Asymptotic stability of perturbation-based extremum-seeking control for nonlinear plants

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***Abstract*—We introduce a perturbation-based extremum-seeking controller for general nonlinear dynamical plants with an arbitrary number of tunable plant parameters. The controller ensures asymptotic convergence of the plant parameters to their performance-optimizing values for any initial plant condition under the assumptions in this work. The key to this result is that the amplitude and the frequencies of the perturbations, as well as other tuning parameters of the controller, are time varying. Remarkably, the time-varying tuning parameters can be chosen such that asymptotic convergence is achieved for all plants that satisfy the assumptions, thereby guaranteeing stability of the resulting closed-loop system of plant and controller regardless of tuning.**

***Index Terms*—Extremum-seeking control, asymptotic stability, time-varying tuning, performance optimization.**

I. Introduction

EXTREMUM-SEEKING control is an adaptive control methodology that optimizes the steady-state performance of a plant by automated tuning of plant parameters. Extremum-seeking methods are model-free optimization techniques driven by measurements. Due to the low requirements for the knowledge about the plant, extremum-seeking control can be applied to many different engineering problems; see for example [1], [29] and references therein. However, due to the relatively slow convergence of extremum-seeking methods, model-based methods are often preferred if an accurate model of the plant is available. Therefore, typical applications of extremum-seeking control are applications for which an accurate model is not available due to the high complexity of the plant, such as for bioreactors [7], [11], [32] and nuclear-fusion reactors [3], [4], [24], or due to time-varying disturbances that are difficult or expensive to measure, such as for wind turbines [5], [8], [14] and solar arrays [2], [9], [18]. Although extremum-seeking methods aim to tune the plant parameters such that the steady-state performance of the plant is optimal, commonly only near-optimal values are obtained due to the effects of plant dynamics, measurement noise and added perturbations. Therefore, practical convergence with respect to the optimal steady-state plant performance is the standard for many extremum-seeking methods; see for example [16], [17], [23], [25], [30].

Asymptotic convergence results are relatively rare. It is shown in [22] that local exponential convergence to the optimal steady-state performance can be achieved for static plants by exponentially decaying the amplitude of the added perturbations once the plant parameters enter a neighborhood of the performance-optimizing values. Similarly, local exponential convergence to the optimal steady-state performance for dynamical plants is claimed in [33] by regulating the perturbation amplitude. In [28], asymptotic convergence for Wiener-Hammerstein-type plants is obtained by letting the perturbation amplitude and the adaptation gain of the controller asymptotically converge to zero as time goes to infinity.

In addition, a few references describe asymptotic behavior for extremum-seeking methods that do not rely on added perturbations; see for example [10], [12]. It is shown in [12] that asymptotic convergence to the optimal plant performance can be obtained with an extremum-seeking controller that uses first-order least-squares fits if the plant is static. Moreover, simulation results for a Hammerstein-type plant indicate that asymptotic convergence can also be obtained for certain dynamical plants. In [10], a simulation example of a Wiener-type plant displays asymptotic convergence to the optimal steady-state performance if the perturbation of the extremum-seeking controller in [10] is omitted.

The main contributions of this work can be summarized as follows. First, we introduce a novel perturbation-based extremum-seeking controller for general nonlinear dynamical plants with an arbitrary number of plant parameters. From the stability analysis in this work, it follows that, under given assumptions and appropriate tuning of the controller, the closed-loop system of plant and controller is globally asymptotically stable with respect to the optimal steady-state plant performance in the sense that the solutions of the closed-loop system are bounded and asymptotically converge to the steady-state values for which the plant performance is optimal for any initial condition of the plant. The key to this result is that the amplitude and the frequencies of the perturbations, as well as other tuning parameters of the controller, are time varying and asymptotically decay to zero as time goes to infinity. To the best of our knowledge, this is the first work about extremum-seeking control in which global asymptotic stability with respect to the optimal steady-state performance of general nonlinear dynamical plants is proved. Second, we prove that global asymptotic stability can even be obtained if the plant is subjected to a time-varying disturbance under the assumption that the perturbations of the controller and the zero-mean component of the disturbance are uncorrelated. Third, there exist time-varying tuning-parameter values of the controller that ensure global asymptotic stability of the closed-loop system for all plants that satisfy the assumptions in this

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work. Application of these values eliminates the necessity (in [17], [30] for example) to tune the extremum-seeking controller in order to obtain a stable closed-loop system.

The organization of this work is as follows. The extremum-seeking problem is formulated in Section II. Our novel extremum-seeking controller is introduced in Section III. The stability analysis of the resulting closed-loop system of plant and controller is given in Section IV. We demonstrate our findings with three simulation examples in Section V, after which this work is concluded in Section VI.

The sets of real numbers and natural numbers (nonnegative integers) are respectively denoted by $\mathbb{R}$ and $\mathbb{N}$. We denote the sets of positive real numbers, nonnegative real numbers and positive integers by $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{N}_{>0}$, respectively. The identity matrix and the zero matrix are denoted by $\mathbf{I}$ and $\mathbf{0}$.

II. Problem formulation

We consider the following mulit-input-single-ouput nonlinear plant:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ y(t) &= h(\mathbf{x}(t), \mathbf{u}(t)) + d(t), \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ is the state, $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$ is the input, $y \in \mathbb{R}$ is the output and where $t \in \mathbb{R}_{\geq 0}$ is the time. The dimensions of the state and the input are denoted by $n_\mathbf{x}, n_\mathbf{u} \in \mathbb{N}_{>0}$, respectively. The input $\mathbf{u}$ can be regarded as a vector of tunable plant parameters. The output of the function h can be seen as a measure for the performance of the plant. We refer to the output of h as the performance cost. The performance cost is measured by the imperfect measurement y. The discrepancy between the performance cost and the measurement is denoted by the disturbance d. Our aim is to find the constant plant-parameter values that optimize the steady-state plant performance by minimizing the steady-state performance cost. However, the exact relation between the plant parameters and the performance cost is unknown, meaning that the state $\mathbf{x}$, the functions $\mathbf{f}$ and h, the state dimension $n_\mathbf{x}$ and the disturbance d are unknown. To identify for which plant-parameter values the steady-state plant performance is optimal, we rely on the plant-parameter values $\mathbf{u}$, the measurement y and a set of general assumptions about the plant, which we introduce next.

Our first assumption is that there exist a constant (unknown) steady-state solution of the plant denoted by $\mathbf{x} = \mathbf{X}(\mathbf{u})$ for each set of constant plant-parameter values $\mathbf{u}$. This is formalized as follows.

Assumption 1. *There exists a twice continuously differentiable map $\mathbf{X} : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}^{n_\mathbf{u}}$ and a constant $L_\mathbf{X} \in \mathbb{R}_{>0}$ such that*

$$\mathbf{0} = \mathbf{f}(\mathbf{X}(\mathbf{u}), \mathbf{u}) \tag{2}$$

and

$$\left\| \frac{d\mathbf{X}}{d\mathbf{u}}(\mathbf{u}) \right\| \leq L_\mathbf{X} \tag{3}$$

for all $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$.

We note that $\mathbf{X}(\mathbf{u})$ is the explicit solution of the implicit equation (2) for any $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$. Our second assumption is that the plant is globally exponentially stable with respect to the steady-state solution $\mathbf{X}(\mathbf{u})$ if $\mathbf{u}$ is constant.

Assumption 2. *There exist constants $\mu_\mathbf{x}, \nu_\mathbf{x} \in \mathbb{R}_{>0}$ such that, for each constant $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$, the solutions of (1) satisfy*

$$\|\tilde{\mathbf{x}}(t)\| \leq \mu_\mathbf{x} \|\tilde{\mathbf{x}}(t_0)\| e^{-\nu_\mathbf{x}(t-t_0)}, \tag{4}$$

with

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{X}(\mathbf{u}), \tag{5}$$

for all $\mathbf{x}(t_0) \in \mathbb{R}^{n_\mathbf{x}}$ and all $t \geq t_0 \geq 0$.

From Assumptions 1 and 2 and the output function of the plant, we obtain that steady-state relation between the plant-parameter values and the performance cost can be written as

$$F(\mathbf{u}) = h(\mathbf{X}(\mathbf{u}), \mathbf{u}). \tag{6}$$

We refer to F as the objective function. In order to minimize the steady-state performance cost and to optimize the steady-state plant performance, we aim to find the plant-parameter values for which the output of objective function is minimal. Because the functions $\mathbf{f}$ and h are unknown, the objective function is also unknown. Nonetheless, we assume that $F(\mathbf{u})$ exhibits a unique minimum for some unknown value $\mathbf{u} = \mathbf{u}^*$ for which the steady-state plant performance is optimal. This is formulated in the following assumption.

Assumption 3. *The objective function $F : \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}$ is twice continuously differentiable and exhibits a unique minimum on the domain $\mathbb{R}^{n_\mathbf{u}}$. Let the corresponding minimizer be denoted by $\mathbf{u}^*$. There exist constants $L_{F1}, L_{F2} \in \mathbb{R}_{>0}$ such that*

$$\frac{dF}{d\mathbf{u}}(\mathbf{u})(\mathbf{u} - \mathbf{u}^*) \geq L_{F1} \|\mathbf{u} - \mathbf{u}^*\|^2 \tag{7}$$

and

$$\left\| \frac{d^2 F}{d\mathbf{u} d\mathbf{u}^T}(\mathbf{u}) \right\| \leq L_{F2} \tag{8}$$

for all $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$.

We note that, although (7) implies that $F(\mathbf{u}^*)$ is a unique minimum of the objective function, it does not imply that the objective function is convex. A similar assumption to (7) for a single-parameter plants is stated in [30].

The existence of a steady-state solution, the stability of the plant and the existence of a minimum of the objective function are common assumptions in the extremum-seeking literature; see for example [17], [30]. Additionally, we require the following bounds on the derivatives of the functions $\mathbf{f}$ and h for analytical purposes.

Assumption 4. *The function $\mathbf{f} : \mathbb{R}^{n_\mathbf{x}} \times \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}^{n_\mathbf{x}}$ and $h : \mathbb{R}^{n_\mathbf{x}} \times \mathbb{R}^{n_\mathbf{u}} \to \mathbb{R}$ are twice continuously differentiable. Moreover, there exist constants $L_{\mathbf{fx}}, L_{\mathbf{fu}}, L_{h\mathbf{x}}, L_{h\mathbf{u}} \in \mathbb{R}_{>0}$ such that*

$$\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u}) \right\| \leq L_{\mathbf{fx}}, \quad \left\| \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}, \mathbf{u}) \right\| \leq L_{\mathbf{fu}} \tag{9}$$

and

$$\left\| \frac{\partial^2 h}{\partial \mathbf{x} \partial \mathbf{x}^T}(\mathbf{x}, \mathbf{u}) \right\| \leq L_{h\mathbf{x}}, \quad \left\| \frac{\partial^2 h}{\partial \mathbf{x} \partial \mathbf{u}^T}(\mathbf{x}, \mathbf{u}) \right\| \leq L_{h\mathbf{u}} \tag{10}$$

for all $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$ and all $\mathbf{u} \in \mathbb{R}^{n_\mathbf{u}}$.

Remark 5. *In this work, we optimize the steady-state plant performance for any initial conditions $\mathbf{x}(0) \in \mathbb{R}^{n_\mathbf{x}}$ and $\mathbf{u}(0) \in$*

$\mathbb{R}^{n_{\mathbf{u}}}$. *For this reason, we require that Assumptions 1-4 are satisfied for all* $\mathbf{x} \in \mathbb{R}^{n_{\mathbf{x}}}$ *and all* $\mathbf{u} \in \mathbb{R}^{n_{\mathbf{u}}}$. *For a local result, it is sufficient to assume that Assumptions 1-4 hold for compact sets of* $\mathbf{x}$ *and* $\mathbf{u}$, *where the steady-state solution* $\mathbf{X}(\mathbf{u})$ *is in the interior of the compact set of* $\mathbf{x}$ *and the minimizer* $\mathbf{u}^*$ *is in the interior of the compact set of* $\mathbf{u}$. *We note that Assumption 4 holds for any compact sets of* $\mathbf{x}$ *and* $\mathbf{u}$ *if the functions* $\mathbf{f}$ *and* h *are twice continuously differentiable.*

Because the objective function is unknown, any information about the objective function is obtained via the measurement y. We note that the measurement y differs from the output of the objective function F (which is equal to the steady-state performance cost) in two ways: first, the measurement is not equal to the performance cost due to the disturbance d; second, the performance cost is not equal to the output of the objective function due to the plant dynamics. Nonetheless, we aim to steer the plant parameters $\mathbf{u}$ to their performance-optimizing values $\mathbf{u}^*$ under the given assumptions by using the measurement y as feedback.

III. Proposed controller

From Assumption 3, it follows that the plant parameters $\mathbf{u}$ converge to their performance-optimizing values $\mathbf{u}^*$ if they are steered in the direction opposite to the gradient of the objective function. Because the objective function is unknown, we estimate (a scaled version of) its gradient and use this gradient estimate to steer $\mathbf{u}$ to $\mathbf{u}^*$. We introduce the following sinusoidal perturbations to provide sufficient excitation to the plant-parameter signals to accurately estimate the gradient of the objective function:

$$\boldsymbol{\omega}(t) = [\omega_1(t),\ \omega_2(t),\ \ldots,\ \omega_{n_{\mathbf{u}}}(t)]^T, \tag{11}$$

with

$$\omega_i(t) = \begin{cases} \sin\left(\dfrac{i+1}{2}\displaystyle\int_0^t \eta_{\boldsymbol{\omega}}(\tau)d\tau\right), & \text{if } i \text{ is odd,} \\ \cos\left(\dfrac{i}{2}\displaystyle\int_0^t \eta_{\boldsymbol{\omega}}(\tau)d\tau\right), & \text{if } i \text{ is even} \end{cases} \tag{12}$$

for $i = 1, 2, \ldots, n_{\mathbf{u}}$, where $\eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ is a time-varying tuning parameter. We note that if $\eta_{\boldsymbol{\omega}}$ is constant, the perturbation signals in (12) are given by $\omega_1 = \sin(\eta_{\boldsymbol{\omega}} t)$, $\omega_2 = \cos(\eta_{\boldsymbol{\omega}} t)$, $\omega_3 = \sin(2\eta_{\boldsymbol{\omega}} t)$, etcetera. The use of sinusoidal perturbations with constant angular frequencies is common in extremum-seeking control; see for example [1], [29] and references therein. The corresponding plant-parameter signals are given by

$$\mathbf{u}(t) = \hat{\mathbf{u}}(t) + \alpha_{\boldsymbol{\omega}}(t)\boldsymbol{\omega}(t), \tag{13}$$

where $\hat{\mathbf{u}} \in \mathbb{R}^{n_{\mathbf{u}}}$ is the nominal value of the plant parameters and $\alpha_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ is the time-varying amplitude of the perturbation signals. The tuning parameters $\alpha_{\boldsymbol{\omega}}$ and $\eta_{\boldsymbol{\omega}}$ satisfy the differential equations

$$\dot{\alpha}_{\boldsymbol{\omega}}(t) = -g_\alpha(t)\alpha_{\boldsymbol{\omega}}(t), \quad \dot{\eta}_{\boldsymbol{\omega}}(t) = -g_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t), \tag{14}$$

with initial conditions $\alpha_{\boldsymbol{\omega}}(0), \eta_{\boldsymbol{\omega}}(0) \in \mathbb{R}_{>0}$ and time-varying parameters $g_\alpha, g_{\boldsymbol{\omega}} \in \mathbb{R}_{\geq 0}$. This is not the first work about extremum-seeking control for which the amplitude of the perturbations is time varying. Sinusoidal perturbations with a time-varying amplitude are also used to optimize the plant performance in the presence of multiple local extrema in [31], to increase the convergence rate of the extremum-seeking controller in [20], to remove steady-state oscillations in [33], to obtain exponential convergence for static plants in [22], and to achieve asymptotic convergence for Wiener-Hammerstein-type plants in [28]. In this work, we utilize sinusoidal perturbations with a time-varying amplitude and time-varying frequencies to obtain asymptotic convergence of the plant parameters to their performance-optimizing values by letting the value of $\alpha_{\boldsymbol{\omega}}$ and $\eta_{\boldsymbol{\omega}}$ asymptotically decay to zero as time goes to infinity. Here, the novelty lies in the decay of the frequencies in addition to the decay of the amplitude of the perturbations, which allows us to extend the results in [28] to the general nonlinear plant in (1).

In this work, we introduce an extremum-seeking controller that asymptotically regulates the nominal plant parameters $\hat{\mathbf{u}}$ to $\mathbf{u}^*$ with the help of an estimate of the gradient of the objective function. To be able to estimate the gradient of the objective function from the measurement y, we impose the following assumption on the disturbance d.

Assumption 6. *The disturbance* $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is integrable. Moreover, there exists a constant* $b_d \in \mathbb{R}$ *for which*

$$b_d = \lim_{T\to\infty} \frac{1}{T}\int_0^T d(t)dt. \tag{15}$$

We define

$$\tilde{d}(t) = d(t) - b_d. \tag{16}$$

In addition, there exists a vector $\mathbf{b}_{\boldsymbol{\omega}d} \in \mathbb{R}^{n_{\mathbf{u}}}$ *for which*

$$\mathbf{b}_{\boldsymbol{\omega}d} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \boldsymbol{\omega}(t)\tilde{d}(t)dt. \tag{17}$$

Furthermore, there exist constants $q_d, q_{\boldsymbol{\omega}d} \in \mathbb{R}_{\geq 0}$ *such that*

$$\left|\int_0^t \tilde{d}(\tau)d\tau\right| \leq q_d \tag{18}$$

and

$$\left\|\int_0^t \left(\boldsymbol{\omega}(\tau)\tilde{d}(\tau) - \mathbf{b}_{\boldsymbol{\omega}d}\right)d\tau\right\| \leq q_{\boldsymbol{\omega}d} \tag{19}$$

for all $t \geq 0$.

