JOTA manuscript No.
(will be inserted by the editor)

Constructive Solution of Inverse Parametric Linear/Quadratic Programming Problems

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the date of receipt and acceptance should be inserted later

Abstract Parametric convex programming has received a lot of attention, since it has many applications in chemical engineering, control engineering, signal processing, etc. Further, inverse optimality plays an important role in many contexts, e.g., image processing, motion planning. This paper introduces a constructive solution of the inverse optimality problem for the class of continuous piecewise affine functions. The main idea is based on the *convex lifting* concept. Accordingly, an algorithm to construct convex liftings of a given *convexly liftable partition* will be put forward. Following this idea, an important result will be presented in this article: any continuous piecewise affine function defined over a polytopic partition is the solution of a parametric linear/quadratic programming problem. Regarding linear optimal control, it will be shown that any continuous piecewise affine control law can be obtained via a linear optimal control problem with the control horizon at most equal to 2 prediction steps.

Keywords Convex liftings · Parametric convex programming · Inverse optimality · Piecewise affine functions

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1 Introduction

Parametric convex programming (PCP) has attracted significant attention due to its relevance in many related areas, e.g., computational geometry, operational research, control theory, etc. In particular, optimal solution of a linear/quadratic programming problem is known to be a piecewise affine (PWA) function defined over a polyhedral partition of the parameter space. In control theory, this structure of control laws appeared in the last decade as an approximation of the classical nonlinear control laws with respect to a predefined error [1–3]. Then, it is shown that this PWA structure is inherited by the exact optimal solution of a linear model predictive control (MPC) problem with respect to a linear/quadratic cost function [4–9].

Inverse parametric linear/quadratic programming aims to construct a linear constraint set and a linear/quadratic cost function such that the optimal solution of their associated optimization problem is equivalent to a given PWA function, defined over a given polyhedral partition. This inverse optimality problem has been investigated for some years and has resulted in interesting results for the general nonlinear continuous functions [10] and recently for continuous PWA functions [11, 12].

The authors in [10] proved that *every continuous feedback law can be obtained by PCP*. This is an insightful mathematical result; however, it remains purely theoretical; neither a *constructive procedure* nor a qualitative interpretation of the dimension of the optimization arguments is provided. The present work is motivated by a comment therein: *A natural question that can arise from this note would be to particularize our results to piecewise linear controllers: can any continuous piecewise linear feedback law be obtained by parametric linear programming?* The answer is positive and one solution to such an inverse optimality problem is recently found in [11] wherein an indirect solution, built upon a decomposition of a continuous PWA function into the difference of two continuous convex functions, is introduced.

In this paper, we present the results obtained using a different approach based on convex lifting. It will be proved that the proposed method can recover the given PWA function with at most one supplementary variable. The major contributions in this direction are: 1) the introduction of the convex lifting concept for use in the inverse optimality problem; 2) the convex liftability related condition for the existence of a solution of the inverse optimality problem; 3) a constructive procedure based on convex liftings for obtaining a solution of the inverse optimality problem.

The most important result related to Linear Optimal Control can be stated as follows: *any continuous piecewise affine control law can be recovered via a linear optimal control problem with a control horizon at most equal to 2 prediction steps.* The key concept used in the developments: the *lifting* can be defined as an inverse operation of orthogonal projection. As underlined by its definition, this operation allows lifting of a given partition onto a higher dimensional space. In particular, a so-called *convex lifting* of a given partition in $\mathbb{R}^d$ amounts to a convex surface in $\mathbb{R}^{d+1}$ such that each pair of neighboring regions are lifted onto two distinct hyperplanes and its image via the orthogonal projection onto $\mathbb{R}^d$ coincides with the given partition.

It is worth reminding that the lifting notion was introduced for the first time in Maxwell's research publications, e.g., [13] some 150 years ago. Later, a plethora of studies were dedicated for the existence conditions of such a *convex lifting* for a given partition [14–20]. However, most of these results are difficult to apply in numerical methods such as those usually employed in linear control design. On the other hand, control theory needs a systematic approach for the use of a lifting procedure in the inverse optimality problem. This aspect will be discussed in details in this paper.

2 Notation and Definitions

$\mathbb{R}, \mathbb{R}_+, \mathbb{N}_{>0}$ denote the field of real numbers, the non-negative real numbers set and the positive integer set, respectively. The following index set is also defined for ease of presentation, with a given $N \in \mathbb{N}_{>0}$, $\mathcal{I}_N := \{1, \ldots, N\}$.

A polyhedron is defined as the intersection of finite number of closed halfspaces. As a sequence, a polyhedron is a closed set. A polytope is defined as a bounded polyhedron. Given a full dimensional polytope $\mathcal{S}$, then $\mathcal{V}(\mathcal{S})$ denotes the set of its vertices, int$(\mathcal{S})$ denotes its interior. By $\dim(\mathcal{S})$, we denote the dimension of the affine hull of a given set $\mathcal{S}$. Also, conv$(\mathcal{S})$ denotes the convex hull of a given set $\mathcal{S}$. If $\mathcal{S}$ is an arbitrary set in $\mathbb{R}^d$ and $\mathbb{S}$ is a subspace of $\mathbb{R}^d$, then $\mathrm{Proj}_{\mathbb{S}}\, \mathcal{S}$ represents the orthogonal projection of $\mathcal{S}$ onto the space $\mathbb{S}$. Further, if $\mathcal{S} \subseteq \mathbb{R}^d$ is a full dimensional polyhedron, a face of $\mathcal{S}$ is the intersection of $\mathcal{S}$ and one of its supporting hyperplanes. A $k-$face represents a face of dimension k. A $0-$face is called a vertex, an $1-$face is called an edge, a $(d-1)-$face is called a facet. $\mathcal{F}(\mathcal{S})$ denotes the set of all facets of the polyhedron $\mathcal{S}$.

For a given $d \in \mathbb{N}_{>0}$, 0_d denotes a vector of dimension d whose elements are equal to 0. Similarly, $0_{m\times n}$ denotes a matrix in $\mathbb{R}^{m\times n}$ composed of the elements equal to 0. Let us recall also some useful definitions.

Definition 2.1 A collection of $N \in \mathbb{N}_{>0}$ full dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^d$, denoted by $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^d$ iff:

1. $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i$,
2. int$(\mathcal{X}_i) \bigcap$ int$(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i,j) \in \mathcal{I}_N^2$.

$(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours iff $(i,j) \in \mathcal{I}_N^2$, $i \neq j$ and dim$(\mathcal{X}_i \cap \mathcal{X}_j) = d-1$. Also, if $\mathcal{X}$ is a polytope then $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is called a *polytopic partition*.

The definition of a *cell complex* was presented by Grünbaum in [21]. For simplicity, a cell complex, in this paper, should be understood as a polyhedral partition whose face-to-face property is fulfilled, i.e., for any pair of regions, the intersection of faces is either empty or a common face. Accordingly, a polyhedral partition of a polyhedron is a cell complex if any pair of neighboring regions share a common facet.

Definition 2.2 For a given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, a *piecewise affine lifting* is described by the function $z : \mathcal{X} \to \mathbb{R}$ with:

$$z(x) = A_i^T x + a_i \ \text{ for any } x \in \mathcal{X}_i, \tag{1}$$

and $A_i \in \mathbb{R}^d$, $a_i \in \mathbb{R}$, $\forall i \in \mathcal{I}_N$.

Definition 2.3 Given a polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, a piecewise affine lifting: $z(x) = A_i^T x + a_i$ for $x \in \mathcal{X}_i$, is called a *convex piecewise affine lifting* iff the following conditions hold true:

– $z(x)$ is continuous over $\mathcal{X}$,
– for each $i \in \mathcal{I}_N$, $z(x) > A_j^T x + a_j$ for all $x \in \mathcal{X}_i \backslash \mathcal{X}_j$ and all $j \neq i,\ j \in \mathcal{I}_N$.

The second condition in the above definition implies that $z(x)$ is a convex function defined over $\mathcal{X}$. Moreover, the strict inequalities ensure that any pair of neighboring regions are lifted onto two distinct hyperplanes.

For ease of presentation, a slight abuse of notation is herefter used: a *convex lifting* is understood as a *convex piecewise affine lifting*. From the above definition, if a polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X}$ admits a convex lifting, then $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ has to be a cell complex. This observation is stated by the following proposition.

Proposition 2.1 *A polyhedral partition of a polyhedron, which admits a convex lifting, is a cell complex.*

Proof: Suppose the given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, which admits a convex lifting, is not a cell complex. By $z(x) = A_i^T x + a_i$ for $x \in \mathcal{X}_i$, we denote this convex lifting of $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$. Then there exists a pair of neighboring regions, denoted by $\mathcal{X}_i, \mathcal{X}_j$, whose facet-to-facet property is not fulfilled.

