, that is, for an algebra A, instead of $\mathbb{K} \otimes A$ we will consider $M_\infty(A) = \varinjlim M_n(A)$ with the standard inclusion in the top left corner. In addition, we will consider an algebraic version of εA, which we will denote by $\varepsilon_a A$. This is obtained by, for an algebra A, letting $A[F]$ be the algebra obtained by adjoining an element F with $F^2 = 1$. Then we let $E_a A$ be the ideal generated by A in $A[F]$ and $\varepsilon_a A$ be the ideal in $A[F]$ generated by $\{aF - Fa | a \in A\}$, essentially the same as the C^*-algebra construction. We then have a canonical short exact sequence

$$0 \longrightarrow \varepsilon_a A \longrightarrow E_a A \longrightarrow A \oplus A \longrightarrow 0$$

For the construction of the second Connes-Chern character in Section 3.2 we note that it was shown in [5] that there is a bijection between traces on $\varepsilon_a A$ and traces on JA, where JA is the standard kernel in the surjection $TA \to A$.

We may now consider the following two algebraic reformulations of KK-theory for arbitrary algebras. In either case we will, for algebras A and B, define $KK_0^{alg}(A, B) = [qA, M_\infty(B)]$. However, due to the KK-equivalence for εA and SA for C^*-algebras, and the bijection between traces on $\varepsilon_a A$ and traces on JA, we may define two versions of $KK_1(A, B)$. We will set $KK_1^{C^*}(A, B) = [SA, M_\infty(B)]$ and $KK_1^{tr}(A, B) = [JA, M_\infty(B)]$. Why $KK_1^{tr}(A, B) = [JA, M_\infty(B)]$ is a natural definition will be discussed in the next section. Moreover, the suspension SA will for us mean the smooth suspension of Section 2.6, and $[C, D]$ will mean differentiable homotopy classes of homomorphisms $C \to D$. These two variations of algebraic KK-theory will give us two bivariant Connes-Chern characters.

3.2 Bivariant Connes-Chern characters

First we construct a bivariant Connes-Chern character from the KK-theory with $KK_0(A, B) = [qA, M_\infty(B)]$ and $KK_1(A, B) = KK_1^{C^*}(A, B) = [SA, M_\infty(B)]$ as discussed in Section 3.1. We will denote by $ch\colon KK_i(A, B) \to HP_i(A, B)$ the Connes-Chern character for $i = 0, 1$. First we need the following result from [12].

Proposition 3.3. *Let A be an algebra and denote by $\pi_0\colon qA \to A$ the map induced by $Id * 0\colon QA \to A$. Then $ch(\pi_0) \in HP_0(qA, A)$ is invertible.*

Remark 3.4. The inverse of $ch(\pi_0)$ is actually $ch(\iota) - ch(\bar{\iota})$.

Let $[\alpha] \in [qA, M_\infty(B)]$. Define

$$ch([\alpha]) = \gamma_0^A \cdot ch(\alpha) \cdot ch(Tr_B^\infty) \tag{3.4}$$

where $Tr_B^\infty \colon M_\infty(B) \to B$ is the trace map of Section 2.4 and γ_0^A is the inverse of $ch(\pi_0)$ from Proposition 3.3. Further, we define for $[\alpha] \in [SA, M_\infty(B)]$

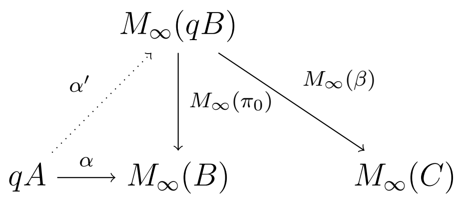
$$ch([\alpha]) = \gamma_1^A \cdot ch(\alpha) \cdot ch(Tr_B^\infty). \tag{3.5}$$

where γ_1^A is the canonical invertible element in $HP_1(A, SA)$, see Section 2.6.

Both of these maps are well-defined as both the left and right sides in either case are defined up to differentiable homotopy.