We note that the disturbance d is allowed to be discontinuous and unbounded as long as the bounds on the integrals in (18) and (19) exist. The constant b_d is a bias in the measurement. We refer to $\tilde{d}$ as the zero-mean component of the disturbance. The vector $\mathbf{b}_{\boldsymbol{\omega}d}$ is a measure for the correlation between $\boldsymbol{\omega}$ and $\tilde{d}$. We refer to $\boldsymbol{\omega}$ and $\tilde{d}$ as uncorrelated if $\mathbf{b}_{\boldsymbol{\omega}d}$ is equal to the zero vector. Uncorrelation between the perturbations and the zero-mean component of the disturbance is used to prove the practical stability results in [1], [29], where (17) is equivalent to the noise assumption in [29] for $\mathbf{b}_{\boldsymbol{\omega}d} = \mathbf{0}$. Similarly, the asymptotic stability result in this work can only be obtained if the perturbations and the zero-mean component of the disturbance are uncorrelated.

A. Model of the input-to-output behavior of the plant

To obtain an estimate of the gradient of the objective function from the measurement signal y, we model the input-to-output behavior of the plant. The state of the model is given by

$$m_1(t) = F(\hat{\mathbf{u}}(t)) + b_d, \quad \mathbf{m}_2(t) = \alpha_{\boldsymbol{\omega}}(t)\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}(t)). \tag{20}$$

By combining the output equation in (1) and the expression for objective function in (6), the measurement y can be expressed as

$$y = h(\mathbf{x}, \mathbf{u}) - h(\mathbf{X}(\mathbf{u}), \mathbf{u}) + F(\mathbf{u}) + d. \tag{21}$$

With the help of Taylor's theorem and (13), the steady-state performance cost can be written as

$$\begin{aligned} F(\mathbf{u}) &= F(\hat{\mathbf{u}} + \alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}) \\ &= F(\hat{\mathbf{u}}) + \alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}}(\hat{\mathbf{u}})\boldsymbol{\omega} \\ &\quad + \alpha_{\boldsymbol{\omega}}^2\boldsymbol{\omega}^T\int_0^1 (1-s)\frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}} + s\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega})ds\boldsymbol{\omega}. \end{aligned} \tag{22}$$

By combining (14), (16) and (20)-(22), we obtain the following input-to-output behavior of the plant:

$$\begin{aligned} \dot{m}_1(t) &= \frac{\dot{\hat{\mathbf{u}}}^T(t)}{\alpha_{\boldsymbol{\omega}}(t)}\mathbf{m}_2(t) \\ \dot{\mathbf{m}}_2(t) &= -g_\alpha(t)\mathbf{m}_2(t) + \alpha_{\boldsymbol{\omega}}^2(t)\mathbf{w}(t) \\ y(t) &= m_1(t) + \boldsymbol{\omega}^T(t)\mathbf{m}_2(t) + \alpha_{\boldsymbol{\omega}}^2(t)v(t) + z(t) + \tilde{d}(t), \end{aligned} \tag{23}$$

with

$$\begin{aligned} \mathbf{w} &= \frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}})\frac{\dot{\hat{\mathbf{u}}}}{\alpha_{\boldsymbol{\omega}}}, \\ v &= \boldsymbol{\omega}^T\int_0^1 (1-s)\frac{d^2F}{d\mathbf{u}d\mathbf{u}^T}(\hat{\mathbf{u}} + s\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega})ds\boldsymbol{\omega}, \\ z &= h(\mathbf{x}, \mathbf{u}) - h(\mathbf{X}(\mathbf{u}), \mathbf{u}). \end{aligned} \tag{24}$$

The signals $\mathbf{w}$, v and z can be regarded as unknown disturbances. The influences of $\mathbf{w}$, v and z on the state and output of the model are small if $\hat{\mathbf{u}}$ is slowly time varying, if $\alpha_{\boldsymbol{\omega}}$ is small and if the state $\mathbf{x}$ of the plant is close to its steady-state value $\mathbf{X}(\mathbf{u})$. We note that the state $\mathbf{m}_2$ in (20) is equal to the gradient of the objective function scaled by the perturbation amplitude $\alpha_{\boldsymbol{\omega}}$. Hence, an estimate of the gradient of the objective function can be obtained from an estimate of the state $\mathbf{m}_2$.

B. Controller design

We introduce an extremum-seeking controller that consists of an observer to estimate the state of the model in (23) and an optimizer that uses the estimate of the state $\mathbf{m}_2$ of the observer to regulate the nominal plant parameters $\hat{\mathbf{u}}$ to their performance-optimizing values $\mathbf{u}^*$. Let the observer be given by

$$\begin{aligned} \dot{\hat{m}}_1(t) &= \eta_{\mathbf{m}}(t)\left(y(t) - \hat{m}_1(t)\right) \\ \dot{\hat{\mathbf{m}}}_2(t) &= -g_\alpha(t)\hat{\mathbf{m}}_2(t) \\ &\quad + \eta_{\mathbf{m}}(t)\mathbf{Q}(t)\boldsymbol{\omega}(t)\left(y(t) - \hat{m}_1(t) - \boldsymbol{\omega}^T(t)\hat{\mathbf{m}}_2(t)\right) \\ \dot{\mathbf{Q}}(t) &= \eta_{\mathbf{m}}(t)\mathbf{Q}(t) - 2g_\alpha(t)\mathbf{Q}(t) \\ &\quad - \eta_{\mathbf{m}}(t)\mathbf{Q}(t)\boldsymbol{\omega}(t)\boldsymbol{\omega}^T(t)\mathbf{Q}(t), \end{aligned} \tag{25}$$

with time-varying tuning parameter $\eta_{\mathbf{m}} \in \mathbb{R}_{>0}$ and state $\hat{m}_1 \in \mathbb{R}$, $\hat{\mathbf{m}}_2 \in \mathbb{R}^{n_{\mathbf{u}}}$ and $\mathbf{Q} \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$, where $\mathbf{Q}$ is symmetric and positive definite. Similar to (14), the tuning parameter $\eta_{\mathbf{m}}$ satisfies the differential equation

$$\dot{\eta}_{\mathbf{m}}(t) = -g_{\mathbf{m}}(t)\eta_{\mathbf{m}}(t), \tag{26}$$

with initial condition $\eta_{\mathbf{m}}(0) \in \mathbb{R}_{>0}$ and time-varying parameter $g_{\mathbf{m}} \in \mathbb{R}_{\geq 0}$. We note that $\hat{m}_1$ and $\hat{\mathbf{m}}_2$ are estimates of m_1 and $\mathbf{m}_2$ in (20), respectively. Therefore, $\hat{\mathbf{m}}_2$ is an estimate of the scaled gradient of the objective function. We define the following gradient-descent optimizer to steer the nominal plant parameters $\hat{\mathbf{u}}$ to their performance optimizing values $\mathbf{u}^*$:

$$\dot{\hat{\mathbf{u}}}(t) = -\lambda_{\mathbf{u}}(t)\frac{\eta_{\mathbf{u}}(t)\hat{\mathbf{m}}_2(t)}{\eta_{\mathbf{u}}(t) + \lambda_{\mathbf{u}}(t)\|\hat{\mathbf{m}}_2(t)\|}, \tag{27}$$

where $\lambda_{\mathbf{u}}, \eta_{\mathbf{u}} \in \mathbb{R}_{>0}$ are time-varying tuning parameters that satisfy the differential equations

$$\dot{\lambda}_{\mathbf{u}}(t) = -g_\lambda(t)\lambda_{\mathbf{u}}(t), \quad \dot{\eta}_{\mathbf{u}}(t) = -g_{\mathbf{u}}(t)\eta_{\mathbf{u}}(t), \tag{28}$$

with initial conditions $\lambda_{\mathbf{u}}(0), \eta_{\mathbf{u}}(0) \in \mathbb{R}_{>0}$ and time-varying parameters $g_\lambda, g_{\mathbf{u}} \in \mathbb{R}_{\geq 0}$. We note that the adaptation gain of the optimizer in (27) is normalized to preclude a finite escape time of the solutions of the closed-loop system of plant and extremum-seeking controller if the estimate $\hat{\mathbf{m}}_2$ is inaccurate.

C. Closed-loop system

The closed-loop system of the plant in (1) and the extremum-seeking controller in (25) and (27) is illustrated in Fig. 1. To accurately estimate the state of the model in (23) with the observer in (25), it is assumed that the following design assumptions are satisfied: first, the plant parameters (that is, the sum of the nominal plant parameters and the perturbations) are slowly time varying with respect to the plant dynamics so that the performance cost remains close to its steady-state value (that is, the disturbance z in (24) is small); second, the observer uses a sufficiently long time history of the perturbation signals and measurement signal to be able to accurately extract the state of the model from these signals, which requires the observer to be slow compared to the perturbations; third, the nominal plant parameters are slowly time varying with respect to the observer so that an accurate state estimate is obtained (that is, the disturbance $\mathbf{w}$ in (24) is small). Under these design assumptions, different time scales can be assigned to the various components of the closed-loop system of plant and controller, similar to [17], [21], [30]. We conclude that the closed-loop system should be tuned to exhibit four time scales under these assumptions:

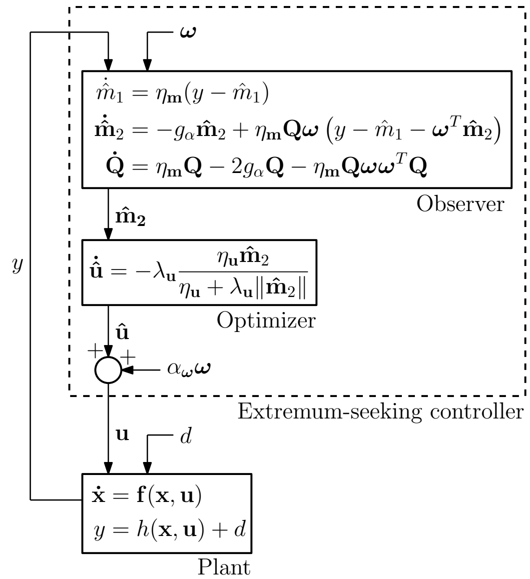

Fig. 1. Closed-loop system of plant and extremum-seeking controller.

- fast – the plant;
- medium fast – the perturbations of the controller;
- medium slow – the observer of the controller;
- slow – the optimizer of the controller.

The time scales of the perturbations, the observer and the optimizer are dependent on the tuning parameters $\alpha_{\boldsymbol{\omega}}$, $\eta_{\boldsymbol{\omega}}$, $\eta_{\mathbf{m}}$, $\lambda_{\mathbf{u}}$ and $\eta_{\mathbf{u}}$. As mentioned above, we let $\alpha_{\boldsymbol{\omega}}$ and $\eta_{\boldsymbol{\omega}}$ asymptotically converge to zero to obtain asymptotic convergence of the plant parameters to their performance-optimizing values. This implies that the perturbations become slower as time progresses. To ensure that the observer and the controller are sufficiently slow compared to the perturbations, the tuning parameters $\eta_{\mathbf{m}}$, $\lambda_{\mathbf{u}}$ and $\eta_{\mathbf{u}}$ are required to be time varying and asymptotically decay to zero as well.

IV. Stability analysis

To investigate under which initial conditions and tuning conditions the plant parameters converge to their performance-optimizing values, we analyse the stability of the closed-loop system of the plant in (1) and the extremum-seeking controller in (25) and (27). Contrary to extremum-seeking controllers with constant tuning parameters in [17], [30], for example, we allow our choice of tuning-parameter values to be bad initially, as long as suitable tuning-parameter values are obtained after a finite time $t_1 \geq 0$. Our main result is presented next.

Theorem 7. *Suppose that the parameters g_α, $g_{\boldsymbol{\omega}}$, $g_{\mathbf{m}}$, g_λ and $g_{\mathbf{u}}$ in (14), (26) and (28) are chosen such that*

$$\begin{gathered}\int_0^\infty e^{-\int_0^t g_{\mathbf{m}}(\tau)d\tau} dt = \infty, \\ \int_0^\infty \min\left\{e^{-\int_0^t (g_\alpha(\tau) + g_\lambda(\tau))d\tau}, e^{-\int_0^t g_{\mathbf{u}}(\tau)d\tau}\right\} dt = \infty\end{gathered} \tag{29}$$

and

$$\max\{g_\alpha(t), g_{\boldsymbol{\omega}}(t), g_{\mathbf{m}}(t), g_\lambda(t), g_{\mathbf{u}}(t)\} \leq c_g \tag{30}$$

for all $t \geq 0$ and some constant $c_g \in \mathbb{R}_{>0}$. Moreover, suppose that

$$\max\left\{\frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)} q_d, \frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)} q_{\boldsymbol{\omega}d}, \frac{1}{\alpha_{\boldsymbol{\omega}}(t)} \|\mathbf{b}_{\boldsymbol{\omega}d}\|\right\} \leq c_d \tag{31}$$

for all $t \geq 0$ and for some constant $c_d \in \mathbb{R}_{>0}$. Let $\alpha_{\boldsymbol{\omega}}(0), \eta_{\boldsymbol{\omega}}(0), \lambda_{\mathbf{m}}(0), \lambda_{\mathbf{u}}(0), \eta_{\mathbf{u}}(0) \in \mathbb{R}_{>0}$. Under these assumptions and Assumptions 1-4 and 6, there exist (sufficiently large) constants $c_1, c_2, \ldots, c_5 \in \mathbb{R}_{>0}$ and (sufficiently small) constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7 \in \mathbb{R}_{>0}$ such that, if there exists a time $t_1 \in \mathbb{R}_{\geq 0}$ for which

$$\begin{aligned} g_\alpha(t) + g_{\boldsymbol{\omega}}(t) &\leq \varepsilon_1, & g_\alpha(t) &\leq \eta_{\mathbf{m}}(t)\varepsilon_2, \\ |g_{\mathbf{m}}(t) - g_{\boldsymbol{\omega}}(t)| &\leq \eta_{\mathbf{m}}(t)\varepsilon_3, & \eta_{\boldsymbol{\omega}}(t) &\leq \varepsilon_4, \\ \eta_{\mathbf{m}}(t) &\leq \eta_{\boldsymbol{\omega}}(t)\varepsilon_5, & \eta_{\mathbf{u}}(t) &\leq \alpha_{\boldsymbol{\omega}}(t)\eta_{\mathbf{m}}(t)\varepsilon_6, \\ \alpha_{\boldsymbol{\omega}}(t)\lambda_{\mathbf{u}}(t) &\leq \eta_{\mathbf{m}}(t)\varepsilon_7 \end{aligned} \tag{32}$$

for all $t \geq t_1$, then the solutions of the closed-loop system of the plant in (1) and the extremum-seeking controller in (25) and (27) are bounded for all $t \geq 0$, all $\mathbf{x}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$, all $\hat{m}_1(0) \in \mathbb{R}$, all $\hat{\mathbf{m}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}}}$, all symmetric positive-definite $\mathbf{Q}(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$ and all $\hat{\mathbf{u}}(0) \in \mathbb{R}^{n_{\mathbf{u}}}$. In addition, the solutions of $\hat{\mathbf{u}}$ satisfy

$$\begin{aligned} \limsup_{t\to\infty} \|\hat{\mathbf{u}}(t) - \mathbf{u}^*\| \leq \limsup_{t\to\infty} \max\Bigg\{ & \alpha_{\boldsymbol{\omega}}(t)c_1, \eta_{\boldsymbol{\omega}}(t)c_2, \\ & \frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)} c_3 q_d, \frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)} c_4 q_{\boldsymbol{\omega}d}, \frac{1}{\alpha_{\boldsymbol{\omega}}(t)} c_5 \|\mathbf{b}_{\boldsymbol{\omega}d}\|\Bigg\}. \end{aligned} \tag{33}$$

We note that the constants $c_1, \ldots, c_5$ and $\varepsilon_1, \ldots, \varepsilon_7$ in Theorem 7 are specific to the plant. The division of time scales in Section III-C is achieved for sufficiently small values of $\varepsilon_4, \ldots, \varepsilon_7$ in (32).