According to the definition of convex liftings, the hyperplane $\mathcal{H}_0$, described by

$$\mathcal{H}_0 := \left\{x \in \mathbb{R}^d : A_i^T x + a_i = A_j^T x + a_j\right\},$$

contains $\mathcal{X}_i \cap \mathcal{X}_j$. Also, due to the violation of the facet-to-facet property, there exists a point, denoted by x_0, such that $x_0 \in \mathcal{H}_0 \cap \mathcal{X}_i$ but $x_0 \notin \mathcal{X}_j$ (an illustration can be found in Fig.1). $x_0 \in \mathcal{H}_0$ implies $A_i^T x_0 + a_i = A_j^T x_0 + a_j$. On the other hand, $x_0 \in \mathcal{X}_i$, $x_0 \notin \mathcal{X}_j$ lead to $A_i^T x_0 + a_i > A_j^T x_0 + a_j$. These two last inclusions are clearly contradictory. Therefore, the partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ has to be a cell complex. □

According to this proposition, a convex lifting is always defined over a cell complex. However, the cell complex characterization of $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is a necessary condition, but not a sufficient condition for the existence of a convex lifting.

Definition 2.4 A given cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ in $\mathbb{R}^d$ has *an affinely equivalent polyhedron* iff there exists a polyhedron $\widetilde{\mathcal{X}} \subset \mathbb{R}^{d+1}$ such that for each $i \in \mathcal{I}_N$:

1. $\exists F_i \in \mathcal{F}(\widetilde{\mathcal{X}})$ satisfying: $\text{Proj}_{\mathbb{R}^d} F_i = \mathcal{X}_i$,
2. if $\underline{z}(x) := \min\limits_z z$ s.t. $\left[x^T\ z\right]^T \in \widetilde{\mathcal{X}}$, then $\left[x^T\ \underline{z}(x)\right]^T \in F_i$ for $x \in \mathcal{X}_i$.

An illustration can be found in Fig.2 where a cell complex in $\mathbb{R}$ consists of the multicolored segments along the horizontal axis. One of its affinely equivalent polyhedra in $\mathbb{R}^2$ is the pink shaded region. Moreover, the lower facets of this polytope are an illustration of the facets F_i appearing in Definition 2.4.

Note that, given a cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, affinely equivalent to a polyhedron $\widetilde{\mathcal{X}} \subset \mathbb{R}^{d+1}$, if z denotes the last coordinate of $\widetilde{\mathcal{X}}$ such that $\left[x^T\ z\right]^T \in \widetilde{\mathcal{X}}$, then $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is nothing other than the cell complex associated with the optimal solution to the following parametric linear programming problem:

$$z^*(x) = \min_z z \text{ subject to } \left[x^T\ z\right]^T \in \widetilde{\mathcal{X}}.$$

Also, $z^*(x)$ represents a convex lifting for this cell complex.

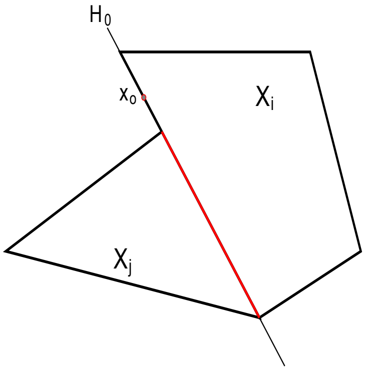

Fig. 1: An illustration for Proposition 2.1.

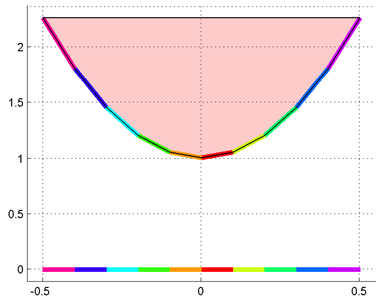

Fig. 2: An illustration of affinely equivalent polyhedron.

3 Problem Statement

3.1 Parametric Linear/Quadratic Programming Problems

Recall (see [4–8]) that a parametric linear/quadratic programming problem is defined as follows with respect to $d_x, d_U \in \mathbb{N}_{>0}$:

$$\min_U f(U,x) \quad \text{subject to:} \quad GU \leq W + Ex, \tag{2}$$

where $x \in \mathbb{R}^{d_x}$ represents the parameter vector, $U \in \mathbb{R}^{d_U}$ represents the decision variable and $f(U,x)$ represents a linear/quadratic cost function in U and x. The above problem has a continuous solution denoted as $U^*(x)$ (see [4]), known to be a piecewise affine function defined over a polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of the parameter space denoted as $\mathcal{X}$, as a polyhedron:

$$U^*(x) = f_{pwa}(x) = F_i x + G_i, \;\; \forall x \in \mathcal{X}_i. \tag{3}$$

Note that the optimal solution to a parametric quadratic programming problem is unique if $f(U,x)$ along U is strictly convex; see [4]. It is already known that this uniqueness may no longer be preserved in the case of a parametric linear programming problem. However, a continuous selection of optimal solution to such a linear problem is shown in [22] to exist. Conversely, given a continuous PWA function defined over a polyhedral partition, the question is whether there exists an optimization problem such that its optimal solution is equivalent to the given PWA function. The answer is shown in [10] to be affirmative, although the numerical construction of such an optimization problem is still open. A possible candidate for this optimization problem may be characterized by a linear/quadratic cost function and a set of linear constraints. For the moment, the definition of an inverse parametric linear/quadratic programming problem is introduced.

3.2 Inverse Parametric Linear/Quadratic Programming Problems

In principle, an inverse parametric linear/quadratic programming problem aims to reconstruct an appropriate optimization problem with respect to a given continuous piecewise affine function $u(x) = f_{pwa}(x)$, defined over a given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of the parameter space $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ such that the optimal solution of this reconstructed problem is equivalent[1] to the given PWA function $f_{pwa}(x)$. This problem can be stated as follows:
Problem statement: For a given polyhedral partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of the parameter space $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, associated with a continuous PWA function $f_{pwa}(x) : \mathcal{X} \to \mathbb{R}^{d_u}$, find a linear/quadratic cost function $J(x,z,u)$ and matrices H_x, H_u, H_z, K such that:

$$f_{pwa}(x) = \text{Proj}_{\mathbb{R}^{d_u}} \arg \min_{[z \ u^T]^T} J(x,z,u) \quad \text{s.t.} \quad H_x x + H_z z + H_u u \leq K. \quad (4)$$

The convex lifting based solution to such an inverse optimality problem is presented in the sequel. A definition of *invertibility* needs to be introduced in order to establish the working assumption of the convex lifting based method.

Definition 3.1 A continuous PWA function defined over a polyhedral partition is called *invertible* iff there exists an appropriate constraint set and a cost function such that their associated parametric convex programming problem admits the given continuous PWA function as its optimal solution.

4 Constructive Convex Lifting based Approach for Inverse Parametric Linear/Quadratic Programming

4.1 Existing Results on Convex Liftings

Many studies dedicated to the existence of convex liftings for the cell complexes in $\mathbb{R}^2$, were investigated, e.g., in [13–15, 19]. These results were then generalized to the cell complexes in the general dimensional space $\mathbb{R}^d$ through different studies, e.g., in [18]. It is shown therein that there exists a convex lifting for a cell complex in $\mathbb{R}^d$ if and only if one of the following holds:

- it admits a strictly positive $d-$stress;
- it is an additively weighted Dirichlet-Voronoi diagram;
- it is an additively weighted Delaunay decomposition;
- it is the section of a $(d+1)$-dimensional Dirichlet-Voronoi partition[2].

The above results cover the general class of cell complexes in $\mathbb{R}^d$. Unfortunately, despite the mathematical completeness of the existing results, the verification of these conditions are expensive. Furthermore, they do not provide any hint for the construction of a convex lifting. The next subsection presents such a construction in the general case of cell complexes.

[1] The equivalence hereafter means that the boundary between two regions of the parameter space partition, corresponding to two different affine functions, must be preserved and a subdivision or refinement of the regions corresponding to the same affine function is possible.

[2] Other related results can be found in Konstantin Rybnikov's thesis [18], equally in [17, 23–25].

4.2 Construction of Convex Liftings

In this subsection, the main objective is to present an algorithm for the construction of a convex lifting for a given cell complex via linear/quadratic programming. This algorithm exploits the continuity and the convexity of a convex lifting for neighboring regions. Note that we restrict our attention in this article to the polytopic partitions. Extensions of the results found here for cell complexes of an unbounded polyhedron have been recently given in [26].

Suppose we want to lift a given cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^d$. Let $\widetilde{\mathcal{X}} \subset \mathbb{R}^{d+1}$ be one of its affinely equivalent polyhedra. For each region $\mathcal{X}_i$ with $i \in \mathcal{I}_N$, the hyperplane $\mathcal{H}_i$, containing the lower facet of $\widetilde{\mathcal{X}}$, whose orthogonal projection onto $\mathbb{R}^d$ coincides with $\mathcal{X}_i$, has the following form:

$$\mathcal{H}_i = \left\{ \left[x^T \ z_i\right]^T \in \mathbb{R}^{d+1} : z_i = A_i^T x + a_i \right\}, \tag{5}$$

for suitable $A_i \in \mathbb{R}^d$, $a_i \in \mathbb{R}$.

