Lyapunov Sufficient Conditions for Uniform Semiglobal Exponential Stability

Kristin Y. Pettersen [a]

[a] *Centre for Autonomous Marine Operations and Systems (NTNU AMOS), Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO7491 Trondheim, Norway*

Abstract

This paper derives Lyapunov sufficient conditions for uniform semiglobal exponential stability (USGES) of parameterized nonlinear time-varying systems. It furthermore investigates the robustness properties that USGES may provide with respect to perturbations.

Key words: Stability analysis; Uniform Semiglobal Exponential Stability; Robustness

1 Introduction

Semiglobal stability of a system may arise from inherent system properties, like high-order nonlinearities, or it may be a result of the control system; the chosen control law or actuator saturation.

An example of physical systems that have the USGES stability property is marine vehicles controlled by the well-known line-of-sight (LOS) or integral line-of-sight (iLOS) guidance laws. It has been shown in Fossen and Pettersen (2014) that the structure of the LOS guidance law prevents the system from having global exponential convergence. In particular, the guidance law can be viewed as a saturated P controller, and this makes the corresponding control gain, and thus the convergence rate, decrease as the norm of the state increases. Therefore, although the system can be shown to have local exponential convergence, the exponential convergence property is not global.

The USGES stability property is also addressed in Orrante-Sakanassi et al. (2015), where a novel tuning procedure is introduced to ensure semi-global exponential stability for the classical PID control of rigid robots. A main motivation for achieving semi-global exponential stability as opposed to previous works reporting semi-global asymptotic stability for rigid robots control systems, is to simultaneously achieve both stability and good performance.

While there exists rigorous theory giving Lyapunov sufficient conditions for uniform semiglobal asymptotic stability (USGAS), given by Teel et al. (1999), Nešić and Loria (2004), and Chaillet and Loria (2008), to the author's best knowledge Lyapunov sufficient conditions for USGES have so far only been considered in Grøtli et al. (2008). In this paper, Lyapunov sufficient conditions are derived, providing a complete proof, and also showing that one of the conditions in Grøtli et al. (2008) can be omitted. In particular, we present a definition of USGES for parameterized nonlinear time-varying systems, and we derive corresponding Lyapunov sufficient conditions for USGES. This provides a Lyapunov analysis tool that can be utilized as part of cascaded systems analysis and control design of USGES systems based on Loria and Panteley (2004).

In addition to guaranteeing stronger convergence properties than asymptotic stability, exponential stability properties are considered beneficial because of the robustness properties they may guarantee. In particular, uniform global exponential stability (UGES), together with an additional condition on the Lyapunov function, guarantees that for all uniformly bounded disturbances, irrespective of magnitude, the solution of the perturbed system will be uniformly bounded (Khalil, 2002, Lemma 9.2). It is therefore interesting to investigate which robustness properties that the stability property USGES can provide.

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Email address: Kristin.Y.Pettersen@itk.ntnu.no (Kristin Y. Pettersen).

The paper is organized as follows. In Section 2 USGES is defined and Lyapunov sufficient conditions for USGES are derived. The robustness properties that USGES may provide with respect to nonvanishing perturbations are discussed in Section 3, while robustness properties with respect to vanishing perturbations are given in Section 4. In Section 5 conclusions are drawn.

2 Uniform Semiglobal Exponential Stability

In this section we present the definition and derive Lyapunov sufficient conditions for USGES.

2.1 Definition of USGES

We consider the parameterized nonlinear time-varying system

$$\dot{x} = f(t, x, \theta) \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$, $x \in \mathbb{R}^n$, $\theta \in \Theta \subset \mathbb{R}^m$ is a constant parameter, and $f(t, x, \theta)$ is locally Lipschitz in x and piecewise continuous in t for all $\theta \in \Theta$. The origin $x = 0$ is an equilibrium point of (1).

A definition of USGES for nonlinear time-varying systems has been given in Loria and Panteley (2004) Def. 2.7. In order to explicitly show the impact that system parameters may have on the USGES property, we will instead use the following definition. This explicitly shows the parameter dependency that USGES may involve and is thus in line with the definition of uniform semiglobal asymptotic stability (USGAS) in Chaillet and Loria (2008) (Def. 8 with $\delta = 0$). This definition of USGES was presented in Grøtli et al. (2008). In the following, B_Δ is defined as the closed ball $\{x \in \mathbb{R}^n : \|x\| \leq \Delta\}$.

Definition 1 (USGES) *Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. System (1) is USGES on Θ if, for any $\Delta > 0$ there exists a parameter $\theta^*(\Delta) \in \Theta$ and positive constants k_Δ, λ_Δ, all independent on t_0, such that $\forall x_0 \in B_\Delta$*

$$\left\|x(t; t_0, x_0, \theta^*)\right\| \leq k_\Delta \|x_0\| \, e^{-\lambda_\Delta (t-t_0)} \quad \forall t \geq t_0 \geq 0 \tag{2}$$

Remark 2 *In other words, the system (1) is USGES if we can choose a parameter value θ^* and find the overshoot and convergence parameters k_Δ and λ_Δ, such that the region of attraction in which the system has exponential convergence, B_Δ, can be made arbitrarily large. If $\Delta \longrightarrow \infty$ then the definition becomes the definition of uniform global exponential stability (Khalil, 2002, Def. 4.5), (Loria and Panteley, 2004, Def. 2.7).*