A. Proof of Theorem 7

To prove Theorem 7, we define the following coordinate transformation:

$$\begin{aligned} \tilde{\mathbf{x}}(t) &= \mathbf{x}(t) - \mathbf{X}(\mathbf{u}(t)), \\ \tilde{m}_1(t) &= \hat{m}_1(t) - m_1(t) \\ &\quad - \eta_{\mathbf{m}}(t)k_1(t) - \frac{\eta_{\mathbf{m}}(t)}{\eta_{\boldsymbol{\omega}}(t)} \mathbf{l}_1^T(t)\mathbf{m}_2(t), \\ \tilde{\mathbf{m}}_2(t) &= \hat{\mathbf{m}}_2(t) - \mathbf{m}_2(t) - \eta_{\mathbf{m}}(t)\mathbf{Q}(t)\mathbf{k}_2(t), \\ \tilde{\mathbf{Q}}(t) &= \mathbf{Q}^{-1}(t) - \frac{1}{2}\mathbf{I} - \frac{\eta_{\mathbf{m}}(t)}{\eta_{\boldsymbol{\omega}}(t)} \mathbf{l}_2(t), \\ \tilde{\mathbf{u}}(t) &= \hat{\mathbf{u}}(t) - \mathbf{u}^*, \end{aligned} \tag{34}$$

with

$$\begin{aligned} k_1(t) &= \int_0^t \tilde{d}(\tau)d\tau, \\ \mathbf{k}_2(t) &= \int_0^t \left(\boldsymbol{\omega}(\tau)\tilde{d}(\tau) - b_{\boldsymbol{\omega}d}\right) d\tau \end{aligned} \tag{35}$$

and

$$\begin{aligned} \mathbf{l}_1(t) &= \int_0^t \eta_{\boldsymbol{\omega}}(\tau)\boldsymbol{\omega}(\tau)d\tau, \\ \mathbf{l}_2(t) &= \int_0^t \eta_{\boldsymbol{\omega}}(\tau)\left(\boldsymbol{\omega}(\tau)\boldsymbol{\omega}^T(\tau) - \frac{1}{2}\mathbf{I}\right) d\tau. \end{aligned} \tag{36}$$

We note that k_1 and $\mathbf{k}_2$ in (35) are bounded; see Assumption 6. Moreover, from the definition of $\boldsymbol{\omega}$ in (11), it follows that $\mathbf{l}_1$ and $\mathbf{l}_2$ in (36) are also bounded. Loosely speaking, the convergence of the closed-loop system can be divided in three stages:

- for $0 \leq t < t_1$, the tuning parameters converge to the bounds in (32), while the state (34) of the closed-loop system may drift;
- for $t_1 \leq t < t_2$, the variables $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{Q}}$ converge to a region of the origin and remain there, while the rest of the state (34) of the closed-loop system may drift;
- for $t \geq t_2$, the variables $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ converge to a region of the origin and remain there.

Next, we derive bounds on solutions of the individual variables in (34) in accordance with the three stages. First, we derive bounds on $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{Q}}$ in Lemmas 8 and 9, respectively.

Lemma 8. *Under the conditions of Theorem 7, there exist constants $c_{\mathbf{x}1}, c_{\mathbf{x}2}, \beta_{\mathbf{x}} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{\mathbf{x}}$ are bounded for all $t \geq 0$ and all $\tilde{\mathbf{x}}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$. Moreover, the solutions of $\tilde{\mathbf{x}}$ satisfy*

$$\|\tilde{\mathbf{x}}(t)\| \leq \max\left\{c_{\mathbf{x}1}\|\tilde{\mathbf{x}}(t_1)\|e^{-\beta_{\mathbf{x}}(t-t_1)}, \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)c_{\mathbf{x}2}\right\} \tag{37}$$

for all $t \geq t_1$.

Proof. See Appendix A. □

Lemma 9. *Under the conditions of Theorem 7, there exist constants $c_{\mathbf{Q}}, \beta_{\mathbf{Q}} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{\mathbf{Q}}$ are bounded for all $t \geq 0$ and all $\tilde{\mathbf{Q}}(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$ for which $\mathbf{Q}(0)$ is symmetric and positive definite. Moreover, the solutions of $\tilde{\mathbf{Q}}$ satisfy*

$$\|\tilde{\mathbf{Q}}(t)\| \leq \max\left\{c_{\mathbf{Q}}\|\tilde{\mathbf{Q}}(t_1)\|e^{-\beta_{\mathbf{Q}}\int_{t_1}^{t}\eta_{\mathbf{m}}(\tau)d\tau}, \frac{1}{8}\right\} \tag{38}$$

for all $t \geq t_1$.

Proof. See Appendix B. □

From Lemmas 8 and 9, we have that the solutions of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{Q}}$ are bounded for all time under the given assumptions. Moreover, it follows that there exists a time $t_2 \geq t_1$ such that $\|\tilde{\mathbf{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)c_{\mathbf{x}2}$ and $\|\tilde{\mathbf{Q}}(t)\| \leq \frac{1}{8}$ for all $t \geq t_2$ under the conditions of Theorem 7. We use these bounds on $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{Q}}$ to obtain the results in Lemmas 10 and 11 regarding the existence of ISS-Lyapunov functions (see for example [26]) for the $\tilde{m}_1$-, $\tilde{\mathbf{m}}_2$- and $\tilde{\mathbf{u}}$-dynamics.

Lemma 10. *Under the conditions of Theorem 7, there exists a time $t_2 \geq t_1$ such that the solutions of $\tilde{m}_1$ and $\tilde{\mathbf{m}}_2$ are bounded for all $0 \leq t \leq t_2$, all $\tilde{m}_1(0) \in \mathbb{R}$ and all $\tilde{\mathbf{m}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}}}$. In addition, there exist a function $V_{\mathbf{m}} : \mathbb{R} \times \mathbb{R}^{n_{\mathbf{u}}} \times \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}} \to \mathbb{R}_{\geq 0}$ and constants $\gamma_{\mathbf{m}1}, \gamma_{\mathbf{m}2}, \ldots, \gamma_{\mathbf{m}5}, c_{\mathbf{m}1}, c_{\mathbf{m}2}, \ldots, c_{\mathbf{m}9} \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned}\max\left\{\gamma_{\mathbf{m}1}|\tilde{m}_1(t)|^2, \gamma_{\mathbf{m}2}\|\tilde{\mathbf{m}}_2(t)\|^2\right\} &\leq W_{\mathbf{m}}(t)\\ &\leq \max\{\gamma_{\mathbf{m}3}|\tilde{m}_1(t)|^2, \gamma_{\mathbf{m}4}\|\tilde{\mathbf{m}}_2(t)\|^2\}\end{aligned} \tag{39}$$

for all $t \geq t_2$, where we used the shorthand notation $W_{\mathbf{m}}(t) = V_{\mathbf{m}}(\tilde{m}_1(t), \tilde{\mathbf{m}}_2(t), \mathbf{Q}(t))$. Moreover, for all $t \geq t_2$, we have that

$$\dot{W}_{\mathbf{m}}(t) \leq -\eta_{\mathbf{m}}(t)\gamma_{\mathbf{m}5}W_{\mathbf{m}}(t) \tag{40}$$

whenever

$$\begin{aligned}W_{\mathbf{m}}(t) \geq \max\Bigg\{&\alpha_{\boldsymbol{\omega}}^4(t)c_{\mathbf{m}1}, \alpha_{\boldsymbol{\omega}}^2(t)\eta_{\boldsymbol{\omega}}^2(t)c_{\mathbf{m}2},\\ &\alpha_{\boldsymbol{\omega}}^2(t)\eta_{\boldsymbol{\omega}}^2(t)c_{\mathbf{m}3}\|\tilde{\mathbf{u}}(t)\|^2, \frac{\alpha_{\boldsymbol{\omega}}^2(t)\eta_{\mathbf{m}}^2(t)}{\eta_{\boldsymbol{\omega}}^2(t)}c_{\mathbf{m}4}\|\tilde{\mathbf{u}}(t)\|^2,\\ &\frac{\alpha_{\boldsymbol{\omega}}^4(t)\lambda_{\mathbf{u}}^2(t)}{\eta_{\mathbf{m}}^2(t)}c_{\mathbf{m}5}\|\tilde{\mathbf{u}}(t)\|^2, \frac{\eta_{\mathbf{u}}^2(t)}{\eta_{\mathbf{m}}^2(t)}c_{\mathbf{m}6}\|\tilde{\mathbf{u}}(t)\|^2,\\ &\eta_{\mathbf{m}}^2(t)c_{\mathbf{m}7}q_d^2, \eta_{\mathbf{m}}^2(t)c_{\mathbf{m}8}q_{\boldsymbol{\omega}d}^2, c_{\mathbf{m}9}\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2\Bigg\}.\end{aligned} \tag{41}$$

Proof. See Appendix C. □

Lemma 11. *Under the conditions of Theorem 7, there exists a time $t_2 \geq t_1$ such that the solutions of $\tilde{\mathbf{u}}$ are bounded for all $0 \leq t \leq t_2$ and all $\tilde{\mathbf{u}}(0) \in \mathbb{R}^{n_{\mathbf{u}}}$. In addition, there exist a function $V_{\mathbf{u}} : \mathbb{R}^{n_{\mathbf{u}}} \to \mathbb{R}_{\geq 0}$ and constants $\gamma_{\mathbf{u}1}, \gamma_{\mathbf{u}2}, \gamma_{\mathbf{u}3}, \gamma_{\mathbf{u}4}, c_{\mathbf{u}1}, c_{\mathbf{u}2} \in \mathbb{R}_{>0}$ such that*

$$\gamma_{\mathbf{u}1}\|\tilde{\mathbf{u}}(t)\|^2 \leq V_{\mathbf{u}}(\tilde{\mathbf{u}}(t)) \leq \gamma_{\mathbf{u}2}\|\tilde{\mathbf{u}}(t)\|^2 \tag{42}$$

for all $t \geq t_2$. Moreover, for all $t \geq t_2$, we have that

$$\begin{aligned}\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}(t)) \leq -\min\Big\{&\alpha_{\boldsymbol{\omega}}(t)\lambda_{\mathbf{u}}(t)\gamma_{\mathbf{u}3}V_{\mathbf{u}}(\tilde{\mathbf{u}}(t)),\\ &\eta_{\mathbf{u}}(t)\gamma_{\mathbf{u}4}\sqrt{V_{\mathbf{u}}(\tilde{\mathbf{u}}(t))}\Big\}\end{aligned} \tag{43}$$

whenever

$$V_{\mathbf{u}}(\tilde{\mathbf{u}}(t)) \geq \max\left\{\frac{1}{\alpha_{\boldsymbol{\omega}}^2(t)}c_{\mathbf{u}1}\|\tilde{\mathbf{m}}_2(t)\|^2, \frac{\eta_{\mathbf{m}}^2(t)}{\alpha_{\boldsymbol{\omega}}^2(t)}c_{\mathbf{u}2}q_{\boldsymbol{\omega}d}^2\right\}. \tag{44}$$

Proof. See Appendix D. □

To prove that the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ remain bounded for all $t \geq t_2$ and to show that bound in (33) holds, we introduce the following Lyapunov-function candidate as proposed in [6], [13], [19]:

$$\begin{aligned}&V(\tilde{m}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{u}}, \mathbf{Q}, \alpha_{\boldsymbol{\omega}})\\ &\quad= \max\left\{V_{\mathbf{u}}(\tilde{\mathbf{u}}), \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})\right\},\end{aligned} \tag{45}$$

where the functions $V_{\mathbf{m}}$ and $V_{\mathbf{u}}$ are defined in Lemmas 10 and 11, respectively. By following similar lines as in [13], we obtain the following result regarding the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$.

Lemma 12. *Under the conditions of Theorem 7, there exist constants $\gamma_{V1}, \gamma_{V2}, \gamma_{V3}, c_{V1}, c_{V2}, \ldots, c_{V5} \in \mathbb{R}_{>0}$ such that the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ are bounded for all $t \geq t_2$, all $\tilde{m}_1(t_2) \in \mathbb{R}$, $\tilde{\mathbf{m}}_2(t_2) \in \mathbb{R}^{n_{\mathbf{u}}}$ and all $\tilde{\mathbf{u}}(t_2) \in \mathbb{R}^{n_{\mathbf{u}}}$, where $t_2 \in \mathbb{R}_{\geq 0}$ is defined in Lemmas 10 and 11. In addition, the solutions of $\tilde{m}_1$, $\tilde{\mathbf{m}}_2$ and $\tilde{\mathbf{u}}$ satisfy*

$$\begin{aligned}&\limsup_{t\to\infty}\max\left\{\frac{\gamma_{V1}}{\alpha_{\boldsymbol{\omega}}(t)}|\tilde{m}_1(t)|, \frac{\gamma_{V2}}{\alpha_{\boldsymbol{\omega}}(t)}\|\tilde{\mathbf{m}}_2(t)\|, \gamma_{V3}\|\tilde{\mathbf{u}}(t)\|\right\}\\ &\leq \limsup_{t\to\infty}\max\Bigg\{\alpha_{\boldsymbol{\omega}}(t)c_{V1}, \eta_{\boldsymbol{\omega}}(t)c_{V2}, \frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)}c_{V3}q_d,\\ &\qquad\frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)}c_{V4}q_{\boldsymbol{\omega}d}, \frac{1}{\alpha_{\boldsymbol{\omega}}(t)}c_{V5}\|\mathbf{b}_{\boldsymbol{\omega}d}\|\Bigg\}.\end{aligned} \tag{46}$$

Proof. See Appendix E. □

The proof of Theorem 7 follows from Lemmas 8-12 and the coordinate transformation in (34).

B. Choice of tuning parameters

We explore the implications of Theorem 7 for different choices of the tuning parameters $\alpha_{\boldsymbol{\omega}}$, $\eta_{\boldsymbol{\omega}}$, $\eta_{\mathbf{m}}$, $\lambda_{\mathbf{u}}$ and $\eta_{\mathbf{u}}$. First, we consider constant tuning parameters, in which case Theorem 7 reduces to the following result.

Corollary 13. *Let the tuning parameters $\alpha_{\boldsymbol{\omega}}, \eta_{\boldsymbol{\omega}}, \eta_{\mathbf{m}}, \lambda_{\mathbf{u}}, \eta_{\mathbf{u}} \in \mathbb{R}_{>0}$ be constant (that is, $g_\alpha = g_{\boldsymbol{\omega}} = g_{\mathbf{m}} = g_\lambda = g_{\mathbf{u}} = 0$). Under Assumptions 1-4 and 6, there exist (sufficiently large) constants $c_1, c_2, \ldots, c_5 \in \mathbb{R}_{>0}$ and (sufficiently small) constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}_{>0}$ such that the solutions of the closed-loop system of the plant in (1) and the extremum-seeking controller in (25) and (27) are bounded for all $t \geq 0$, all $\mathbf{x}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$, all $\hat{m}_1(0) \in \mathbb{R}$, all $\hat{\mathbf{m}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}}}$, all symmetric positive-definite $\mathbf{Q}(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$, all $\hat{\mathbf{u}}(0) \in \mathbb{R}^{n_{\mathbf{u}}}$ and all $\alpha_{\boldsymbol{\omega}}, \eta_{\boldsymbol{\omega}}, \eta_{\mathbf{m}}, \lambda_{\mathbf{u}}, \eta_{\mathbf{u}} \in \mathbb{R}_{>0}$ that satisfy $\eta_{\boldsymbol{\omega}} < \varepsilon_1$, $\eta_{\mathbf{m}} < \eta_{\boldsymbol{\omega}} \varepsilon_2$, $\eta_{\mathbf{u}} < \alpha_{\boldsymbol{\omega}} \eta_{\mathbf{m}} \varepsilon_3$ and $\alpha_{\boldsymbol{\omega}} \lambda_{\mathbf{u}} < \eta_{\mathbf{m}} \varepsilon_4$. In addition, the solutions of $\hat{\mathbf{u}}$ satisfy*

$$\limsup_{t\to\infty} \|\hat{\mathbf{u}}(t) - \mathbf{u}^*\| \leq \max\Bigg\{ \alpha_{\boldsymbol{\omega}} c_1, \eta_{\boldsymbol{\omega}} c_2, \frac{\eta_{\mathbf{m}}}{\alpha_{\boldsymbol{\omega}}} c_3 q_d, \frac{\eta_{\mathbf{m}}}{\alpha_{\boldsymbol{\omega}}} c_4 q_{\boldsymbol{\omega} d}, \frac{1}{\alpha_{\boldsymbol{\omega}}} c_5 \|\mathbf{b}_{\boldsymbol{\omega} d}\| \Bigg\}. \tag{47}$$

Proof. The proof follows directly from Theorem 7 for $g_\alpha = g_{\boldsymbol{\omega}} = g_{\mathbf{m}} = g_\lambda = g_{\mathbf{u}} = 0$. □

From Corollary 13, we obtain that $\hat{\mathbf{u}}$ converges to a region of performance-optimizing value $\mathbf{u}^*$, where the size of the region is dependent on the tuning parameters $\alpha_{\boldsymbol{\omega}}$, $\eta_{\boldsymbol{\omega}}$ and $\eta_{\mathbf{m}}$ and the disturbance-related constants q_d, $q_{\boldsymbol{\omega} d}$ and $\mathbf{b}_{\boldsymbol{\omega} d}$. If the perturbations and the zero-mean component of the disturbance are uncorrelated (that is, $\mathbf{b}_{\boldsymbol{\omega} d} = \mathbf{0}$), the size of the region of $\mathbf{u}^*$ to which $\hat{\mathbf{u}}$ converges can be made arbitrarily small by selecting suitable tuning parameters. This result is similar to the results for plants with output disturbances in [1], [29]. It is generally not possible to make the size of the region of $\mathbf{u}^*$ to which $\hat{\mathbf{u}}$ converges arbitrarily small if the perturbations and the zero-mean component of the disturbance are correlated. We note that, because $\mathbf{b}_{\boldsymbol{\omega} d}$ depends on the tuning parameter $\eta_{\boldsymbol{\omega}}$ (see Assumption 6), correlation of the perturbations and the zero-mean component of the disturbance may be avoided by choosing a different value of $\eta_{\boldsymbol{\omega}}$.

Now, let us consider time-varying tuning parameters. In particular, let the time-varying parameters g_α, $g_{\boldsymbol{\omega}}$, $g_{\mathbf{m}}$, g_λ and $g_{\mathbf{u}}$ be defined as follows.