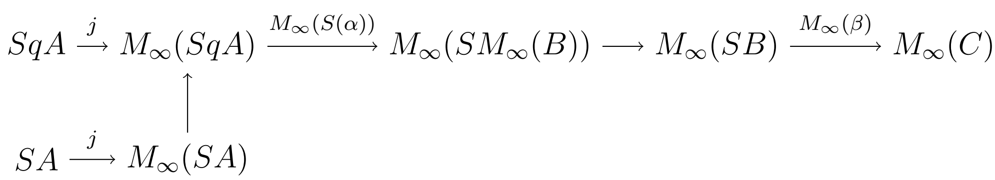
We wish for the transformation $KK_*(A, B) \to HP_*(A, B)$ to respect products. In order to discuss this, we must discuss when products should exist. If we were working with C^*-algebras (and changing back from $M_\infty(\cdot)$ to $\mathbb{K}\otimes$ and from ε_a to ε, taking C^*-tensor products) the products $KK_i \times KK_j \to KK_{i+j}$ (addition modulo 2) would always exist. This is due to canonical homomorphisms $[q^2A, \mathbb{K}\otimes B] \to [qA, \mathbb{K}\otimes B]$, $[\varepsilon(\varepsilon A), \mathbb{K}\otimes B] \to [qA, \mathbb{K}\otimes B]$ and so on [15]. However, this requires special properties of C^*-algebras, and we cannot hope this holds in general. For example, the product $[qA, \mathbb{K} \otimes B] \times [qB, \mathbb{K} \otimes C] \to [qA, \mathbb{K} \otimes C]$ exists and is well defined because of Pedersen's derivation lifting property [4].

Now let A, B and C be algebras, and let α represent an element $[\alpha] \in [qA, M_\infty(B)]$ and β represent an element $[\beta] \in [qB, M_\infty(C)]$. The product of $[\alpha]$ and $[\beta]$ is said to exist if there exists α' unique up to differentiable homotopy such that the following diagram commutes up to differentiable homotopy



that is, $M_\infty(\pi_0) \circ \alpha'$ is differentiably homotopic to α. The product of $[\alpha]$ and $[\beta]$ is then defined to be $[M_\infty(\beta) \circ \alpha'] \in [qA, M_\infty(C)]$. We have used that $M_\infty(M_\infty(C)) \cong M_\infty(C)$, a fact we will continue to use without mention.

Now alter the setting above so that $[\beta] \in [SB, M_\infty(C)]$. We then say that the product of $[\alpha]$ and $[\beta]$ exists if there is a differentiable homotopy equivalence $M_\infty(SA) \to M_\infty(SqA)$. Then we may represent the product with the following diagram



where the maps j are just sending an element to the same element tensored with a minimal idempotent in $M_\infty(\mathbb{C})$. That is, $j \colon a \mapsto a \otimes p$, where p is a minimal idempotent in $M_\infty(\mathbb{C})$. Any two choices of minimal idempotents are connected by a differentiable homotopy, so this will be well-defined in HP. We have suppressed a map here. The map $M_\infty(SM_\infty(B)) \to M_\infty(SB)$ is the composition of natural maps $M_\infty(SM_\infty(B)) \to M_\infty(M_\infty(SB)) \cong M_\infty(SB)$. We then set the product of $[\alpha]$ and $[\beta]$ to be the differentiable homotopy class in $[SA, M_\infty(C)]$ of the composition in the diagram from SA to $M_\infty(C)$.

If instead $[\alpha] \in [SA, M_\infty(B)]$ and $[\beta] \in [qB, M_\infty(C)]$, we say the product of $[\alpha]$ and $[\beta]$ exists if there exists a map α' unique up to differentiable homotopy such that the following diagram commutes up to differentiable homotopy.

$$\begin{array}{ccccc} & & M_\infty(qB) & & \\ & \overset{\alpha'}{\nearrow} & \Big\downarrow{\scriptstyle M_\infty(\pi_0)} & \overset{M_\infty(\beta)}{\searrow} & \\ SA & \xrightarrow{\alpha} & M_\infty(B) & & M_\infty(C) \end{array}$$

If such an α' exists we set the product of $[\alpha]$ and $[\beta]$ to be $[M_\infty(\beta) \circ \alpha'] \in [SA, M_\infty(C)]$.