Let $(i,j) \in \mathcal{I}_N^2$ be an index pair such that $(\mathcal{X}_i, \mathcal{X}_j)$ are neighbors. The *continuity conditions* between them are described as follows:

$$\forall x \in \mathcal{X}_i \cap \mathcal{X}_j, \ i \neq j, \ z_i(x) = z_j(x). \tag{6}$$

Moreover, the *convexity conditions* between them can be handled as:

$$\forall x \in \mathcal{X}_i \backslash (\mathcal{X}_i \cap \mathcal{X}_j), \ z_i(x) > z_j(x). \tag{7}$$

Algorithm 1 summarizes the constructive procedure which allows for the computation of the gains $(A_i,\, a_i)$, $\forall i \in \mathcal{I}_N$ of a convex lifting.

The following theorem serves as an explanation of this algorithm.

Theorem 4.1 *If the optimization problem* (10) *is feasible, then the function*

$$z(x) = A_i^T x + a_i \ \textit{ for } \ x \in \mathcal{X}_i$$

represents a convex lifting for the given cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$.

Proof: If the optimization problem (10) is feasible, then the continuity conditions on the function $z(x)$ and the convexity conditions of its epigraph are all fulfilled. Accordingly, for two neighboring regions $(\mathcal{X}_i, \mathcal{X}_j)$, it follows that:

$$\begin{aligned} A_i^T x + a_i &= A_j^T x + a_j \ \text{ for all } \ x \in \mathcal{X}_i \cap \mathcal{X}_j, \\ A_i^T x + a_i &> A_j^T x + a_j \ \text{ for all } \ x \in \mathcal{X}_i \backslash \mathcal{X}_j. \end{aligned} \tag{11}$$

The same inclusion holds for the other pairs of neighboring regions. This leads to the continuity of $z(x)$ and for each $i \in \mathcal{I}_N$:

$$A_i^T x + a_i > A_j^T x + a_j \ \text{ for all } \ x \in \mathcal{X}_i \backslash \mathcal{X}_j, \ \forall j \neq i, \, j \in \mathcal{I}_N. \tag{12}$$

Therefore, function $z(x) = A_i^T x + a_i$ for $x \in \mathcal{X}_i$ is a convex lifting defined over the cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, as defined in Definition 2.3. □

Algorithm 1 Construction of a convex lifting for a given cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^d$.

Input: $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ and a given constant $c > 0$.
Output: Gains $(A_i,\, a_i)$, $\forall i \in \mathcal{I}_N$.

1: Register all pairs of neighboring regions in $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$.
2: For each pair of neighboring regions $(\mathcal{X}_i,\, \mathcal{X}_j)$, $(i,\, j) \in \mathcal{I}_N^2$:
 – Add continuity conditions $\forall v \in \mathcal{V}(\mathcal{X}_i \cap \mathcal{X}_j)$:

$$A_i^T v + a_i = A_j^T v + a_j. \tag{8}$$

 – Add convexity conditions $\forall u \in \mathcal{V}(\mathcal{X}_i), u \notin \mathcal{V}(\mathcal{X}_j)$:

$$A_i^T u + a_i \geq A_j^T u + a_j + c. \tag{9}$$

3: Solve the following convex optimization problem by minimizing a chosen cost function, e.g.,

$$\min_{A_i,\, a_i} \sum_{i=1}^{N} (A_i^T A_i + a_i^2) \text{ subject to } (8), (9). \tag{10}$$

Note that the cost function chosen in (10) aims to avoid the unboundedness of the optimal solution. Other choices of this cost function are possible as long as the boundedness of the optimal solution is guaranteed. Also, as seen in (9), the strict convexity condition (7) can easily be transformed into inequality constraints in an optimization problem by adding a positive constant c on the right-hand side of (9), thus $>$ can be replaced with $\geq$. Theoretically, if the given cell complex is convexly liftable, then any choice of this positive constant does not have any effect on the feasibility of the optimization problem (10). Since (8) and (9) amount to

$$(\alpha A_i)^T v + (\alpha a_i) = (\alpha A_j)^T v + (\alpha a_j) \text{ for } v \in \mathcal{V}(\mathcal{X}_i \cap \mathcal{X}_j)$$
$$(\alpha A_i)^T u + (\alpha a_i) \geq (\alpha A_j)^T u + (\alpha a_j) + \alpha c \text{ for } u \in \mathcal{V}(\mathcal{X}_i), u \notin \mathcal{V}(\mathcal{X}_j),$$

for any $\alpha > 0$. In other words, $\widetilde{\ell}(x) = (\alpha A_i)^T x + (\alpha a_i)$ for $x \in \mathcal{X}_i$ also represents a convex lifting of cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ which may be resulted from Algorithm 1 with a given constant αc. Therefore, the optimization problem (10) is still feasible with the constant $\alpha c > 0$. Accordingly, the feasibility of the optimization problem (10) can serve as another necessary and sufficient condition for the existence of a convex lifting of a given cell complex. Furthermore, according to Proposition 2.1, the optimization problem (10) is infeasible for the polytopic partitions of polytopes whose facet-to-facet property is not fulfilled.

Note also that this construction of convex liftings relies on the vertices of the polytopes in $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, therefore the computation of its vertices, called vertex enumeration, is required. Although vertex enumeration does not scale exponentially with the dimension of the associated polytopes, see [27], its computation however becomes demanding as the dimension increases. Fortunately, this computation is performed offline, thus it is reasonable to assume that ample computational resources are available.

To illustrate Algorithm 1, a cell complex in $\mathbb{R}^2$ is presented in Fig.3. One of its affinely equivalent polyhedra is the shaded polytope with the lower facets multicolored.

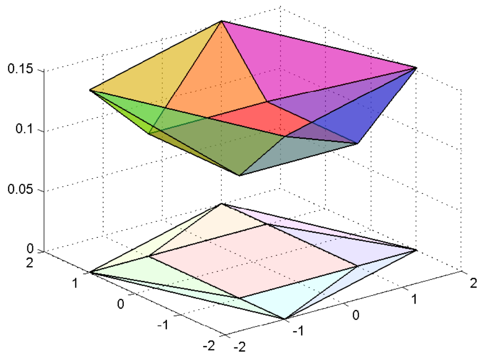

Fig. 3: A cell complex in $\mathbb{R}^2$ and one of its affinely equivalent polyhedra in $\mathbb{R}^3$.

4.3 Convexly Non-liftable Polyhedral Partitions

It is already known that the parameter partition, associated with the optimal solution to a parametric quadratic programming problem, may not be a cell complex but a polyhedral partition. This case usually takes place in linear model predictive control problems with respect to quadratic cost functions. Therefore, to solve the inverse optimality problem via convex liftings, it is necessary to treat such singular partitions in order that their convex liftability is retrieved. It is shown in [20, 28] that any polyhedral partition can be subdivided into a convexly liftable one provided that its internal boundaries are still preserved. This result is recalled here for completeness.

Theorem 4.2 ([20,28]) *Given a convexly non-liftable polyhedral partition* $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ *of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^d$*, there exists at least one subdivision, preserving the internal boundaries of this partition such that the new cell complex is convexly liftable.*

Interested readers can find details about the proof in [20, 28]. According to its proof, the *hyperplane arrangement* technique can be of use to carry out this goal. Practically, hyperplane arrangement is only one approach to show the existence of modifications for the given convexly non-liftable polyhedral partition into a convexly liftable one. In control theory, such a modification can increase the complexity of PWA control laws in the implementation. However, such a *complete* refinement may not be necessary in practice. Many different refinement techniques exist. We refer to [29] for an alternative technique for a class of particular cases in control theory.

5 Solution to Inverse Parametric Linear/Quadratic Programming Problems

The definition of an inverse parametric linear/quadratic programming problem has been introduced in Subsection 3.2. The solution to such inverse optimality problems is built in this paper upon the convex lifting approach. First, some regularity assumptions need to be stated to make this approach reasonable from a construction point of view. These assumptions are stated with respect to the notation in Subsection 3.2.

Assumption 5.1 *The parametric linear/quadratic programming problems are exclusively considered as possible candidates for solutions to the inverse optimality problem. As a consequence, the cost function has the following form:*

$$J(x,z,u) = \left[x^T\ z\ u^T\right] Q \left[x^T\ z\ u^T\right]^T + C^T \left[x^T\ z\ u^T\right]^T, \tag{13}$$

with positive semidefinite matrix $Q^T = Q \geq 0$.