Corollary 14. *Let the parameters g_α, $g_{\boldsymbol{\omega}}$, $g_{\mathbf{m}}$, g_λ and $g_{\mathbf{u}}$ in (14), (26) and (28) be given by*

$$\begin{aligned} g_\alpha(t) &= \frac{r_\alpha}{r_0 + t}, \quad g_{\boldsymbol{\omega}}(t) = \frac{r_{\boldsymbol{\omega}}}{r_0 + t}, \quad g_{\mathbf{m}}(t) = \frac{r_{\mathbf{m}}}{r_0 + t}, \\ g_\lambda(t) &= \frac{r_\lambda}{r_0 + t}, \quad g_{\mathbf{u}}(t) = \frac{r_{\mathbf{u}}}{r_0 + t}, \end{aligned} \tag{48}$$

where the constants $r_0 \in \mathbb{R}_{>0}$ and $r_\alpha, r_{\boldsymbol{\omega}}, r_{\mathbf{m}}, r_\lambda, r_{\mathbf{u}} \in \mathbb{R}_{\geq 0}$ satisfy

$$\begin{aligned} 0 < r_\alpha < r_{\mathbf{m}}, &\qquad 0 < r_{\boldsymbol{\omega}} < r_{\mathbf{m}}, \\ r_{\mathbf{m}} < r_\alpha + r_\lambda \leq 1, &\quad r_\alpha + r_{\mathbf{m}} < r_{\mathbf{u}} \leq 1. \end{aligned} \tag{49}$$

Suppose that the perturbations and the zero-mean component of the disturbance are uncorrelated (that is, $\mathbf{b}_{\boldsymbol{\omega} d} = \mathbf{0}$). Under this assumption and Assumptions 1-4 and 6, the solutions of the closed-loop system of the plant in (1) and the extremum-seeking controller in (25) and (27) are bounded for all $t \geq 0$, all $\mathbf{x}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$, all $\hat{m}_1(0) \in \mathbb{R}$, all $\hat{\mathbf{m}}_2(0) \in \mathbb{R}^{n_{\mathbf{u}}}$, all symmetric positive-definite $\mathbf{Q}(0) \in \mathbb{R}^{n_{\mathbf{u}} \times n_{\mathbf{u}}}$, all $\hat{\mathbf{u}}(0) \in \mathbb{R}^{n_{\mathbf{u}}}$ and all $\alpha_{\boldsymbol{\omega}}(0), \eta_{\boldsymbol{\omega}}(0), \eta_{\mathbf{m}}(0), \lambda_{\mathbf{u}}(0), \eta_{\mathbf{u}}(0) \in \mathbb{R}_{>0}$. In addition, the solutions of $\hat{\mathbf{u}}$ satisfy $\lim_{t\to\infty} \hat{\mathbf{u}}(t) = \mathbf{u}^*$.

Proof. The proof follows from Theorem 7 for g_α, $g_{\boldsymbol{\omega}}$, $g_{\mathbf{m}}$, g_λ and $g_{\mathbf{u}}$ defined in (48) and (49). We note that

$$\begin{aligned} \alpha_{\boldsymbol{\omega}}(t) &= \frac{r_0^{r_\alpha} \alpha_{\boldsymbol{\omega}}(0)}{(r_0 + t)^{r_\alpha}}, \quad \eta_{\boldsymbol{\omega}}(t) = \frac{r_0^{r_{\boldsymbol{\omega}}} \eta_{\boldsymbol{\omega}}(0)}{(r_0 + t)^{r_{\boldsymbol{\omega}}}}, \\ \eta_{\mathbf{m}}(t) &= \frac{r_0^{r_{\mathbf{m}}} \eta_{\mathbf{m}}(0)}{(r_0 + t)^{r_{\mathbf{m}}}}, \quad \lambda_{\mathbf{u}}(t) = \frac{r_0^{r_\lambda} \lambda_{\mathbf{u}}(0)}{(r_0 + t)^{r_\lambda}}, \\ \eta_{\mathbf{u}}(t) &= \frac{r_0^{r_{\mathbf{u}}} \eta_{\mathbf{u}}(0)}{(r_0 + t)^{r_{\mathbf{u}}}}, \end{aligned} \tag{50}$$

which follows from (14), (26), (28) and (48). Hence, for any $\alpha_{\boldsymbol{\omega}}(0), \eta_{\boldsymbol{\omega}}(0), \eta_{\mathbf{m}}(0), \lambda_{\mathbf{u}}(0), \eta_{\mathbf{u}}(0) \in \mathbb{R}_{>0}$, there exists a time $t_1 \in \mathbb{R}_{\geq 0}$ such that (32) in Theorem 7 holds for all $t \geq t_1$ under the conditions in (48) and (49). Moreover, from (49) and (50), we obtain $\lim_{t\to\infty} \alpha_{\boldsymbol{\omega}}(t) = \lim_{t\to\infty} \eta_{\boldsymbol{\omega}}(t) = \lim_{t\to\infty} \frac{\eta_{\mathbf{m}}(t)}{\alpha_{\boldsymbol{\omega}}(t)} = 0$ so that the right-hand side of (33) in Theorem 7 reduces to zero for $\mathbf{b}_{\boldsymbol{\omega} d} = \mathbf{0}$. □

Under the conditions of Corollary 14, $\hat{\mathbf{u}}$ converges to $\mathbf{u}^*$, even in the presence of an unknown disturbance (if the perturbations and the zero-mean component of the disturbance are uncorrelated). It is not difficult to show that the state $\mathbf{x}$ of the plant converges to $\mathbf{X}(\mathbf{u}^*)$ under the conditions of Corollary 14, which implies that the plant performance converges to the optimal steady-state performance as time goes to infinity. We note that the closed-loop system is globally asymptotically stable with respect to the optimal steady-state plant performance under the conditions of Corollary 14 in the sense that the solutions of the closed-loop system are bounded and asymptotically converge to the steady-state values for which the plant performance is optimal for any initial condition of the plant. To the best of our knowledge, this is the first work about extremum-seeking control in which global asymptotic stability to the optimal steady-state performance of the general nonlinear plant in (1) is proved. Because global asymptotic stability with respect to the optimal steady-state plant performance is ensured for any plant that satisfies the assumptions in Corollary 14, selecting any set of tuning parameters that satisfy (48) and (49) eliminates the necessity (in [17], [30] for example) to tune the extremum-seeking controller in order to guarantee stability of the resulting closed-loop system. Nonetheless, plant-specific tuning of the controller is often desirable as suitably chosen tuning parameters can significantly enhance the overall convergence rate of the extremum-seeking scheme. Moreover, we note that $\mathbf{b}_{\boldsymbol{\omega}d}$ is the time average of the product of the perturbations, whose frequencies asymptotically converge to zero, and the zero-mean component of the disturbance. Hence, $\mathbf{b}_{\boldsymbol{\omega}d} = \mathbf{0}$ for a large class of disturbances. Corollary 14 does not guarantee convergence or boundedness of the solutions of the closed-loop system if $\mathbf{b}_{\boldsymbol{\omega}d} \neq \mathbf{0}$. To guarantee robustness of the closed-loop system for time-varying tuning of the controller if $\mathbf{b}_{\boldsymbol{\omega}d} \neq \mathbf{0}$, the perturbation amplitude should be chosen such that $\lim_{t\to\infty} \alpha_{\boldsymbol{\omega}}(t) > 0$, which precludes asymptotic convergence to the optimal steady-state plant performance. Additionally, we note that the tuning parameters of the controller should remain positive as time goes to infinity to be able to track changes in the performance-optimizing plant-parameter values if slowly time-varying plants are considered as in [1].

V. Simulation examples

A. Example 1

Consider the following double-input-single-output plant

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u_2^2(t) \\ \dot{x}_2(t) &= -x_2(t) + u_1(t) \\ \dot{x}_3(t) &= -x_3(t) + u_2(t)x_2(t) \\ y(t) &= 2(x_1(t) + x_2(t) - u_1(t)) + (x_2(t) + x_3(t))^2, \end{aligned} \tag{51}$$

with state $\mathbf{x} = [x_1, x_2, x_3]^T$ and plant-parameter vector $\mathbf{u} = [u_1, u_2]^T$. The corresponding objective function of the plant is given by $F(\mathbf{u}) = (1 + u_2)^2 u_1^2 + 2u_2^2$. We apply the extremum-seeking controller in Section III to the plant (51). The tuning parameters of the controller are chosen as defined in Corollary 13 and Corollary 14, where the tuning constants in Corollary 14 are set to $r_0 = 200$, $r_\alpha = 0.4$, $r_{\boldsymbol{\omega}} = 0.4$, $r_{\mathbf{m}} = 0.45$, $r_\lambda = 0.1$ and $r_{\mathbf{u}} = 0.9$. The initial tuning-parameter values are set to $\alpha_{\boldsymbol{\omega}}(0) = 0.1$, $\eta_{\boldsymbol{\omega}}(0) = 1$, $\eta_{\mathbf{m}}(0) = 1$, $\lambda_{\mathbf{u}}(0) = 0.5$ and $\eta_{\mathbf{u}}(0) = 0.04$ for both tuning conditions. The trajectories of the plant parameters are illustrated in Fig. 2. Fig. 3 displays the corresponding measurement y of the performance cost for the first 2000 time units. From Fig. 2, we obtain that the plant parameters asymptotically converge to the performance-optimizing values $\mathbf{u}^* = \mathbf{0}$ if the time-varying tuning in Corollary 14 is applied. The corresponding measurement y of the performance cost in Fig. 3 asymptotically converges to the minimum $F(\mathbf{u}^*) = 0$ of the objective function. This implies that the optimal steady-state performance of the plant is obtained as time goes to infinity. Contrarily, the plant parameters converge to a region of $\mathbf{u}^* = \mathbf{0}$ for the constant tuning in Corollary 13 (see Fig. 2) for which the obtained plant performance is suboptimal. As a result, we observe in Fig. 3 that the measurement y converges to the value 0.5 instead of zero.

B. Example 2

To illustrate the influence of a time-varying disturbance on the convergence of the plant parameters for the time-varying tuning in Corollary 14, we consider the plant

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \\ y(t) &= (x(t) - 1)^2 + d(t), \end{aligned} \tag{52}$$

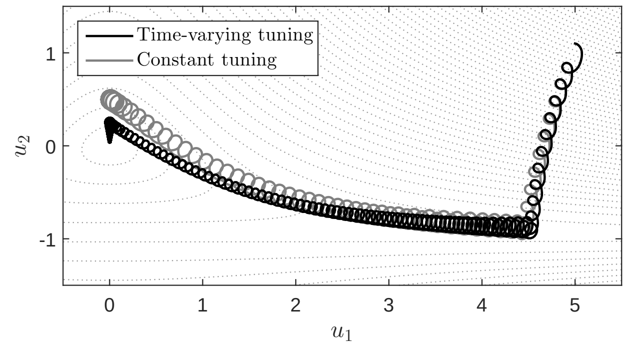

Fig. 2. Trajectory of the plant parameters $\mathbf{u} = [u_1, u_2]^T$ for Example 1 using the constant tuning in Corollary 13 and the time-varying tuning in Corollary 14.

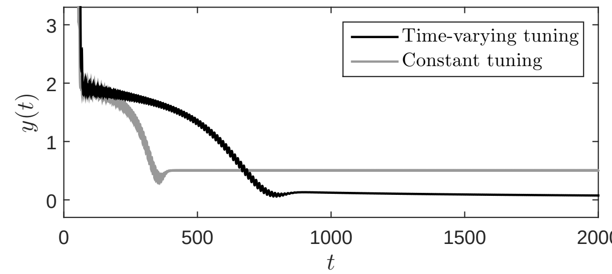

Fig. 3. Measurement y of the performance cost as a function of time for Example 1 using the constant tuning in Corollary 13 and the time-varying tuning in Corollary 14.

with disturbance $d(t) = \sin(0.2t)$. The objective function is given by $F(u) = (u - 1)^2$. We note that the perturbation ω in (11) and the zero-mean component of the disturbance $\tilde{d} = d$ are uncorrelated for any values $r_0, r_{\boldsymbol{\omega}} > 0$ in Corollary 14. We let $r_0 = 10$, $r_\alpha = 0.15$, $r_\omega = 0.25$, $r_{\mathbf{m}} = 0.4$, $r_\lambda = 0.3$ and $r_u = 0.6$. Figs. 4 and 5 illustrate the evolution of the plant parameter u, the performance cost $(x-2)^2$ and the measurement y as a function of time for the initial tuning-parameter values $\alpha_\omega(0) = 0.2$, $\eta_\omega(0) = 0.8$, $\eta_{\mathbf{m}}(0) = 0.6$, $\lambda_u(0) = 0.2$ and $\eta_u(0) = 0.4$. We observe in Fig. 4 that the plant parameter u converges to its performance-optimizing values $u^* = 1$ as time progresses. However, the convergence of the plant parameter is momentarily disrupted when the angular frequency η_ω of the perturbation is equal to the angular frequency of the disturbance (that is, $\eta_\omega = 0.2$). A similar observation can be made in Fig. 5 where the performance cost rises as η_ω reaches the value 0.2. We note that this disruption can be contributed to a "momentary correlation" of the perturbation and the zero-mean component of the disturbance for $\eta_\omega = 0.2$. We note that the effect of the momentary correlation can be diminished by increasing the perturbation amplitude. Alternatively, the disruption can be prevented by choosing $\eta_\omega(0)$ smaller than 0.2. Fig. 5 shows that the performance cost converges to the optimal value $F(u^*) = 0$ as time elapses. This implies that the optimal steady-state performance is achieved despite that the measurement y of the performance cost is corrupted by the disturbance d.

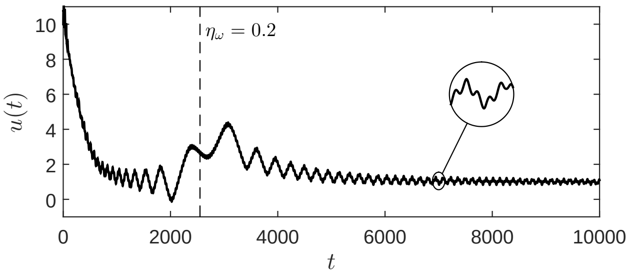

Fig. 4. Plant parameter u as a function of time for Example 2.

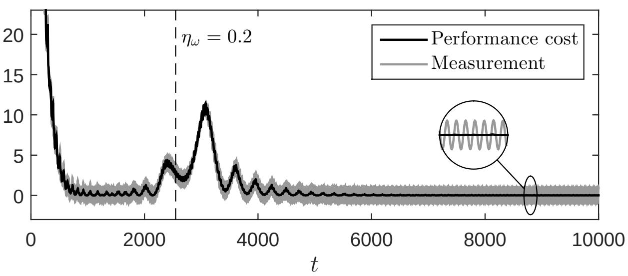

Fig. 5. Performance cost $(x-1)^2$ and measurement y as a function of time for Example 2.

C. Example 3

We apply the extremum-seeking controller in [33] and the presented controller with the tuning in Corollary 14 to the plant

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2u(t) + 2 \\ \dot{x}_2(t) &= 3x_1(t) - x_2(t) + u^2(t) \\ y(t) &= x_1(t) + x_2(t) + 2, \end{aligned} \tag{53}$$

with state $\mathbf{x} = [x_1, x_2]^T$ and objective function $F(u) = (u+4)^2 - 6$. Let the tuning constants in Corollary 14 be given by $r_0 = 100$, $r_\alpha = 0.3$, $r_{\boldsymbol{\omega}} = 0.6$, $r_{\mathbf{m}} = 0.65$, $r_\lambda = 0.4$ and $r_{\mathbf{u}} = 1$, with initial parameter values $\alpha_{\boldsymbol{\omega}}(0) = 0.1$, $\eta_{\boldsymbol{\omega}}(0) = 0.5$, $\eta_{\mathbf{m}}(0) = 0.1$, $\lambda_{\mathbf{u}}(0) = 0.2$ and $\eta_{\mathbf{u}}(0) = 0.05$. The tuning parameters of the controller in [33] are set to $k = 0.05$, $\omega = 0.5$, $\omega_h = 0.2$, $\omega_l = 0.01$ and $r = 1$, with initial perturbation amplitude $a(0) = 0.1$. We observe in Fig. 6 that the plant parameter asymptotically converges to the performance-optimizing value $u^* = -4$ using the presented controller. The corresponding measurement of the performance cost asymptotically converges to the minimum $F(u^*) = -6$ of the objective function, as shown in Fig. 7. The controller in [33] regulates the perturbation amplitude while the perturbation frequency is kept constant; see Fig. 8. The plant parameter in Fig. 6 converges to a constant value (that is, $u \approx -3.2$) in a region of the performance-optimizing value u^* using the controller in [33]. This region can be made arbitrarily small by choosing a sufficiently small perturbation frequency, which implies practical convergence. Asymptotic convergence is only achieved with the presented controller for this example.

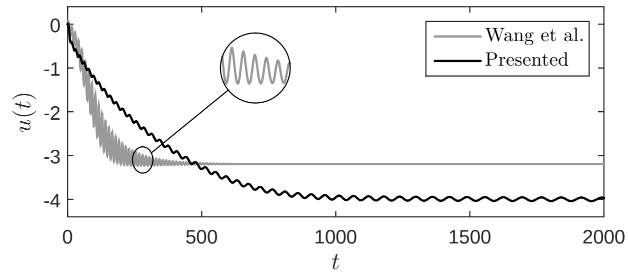

Fig. 6. Plant parameter u as a function of time for Example 3 using the controller by Wang et al. [33] and the presented controller.