Lastly, if $[\alpha] \in [SA, M_\infty(B)]$ and $[\beta] \in [SB, M_\infty(C)]$ we say the product exists if there is a differentiable homotopy equivalence $M_\infty(qA) \to M_\infty(S^2A)$, where $S^2A = S(SA)$. In this case we get the following diagram

$$\begin{array}{ccccccccccc} qA & \xrightarrow{j} & M_\infty(qA) & \longrightarrow & M_\infty(S^2A) & \xrightarrow{M_\infty S(\alpha)} & M_\infty(SM_\infty(B)) & \longrightarrow & M_\infty(SB) & \xrightarrow{M_\infty(\beta)} & M_\infty(C) \\ & & & & \Big\uparrow{\scriptstyle j} & & & & & & \\ & & & & S^2A & & & & & & \end{array}$$

where j is the map above and the map $M_\infty(SM_\infty(B)) \to M_\infty(SB)$ is the map $M_\infty(SM_\infty(B)) \to M_\infty(M_\infty(SB)) \cong M_\infty(SB)$ as before. If the product exists we set it to be the equivalence class of the total composition $qA \to M_\infty(C)$ in the diagram above. This is an element of $[qA, M_\infty(C)]$.

It now remains to show that our transformation $ch\colon KK_i(A,B) \to HP_i(A,B)$ for $i = 0, 1$ is compatible with these products. The cases are very similar, so we include just two of the verifications.

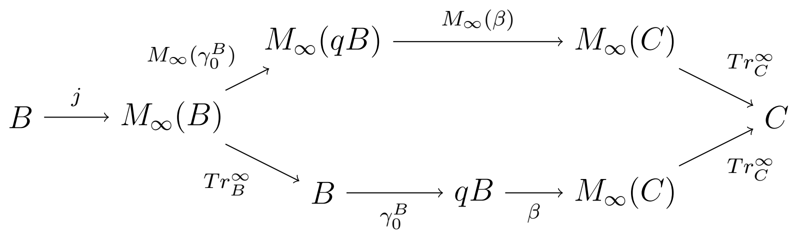
Let α represent $[\alpha] \in [qA, M_\infty(B)]$ and β represent $[\beta] \in [qB, M_\infty(C)]$, and suppose the product $[\alpha][\beta]$ exists. Denote the product by $[\eta] \in [qA, M_\infty(C)]$. Then

$$ch([\alpha])ch([\beta]) = \gamma_0^A ch(\alpha) ch(Tr_B^\infty)\gamma_0^B ch(\beta) ch(Tr_C^\infty) \tag{3.6}$$

while

$$ch([\eta]) = \gamma_0^A ch(M_\infty(\beta) \circ \alpha') ch(Tr_C^\infty) = \gamma_0^A ch(\alpha') ch(M_\infty(\beta)) ch(Tr_C^\infty) \tag{3.7}$$

where α' is the lift of α so that $M_\infty(\pi_0) \circ \alpha'$ is differentiably homotopic to α. Indeed we may write $ch(\alpha') = ch(\alpha) ch(M_\infty(\gamma_0^B))$. Checking that the two products are equal now reduces to checking that $ch(M_\infty(\gamma_0^B)) ch(M_\infty(\beta)) ch(Tr_C^\infty)$ is equal to $ch(Tr_B^\infty)\gamma_0^B ch(\beta) ch(Tr_C^\infty)$. This follows from that the two paths through the following diagram give the same HP-class, as well as that $ch(j)$ is invertible,



where we have written γ_0^B also for a representative of the equivalence class. The verification of the product $[SA, M_\infty(B)] \times [qB, M_\infty(C)] \to [SA, M_\infty(C)]$ being well-defined follows from a very similar argument.

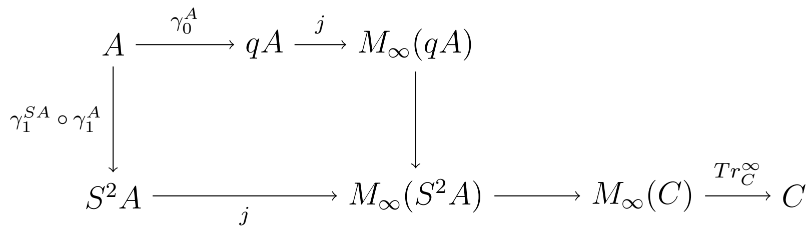
We also include the verification for the product $[SA, M_\infty(B)] \times [SB, M_\infty(C)] \to [qA, M_\infty(C)]$. Let α represent $[\alpha] \in [SA, M_\infty(B)]$ and β represent $[\beta] \in [SB, M_\infty(C)]$. Suppose the product $[\alpha][\beta]$ exists and denote it by $[\eta]$. We have

$$ch([\alpha])ch([\beta]) = \gamma_1^A ch(\alpha) ch(Tr_B^\infty)\gamma_1^B ch(\beta) ch(Tr_C^\infty) \tag{3.8}$$

