Abstract

Group objects of categories have been heavily studied in a general setting, but racks are mostly treated explicitly. Since rack structures are more general than groups, this thesis aims to explore the properties of general rack objects and use the tools of category theory to put topological racks in a new light.

Sammendrag

Kategoriske gruppeobjekter har blitt godt studert, men racks håndteres som regel eksplisitt. Siden racks er mer generelle strukturer enn grupper forsøker denne tesen å utforske egenskapene til generaliserte rack-objekter i håp om å bruke kategoriteori til å kaste nytt lys over topologiske racks.

The Rack Roll

We're no strangers to knots,
You know the rules, and so do I.
A faithful functor's what I'm thinking of,
You wouldn't get this x from any other y.
I just want to tell you 'bout my thesis.
Gonna make you understand:

For every single Abelian cat,
every single formal rack,
doesn't really matter what:
there's a quandle.
For every single pointed space,
every time you choose a base,
whenever you compute a trace:
there's a quandle.

Contents

Preface

This thesis was written in 2016 under the supervision of professor Markus Szymik at the Norwegian University of Science and Technology. It concerns itself with racks, or self-distributive and invertible binary operations. These structures show up across a plethora of mathematical disciplines, though most often they are described as invariants of knot theory or as the conjugation of groups. In the latter case, we get applications for Lie groups (Adjoint action), vector spaces (Choice of basis) and linear operators (Trace being invariant under conjugation).

In particular, this thesis focuses on the categorification of racks, inspired by the more commonly appreciated groups and motivated by a general desire to introduce racks to category theory after reading about Kauffman's knot-sets[9] during a course on the Foundations of Mathematics. Ultimately this thesis did not go the foundational route, choosing to approach racks from categories rather than categories from racks.

Chapter 1 contains an introduction to classical racks and quandles, including the special properties of symmetry, isotropy and involutivity which they may or may not exhibit. After showing that these three properties obey a 2-out-of-3 rule, section 1.2 then defines the category **Racks** and provides a primer on the terminology used. Section 1.3 reviews the relation of racks and quandles to the knot theory that popularized them. In particular, we construct the action of the braid groups on product racks and relate them to colorings of the knot diagrams generated by the braid closure.

Chapter 2 reviews the notion of group object in definition 2.1, then introduces the analogous notion of rack object in definition 2.2. We show that categorical racks admit an action of the braid groups, and prove that the 2-out-of-3 property of classical racks holds for general categorical racks in corollary 2.1.

In section 2.1 and 2.2 we extend the vocabulary of racks beyond the introductory and note that mapping racks to their inner automorphism group defines a functor $\mathbf{Inn}_{\triangleright}$ not from **Racks** to **Grp** but instead from **Racks** to **Racks**.

In section 2.3 we briefly discuss topological racks and the isotopy classes of the inner automorphisms.

In section 2.4 we introduce the notion of Rack-rack and prove various exclusivity statements about the existence of certain elements in a Rack-rack. We also introduce the **Hom**-rack on the **Hom**-sets into any categorical rack.

In section 2.5 and 2.6 we thoroughly investigate rack objects in **Grp** and **Ab**. In theorem 2.23 we derive the category isomorphism **GrpRacks** = **RacksGrp** between rack objects in **Grp** and group objects in **Racks**. This generalizes a result of Bourn [4], which proved the isomorphism between Alexander quandles in **Ab** and Abelian group objects in the category of quandles. In theorem 2.32 we further derive an isomorphism between the category of rack objects in

Ab and the module category of $\mathbb{Z}[f, g, g^{-1}]/(f^2 - f + fg)$, with Bourn's result corresponding to the module category of $\mathbb{Z}[f, g, g^{-1}]/(f - 1 + g) \simeq \mathbb{Z}[g, g^{-1}]$. In lemma 2.36 and lemma 2.37 we classify the (co-)symmetric, bi-symmetric and isotropic group quandles by the module categories $\mathbf{Mod}\mathbb{Z}[2^{-1}], \mathbf{Mod}\mathbb{Z}_3$ and $\mathbf{Mod}\mathbb{Z}[g]/(g^2 - g + 1)$ respectively.

Along the way we prove in lemma 2.25 that every group rack is a Rack-rack in the sense of section 2.4 and in corollary 2.26 that every group quandle is a crossed set in the sense of [11]. We also show a quandle-mirroring symmetry for the set of Abelian rack structures in theorem 2.34.

In section 2.7, we introduce a way for rack objects to act internal to their category, analogous to how group objects can. We then introduce the notion of crossed modules [1] and the functor pair (**Conj**, **As**) to show that any internal rack action in concrete categories gives rise to a group action of the associated group. This can then be used to define a monoidal action across categories, just like with groups.

In section 2.8 we introduce the conjugate equivalence relation on racks and show in theorem 2.43 that each rack object is conjugate to a unique quandle object. In lemma 2.44 we use this to classify all conjugation classes of group racks. This allows us to place restrictions on the set of group rack structures and fully classify the set of group quandles on any given Abelian group, which we do in corollary 2.45.

Finally, we generalize the n-racks of G.R. Biyogmam [5] to a categorical context in section 2.9. By the process of iteration and diagonalization, this lets us construct non-trivial rack structures on the set-theoretic product that are distinct from the standard product rack structure. We exemplify this process with **Ab**-n-racks, and prove in lemma 2.49 that the only **Ab**-quandles in the image of the diagonalization functor (from $n > 2$) are the trivial quandles. As a corollary this shows that for $n > 2$ there are no non-trivial strong n-quandles in **Ab**.

Chapter 3 investigates how rack structures can be used to produce novel invariants outside of knot theory. The main results, based on theorem 3.1 on the preservation of rack object structures by product-preserving functors, lead to useful computational tools in theorem 3.7, on the homotopy groups of topological racks. We use this to prove that there exist no symmetric or isotropic topological racks on the spheres or projective spaces in corollary 3.8 nor on many of the classical Lie groups in corollary 3.9. Furthermore, we show in lemma 3.10 that $\triangleleft \times \triangleright$ cannot be a homeomorphism for any isotropic or bisymmetric topological rack on a non-contractible space.

In section 3.2, we review a known homology theory from [7] defined on the category of racks, and note that we may give it coefficients in rack objects of **Ab**. We prove that the only **Ab**-quandle $(R, \triangleright)$ for which $\triangleright$ is an R-valued 2-cocycle is the trivial quandle, and suggest a path forward for future research.

Notation

While reading articles on racks one gets the impression that there are as many different notations as there are authors. Depending on your source the self-distributivity of a rack may be presented as follows

$$\begin{aligned} a \rhd (b \rhd c) &= (a \rhd b) \rhd (a \rhd c) && [11] \\ a \lhd (b \lhd c) &= (a \lhd b) \lhd (a \lhd c) && [\textbf{Wikipedia}] \\ (a \rhd b) \rhd c &= (a \rhd c) \rhd (b \rhd c) && [3][4] \\ (a \lhd b) \lhd c &= (a \lhd c) \lhd (b \lhd c) && [10][1] \\ a \circ (b \circ c) &= (a \circ b) \circ (a \circ c) && [5] \\ (a \star b) \star c &= (a \star c) \star (b \star c) && [14] \\ (a * b) * c &= (a * c) * (b * c) && [7] \\ a^{bc} &= a^{cb^c} && [13] \end{aligned}$$

If the inverse operation is mentioned at all, perhaps it is represented by

$$\begin{aligned} a \rhd (b \lhd a) &= b && [\textbf{nLab}] \\ a \lhd (b \rhd a) &= b && [\textbf{Wikipedia}] \\ (b \rhd a) \rhd^{-1} a &= b && [3][4] \end{aligned}$$

The ambiguity of symbols and variable placement is even worse when dealing with the conjugation quandle. $a \rhd b$ could mean $aba^{-1}, a^{-1}ba, b^{-1}ab$ or bab^{-1}.

I have decided to adopt the notation where the operation distributes when it moves through a parenthesis in the direction of the arrow, and variables coming in from the right act inversely to the ones coming in from the left.

In order to limit opportunities for confusion, I have attempted to eliminate all references to "left" and "right" if ambiguity could arise.

Chapter 1

Racks and quandles

1.1 Basic definitions

Let us start by formulating the standard definition of a rack [10] as a set:

Definition 1.1: A **rack** is a set R with two binary operations $\triangleright, \triangleleft$: $R \times R \to R$ satisfying for all a, b, c in R:

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c) \qquad \textbf{(Self-distributivity of } \triangleright\textbf{)} \tag{1.1}$$
$$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \qquad \textbf{(Self-distributivity of } \triangleleft\textbf{)} \tag{1.2}$$
$$(a \triangleright b) \triangleleft a = a \triangleright (b \triangleleft a) = b \qquad \textbf{(Inversion axiom)} \tag{1.3}$$

The inversion axiom is equivalent to the statement "for all (a, b) there exists a unique c such that $a \triangleright c = b$." This unique c can then be denoted by $c = b \triangleleft a$ and so one often considers only one of the rack operations.

Racks are given special names if for all a, b the operation satisfies

$$a \triangleright b = b \triangleright a \qquad \textbf{(Symmetric rack)} \tag{1.4}$$
$$a \triangleleft b = b \triangleleft a \qquad \textbf{(Co-symmetric rack)} \tag{1.5}$$
$$a \triangleright b = a \triangleleft b \qquad \textbf{(Isotropic rack)} \tag{1.6}$$
$$a \triangleright b = b \triangleleft a \qquad \textbf{(Involutive rack)} \tag{1.7}$$
$$a \triangleright a = a \triangleleft a = a \qquad \textbf{(Quandle)} \tag{1.8}$$

The latter two are by far the most common properties, and an involutive quandle is sometimes called a **kei**. If the rack is both symmetric and co-symmetric it is called **bi-symmetric**.

Lemma 1.1: 1) An isotropic, symmetric, or co-symmetric rack is a quandle.
2) (Symmetry, isotropy, involutivity) and (Co-symmetry, isotropy, involutivity) obey a 2-out-of-3 rule.

Proof. 1) We prove isotropic or symmetric racks are quandles. The argument is analogous for co-symmetric racks.

$$a \triangleleft (a \triangleright b) = a \triangleright (a \triangleright b) = (a \triangleright a) \triangleright (a \triangleright b) = (a \triangleright a) \triangleleft (a \triangleright b) \tag{1.9}$$
$$(a \triangleright b) \triangleright a = a \triangleright (a \triangleright b) = (a \triangleright a) \triangleright (a \triangleright b) = (a \triangleright b) \triangleright (a \triangleright a) \tag{1.10}$$

Since $(a \triangleright b) \triangleright -$ and $- \triangleleft (a \triangleright b)$ are bijections, $a \triangleright a = a$.
2) Assume co-symmetry and involutivity. Then $a \triangleright b = b \triangleleft a = a \triangleleft b$.

Assume co-symmetry and isotropy. Then $a \triangleright b = a \triangleleft b = b \triangleleft a$.

Assume involutivity and isotropy. Then $b \triangleleft a = a \triangleright b = a \triangleleft b = b \triangleright a$, and bi-symmetry is clear. □

Example 1.1: Let G be a group. The **conjugation quandle** over G is defined by $x \triangleright y = xyx^{-1}$ for $x, y \in G$. This is considered the archetypical example of a quandle.

Definition 1.2: Let X be a set and R be a rack. A **(rack) action** $\cdot$ of R on X is an action by bijections such that for all $r_1, r_2 \in R, x \in X$, we have

$$r_1 \cdot (r_2 \cdot x) = (r_1 \triangleright r_2) \cdot (r_1 \cdot x) \tag{1.11}$$

Example 1.2: Any rack R acts on its underlying set by its binary operation.

Example 1.3: Any group action defines a rack action of the group's conjugation quandle, since $a \cdot (b \cdot x) = (aba^{-1})(a(x)) = a(b(x)) = (ab)(x)$.

Definition 1.3: Let R be a rack. A **subrack** $Y \subset R$ is a subset satisfying $y_1 \triangleright y_2 \in Y$ whenever $y_1, y_2 \in Y$.

Definition 1.4: An **ideal** of R is a subrack I such that $R \triangleright I \subset I$.

1.2 The category of racks

One defines the category of racks, denoted **Racks**, by introducing **rack homomorphisms**. These are functions compatible with the binary operations:

$$\begin{aligned} f : (X, \triangleright_X) &\to (Y, \triangleright_Y) \\ f(a \triangleright_X b) &= f(a) \triangleright_Y f(b) \end{aligned} \tag{1.12}$$

As an example, let x be a fixed element of the rack R and define $f(-) = x \triangleright -$.

$$f(a \triangleright b) = x \triangleright (a \triangleright b) = (x \triangleright a) \triangleright (x \triangleright b) = f(a) \triangleright f(b) \tag{1.13}$$

From the axioms we know that this has an inverse function $- \triangleleft x$, but it is not immediately clear that a function which distributes over $\triangleleft$ automatically distributes over $\triangleright$. However, we can choose to extract a common factor:

$$f(a \triangleleft b) = f((c \triangleleft x) \triangleleft (d \triangleleft x)) = f((c \triangleleft d) \triangleleft x) = c \triangleleft d = f(a) \triangleleft f(b) \tag{1.14}$$

This shows that $\triangleright$ and $\triangleleft$ distribute over one-another in the direction of the arrows and thus $x \triangleright -$ is a rack automorphism. In general we call these **inner automorphisms**, and we will see in chapter 2 that this self-action is closely related to the inner automorphisms seen in group theory.

To summarize, in order for morphism composition to be compatible with inner automorphisms, any rack homomorphism must be compatible with both $\triangleright$ and $\triangleleft$, and this follows from compatibility with either. This lets us conclude that the mapping $(X, \triangleright_X) \mapsto (X, \triangleleft_X)$ is a functorial duality.

Each special rack property defines a subcategory of **Racks**. Denote these by **SymRacks**, **IsoRacks**, **InvRacks**, **Quandles**, and their intersection by **IsoKei**.

Proposition 1.2: The image of a rack homomorphism is a subrack.

Proof. Let $a = f(x)$ and $b = f(y)$. Then $x \rhd y = f(x) \rhd f(y) = f(x \rhd y)$ is also in the image of f. $\square$

As the inclusion of a subrack is a rack homomorphism, this can be taken as an alternative definition of subrack. This mirrors the notion of subgroups in group theory.

Proposition 1.3: For any rack homomorphism, the preimage of a subrack is a subrack.

Proof. Let x, y be in the preimage of the subrack S. Then $x \rhd y \mapsto f(x \rhd y) = f(x) \rhd f(y) \in S$. $\square$

Proposition 1.4: For any rack homomorphism, the preimage of an ideal is an ideal.

Proof. Let $f(y) \in I$. Then $f(x \rhd y) = f(x) \rhd f(y) \in I$ for any x. $\square$

It's worth noting that any element of a quandle can be considered the image of a singleton and is thus a subquandle. In particular, we have the following useful result:

Corollary 1.5: Let $(R, \rhd_R)$ be a rack, $(Q, \rhd_Q)$ be a quandle, and $f : R \to Q$ be a rack homomorphism. For any $q \in Q$, the preimage $f^{-1}(q)$ is a subrack of R.

Proof. Let $a, b \in f^{-1}(q)$.

$$f(a \rhd_R b) = f(a) \rhd_Q f(b) = q \rhd_Q q = q \tag{1.15}$$

$\square$

1.2.1 Special objects and universal properties

Like groups or topological spaces, racks are usually considered as sets enriched with additional structure. Like in **Top** we have the empty rack as the initial object and the singleton quandle as the final object, and every set has at least one rack structure available to it.

Definition 1.5: Let X be a set. The **trivial quandle** on X is given by $a \rhd b = b$ for all $a, b \in X$ and is denoted (X, π_2).

This can be described as a faithful functor TQ : **Set** $\to$ **Racks**.

The universal product of sets with coordinate-wise rack operation recovers a universal product in the category of racks.

Definition 1.6: Let $(X,\rhd_X)$ and $(Y,\rhd_Y)$ be racks. Their universal product is given by $X \times Y$ with rack operation

$$(x_1, y_1) \rhd_{X \times Y} (x_2, y_2) = (x_1 \rhd_X x_2, y_1 \rhd_Y y_2) \tag{1.16}$$

On the other hand, the disjoint union admits a plethora of inequivalent rack structures [11], and having the components act trivially on one-another does not work, as seen by the following diagram where R isn't a quandle:

$$\begin{tikzcd} & R & \\ R \arrow[ur, "id"] \arrow[r, hook] & R\coprod R \arrow[u, dashed, "?"] & R \arrow[ul, "id"'] \arrow[l, hook'] \end{tikzcd} \tag{1.17}$$

Instead, we construct the coproduct entirely analogously to the free product in the category of groups. This first requires us to define free racks.

Definition 1.7: Let X be a set. The **free rack on X**, denoted $\mathbf{FR}(X)$ consists of words generated by letters in X and the binary operations $(-\triangleright-)$ and $(-\triangleleft-)$, reduced by the relations imposed by the rack axioms. The **free quandle on X**, $\mathbf{FQ}(X)$, is obtained by imposing the quandle relation on $\mathrm{FR}(X)$.

Example 1.4: Let X be a set, and let $FG(X)$ denote the free group on X. Since $\mathbf{Inn}_{\triangleright}(FR(X)) \simeq FG(X)$, we may formally represent $FR(X)$ as $FG(X) \times X$. $FQ(X)$ can be represented as the conjugation subquandle of $FG(X)$ given by the orbit of X with respect to the conjugation action.

$$FR(X) \mapsto FQ(X) \simeq \{g \triangleright x | g \in FG(X), x \in X\} \subset FG(X) \tag{1.18}$$

Definition 1.8: Let X and Y be racks. The coproduct of X and Y is given by $\mathrm{FR}(X \coprod Y)/\!\sim$, where $\sim$ represents the relations imposed by X and Y.