Assumption 5.2 *The polytopic partition* $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$*, associated with a given continuous PWA function, is convexly liftable.*

Assumption 5.3 *The parameter space* $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i$ *is a polytope.*

Note that Assumption 5.1 provides a manageable framework for the constructive inverse optimality procedure. Larger classes of objective functions can provide more degrees of freedom, but the linearity of such parametric convex programming problems is lost. Assumption 5.3 restricts the inverse optimality problem to bounded feasible region given by a polytope, since linear constraints are exclusively of interest. Also, the construction presented below can easily be extended to polyhedral partitions of polyhedra, see [26]. Assumption 5.2 is not restrictive, since the convex liftability of the given polytopic partition can be enforced by refinement procedures according to Theorem 4.2. Note also that in the scope of this paper, we restrict our attention to the class of continuous PWA functions. It will be shown that due to this continuity property, the optimal solution to the recovered optimization problem is unique. Inverse optimality for the class of discontinuous PWA functions is studied in [30]. In this case, it is however shown that the uniqueness of the optimal solution to the recovered optimization problem is lost.

The following intermediate result is necessary for the development of a constructive solution to the inverse optimality problem.

Proposition 5.1 *Let* $\Gamma_s \subset \mathbb{R}^{d_s}$ *be a full dimensional polytope with the set of vertices* $\mathcal{V}(\Gamma_s) = \{s^{(1)}, \ldots, s^{(q)}\}$*. For any finite set of points* $\{t^{(1)}, \ldots, t^{(q)}\} \subset \mathbb{R}^{d_t}$ *defining a full dimensional polytope in* $\mathbb{R}^{d_t}$*, an extension of the family* $\mathcal{V}(\Gamma_s)$ *can be obtained* *in higher dimensional space* $\mathbb{R}^{d_s+d_t}$ *for the concatenated vectors* $\begin{bmatrix} s^T & t^T \end{bmatrix}^T$ *defining the set:*

$$V_{[s^T \ t^T]^T} := \left\{ \begin{bmatrix} s^{(1)} \\ t^{(1)} \end{bmatrix}, \dots, \begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix} \right\}.$$

The polytope $\Gamma_{[s^T \ t^T]^T} = \text{conv}(V_{[s^T \ t^T]^T})$ *satisfies:* $V_{[s^T \ t^T]^T} = \mathcal{V}(\Gamma_{[s^T \ t^T]^T})$.

Proof: Geometrically, this proposition shows that if $s^{(i)}$ is a vertex of $\Gamma_s \subset \mathbb{R}^{d_s}$, then with any complementary vector $t^{(i)} \in \mathbb{R}^{d_t}$ yielding vector $\begin{bmatrix} s^{(i)} \\ t^{(i)} \end{bmatrix} \in \mathbb{R}^{d_s+d_t}$, this vector represents a vertex of the new polytope $\Gamma_{[s^T \ t^T]^T}$ in $\mathbb{R}^{d_s+d_t}$ defined as the convex hull of the extended set of points $V_{[s^T \ t^T]^T}$. By construction, it can be observed that $\mathcal{V}(\Gamma_{[s^T \ t^T]^T}) \subseteq V_{[s^T \ t^T]^T}$. Therefore, in order to prove this claim, we will show that $\mathcal{V}(\Gamma_{[s^T \ t^T]^T}) \subset V_{[s^T \ t^T]^T}$ leads to a contradiction.

In fact, suppose $\mathcal{V}(\Gamma_{[s^T \ t^T]^T}) \subset V_{[s^T \ t^T]^T}$. According to this assumption, there exists a point in $V_{[s^T \ t^T]^T}$ which lies in the interior of the polytope $\Gamma_{[s^T \ t^T]^T}$ or can be described by a convex combination of the other points. Without loss of generality, let $\begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix}$ denote this point, then there exists a vector $\alpha \in \mathbb{R}_+^{q-1}$ such that:

$$\begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix} = \sum_{i=1}^{q-1} \alpha_i \begin{bmatrix} s^{(i)} \\ t^{(i)} \end{bmatrix}, \quad \sum_{i=1}^{q-1} \alpha_i = 1. \tag{14}$$

One can easily see from (14) that $s^{(q)}$, as a vertex of Γ_s, is described by a convex combination of the other vertices of Γ_s. This inclusion is contradictory to the definition of a vertex of a convex set, see [21]. In other words, all elements of $V_{[s^T \ t^T]^T}$ are the vertices of $\Gamma_{[s^T \ t^T]^T}$. □

Remark 5.1 Note also that this proposition remains valid for the degenerate case where all points $\{t^{(1)}, \dots, t^{(q)}\}$ are placed on a hyperplane in $\mathbb{R}^{d_t}$. In this case, the dimension of polytope $\Gamma_{[s^T \ t^T]^T}$ however becomes lower than $d_s + d_t$. These particular cases of values $t^{(i)}, \forall i \in \mathcal{I}_q$, are excluded in the previous result as not relevant for the scope of this paper, even though the mathematical result holds.

Consider a given cell complex $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^{d_x}$ satisfying Assumption 5.2 and a continuous PWA function $f_{pwa}(x) : \mathcal{X} \to \mathbb{R}^{d_u}$ defined over this cell complex. For ease of presentation, let $\ell(x)$ denote a convex lifting for $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$. Define also the following sets:

$$\begin{aligned}
\Pi_{[x^T \ z]^T} &:= \text{conv} \left\{ \begin{bmatrix} v^T & \ell(v) \end{bmatrix}^T : v \in \bigcup_{i \in \mathcal{I}_N} \mathcal{V}(\mathcal{X}_i) \right\}, \\
V_{[x^T \ z \ u^T]^T} &:= \left\{ \begin{bmatrix} v^T & \ell(v) & f_{pwa}^T(v) \end{bmatrix}^T : v \in \bigcup_{i \in \mathcal{I}_N} \mathcal{V}(\mathcal{X}_i) \right\}, \\
\Pi_{[x^T \ z \ u^T]^T} &:= \text{conv} \left(V_{[x^T \ z \ u^T]^T} \right).
\end{aligned} \tag{15}$$

With respect to the above notation, the solution to an inverse parametric linear/quadratic programming problem can be stated as follows.

Theorem 5.4 *Given a continuous PWA function $f_{pwa}(x)$ defined over a polytopic partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ satisfying Assumptions 5.2, 5.3 and the sets defined in* (15)*, the following hold true:*

1. $V_{[x^T\ z\ u^T]^T} = \mathcal{V}(\Pi_{[x^T\ z\ u^T]^T})$ *and* $\Pi_{[x^T\ z]^T} = \mathrm{Proj}_{[x^T\ z]^T}\Pi_{[x^T\ z\ u^T]^T}$,
2. *The given PWA function $f_{pwa}(x)$ is the image via the orthogonal projection onto $\mathbb{R}^{d_u}$ of the optimal solution to the optimization problem below:*

$$\min_{[z\ u^T]^T} z \ \ s.t. \ \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T}. \tag{16}$$

Proof: 1. The first claim: $V_{[x^T\ z\ u^T]^T} = \mathcal{V}(\Pi_{[x^T\ z\ u^T]^T})$, is directly deduced from Proposition 5.1. The second claim follows from the construction of $\Pi_{[x^T\ z\ u^T]^T}$ having all its vertices as extended vectors of the vertices of $\Pi_{[x^T\ z]^T}$.
2. It is known that $\Pi_{[x^T\ z]^T} \subset \mathbb{R}^{d_x+1}$ represents an affinely equivalent polyhedron of the partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$. Let $F^{(i)}_{[x^T\ z]^T}$ for $i \in \mathcal{I}_N$ denote the lower facet of $\Pi_{[x^T\ z]^T}$ such that: $\mathrm{Proj}_x F^{(i)}_{[x^T\ z]^T} = \mathcal{X}_i$ and for any $x \in \mathcal{X}_i$, $\left[x^T\ \ell(x)\right]^T \in F^{(i)}_{[x^T\ z]^T}$ satisfies

$$\ell(x) = \min_z z \ \text{ s.t. } \ \left[x^T\ z\right]^T \in \Pi_{[x^T\ z]^T}.$$

Such a facet $F^{(i)}_{[x^T\ z]^T}$ can be represented by:

$$F^{(i)}_{[x^T\ z]^T} = \mathrm{conv}\left\{\left[v^T\ \ell(v)\right]^T : v \in \mathcal{V}(\mathcal{X}_i)\right\}.$$

Also, there exists, in higher dimensional space $\mathbb{R}^{d_x+d_u+1}$, a d_x-face denoted as $F^{(i)}_{[x^T\ z\ u^T]^T}$ of $\Pi_{[x^T\ z\ u^T]^T}$ such that: $\mathrm{Proj}_{[x^T\ z]^T} F^{(i)}_{[x^T\ z\ u^T]^T} = F^{(i)}_{[x^T\ z]^T}$. Thus, a point $\left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T}$ satisfying $x \in \mathcal{X}_i$ has the minimal value of z if and only if this point locates on $F^{(i)}_{[x^T\ z\ u^T]^T}$. It is worth stressing that the face $F^{(i)}_{[x^T\ z\ u^T]^T}$ is defined as follows:

$$F^{(i)}_{[x^T\ z\ u^T]^T} := \mathrm{conv}\left\{\left[v^T\ \ell(v)\ f^T_{pwa}(v)\right]^T : v \in \mathcal{V}(\mathcal{X}_i)\right\}.$$