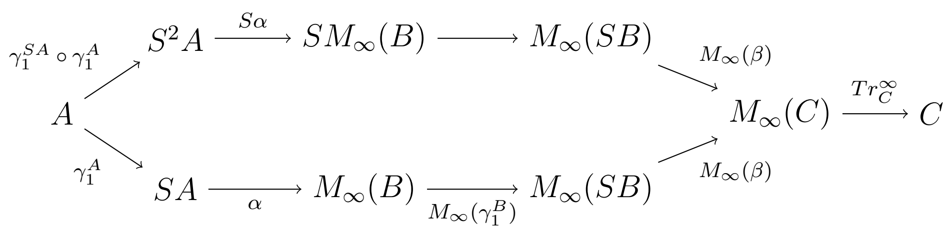
and

$$ch([\eta]) = \gamma_0^A ch([qA \to M_\infty(C)]) ch(Tr_C^\infty) \tag{3.9}$$

where $qA \to M_\infty(C)$ is the composition of the top row in the definition of this product. All maps in the square in the following diagram are invertible in *HP*, hence the two paths through the diagram gives the same *ch*-class



where we once again have let γ_1^{SA} and γ_1^A also denote maps that represent the equivalence classes. Here $M_\infty(S^2A) \to M_\infty(C)$ is the composition in the definition of the product. The two paths through the following diagram give the same *HP*-class.



This can be verified by realizing that any map in the diagram not involving α or β gives an invertible *HP*-class. It follows that the product is well defined. Checking that the product $[qA, M_\infty(B)] \times [SB, M_\infty(C)]$ is compatible with *ch* whenever it is defined can be done in essentially the same way. We summarize the results of this section so far in the following theorem, where the identifications $KK_0(A, B) = [qA, M_\infty(B)]$ and $KK_1(A, B) = [SA, M_\infty(B)]$ have been made.

Theorem 3.5. *The map* $ch\colon KK_i(A, B) \to HP_i(A, B)$, $i = 0, 1$, *is compatible with the product structure of both* KK *and* HP. *That is, for elements* $[\alpha] \in$

$KK_i(A,B)$ and $[\beta] \in KK_j(B,C)$ for which the product $[\alpha][\beta] \in KK_{i+j}(A,C)$ exists, we have

$$ch([\alpha][\beta]) = ch([\alpha])ch([\beta]) \in HP_{i+j}(A,C) \tag{3.10}$$

with addition modulo 2.

We now turn to the other bivariant Connes-Chern character construction as is done in [12]. Essentially everything is defined and proved in an analogous way as for the character above, so the treatment will be brief. When defining the bivariant Connes-Chern character above, the last thing we multiplied with was the map Tr_B^∞, in order for $ch([\alpha]), [\alpha] \in [qA, M_\infty(B)]$ or $[\alpha] \in [SA, M_\infty(B)]$, to define an element of $HP_i(A,B)$ rather than an element of $HP_i(A, M_\infty(B))$. Using the observation from [5] that there is a bijection between traces on $\varepsilon_a A$ and JA, the definition $KK_1(A,B) = KK_1^{tr}(A,B) = [JA, M_\infty(B)]$ seems fitting.

Once again for $[\alpha] \in [qA, M_\infty(B)]$ we set

$$ch([\alpha]) = \gamma_0^A ch(\alpha) ch(Tr_B^\infty) \tag{3.11}$$

and for $[\beta] \in [JA, M_\infty(B)]$ we set

$$ch([\beta]) = \bar{\gamma}_1^A ch(\beta) ch(Tr_B^\infty) \tag{3.12}$$

where $\bar{\gamma}_1^A$ is the canonical invertible element in $HP_1(A, JA)$, see Section 2.7. The products of elements in the sets of differentiable homotopy classes are said to be defined if they satisfy essentially the same conditions as for the bivariant Connes-Chern character above. That is, we still require existence of the appropriate lifts, differential homotopy equivalence between $M_\infty(qA)$ and $M_\infty(J^2A) = M_\infty(J(JA))$ (instead of between $M_\infty(qA)$ and $M_\infty(S^2A)$), and differential homotopy equivalence between $M_\infty(JA)$ and $M_\infty(JqA)$ (instead of $M_\infty(SA)$ and $M_\infty(SqA)$). Compatibility of $ch\colon KK_*(A,B) \to HP_*(A,B)$ with the respective product structures are proved as for our first bivariant Connes-Chern character.