1.3 Racks in knot theory

To motivate the concept of racks and quandles we turn to knot theory, where they turn out to be quite useful as their relations are similar to the generators of the braid group, see 1.3.2. Since Markov's theorem [2] states that any knot or link diagram can be represented as the closure of a braid, one might not be surprised to learn that one can define a rack structure from such diagrams.

1.3.1 Racks of knot diagrams

A simple way to encode the information of a knot diagram is to label each directed arc component, and at each crossing write down which arc component crosses over and which two end there.

Given a knot diagram K, we may define the **fundamental rack of** K from it by considering the free rack generated by the set of arc components in K, modulo the interpretation of $a \triangleright b$ as "the other side of a seen from b, using the right-hand rule for the directed crossing" and $a \triangleleft b$ inversely by using the left hand rule for the crossing.

Typically one also enforces invariance under the first Reidemeister move by assuming the quandle axiom. The resulting **fundamental quandle**, which we will denote $\mathrm{FunQ}(K)$, is a famous knot invariant discovered by Joyce in [3].

Definition 1.9: Let K be a knot diagram and let Q be any quandle. A **coloring** of K is a rack homomorphism $\mathrm{FunQ}(K) \to Q$. It is displayed by labeling the arc components of K by their image in Q.

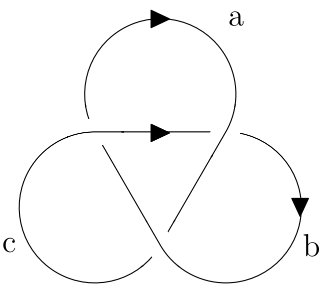


Figure 1.1: Labeling of directed arc components of a trefoil diagram.

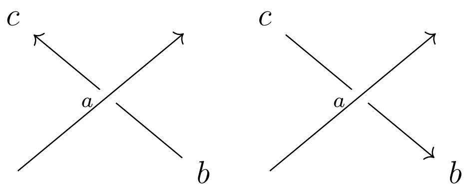
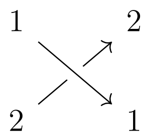


Figure 1.2: Visualization of the calculations $a \triangleright b = c$ and $a \triangleleft b = c$.

1.3.2 Braiding of racks

The braid group B_n is related to the symmetry group S_n. Whereas the elements of S_n can be thought of as formally swapping labeled points, B_n acts on n *strands* by generators $\{\sigma_i | 1 \leq i < n\}$ generators that diagrammatically cross one strand over another.

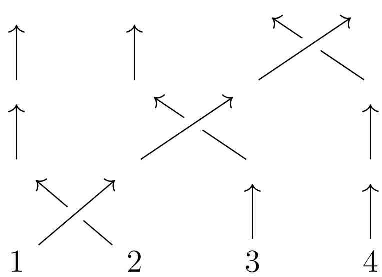
$$\sigma_1 : \begin{array}{c} 1 \longrightarrow 1 \\ \\ 2 \longrightarrow 2 \end{array} \mapsto \tag{1.19}$$



The resulting labeling corresponds to an element of S_n, but the braid group further takes into account *how* they got there. A topological interpretation of braid elements are as isotopy classes of parametrized paths in $\mathbb{R}^2$ connecting a set of n labeled points to their configuration after applying an element of S_n.

The group multiplication then equates to sequencing the braids.

$$\sigma_3\sigma_2\sigma_1 = \tag{1.20}$$



By the previous interpretation of crossings as rack operations, this defines a group action of B_n on the free rack on n generators.

$$B_n \to \mathbf{Aut}_{\mathbf{Racks}}(FR(\mathbb{Z}_n)) \tag{1.21}$$

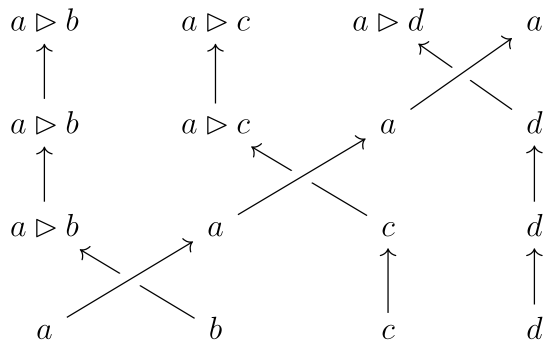
In the procedure called **closure** one identifies the top and bottom of a braid to get a knot diagram. Doing so reveals that closed braids naturally encode the relevant Reidemeister moves in that adjacent crossings don't commute, yet distant crossings do.

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } |i-j| = 1 \tag{1.22}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{otherwise.} \tag{1.23}$$

Invariance under the Reidemeister moves then equates to taking the relations that turn the free rack into the closure's fundamental quandle. Note that we may now use the braids' actions on the free rack to act on any product rack R^n in a natural way:

$$\begin{aligned} \sigma_i : R^n &\to R^n \\ (x_1, ..., x_i, x_{i+1}, ..., x_n) &\mapsto (x_1, ..., x_i \rhd x_{i+1}, x_i, ..., x_n) \\ \sigma_i^{-1} : R^n &\to R^n \\ (x_1, ..., x_i, x_{i+1}, ..., x_n) &\mapsto (x_1, ..., x_{i+1}, x_i \lhd x_{i+1}, ..., x_n) \end{aligned}$$

For instance, the previous diagram acts on $(a, b, c, d) \in R^4$ as follows:



$$\sigma_3\sigma_2\sigma_1(a, b, c, d) = (a \rhd b, a \rhd c, a \rhd d, a)$$

Without loss of generality we may verify that this is an action of the braid group by computing the actions of B_3's element $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$.

$$\begin{aligned} &\sigma_2\sigma_1\sigma_2(a, b, c) \\ &= \sigma_2\sigma_1(a, b \rhd c, b) \\ &= \sigma_2(a \rhd (b \rhd c), a, b) \\ &= (a \rhd (b \rhd c), a \rhd b, a) \end{aligned} \qquad \begin{aligned} &\sigma_1\sigma_2\sigma_1(a, b, c) \\ &= \sigma_1\sigma_2(a \rhd b, a, c) \\ &= \sigma_1(a \rhd b, a \rhd c, a) \\ &= ((a \rhd c) \rhd (b \rhd c), a \rhd b, a) \end{aligned}$$

By the self-distributive property the two results are equal.

One then sees that coloring the closure of a braid diagram necessitates that the braid's action leaves each strand's color invariant. Again, this is because if one considers each strand as a generator for the free rack on n generators, enforcing the braid's relations constitutes defining the fundamental rack, and like for groups, any rack satisfying additional relations may then be taken as an image of the fundamental rack.

In [14], Rubinsztein restricted himself to considering topological quandles and condensed the previous statements to theorem form. The following theorems have been modified to not rely on the topological structure of the quandles.

Theorem 1.6: Let Q be a quandle, and let $\sigma \in B_n$ be a braid of n strands. Let $J_Q(\sigma)$ denote the fixed points of the action of $\sigma : Q^n \to Q^n$. Then,
1) $J_Q(\sigma)$ is an isotopy invariant of the closure of σ.
2) For a given σ, the assignment $Q \mapsto J_Q(\sigma)$ is functorial.

Proof.
1) By Markov's theorem [2], the braid closure is invariant under extension to B_{n+1} by $\sigma \mapsto \sigma_n \circ \sigma$, and conjugation by other braids, $\sigma \mapsto f\sigma f^{-1}$. If we can show that $J_Q(\sigma)$ also is invariant under these operations, we are done.

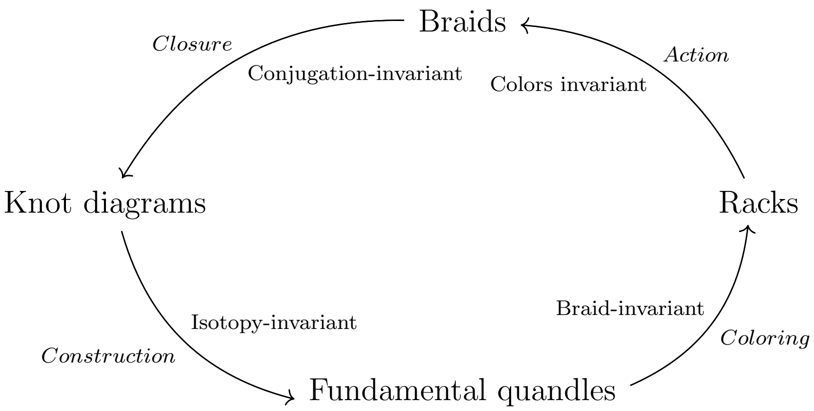
Since σ_n acts by $(..., x_n, x_{n+1}) \mapsto (..., x_n \triangleright x_{n+1}, x_n)$, a fixed point in Q^{n+1} necessarily has $x_n = x_{n+1}$ and thus the fixed points are in bijection with the ones in Q^n.

The conjugation $f\sigma f^{-1}$ bijectively maps $f(J_Q(\sigma))$ to $J_Q(\sigma)$, where σ does nothing, then it returns each fixed point whence it came while sending no non-fixed point whence it came. In other words, up to isomorphism the fixed points are the same.
2) Given $f : A \to B$, $J_f(\sigma)$ is simply the restriction of $\prod_i^n f$ to $J_A(\sigma) \subset A^n$. Since f is a rack morphism, $\prod_i^n f$ necessarily maps invariants to invariants. $\square$

Lemma 1.7: Let σ be a braid, let Q be a quandle, and let $\mathrm{FunQ}(\sigma)$ denote the fundamental quandle of the closure of σ. The set of Q-colorings of σ, $\mathbf{Hom}_{\mathbf{Racks}}(\mathrm{FunQ}(\sigma), Q)$, is isomorphic to $J_Q(\sigma)$.

Proof. Each $\phi \in \mathbf{Hom}_{\mathbf{Racks}}(\mathrm{FunQ}(\sigma), Q)$ assigns an element from Q to each arc of the diagram in such a way to be consistent with the action of each crossing. Since the closure identifies the bottom of the braid diagram with the top, this equates to picking an element in $J_Q(\sigma)$, represented as a subset of Q^n, to label the n strands of σ. The arc labels follow uniquely from the strand labels. $\square$

To summarize, we now see that racks and knots are related by braids, and together they produce invariants of knots and racks alike.



Chapter 2

Rack objects in categories

Analogous to how one can define abstract group objects in a category, we want to define rack objects in a category $\mathcal{C}$ which has products. To help motivate this construction, let us first take a look at group objects.

Definition 2.1: Let $\mathcal{C}$ be a category with products and terminal object $\star$. A **group object in** $\mathcal{C}$, or $\mathcal{C}$**-group**, is a quadruplet $(G, 1, i, \mu)$ where G is an object, $1 : \star \to G$ is called the **unit morphism**, $i : G \to G$ is called the **inversion morphism** and $\mu : G \times G \to G$ is called the **multiplication morphism**. They satisfy the following properties:
1. Associativity: The following diagram commutes:

$$\begin{tikzcd} G^3 \arrow[r, "\pi_1\times\mu(\pi_2\times\pi_3)"] \arrow[d, "\mu(\pi_1\times\pi_2)\times\pi_3"'] & G^2 \arrow[d, "\mu"] \\ G^2 \arrow[r, "\mu"'] & G \end{tikzcd} \tag{2.1}$$

2. Two-sided identity of unit: The following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}\times 1"] \arrow[d, "1\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G^2 \arrow[d, "\mu"] \\ G^2 \arrow[r, "\mu"'] & G \end{tikzcd} \tag{2.2}$$

3. Invertibility: The following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{id}\times i"] \arrow[d, "i\times\mathrm{id}"'] \arrow[dr, "1"] & G^2 \arrow[d, "\mu"] \\ G^2 \arrow[r, "\mu"'] & G \end{tikzcd} \tag{2.3}$$

$\mathcal{C}$-group homomorphisms $f : (A, 1_A, i_A, \mu_A) \to (B, 1_B, i_B, \mu_B)$ satisfy

$$f \circ \mu_A = \mu_B \circ ((f \circ \pi_1) \times (f \circ \pi_2)) \tag{2.4}$$
$$f \circ i_A = i_B \circ f \tag{2.5}$$
$$f \circ 1_A = 1_B \tag{2.6}$$

$\mathcal{C}$-group anti-homomorphisms satisfy $f \circ \mu_A = \mu_B \circ ((f \circ \pi_2) \times (f \circ \pi_1))$ instead of the first of the above equations.

Given a category $\mathcal{C}$ with products and terminal object, denote by $\mathcal{C}$**Grp** the category of $\mathcal{C}$-groups and $\mathcal{C}$-group homomorphisms.

$\mathcal{C}$-groups have been abundantly researched, with **Set**-groups being normal everyday groups, **Top**-groups being topological groups, **Diff**-groups being Lie groups, and **Grp**-groups being Abelian groups. In each case, most texts on the subject are written without ever taking into account the categorical view, yet it's quite useful to have a generalization once functors and universal properties get involved.

In a similar vein, topological racks and racks based on group structure have been popular research topics ever since their applications in knot theory were discovered. However, a full generalization to the world of category theory has not - to the knowledge of the author - been done before, and this will be the primary topic for the rest of this text.

The axioms we need to address are self-distributivity and invertibility, and our construction should be as unrestrictive as it can be while still recovering racks in **Set**. As we have seen, racks generally allow an action of the braid group, and it is this that motivates the choice of notation used in the following generalization.

Definition 2.2: Let $\mathcal{C}$ be a category with products. A **rack object in** $\mathcal{C}$, or $\mathcal{C}$**-rack**, is an object R and a pair of morphisms $\rhd, \lhd : R^2 \to R$ satisfying
1. Invertibility: The following diagram commutes:

$$\begin{tikzcd} R^2 \arrow[r, "\pi_2\times\lhd"] \arrow[d, "\rhd\times\pi_1"'] \arrow[dr, "\mathrm{id}"] & R^2 \arrow[d, "\rhd\times\pi_1"] \\ R^2 \arrow[r, "\pi_2\times\lhd"'] & R^2 \end{tikzcd} \tag{2.7}$$

2. Self-distributivity: The following diagrams commute:

$$\begin{tikzcd} R^3 \arrow[rr, "(\rhd\times\pi_1)\times\pi_3"] \arrow[d, "\pi_1\times\rhd"] & & R^3 \arrow[d, "\pi_1\times\rhd"] \\ R^2 \arrow[r, "\rhd"] & R & R^2 \arrow[l, "\rhd"'] \end{tikzcd} \quad \begin{tikzcd} R^3 \arrow[rr, "\pi_1\times(\pi_3\times\lhd)"] \arrow[d, "\lhd\times\pi_3"] & & R^3 \arrow[d, "\lhd\times\pi_3"] \\ R^2 \arrow[r, "\lhd"] & R & R^2 \arrow[l, "\lhd"'] \end{tikzcd} \tag{2.8}$$

Rack homomorphisms between two rack objects $(R, \rhd, \lhd)$ and $(S, \rhd', \lhd')$ are morphisms $R \xrightarrow{f} S$ for which the following diagram commutes:

$$\begin{tikzcd} R \arrow[d, "f"] & R\times R \arrow[l, "\lhd"] \arrow[r, "\rhd"] \arrow[d, "f\times f"] & R \arrow[d, "f"] \\ S & S\times S \arrow[l, "\lhd'"] \arrow[r, "\rhd'"] & S \end{tikzcd} \tag{2.9}$$

Note that this definition guarantees an action of the braid group by mapping a generator σ_i to $\pi_1 \times ... \times \pi_{i-1} \times ((\rhd \times \pi_1) \circ (\pi_i \times \pi_{i+1})) \times \pi_{i+2} \times ... \times \pi_n$ and its inverse to $\pi_1 \times ... \times \pi_{i-1} \times ((\pi_2 \times \lhd) \circ (\pi_i \times \pi_{i+1})) \times \pi_{i+2} \times ... \times \pi_n$. Mutual distributivity of $\rhd$ and $\lhd$ is then guaranteed by the braid equation

$$\sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1 \tag{2.10}$$

which can be recovered by the following commutative diagram

$$
\begin{tikzcd}
(a \rhd b, a, c) \arrow[d, "\pi_1\times(\pi_3\times\lhd)"'] & (a,b,c) \arrow[l, "(\rhd\times\pi_1)\times\pi_3"'] \arrow[r, "\pi_1\times(\pi_3\times\lhd)"] & (a, c, b \lhd c) \arrow[d, "(\pi_2\times\lhd)\times\pi_3"] \\
(a \rhd b, c, a \lhd c) \arrow[d, "(\pi_2\times\lhd)\times\pi_3"'] & & (c, a \lhd c, b \lhd c) \arrow[d, "\pi_1\times(\rhd\times\pi_2)"] \\
(c, (a \rhd b) \lhd c, a \lhd c) \arrow[r, "\pi_2"'] & R & (c, (a \lhd c) \rhd (b \lhd c), a \lhd c) \arrow[l, "\pi_2"]
\end{tikzcd}
$$

Similarly, (2.8) could be derived from the relation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$.