Accordingly, if $x \in \mathcal{X}_i$ is represented by $x = \sum_{v\in\mathcal{V}(\mathcal{X}_i)} \alpha(v)v$ for $\alpha(v) \geq 0$ and $\sum_{v\in\mathcal{V}(\mathcal{X}_i)} \alpha(v) = 1$, then the optimal solution $\left[z^*(x)\ (u^*)^T(x)\right]^T$ to the problem (16) is expressed by:

$$\begin{bmatrix} z^*(x) \\ u^*(x) \end{bmatrix} = \sum_{v\in\mathcal{V}(\mathcal{X}_i)} \alpha(v) \begin{bmatrix} \ell(v) \\ f_{pwa}(v) \end{bmatrix} = \begin{bmatrix} \ell(x) \\ f_{pwa}(x) \end{bmatrix}, \ \ \forall x \in \mathcal{X}_i.$$

Clearly, $f_{pwa}(x)$ is a sub-component of this optimal solution.

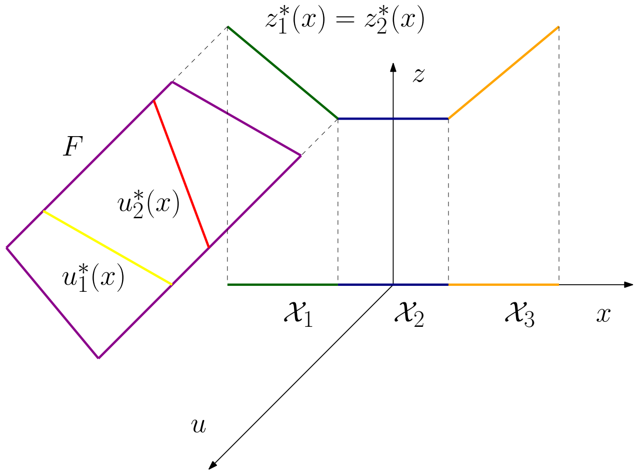

Fig. 4: An illustration of two different optimal solutions.

To complete the proof, the uniqueness of such an optimal solution needs to be clarified. Suppose there exist two different optimal solutions to (16):

$$\left[z_1^*(x)\ (u_1^*)^T(x)\right]^T = \arg\min_{[z\ u^T]^T} z, \quad \text{s.t.} \quad \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T},$$
$$\left[z_2^*(x)\ (u_2^*)^T(x)\right]^T = \arg\min_{[z\ u^T]^T} z, \quad \text{s.t.} \quad \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T},$$

then it is clear that $z_1^*(x) = z_2^*(x) = \ell(x)$. Accordingly, if $u_1^*(x) \neq u_2^*(x)$ for $x \in \mathcal{X}_i$, there exists a $(d_x+1)-$face denoted as F of $\Pi_{[x^T\ z\ u^T]^T}$ (illustrated in Fig.4) to which two optimal solutions $\left[z_1^*(x)\ (u_1^*)^T(x)\right]^T$ and $\left[z_2^*(x)\ (u_2^*)^T(x)\right]^T$ belong such that F is perpendicular to the space $\left[x^T\ z\right]^T$. This implies that the value of $f_{pwa}(v)$ is not uniquely defined for vertices $v \in \mathcal{V}(\mathcal{X}_i)$. This consequence contradicts the construction of the constraint set $\Pi_{[x^T\ z\ u^T]^T}$ presented in (15). Therefore, such two optimal solutions have to be identical, leading to the uniqueness. □

The constructive procedure towards recovering a continuous PWA function defined over a convexly liftable polytopic partition is summarized through Algorithm 2.

Theorem 5.4 proves the existence of an optimization problem with respect to a linear cost function which admits a given PWA function as a sub-component of the optimal solution. The following theorem shows the existence of equivalent optimization problem with respect to a *quadratic cost function*.

Theorem 5.5 *Consider a continuous PWA function $f_{pwa}(x)$ defined over a polytopic partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ satisfying Assumptions 5.2, 5.3 and the sets defined in* (15). *Function $f_{pwa}(x)$ is the image via the orthogonal projection onto $\mathbb{R}^{d_u}$ of the optimal solution to the following optimization problem:*

$$\min_{[z\ u^T]^T} (z - \sigma(x))^2 \ \ \textit{s.t.} \ \ \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T}, \tag{17}$$

Algorithm 2 Linear equivalent optimization problem

Input: A continuous PWA function $f_{pwa}(x)$ defined over a convexly liftable polytopic partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^{d_x}$.
Output: $\Pi_{[x^T\ z\ u^T]^T}$ and $J(x,z,u)$.

1: Construct a convex lifting $\ell(x)$ for $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ via Algorithm 1.
2: Compute $\Pi_{[x^T\ z\ u^T]^T}$ as in (15).
3: Define $J(x,z,u) = z$.
4: Solve the following parametric linear programming problem:

$$\left[z^*(x)\ (u^*)^T(x)\right]^T = \arg\min_{[z\ u^T]^T} z \ \text{ subject to } \ \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T}.$$

5: Obtain the given PWA function: $\text{Proj}_u \begin{bmatrix} z^*(x) \\ u^*(x) \end{bmatrix} = f_{pwa}(x)$.

where $\sigma(x) : \mathcal{X} \to \mathbb{R}$ *denotes any function satisfying:* $\sigma(x) \le \ell(x)$.

Proof: Consider an affinely equivalent polyhedron $\Pi_{[x^T\ z]^T}$ defined as in (15). According to its definition, we obtain:

$$\ell(x) = \min_z z \ \text{ subject to } \ \left[x^T\ z\right]^T \in \Pi_{[x^T\ z]^T}.$$

Therefore, for any function $\sigma(x) : \mathcal{X} \to \mathbb{R}$ satisfying $\sigma(x) \le \ell(x)$, the minimization of $(z-\sigma(x))^2$ amounts to the minimization of z subject to the same set of constraints $\Pi_{[x^T\ z\ u^T]^T}$. According to Theorem 5.4, the given continuous PWA function $f_{pwa}(x)$ is a sub-component of the optimal solution to (16) as well as (17). □

Algorithm 3 summarizes the constructive procedure of an equivalent optimization problem with respect to a quadratic cost function.

Algorithm 3 Quadratic equivalent optimization problem

Input: A continuous PWA function $f_{pwa}(x)$ defined over a convexly liftable polytopic partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^{d_x}$.
Output: $\Pi_{[x^T\ z\ u^T]^T}$ and $J(x,z,u)$.

1: Construct a convex lifting $\ell(x)$ for $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ via Algorithm 1.
2: Compute $\Pi_{[x^T\ z\ u^T]^T}$ as in (15).
3: Choose a function[3] $\sigma(x) : \mathcal{X} \to \mathbb{R}$ such that $\sigma(x) \le \ell(x)$.
4: Define $J(x,z,u) = (z-\sigma(x))^2$.
5: Solve the following parametric quadratic programming problem:

$$\left[z^*(x)\,(u^*)^T(x)\right]^T = \arg\min_{[z\ u^T]^T} (z-\sigma(x))^2 \ \text{ s.t. } \ \left[x^T\ z\ u^T\right]^T \in \Pi_{[x^T\ z\ u^T]^T}.$$

6: Project the optimal solution onto $\mathbb{R}^{d_u}$: $\text{Proj}_u \begin{bmatrix} z^*(x) \\ u^*(x) \end{bmatrix} = f_{pwa}(x)$.

Remark 5.2 Theorem 5.5 proposes a cost function of z and x. To return this cost function to the form (13), function $\sigma(x)$ should be chosen as an affine function of x.

We will present in the sequel the important properties of the solution to inverse parametric linear/quadratic programming problems via convex liftings, i.e., the *invertibility* and the *complexity* of the above constructive procedures.

Theorem 5.6 *(Invertibility) Given a polytopic partition $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^{d_x}$, then any continuous PWA function $f_{pwa}(x) : \mathcal{X} \to \mathbb{R}^{d_u}$, defined over $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, is invertible.*

The proof of Theorem 5.6 is omitted, as it summarizes the above results. The complexity of an inverse parametric linear/quadratic programming problem based on convex liftings is also stated as follows:

Theorem 5.7 *(Complexity) Any continuous PWA function defined over a polytopic partition of a polytope can be equivalently obtained by a parametric linear/quadratic programming problem with at most one auxiliary $1-$dimensional variable.*

Proof: Let $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ denote this given polytopic partition of a polytope $\mathcal{X}$. If $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$ is convexly liftable, this $1-$dimensional variable describes the convex lifting in the recovered optimization problem. Theorems 5.4, 5.5 show that this PWA function is invertible through the convex lifting based approach.