3.3 Topological algebras

So far everything we have done has been without regard to any topology on our algebras, and we have not imposed continuity on any of our morphisms or linear splittings. It turns out [7], [9] essentially the entire theory carries over to a certain type of topological algebras.

Definition 3.6. A topological algebra A is an algebra equipped with a topology such that multiplication $A \times A \to A$ is jointly continuous.

We want to carry over as much of our theory as possible to topological algebras. This includes the incredibly important excision theorem. We based the excision theorem on the proof of Cuntz and Quillen from [12], which relies on the existence of canonical quasi-free algebras so that for any algebra A we could find a quasi-free extension of A. The algebra used for this was the tensor algebra TA. However, general tensor algebras do not exist in the category of topological algebras (with continuous algebra homomorphisms as morphisms). To make things easier for us, and to continue to use most of the same theory, we specialize to a subcategory of the category of topological algebras where tensor algebras exist. The (full) subcategory we specialize to is the one of *m-algebras*.

Definition 3.7. An m-algebra is a topological algebra with topology defined by a family of submultiplicative seminorms. Equivalently, it can be written as the projective limit of a system of Banach algebras.

For any m-algebra A the corresponding tensor algebra TA is also an m-algebra (see Appendix A.2).

We may now carry over all of our theory on the universal differential algebra ΩA to the category of m-algebras. The essence of adapting the theory to the topological case is to make all tensor products into completed projective tensor products, as well as making all homomorphisms and linear splittings continuous. Using the completed projective tensor products in both the cyclic bicomplex version of (periodic) cyclic homology, and in the $(B, -b)$-bicomplex version of (periodic) cyclic homology, we find that all the basic operators (in both frameworks) are continuous and extend to the completions. In applying the homology functor to obtain (bivariant) periodic cyclic homology, we take the quotient by the image of the differential, not by the closure of the image.

The definition of differentiable homotopy requires a similar (completed) projective tensor product. A differential homotopy from an algebra A to an algebra B is now defined as a continuous map from A into $B\hat{\otimes}\mathcal{C}^\infty([0,1])$. With this definition of differentiable homotopy, both homotopy invariance under differentiable homotopies and Morita invariance holds for *HP* applied to the category of m-algebras. We mention that a slight modification has to be made to Morita invariance, specifically that it is the completions of the subalgebras generated by the products of the linear subspaces that will be *HP*-equivalent. The all-important property of excision also holds. Hence we have (bivariant) periodic cyclic homology on the category of m-algebras, and all the previously covered properties still hold.

With this we have a proper receptacle for a bivariant Connes-Chern character from the category of m-algebras. However, our definitions of the algebraic *KK*-groups must now be made topological. First note that if A is an m-algebra, then both qA and JA are m-algebras (see Appendix A.1 and Appendix A.2). Further, TA is still differentiably contractible. Hence there is a canonical invertible element $\bar{\gamma}_1^A \in HP_1(A, JA)$ for any m-algebra A. The map $\pi_0\colon qA \to A$ is also continuous for any m-algebra A and its inverse gives an invertible element $\gamma_0^A \in HP_0(A, qA)$.

For an m-algebra A, define the cone $\hat{C}A$ of A by

$$\hat{C}A = \bar{\mathcal{C}}_0^\infty((0,1])\hat{\otimes}A \tag{3.13}$$

where $\bar{\mathcal{C}}_0^\infty((0,1])$ is the m-algebra of smooth functions on $[0,1]$ vanishing at 0 whose derivatives to arbitrary orders vanish at both 0 and 1. From this we also get the suspension of A

$$\hat{S}A = Ker[ev_1\colon \hat{C}A \to A] \tag{3.14}$$

where ev_1 is the evaluation in 1. $\hat{S}A$ can be realized as $\bar{\mathcal{C}}_0^\infty((0,1))\hat{\otimes}A$ with the obvious corresponding vanishing conditions on the derivatives of functions in $\bar{\mathcal{C}}_0^\infty((0,1))$. Also, $\hat{C}A$ is differentiably contractible by the same argument as before. Hence there is a canonical invertible element $\gamma_1^A \in HP_1(A, SA)$.