The special kinds of rack can be categorized as follows.

$$\rhd \circ (\pi_2 \times \pi_1) = \rhd \qquad \textbf{(Symmetric rack object)} \tag{2.11}$$
$$\rhd = \lhd \qquad \textbf{(Isotropic rack object)} \tag{2.12}$$
$$(\pi_1 \times \rhd)^2 = \mathrm{id} \qquad \textbf{(Involutive rack object)} \tag{2.13}$$
$$\rhd \circ (\mathrm{id} \times \mathrm{id}) = \mathrm{id} \qquad \textbf{(Quandle object)} \tag{2.14}$$

Corollary 2.1: The 2-out-of-3 rule of lemma 1.1 holds for any categorical rack object as well.

Proof. Denote by $T = \pi_2 \times \pi_1$ the transposition morphism. Note that $T^2 = \mathrm{id}$, and that involution is equivalent to the useful relation $\rhd \circ T = \lhd$:

$$T \circ (\pi_1 \times \rhd)^2 \circ T = (\rhd \times \pi_1) \circ (\pi_2 \times (\rhd \circ T)) = (\rhd \times \pi_1) \circ (\pi_2 \times \lhd) \tag{2.15}$$

Assume symmetry and isotropy. By the above, $\rhd = \lhd = \rhd \circ T$.
Assume symmetry and involutivity. By the above, $\rhd = \rhd \circ T = \lhd$.
Assume isotropy and involutivity. By the above, $\rhd \circ T = \lhd = \rhd$. □

Example 2.1: Trivial quandles and permutation racks.
Every category $\mathcal{C}$ with finite products allows a rack structure on any object X given by $(\rhd, \lhd) = (\pi_2, \pi_1)$. We call this the **trivial quandle on** X, as the action of the braid group on the self-product of such a rack reduces to an action of the symmetry group.

Furthermore, for any $\mathcal{C}$-automorphism g, $(\rhd, \lhd) = ((g \circ \pi_2), (g^{-1} \circ \pi_1))$ is a rack object. We call this the **permutation rack** of g.

Example 2.2: Alexander quandles. [14]
For any $\mathcal{C}$-group G, one can define $\mathcal{C}$-quandle structures on G that recover the following structures based on conjugation:

$$a \rhd b = aba^{-1} : \qquad G^2 \xrightarrow{\pi_1\times\pi_2\times(i\circ\pi_1)} G^3 \xrightarrow{\pi_1\times\mu(\pi_2\times\pi_3)} G^2 \xrightarrow{\mu} G \tag{2.16}$$
$$a \rhd b = a\phi(ba^{-1}) : \qquad G^2 \xrightarrow{\pi_1\times\pi_2\times(i\circ\pi_1)} G^3 \xrightarrow{\pi_1\times\phi\mu(\pi_2\times\pi_3)} G^2 \xrightarrow{\mu} G \tag{2.17}$$
$$a \rhd b = ab^{-1}a : \qquad G^2 \xrightarrow{\pi_1\times(i\circ\pi_2)\times\pi_1} G^3 \xrightarrow{\pi_1\times\mu(\pi_2\times\pi_3)} G^2 \xrightarrow{\mu} G \tag{2.18}$$
$$a \rhd b = a\eta(b^{-1})\eta(a) : \qquad G^2 \xrightarrow{\pi_1\times(i\circ\pi_2)\times\pi_1} G^3 \xrightarrow{\pi_1\times\eta\mu(\pi_3\times\pi_2)} G^2 \xrightarrow{\mu} G \tag{2.19}$$

Where ϕ can be any $\mathcal{C}$**Grp**-automorphism and η any $\mathcal{C}$**Grp**-anti-automorphism. These variants of conjugation are called **(Anti-)Alexander quandles**.

For a category $\mathcal{C}$ with finite products we will denote the category of rack objects in $\mathcal{C}$ by $\mathcal{C}$**Racks**, and for a particular object R we denote the subcategory of rack structures on R by $\mathcal{C}\mathbf{Racks}(R)$.

Remark 1: There are functors $\mathbf{Conj}_{\mathcal{C}}$ and $\mathbf{Alex}_{\mathcal{C}} : \mathcal{C}\mathbf{Grp} \to \mathcal{C}\mathbf{Racks}$ that take each $\mathcal{C}$-group to the operations defined in (2.16) and (2.18), respectively.

We will now investigate $\mathcal{C}$**Racks** in various familiar categories and see what we may learn from each case.

2.1 Set

In the category of sets, the ability to fix either argument by precomposition with constant maps means that any **Set**-rack R can naturally be mapped into its endomorphism and automorphism sets:

$$\begin{aligned} \Phi^{\triangleright}, \Phi^{\triangleleft} &: R \to \mathbf{Aut}(R) & \epsilon^{\triangleright}, \epsilon^{\triangleleft} &: R \to \mathbf{End}(R) \\ \Phi^{\triangleright}(a) &= (a \triangleright -) & \epsilon^{\triangleright}(a) &= (- \triangleright a) \\ \Phi^{\triangleleft}(a) &= (- \triangleleft a) & \epsilon^{\triangleleft}(a) &= (a \triangleleft -) \end{aligned}$$

Note that these mappings need neither be injective nor surjective.

Example 2.3: Let X be a three-point set. Then $|\mathbf{Aut}(X)| = |S_3| = 6$ and $|\mathbf{End}(X)| = 3^3 = 27$. We see that the maps cannot be surjective.

The trivial rack given by $\triangleright = \pi_2$ lifts each element to the identity and a constant map. The first mapping is then not injective.

Definition 2.3: Let $(R, \triangleright)$ be a rack. The **Inner Automorphism Group**, denoted $\mathbf{Inn}_{\triangleright}(R)$, is the subgroup of $\mathbf{Aut}(R)$ generated by $\{\Phi^{\triangleright}(r) | r \in R\}$. Unless otherwise specified, it is considered a conjugation subquandle of $\mathbf{Aut}(R)$.

Lemma 2.2: The map $(R, \triangleright_R) \to (\mathbf{Inn}_{\triangleright}(R), \triangleright_c)$ is a rack homomorphism.

Proof. Let α, β be short-hand notation for $a \triangleright_R -$ and $b \triangleright_R -$.

$$(a \triangleright_R b) \triangleright_R - = a \triangleright_R (b \triangleright_R (- \triangleleft_R a)) = \alpha(\beta(\alpha^{-1}(-))) = (\alpha \triangleright_c \beta)(-) \tag{2.20}$$

□

Lemma 2.3: If $\mathbf{Inn}_{\triangleright}(R)$ is Abelian and R is isotropic or symmetric, R is the singleton.

Proof. Recall that $a \triangleright_R -$ is a bijection. We will show $b = c$ for arbitrary b, c.

$$\begin{aligned} a \triangleright_R (b \triangleright_R c) &= a \triangleleft_R (\beta \triangleright_c \gamma) = a \triangleleft_R c = a \triangleright_R c \\ a \triangleright_R (b \triangleright_R c) &= a \triangleleft_R (\beta \triangleleft_c \gamma) = a \triangleleft_R b = a \triangleright_R b \end{aligned} \tag{2.21}$$

$$\begin{aligned} a \triangleright_R (b \triangleright_R c) &= (\beta \triangleright_c \gamma) \triangleright_R a = c \triangleright_R a = a \triangleright_R c \\ a \triangleright_R (b \triangleright_R c) &= (\gamma \triangleright_c \beta) \triangleright_R a = b \triangleright_R a = a \triangleright_R b \end{aligned} \tag{2.22}$$

□

Corollary 2.4: The preimage of any $f \in \mathbf{Aut}(R)$ is a subrack of R. The image of R in $\mathbf{Inn}(R)$ is an ideal of the conjugation quandle.

Example 2.4: For any bijection g, $\triangleright = g \circ \pi_2$ defines a rack.

Definition 2.4: Define an equivalence relation on a rack X as follows: $a \sim b$ if there exists $f \in \mathbf{Inn}_\triangleright(X)$ such that $f(a) = b$. The equivalence class of a is called the **orbit** of a, and there is a functor **Orb**: **Racks** $\to$ **Quandles**, $X \mapsto X/\sim$.

Orb(X) is equipped with a trivial quandle structure, and represents the biggest trivial quandle to which there exists an epimorphism from X.

Corollary 2.5: Let $X \xrightarrow{f} Y$ be a rack homomorphism. Then there exists a quandle homomorphism $\mathbf{Inn}_\triangleright(X) \xrightarrow{\mathbf{Inn}f} \mathbf{Inn}_\triangleright(Y)$ given by $\mathbf{Inn}f(a \triangleright -) = f(a) \triangleright -$, which makes the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\triangleright_X"] & Y \arrow[d, "\triangleright_Y"] \\
\mathbf{Inn}_\triangleright(X) \arrow[r, "\mathbf{Inn}f"] & \mathbf{Inn}_\triangleright(Y)
\end{tikzcd} \tag{2.23}$$

Thus $\mathbf{Inn}_\triangleright$ is a functor **Racks** $\to$ **Quandles**. If f is surjective, $\mathbf{Inn}f$ extends to a group homomorphism, but as demonstrated by the inclusion of the singleton to a non-trivial quandle, this need not happen in general.

It will be convenient to think of $a \triangleright -$ as a *acting* by an inner automorphism, and $- \triangleright a$ as *evaluating* at the argument a.

Definition 2.5: Let $(R, \triangleright)$ be a rack and let $r \in R$.
If $r \triangleright r = r$, we call r an *idempotent element.*
If $r \triangleright -$ is the identity, we call r a *trivial element.*
If $(r \triangleright -)^2$ is the identity, we call r an *involutive element.*
If $- \triangleright r$ is a monomorphism of sets, we call r a *monic argument.*
If $- \triangleright r$ is an epimorphism of sets, we call r an *epic argument.*
If $- \triangleright r$ is an isomorphism of sets, we call r an *iso argument.*
If $- \triangleright r$ is a constant map, we call r a *constant argument.*

If every argument of a rack is monic, epic or iso, we call it respectively a monic, epic or iso rack.

Corollary 2.6: If R is not the singleton and $\mathbf{Inn}_\triangleright(R)$ is an Abelian group, then R has no monic arguments.

Proof. Conjugation of an Abelian group is trivial, so $(a \triangleright b) \triangleright - = b \triangleright -$ for all a, b. Thus if there existed a monic argument x then R must be trivial. Choosing $b = x$ reveals that R must actually be the singleton. $\square$

Corollary 2.7: In a finite rack an argument is epic if and only if it is monic.

Corollary 2.8: A symmetric, cosymmetric or isotropic rack is an iso quandle.

Corollary 2.9: A rack that contains an epic argument has only one orbit.

2.2 PSet

The objects in the category of pointed sets consist of sets augmented with a distinguished element, with any morphisms taking distinguished elements to distinguished elements. The product is given by $(A, a) \times (B, b) = (A \times B, (a, b))$.

If we denote a pointed rack by $(R, r, \rhd, \lhd)$, the distinguished element then imposes the relation

$$r \rhd r = r \qquad\qquad r \lhd r = r \tag{2.24}$$

This means that the pointed rack acts as a quandle at the distinguished point.

If we were to require the ability to map the elements to morphisms as we did in **Set**, distinguishing the identity of $\mathbf{Aut}(R, r)$ and the constant map of $\mathbf{End}(R, r)$ restricts us further to consider structures satisfying

$$a \rhd r = r \qquad\qquad a \lhd r = a \tag{2.25}$$

$$r \rhd a = a \qquad\qquad r \lhd a = r \tag{2.26}$$

for all $a \in R$, which means that it acts as the *trivial* quandle at r. This construction is called a **unital rack**, as it coincides with the unit of the group conjugation quandle.

$$1 \rhd x = 1x1^{-1} = x \qquad\qquad x \rhd 1 = x1x^{-1} = 1 \tag{2.27}$$

Note that unlike a group, a unital rack may have several units, as exemplified by the trivial quandle.

Proposition 2.10: Let R be a unital rack. If R is also symmetric, co-symmetric or isotropic, then R is the singleton.

Proof. Symmetric, co-symmetric and isotropic racks are iso racks, but the unit is a constant argument. The constant being an isomorphism implies that R is the singleton. $\square$

The categorification of the unital structure may be formulated somewhat similarly to how it is implemented for a categorical group.

Definition 2.6: Let R be a $\mathcal{C}$-rack and $*$ be a terminal object in $\mathcal{C}$. A **unit** of R is a morphism $1 : * \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
& R \arrow[dl, bend right, "1"'] \arrow[d, "1\times\mathrm{id}"] \arrow[dr, bend left, "\mathrm{id}"] & \\
R & R^2 \arrow[l, "\lhd"'] \arrow[r, "\rhd"] & R \\
& R \arrow[ul, bend left, "\mathrm{id}"] \arrow[u, "\mathrm{id}\times 1"'] \arrow[ur, bend right, "1"'] &
\end{tikzcd} \tag{2.28}$$

If such a unit exists, we call R a **unital $\mathcal{C}$-rack**.

2.3 Top

A **topological rack** $(R, \triangleright, \triangleleft)$ is a rack object with topology such that the maps

$$\begin{aligned} (x,y) &\mapsto x \triangleright y & (x,y) &\mapsto x \triangleleft y \\ x &\mapsto a \triangleright x & x &\mapsto x \triangleleft a \\ x &\mapsto x \triangleright a & x &\mapsto a \triangleleft x \end{aligned}$$

are all continuous for any a. While this is entirely analogous to racks in the category of sets, we will see that this is actually more structure than is normally available. This is granted to us by the fact that any element is selectable by a constant map - which is still a morphism in **Top** - which may then be composed with the required morphism $\triangleright$ to produce inner automorphisms and argument evaluations that remain internal to **Top**.

Most texts deal with rack structures on topological manifolds, which have a standardized topological structure. This is useful as it may allow us to envision $\mathbf{Inn}_\triangleright(R)$ by introducing the equivalence relation $a \sim b$ whenever $a \triangleright - = b \triangleright -$. Then $\mathbf{Inn}_\triangleright(R)$ is generated by the quotient space $R/\sim$, and obviously for a path-connected involutive rack, the words of length 3 are on the same path component as the original generators.

Example 2.5: On the unit circle $S^1 \simeq \mathbb{R}/(2\pi\mathbb{Z})$, let $a \triangleright b = 2a - b$. Then $\mathbf{Inn}_\triangleright(S^1)$ can be identified with $S^1 \coprod S^1$. The first copy consists of orientation-reversing functions $\theta_1(x) = \theta - x$, and the other copy consists of orientation-preserving functions $\theta_2(x) = \theta + x$.

Example 2.6: [14] Consider the unit sphere of a complex Hilbert space H. The following linear operators, defined for $a \neq 0 \in H, \theta \in \mathbb{R}$,

$$a \triangleright_\theta - = e^{i\theta} + \frac{(1-e^{i\theta})}{\langle a, a\rangle}|a\rangle\langle a| \tag{2.29}$$

$$\Leftrightarrow a \triangleright_\theta b = e^{i\theta} b + (1-e^{i\theta})\frac{\langle a, b\rangle}{\langle a, a\rangle} a \tag{2.30}$$

restrict to topological quandle operations on the unit sphere. For $\theta = \pi$ this also defines an involutive operation on the real spheres.

Example 2.7: The conjugation quandle of any topological group is a unital topological quandle.

Lemma 2.11: Let $(R, \triangleright)$ be a path-connected topological rack. Then all the basic inner automorphisms on the form $a \triangleright -$ are isotopic.

Proof. Let $a, b \in R$ and let $\gamma : [0,1] \to R$ be a path such that $\gamma(0) = a$ and $\gamma(1) = b$. Then $H(t, x) = \triangleright(\gamma(t), x)$ is a homotopy between $a \triangleright -$ and $b \triangleright -$, with every fixed t producing a homeomorphism. $\square$

Corollary 2.12: Let $(R, \triangleright)$ be a path-connected, unital topological rack. Then every inner automorphism is isotopic to the identity.

Proof. Words of length ± 1 are all isotopic to $1 \triangleright - = - \triangleleft 1 = \mathrm{id}$, of length 0. By induction any word of any length is isotopic to id. $\square$

Corollary 2.13: Let $(R, \triangleright)$ be a path-connected unital rack. If R contains an involutive element, there are at most two isotopy classes in $\mathbf{Inn}_\triangleright(R)$.

2.4 Racks

A rack object in **Racks** corresponds to a rack $(R, \rhd)$ with a rack homomorphism $\rhd' : R \times R \to R$. In other words, for $a, b, c, d \in R$ we have:

$$\begin{aligned}(a \rhd b) \rhd' (c \rhd d) &= (a \rhd' c) \rhd (b \rhd' d)\\ (a \lhd b) \rhd' (c \lhd d) &= (a \rhd' c) \lhd (b \rhd' d)\end{aligned} \tag{2.31}$$

Example 2.8: For any $g \in \mathbf{Aut}_{\mathbf{Racks}}(X)$ we have a rack object $\rhd' = g \circ \pi_2$.

For $\rhd = \rhd'$, this property is sometimes referred to as "Abelian"[11]. However, since we soon will reserve this name for rack objects in **Ab**, we will instead refer to racks satisfying this property as Rack-racks.

For any category $\mathcal{C}$ with products and any $\mathcal{C}$-rack $(R, \rhd)$ we may evaluate whether $\rhd$ is a $\mathcal{C}$**Racks**-homomorphism. If it is, we call it a ($\mathcal{C}$-)Rack-rack.

Proposition 2.14: Let $(R, \rhd)$ be a Rack-rack and let $r \in R$ be an idempotent element. Then $- \rhd r \in \mathbf{End}_{\mathbf{Racks}}(R)$.