Otherwise, in case the given partition is not convexly liftable, Theorem 4.2 shows that there exists at least one way to subdivide the given convexly non-liftable polytopic partition into a convexly liftable cell complex, denoted by $\left\{\widetilde{\mathcal{X}}_i\right\}_{i\in\mathcal{I}_{\widetilde{N}}}$, meanwhile the internal boundaries are maintained. According to this subdivision, the given PWA function $f_{pwa}(x)$ is also subdivided. This new PWA function, say $\widetilde{f_{pwa}}(x)$, is equivalent to $f_{pwa}(x)$ and defined over a convexly liftable cell complex $\left\{\widetilde{\mathcal{X}}_i\right\}_{i\in\mathcal{I}_{\widetilde{N}}}$. Therefore, similar to the first case, a convex lifting of $\left\{\widetilde{\mathcal{X}}_i\right\}_{i\in\mathcal{I}_{\widetilde{N}}}$, represents the $1-$dimensional auxiliary variable. Also, as proved in Theorems 5.4, 5.5, $\widetilde{f_{pwa}}(x)$, associated with $\left\{\widetilde{\mathcal{X}}_i\right\}_{i\in\mathcal{I}_{\widetilde{N}}}$, is invertible via the convex lifting based method. □

6 Applications to Linear Optimal Control

Consider a linear, discrete time-invariant system:

$$x_{k+1} = Ax_k + Bu_k$$

where x_k, u_k denote the state and control variables at time k and A, B represent matrices of suitable dimension. A linear optimal control problem with a prediction horizon $N \in \mathbb{N}_{>0}$ for the above system can be written in the following form:

$$\begin{aligned} U^*(x) = &\operatorname*{argmin}_{U} F(x_{k|k}, \ldots, x_{k+N|k}, u_{k|k}, \ldots, u_{k+N-1|k}) \\ &\text{s.t. } G(x_{k|k}, \ldots, x_{k+N|k}, u_{k|k}, \ldots, u_{k+N-1|k}) \le 0, \end{aligned} \tag{18}$$

[3] One can choose $\sigma(x)$ to be an affine function composing $\ell(x)$, i.e., $\sigma(x) = A_i^T x + a_i$.

where $x_{k+i|k} \in \mathbb{R}^{d_x}, u_{k+i|k} \in \mathbb{R}^{d_u}$ are the state variable, control variable, respectively, at time $k+i$, predicted at time k; $U = \begin{bmatrix} u_{k|k}^T \dots u_{k+N-1|k}^T \end{bmatrix}^T$ denotes the decision variable composed of the control variables over the prediction horizon; and $x_{k|k} = x_k, u_{k|k} = u_k$. Also, $G(x_{k|k}, \dots, x_{k+N|k}, u_{k|k}, \dots, u_{k+N-1|k})$ represents a linear function along its variables, $F(x_{k|k}, \dots, x_{k+N|k}, u_{k|k}, \dots, u_{k+N-1|k})$ represents a linear/quadratic, real-valued function along its variables. In implementation, the interest of the optimal solution $U^*(x)$ is restricted to the first part, i.e., u_k. The main message of this paper is stated in the sequel.

Proposition 6.1 *Any continuous PWA control law is the optimal solution to a parametric linear/quadratic programming problem.*

Proof: The proof directly follows from Theorem 5.6 since a continuous PWA control law is a continuous PWA function defined over a polyhedral partition.

Theorem 6.1 *Any continuous PWA control law can be equivalently obtained through a linear optimal control problem with a linear or quadratic cost function and the control horizon at most equal to 2 prediction steps.*

Proof: Let $u(x) : \mathbb{R}^{d_x} \to \mathbb{R}^{d_u}$ denote a given continuous PWA control law defined over a state space partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^{d_x}$. If $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ is not convexly liftable, it can be subdivided into a convexly liftable cell complex according to Theorem 4.2. Therefore, one can exclusively focus on the case $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ is convexly liftable.

Now, let $\Pi_{\left[x_k^T \ z \ u_k^T\right]^T}$ denote the set of constraints in the recovered optimization problem, i.e.,

$$\min_{\left[z \ u_k^T\right]^T} z \ \text{ s.t. } \ \left[x_k^T \ z \ u_k^T\right]^T \in \Pi_{\left[x_k^T \ z \ u_k^T\right]^T}. \tag{19}$$

For ease of presentation, let $\Pi_{\left[x_k^T \ z \ u_k^T\right]^T}$ be given in the following form:

$$H_x x_k + H_z z + H_u u_k \leq K.$$

If $d_u = 1$, then it suffices to use z as the second predicted controller, i.e., $u_{k+1|k} = z$. Otherwise, the set of constraints $H_x x_k + H_u u_k + H_z z \leq K$ amounts to the following constraints:

$$H_x x_k + H_u u_k + [H_z \ 0] \left[z \ \ s^T\right]^T \leq K, \tag{20}$$

where 0 denotes a matrix of appropriate dimension, composed of zeros with the number of columns equal to $d_u - 1$. Also, $s \in \mathbb{R}^{d_u - 1}$ denotes auxiliary variable. Again, apply $\begin{bmatrix} z \\ s \end{bmatrix}$ for the next predicted control variable, i.e., $u_{k+1|k} = \begin{bmatrix} z \\ s \end{bmatrix}$. Accordingly, (19) can be written as follows:

$$\min_{\left[u_k^T \ u_{k+1|k}^T\right]^T} \left[0_{d_u}^T \ 1 \ 0_{d_u-1}^T\right] \begin{bmatrix} u_k \\ u_{k+1|k} \end{bmatrix} \ \text{ s.t. } \ H_x x_k + [H_u \ H_z \ 0] \begin{bmatrix} u_k \\ u_{k+1|k} \end{bmatrix} \leq K,$$

known to be a linear optimal control problem with respect to a linear cost function.

On the other hand, according to Theorem 5.5, the recovered optimization problem with a quadratic cost function can also be written in the following form:

$$\min_{\left[z\ u_k^T\right]^T} (z - \sigma(x_k))^2 \text{ s.t. } \left[x_k^T\ z\ u_k^T\right]^T \in \Pi_{\left[x_k^T\ z\ u_k^T\right]^T}, \tag{21}$$

where $\sigma(x_k) \leq \ell(x_k)$, $\ell(x_k)$ denotes the convex lifting for the given cell complex $\{\mathcal{X}_i\}_{i\in\mathcal{I}_N}$, used to compute $\Pi_{\left[x_k^T\ z\ u_k^T\right]^T}$. Suppose $\ell(x_k) = A_i^T x_k + a_i$ for $x_k \in \mathcal{X}_i$, it suffices to choose $\sigma(x_k) = A_i^T x_k + a_i$ over $\mathcal{X}$. Accordingly, similar to the case of linear optimal control with a linear cost function, (21) can be easily written in the form of linear optimal control with respect to a quadratic function of $\left[u_k^T\ u_{k+1|k}^T\right]^T$.

To complete the proof, we will show that (20) can be described as the constraint set in (18). In fact, as $G(x_{k|k}, \ldots, x_{k+N|k}, u_{k|k}, \ldots, u_{k+N-1|k})$ is a linear function of its variables, with a prediction horizon 2, this constraint set can be written as follows: $H_1 x_k + H_2 u_k + H_3 x_{k+1|k} + H_4 u_{k+1|k} + H_5 x_{k+2|k} \leq H_6$.

According to the given dynamics and some trivial transformations, (20) yields:

$$\begin{aligned} &H_1 + H_3 A + H_5 A^2 = H_x,\ H_4 + H_5 B = [H_z\ 0]\,, \\ &H_2 + H_3 B + H_5 AB = H_u,\ H_6 = K. \end{aligned} \tag{22}$$

The feasibility of the equations in (22) is obvious according to a trivial solution $H_3 = 0, H_5 = 0$. The choice of matrices H_3, H_5 of suitable dimension represents a degree of freedom in the recovered optimization problem and its equivalent form of an optimal control problem. The proof is complete. □

Remark 6.1 Note that the second predicted control variable $u_{k+1|k}$ in the recovered optimization problem does not necessarily satisfy the given constraints on the control variable, as it will not be implemented. Therefore, we can use it as an *artificial* variable. However, making $u_{k+1|k}$ fulfill the constraints on control variable, usually in the form: $u_{\min} \leq u_{k+i|k} \leq u_{\max}$, can be adapted from the construction of convex lifting by fixing the upper and lower bounds for its value.