Finally we need an m-algebra analogue of $M_\infty(A) = M_\infty(\mathbb{C}) \otimes A$. This role will be played by the m-algebra $\mathfrak{K}$ of smooth compact operators, which can be realized as $\mathbb{N} \times \mathbb{N}$-matrices over $\mathbb{C}$ with rapidly decreasing coefficients. In particular its topology is defined by the family of multiplicative norms given by

$$p_n((a_{ij})) = \sum_{i,j\in\mathbb{N}} |1+i+j|^n |a_{ij}| \tag{3.15}$$

There is a canonical inclusion $j^A \colon A \to \mathfrak{K}\hat{\otimes}A$ which by Morita equivalence is invertible in $HP_0(A, \mathfrak{K}\hat{\otimes}A)$. We denote its inverse in HP by $ch(Tr_A)$.

With this we obtain two Connes-Chern characters from our KK-theory for m-algebras with values in HP.

We base the first character on the suspension, as we did in the algebraic case. Now let $[C, D]$ be the set of differentiable homotopy classes of continuous homomorphisms from C to D. Define

$$KK_0(A,B) = [qA, \mathfrak{K}\hat{\otimes}B], \quad KK_1(A,B) = [\hat{S}A, \mathfrak{K}\hat{\otimes}B] \tag{3.16}$$

The existence of products is the same as in Section 3.2 with the obvious adjustments for continuity, and $\mathfrak{K}$ instead of $M_\infty(\mathbb{C})$. We then define the bivariant Connes-Chern character as in Section 3.2. For $[\alpha] \in KK_0(A,B)$ with a representative α, we set

$$ch([\alpha]) = \gamma_0^A \cdot ch(\alpha) \cdot ch(Tr_B) \tag{3.17}$$

and for $[\beta] \in KK_1(A,B)$ with a representative β, we set

$$ch([\beta]) = \gamma_1^A \cdot ch(\beta) \cdot ch(Tr_B) \tag{3.18}$$

The compatibility of ch with the respective product structures is verified as in Section 3.2, adjusting for continuity, and $\mathfrak{K}$ instead of $M_\infty(\mathbb{C})$.

Likewise we may construct the topological analogue of the second Connes-Chern character of Section 3.2 by setting

$$KK_0(A,B) = [qA, \mathfrak{K}\hat{\otimes}B], \quad KK_1(A,B) = [JA, \mathfrak{K}\hat{\otimes}B] \tag{3.19}$$

The product structure is the same as in the previous section adjusting for continuity, and $\mathfrak{K}$ instead of $M_\infty(\mathbb{C})$. The second bivariant Connes-Chern character can then be defined from our KK-theory for m-algebras with values in HP in the following way. Let $[\alpha] \in KK_0(A,B)$ and let α be a representative. Then set

$$ch([\alpha]) = \gamma_0^A \cdot ch(\alpha) \cdot ch(Tr_B) \tag{3.20}$$

and for $[\beta] \in KK_1(A,B)$ with representative β set

$$ch([\beta]) = \bar{\gamma}_1^A \cdot ch(\beta) \cdot ch(Tr_B) \tag{3.21}$$

Once again, verifying the compatibility of ch with the respective product structures is just as in Section 3.2 adjusting for continuity, and $\mathfrak{K}$ instead of $M_\infty(\mathbb{C})$.

We have here only adapted HP to a topological setting and imposed continuity on the homomorphisms in our (algebraic) KK-theory. From this we easily got two Connes-Chern characters. There is however no reason why this should be particularly well-behaved in the case of m-algebras. Indeed, for m-algebras it it possible to rather alter the bivariant K-theory we use as the source of our Connes-Chern character. Such a construction was done by Cuntz in [6]. Doing this we may construct an altered category of m-algebras where differential homotopy classes of m-algebra homomorphisms are identified, and it turns out there exists a unique multiplicative Connes-Chern character from this category with values in HP.

Appendix A

Some Algebraic Constructions

A.1 The free product of algebras

The *(nonunital) free product* of two algebras, denoted $A_1 * A_2$, is the universal algebra equipped with two homomorphisms ι_1 and ι_2 such that for any pair of algebra homomorphisms $\phi_1 \colon A_1 \to E$ and $\phi_2 \colon A_2 \to E$ into an algebra E, there exists a unique algebra homomorphism $\phi_1 * \phi_2 \colon A_1 * A_2 \to E$ such that $\phi_j = \big(\phi_1 * \phi_2\big) \circ \iota_j$, $j = 1, 2$.