Proof. Let $a, b \in R$.

$$(a \rhd b) \rhd r = (a \rhd b) \rhd (r \rhd r) = (a \rhd r) \rhd (b \rhd r) \tag{2.32}$$

□

Proposition 2.15: Let R be a Rack-rack. If R contains a trivial element then $\mathbf{Inn}_{\rhd}(R)$ is an Abelian group with a trivial conjugation quandle.

Proof. Let $1 \in R$ be a trivial element. Then for all $a, b, c \in R$ we have

$$\begin{aligned}(a \rhd b) \rhd c &= (a \rhd b) \rhd (1 \rhd c) = (a \rhd 1) \rhd (b \rhd c)\\ &= ((a \rhd -) \rhd \mathrm{id})(b \rhd c) = \mathrm{id}(b \rhd c) = b \rhd c\end{aligned} \tag{2.33}$$

In other words $(a \rhd b) \rhd (-) = b \rhd (-)$. □

Proposition 2.16: Let R be a Rack-rack with more than one element. If R contains a monic argument, then R is a quandle, $\mathbf{Inn}_{\rhd}(R)$ is non-Abelian, and there are no trivial elements.

Proof. Let $b \rhd c$ be a monic argument. Then we have that

$$a \rhd (b \rhd c) = (a \rhd b) \rhd (a \rhd c) = (a \rhd a) \rhd (b \rhd c) \tag{2.34}$$

By the mono property, $a \rhd a = a$ for all a. The rest of the statement follows from corollary 2.6. □

Proposition 2.17: Let $(R, \rhd)$ be a Rack-rack and denote by $\hat{R}$ the subset of epic arguments in R. Then $\hat{R}$ is a subrack of R. If $|\hat{R}| > 1$ it is non-unital.

Proof. Let $x \in R$ and $a, b \in \hat{R}$. We must show that $a \rhd b \in \hat{R}$.

$$\begin{aligned}\exists d &: d \rhd b = b\\ \exists e &: e \rhd b = x\\ \exists c &: c \rhd a = e\\ (c \rhd d) \rhd (a \rhd b) = (c \rhd a) \rhd (d \rhd b) &= e \rhd b = x\end{aligned} \tag{2.35}$$

Units obviously do not give surjective maps, proving the last statement. □

Corollary 2.18: Let R be a finite Rack-rack such that the epic subrack is non-empty. Then R is a quandle without trivial elements.

As we soon will encounter a plethora of Rack-racks, let us briefly consider a set-rack which is *not* a Rack-rack.

Example 2.9: Let X be the free group generated by $\{a, b\}$ and let $x \triangleright y = xyx^{-1}$ be its quandle operation. Then,

$$\begin{aligned} (a \triangleright b) \triangleright (1 \triangleright b) = (aba^{-1}) \triangleright b = aba^{-1}bab^{-1}a^{-1} \\ (a \triangleright 1) \triangleright (b \triangleright b) = (1) \triangleright (b) = b \end{aligned} \tag{2.36}$$

These are distinct elements of X, so $(X, \triangleright)$ is not a Rack-rack.

Definition 2.7: Let $(Y, \triangleright_Y)$ be a $\mathcal{C}$-rack. For any object X in $\mathcal{C}$ define the **Hom-rack** of X and Y, denoted $(\mathbf{Hom}_{\mathcal{C}}(X, Y), \triangleright_{\mathbf{Hom}})$, by

$$\begin{aligned} f \triangleright_{\mathbf{Hom}} f' = \triangleright_Y \circ (f \times f') \\ (f \triangleright_{\mathbf{Hom}} f')(x) = f(x) \triangleright_Y f'(x) \end{aligned} \tag{2.37}$$

Note that the Hom-rack need not be a $\mathcal{C}$-rack in general, since $\mathbf{Hom}_{\mathcal{C}}(X, Y)$ need not be an object in $\mathcal{C}$.

Proposition 2.19: The $\mathbf{Hom}_{\mathbf{Racks}}$-rack of a rack X and a Rack-rack Y is
1) A Rack-rack.
2) A quandle if either X or Y is a quandle.
3) Involutive if either X or Y is involutive.
4) Unital if Y is unital.

Proof. 1) This follows from Y being a Rack-rack.
2) Whether X or Y is a quandle, the image of any morphism must consist of only idempotent elements, and the claim follows.
3) Whether X or Y is involutive, the image of any morphism must consist of only involutive elements, and the claim follows.
4) Each constant morphism mapping all of X to a unit of Y defines a unit in the Hom-rack. □

Proposition 2.20: Let X be a rack and let Y be a Rack-rack. Evaluation in X produces a rack homomorphism $(-)^* : X \to \mathbf{Hom}_{\mathbf{Racks}}(\mathbf{Hom}_{\mathbf{Racks}}(X, Y), Y)$ such that x^* maps $f \mapsto f(x)$.

Proof.

$$\begin{aligned} (x \triangleright_X y)^*(f \triangleright_{\mathbf{Hom}} g) &= f(x \triangleright_X y) \triangleright_Y g(x \triangleright_X y) \\ &= (f(x) \triangleright_Y f(y)) \triangleright_Y (g(x) \triangleright_Y g(y)) \\ &= (f(x) \triangleright_Y g(x)) \triangleright_Y (f(y) \triangleright_Y g(y)) \\ &= (x^*(f \triangleright_{\mathbf{Hom}} g)) \triangleright_Y (y^*(f \triangleright_{\mathbf{Hom}} g)) \\ &= (x^* \triangleright_{\mathbf{Hom}} y^*)(f \triangleright_{\mathbf{Hom}} g) \end{aligned} \tag{2.38}$$

□

Corollary 2.21: Let $(X, \triangleright)$ be a Rack-quandle. Then $\mathbf{Inn}_{\triangleright}(X)$ is a subquandle of $\mathbf{Hom}_{\mathbf{Racks}}(X, X)$.

Proof. Let $a, b \in \mathbf{Inn}_{\triangleright}(X) \subset \mathbf{Hom}_{\mathbf{Racks}}(X, X)$. Then for any $x \in X$ we have

$$x^*(a \triangleright_{\mathbf{Hom}} b) = (a \triangleright x) \triangleright (b \triangleright x) = (a \triangleright b) \triangleright (x \triangleright x) = (a \triangleright b) \triangleright x \tag{2.39}$$

□

Corollary 2.22: Let X, Y be rack objects in a concrete category $\mathcal{C}$, that is to say a category with a faithful functor $\mathcal{C} \to \mathbf{Set}$ taking each object to its underlying set of elements. If Y satisfies the $\mathcal{C}$Rack-rack property, then $\mathbf{Hom}_{\mathcal{C}\mathbf{Racks}}(X, Y)$ is a subrack of $\mathbf{Hom}_{\mathbf{Racks}}(X, Y)$.

2.5 Grp

In the category of groups, a rack object is equipped with a group homomorphism

$$R \times R \xrightarrow{\triangleright} R$$

Slightly abusing the notation for $\triangleright$ both as a binary operation and as a function, the homomorphism requirement on the direct product produces relations

$$(a \triangleright b) \cdot (c \triangleright d) = \triangleright((a, b) \cdot (c, d)) = ac \triangleright bd \tag{2.40}$$

$$(a \triangleright 1) \cdot (1 \triangleright d) = a1 \triangleright 1d = a \triangleright d = 1a \triangleright d1 = (1 \triangleright d) \cdot (a \triangleright 1) \tag{2.41}$$

Since there is a constant homomorphism - let's denote it by the same "1" used for the identity element - we can include the group in its product through $(1, \mathrm{id}(x))$ or $(\mathrm{id}(x), 1)$ and attain a pair of endomorphisms,

$$f = \triangleright(\mathrm{id} \times 1) : x \mapsto x \triangleright 1 \tag{2.42}$$

$$g = \triangleright(1 \times \mathrm{id}) : x \mapsto 1 \triangleright x \tag{2.43}$$

and by the inversion axiom we see that g is bijective and thus an automorphism. The homomorphism property then lets us conclude that $\triangleright$ factorizes, and the factors commute with respect to the group multiplication.

$$\triangleright(a, b) = f(a) \cdot g(b) = g(b) \cdot f(a) \tag{2.44}$$

With this factorization, the self-distributivity produces another set of relations.

$$a \triangleright (b \triangleright c) = f(a) \cdot gf(b) \cdot g^2(c) \tag{2.45}$$

$$(a \triangleright b) \triangleright (a \triangleright c) = f^2(a) \cdot fg(b) \cdot gf(a) \cdot g^2(c) \tag{2.46}$$

which we may evaluate at the identity elements to get

$$\begin{aligned} f(a) &= f^2(a) \cdot gf(a) \\ gf(b) &= fg(b). \end{aligned} \tag{2.47}$$

And so f and g also commute in the sense of composition.

Definition 2.8: A group rack morphism is a group homomorphism satisfying

$$\begin{aligned} \phi : (R, f, g) \to (R', f', g') \\ \phi f(x) \cdot \phi g(y) = f'\phi(x) \cdot g'\phi(y) \end{aligned} \tag{2.48}$$

In particular, a group rack endomorphism commutes with f and g, which means that f and g is a group rack endomorphism and group rack automorphism, respectively. Thus, group rack isomorphisms act by conjugation of the morphisms:

$$(f', g') = (\phi f \phi^{-1}, \phi g \phi^{-1}) \tag{2.49}$$

The previous calculations would hold for the other rack operation, and so we may represent the inverse operation by (k, h) to investigate the inversion axiom:

$$\begin{aligned} (a \rhd b) \lhd a =& k f(a) k g(b) h(a) = b \\ a \rhd (b \lhd a) =& f(a) g k(b) g h(a) = b \\ a = 1 \Rightarrow & k = g^{-1} \\ b = 1 \Rightarrow & f(a) \cdot g h(a) = 1 \end{aligned} \tag{2.50}$$

Theorem 2.23: There is an isomorphism of categories **RacksGrp** = **GrpRacks**.

Proof. A group object $(G, 1, i, \mu)$ in **Racks** is a **Set**-group G such that $1, i$ and μ are rack homomorphisms.

$$\mu((\pi_1, \pi_2) \rhd (\pi_3, \pi_4)) = \mu((\pi_1 \rhd \pi_3), (\pi_2 \rhd \pi_4)) = \mu(\pi_1, \pi_2) \rhd \mu(\pi_3, \pi_4) \tag{2.51}$$

However, labeling $(\pi_1, \pi_2, \pi_3, \pi_4)$ with placeholder variables (a, c, b, d) reveals that this is precisely the criterion that $\rhd$ be a group homomorphism. Furthermore, the unit of a group is always an idempotent rack element, so its inclusion is a rack homomorphism. The inversion being a rack homomorphism then follows from the rack operation being a group homomorphism, though it is worth noting that i is part of the object information in **RacksGrp**, and is not necessarily a group homomorphism.

That group rack homomorphisms are rack group homomorphisms is proven in an entirely analogous fashion. □

Lemma 2.24: For any group rack (G, f, g), $\mathrm{Im} f \subseteq Z(G)$.

Proof. By Eq. (2.44), $\mathrm{Im} f$ must commute with $\mathrm{Im}\ g$. However, by Eq. (2.43), g is an isomorphism. In particular it is surjective, and so $\mathrm{Im} f$ commutes with every element. □

This means we may freely introduce the notation

$$gh = -f : a \mapsto (f(a))^{-1} \tag{2.52}$$

as this will always be a valid homomorphism, not an antihomomorphism.

Lemma 2.25: Each group rack is also a Rack-rack.

Proof. We need to verify that (2.31) holds. Using that $fg = gf$ and that the image of f is central, we have

$$\begin{aligned} (a \rhd b) \rhd (c \rhd d) =& f(a) g(b) \rhd f(c) g(d) \\ = f(f(a) g(b)) g(f(c) g(d)) =& f^2(a) f g(b) g f(c) g^2(d) \\ = f^2(a) f g(c) f g(b) g^2(d) =& (a \rhd c) \rhd (b \rhd d) \end{aligned}$$

□

Group quandles

The quandle axiom yields

$$a \triangleright a = f(a)g(a) = a \tag{2.53}$$
$$f(a) = a \cdot g(a^{-1}) \tag{2.54}$$

Note the slight distinction from the Alexander quandles, and that the kernel of f corresponds to the fixed points of g.

Remark: For a chosen g, requiring quandle structure uniquely determines f by (2.54). Let us henceforth denote this unique endomorphism by the letter q.

Corollary 2.26: Let $(R, \triangleright)$ be a group quandle. For any $a, b \in R$, we have that $a \triangleright b = b$ if and only if $b \triangleright a = a$.

Proof. Assume $a \triangleright b = f(a)g(b) = ag(a^{-1})g(b) = b$. Divide by $g(b)$ on both sides and we find that $f(a) = f(b)$. Thus, $b \triangleright a = f(b)g(a) = f(a)g(a) = a$. □

Lemma 2.27: Let (R, f, g) be a group rack. If R contains an epic (or monic, iso) argument, then:
1) f is an epimorphism (monomorphism, isomorphism).
2) R is an epic (monic, iso) rack.
3) R is a quandle.
4) R is an Abelian group.

Proof. 1) An argument b maps $a \mapsto f(a) \cdot g(b)$. Since multiplication with a fixed $g(b)$ is a bijection, it is clear that f must be the right kind of morphism.
2) From 1), b is interchangeable for any other element.
3) Since f and g commute, any cancellative property lets us conclude from $f(-) = f(f(-) \cdot g(-)) = (f \cdot g) \circ f(-)$ that $\mathrm{id} = f \cdot g$.
4) Finally, we have that up to isomorphism $Z(R) \supset \mathrm{Im} f \simeq R$, and thus R must be Abelian. □

Corollary 2.28: The only non-quandle **Grp**-rack structures on $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{Z}_p$ where p is a prime are the permutation racks.

Proof. Any $f \neq 0$ is a monomorphism on these groups. By lemma 2.27, (f, g) is then a quandle. □

Involutive group racks

In the special case of the rack operation being involutive, (2.47) yields

$$\begin{aligned} a \triangleright (a \triangleright b) = f(a)gf(a)g^2(b) &= b \\ f &= -gf \\ f^2 &= 2f \\ g^2 &= \mathrm{id} \end{aligned} \tag{2.55}$$

From corollary 2.8 and lemma 2.27 we know symmetric and isotropic group racks must be quandles on Abelian groups. We will tackle these properties after properly introducing Abelian quandles in section 2.6.

Unital group racks

A unital rack is not very interesting in this category. The categorified unit from definition 2.6 must be the identity element since that is the only terminal morphism in **Grp**. The relations $a \triangleright 1 = f(a) = 1$ and $1 \triangleright a = g(a) = a$ for any a imply that (f, g) must be $(1, \mathrm{id})$, the trivial quandle.

Corollary 2.29: The only group quandle to contain a trivial element is the trivial quandle.

Proof. Let r be a trivial element. Since $r \triangleright 1 = 1$, $r \in \ker q$ and so, just like in the unital case, we get $g = \mathrm{id}$. The quandle property then enforces $q = 1$. □

Note that this does not prevent trivial elements from existing for non-quandle group racks.

Corollary 2.30: If G is a centerless group, the only quandle structure on it is the trivial quandle.

Proof. $\mathrm{Im} q = \{1\} \Rightarrow q(a) = a \cdot g(a^{-1}) = 1 = aa^{-1} \Rightarrow g = \mathrm{id}$ □

Corollary 2.31: If G is a perfect group, that is to say every group element can be written as a commutator $aba^{-1}b^{-1}$ for some a, b, then the only quandle structure on G is the trivial quandle.

Proof.

$$q(x) = q(aba^{-1}b^{-1}) = q(a)q(b)q(a^{-1})q(b^{-1}) = 1 \tag{2.56}$$

□

2.6 Ab

In the category of Abelian groups endomorphism centrality is trivial, and by the product being a biproduct, our investigation into potential rack structures becomes simplified. We can now comfortably represent the operation as a matrix.

$$\triangleright(a, b) \equiv a \triangleright b = \begin{pmatrix} f, g \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.57}$$

Let's briefly reformulate the group relations of the previous section in terms of addition, where 1 now denotes the identity automorphism rather than the identity element. The self-distributive axiom yields

$$\begin{aligned} f^2 =& (1 - g)f \\ gf =& fg \end{aligned} \tag{2.58}$$

Similar results hold for the inverse operation, so it too can be characterized by a representative matrix $\triangleleft = (k, h)$ satisfying

$$\begin{aligned} k =& g^{-1} \\ h =& -g^{-1}f \end{aligned} \tag{2.59}$$

From this we verify that a general rack operation on an Abelian group is fully characterized by two endomorphisms, one of which is an automorphism. We may formulate this in the following equivalent way:

Theorem 2.32: Define the commutative ring $\Gamma = \mathbb{Z}[f, g, g^{-1}]/(f^2 - f + fg)$. The category **AbRacks** of Abelian racks is isomorphic to **Mod**Γ.

Proof. Let M be a Γ-module. $\triangleright(m, m') = f \cdot m + g \cdot m'$ defines an Abelian rack structure on M. Conversely, given any Abelian rack (R, f', g'), the relations $((f')^2 - f' + f'g')$ and $f'g' = g'f'$ are satisfied and thus mapping $(f, g) \mapsto (f', g')$ gives R a Γ-module structure. Group rack homomorphisms are easily seen to be precisely the module homomorphisms. □

Corollary 2.33: **AbRacks** is an Abelian subcategory of **Ab**. This entails:
$\mathbf{Hom}_{\mathbf{AbRacks}}(X, Y)$ is an **Ab**-rack.
For any $f \in \mathbf{Hom}_{\mathbf{AbRacks}}(X, Y)$, im$f$, ker$f$ and cok$f = Y/$im$f$ are **Ab**-racks.
Pullbacks and pushouts of **AbRacks**-morphisms are **Ab**-racks.