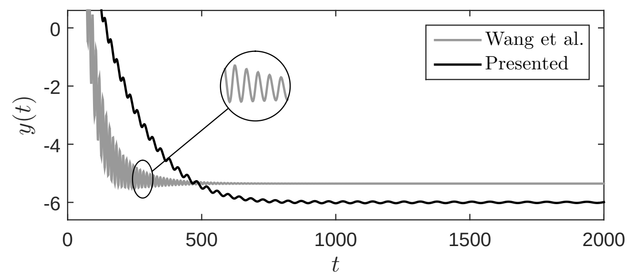

Fig. 7. Measurement y of the performance cost as a function of time for Example 3 using the controller by Wang et al. [33] and the presented controller.

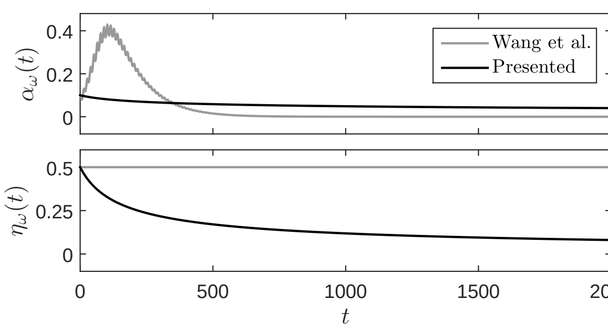

Fig. 8. Amplitude $\alpha_{\boldsymbol{\omega}}$ (a in [33]) and angular frequency $\eta_{\boldsymbol{\omega}}$ (ω in [33]) of the perturbation as a function of time for Example 3 using the controller by Wang et al. [33] and the presented controller.

VI. Conclusion

In this work, we have introduced a perturbation-based extremum-seeking controller to optimize the steady-state performance of nonlinear dynamical plants. We have shown that global asymptotic stability of the closed-loop system of plant and controller with respect to the optimal steady-state plant performance can be obtained for any plant that satisfies the assumptions in the work. The key to this result is that the tuning parameters of the controller are time varying and asymptotically decay to zero as time goes to infinity. We note that global asymptotic stability can even be obtained if the plant is subjected to a time-varying disturbance under the assumption that the perturbations of the controller and the zero-mean component of the disturbance are uncorrelated. Moreover, we have identified time-varying tuning-parameter values of the controller for which the closed-loop system is globally asymptotically stable for all plants that satisfy the assumptions in this work. Three simulation examples illustrate the effectiveness of the proposed extremum-seeking controller.

Appendix A
Proof of Lemma 8

From (1) and (34), we obtain that the state equation for $\tilde{\mathbf{x}}$ is given by

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{X}(\mathbf{u}), \mathbf{u}) - \frac{d\mathbf{X}}{d\mathbf{u}}(\mathbf{u})\dot{\mathbf{u}}. \tag{54}$$

Because the plant is globally exponentially stable with respect to the steady-state solution $\mathbf{X}(\mathbf{u})$ for constant $\mathbf{u}$, the following converse lemma holds.

Lemma 15. *Under Assumptions 1, 2 and 4, there exists a function $V_{\mathbf{x}} : \mathbb{R}^{n_{\mathbf{x}}} \times \mathbb{R}^{n_{\mathbf{u}}} \to \mathbb{R}$ and constants $\gamma_{\mathbf{x}1}, \gamma_{\mathbf{x}2}, \gamma_{\mathbf{x}3}, \gamma_{\mathbf{x}4}, \gamma_{\mathbf{x}5} \in \mathbb{R}_{>0}$ such that the inequalities*

$$\gamma_{\mathbf{x}1}\|\tilde{\mathbf{x}}\|^2 \le V_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{u}) \le \gamma_{\mathbf{x}2}\|\tilde{\mathbf{x}}\|^2, \tag{55}$$

$$\frac{\partial V_{\mathbf{x}}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{x}}, \mathbf{u})\mathbf{f}(\tilde{\mathbf{x}} + \mathbf{X}(\mathbf{u}), \mathbf{u}) \le -\gamma_{\mathbf{x}3}\|\tilde{\mathbf{x}}\|^2 \tag{56}$$

and

$$\left\|\frac{\partial V_{\mathbf{x}}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{x}}, \mathbf{u})\right\| \le \gamma_{\mathbf{x}4}\|\tilde{\mathbf{x}}\|, \quad \left\|\frac{\partial V_{\mathbf{x}}}{\partial \mathbf{u}}(\tilde{\mathbf{x}}, \mathbf{u})\right\| \le \gamma_{\mathbf{x}5}\|\tilde{\mathbf{x}}\| \tag{57}$$

are satisfied for all $\tilde{\mathbf{x}} \in \mathbb{R}^{n_{\mathbf{x}}}$ and all $\mathbf{u} \in \mathbb{R}^{n_{\mathbf{u}}}$.

Proof. The proof follows similar steps as the proof of [15, Lemma 9.8]. □

We use the function $V_{\mathbf{x}}$ as a Lyapunov-function candidate for the $\tilde{\mathbf{x}}$-dynamics for time-varying plant parameters $\mathbf{u}$. By using (54), the time derivative of $V_{\mathbf{x}}$ for time-varying plant parameters can be written as

$$\begin{aligned}\dot{V}_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{u}) = & \frac{\partial V_{\mathbf{x}}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{x}}, \mathbf{u})\mathbf{f}(\tilde{\mathbf{x}} + \mathbf{X}(\mathbf{u}), \mathbf{u}) \\ & + \left(\frac{\partial V_{\mathbf{x}}}{\partial \mathbf{u}}(\tilde{\mathbf{x}}, \mathbf{u}) - \frac{\partial V_{\mathbf{x}}}{\partial \tilde{\mathbf{x}}}(\tilde{\mathbf{x}}, \mathbf{u})\frac{d\mathbf{X}}{d\mathbf{u}}(\mathbf{u})\right)\dot{\mathbf{u}}.\end{aligned} \tag{58}$$

From Assumption 1 and Lemma 15, we obtain that the time derivative of $V_{\mathbf{x}}$ can be bounded by

$$\dot{V}_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{u}) \le -\gamma_{\mathbf{x}3}\|\tilde{\mathbf{x}}\|^2 + (\gamma_{\mathbf{x}5} + \gamma_{\mathbf{x}4}L_{\mathbf{X}})\|\tilde{\mathbf{x}}\|\|\dot{\mathbf{u}}\|. \tag{59}$$

Subsequently, from Lemma 15 and Young's inequality, it follows that

$$\dot{V}_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{u}) \le -\frac{\gamma_{\mathbf{x}3}}{2\gamma_{\mathbf{x}2}}V_{\mathbf{x}}(\tilde{\mathbf{x}}, \mathbf{u}) + \frac{1}{2\gamma_{\mathbf{x}3}}(\gamma_{\mathbf{x}5} + \gamma_{\mathbf{x}4}L_{\mathbf{X}})^2\|\dot{\mathbf{u}}\|^2. \tag{60}$$

From (60) and the comparison lemma [15, Lemma 3.4], we obtain

$$\begin{aligned}V_{\mathbf{x}}(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) \le & V_{\mathbf{x}}(\tilde{\mathbf{x}}(t_0), \mathbf{u}(t_0))e^{-\frac{\gamma_{\mathbf{x}3}}{2\gamma_{\mathbf{x}2}}(t-t_0)} \\ & + \frac{1}{2\gamma_{\mathbf{x}3}}(\gamma_{\mathbf{x}5} + \gamma_{\mathbf{x}4}L_{\mathbf{X}})^2\int_{t_0}^{t} e^{-\frac{\gamma_{\mathbf{x}3}}{2\gamma_{\mathbf{x}2}}(t-\tau)}\|\dot{\mathbf{u}}(\tau)\|^2 d\tau\end{aligned} \tag{61}$$

for all $t \ge t_0 \ge 0$. To find an upper bound for $\|\dot{\mathbf{u}}\|$, we note that it follows from (13) and (14) that

$$\dot{\mathbf{u}} = \dot{\hat{\mathbf{u}}} - g_\alpha\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega} + \alpha_{\boldsymbol{\omega}}\dot{\boldsymbol{\omega}}. \tag{62}$$

From the definition of $\boldsymbol{\omega}$ in (11), it follows that there exist constants $L_{\boldsymbol{\omega}1}, L_{\boldsymbol{\omega}2} \in \mathbb{R}_{>0}$ such that

$$\|\boldsymbol{\omega}\| \le L_{\boldsymbol{\omega}1}, \quad \|\dot{\boldsymbol{\omega}}\| \le \eta_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}. \tag{63}$$

Moreover, from (27), we have that $\|\dot{\hat{\mathbf{u}}}\| \le \eta_{\mathbf{u}}$. Therefore, from (62), (63) and $\|\dot{\hat{\mathbf{u}}}\| \le \eta_{\mathbf{u}}$, we obtain

$$\|\dot{\mathbf{u}}\| \le \eta_{\mathbf{u}} + \alpha_{\boldsymbol{\omega}}g_\alpha L_{\boldsymbol{\omega}1} + \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}2}. \tag{64}$$

Because $\alpha_{\boldsymbol{\omega}}$, $\eta_{\boldsymbol{\omega}}$ and $\eta_{\mathbf{u}}$ are nonincreasing (see (14) and (28)), from (30) in Theorem 7 and (64), it follows that

$$\|\dot{\mathbf{u}}(t)\| \le \eta_{\mathbf{u}}(0) + \alpha_{\boldsymbol{\omega}}(0)c_g L_{\boldsymbol{\omega}1} + \alpha_{\boldsymbol{\omega}}(0)\eta_{\boldsymbol{\omega}}(0)L_{\boldsymbol{\omega}2} \tag{65}$$

for all $t \ge 0$. By substituting (64) in (61), we obtain

$$\begin{aligned}V_{\mathbf{x}}(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) \le & V_{\mathbf{x}}(\tilde{\mathbf{x}}(0), \mathbf{u}(0)) + \frac{\gamma_{\mathbf{x}2}}{\gamma_{\mathbf{x}3}^2}(\gamma_{\mathbf{x}5} + \gamma_{\mathbf{x}4}L_{\mathbf{X}})^2 \\ & \times (\eta_{\mathbf{u}}(0) + \alpha_{\boldsymbol{\omega}}(0)c_g L_{\boldsymbol{\omega}1} + \alpha_{\boldsymbol{\omega}}(0)\eta_{\boldsymbol{\omega}}(0)L_{\boldsymbol{\omega}2})^2\end{aligned} \tag{66}$$

for all $t \ge 0$. From (32) in Theorem 7 and (64), it follows that

$$\|\dot{\mathbf{u}}(t)\| \le \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)(\varepsilon_5\varepsilon_6 + \varepsilon_2\varepsilon_5 L_{\boldsymbol{\omega}1} + L_{\boldsymbol{\omega}2}) \tag{67}$$

for all $t \ge t_1$, all $g_\alpha \le \eta_{\mathbf{m}}\varepsilon_2$, all $\eta_{\mathbf{m}} \le \eta_{\boldsymbol{\omega}}\varepsilon_5$ and all $\eta_{\mathbf{u}} \le \alpha_{\boldsymbol{\omega}}\eta_{\mathbf{m}}\varepsilon_6$. From (14), we have that

$$\alpha_{\boldsymbol{\omega}}(t) = \alpha_{\boldsymbol{\omega}}(\tau)e^{-\int_\tau^t g_\alpha(s)ds}, \quad \eta_{\boldsymbol{\omega}}(t) = \eta_{\boldsymbol{\omega}}(\tau)e^{-\int_\tau^t g_{\boldsymbol{\omega}}(s)ds} \tag{68}$$

for any $t \ge \tau \ge 0$. Without loss of generality, we assume that ε_1 in Theorem 7 is sufficiently small such that it follows from (32) and (68) that

$$\begin{aligned}\alpha_{\boldsymbol{\omega}}(\tau)\eta_{\boldsymbol{\omega}}(\tau) &= \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)e^{\int_\tau^t(g_\alpha(s)+g_{\boldsymbol{\omega}}(s))ds} \\ &\le \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)e^{\frac{\gamma_{\mathbf{x}3}}{8\gamma_{\mathbf{x}2}}(t-\tau)}.\end{aligned} \tag{69}$$

for all $t \ge \tau \ge t_1$ and all $g_\alpha + g_{\boldsymbol{\omega}} \le \varepsilon_1$. From (67) and (69), we have

$$\begin{aligned}&\int_{t_1}^{t} e^{-\frac{\gamma_{\mathbf{x}3}}{2\gamma_{\mathbf{x}2}}(t-\tau)}\|\dot{\mathbf{u}}(\tau)\|^2 d\tau \\ &\quad \le 4\alpha_{\boldsymbol{\omega}}^2(t)\eta_{\boldsymbol{\omega}}^2(t)\frac{\gamma_{\mathbf{x}2}}{\gamma_{\mathbf{x}3}}(\varepsilon_5\varepsilon_6 + \varepsilon_2\varepsilon_5 L_{\boldsymbol{\omega}1} + L_{\boldsymbol{\omega}2})^2\end{aligned} \tag{70}$$

for all $t \ge t_1$. Therefore, from (61) and (70), we obtain

$$\begin{aligned}V_{\mathbf{x}}(\tilde{\mathbf{x}}(t), \mathbf{u}(t)) \le \max\Big\{ & 2V_{\mathbf{x}}(\tilde{\mathbf{x}}(t_1), \mathbf{u}(t_1))e^{-\frac{\gamma_{\mathbf{x}3}}{2\gamma_{\mathbf{x}2}}(t-t_1)}, \\ & 4\alpha_{\boldsymbol{\omega}}^2(t)\eta_{\boldsymbol{\omega}}^2(t)\frac{\gamma_{\mathbf{x}2}}{\gamma_{\mathbf{x}3}^2}(\gamma_{\mathbf{x}5} + \gamma_{\mathbf{x}4}L_{\mathbf{X}})^2 \\ & \times (\varepsilon_5\varepsilon_6 + \varepsilon_2\varepsilon_5 L_{\boldsymbol{\omega}1} + L_{\boldsymbol{\omega}2})^2\Big\}\end{aligned} \tag{71}$$

for all $t \ge t_1$. From (55) in Lemma 15 and (66), it follows that the solutions $\tilde{\mathbf{x}}(t)$ are bounded for all $t \ge 0$ and all $\tilde{\mathbf{x}}(0) \in \mathbb{R}^{n_{\mathbf{x}}}$. The bound in (37) of Lemma 8 follows from (55) and (71).

Appendix B
Proof of Lemma 9

We note that $\tilde{\mathbf{Q}}$ in (34) is well defined if $\mathbf{Q}^{-1}$ exists. First we will show that the solution $\mathbf{Q}(t)$ of (25) is invertible for all $t \ge 0$ and all symmetric and positive-definite $\mathbf{Q}(0)$. Let $[0, t_{\mathbf{Q}})$ be the maximal interval of existence of $\mathbf{Q}^{-1}(t)$, with $t_{\mathbf{Q}} \in \mathbb{R}_{\ge 0} \cup \{\infty\}$. We note that $\mathbf{Q}^{-1}(t)$ is positive definite for all $t_{\mathbf{Q}} \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ because $\mathbf{Q}(0)$ is positive definite. From (25), it follows that the time derivative of $\mathbf{Q}^{-1}$ is given by

$$\frac{d}{dt}\left(\mathbf{Q}^{-1}\right) = -\eta_{\mathbf{m}}\mathbf{Q}^{-1} + 2g_\alpha\mathbf{Q}^{-1} + \eta_{\mathbf{m}}\boldsymbol{\omega}\boldsymbol{\omega}^T \tag{72}$$

for all $t \in [0, t_{\mathbf{Q}})$, where we omitted the time index t for brevity. From (72), we obtain

$$-\eta_{\mathbf{m}}\mathbf{Q}^{-1} \preceq \frac{d}{dt}\left(\mathbf{Q}^{-1}\right) \preceq 2g_\alpha \mathbf{Q}^{-1} + \eta_{\mathbf{m}}\|\boldsymbol{\omega}\|^2\mathbf{I} \tag{73}$$

for all $t \in [0, t_{\mathbf{Q}})$. Because $\eta_{\mathbf{m}}$ is nonincreasing (see (26)), we have from (30) in Theorem 7, (63) in the proof of Lemma 8 and (73) that

$$-\eta_{\mathbf{m}}(0)\mathbf{Q}^{-1}(t) \preceq \frac{d}{dt}\left(\mathbf{Q}^{-1}(t)\right) \preceq 2c_g\mathbf{Q}^{-1}(t) + \eta_{\mathbf{m}}(0)L_{\boldsymbol{\omega}1}^2\mathbf{I} \tag{74}$$

for all $t \in [0, t_{\mathbf{Q}})$. Subsequently, using the comparison lemma [15, Lemma 3.4], we obtain

$$\mathbf{Q}^{-1}(0)e^{-\eta_{\mathbf{m}}(0)t} \preceq \mathbf{Q}^{-1}(t) \preceq \mathbf{Q}^{-1}(0)e^{2c_g t} + \frac{\eta_{\mathbf{m}}(0)}{2c_g}L_{\boldsymbol{\omega}1}^2\mathbf{I} \tag{75}$$

for all $t \in [0, t_{\mathbf{Q}})$. From (75) and the continuity of the solutions of $\mathbf{Q}^{-1}$, it follows that $\mathbf{Q}^{-1}(t)$ is defined for all $t \geq 0$ and all positive definite $\mathbf{Q}(0)$. Hence, $t_{\mathbf{Q}} = \infty$. Moreover, from (75), we have that $\mathbf{Q}^{-1}(t)$ is positive definite for all $t \geq 0$ and all positive definite $\mathbf{Q}(0)$.