Remark 6.2 In the context of generalized predictive control (GPC), it has been shown in [31] that a large prediction horizon can be beneficial for complex plants. In general, the nature of GPC does not take constraints into account, closed-loop stability guarantee being verified at post-design stage. Therefore, the quality of control algorithm relies mainly on the prediction horizon. It is worth noting that inverse optimality problem for such a GPC design can easily be obtained, as the controller resulted from a GPC problem possesses a linear structure. On the other hand, model predictive control guarantees closed-loop stability due to suitable terminal constraints imposed at the end of prediction horizon and only the first part of its optimal solution is of interest in implementation. Accordingly, the solution of inverse optimality problem presents *in a compact manner* a set of constraints, considering the input to be implemented and an auxiliary variable as the second control action. That is the main reason for which the inverse optimality problem only needs a prediction horizon of 2 steps.

Remark 6.3 The reduction of prediction horizon to 2 has some consequences in the computation of the associated controller. A remarkable one is that a linear programming problem shares the same first part of its optimal solution with a quadratic programming problem. It is known that a quadratic programming problem is in general more computationally demanding than a linear one. To illustrate this point, we refer the reader to [29, 32] for more detailed analyses and case studies. Note however that this computational complexity reduction is sometimes obtained at the price of a larger number of constraints as seen in the numerical example in Subsection 7.2. As for the implementation of PWA controllers, the constructed convex lifting turns out to be helpful to avoid storing state-space partition at the hardware level and facilitate the point-location problem [28, 33]. Accordingly, it allows for implementation of PWA controllers onto embedded systems with low computational performance and memory footprint.

Remark 6.4 It is shown in [4–6] that in the case of non-degeneracies, the optimal cost function of a parametric quadratic programming problem represents a strictly convex, piecewise quadratic function (this can also be proven using dynamic programming). In the context of inverse optimality, the idea of using this optimal cost function as an auxiliary variable is proposed in [10] and is recalled here. For ease of presentation, let $\mu(x)$ denote the controller defined over a convex set Ω to be recovered and $g^0(x)$ represent a strictly convex function over Ω. Accordingly, the set of constraints denoted therein by E for inverse optimality problem can be computed as follows:

$$S := \left\{(x, \mu(x), t) : x \in \Omega, g^0(x) \leq t\right\}, \quad E := \mathrm{conv}(S). \tag{23}$$

It can be seen that taking the present controller into account, t as an auxiliary variable is also used to represent the second control action. In the case of non-degenerate parametric quadratic programming, its optimal cost function, being strictly convex and piecewise quadratic, can be used as $g^0(x)$. Note also that the controller $\mu(x)$ in this case is a continuous PWA function. Unfortunately, it becomes very difficult to be able to obtain an explicit representation of the constraint set E as in (23). First, it is not clear whether E could be described by a set of quadratic constraints. Second, to our best knowledge, solvers for parametric programming problems subject to quadratic constraints are still unavailable; instead, only approximate optimal solution can be obtained. These obstacles prevent this optimal cost function from the application in the inverse optimality problem. It is worth stressing that a convex lifting used in this paper is not strictly convex, but convex, as there does not exist a strictly convex, piecewise affine function. Moreover, the recovered optimization problem is shown to have the unique optimal solution which was not treated in [10].

7 Illustrative Examples

This section considers numerical examples to illustrate the above results.

7.1 Double Integrator System

To illustrate the above results, consider a double integrator system:

$$x_{k+1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \tag{24}$$

An MPC problem, constructed with the minimization of a quadratic cost function over a prediction horizon $N = 5$, is presented as follows, with respect to weighting matrices $Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, $R = 0.5$:

$$J = \sum_{i=0}^{4} (x_{k+i|k}^T Q x_{k+i|k}^T + u_{k+i|k}^T R u_{k+i|k}) + x_{k+5|k}^T P x_{k+5|k},$$

where P is computed via the Riccati equation. Constraints on the present control variable and output signal are given by:

$$u_k \in [-2,\ 2]\,,\ y_k \in [-5,\ 5]\,. \tag{25}$$

The terminal constraint is chosen as the maximal output admissible set as shown in [34]. The feedback control law is depicted in Fig.7. Its inverse optimization problem in form (4) can be composed of the components presented in (27). Accordingly, the new linear optimal control problem with the control horizon of 2 is described as follows:

$$\min_{\begin{bmatrix} u_k & u_{k+1|k} \end{bmatrix}^T} [0\ 1] \begin{bmatrix} u_k \\ u_{k+1|k} \end{bmatrix} \quad \text{s.t.} \;\; H_x x_k + H_u u_k + H_z u_{k+1|k} \le K, \tag{26}$$

where H_x, H_u, H_z, K are presented in (27).

Consider the same MPC problem but with a prediction horizon 3, we obtain the inverse optimization problem in form (4) associated with the ingredients in (28). Finally, an equivalent linear optimal control problem with the prediction horizon 2 is represented as in (26) with the information in (28).

$$J(x, z, u) = z$$

$$H_x = \begin{bmatrix} 0.5423 & 0.8255 \\ 0 & 0 \\ 0.5411 & 0.8266 \\ -0.4193 & -0.8030 \\ -0.4442 & -0.7078 \\ 0.4378 & 0.8465 \\ 0 & 0 \\ -0.4444 & -0.7067 \\ 0.4333 & 0.8472 \\ -0.4161 & -0.8051 \\ 0.4313 & 0.8475 \\ -0.4144 & -0.8057 \\ 0.4293 & 0.8477 \\ -0.4128 & -0.8061 \\ -0.1863 & -0.0931 \\ -0.1622 & -0.1622 \\ -0.1339 & -0.2008 \\ -0.0621 & -0.2527 \\ 0.1961 & 0 \\ 0.1863 & 0.0931 \\ 0.1622 & 0.1622 \\ 0.1339 & 0.2008 \\ 0.0621 & 0.2527 \\ -0.1961 & 0 \end{bmatrix},\ H_z = \begin{bmatrix} -0.1565 \\ 0 \\ -0.1546 \\ -0.0266 \\ -0.0881 \\ -0.0308 \\ 0 \\ -0.0887 \\ -0.0295 \\ -0.0256 \\ -0.0291 \\ -0.0253 \\ -0.0288 \\ -0.0251 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ H_u = \begin{bmatrix} 0.0039 \\ -1 \\ 0.0087 \\ -0.4228 \\ -0.5422 \\ 0.3014 \\ 1 \\ -0.5434 \\ 0.3058 \\ -0.4220 \\ 0.3080 \\ -0.4224 \\ 0.3102 \\ -0.4232 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ K = \begin{bmatrix} -3.9966 \\ 2 \\ -3.9564 \\ -0.2669 \\ -2.2547 \\ -0.3090 \\ 2 \\ -2.2667 \\ -0.2489 \\ -0.2163 \\ -0.2256 \\ -0.1964 \\ -0.2045 \\ -0.1782 \\ 0.9781 \\ 0.9733 \\ 0.9704 \\ 0.9655 \\ 0.9806 \\ 0.9781 \\ 0.9733 \\ 0.9704 \\ 0.9655 \\ 0.9806 \end{bmatrix}. \tag{27}$$

$$J(x,z,u) = z$$

$$H_x = \begin{bmatrix} -0.1786 & 0.8307 \\ 0 & 0 \\ -0.4566 & -0.7939 \\ 0.4239 & 0.8322 \\ 0.7029 & 0.5095 \\ -0.4246 & -0.8220 \\ 0 & 0 \\ 0.7070 & 0.4889 \\ -0.4215 & -0.8233 \\ 0.4209 & 0.8331 \\ -0.1863 & -0.0931 \\ -0.1111 & -0.3407 \\ 0.1961 & 0 \\ 0.1863 & 0.0931 \\ 0.1111 & 0.3407 \\ -0.1961 & 0 \end{bmatrix}, H_z = \begin{bmatrix} -0.5271 \\ 0 \\ -0.0658 \\ -0.0230 \\ -0.4955 \\ -0.0244 \\ 0 \\ -0.5108 \\ -0.0235 \\ -0.0222 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, H_u = \begin{bmatrix} -0.0132 \\ 1 \\ -0.3961 \\ 0.3567 \\ 0.0279 \\ -0.3788 \\ -1 \\ 0.0128 \\ -0.3796 \\ 0.3582 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, K = \begin{bmatrix} -6.5119 \\ 2 \\ -1.4443 \\ 0.0650 \\ -10.8792 \\ -0.1638 \\ 2 \\ -11.1891 \\ -0.1209 \\ 0.0979 \\ 0.9781 \\ 0.9336 \\ 0.9806 \\ 0.9781 \\ 0.9336 \\ 0.9806 \end{bmatrix}. \tag{28}$$

7.2 Non-Minimum Phase System

To illustrate the comparison between GPC design [35, 36] and inverse optimality in terms of prediction horizon, we consider the following non-minimum phase system in the discrete-time domain:

$$H(z) = \frac{-0.375z + 0.625}{z^2 - 2z + 1}. \tag{29}$$

The parameters in GPC design are chosen below: the minimum output horizon $N_1 = 1$; the maximum output horizon $N_2 = 5$; the control horizon $N_u = 5$ and weighting value $\lambda = 1$. The reference signal (the green line) is represented by a square wave as shown in Fig.5. Clearly, the controller designed with the above parameters is not stabilizing, as the output signal (the blue one) diverges from the reference signal. As a consequence, if $N_2 \leq 5$, the obtained controller cannot stabilize the system. For now, we increase the maximum output horizon $N_2 = 6$, whereas the other parameters are still the same. The result presented in Fig.6 verifies that the controller designed with sufficiently large horizon stabilizes the given system. Note also that the green and blue signals in Fig.6 represent the reference and output, respectively.