Abelian quandles

Just like in **Grp**, the quandle of an automorphism is uniquely determined by being a left inverse to the diagonal.

$$\triangleright(\Delta(a)) = a \tag{2.60}$$
$$q + g = 1 \tag{2.61}$$

There are some things to note here. First, this quandle structure exists for any automorphism g, unlike in the non-commutative group case where centerless quandles are trivial, so in terms of theorem 2.32 the category of **Ab**-quandles is simply the module category of the Laurent polynomial ring $\mathbb{Z}[g, g^{-1}]$. Secondly, this always holds on the image of a rack's f according to (2.58), which means Imf is a subquandle of any Abelian rack. Finally, for a quandle, $g^2 = 1$ always implies $q^2 = 2q$ and is therefore sufficient for involution.

Theorem 2.34: If (f, g) is an Abelian rack structure, so is its quandle-mirror $(q - f, g)$.

Proof. Insert $f = q - x$ into (2.58)

$$f^2 = (q - x)f = qf \Rightarrow xf = 0 \Rightarrow x^2 = qx \tag{2.62}$$

Meaning that $(x, g) = (q - f, g)$ also is a rack structure. □

Corollary 2.35: Any automorphism $g \neq 1$ on $\mathbb{Z}^n$ has either infinite or an even number of available rack structures. In particular, you always have q and 0.
If $g = 1$ there are infinitely many rack structure satisfying $f^2 = 0$ if $n > 1$.

Proof. The only way f is not part of a distinct pair is if $q - f = f$.

$$q - f = f \Rightarrow 2f = q = 1 - g \Rightarrow f^2 = (2f)f = 2f^2 \Rightarrow f^2 = 0 \tag{2.63}$$

However, this implies that (nf, g) is an eligible rack structure for any $n \in \mathbb{N}$, and thus there are either infinite rack structures or $f = 0$. □

Example 2.10: $\mathbb{Z}$
The only automorphisms are ± 1, and so the only allowed rack operations are given by $(f, \pm 1)$, where $f^2 = (1 \mp 1)f$.
The pair $(0, 1)$ is obviously the trivial kei structure.
The pair $(0, -1)$ is an involutive rack but not a quandle.
The pair $(2, -1)$ is a kei.
Since f must be an integer, there are no further solutions.

Example 2.11: $\mathbb{Z}^n$
The automorphism g is an integer matrix, and thus must have determinant ± 1.
The endomorphism f is required to commute with g as a matrix.
Obviously, diagonal matrices with pairwise entries from the one-dimensional case qualify, and preserve any additional structure if it is present in every entry.

For $n = 2$, we can diagonally construct
$(f, g) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ is a kei from two kei.
$(f, g) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ is an involutive rack from two involutive racks.
$(f, g) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ is a kei from two kei.
$(f, g) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right)$ is an involutive rack but not a quandle.
We can also verify the existence of
$(f, g) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right)$ is only a rack.
$(f, g) = \left(\begin{pmatrix} -1 & -3 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right)$ is the quandle.[1]
However, we cannot yet construct more than one of the infinite nilpotent family
$(f, g) = \left(\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$ are all rack structures.

Example 2.12: Torus racks in **Ab** and **Top**.
The standard tori $T^n = (S^1)^n$ are examples of nice topological groups with a group structure inherited from the addition of the reals. Group homomorphisms are restricted to integer matrices by periodicity, and so the set of Abelian racks on T^n is in 1-1 correspondence with the set of Abelian racks on $\mathbb{Z}^n$.

However, were we to consider the same rack structures in the category **Top**, there are suddenly far more rack-compatible morphisms available than in **Ab**. Not only do we have the compatible integer matrices, we also have all the inner automorphisms and arbitrary translations for quandles.

Example 2.13: Positive-definite involutive racks on $\mathbb{Z}^n$.
Let g be a positive-definite involution. q then has no non-zero eigenvalues and is therefore nilpotent. By (2.58), this means that any rack's f must also be

[1]Note that even if g is an integer matrix, not every f that fits the rack criteria needs to be. For instance, for $g = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $f = (1 - g)/2 \pm \frac{1}{2\sqrt{3}}(1 + g)$ would give a perfectly acceptable rack structure acting on the real plane.

nilpotent.

$$\begin{aligned} 0 = f^n = q^{n-1}f = 2^{n-2}qf \\ \Rightarrow f^2 = qf = 0 \end{aligned} \tag{2.64}$$

In particular, the unique quandle structure of g satisfies $q^2 = 2q = 0 \Rightarrow g = 1$. By uniqueness, we have in a roundabout way proven that the identity matrix is the only positive-definite involution.

Lemma 2.36: The category of (co-)symmetric **Grp**-quandles is $\mathbf{Mod}\mathbb{Z}[2^{-1}]$ and the category of bi-symmetric **Grp**-quandles is $\mathbf{Mod}\mathbb{Z}_3$.

Proof. Let $(X, \triangleright)$ be a group quandle such that $\triangleright$ is symmetric, i.e. $x \triangleright y = y \triangleright x$ for all $x, y \in X$. All arguments are iso, so R is an Abelian quandle. Then,

$$\begin{aligned} (1-g)(a) + g(0) = (1-g)(0) + g(a) \\ 2g = (1-g) + g = 1 \end{aligned} \tag{2.65}$$

In other words, the symmetric quandles $(R, 2^{-1}, 2^{-1})$ only exist for groups where multiplication by 2 is invertible, i.e. $\mathbf{Mod}\mathbb{Z}[2^{-1}]$. Since the dual quandle is $\triangleleft = (2, -1)$, the symmetric quandle is bi-symmetric if and only if $-1 = 2 = 2^{-1}$, which means the group is of order 3. $\square$

It is worth noting that the bi-symmetric quandle is both involutive and isotropic. By the two-out-of-three property, this lets us conclude that for any group of order different from 3, symmetry, cosymmetry, involutivity and isotropy are exclusive properties, meaning at most one can hold for any rack structure on such a group.

Lemma 2.37: The category of isotropic **Ab**-quandles is $\mathbf{Mod}\mathbb{Z}[g]/(g^2 - g + 1)$.

Proof. Let $(X, \triangleright)$ be a group quandle where $\triangleright = \triangleleft$, i.e. $x \triangleright y = x \triangleleft y$ for all $x, y \in X$. Again all arguments are iso, so it is an Abelian quandle.

$$\begin{aligned} f(a) + g(b) &= g^{-1}(a) - fg^{-1}(b) \\ f &= g^{-1} \\ g^{-1} + g &= 1 \end{aligned} \tag{2.66}$$

By inserting (g, g^{-1}) into theorem 2.32 we see that as a subcategory of $\mathbf{Mod}\Gamma$, the category of isotropic **Ab**-quandles is equivalent to $\mathbf{Mod}\mathbb{Z}[g]/(g^2 - g + 1)$. $\square$

By way of the characteristic polynomial $g^2 - g + 1 = (g-2)(g+1) + 3 = 0$, we find $g = 2$ on $\mathbb{Z}_3$, $g = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3}$ on $\mathbb{C}$, and eight isotropic quandles on $\mathbb{Z}^2$:

$$g = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.67}$$

Flipping the first $\pm$ symbol constitutes reordering the diagonal, flipping all three constitutes inversion, and transposition flips the last.

Example 2.14: Alexander quandles revisited.
Recall the Alexander quandles from example 2.2.

$$\begin{aligned} &a \triangleright b = aba^{-1} && b \triangleleft a = a^{-1}ba && a \triangleright b = b \triangleleft a^{-1} && (2.68)\\ &a \triangleright b = a\phi(ba^{-1}) && b \triangleleft a = \phi^{-1}(a^{-1}b)a && a \triangleright b = \phi^2(b) \triangleleft \phi(a^{-1}) && (2.69)\\ &a \triangleright b = ab^{-1}a && b \triangleleft a = ab^{-1}a && a \triangleright b = b \triangleleft a && (2.70)\\ &a \triangleright b = a\eta(b^{-1})\eta(a) && b \triangleleft a = \eta^{-1}(a)\eta^{-1}(b^{-1})a && a \triangleright b = \eta^2(b) \triangleleft \eta(a) && (2.71) \end{aligned}$$

Here, ϕ can be any group automorphism and η any anti-automorphism. Since the standard interpretation of **Grp** does not include the anti-homomorphisms, these are generally only **Set**-racks defined from **Set**-groups, not **Grp**-racks. However, since Abelian groups are **Grp**-groups, the Alexander quandle objects that *are* in **Grp** *do* define the **Ab**-quandles $(1 - \phi, \phi)$.

2.7 Internal rack action and crossed modules

Every rack we have seen so far obviously acts on their underlying set, and we can discuss how each individual rack element acts. This is because our examples so far have all been concrete racks, but in principle a categorical rack object does not need to be. We now want to generalize the rack action to abstract racks in such a way that each rack object acts on its underlying object.

Recall that a concrete rack R can act on a set X by bijections such that for all $r_1, r_2 \in R, x \in X$, we have

$$r_1 \cdot (r_2 \cdot x) = (r_1 \triangleright r_2) \cdot (r_1 \cdot x) \tag{2.72}$$

Categorically, this constitutes a **Set**-morphism $\cdot : R \times X \to X$ satisfying

$$\cdot \circ (\pi_1 \times \cdot) = \cdot \circ ((\pi_1 \times \cdot) \circ (\triangleright \times \pi_1 \times \pi_3)) \tag{2.73}$$

This is a slight variation of the self-distributive diagram where one factor is distinguished:

$$\begin{tikzcd} R^2 \times X \arrow[rr, "(\triangleright \times \pi_1) \times \pi_3"] \arrow[d, "\pi_1 \times \cdot"] & & R^2 \times X \arrow[d, "\pi_1 \times \cdot"] \\ R \times X \arrow[r, "\cdot"] & X & R \times X \arrow[l, "\cdot"'] \end{tikzcd} \tag{2.74}$$

That the action is by bijections comes from the inversion axiom, and necessitates a second morphism to act by the inverses. If we label the morphisms respectively $\cdot_\triangleright$ and $\cdot_\triangleleft$, their action on a distinguished X is given by

$$\begin{tikzcd} R \times X \arrow[r, "\pi_1 \times \cdot_\triangleleft"] \arrow[d, "\pi_1 \times \cdot_\triangleright"'] \arrow[dr, "\mathrm{id}"] & R \times X \arrow[d, "\pi_1 \times \cdot_\triangleright"] \\ R \times X \arrow[r, "\pi_1 \times \cdot_\triangleleft"'] & R \times X \end{tikzcd} \tag{2.75}$$

Clearly, $\cdot_\triangleright = \triangleright$ defines such an action of any rack object on itself, but it is worth noting that the inverse action is $\triangleleft \circ (\pi_2 \times \pi_1)$, thanks to the distinguished factor. One could equivalently formulate $\cdot_{\triangleleft}$ as an action from the right through a morphism $X \times R \to X$, with slightly different diagrams.

We can verify that fixing the first argument of the conjugation $\triangleright$ produces the kind of invertible group homomorphism we call an inner automorphism.

$$a \triangleright (bc) = abca^{-1} = aba^{-1} \cdot aca^{-1} = (a \triangleright b) \cdot (a \triangleright c) \tag{2.76}$$

Evaluation of inner automorphisms at a fixed second argument is not a group homomorphism, but rather an action:

$$(ab) \triangleright c = ab \cdot c \cdot b^{-1}a^{-1} = a \triangleright (b \triangleright c) \neq (a \triangleright c) \cdot (b \triangleright c) \tag{2.77}$$

If the group is not Abelian - which would make the rack trivial - the above facts violate our definition of a rack object internal to **Grp**. Instead, the conjugation quandle is an example of a *crossed module*[1].

Definition 2.9: A **crossed module** of groups (resp. racks) is a quadruple $(G, H, \mu, \cdot)$ where G and H both are groups (or racks), $\cdot$ is an action of H on G, and μ is a group (or rack) homomorphism from G to H such that

$$\begin{aligned} \mu(g) \cdot x =& g \triangleright x \in G \\ \mu(h \cdot x) =& h \triangleright \mu(x) \in H \end{aligned} \tag{2.78}$$

where $\triangleright$ in the group case is assumed to be the conjugation action.

Let us draw a diagram to show how this differs from categorical rack objects:

$$\begin{tikzcd} G \times G \arrow[r, "\mu \times \pi_2"] \arrow[dr, "\triangleright"] & H \times G \arrow[r, "\pi_1 \times \mu"] \arrow[d, "\cdot"] & H \times H \arrow[d, "\triangleright"] \\ & G \arrow[r, "\mu"] & H \end{tikzcd} \tag{2.79}$$

In the categorical definition of group rack, these arrows would be group homomorphisms. However, if $\triangleright$ is the conjugation action this fails to be the case, and there is no general reason to expect the non-horizontal maps to be morphisms just because the horizontal ones are. Let us investigate this distinction further.

Example 2.15: If R is a rack, $(R, R, \mathrm{id}, \triangleright)$ and $(R, \mathbf{Aut}_{\mathbf{Racks}}, (-\triangleright), eval)$ are both crossed modules of racks yielding the natural rack action of R on itself.

Example 2.16: If G is a group, $(G, \mathbf{Aut}_{\mathbf{Grp}}(G), \Phi_G, eval)$ is a crossed module of groups representing the conjugation action of G on itself.

We can now motivate the construction of a functor **Conj** : **Grp** $\to$ **Racks** sending each group to its conjugation quandle. This has an adjoint functor **As** : **Racks** $\to$ **Grp** associating the free group generated by the rack elements modulo the relation $x \triangleright y \sim xyx^{-1}$.

Note that **Conj** always produces a unital quandle with the identity being a unit, and there is a canonical rack morphism $i : R \to \mathbf{Conj}(\mathbf{As}(R))$. This need not be injective nor surjective, however.

Example 2.17: Let R be a rack that is not a quandle. Then for some x we have $x \triangleright x = y \neq x$. However, by construction $\mathbf{As}(R)$ identifies

$$y = x \triangleright x \sim xxx^{-1} = x$$

We see that i is not injective.

Example 2.18: Let R be the trivial quandle with n elements. The associated group is then the free Abelian group $\mathbb{Z}^n$ and the conjugation quandle is the trivial quandle on countably many elements. Thus, i is not surjective.

Example 2.19: Let R be a finite involutive rack. Then in $\mathbf{Conj}(\mathbf{As}(R))$ we have $b = a \triangleright (a \triangleright b) = a^2 \triangleright b$ and so $a^2 \in Z(\mathbf{As}(R))$ for all $a \in R$. Furthermore,

$$\begin{aligned} a \triangleright b =& b \triangleleft a \\ aba^{-1} =& a^{-1}ba \\ (a \triangleright b)^2 \sim aba^{-2}ba =& ab^2aa^{-2} = b^2 \end{aligned}$$

Each square is a unit in the conjugation quandle, and by the free construction there is at least one for each element of R. Thus i is not surjective.

Example 2.20: Consider the Abelian involutive quandle structure $(\mathbb{Z}_5, (2, -1))$. The group associated to this quandle is

$$\mathbb{Z}(\mathbb{Z}_5)/((2x - y \mod 5) \sim xyx^{-1}) \tag{2.80}$$

Note that 0 is not the identity in the associated group. Rewriting the relation as $xy \sim (2x - y)x$ we can see that this identifies a lot of binary products:

$$\begin{aligned} 01 &\sim 40 \sim 34 \sim 23 \sim 12 \\ 02 &\sim 30 \sim 13 \sim 41 \sim 24 \\ 03 &\sim 20 \sim 42 \sim 14 \sim 31 \\ 04 &\sim 10 \sim 21 \sim 32 \sim 43 \\ nm \sim (n+x)(m+x)&|x, n, m \in \mathbb{Z}_5, n \neq m \end{aligned}$$

We see the rack morphism into the associated conjugation quandle is injective but obviously not surjective. However, due to the involution we know from the previous example that squares are central. By the relation table above this means that any product of three elements can be written as a central element times a basis element, so not only is i injective, but the only inner automorphism in $\mathbf{Conj}\left(\mathbf{As}(\mathbf{R})/\mathbf{Z}(\mathbf{As}(\mathbf{R}))\right)$ not hit is the unit introduced by the identity.

Lemma 2.38: Let R be a rack and G be a group. For any rack morphism $f : R \to \mathbf{Conj}(G)$, there exists a unique group homomorphism $g : \mathbf{As}(R) \to G$ such that $f = \mathbf{Conj}(g) \circ i$.

Proof. By definition, the subgroup of G generated by the image of f satisfies the conjugation relation of the associated group. Since any group can be realized as a quotient of a free group, this equates to saying that the generated subgroup is isomorphic to a quotient of $\mathbf{As}(R)$. By the first isomorphism theorem we may realize our desired g via this quotient by composing the canonical projection map with its inclusion as the subgroup, and we see that $\mathbf{Conj}(g) \circ i$ indeed recovers f.

To prove uniqueness, we see that if $g \neq h$ there exists some formal product $x = \prod_i r_i$ of elements in R such that $g(x) \neq h(x)$. However, this implies that at least one r_i has $g(r_i) \neq h(r_i)$, which means they do not induce the same f. □

We may now revisit what happened when we lifted set elements to bijections.