Now, from (26), (28), (34), (35) and (72), we obtain that the state equation for $\tilde{\mathbf{Q}}$ is given by

$$\dot{\tilde{\mathbf{Q}}} = -\eta_{\mathbf{m}}\tilde{\mathbf{Q}} + 2g_\alpha\tilde{\mathbf{Q}} + g_\alpha\mathbf{I} + \frac{\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\left(2g_\alpha + g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}}\right)\mathbf{l}_2. \tag{76}$$

Because $\mathbf{Q}(0)$ is symmetric and $\mathbf{l}_2$ in (36) is a symmetric function, we obtain from (34) that $\tilde{\mathbf{Q}}(0)$ is symmetric as well. Subsequently, from (76), it follows that $\tilde{\mathbf{Q}}(t)$ remains symmetric for all $t \geq 0$. We define the following Lyapunov-function candidate for the $\tilde{\mathbf{Q}}$-dynamics:

$$V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) = \operatorname{tr}\left(\tilde{\mathbf{Q}}^2\right). \tag{77}$$

From (76), it follows that the time derivative of $V_{\mathbf{Q}}$ can be written as

$$\begin{aligned}\dot{V}_{\mathbf{Q}}(\tilde{\mathbf{Q}}) &= -2\eta_{\mathbf{m}}\operatorname{tr}\left(\tilde{\mathbf{Q}}^2\right) + 4g_\alpha\operatorname{tr}\left(\tilde{\mathbf{Q}}^2\right) + 2g_\alpha\operatorname{tr}\left(\tilde{\mathbf{Q}}\right)\\ &\quad + \frac{2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}(2g_\alpha + g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}})\frac{\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\operatorname{tr}\left(\tilde{\mathbf{Q}}\mathbf{l}_2\right).\end{aligned} \tag{78}$$

From Young's inequality, (77) and (78), we obtain

$$\begin{aligned}\dot{V}_{\mathbf{Q}}(\tilde{\mathbf{Q}}) &\leq -\eta_{\mathbf{m}}V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) + 4g_\alpha V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) + \frac{2}{\eta_{\mathbf{m}}}g_\alpha^2\operatorname{tr}(\mathbf{I})\\ &\quad + \frac{2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}(2g_\alpha + g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}})^2\operatorname{tr}\left(\mathbf{l}_2^2\right).\end{aligned} \tag{79}$$

We note that $\operatorname{tr}(\mathbf{I}) = n_{\mathbf{u}}$. Moreover, from the definition of $\mathbf{l}_2$ in (35), it follows that there exists a constant $L_{\mathbf{l}2} \in \mathbb{R}_{>0}$ such that

$$\|\mathbf{l}_2\| \leq L_{\mathbf{l}2}, \tag{80}$$

which implies that $\operatorname{tr}\left(\mathbf{l}_2^2\right) \leq n_{\mathbf{u}}L_{\mathbf{l}2}^2$. We therefore obtain that

$$\begin{aligned}\dot{V}_{\mathbf{Q}}(\tilde{\mathbf{Q}}) &\leq -\eta_{\mathbf{m}}V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) + 4g_\alpha V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) + \frac{2}{\eta_{\mathbf{m}}}g_\alpha^2 n_{\mathbf{u}}\\ &\quad + \frac{2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}(2g_\alpha + |g_{\mathbf{m}} - g_{\boldsymbol{\omega}}| + \eta_{\mathbf{m}})^2 n_{\mathbf{u}}L_{\mathbf{l}2}^2.\end{aligned} \tag{81}$$

From (30) in Theorem 7, (14) and (26), it follows that

$$\eta_{\boldsymbol{\omega}}(0)e^{-c_g t} \leq \eta_{\boldsymbol{\omega}}(t), \quad \eta_{\mathbf{m}}(0)e^{-c_g t} \leq \eta_{\mathbf{m}}(t) \tag{82}$$

for all $t \geq 0$. Because $\eta_{\mathbf{m}}$ is nonincreasing (see (26)), from (30) in Theorem 7, (81) and (82), we obtain that

$$\begin{aligned}\dot{V}_{\mathbf{Q}}(\tilde{\mathbf{Q}}(t)) &\leq 4c_g V_{\mathbf{Q}}(\tilde{\mathbf{Q}}(t)) + \frac{2}{\eta_{\mathbf{m}}(0)}c_g^2 n_{\mathbf{u}}e^{c_g t}\\ &\quad + \frac{2\eta_{\mathbf{m}}(0)}{\eta_{\boldsymbol{\omega}}(0)^2}(3c_g + \eta_{\mathbf{m}}(0))^2 n_{\mathbf{u}}L_{\mathbf{l}2}^2 e^{3c_g t}\end{aligned} \tag{83}$$

for all $t \geq 0$. Applying the comparison lemma [15, Lemma 3.4] gives

$$\begin{aligned}V_{\mathbf{Q}}(\tilde{\mathbf{Q}}(t)) &\leq V_{\mathbf{Q}}(\tilde{\mathbf{Q}}(0))e^{4c_g t} + \frac{2}{3\eta_{\mathbf{m}}(0)}c_g n_{\mathbf{u}}e^{c_g t}\\ &\quad + \frac{2\eta_{\mathbf{m}}(0)}{\eta_{\boldsymbol{\omega}}(0)^2 c_g}(3c_g + \eta_{\mathbf{m}}(0))^2 n_{\mathbf{u}}L_{\mathbf{l}2}^2 e^{3c_g t}\end{aligned} \tag{84}$$

for all $t \geq 0$. Without loss of generality, we assume that ε_2, ε_3 and ε_5 in Theorem 7 are sufficiently small such that it follows from (32) and (81) that

$$\dot{V}_{\mathbf{Q}}(\tilde{\mathbf{Q}}) \leq -\frac{\eta_{\mathbf{m}}}{2}V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) + \frac{\eta_{\mathbf{m}}}{256} \tag{85}$$

for all $t \geq t_1$, all $g_\alpha \leq \eta_{\mathbf{m}}\varepsilon_2$, all $|g_{\mathbf{m}} - g_{\boldsymbol{\omega}}| \leq \eta_{\mathbf{m}}\varepsilon_3$ and all $\eta_{\mathbf{m}} \leq \eta_{\boldsymbol{\omega}}\varepsilon_5$. Use of the comparison lemma [15, Lemma 3.4] yields

$$V_{\mathbf{Q}}(\tilde{\mathbf{Q}}(t)) \leq \max\left\{2V_{\mathbf{Q}}(\tilde{\mathbf{Q}}(t_1))e^{-\frac{1}{2}\int_{t_1}^t \eta_{\mathbf{m}}(\tau)d\tau}, \frac{1}{64}\right\} \tag{86}$$

for all $t \geq t_1$. We note that, from (77), it follows that

$$\|\tilde{\mathbf{Q}}\|^2 \leq V_{\mathbf{Q}}(\tilde{\mathbf{Q}}) \leq n_{\mathbf{u}}\|\tilde{\mathbf{Q}}\|^2. \tag{87}$$

The boundedness of the solutions $\tilde{\mathbf{Q}}(t)$ follows from (84) and (87) for $0 \leq t \leq t_1$ and from (86) and (87) for $t \geq t_1$. The bound in (38) of Lemma 9 follows from (86) and (87).

Appendix C
Proof of Lemma 10

From (23), (25), (34) and (35), we obtain that the state equations for $\tilde{m}_1$ and $\tilde{\mathbf{m}}_2$ are given by

$$\begin{aligned}\dot{\tilde{m}}_1 &= -\eta_{\mathbf{m}}\tilde{m}_1 - \frac{\dot{\hat{\mathbf{u}}}^T}{\alpha_{\boldsymbol{\omega}}}\mathbf{m}_2 + \eta_{\mathbf{m}}\left(g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}}\right)k_1\\ &\quad + \frac{\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\left(g_\alpha + g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}}\right)\mathbf{l}_1^T\mathbf{m}_2\\ &\quad - \frac{\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\mathbf{l}_1^T\mathbf{w} + \alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}v + \eta_{\mathbf{m}}z\end{aligned} \tag{88}$$

and

$$\begin{aligned}\dot{\tilde{\mathbf{m}}}_2 &= -g_\alpha\tilde{\mathbf{m}}_2 - \eta_{\mathbf{m}}\mathbf{Q}\boldsymbol{\omega}\tilde{m}_1 - \eta_{\mathbf{m}}\mathbf{Q}\boldsymbol{\omega}\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2\\ &\quad - \eta_{\mathbf{m}}^2\mathbf{Q}\boldsymbol{\omega}k_1 - \frac{\eta_{\mathbf{m}}^2}{\eta_{\boldsymbol{\omega}}}\mathbf{Q}\boldsymbol{\omega}\mathbf{l}_1^T\mathbf{m}_2\\ &\quad + \eta_{\mathbf{m}}\left(g_\alpha + g_{\mathbf{m}} - g_{\boldsymbol{\omega}} - \eta_{\mathbf{m}}\right)\mathbf{Q}\mathbf{k}_2\\ &\quad - \alpha_{\boldsymbol{\omega}}^2\mathbf{w} + \alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}\mathbf{Q}\boldsymbol{\omega}v + \eta_{\mathbf{m}}\mathbf{Q}\boldsymbol{\omega}z + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{b}_{\boldsymbol{\omega}d}.\end{aligned} \tag{89}$$

We introduce the following Lyapunov-function candidate for the $\tilde{m}_1, \tilde{\mathbf{m}}_2$-dynamics:

$$V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) = \tilde{m}_1^2 + \tilde{\mathbf{m}}_2^T\mathbf{Q}^{-1}\tilde{\mathbf{m}}_2. \tag{90}$$

We note that

$$\begin{aligned}\max\left\{|\tilde{m}_1|^2, \lambda_{\min}(\mathbf{Q}^{-1})\|\tilde{\mathbf{m}}_2\|^2\right\} &\leq V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})\\ &\leq \max\left\{2|\tilde{m}_1|^2, 2\lambda_{\max}(\mathbf{Q}^{-1})\|\tilde{\mathbf{m}}_2\|^2\right\},\end{aligned} \tag{91}$$

where $\lambda_{\min}(\mathbf{Q}^{-1})$ and $\lambda_{\max}(\mathbf{Q}^{-1})$ are the smallest and largest eigenvalue of $\mathbf{Q}^{-1}$, respectively. From (25) (see also (72) in the proof of Lemma 9), (88) and (89), it follows that the time derivative of $V_{\mathbf{m}}$ can be written as

$$\begin{aligned}\dot{V}_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}) &= -\eta_{\mathbf{m}}\tilde{m}_1^2 - \eta_{\mathbf{m}}\tilde{\mathbf{m}}_2^T\mathbf{Q}^{-1}\tilde{\mathbf{m}}_2\\ &- \eta_{\mathbf{m}}(\tilde{m}_1+\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2)^2 - \frac{2}{\alpha_{\boldsymbol{\omega}}}\tilde{m}_1\dot{\hat{\mathbf{u}}}^T\mathbf{m}_2\\ &+ 2\eta_{\mathbf{m}}\left(g_{\mathbf{m}}-g_{\boldsymbol{\omega}}\right)\tilde{m}_1k_1 - 2\eta_{\mathbf{m}}^2(\tilde{m}_1+\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2)k_1\\ &+ \frac{2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\left(g_\alpha+g_{\mathbf{m}}-g_{\boldsymbol{\omega}}\right)\tilde{m}_1\mathbf{l}_1^T\mathbf{m}_2\\ &- \frac{2\eta_{\mathbf{m}}^2}{\eta_{\boldsymbol{\omega}}}(\tilde{m}_1+\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2)\mathbf{l}_1^T\mathbf{m}_2\\ &+ 2\eta_{\mathbf{m}}\left(g_\alpha+g_{\mathbf{m}}-g_{\boldsymbol{\omega}}-\eta_{\mathbf{m}}\right)\tilde{\mathbf{m}}_2^T\mathbf{k}_2\\ &- \frac{2\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}\tilde{m}_1\mathbf{l}_1^T\mathbf{w} - 2\alpha_{\boldsymbol{\omega}}^2\tilde{\mathbf{m}}_2^T\mathbf{Q}^{-1}\mathbf{w} + 2\eta_{\mathbf{m}}\tilde{\mathbf{m}}_2^T\mathbf{b}_{\boldsymbol{\omega}d}\\ &+ 2\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}(\tilde{m}_1+\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2)v + 2\eta_{\mathbf{m}}(\tilde{m}_1+\boldsymbol{\omega}^T\tilde{\mathbf{m}}_2)z.\end{aligned} \tag{92}$$

By applying Young's inequality and using (90), we obtain

$$\begin{aligned}\dot{V}_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}) &\le -\frac{\eta_{\mathbf{m}}}{2}V_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q})\\ &+ \frac{8}{\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}}\|\dot{\hat{\mathbf{u}}}\|^2\|\mathbf{m}_2\|^2 + \frac{4\eta_{\mathbf{m}}^3}{\eta_{\boldsymbol{\omega}}^2}\|\mathbf{l}_1\|^2\|\mathbf{m}_2\|^2\\ &+ 8\eta_{\mathbf{m}}|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|^2|k_1|^2 + 4\eta_{\mathbf{m}}^3|k_1|^2\\ &+ \frac{8\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}\left(g_\alpha+|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|\right)^2\|\mathbf{l}_1\|^2\|\mathbf{m}_2\|^2\\ &+ 6\eta_{\mathbf{m}}\left(g_\alpha+|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|+\eta_{\mathbf{m}}\right)^2\|\mathbf{Q}\|\|\mathbf{k}_2\|^2\\ &+ \frac{8\alpha_{\boldsymbol{\omega}}^4\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}\|\mathbf{l}_1\|^2\|\mathbf{w}\|^2 + \frac{6\alpha_{\boldsymbol{\omega}}^4}{\eta_{\mathbf{m}}}\|\mathbf{Q}^{-1}\|\|\mathbf{w}\|^2\\ &+ 4\alpha_{\boldsymbol{\omega}}^4\eta_{\mathbf{m}}|v|^2 + 4\eta_{\mathbf{m}}|z|^2 + 6\eta_{\mathbf{m}}\|\mathbf{Q}\|\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2.\end{aligned} \tag{93}$$

From Assumption 3 and (20), we have

$$\|\mathbf{m}_2\| \le \alpha_{\boldsymbol{\omega}}L_{F2}\|\tilde{\mathbf{u}}\|. \tag{94}$$

From Assumption 6 and (35), it follows that

$$|k_1| \le q_d, \quad \|\mathbf{k}_2\| \le q_{\boldsymbol{\omega}d}. \tag{95}$$

From the definition of $\mathbf{l}_1$ in (35), it follows that there exists a constant $L_{\mathbf{l}1} \in \mathbb{R}_{>0}$ such that

$$\|\mathbf{l}_1\| \le L_{\mathbf{l}1}. \tag{96}$$

From Assumption 3 and (24), we obtain

$$\|\mathbf{w}\| \le \frac{1}{\alpha_{\boldsymbol{\omega}}}L_{F2}\|\dot{\hat{\mathbf{u}}}\|. \tag{97}$$

Similarly, from Assumption 3, (63) in the proof of Lemma 8 and (24), we obtain

$$|v| \le \frac{1}{2}L_{F2}L_{\boldsymbol{\omega}1}^2. \tag{98}$$

Furthermore, to obtain a bound on $|z|$, from (24), we have

$$\begin{aligned}|z| &\le \left|\int_0^1\left(\frac{\partial h}{\partial \mathbf{x}}(\sigma\tilde{\mathbf{x}}+\mathbf{X}(\mathbf{u}),\mathbf{u}) - \frac{\partial h}{\partial \mathbf{x}}(\mathbf{X}(\mathbf{u}),\mathbf{u})\right)d\sigma\tilde{\mathbf{x}}\right|\\ &+ \left|\left(\frac{\partial h}{\partial \mathbf{x}}(\mathbf{X}(\mathbf{u}),\mathbf{u}) - \frac{\partial h}{\partial \mathbf{x}}(\mathbf{X}(\mathbf{u}^*),\mathbf{u}^*)\right)\tilde{\mathbf{x}}\right|\\ &+ \left|\frac{\partial h}{\partial \mathbf{x}}(\mathbf{X}(\mathbf{u}^*),\mathbf{u}^*)\tilde{\mathbf{x}}\right|\end{aligned} \tag{99}$$

From Assumption 4, it follows that

$$\begin{aligned}&\left\|\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}_1,\mathbf{u}_1) - \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}_2,\mathbf{u}_2)\right\|\\ &\quad\le L_{h\mathbf{x}}\|\mathbf{x}_1-\mathbf{x}_2\| + L_{h\mathbf{u}}\|\mathbf{u}_1-\mathbf{u}_2\|\end{aligned} \tag{100}$$

for all $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^{n_{\mathbf{x}}}$ and all $\mathbf{u}_1,\mathbf{u}_2 \in \mathbb{R}^{n_{\mathbf{u}}}$. By applying the bound in (100) to (99), we obtain

$$\begin{aligned}|z| &\le \frac{L_{h\mathbf{x}}}{2}\|\tilde{\mathbf{x}}\|^2 + L_{h\mathbf{x}}\|\mathbf{X}(\mathbf{u})-\mathbf{X}(\mathbf{u}^*)\|\|\tilde{\mathbf{x}}\|\\ &\quad+ L_{h\mathbf{u}}\|\mathbf{u}-\mathbf{u}^*\|\|\tilde{\mathbf{x}}\| + L_{h*}\|\tilde{\mathbf{x}}\|,\end{aligned} \tag{101}$$

with $L_{h*} = \|\frac{\partial h}{\partial \mathbf{x}}(\mathbf{X}(\mathbf{u}^*),\mathbf{u}^*)\|$. Subsequently, from Assumption 1, it follows that

$$|z| \le \frac{L_{h\mathbf{x}}}{2}\|\tilde{\mathbf{x}}\|^2 + (L_{h\mathbf{x}}L_{\mathbf{X}}+L_{h\mathbf{u}})\|\mathbf{u}-\mathbf{u}^*\|\|\tilde{\mathbf{x}}\| + L_{h*}\|\tilde{\mathbf{x}}\|. \tag{102}$$