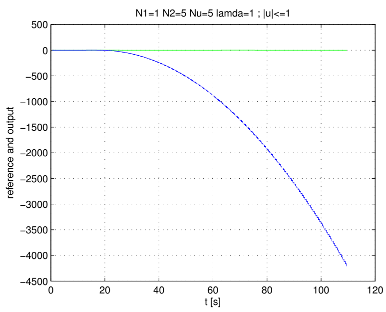

Fig. 5: Reference tracking designed by GPC with $N_2 = 5$.

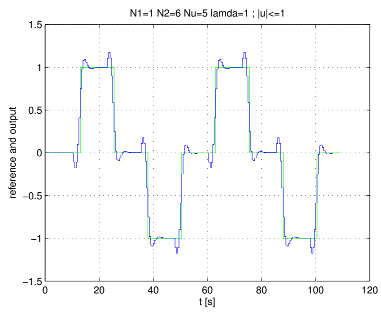

Fig. 6: Reference tracking designed by GPC with $N_2 = 6$.

To clarify the advantage of inverse optimality, we consider model predictive control problem for the same system with the same prediction horizon, i.e., $N = 5$. Note

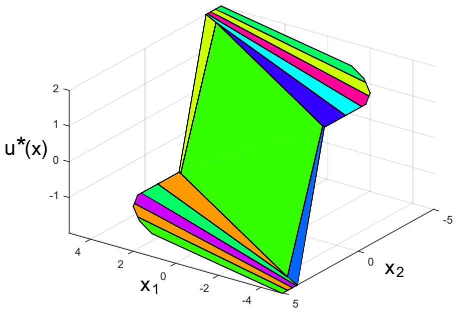

Fig. 7: The piecewise affine controller to recover in Subsection 7.1 with $N = 5$.

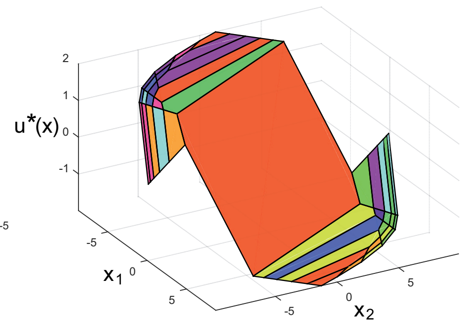

Fig. 8: Controller obtained from the MPC problem in Subsection 7.2.

that a state-space representation of the discrete system (29) is described by:

$$x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} u_k, \quad y_k = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix} x_k.$$

We choose the weighting matrices $Q = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}$, $R = 1$. The cost function is as in Subsection 7.1; constraints on present output and control variable are as in (25). Fig.8 depicts the controller obtained by the above MPC problem. Accordingly, a linear optimal control problem in form (26) with prediction horizon 2, which admits this controller as its optimal solution, is composed of the matrices defined in (30).

8 Conclusions

This article presents a method to solve inverse parametric linear/quadratic programming problems. This method relies on convex lifting. It is shown that for any continuous PWA function defined over a polytopic partition, an appropriate equivalence of this function can be obtained by another parametric linear/quadratic programming problem with a supplementary variable of dimension equal to 1. In view of linear optimal control, it has been shown that any continuous PWA control law can be obtained via a linear optimal control problem with the prediction horizon equal to 2 prediction steps. Several numerical examples prove the effectiveness of this method.

$$H_x = \begin{bmatrix} -0.1125 & -0.0466 \\ 0.1052 & 0.0613 \\ 0.3714 & -0.7428 \\ 0 & 0 \\ -0.3714 & 0.7428 \\ 0 & 0 \\ -0.2257 & -0.2896 \\ 0.2211 & 0.1060 \\ 0.2198 & 0.3048 \\ -0.2252 & -0.0913 \\ -0.3634 & -0.4753 \\ 0.3091 & 0.1311 \\ 0.3651 & 0.2709 \\ 0.3599 & 0.4895 \\ -0.3654 & -0.2557 \\ -0.3096 & -0.1171 \\ -0.4671 & -0.6162 \\ 0.3691 & 0.1410 \\ 0.4129 & 0.2241 \\ 0.4744 & 0.3645 \\ 0.4654 & 0.6274 \\ -0.4721 & -0.3517 \\ -0.4101 & -0.2123 \\ -0.3671 & -0.1286 \\ -0.5205 & -0.6898 \\ 0.4000 & 0.1391 \\ 0.3961 & 0.1733 \\ 0.4590 & 0.2548 \\ 0.5311 & 0.4153 \\ 0.5197 & 0.6972 \\ -0.5287 & -0.4063 \\ -0.4561 & -0.2466 \\ -0.3934 & -0.1653 \\ -0.3969 & -0.1288 \\ -0.1204 & -0.0535 \\ -0.1148 & -0.0656 \\ -0.0994 & -0.0796 \\ -0.0990 & -0.0930 \\ -0.0697 & -0.1014 \\ -0.0253 & -0.0990 \\ 0.0990 & 0.0484 \\ 0.1208 & 0.0535 \\ 0.1204 & 0.0656 \\ 0.1148 & 0.0796 \\ 0.0994 & 0.0930 \\ 0.0697 & 0.1014 \\ 0.0253 & 0.0990 \\ -0.1208 & -0.0484 \end{bmatrix}, \ H_z = \begin{bmatrix} -0.9747 \\ -0.9821 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.9004 \\ -0.9118 \\ -0.9064 \\ -0.8983 \\ -0.7387 \\ -0.8002 \\ -0.7547 \\ -0.7440 \\ -0.7427 \\ -0.7850 \\ -0.5224 \\ -0.6666 \\ -0.5513 \\ -0.5355 \\ -0.5254 \\ -0.5280 \\ -0.5417 \\ -0.6532 \\ -0.3327 \\ -0.5367 \\ -0.3816 \\ -0.3499 \\ -0.3413 \\ -0.3340 \\ -0.3380 \\ -0.3455 \\ -0.3761 \\ -0.5267 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ , H_u = \begin{bmatrix} -0.1875 \\ 0.1437 \\ 0.5571 \\ -1 \\ -0.5571 \\ 1 \\ -0.2334 \\ 0.3295 \\ 0.1932 \\ -0.3660 \\ -0.3103 \\ 0.4969 \\ 0.4730 \\ 0.2782 \\ -0.4996 \\ -0.5237 \\ -0.3595 \\ 0.6321 \\ 0.6894 \\ 0.5961 \\ 0.3373 \\ -0.6121 \\ -0.7024 \\ -0.6496 \\ -0.3776 \\ 0.7298 \\ 0.8170 \\ 0.7759 \\ 0.6550 \\ 0.3637 \\ -0.6642 \\ -0.7821 \\ -0.8225 \\ -0.7406 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ K = \begin{bmatrix} -7.8134 \\ -7.8728 \\ 7.4278 \\ 2 \\ 7.4278 \\ 2 \\ -5.2366 \\ -7.1781 \\ -5.2714 \\ -7.0721 \\ -2.2718 \\ -6.0973 \\ -4.5081 \\ -2.2878 \\ -4.4358 \\ -5.9812 \\ 0.9457 \\ -4.7929 \\ -3.2222 \\ -1.7351 \\ 0.9511 \\ -1.7108 \\ -3.1659 \\ -4.6971 \\ 3.4171 \\ -3.5266 \\ -2.1733 \\ -1.0872 \\ 0.5602 \\ 3.4299 \\ 0.5548 \\ -1.0737 \\ -2.1417 \\ -3.4612 \\ 0.9913 \\ 0.9912 \\ 0.9919 \\ 0.9932 \\ 0.9945 \\ 0.9901 \\ 0.9915 \\ 0.9913 \\ 0.9912 \\ 0.9919 \\ 0.9932 \\ 0.9945 \\ 0.9901 \\ 0.9915 \end{bmatrix}. \tag{30}$$

Acknowledgements The authors would like to thank the Associate Editor and reviewers for constructive and helpful comments. The research leading to these results has benefited from the financial support of the European Unions 7th Framework Programme under EC-GA No. 607957 TEMPO - Training in Embedded Model Predictive Control and Optimization. The first author would like to thank Dr. Martin Gulan and Prof. Luigi Del Re for their support and fruitful discussions. The last author acknowledges support from UEFISCDI Romania, PNII-RU-TE, project MoCOBiDS, no. 176/01.10.2015.

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