Corollary 2.39: If R is a rack acting on a set X, we may factorize this action through the conjugacy of **Set**-bijections: $R \to \mathbf{Conj}(\mathbf{Aut}_{\mathbf{Set}}(X))$ where

$$a \rhd (b \rhd x) = \alpha(\beta(x)) = (\alpha\beta\alpha^{-1})(\alpha(x)) = (a \rhd b) \rhd (a \rhd x) \tag{2.81}$$

$$\begin{tikzcd} & \mathbf{Conj}(\mathbf{Aut}_{\mathbf{Set}}(X)) \times X \arrow[dl, "Evaluation"'] & \\ X & & (R, \rhd) \times X \arrow[ul, "Morphism"'] \arrow[ll, "\cdot"] \end{tikzcd} \tag{2.82}$$

Proof. By the inversion axiom, the action of a rack element a has to correspond to a bijection $a \rhd - = \alpha(-)$ of X. Self-distributivity implies that $(a \rhd b) \rhd -$ takes all $\alpha(x)$ to $\alpha(\beta(x))$, but this defines precisely the bijection $\alpha\beta\alpha^{-1}$. □

We now see that the rack objects in **Ab** and **Grp** define internal actions on the underlying sets by $\mathbf{Aut}_{\mathbf{Set}}$, not necessarily $\mathbf{Aut}_{\mathbf{Grp}}$. Indeed, the only group racks that act by $\mathbf{Aut}_{\mathbf{Grp}}$ are the permutation racks of the form $\rhd = (1, g)$, as the action otherwise would not map the identity to itself. Furthermore, combining this with lemma 2.38 lets us derive the adjoint of example 1.3.

Corollary 2.40: Any internal rack action of a **Set**-rack R defines a **Set**-group action of $\mathbf{As}(R)$. In particular, for any concrete rack there is a surjective group homomorphism $\mathbf{As}(R) \to \mathbf{Inn}_{\rhd}(R)$ owing to the self-action of R.

Proposition 2.41: As and **Conj** send crossed modules to crossed modules.

Proof. Let $(X, A, \mu, \cdot)$ be a crossed module of racks. By corollary 2.39 we know that both the self-action $\rhd$ and the action $\cdot$ factor through $\mathbf{Conj}(\mathbf{Aut}_{\mathbf{Set}}(X))$ by some rack homomorphisms ξ, σ which by lemma 2.38 are induced by group homomorphisms Ξ, Σ:

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "\xi"'] & A \arrow[d, "\sigma"] \\ & \mathbf{Conj}(\mathbf{Aut}_{\mathbf{Set}}(X)) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{As}(X) \arrow[r, "\mathbf{As}(\mu)"] \arrow[dr, "\Xi"'] & \mathbf{As}(A) \arrow[d, "\Sigma"] \\ & \mathbf{Aut}_{\mathbf{Set}}(X) \end{tikzcd} \tag{2.83}$$

By the functoriality of the free group construction we see that $\mathbf{Aut}_{\mathbf{Set}}(X)$ acts naturally on it by group automorphisms. This passes to $\mathbf{As}(X)$ by satisfying

$$f(x \rhd y) = f(x)f(y)f(x^{-1}) = f(x) \rhd f(y), \tag{2.84}$$

thus making it well-defined.

We may now verify that it satisfies the crossed module requirements:

$$\begin{aligned} (\Sigma \circ \mathbf{As}(\mu))(a) : xy \mapsto (a \rhd x)(a \rhd y) \\ \mathbf{As}(\mu)(a) \cdot b = a \rhd b \in \mathbf{As}(R) \end{aligned} \tag{2.85}$$

$$\begin{aligned} \mathbf{As}(\mu)(x \cdot a) = \mathbf{As}(\mu)(\Sigma(x)(a)) \\ = \mathbf{As}(\mu)\left(\prod_i (x_i) \cdot (a)\right) = \left(\prod_i x_i\right) \rhd \mathbf{As}(\mu)(a) = x \rhd \mathbf{As}(\mu)(a) \end{aligned} \tag{2.86}$$

That **Conj** preserves crossed modules is demonstrated in example 1.3. □

Definition 2.10: Let R be a **Set**-rack. The **associated monoid** of R is the category with a single object $*$ and $\mathbf{End}(*) = \mathbf{As}(R)$. The **inner monoid** of R is the category with a single object $*$ and $\mathbf{End}(*) = \mathbf{Inn}_{\triangleright}(R)$.

Such a monoid can act on objects in other categories.

Definition 2.11: Let M be a monoid, i.e. a category with a single object $*$. A **categorical action** of M on an object X in another category $\mathcal{C}$ is a functor $F : M \to \mathcal{C}$ such that $F(*) = X$.

2.8 Conjugated racks

We have seen that any rack R together with $\mathbf{Aut}_{\mathbf{Racks}}(R)$ defines a crossed module. In particular this means that for any rack automorphism f we have

$$\Phi^{\triangleright}_{f(a)} = f \triangleright \Phi^{\triangleright}_{a} = f\Phi^{\triangleright}_{a} f^{-1} \tag{2.87}$$

even if f is not an inner automorphism. In other words the subgroup $\mathbf{Inn}(R)$ generated by the inner automorphisms is a normal subgroup of $\mathbf{Aut}_{\mathbf{Racks}}(R)$. One can then define the centralizer $C_{\triangleright}$ of $\mathbf{Inn}(R)$ as a subgroup of $\mathbf{Aut}_{\mathbf{Racks}}(R)$:

$$f \in C_{\triangleright} \Rightarrow a \triangleright f(b) = \left(f \circ (a \triangleright -) \circ f^{-1}\right)(f(b)) = f(a \triangleright b) \tag{2.88}$$

Each such f defines a new rack structure on the underlying set of R,[11]

$$a \triangleright_f b = a \triangleright f(b) \tag{2.89}$$

We say that $\triangleright_f$ is **conjugated** to $\triangleright$ by f, and one may easily verify that this is an equivalence relation. An intuitive interpretation of this process can be attained by considering $C_{\triangleright}$ to consist of symmetries of the rack action, since $f(a) \triangleright - = a \triangleright -$ for all a. One of these is of special interest.

Definition 2.12: For a rack $(R,\triangleright)$, the $\triangleleft$-square, denoted ι, is given by

$$a \triangleright \iota(a) = a \Leftrightarrow \iota(a) = a \triangleleft a \tag{2.90}$$

Proposition 2.42: $\iota \in C_{\triangleright}$.

Proof. We need to show that ι a distributive bijection that satisifes (2.88) for any $a, b \in R$. Centrality is easy:

$$(a \triangleright b) \triangleright \iota(a \triangleright b) = a \triangleright b = a \triangleright (b \triangleright \iota(b)) = (a \triangleright b) \triangleright (a \triangleright \iota(b)) \tag{2.91}$$

This immediately implies surjectivity by

$$a = a \triangleright (a \triangleleft a) = a \triangleright \iota(a) = \iota(a \triangleright a) \tag{2.92}$$

and injectivity follows by $\triangleright$ and $\triangleleft$ distributing over one-another:

$$\iota(a) \triangleright \iota(a) = (a \triangleleft a) \triangleright (a \triangleleft a) = (a \triangleright a) \triangleleft a = a \tag{2.93}$$

Finally we may verify distributivity by using the inversion axiom:

$$\begin{aligned} \iota(a \triangleright b) &= (a \triangleright \iota(a \triangleright b)) \triangleleft a = (a \triangleright (a \triangleright \iota(b))) \triangleleft a \\ &= ((a \triangleright \iota(a)) \triangleright (a \triangleright \iota(b))) \triangleleft a = (a \triangleright (\iota(a) \triangleright \iota(b))) \triangleleft a \\ &= \iota(a) \triangleright \iota(b) \end{aligned} \tag{2.94}$$

□

Theorem 2.43: For any categorical rack $(R, \triangleright)$, the conjugated rack $(R, \triangleright_\iota)$ is a categorical quandle.

Proof. First note that ι and $\triangleright_\iota$ are automatically morphisms.

$$\begin{aligned} \iota &= \triangleleft \circ \Delta \in \mathbf{Hom}_C(R, R) \\ \triangleright_\iota &= \triangleright \circ (\pi_1 \times (\triangleleft \circ \Delta)) \in \mathbf{Hom}_C(R \times R, R) \end{aligned}$$

The quandle property can be written $a \triangleright_\iota a = a \triangleright (a \triangleleft a) = a$, or alternatively

$$\begin{aligned} \triangleright_\iota \circ \Delta &= \pi_1 \circ (\triangleright \times \pi_1) \circ (\pi_1 \times (\triangleleft \circ \Delta)) \circ \Delta \\ &= \pi_1 \circ (\triangleright \times \pi_1) \circ (\mathrm{id} \times (\triangleleft \circ (\mathrm{id} \times \mathrm{id}))) \\ &= \pi_1 \circ (\triangleright \times \pi_1) \circ (\pi_2 \times \triangleleft) \circ \Delta \\ &= \pi_1 \circ \mathrm{id} \circ \Delta = \mathrm{id} \end{aligned}$$

Verifying the new rack structure's self-distributivity and invertibility in an element-independent manner is left as an exercise to the reader. □

Example 2.21: Every permutation $\mathcal{C}$-rack is conjugate to the trivial quandle.

Lemma 2.44: Let R be a group. Two **Grp**-rack structures $(f, g), (f', g')$ on R are conjugate if and only if $f = f'$.

Proof. If they are conjugate, then they must also be conjugate to the same unique quandle structure. Computing the $\triangleleft$-square yields a candidate

$$\begin{aligned} \triangleright = (f, g), \triangleleft &= (g^{-1}, -g^{-1}f) \\ \Rightarrow \iota = g^{-1}(\mathrm{id} - f) \Rightarrow \triangleright_\iota &= (f, \mathrm{id} - f) \end{aligned} \tag{2.95}$$

We may verify that ι is indeed a central rack homomorphism, i.e. $f\iota = f$.

$$\begin{aligned} f^2 &= (\mathrm{id} - g)f \\ \Rightarrow f\iota = fg^{-1}(\mathrm{id} - f) = g^{-1}(f - f^2) = g^{-1}gf &= f \end{aligned} \tag{2.96}$$

□

We now have two non-trivial ways to create new group rack structures:
1) Quandle-mirroring: $(f, g) \mapsto (1 - g - f, g)$
2) Quandlization: $(f, g) \mapsto (f, 1 - f)$ or $(f, g) \mapsto (1 - g, g)$.
We therefore realize that for any (f, g) the composition $(1-g-f, f+g)$ is a quandle which is trivial if and only if the original rack was a quandle. We may now improve on corollary 2.35.

Corollary 2.45: Let R be an Abelian group. Then:
1) The set of **Ab**-quandle structures on R is equal to $\mathbf{Aut}_{\mathbf{Ab}}(R)$.
2) Each **Ab**-rack structure on R can be uniquely written as $(1 - g, g')$ for some commuting automorphism pair (g, g').

Proof. 1) This claim follows from the existence and uniqueness of $(1 - g, g)$ for each automorphism g.
2) Any given rack structure (f, g) is conjugate to a unique quandle $(f, 1-f)$. In particular, this means that $(1 - f) \equiv g'$ is an automorphism.

□

In fact, we can rewrite the rack equation to find yet another symmetry:

$$f^2 = f(1-g) = f(1-1+f') = ff' \tag{2.97}$$

In other words, if $(f, 1-f')$ is a rack, then $(f', 1-f)$ is also a rack if and only if $f^2 = (f')^2$. Alternatively one could write the rack equation of $(1-g, h)$ as

$$(1-g)^2 = 1 - 2g + g^2 = (1-g)(1-h) = 1 - g - h + gh \tag{2.98}$$

and conclude that $g^2 - hg = g - h$. Then $(1-h, g)$ is a rack if and only if $h - g = h^2 - hg$, which we can combine to get that $g^2 - 2gh + h^2 = (g-h)^2 = 0$.

Corollary 2.46: Assume $n \in \mathbb{N}$ has no prime factors with multiplicity greater than 1. If $(\mathbb{Z}_n, (1-g, h))$ is a non-quandle **Ab**-rack, then $(\mathbb{Z}_n, (1-h, g))$ is not a rack.

Proof. By the Chinese remainder theorem, $\mathbb{Z}_n$ is isomorphic to the product of $\mathbb{Z}_{p^k}$ over its prime factors p with multiplicity k, and on this form $\triangleright$ becomes a diagonal matrix. The only non-quandle racks on the finite simple groups are the permutation racks, so any non-quandle on the product must have a permutation rack factor. However, the restriction of $g-h$ to that factor leaves only $-h$, which is an automorphism and does not square to 0. $\square$

Example 2.22: There are two families of racks on $\mathbb{Z}^2$ such that $f^2 = 0$.

$$(f, g) = \left(\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right) \tag{2.99}$$

$$(f, g) = \left(\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \right) \tag{2.100}$$

These two families are only compatible if either $n = 0$ or $m = 0$.

2.9 Categorical n-Racks

Aside from the categorification we've done, there exists another generalization of racks in the literature, the so-called n-Racks [5], which concerns operations that are n-ary rather than binary. Representing it as an action of R^{n-1} on R, the axioms look quite similar to the binary case:

Self-distributivity: $\vec{a} \triangleright (\vec{b} \triangleright c) = (\vec{a} \triangleright b_1, ..., \vec{a} \triangleright b_{n-1}) \triangleright (\vec{a} \triangleright c)$.

Invertibility For $\vec{a} \in R^{n-1}, x \in R$ there is a unique $b \in R$ so that $\vec{a} \triangleright b = x$. Again we introduce $\triangleleft$, and declare invertibility by $\vec{a} \triangleright (x \triangleleft \vec{a}) = (\vec{a} \triangleright x) \triangleleft \vec{a} = x$.

Some of the special properties of binary racks also generalize.

An **involutive** n**-rack** satisfies $\vec{a} \triangleright (\vec{a} \triangleright x) = x$ for all $\vec{a}, x$.

A k**-volutive** n**-rack** satisfies $(\vec{a} \triangleright -)^k = \mathrm{id}(-)$ for all $\vec{a}$.

An **isotropic** n**-rack** satisfies $(a_1, ..., a_{n-1}) \triangleright a_n = a_1 \triangleleft (a_2, ..., a_n)$.

A **symmetric** n**-rack** satisfies $\triangleright(\vec{a}) = \triangleright(\sigma(\vec{a}))$ for any $\sigma \in S_n$ and $\vec{a} \in R^n$.

A **unital** n**-rack** contains a unit 1 satisfying $(1, 1, ..., 1) \triangleright x = x$ and $\vec{a} \triangleright 1 = 1$.

A **weak** n**-quandle** is a one-sided inverse to the diagonal: $(x, ..., x) \triangleright x = x$.

A **strong** n**-quandle** satisfies $\vec{a} \triangleright x = x$ if $a_i = x$ for some i.

A **permutation** n**-rack** satisfies $\vec{a} \triangleright - = g \in \mathbf{Aut}(R)$ for all $\vec{a} \in R^{n-1}$.

An n**-rack homomorphism** is a morphism that commutes with $\rhd$:

$$f \circ \rhd = \rhd \circ (\prod_{i=1}^{n} f) \Leftrightarrow f(\vec{a} \rhd b) = (f(a_1), ..., f(a_{n-1})) \rhd f(b) \tag{2.101}$$

We will soon cover the categorical description of n-racks, but for the sake of keeping our equations reasonably compact, let us introduce some new notation.

Notation Let $(R, \rhd)$ be an n-rack.

For each $i \leq n$ define $\rhd^i : R^{n-1} \times R^m \to R^m$ and $\lhd^i : R^m \times R^{n-1} \to R^m$ by

$$\vec{a} \rhd^i \vec{x} = (x_1, ..., x_i, \vec{a} \rhd x_{i+1}, ..., \vec{a} \rhd x_m) \tag{2.102}$$

$$\vec{x} \lhd^i \vec{a} = (x_1 \lhd \vec{a}, ..., x_i \lhd \vec{a}, x_{i+1}, ..., x_m) \tag{2.103}$$

i.e. $\rhd^i$ applies $\rhd$ on the last $m - i$ entries and $\lhd^i$ applies $\lhd$ on the first i.

Denote by $\pi_{a,b}$ the product projection morphism $\prod_{i=a}^{b} \pi_i : R^n \to R^{b-a+1}$.

Definition 2.13: Let $(R, \rhd)$ be a binary rack. The **iteration** n**-Rack** of R uses the n-ary operation $\rhd_n$ given by the iteration of $\rhd$ on the rightmost entry:

$$\vec{a} \rhd_n b = a_1 \rhd (a_2 \rhd (... \rhd (a_{n-1} \rhd b)..) \tag{2.104}$$

The inverse operation $\lhd_n$ is given by iteration by $\lhd$ on the leftmost entry. Mapping a binary rack to its iteration rack defines a functor $\mathbf{Iter}_n$, as it takes rack homomorphisms to n-rack homomorphisms.

Corollary 2.47: Some special rack properties are preserved by iteration.
1) $\mathbf{Inn}_{\rhd_n}(R) \subseteq \mathbf{Inn}_{\rhd}(R)$ as an ideal subquandle.
2) If R is involutive and $\mathbf{Inn}_{\rhd}(R)$ is Abelian, $\mathbf{Iter}_n(R)$ is involutive.
3) The iteration n-rack of a unital rack is unital.
4) The iteration n-rack of a quandle is a weak n-quandle.
5) The iteration 3-rack of a symmetric kei is 3-volutive.
6) The iteration n-rack of a permutation rack is a permutation n-rack.