From (13), (34) and (63) in the proof of Lemma 8, we have

$$\|\mathbf{u}-\mathbf{u}^*\| \le \|\tilde{\mathbf{u}}\| + \alpha_{\boldsymbol{\omega}}L_{\boldsymbol{\omega}1}. \tag{103}$$

By substituting (103) in (102), we obtain the following bound on $|z|$:

$$\begin{aligned}|z| &\le \frac{L_{h\mathbf{x}}}{2}\|\tilde{\mathbf{x}}\|^2 + (L_{h\mathbf{x}}L_{\mathbf{X}}+L_{h\mathbf{u}})\|\tilde{\mathbf{u}}\|\|\tilde{\mathbf{x}}\|\\ &\quad+ \alpha_{\boldsymbol{\omega}}(L_{h\mathbf{x}}L_{\mathbf{X}}+L_{h\mathbf{u}})L_{\boldsymbol{\omega}1}\|\tilde{\mathbf{x}}\| + L_{h*}\|\tilde{\mathbf{x}}\|.\end{aligned} \tag{104}$$

From (27), it follows that $\|\dot{\hat{\mathbf{u}}}\| \le \eta_{\mathbf{u}}$. By substituting $\|\dot{\hat{\mathbf{u}}}\| \le \eta_{\mathbf{u}}$ and the bounds in (94)-(98) and (104) in (93), we obtain

$$\begin{aligned}\dot{V}_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q}) &\le -\frac{\eta_{\mathbf{m}}}{2}V_{\mathbf{m}}(\tilde{m}_1,\tilde{\mathbf{m}}_2,\mathbf{Q})\\ &+ \frac{8\eta_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}L_{F2}^2\|\tilde{\mathbf{u}}\|^2 + \frac{4\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}^3}{\eta_{\boldsymbol{\omega}}^2}L_{\mathbf{l}1}^2L_{F2}^2\|\tilde{\mathbf{u}}\|^2\\ &+ 8\eta_{\mathbf{m}}|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|^2q_d^2 + 4\eta_{\mathbf{m}}^3q_d^2\\ &+ \frac{8\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}\left(g_\alpha+|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|\right)^2L_{\mathbf{l}1}^2L_{F2}^2\|\tilde{\mathbf{u}}\|^2\\ &+ 6\eta_{\mathbf{m}}\left(g_\alpha+|g_{\mathbf{m}}-g_{\boldsymbol{\omega}}|+\eta_{\mathbf{m}}\right)^2\|\mathbf{Q}\|q_{\boldsymbol{\omega}d}^2\\ &+ \frac{8\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}^2}L_{\mathbf{l}1}^2L_{F2}^2\|\dot{\hat{\mathbf{u}}}\|^2 + \frac{6\alpha_{\boldsymbol{\omega}}^2}{\eta_{\mathbf{m}}}\|\mathbf{Q}^{-1}\|L_{F2}^2\|\dot{\hat{\mathbf{u}}}\|^2\\ &+ \alpha_{\boldsymbol{\omega}}^4\eta_{\mathbf{m}}L_{F2}^2L_{\boldsymbol{\omega}1}^4 + 4\eta_{\mathbf{m}}\Big((L_{h\mathbf{x}}L_{\mathbf{X}}+L_{h\mathbf{u}})\|\tilde{\mathbf{u}}\|\|\tilde{\mathbf{x}}\|\\ &+ \alpha_{\boldsymbol{\omega}}(L_{h\mathbf{x}}L_{\mathbf{X}}+L_{h\mathbf{u}})L_{\boldsymbol{\omega}1}\|\tilde{\mathbf{x}}\| + L_{h*}\|\tilde{\mathbf{x}}\| + \frac{L_{h\mathbf{x}}}{2}\|\tilde{\mathbf{x}}\|^2\Big)^2\\ &+ 6\eta_{\mathbf{m}}\|\mathbf{Q}\|\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2.\end{aligned} \tag{105}$$

We note that if the right-hand side of (105) is bounded and $\mathbf{Q}^{-1}$ is positive definite and bounded for all $0 \le t \le t_2$, where $t_2 \ge t_1$ is a finite time, then it follows from (91) and (105) that the solutions $\tilde{m}_1(t)$ and $\tilde{\mathbf{m}}_2(t)$ are bounded for all $0 \le t \le t_2$ using the same arguments as applied in the proofs of Lemmas 8 and 9. From (75) in the proof of Lemma 9, it follows that

$$\begin{aligned}&\lambda_{\min}(\mathbf{Q}^{-1}(0))e^{-\eta_{\mathbf{m}}(0)t} \le \lambda_{\min}(\mathbf{Q}^{-1}(t)),\\ &\lambda_{\max}(\mathbf{Q}^{-1}(t)) \le \lambda_{\max}(\mathbf{Q}^{-1}(0))e^{2c_gt} + \frac{\eta_{\mathbf{m}}(0)}{2c_g}L_{\boldsymbol{\omega}1}^2\end{aligned} \tag{106}$$

for all $t \ge 0$, which implies that $\mathbf{Q}^{-1}$ is positive definite and bounded for all $0 \le t \le t_2$. The boundedness of the right-hand side of (105) for all $0 \leq t \leq t_2$ follows from the bounds on g_α, $g_{\boldsymbol{\omega}}$ and $g_{\mathbf{m}}$ in (30) of Theorem 7, from the lower bound on $\eta_{\boldsymbol{\omega}}$ and $\eta_{\mathbf{u}}$ in (82) in the proof of Lemma 9 and the fact that $\alpha_{\boldsymbol{\omega}}$, $\eta_{\mathbf{m}}$ and $\eta_{\mathbf{u}}$ are nonincreasing (see (14) and (28)), from the boundedness of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{u}}$ for $0 \leq t \leq t_2$ in Lemmas 8 and 11, respectively, from $\|\dot{\hat{\mathbf{u}}}\| \leq \eta_{\mathbf{u}}$ (see (27)) and from the bounds in (106), which imply that $\|\mathbf{Q}^{-1}(t)\| = \lambda_{\max}(\mathbf{Q}^{-1}(t))$ and $\|\mathbf{Q}(t)\| = \frac{1}{\lambda_{\min}(\mathbf{Q}^{-1}(t))}$ are bounded for all $0 \leq t \leq t_2$. Further details regarding the boundedness of the solutions $\tilde{m}_1(t)$ and $\tilde{\mathbf{m}}_2(t)$ for $0 \leq t \leq t_2$ are left to the reader.

Let us define $t_2 \geq t_1$ such that

$$\|\tilde{\mathbf{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)c_{\mathbf{x}2}, \quad \|\tilde{\mathbf{Q}}(t)\| \leq \frac{1}{8} \tag{107}$$

for all $t \geq t_2$. The existence of a finite time $t_2 \geq t_1$ such that $\|\tilde{\mathbf{x}}(t)\| \leq \alpha_{\boldsymbol{\omega}}(t)\eta_{\boldsymbol{\omega}}(t)c_{\mathbf{x}2}$ for all $t \geq t_2$ follows from Lemma 8, where we assume without loss of generality that ε_1 in Theorem 7 is sufficiently small such that $g_\alpha(t)+g_{\boldsymbol{\omega}}(t) < \beta_{\mathbf{x}}$ for all $t \geq t_1$ and all $g_\alpha + g_{\boldsymbol{\omega}} \leq \varepsilon_1$. Similarly, the existence of a constant $t_2 \geq t_1$ such that $\|\tilde{\mathbf{Q}}(t)\| \leq \frac{1}{8}$ for all $t \geq t_2$ follows from Lemma 9 if $\int_{t_1}^{\infty} \eta_{\mathbf{m}}(t)dt = \infty$, which implies that first term in the right-hand side of (38) becomes smaller than $\frac{1}{8}$ as time goes to infinity. From (26) and the first equation in (29) of Theorem 7, we have

$$\begin{aligned}\int_{t_1}^{\infty} \eta_{\mathbf{m}}(t)dt &= \eta_{\mathbf{m}}(0)\int_{t_1}^{\infty} e^{-\int_0^t g_{\mathbf{m}}(\tau)d\tau}dt\\ &= \eta_{\mathbf{m}}(0)\left(\underbrace{\int_0^{\infty} e^{-\int_0^t g_{\mathbf{m}}(\tau)d\tau}dt}_{=\infty} - \underbrace{\int_0^{t_1} e^{-\int_0^t g_{\mathbf{m}}(\tau)d\tau}dt}_{\leq t_1}\right)\\ &= \infty\end{aligned} \tag{108}$$

for all $\eta_{\mathbf{m}}(0) \in \mathbb{R}_{>0}$. Hence, there exist a time $t_2 \geq t_1$ such that (107) holds for all $t \geq t_2$.

Now, from (34) and (80) in the proof of Lemma 9, it follows that

$$\left\|\mathbf{Q}^{-1} - \frac{1}{2}\mathbf{I}\right\| \leq \|\tilde{\mathbf{Q}}\| + \frac{\eta_{\mathbf{m}}}{\eta_{\boldsymbol{\omega}}}L_{12}. \tag{109}$$

Without loss of generality, we assume that ε_5 in Theorem 7 is sufficiently small such that it follows from (32), Lemma 9 and (107) that

$$\frac{1}{4}\mathbf{I} \preceq \mathbf{Q}^{-1} \preceq \frac{3}{4}\mathbf{I} \tag{110}$$

for all $t \geq t_2$ and all $\eta_{\mathbf{m}} \leq \eta_{\boldsymbol{\omega}}\varepsilon_5$. Subsequently, from (91) and (110), we obtain

$$\begin{aligned}\max\left\{|\tilde{m}_1|^2, \frac{1}{4}\|\tilde{\mathbf{m}}_2\|^2\right\} &\leq V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})\\ &\leq \max\left\{2|\tilde{m}_1|^2, \frac{3}{2}\|\tilde{\mathbf{m}}_2\|^2\right\}\end{aligned} \tag{111}$$

for $t \geq t_2$. Moreover, from (110), it follows that

$$\|\mathbf{Q}^{-1}\| \leq \frac{3}{4}, \quad \|\mathbf{Q}\| \leq 4 \tag{112}$$

for all $t \geq t_2$. From (27) and (34), we have that

$$\|\dot{\hat{\mathbf{u}}}\| \leq \lambda_{\mathbf{u}}\|\hat{\mathbf{m}}_2\| \leq \lambda_{\mathbf{u}}\left(\|\tilde{\mathbf{m}}_2\| + \|\mathbf{m}_2\| + \eta_{\mathbf{m}}\|\mathbf{Q}\|\|\mathbf{k}_2\|\right). \tag{113}$$

Subsequently, from (94), (95) and (112), we obtain

$$\|\dot{\hat{\mathbf{u}}}\| \leq \lambda_{\mathbf{u}}\left(\|\tilde{\mathbf{m}}_2\| + \alpha_{\boldsymbol{\omega}}L_{F2}\|\tilde{\mathbf{u}}\| + 4\eta_{\mathbf{m}}q_{\boldsymbol{\omega}d}\right). \tag{114}$$

for all $t \geq t_2$. From (111) and (114), it follows that

$$\begin{aligned}\|\dot{\hat{\mathbf{u}}}\|^2 \leq\ & 12\lambda_{\mathbf{u}}^2 V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})\\ &+ 3\alpha_{\boldsymbol{\omega}}^2\lambda_{\mathbf{u}}^2 L_{F2}^2\|\tilde{\mathbf{u}}\|^2 + 48\eta_{\mathbf{m}}^2\lambda_{\mathbf{u}}^2 q_{\boldsymbol{\omega}d}\end{aligned} \tag{115}$$

for all $t \geq t_2$. Without loss of generality, we assume that ε_2, ε_3, ε_4, ε_5 and ε_7 in Theorem 7 are sufficiently small such that we obtain from (32), (105), (107), (112) and (115) that

$$\begin{aligned}\dot{V}_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) \leq\ & -\frac{\eta_{\mathbf{m}}}{4}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})\\ &+ 2\alpha_{\boldsymbol{\omega}}^4\eta_{\mathbf{m}}L_{F2}^2L_{\boldsymbol{\omega}1}^4 + 16\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}c_{\mathbf{x}2}^2L_{h*}^2\\ &+ 16\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}c_{\mathbf{x}2}^2(L_{h\mathbf{x}}L_{\mathbf{X}} + L_{h\mathbf{u}})^2\|\tilde{\mathbf{u}}\|^2\\ &+ \frac{8\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}^3}{\eta_{\boldsymbol{\omega}}^2}L_{11}^2L_{F2}^2\|\tilde{\mathbf{u}}\|^2 + \frac{27\alpha_{\boldsymbol{\omega}}^4\lambda_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}L_{F2}^4\|\tilde{\mathbf{u}}\|^2\\ &+ \frac{8\eta_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}L_{F2}^2\|\tilde{\mathbf{u}}\|^2 + 8\eta_{\mathbf{m}}^3q_d^2 + 96\eta_{\mathbf{m}}^3q_{\boldsymbol{\omega}d}^2 + 24\eta_{\mathbf{m}}\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2\end{aligned} \tag{116}$$

for all $t \geq t_2$, all $g_\alpha \leq \eta_{\mathbf{m}}\varepsilon_2$, all $|g_{\mathbf{m}} - g_{\boldsymbol{\omega}}| \leq \eta_{\mathbf{m}}\varepsilon_3$, all $\eta_{\boldsymbol{\omega}} \leq \varepsilon_4$, $\eta_{\mathbf{m}} \leq \eta_{\boldsymbol{\omega}}\varepsilon_5$ and all $\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}} \leq \eta_{\mathbf{m}}\varepsilon_7$. From this, it follows that

$$\dot{V}_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) \leq -\frac{\eta_{\mathbf{m}}}{8}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) \tag{117}$$

whenever

$$\begin{aligned}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) \geq 72\max\Big\{& 2\alpha_{\boldsymbol{\omega}}^4\eta_{\mathbf{m}}L_{F2}^2L_{\boldsymbol{\omega}1}^4,\\ & 16\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}c_{\mathbf{x}2}^2L_{h*}^2, 16\alpha_{\boldsymbol{\omega}}^2\eta_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}c_{\mathbf{x}2}^2(L_{h\mathbf{x}}L_{\mathbf{X}} + L_{h\mathbf{u}})^2\|\tilde{\mathbf{u}}\|^2,\\ & \frac{8\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}^3}{\eta_{\boldsymbol{\omega}}^2}L_{11}^2L_{F2}^2\|\tilde{\mathbf{u}}\|^2, \frac{27\alpha_{\boldsymbol{\omega}}^4\lambda_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}L_{F2}^4\|\tilde{\mathbf{u}}\|^2,\\ & \frac{8\eta_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}L_{F2}^2\|\tilde{\mathbf{u}}\|^2, 8\eta_{\mathbf{m}}^3q_d^2, 96\eta_{\mathbf{m}}^3q_{\boldsymbol{\omega}d}^2, 24\eta_{\mathbf{m}}\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2\Big\}\end{aligned} \tag{118}$$

for all $t \geq t_2$. The bounds in (39), (40) and (41) of Lemma 10 follow from (111), (117) and (118), respectively.