Explicitly this algebra is realized as

$$A_1 * A_2 = A_1 + A_2 + A_1 \otimes A_2 + A_2 \otimes A_1 + A_1 \otimes A_2 \otimes A_1 + A_2 \otimes A_1 \otimes A_2 + \cdots \quad \text{(A.1)}$$

with multiplication being concatenation and simplification of tensors using the multiplication in A_1 and A_2.

We shall be particularly interested in $QA = A * A$ for an algebra A. In particular we note that there is a canonical ideal qA in QA, called the *folding ideal*, which is the kernel of the map $QA \to A$ obtained from the universal property applied to the identity map $A \to A$ in both factors. This map is called the *folding map*. In case of the free product QA we will denote the canonical homomorphisms ι and $\bar{\iota}$.

QA may equivalently be described as the universal algebra generated by symbols x, $q(x)$ satisfying

$$q(xy) = xq(y) + q(x)y - q(x)q(y) \quad \text{(A.2)}$$

where the canonical homomorphisms ι, $\bar{\iota}$ are given by $\iota(x) = x$ and $\bar{\iota}(x) = x - q(x)$ [4].

The free product of two m-algebras A_1 and A_2 (see Section 3.3) equipped with families of seminorms $\{p_1^i\}$ and $\{p_2^j\}$ respectively, is

$$A_1 \hat{*} A_2 = A_1 + A_2 + A_1 \hat{\otimes} A_2 + A_2 \hat{\otimes} A_1 + A_1 \hat{\otimes} A_2 \hat{\otimes} A_1 + A_2 \hat{\otimes} A_1 \hat{\otimes} A_2 + \cdots \quad \text{(A.3)}$$

where the tensor products are completed projective tensor products, and it becomes an m-algebra when equipped with the family of seminorms $p_1^i \hat{*} p_2^j$ defined by

$$p_1^i \hat{*} p_2^j = p_1^i + p_2^j + p_1^i \hat{\otimes} p_2^j + p_2^j \hat{\otimes} p_1^i + \cdots \quad \text{(A.4)}$$

The folding ideal is also an m-algebra with the induced topology.

The free product of two m-algebras is the free product in the category of m-algebras.

A.2 The tensor algebra

For any vector space V we may consider the tensor algebra over V, denoted TV, which is the universal algebra equipped with an inclusion $\rho\colon V \to TV$ such that for any linear map $\phi\colon V \to E$ into an algebra E, there is a unique algebra homomorphism $\Phi\colon TV \to E$ such that $\phi = \Phi \circ \rho$.

Explicitly this algebra can be realized as

$$TV = V + V^{\otimes 2} + V^{\otimes 3} + \cdots \tag{A.5}$$

with multiplication being concatenation of tensors. The canonical inclusion ρ is then inclusion into the first summand. Given a linear map $\phi\colon V \to E$ into an algebra E as above, the unique algebra homomorphism Φ obtained from the universal property of TV can then be described as the linear extension of

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \phi(x_1)\phi(x_2)\cdots\phi(x_n) \tag{A.6}$$

for $x_i \in V$.

Specializing to algebras we can consider the tensor algebra TA over an algebra A. There is then a canonical ideal JA in TA which is the kernel of the natural surjective algebra homomorphism $TA \to A$ obtained by the universal property of TA applied to the identity $A \to A$.

For a complete locally convex vector space V with topology defined by a family of seminorms $\{p_i\}$ the tensor algebra TV becomes an m-algebra under completed projective tensor products

$$TV = V + V^{\hat{\otimes} 2} + V^{\hat{\otimes} 3} + \cdots \tag{A.7}$$

with respect to the family of seminorms

$$\hat{p}_i = p_i + p_i^{\hat{\otimes} 2} + p_i^{\hat{\otimes} 3} + \cdots \tag{A.8}$$

The canonical inclusion $\rho\colon V \to TV$ is continuous and for a continuous linear map $V \to E$ into an m-algebra E the resulting map, (A.6), obtained by the universal property will be continuous.

In the case of an m-algebra A, the resulting canonical ideal JA is also an m-algebra with the induced topology from TA.

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