Proof. 1) $\mathbf{Inn}_{\rhd_n}(R)$ is generated by words of length $n-1$. For any $f \in \mathbf{Inn}_{\rhd}(R)$,

$$f \rhd (x_1 \cdot x_2 \cdot ... \cdot x_{n-1}) = (f \rhd x_1) \cdot (f \rhd x_2) \cdot ... \cdot (f \rhd x_{n-1}) \tag{2.105}$$

This preserves word length and the claim follows.
2) $\vec{x} \rhd (\vec{x} \rhd -) = (\prod_i x_i) \circ (\prod_i x_i)(-) = (\prod_i x_i^2)(-) = \mathrm{id}(-)$.
3) By 1) each constant argument of $\rhd$ is a constant argument of $\rhd_n$. Furthermore, for any trivial element 1, $(1, 1, ..., 1)$ also acts trivially.
4) Any expression formulated with only $\rhd$ and $\lhd$ holds trivially for any single idempotent element.
5) For any a, b, c we have

$$\begin{aligned} a \rhd (b \rhd (a \rhd (b \rhd c))) \\ = (a \rhd b) \rhd (b \rhd c) \\ = (b \rhd a) \rhd (b \rhd c) \\ = b \rhd (a \rhd c) \\ = (c \lhd a) \lhd b \end{aligned} \tag{2.106}$$

6) Without loss of generality, the iteration of $\lhd = g \circ \pi_1$ is just $g^{n-1} \circ \pi_1$. $\square$

Example 2.23: The iteration n-rack of the binary conjugation quandle is a weak n-quandle but not a strong n-quandle.

Definition 2.14: Let $(R, \rhd)$ be an n-rack. The **diagonal** of R is the binary rack $(R^{n-1}, \rhd^0)$. This defines a functor **Diag**.

Corollary 2.48: Some rack properties are preserved by diagonalization.
1) **Diag**(R) is k-volutive if and only if R is a k-volutive n-rack.
2) **Diag**(R) is unital if and only if R is a unital n-rack.
3) **Diag**(R) is a quandle if and only if R is a strong n-quandle.
4) **Diag**(R) is a permutation rack if and only if R is a permutation n-rack.

Definition 2.15: An n**-Rack object** $(R, \rhd)$ is an object R and a morphism pair $\rhd, \lhd :\ R \xleftarrow[\lhd]{} R^n \xrightarrow{\rhd} R$ such that the following diagrams commute:

$$
\begin{tikzcd}
R^{2n-1} \arrow[r, "\rhd^{n-1}"] \arrow[d, "\pi_{1,n-1}\times\rhd"'] & R^n \arrow[d, "\rhd"] \\
R^n \arrow[r, "\rhd"'] & R
\end{tikzcd}
\quad
\begin{tikzcd}
R^n \arrow[r, "\pi_{2,n}\times\lhd"] \arrow[d, "\rhd\times\pi_{1,n-1}"'] \arrow[dr, "\mathrm{id}"] & R^n \arrow[d, "\rhd\times\pi_{1,n-1}"] \\
R^n \arrow[r, "\pi_{2,n}\times\lhd"'] & R^n
\end{tikzcd}
\quad
\begin{tikzcd}
R^n \arrow[d, "\lhd"'] & R^{2n-1} \arrow[l, "\lhd^n"'] \arrow[d, "\lhd\times\pi_{n+1,2n-1}"] \\
R & R^n \arrow[l, "\lhd"]
\end{tikzcd}
\tag{2.107}
$$

It is clear that any categorical rack object gives rise to n-rack objects by iteration and any n-rack object gives rise to binary rack objects by diagonalization.

Example 2.24: As in section 2.6 let $\Gamma = \mathbb{Z}[f, g, g^{-1}]/(f^2 - f + fg)$. Any Γ-module has an Abelian $(n+1)$-rack structure by the iteration rack with $\rhd = (f, gf, g^2 f, ..., g^{n-1} f, g^n)$ of the binary rack (f, g). In this case invertibility is trivial. Let us verify self-distributivity:

$$
\begin{aligned}
&(\vec{a} \rhd^0 \vec{b}) \rhd (\vec{a} \rhd c) \\
&= \sum_{j=1}^{n} fg^{j-1} \left(g^n(b_j) + \sum_{i=1}^{n} fg^{i-1}(a_i) \right) + \left(\sum_{i=1}^{n} fg^{n+i-1}(a_i) \right) + g^{2n}(c) \\
&= \sum_{i,j=1}^{n} f^2 g^{j+i-2}(a_i) + \sum_{j=1}^{n} fg^{n+j-1}(b_j) + \sum_{i=1}^{n} fg^{n+i-1}(a_i) + g^{2n}(c) \\
&= \sum_{i=1}^{n} fg^{n+i-1}(a_i) + \sum_{i,j=1}^{n} f^2 g^{j+i-2}(a_i) + 0 \rhd (\vec{b} \rhd c) \\
&= \sum_{i=1}^{n} g^{i-1} \left(fg^n + \sum_{j=1}^{n} f^2 g^{j-1} \right) (a_i) + 0 \rhd (\vec{b} \rhd c)
\end{aligned}
\tag{2.108}
$$

We may now use induction on the relation $fg^n = (fg)g^{n-1} = fg^{n-1} - f^2 g^{n-1}$ to cancel the sum over j, leaving only an fg^0-term.

$$
\sum_{i=1}^{n} g^{i-1} f(a_i) + 0 \rhd (\vec{b} \rhd c) = \vec{a} \rhd (\vec{b} \rhd c) \tag{2.109}
$$

Example 2.25: Let (R, f, g) be an Abelian rack on a group of order n. Then for any $m \in \mathbb{N}$ there are $(mn+2)$-racks given by $\triangleright = (f, f, ..., f, g)$. Once more we verify self-distributivity since invertibility is obvious:

$$\begin{aligned}
&(\vec{a} \triangleright^0 \vec{b}) \triangleright (\vec{a} \triangleright c) \\
&= g^2(c) + \sum_{i=1}^{mn+1} \left(fg(b_i) + \sum_{j=1}^{mn+1} f^2(a_j) \right) + \sum_{j=1}^{mn+1} gf(a_j) \\
&= g^2(c) + \sum_{i=1}^{mn+1} \big(((mn+1)f^2 + fg)(a_i) + fg(b_i) \big) \\
&\equiv g^2(c) + \sum_{i=1}^{mn+1} \big((f^2 + fg)(a_i) + fg(b_i) \big) \mod n \\
&= g^2(c) + \sum_{i=1}^{mn+1} (f(a_i) + fg(b_i)) \\
&= \vec{a} \triangleright (\vec{b} \triangleright c)
\end{aligned} \tag{2.110}$$

The diagonal rack is given by two $mn+1$ by $mn+1$ block matrices F and G, where the first has f in every entry and the second has g on every diagonal entry. One easily verifies that $F^2 = (mn+1)Ff = Ff = (1-g)F$.

Example 2.26: Let $(R, (f, g))$ be an **Ab**-rack. Then $\mathbf{Diag}(\mathbf{Iter}_3(R)) = (R^2, (F, G))$, where $F = \begin{pmatrix} f & fg \\ f & fg \end{pmatrix} = f \begin{pmatrix} 1 & g \\ 1 & g \end{pmatrix}$ and $G = \begin{pmatrix} g^2 & 0 \\ 0 & g^2 \end{pmatrix}$. We may verify that this is an **Ab**-rack by computing $F^2 - F + GF$:

$$F^2 - F + GF = (f(1+g) - 1 + g^2)F \tag{2.111}$$
$$f^2 - f + f^2 g + g^2 f = -fg + f^2 g + g(f - f^2) = 0 \tag{2.112}$$

However, this is not a quandle for any $f \neq 0$, not even if $f = 1 - g$.

Lemma 2.49: Let $n > 2$ and let $(R, (\vec{f}, g))$ be a non-permutation **Ab**-n-rack. Then $\mathbf{Diag}(R)$ is a non-permutation, non-quandle **Ab**-rack.

Proof. Since the resulting automorphism is diagonal, it is sufficient to see that the endomorphism $F = \begin{pmatrix} \vec{f} \\ ... \\ \vec{f} \end{pmatrix}$ created by this process has non-diagonal terms if $\vec{f}$ has non-zero terms. □

Corollary 2.50: There are no non-trivial strong n-quandles in **Ab** for $n > 2$.

Corollary 2.51: Let R be a path-connected topological rack. If there exists an n such that $\mathbf{Diag}(\mathbf{Iter}_n(R))$ contains a trivial element, then $\mathbf{Inn}_{\triangleright}(R)$ contains less than n isotopy classes.

Proof. If there exists $\vec{x} \in R^{n-1}$ such that $\vec{x} \triangleright_n - = \mathrm{id}(-)$, then every word of length n in $\mathbf{Inn}_{\triangleright}(R)$ is isotopic to a word of length 1. □

Chapter 3

Invariants of rack objects

We have already seen a few functors that relate sets or groups to racks. For instance, for a category $\mathcal{C}$ with products, there are always the trivial quandles that can be assigned functorially:

$$\begin{aligned} TQ_{\mathcal{C}} : \mathcal{C} &\to \mathcal{C}\textbf{Racks} \\ X &\mapsto (X, \pi_2, \pi_1) \end{aligned} \tag{3.1}$$

It is clear that this functor commutes with any product-preserving functor, meaning trivial quandles are mapped to trivial quandles. As it turns out, product-preserving functors are precisely the ones that preserve rack object structures.

Theorem 3.1: Let $\mathcal{C}, \mathcal{D}$ be categories with products and let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving products. If $(R, \rhd, \lhd)$ is a rack object in $\mathcal{C}$, then $(F(R), F(\rhd), F(\lhd))$ is a rack object in $\mathcal{D}$.

Proof. In condensed form, the criteria for a rack object are contained in the following commutative diagrams, where the first one is simply a definition:

$$\begin{tikzcd}
& R^2 \arrow[dl, "\rhd"'] \arrow[dr, "\pi_1"] \arrow[d, dashed, "\rhd\times\pi_1"] & \\
R & R^2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"] & R \\
& R^2 \arrow[ul, "\pi_2"] \arrow[ur, "\lhd"'] \arrow[u, dashed, "\pi_2\times\lhd"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R^2 \arrow[r, "\pi_2\times\lhd"] \arrow[d, "\rhd\times\pi_1"'] \arrow[dr, "\mathrm{id}"] & R^2 \arrow[d, "\rhd\times\pi_1"] \\
R^2 \arrow[r, "\pi_2\times\lhd"'] & R^2
\end{tikzcd}
\qquad
\begin{tikzcd}
R^3 \arrow[rr, "(\rhd\times\pi_1)\times\pi_3"] \arrow[d, "\pi_1\times\rhd"] & & R^3 \arrow[d, "\pi_1\times\rhd"] \\
R^2 \arrow[r, "\rhd"] & R & R^2 \arrow[l, "\rhd"']
\end{tikzcd}$$

Since a product-preserving functor sends projections to projections, it follows by the uniqueness of the product's universal property that products of morphisms are also preserved. It is then immediate that each of the above diagrams are preserved by the functor. $\square$

Indeed, we see that any morphism equation formulated through the categorical products and compositions of $\rhd, \lhd, \mathrm{id}$ and projections is automatically preserved by product-preserving functors.

Proposition 3.2: Limit-preserving functors preserve the categorical unitality, (co-)symmetry, involutivity, isotropy and quandle properties that may apply to a given rack.

Proof. A terminal object can be considered the limit of the empty diagram, and is thus preserved. In other words, the defining diagram of unitality is preserved:

$$\begin{tikzcd}
& R \arrow[dl, bend right, "1"'] \arrow[d, "1\times\mathrm{id}"] \arrow[dr, bend left, "\mathrm{id}"] & \\
R & R^2 \arrow[l, "\lhd"'] \arrow[r, "\rhd"] & R \\
& R \arrow[ul, bend left, "\mathrm{id}"] \arrow[u, "\mathrm{id}\times 1"'] \arrow[ur, bend right, "1"'] &
\end{tikzcd}$$

The rest of the properties can be formulated as

$$\begin{aligned}
\rhd \circ (\pi_2 \times \pi_1) &= \rhd & \qquad \rhd &= \lhd \\
(\pi_1 \times \rhd)^2 &= \mathrm{id} & \qquad \rhd \circ (\mathrm{id} \times \mathrm{id}) &= \mathrm{id}
\end{aligned}$$

□

Corollary 3.3: If A and B are conjugate rack objects and F is a product-preserving functor, then $F(A)$ and $F(B)$ are also conjugate rack objects.

Proof. Follows from the formulation used in the proof of theorem 2.43. □

This result is of particular interest when it comes to calculating invariants of topological racks. In the event that two topological spaces X and Y have the same invariant $F(X) = F(Y)$, we may not be able to distinguish them. However, if we can construct a rack structure $(X, \rhd)$, calculate the invariant rack object $(F(X), F(\rhd))$, and then prove that no topological rack structures on Y can induce $F(\rhd_X)$, we *can* distinguish them.

Example 3.1: The functor T_1 that takes a Lie group to its Lie algebra preserves products. Thus, any Lie rack can be described by some rack of vector spaces satisfying the appropriate Lie bracket.

Left exact functors preserve finite limits and so they preserve both rack and group object structures. Examples include the $\mathbf{Hom}_C(A, -)$ and $\mathbf{Hom}_C(-, A)$ for any object A in an Abelian category C. Unfortunately our favorite functors are not always left exact. However, with a few restrictions, we may still create interesting rack structures with them. Of particular interest are the homotopy and homology functors, as these have many applications in algebraic topology.

3.1 Homotopy and Singular Homology racks

By construction the homotopy group functors preserve products of path-connected topological spaces, which means that a topological rack yields a group rack structure on each homotopy group. Note that as long as we are dealing with the homotopy groups π_n we will denote projection morphisms by p instead of π.

Proposition 3.4: Let $(R, \rhd)$ be a path-connected topological rack. Then $\mathbf{Inn}_{\rhd}(R)$ maps to a singly generated subgroup of $\mathbf{Aut}_{\mathbf{Grp}}(\pi_n(R))$ for each n.

Proof. From section 2.3 we know that all generators of $\mathbf{Inn}_{\triangleright}(R)$ are isotopic. Since homotopic maps are identified on the level of homotopy, the claim follows. $\square$

Example 3.2: The topological Abelian rack $(S^1, (2, -1))$ can be seen to yield $(\mathbb{Z}, (2, -1))$ on its fundamental group. The basic inner automorphisms of S^1 map to $-\mathrm{id} \in \mathbf{Aut}_{\mathbf{Grp}}(\mathbb{Z})$, while the isotopy class of the identity maps to id.

Lemma 3.5: Let $(R, \triangleright)$ be a topological rack which is conjugate to a unital rack. Then $(\pi_n(R), \pi_n(\triangleright))$ is a permutation rack for all n.

Proof. We know from section 2.5 that the only unital **Grp**-racks are the trivial ones, which are conjugate to any permutation **Grp**-rack. Apply proposition 3.2 and corollary 3.3 to see that the Homotopy racks must be on the form $\triangleright = (1, g)$. $\square$

Corollary 3.6: Let $(R, \triangleright)$ be the conjugation quandle of a topological group. Then $(\pi_n(R), \pi_n(\triangleright))$ is a trivial **Ab**-quandle for all n.

Proof. **Top**-groups get sent to **Grp**-groups, which are Abelian, and unital racks produce trivial structures. $\square$

We may now utilize the exclusivity from lemma 2.36.

Theorem 3.7: Let R be a path-connected topological space.
If there exists a symmetric topological quandle on R, then $\pi_n(R)$ is either 0 or an Abelian group of odd order for all n.
If there exists a bisymmetric topological quandle on R, then $\pi_n(R)$ is either 0 or an Abelian group of order 3 for all n.
If there exists an isotropic topological quandle on R, then $\pi_n(R)$ is a module of the ring $\mathbb{Z}[g]/(g^2 - g + 1)$ for all n.

Corollary 3.8: The standard n-spheres S^n, the real projective spaces $\mathbb{R}P^n$ and complex projective spaces $\mathbb{C}P^n$ admit no symmetric or isotropic quandle structures for any n.

Proof. However complicated the other groups are, we always have $\pi_n(S^n) = \pi_n(\mathbb{R}P^n) = \pi_{2n}(\mathbb{C}P^n) = \mathbb{Z}$. Since $\mathbb{Z}$ does not admit any symmetric or isotropic quandle structures, none may exist on the topological spaces either. $\square$

Corollary 3.9: The Lie groups $GL(n, \mathbb{C})$, $U(n)$, $Sp(n, \mathbb{R})$, $SO(n, \mathbb{R})$ admit no symmetric or isotropic quandle structures for any n.

Proof. These all have fundamental groups that are either $\mathbb{Z}$ or $\mathbb{Z}_2$, neither of which admit symmetric or isotropic **Ab**-quandle structure. $\square$

Lemma 3.10: Let $(R, \triangleright, \triangleleft)$ be a bisymmetric or isotropic **Top**-quandle on a path-connected, non-contractible space R. Then $\triangleleft \times \triangleright : R^2 \to R^2$ is not invertible in **Top**.