Appendix D
Proof of Lemma 11

From (20), (27) and (34), we obtain that the state equation for $\tilde{\mathbf{u}}$ is given by

$$\dot{\tilde{\mathbf{u}}} = -\lambda_{\mathbf{u}}\frac{\eta_{\mathbf{u}}\left(\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right)}{\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|}. \tag{119}$$

From (119), it follows that $\|\dot{\tilde{\mathbf{u}}}\| \leq \eta_{\mathbf{u}}$, from which we obtain that

$$\|\tilde{\mathbf{u}}(t)\| \leq \|\tilde{\mathbf{u}}(0)\| + \eta_{\mathbf{u}}(0)t \tag{120}$$

for all $t \geq 0$. We define the following Lyapunov-function candidate for the $\tilde{\mathbf{u}}$-dynamics:

$$V_{\mathbf{u}}(\tilde{\mathbf{u}}) = \|\tilde{\mathbf{u}}\|^2. \tag{121}$$

From (119) and (121), it follows that the time derivative of $V_{\mathbf{u}}$ is given by

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) = -2\lambda_{\mathbf{u}}\frac{\eta_{\mathbf{u}}\left(\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}}(\hat{\mathbf{u}})\tilde{\mathbf{u}} + \tilde{\mathbf{u}}^T\tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\tilde{\mathbf{u}}^T\mathbf{Q}\mathbf{k}_2\right)}{\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|}. \tag{122}$$

From Assumption 3, we subsequently obtain that

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) \leq -\frac{2\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\eta_{\mathbf{u}}L_{F1}\|\tilde{\mathbf{u}}\|^2}{\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|} + \frac{2\lambda_{\mathbf{u}}\eta_{\mathbf{u}}\|\tilde{\mathbf{u}}\|\left(\|\tilde{\mathbf{m}}_2\| + \eta_{\mathbf{m}}\|\mathbf{Q}\|\|\mathbf{k}_2\|\right)}{\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|}. \tag{123}$$

By applying Young's inequality, it follows that

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) \leq -\frac{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\eta_{\mathbf{u}}L_{F1}\|\tilde{\mathbf{u}}\|^2}{\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|} + \frac{4\lambda_{\mathbf{u}}\eta_{\mathbf{u}}\max\left\{\|\tilde{\mathbf{m}}_2\|^2, \eta_{\mathbf{m}}^2\|\mathbf{Q}\|^2\|\mathbf{k}_2\|^2\right\}}{\alpha_{\boldsymbol{\omega}}L_{F1}\left(\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|\right)}. \tag{124}$$

If $V_{\mathbf{u}}(\tilde{\mathbf{u}}) \geq \frac{8}{\alpha_{\boldsymbol{\omega}}^2 L_{F1}^2}\max\left\{\|\tilde{\mathbf{m}}_2\|^2, \eta_{\mathbf{m}}^2\|\mathbf{Q}\|^2\|\mathbf{k}_2\|^2\right\}$, then from (121) and (124), it follows that

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) \leq -\frac{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\eta_{\mathbf{u}}L_{F1}\|\tilde{\mathbf{u}}\|^2}{2\left(\eta_{\mathbf{u}} + \lambda_{\mathbf{u}}\left\|\alpha_{\boldsymbol{\omega}}\frac{dF}{d\mathbf{u}^T}(\hat{\mathbf{u}}) + \tilde{\mathbf{m}}_2 + \eta_{\mathbf{m}}\mathbf{Q}\mathbf{k}_2\right\|\right)}. \tag{125}$$

From Assumption 3, (121) and (125), we obtain that

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) \leq -\frac{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\eta_{\mathbf{u}}L_{F1}V_{\mathbf{u}}(\tilde{\mathbf{u}})}{2\left(\eta_{\mathbf{u}} + \alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\left(L_{F2} + \frac{L_{F1}}{\sqrt{2}}\right)\sqrt{V_{\mathbf{u}}(\tilde{\mathbf{u}})}\right)}, \tag{126}$$

whenever $V_{\mathbf{u}}(\tilde{\mathbf{u}}) \geq \frac{8}{\alpha_{\boldsymbol{\omega}}^2 L_{F1}^2}\max\left\{\|\tilde{\mathbf{m}}_2\|^2, \eta_{\mathbf{m}}^2\|\mathbf{Q}\|^2\|\mathbf{k}_2\|^2\right\}$. From Assumption 6, from (95) and (112) in the proof of Lemma 10 and from (126), it follows that, for all $t \geq t_2$,

$$\dot{V}_{\mathbf{u}}(\tilde{\mathbf{u}}) \leq -\frac{1}{4}\min\left\{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}L_{F1}V_{\mathbf{u}}(\tilde{\mathbf{u}}), \eta_{\mathbf{u}}\frac{\sqrt{2}L_{F1}}{\sqrt{2}L_{F2} + L_{F1}}\sqrt{V_{\mathbf{u}}(\tilde{\mathbf{u}})}\right\}, \tag{127}$$

whenever $V_{\mathbf{u}}(\tilde{\mathbf{u}}) \geq \frac{8}{\alpha_{\boldsymbol{\omega}}^2 L_{F1}^2}\max\left\{\|\tilde{\mathbf{m}}_2\|^2, 16\eta_{\mathbf{m}}^2 q_{\boldsymbol{\omega}d}^2\right\}$. The boundedness of the solutions $\tilde{\mathbf{u}}(t)$ for all $0 \leq t \leq t_2$ follows from (120). The bounds in (42), (43) and (42) of Lemma 11 follow from (121), (127) and $V_{\mathbf{u}}(\tilde{\mathbf{u}}) \geq \frac{8}{\alpha_{\boldsymbol{\omega}}^2 L_{F1}^2}\max\left\{\|\tilde{\mathbf{m}}_2\|^2, 16\eta_{\mathbf{m}}^2 q_{\boldsymbol{\omega}d}^2\right\}$, respectively.

Appendix E
Proof of Lemma 12

For notational convenience, we introduce the shorthand notation $W(t) = V(\tilde{m}_1(t), \tilde{\mathbf{m}}_2(t), \tilde{\mathbf{u}}(t), \mathbf{Q}(t), \alpha_{\boldsymbol{\omega}}(t))$. We note that the function V in (45) is not continuously differentiable with respect to time due to the use of the maximum function. Let the upper right-hand time derivative of V (see for example [15]) be denoted by $D^+W(t)$ using the shorthand notation above. Let us consider the following three cases, similar to [13].

Case 1: $V_{\mathbf{u}}(\tilde{\mathbf{u}}) > \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$. We note that $W = V_{\mathbf{u}}(\tilde{\mathbf{u}})$ for Case 1. Therefore, we obtain from Lemma 11 that, for all $t \geq t_2$,

$$D^+W \leq -\min\left\{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\gamma_{\mathbf{u}3}W, \eta_{\mathbf{u}}\gamma_{\mathbf{u}4}\sqrt{W}\right\} \tag{128}$$

whenever

$$W \geq \max\left\{\frac{1}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{u}1}\|\tilde{\mathbf{m}}_2\|^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{u}2}q_{\boldsymbol{\omega}d}^2\right\}. \tag{129}$$

It follows from Lemma 10 that

$$\frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q}) \geq \frac{1}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{u}1}\|\tilde{\mathbf{m}}_2\|^2. \tag{130}$$

Because $W > \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$ for Case 1, we conclude from (129) and (130) that, for all $t \geq t_2$, (128) holds whenever

$$W \geq \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{u}2}q_{\boldsymbol{\omega}d}^2. \tag{131}$$

Case 2: $V_{\mathbf{u}}(\tilde{\mathbf{u}}) < \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$. We note that $W = \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$ for Case 2. Therefore, it follows from (14) and Lemma 10 that, for all $t \geq t_2$,

$$D^+W \leq -(\eta_{\mathbf{m}}\gamma_{\mathbf{m}5} - 2g_\alpha)W \tag{132}$$

whenever

$$W \geq \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}\max\left\{\alpha_{\boldsymbol{\omega}}^2c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2c_{\mathbf{m}2}, \eta_{\boldsymbol{\omega}}^2c_{\mathbf{m}3}\|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{m}}^2}{\eta_{\boldsymbol{\omega}}^2}c_{\mathbf{m}4}\|\tilde{\mathbf{u}}\|^2, \frac{\alpha_{\boldsymbol{\omega}}^2\lambda_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}c_{\mathbf{m}5}\|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{u}}^2}{\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}^2}c_{\mathbf{m}6}\|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{m}7}q_d^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{m}8}q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2}c_{\mathbf{m}9}\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2\right\}. \tag{133}$$

Without loss of generality, we assume that ε_2 in Theorem 7 is sufficiently small such that we obtain from (32) and (132) that

$$D^+W \leq -\frac{\eta_{\mathbf{m}}}{2}\gamma_{\mathbf{m}5}W \tag{134}$$

for all $g_\alpha \leq \eta_{\mathbf{m}}\varepsilon_2$. Moreover, without loss of generality, we assume that ε_4, ε_5, ε_6 and ε_7 in Theorem 7 are sufficiently small such it follows from (32) and Lemma 11 that

$$V_{\mathbf{u}}(\tilde{\mathbf{u}}) \geq \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}\max\left\{\eta_{\boldsymbol{\omega}}^2c_{\mathbf{m}3}\|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{m}}^2}{\eta_{\boldsymbol{\omega}}^2}c_{\mathbf{m}4}\|\tilde{\mathbf{u}}\|^2, \frac{\alpha_{\boldsymbol{\omega}}^2\lambda_{\mathbf{u}}^2}{\eta_{\mathbf{m}}}c_{\mathbf{m}5}\|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{u}}^2}{\alpha_{\boldsymbol{\omega}}^2\eta_{\mathbf{m}}^2}c_{\mathbf{m}6}\|\tilde{\mathbf{u}}\|^2\right\} \tag{135}$$

for all $\eta_{\boldsymbol{\omega}} \leq \varepsilon_4$, $\eta_{\mathbf{m}} \leq \eta_{\boldsymbol{\omega}}\varepsilon_5$, $\eta_{\mathbf{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\mathbf{m}}\varepsilon_6$ and $\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}} \leq \eta_{\mathbf{m}}\varepsilon_7$. Because $W > V_{\mathbf{u}}(\tilde{\mathbf{u}})$ for Case 2, we conclude from (133) and (135) that, for all $t \geq t_2$, (134) holds whenever

$$W \geq \max\left\{\alpha_{\boldsymbol{\omega}}^2\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}c_{\mathbf{m}2}, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}c_{\mathbf{m}7}q_d^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}c_{\mathbf{m}8}q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}c_{\mathbf{m}9}\|\mathbf{b}_{\boldsymbol{\omega}d}\|^2\right\}. \tag{136}$$

Case 3: $V_{\mathbf{u}}(\tilde{\mathbf{u}}) = \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$. We note that $W = V_{\mathbf{u}}(\tilde{\mathbf{u}}) = \frac{1}{\alpha_{\boldsymbol{\omega}}^2}\frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}}V_{\mathbf{m}}(\tilde{m}_1, \tilde{\mathbf{m}}_2, \mathbf{Q})$ for Case 3. Therefore, we obtain from (14) and Lemmas 10 and 11 that, for all $t \geq t_2$,

$$D^+W \leq -\min\left\{\alpha_{\boldsymbol{\omega}}\lambda_{\mathbf{u}}\gamma_{\mathbf{u}3}W, \eta_{\mathbf{u}}\gamma_{\mathbf{u}4}\sqrt{W}, (\eta_{\mathbf{m}}\gamma_{\mathbf{m}5} - 2g_\alpha)W\right\} \tag{137}$$

whenever

$$W \geq \max\Bigg\{ \frac{1}{\alpha_{\boldsymbol{\omega}}^2} c_{\mathbf{u}1} \|\tilde{\mathbf{m}}_2\|^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2 \eta_{\boldsymbol{\omega}}^2} c_{\mathbf{u}2} q_{\boldsymbol{\omega}d}^2, \alpha_{\boldsymbol{\omega}}^2 \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2 \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}2}, \eta_{\boldsymbol{\omega}}^2 \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}3} \|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{m}}^2}{\eta_{\boldsymbol{\omega}}^2} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}4} \|\tilde{\mathbf{u}}\|^2, \frac{\alpha_{\boldsymbol{\omega}}^2 \lambda_{\mathbf{u}}^2}{\eta_{\mathbf{m}}} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}5} \|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{u}}^2}{\alpha_{\boldsymbol{\omega}}^2 \eta_{\mathbf{m}}^2} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}6} \|\tilde{\mathbf{u}}\|^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}7} q_d^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}8} q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2} c_{\mathbf{m}9} \|\mathbf{b}_{\boldsymbol{\omega}d}\|^2 \Bigg\}. \tag{138}$$

By following the same steps as for Case 1 and Case 2, we obtain from (137) and (138) that, for all $t \geq t_2$,

$$D^+ W \leq -\min\Big\{ \alpha_{\boldsymbol{\omega}} \lambda_{\mathbf{u}} \gamma_{\mathbf{u}3} W, \eta_{\mathbf{u}} \gamma_{\mathbf{u}4} \sqrt{W}, \frac{\eta_{\mathbf{m}}}{2} \gamma_{\mathbf{m}5} W \Big\} \tag{139}$$

whenever

$$W \geq \max\Bigg\{ \alpha_{\boldsymbol{\omega}}^2 \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2 \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}2}, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}7} q_d^2, \frac{\eta_{\mathbf{m}}^2}{\alpha_{\boldsymbol{\omega}}^2} \max\left\{ c_{\mathbf{u}2}, \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}8} \right\} q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2} c_{\mathbf{m}9} \|\mathbf{b}_{\boldsymbol{\omega}d}\|^2 \Bigg\}. \tag{140}$$

We note that both (128) and (134) imply that (139) holds. Moreover, the inequalities in (131) and (136) are satisfied if (140) is satisfied. Hence, for all three cases and for all $t \geq t_2$, we have that the (139) holds if the inequality in (140) is satisfied. From (32) in Theorem 7 and (139), we obtain that, for all $t \geq t_2$ (with $t_2 \geq t_1$),

$$D^+ W \leq -\min\left\{ \alpha_{\boldsymbol{\omega}} \lambda_{\mathbf{u}}, \eta_{\mathbf{u}} \right\} \beta_V \min\left\{ W, \sqrt{W} \right\} \tag{141}$$

for all $\alpha_{\boldsymbol{\omega}} \lambda_{\mathbf{u}} \leq \eta_{\mathbf{m}} \varepsilon_7$ whenever (140) holds, with $\beta_V = \min\left\{ \gamma_{\mathbf{u}3}, \gamma_{\mathbf{u}4}, \frac{\gamma_{\mathbf{m}5}}{2\varepsilon_7} \right\}$. By applying the same reasoning as for (108) in the proof of Lemma 10, it follows from the second equation in (29) that

$$\int_{t_2}^{\infty} \min\left\{ \alpha_{\boldsymbol{\omega}}(\tau) \lambda_{\mathbf{u}}(\tau), \eta_{\mathbf{u}}(\tau) \right\} d\tau = \infty. \tag{142}$$

Now, from (141), (142) and the comparison lemma [15, Lemma 3.4], we obtain that the solutions $W(t)$ monotonically converge to zero as time goes to infinity for any initial condition $W(t_2) \geq 0$ if the right-hand side of (140) is zero. By using similar arguments as in the proof of [15, Theorem 4.18], we obtain from (140), (141) and (142) that

$$\sup_{t \geq t_2} W(t) \leq \sup_{t \geq t_2} \max\Bigg\{ W(t_2), \alpha_{\boldsymbol{\omega}}^2(t) \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2(t) \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}2}, \frac{\eta_{\mathbf{m}}^2(t)}{\alpha_{\boldsymbol{\omega}}^2(t)} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}7} q_d^2, \frac{\eta_{\mathbf{m}}^2(t)}{\alpha_{\boldsymbol{\omega}}^2(t)} \max\left\{ c_{\mathbf{u}2}, \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}8} \right\} q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2(t)} c_{\mathbf{m}9} \|\mathbf{b}_{\boldsymbol{\omega}d}\|^2 \Bigg\} \tag{143}$$

and

$$\limsup_{t \to \infty} W(t) \leq \limsup_{t \to \infty} \max\Bigg\{ \alpha_{\boldsymbol{\omega}}^2(t) \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}1}, \eta_{\boldsymbol{\omega}}^2(t) \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}2}, \frac{\eta_{\mathbf{m}}^2(t)}{\alpha_{\boldsymbol{\omega}}^2(t)} \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}7} q_d^2, \frac{\eta_{\mathbf{m}}^2(t)}{\alpha_{\boldsymbol{\omega}}^2(t)} \max\left\{ c_{\mathbf{u}2}, \frac{c_{\mathbf{u}1}}{\gamma_{\mathbf{m}2}} c_{\mathbf{m}8} \right\} q_{\boldsymbol{\omega}d}^2, \frac{1}{\alpha_{\boldsymbol{\omega}}^2(t)} c_{\mathbf{m}9} \|\mathbf{b}_{\boldsymbol{\omega}d}\|^2 \Bigg\}, \tag{144}$$

where we applied [27, Lemma II.1] to obtain the limit superior in the right-hand side of (144). Because $\alpha_{\boldsymbol{\omega}}$ and $\eta_{\boldsymbol{\omega}}$ are nonincreasing (see (14)), it follows that the second and third term in the right-hand side of (143) are bounded. Moreover, from (31) in Theorem 7, we have that the fourth, fifth and sixth term in the right-hand side of (143) are bounded. Hence, we obtain from (143) that the solutions $W(t)$ are bounded for all $t \geq t_2$. From Lemmas 10 and 11 and from the definition of V in (45), we have that

$$\max\left\{ \frac{c_{\mathbf{u}1}}{\alpha_{\boldsymbol{\omega}}^2} \frac{\gamma_{\mathbf{m}1}}{\gamma_{\mathbf{m}_2}} |\tilde{m}_1|^2, \frac{c_{\mathbf{u}1}}{\alpha_{\boldsymbol{\omega}}^2} \|\tilde{\mathbf{m}}_2\|^2, \gamma_{\mathbf{u}1} \|\tilde{\mathbf{u}}\|^2 \right\} \leq W \leq \max\left\{ \frac{c_{\mathbf{u}1}}{\alpha_{\boldsymbol{\omega}}^2} \frac{\gamma_{\mathbf{m}3}}{\gamma_{\mathbf{m}_2}} |\tilde{m}_1|^2, \frac{c_{\mathbf{u}1}}{\alpha_{\boldsymbol{\omega}}^2} \frac{\gamma_{\mathbf{m}4}}{\gamma_{\mathbf{m}_2}} \|\tilde{\mathbf{m}}_2\|^2, \gamma_{\mathbf{u}2} \|\tilde{\mathbf{u}}\|^2 \right\} \tag{145}$$

for $t \geq t_2$, where we used the shorthand notation $W = V(\tilde{m}_1, \tilde{\mathbf{m}}_2, \tilde{\mathbf{u}}, \mathbf{Q}, \alpha_{\boldsymbol{\omega}})$. From (143) and (145), it follows that the solutions $\tilde{m}_2(t)$, $\tilde{\mathbf{m}}_2(t)$ and $\tilde{\mathbf{u}}(t)$ are bounded for all $t \geq t_2$, all $\tilde{m}_1(t_2) \in \mathbb{R}$, $\tilde{\mathbf{m}}_2(t_2) \in \mathbb{R}^{n_{\mathbf{u}}}$ and all $\tilde{\mathbf{u}}(t_2) \in \mathbb{R}^{n_{\mathbf{u}}}$. The bound in (46) of Lemma 12 follows from (144) and (145).

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