Proof. In both cases, exclusivity entails that each homotopy rack $(\pi_n(R), (f, g), (k, h))$ is isotropic.

$$\det\left(\pi_n(\triangleleft) \times \pi_n(\triangleright)\right) = \det \begin{pmatrix} k & h \\ f & g \end{pmatrix} = \det \begin{pmatrix} g^{-1} & g \\ g^{-1} & g \end{pmatrix} = 0 \tag{3.2}$$

An inverse in **Top** would be preserved by functoriality, and so it cannot exist. $\square$

3.1.1 Singular Homology racks

Singular Homology is, unfortunately, not a left exact functor. However, thanks to the Hurewicz theorem and the Künneth formula, we can identify some cases where *some* products are preserved.

Definition 3.1: A path-connected topological space X is said to be n**-connected** if $\pi_i(X) = 0$ for $1 \leq i \leq n$.

Theorem 3.11: The Hurewicz Theorem[15] Let X be a path-connected topological space. For any $k \in \mathbb{N}$ there exists a natural transformation $h_k : \pi_k(X) \to H_k(X)$, which is called the **Hurewicz homomorphism**. If X is $(n-1)$-connected, then the Hurewicz homomorphism is:
1) If $(n > 1)$: An isomorphism for all $k \leq n$ and an epimorphism for $k = n+1$.
2) If $(n = 1)$: Equal to the Abelianization of $\pi_1(X)$.

Let R be an $(n-1)$-connected topological rack. By the Künneth formula,

$$H_n(R \times R) \simeq (H_0(R) \otimes H_n(R)) \oplus (H_n(R) \otimes H_0(R)) = H_n(R) \oplus H_n(R)$$

since all the other terms of the formula disappear. For $n > 1$ this also follows from the Hurewicz theorem.

The topological rack operation $R \times R \xrightarrow{\rhd} R$ therefore induces a group homomorphism $H_n(R) \oplus H_n(R) \xrightarrow{H_n(\rhd)} H_n(R)$ which turns $(H_n(R), H_n(\rhd))$ into an Abelian rack.

Definition 3.2: Let $(R, \rhd)$ be an $(n-1)$-connected topological rack. The **first Homology rack on** R is given by $(H_n(R), H_n(\rhd))$.

Example 3.3: By the Hurewicz homomorphism being a natural transformation, the rack $(S^1, 2, -1)$ yields the first Homology rack $(\mathbb{Z}, 2, -1)$.

The statements of theorem 3.7 carry over to singular Homology.

Corollary 3.12: Let R be an $(n-1)$-connected topological space.
If there exists a symmetric quandle on R, then $H_n(R)$ is a group of odd order.
If there exists a bisymmetric quandle on R, then $H_n(R)$ is a group of order 3.
If there exists an isotropic quandle on R, then $H_n(R)$ is a $\mathbb{Z}[g]/(g^2 - g + 1)$-module.

Example 3.4: The Klein bottle has a fundamental group $\langle a, b \rangle/(abab^{-1})$, and since this is not Abelian, we may immediately rule out the existence of isotropic or symmetric quandles on the Klein bottle. Since we may use $ab = ba^{-1}$ to represent every element as $a^i b^j$, we see that it has a center consisting of elements of the form b^{2k}.

Group quandles $(\pi_1(K), q, g)$ satisfy $q(a^{-1}) = q(bab^{-1}) = q(a)$, so $q(a) = 1$. Furthermore $q(b) = b \cdot g(b^{-1}) = b^{2k}$, so either $g = \text{id}$ or $g(b) = b^{-1}$.

The fundamental group Abelianizes by Hurewicz to $\mathbb{Z}[a, b]/(2a) = \mathbb{Z}_2 \times \mathbb{Z}$. The automorphisms of this group are the following upper triangular matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

where $m(x) = x \mod 2$. The latter two are not the Abelianizations of any group quandles on the fundamental group, and so cannot be the first Homology racks of any topological quandle structures on the Klein bottle.

Note that this does not prevent them from appearing as the Abelianization of non-quandle group rack structures, for instance from group racks of the form $(1, g)$, where g is the extension of $g(a) = a, g(b) = ab$. The image of q_g is not central and so its quandle is disqualified as a group rack structure. By the preservation of the conjugate equivalence relation, this prevents any topological rack on the Klein bottle from having $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & m \\ 0 & 2 \end{pmatrix}$ as its first Homology rack's endomorphism, and any topological space which does admit such rack structures must be homotopy-inequivalent to the Klein bottle.

Lemma 3.13: Let (R, f, g) be an Abelian rack. Any chain or cochain complex with coefficients in R can be given structure as a complex of **AbRacks**.

$$(\sum_x a_x \otimes x) \triangleright (\sum_y b_y \otimes y) =(\sum_x a_x \otimes f(x)) + (\sum_y b_y \otimes g(y)) \tag{3.4}$$

$$(\alpha \triangleright_{\mathbf{Hom}} \beta)(x) =f\alpha(x) + g\beta(x) \tag{3.5}$$

This turns ∂ into an **AbRacks**-homomorphism.

By corollary 2.33 we conclude that any Homology or Cohomology group with coefficients in an **Ab**-rack is an **Ab**-rack, whether singular or otherwise.

3.2 Rack homology

We will now review a Homology theory [5] [6] [7] specifically based on racks rather than topological structure. In [7], the authors use the rack cocycles from this theory to construct state-sum invariants of knots and knotted surfaces, and prove that cohomologous cocycles give rise to the same invariant. This motivates calculation of the rack cohomology groups.

Recall the notation from 2.9. Denote for each vector $\vec{x} = (x_1, ..., x_n) \in R^n$ and each $i \leq n$ the vector $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ by $\vec{x}_{[i]}$.

Let $C_n(R)$ be the free Abelian group generated by elements of R^n and define homomorphisms $\partial_n^{\triangleright}, \partial_n^{\triangleleft} : C_n(R) \to C_{n-1}(R)$ by 0 if $n < 2$ and otherwise by

$$\partial_n^{\triangleright}(\vec{x}) =\sum_{i=1}^{n} (-1)^i \left(\vec{x} - x_i \triangleright^i \vec{x}\right)_{[i]} \tag{3.6}$$

$$\partial_n^{\triangleleft}(\vec{x}) =\sum_{i=1}^{n} (-1)^i \left(\vec{x} - \vec{x} \triangleleft^i x_i\right)_{[i]} \tag{3.7}$$

These define chain complex structures on $C_*(R)$. One could use either, but we will primarily use $\partial^{\triangleright}$ unless otherwise noted.

Remark 2: The initial term of (3.7) and the final term of (3.6) are trivially zero, so these formulations are equivalent to the definitions from [5], [6] and [7].

To see that $\partial^2 = 0$, note that for $i > j$ we have on any basis element $\vec{x} \in R^n$

$$\vec{x}_{[i,j]} = \vec{x}_{[j,i-1]} \tag{3.8}$$
$$(x_i \rhd^i \vec{x})_{[i,j]} = (x_i \rhd^{i-1} \vec{x}_{[i]})_{[j]} = x_i \rhd^{i-2} \vec{x}_{[i,j]} \tag{3.9}$$
$$(x_j \rhd^j \vec{x})_{[j,i-1]} = (x_j \rhd^{j-1} \vec{x}_{[j]})_{[i-1]} = x_j \rhd^{j-1} \vec{x}_{[i,j]} \tag{3.10}$$

In other words, we may split the sum.

$$\begin{aligned}\sum_{j,i}(-1)^{i+j}(\vec{x}_{[i,j]} - (x_i \rhd^i \vec{x})_{[i,j]} - (x_{[i],j} \rhd^j \vec{x}_{[i]})_{[j]}\\ +((x_i \rhd^i \vec{x})_{[i],j} \rhd^j (x_i \rhd^i \vec{x})_{[i]})_{[j]})\\ = \sum_{i>j}(-1)^{i+j}((\ldots)_{i,j} - (\ldots)_{j,i-1})\end{aligned} \tag{3.11}$$

Split this way, the first term cancels itself,

$$(-1)^{i+j}\vec{x}_{[i,j]} + (-1)^{i+j-1}\vec{x}_{[j,i-1]} = 0 \tag{3.12}$$

After some simplification the second term is cancelled by the third term,

$$\begin{aligned}(x_{[i],j} \rhd^j \vec{x}_{[i]})_{[j]} - (x_{[j],i-1} \rhd^{i-1} \vec{x}_{[j]})_{[i-1]}\\ = (x_j \rhd^j \vec{x})_{[i,j]} - (x_i \rhd^i \vec{x})_{[i,j]}\end{aligned} \tag{3.13}$$

Still assuming $i > j$, the last term can be simplified to cancel itself,

$$\begin{aligned}((x_i \rhd^i \vec{x})_{[i],j} \rhd^j (x_i \rhd^i \vec{x})_{[i]})_{[j]}\\ = (x_j \rhd^j (x_i \rhd^i \vec{x}))_{[i,j]}\end{aligned} \tag{3.14}$$

$$\begin{aligned}((x_j \rhd^j \vec{x})_{[j],i-1} \rhd^{i-1} (x_j \rhd^j \vec{x})_{[j]})_{[i-1]}\\ = ((x_j \rhd x_i) \rhd^i (x_j \rhd^j \vec{x}))_{[j,i-1]}\\ = (x_j \rhd^j (x_i \rhd^i \vec{x}))_{[i,j]}\end{aligned} \tag{3.15}$$

There are two closely related and computationally useful complexes which inherit their chain complex structure from $C_*(R)$ whenever R is a quandle.

Definition 3.3: The **degeneration complex** $C_n^D(R) \subset C_n(R)$ consists of subgroups generated by vectors $\vec{a}$ with at least one repetition $a_i = a_{i+1}$.
The **quandle complex** $C_n^Q(R)$ consists of the quotients $C_n(R)/C_n^D(R)$.

If R is a quandle, it is easy to verify that $C_*^D(R)$ is a subcomplex of $C_*(R)$ since each term of $\partial(\vec{a})$ that eliminates a repeated entry is cancelled by the adjacent term. Thus, we may describe the inclusion and projection by chain complex homomorphisms, and we have a short exact sequence of chain complexes,

$$0 \longrightarrow C_*^D(X) \longrightarrow C_*(X) \longrightarrow C_*^Q(X) \longrightarrow 0 \tag{3.16}$$

From here on the usual rules for chain complexes apply. The cycles are denoted $Z_n^{\rhd}(R) = \ker(\partial_n^{\rhd})$ and the boundaries $B_n^{\rhd}(R) = \mathrm{Im}(\partial_{n+1}^{\rhd})$. Given an Abelian group A, it can be taken as coefficient group by the tensor product $C_n(R) \otimes A$, and the cochain complex is given $C^*(R; A) = \mathbf{Hom_{Ab}}(C_*(R), A)$, with its induced cocycles and coboundaries.

Example 3.5: For the trivial quandle, every chain is a cycle and every cochain is a cocycle.

Definition 3.4: Let $(R, \triangleright)$ be a binary rack.
The **Rack Homology** of R is the homology of the chain complex $C_*(R)$, given by $H_n^{\triangleright}(R) = Z_n^{\triangleright}(R)/B_n^{\triangleright}(R)$.
If R is a quandle, the **Degeneration Homology** of R is $H_n^D(R) = Z_n^D(R)/B_n^D(R)$ and the **Quandle Homology** of R is given by $H_n^Q(R) = Z_n^Q(R)/B_n^Q(R)$.

Definition 3.5: The rack, degeneration or quandle homology of an n-rack $(R, \triangleright)$ is the rack, degeneration or quandle homology of the diagonal R^{n-1}.

Note that since the chain complexes are free, the universal coefficient theorem holds, and if X is a quandle, there is a long exact sequence of Rack Homology:

$$... \to H_{n+1}^Q(X) \to H_n^D(X) \to H_n^{\triangleright}(X) \to H_n^Q(X) \to H_{n-1}^D(X) \to ... \quad (3.17)$$

induced by the short exact sequence of chain complexes in (3.16).

Rack, degeneration and quandle cohomologies are dually defined from their respective cochain complexes.

Example 3.6: A 2-cocycle ϕ satisfies

$$\phi(\partial_2^{\triangleright}(x,y,z)) = \phi(-(y,z) + (x \triangleright y, x \triangleright z) + (x,z) - (x, y \triangleright z)) = 0 \quad (3.18)$$
$$\phi(x,z) - \phi(y,z) = \phi(x, y \triangleright z) - \phi(x \triangleright y, x \triangleright z) \quad (3.19)$$

If R is an Abelian rack, one may take R as the coefficient group. Inserting $\phi = \triangleright$ for such a rack, we get

$$x \triangleright z - y \triangleright z = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright (x \triangleright z) = 0 \quad (3.20)$$

And so we must have $f = 0$. In other words, the only Abelian quandle structure that is its own 2-cocycle is the trivial quandle.

Remark 3: Computations are often simplified for finite racks, and simplified further for finite quandles. If R is a rack with m elements, then $C_n(R)$ has m^n generators, $C_n^Q(R)$ has $m(m-1)^{n-1} = (m-1)^{n-1} + (m-1)^n$ generators, and the rank of $C_n^D(R)$ is simply $m^n - m(m-1)^{n-1}$.

Example 3.7: Consider the symmetric quandle on $\mathbb{Z}_3$. We will show that $H_Q^2(\mathbb{Z}_3, 2, 2; \mathbb{Z})$ is trivial. First note that $C_2^Q(\mathbb{Z}_3, 2, 2)$ only has 6 generators since elements of the form (x, x) are degenerate. For each of these there is a generator of $C_Q^2(\mathbb{Z}_3, 2, 2; \mathbb{Z})$ given by $\delta_{(i,j)}(x,y) = 1$ if $(x,y) = (i,j)$ and 0 otherwise.

The 2-coboundaries are spanned by functions of the form

$$\phi(x,y) = (a\delta_0 + b\delta_1 + c\delta_2)(x \triangleright y - y) \quad (3.21)$$
$$\partial^1 \delta_i = \delta_{(i+1,i+2)} + \delta_{(i+2,i+1)} - \delta_{(i+1,i)} - \delta_{(i+2,i)} \quad (3.22)$$

Without loss of generality, let x, y, z be distinct labels from $\mathbb{Z}_3$. A cocycle satisfies

$$\phi(x,z) - \phi(y,z) = \phi(x, 2y+2z) - \phi(2x+2y, 2x+2z) = 0 - \phi(z,y) \quad (3.23)$$
$$\phi(x,z) + \phi(z,y) = \phi(y,z) = (\phi(y,z) + \phi(z,x)) + \phi(z,y) \quad (3.24)$$

In other words, the swapping of one argument is subject to relations

$$\phi(z,x) + \phi(z,y) = 0 \tag{3.25}$$
$$\phi(x,z) - \phi(y,z) = -\phi(z,y) \tag{3.26}$$

The first relation mandates that every ϕ can be written with three parameters:

$$a_0\delta_{(0,1)} - a_0\delta_{(0,2)} + a_1\delta_{(1,2)} - a_1\delta_{(1,0)} + a_2\delta_{(2,0)} - a_2\delta_{(2,1)} \tag{3.27}$$

The second relation lets us eliminate one parameter by way of $a_0 + a_1 + a_2 = 0$.

$$a_1\left(\delta_{(0,2)} - \delta_{(0,1)} + \delta_{(1,2)} - \delta_{(1,0)}\right) + a_2\left(\delta_{(2,1)} - \delta_{(2,0)} + \delta_{(0,1)} - \delta_{(0,2)}\right) \tag{3.28}$$
$$= -\partial^1(a_1\delta_2 + a_2\delta_1) \tag{3.29}$$

In other words, every cocycle is a coboundary, and thus the cohomology is trivial.

Proposition 3.14: Let X be a quandle with finitely many orbits, i.e. such that $|\mathbf{Orb}(X)| = m$. Then $H_1^Q(X) = H_1^{\triangleright}(X) = \mathbb{Z}^m$.

Proof. Note that $H_1^Q(X) = H_1^{\triangleright}(X)$ is true in general since $C_1^D(X) = 0$. While every element of $C_1^{\triangleright}(X)$ is trivially a cycle, $B_1^{\triangleright}(X)$ is generated by elements of the form $x - y \triangleright x$ or $x - x \triangleleft y$ depending on whether you use $\partial^{\triangleright}$ or $\partial^{\triangleleft}$. In either case, taking the quotient identifies elements on the same orbit, leaving one representative from each orbit class to generate $H_1^{\triangleright}(X)$. □

Proposition 3.15: Let X be a quandle and assume $|\mathbf{Orb}(X)| = m$. Then $H_2^D(X) \simeq \mathbb{Z}^m$.

Proof. Since $C_1^D(X) = 0$, every element of C_2^D is a cycle. Using $\partial^{\triangleright}$ for the calculation, every element of the form $(x,x,y) \in \ker \partial_3^{\triangleright}$, so $B_2^D(X)$ is generated by elements of the form $\partial_3^{\triangleright}(x,y,y) = (x \triangleright y, x \triangleright y) - (y,y)$. Once more, taking the quotient identifies elements on the same orbit. □

Corollary 3.16: Let X be a quandle. If X contains an epic argument, then $H_1^Q(X) = H_1^{\triangleright}(X) = H_2^D(X) = \mathbb{Z}$.

In particular this holds for any symmetric, co-symmetric or isotropic quandle, and for any non-trivial **Ab**-quandle on $\mathbb{Z}_p$, where p is prime.

In [12], Etingof and Graña used modules of $\mathbf{As}(X)$ to prove that for any finite rack X the Betti numbers of X match those of $\mathbf{Orb}(X)$. Using coefficients in **Ab**-racks could grant us insight into any possible torsion terms. Since taking homotopy produces actions of any topological rack on its homotopy racks, it is an exciting prospect to look into the rack homology of topological racks by using coefficients in the homotopy racks. This is a prime opportunity for further research.

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