Remark 3 *Definition 1 is a special case of Def. 2.7 of USGES in Loria and Panteley (2004) in which we explicitly show the parameter dependency that may be part of the USGES property. Systems satisfying Definition 1 of USGES therefore also satisfy Loria and Panteley (2004) Def. 2.7 of USGES. The cascaded systems theory result for USGES systems in Loria and Panteley (2004) Proposition 2.3 may thus be applied, and the Lyapunov sufficient conditions derived in the next section can be utilized in such a cascaded systems analysis.*

Remark 4 *For the definition of USGAS there exist several different versions in the literature, see Sepulchre et al. (1997), Teel et al. (1999), Nešić and Loria (2004), Tan et al. (2006), and Chaillet and Loria (2008). Similarly, we could have given a stronger definition of USGES by requiring that the overshoot and convergence parameters k and λ should be uniform in Δ, i.e. should not be allowed to depend on the size of the region of attraction. Since both overshoot and convergence in practice typically depend on the tuning of the system, we have chosen the more relaxed definition allowing a dependence on Δ, which is in line with the definition of USGAS in Chaillet and Loria (2008). In particular, since the overshoot and convergence parameters in practice often will depend on the tuning parameters of the system, we then have $k(\theta^*(\Delta))$ and $\lambda(\theta^*(\Delta))$ such that the overshoot and convergence parameters naturally depend on Δ.*

2.2 Lyapunov sufficient conditions for USGES

The following theorem gives Lyapunov sufficient conditions for USGES.

Theorem 5 *Consider the system given in (1). If for any $\Delta > 0$ there exist a parameter $\theta^*(\Delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\Delta : \mathbb{R}_{\geq 0} \times B_\Delta \longrightarrow \mathbb{R}_{\geq 0}$, and positive constants $k_{1_\Delta}, k_{2_\Delta}, k_{3_\Delta}, a$, such that $\forall x \in B_\Delta$*

$$k_{1_\Delta} \|x\|^a \leq V_\Delta(t, x) \leq k_{2_\Delta} \|x\|^a \tag{3}$$

$$\frac{\partial V_\Delta}{\partial t} + \frac{\partial V_\Delta}{\partial x} f(t, x, \theta^*) \leq -k_{3_\Delta} \|x\|^a \tag{4}$$

$\forall t \geq t_0 \geq 0$, then the origin of the system (1) is USGES on Θ.

Here $\|x\|$ denotes the Euclidean norm.
Proof:
Let Δ be any given positive constant and let V_Δ and $\theta^*(\Delta)$ be generated by the assumptions of Theorem 5.

The proof follows along the lines of the proofs of (Khalil, 2002, Theorems 4.8 and 4.10), while particular care is taken to show that the estimate of the region of attraction in which the system is shown to have exponential convergence can be made arbitrarily large by increasing Δ. Choose the positive constant $c_\Delta = \alpha k_{1_\Delta} \Delta^a$ where $0 < \alpha < 1$. Define the time-dependent set

$$\Omega_{t,c_\Delta} = \{x \in B_\Delta : V_\Delta(t, x) \leq c_\Delta\} \tag{5}$$

This set Ω_{t,c_Δ} contains the set $\Omega_2 = \{x \in B_\Delta : k_{2_\Delta} \|x\|^a \leq c_\Delta\}$ since $(k_{2_\Delta} \|x\|^a \leq c_\Delta) \implies (V_\Delta(t, x) \leq$

c_Δ) by Condition (3). Furthermore, the set $\Omega_{t,c_\Delta} \subset \Omega_1 = \{x \in B_\Delta : k_{1_\Delta}\|x\|^a \le c_\Delta\}$ since $V_\Delta(t,x) \le c_\Delta$ by (3) implies that $k_{1_\Delta}\|x\|^a \le c_\Delta$. We thus have the following nested sets:

$$\Omega_2 \subset \Omega_{t,c_\Delta} \subset \Omega_1 \subset B_\Delta \tag{6}$$

$\forall t \ge t_0 \ge 0$. Since $\dot{V}_\Delta(t,x) < 0$ on B_Δ according to (4), for any $t_0 \ge 0$ and any $x_0 \in \Omega_{t_0,c_\Delta}$ the solution starting at (t_0, x_0) will stay in Ω_{t,c_Δ} for all $t \ge t_0$. Therefore, any solution that starts in Ω_2 will stay in Ω_{t,c_Δ} and consequently in Ω_1 for all future time, which by the choice of c_Δ implies that $\|x\| < \Delta$ for all future time. Hence, (1) has a unique and bounded solution defined for all $t \ge t_0 > 0$, whenever $x_0 \in \Omega_2$. To sum up, we thus have that $\forall t_0 \in \mathbb{R}_{\ge 0}$ the solutions of (1) satisfy

$$\|x_0\| \le \left(\frac{\alpha k_{1_\Delta}}{k_{2_\Delta}}\right)^{1/a}\Delta \implies \left\|x(t;t_0,x_0,\theta^*)\right\| < \Delta \quad \forall t \ge t_0 \ge 0 \tag{7}$$

We define $\tilde{\Delta} = (\frac{\alpha k_{1_\Delta}}{k_{2_\Delta}})^{1/a}\Delta$ and note that $\tilde{\Delta} < \Delta$. Condition (4) together with (7) gives that

$$\|x_0\| \le \tilde{\Delta} \implies \dot{V}_\Delta \le -k_{3_\Delta}\|x\|^a \quad \forall t \ge t_0 \ge 0 \tag{8}$$

It follows from (8) and (3) that $\forall x_0 \in B_{\tilde{\Delta}}$, V_Δ satisfies the differential inequality

$$\dot{V}_\Delta \le -\frac{k_{3_\Delta}}{k_{2_\Delta}}V_\Delta \quad \forall t \ge t_0 \ge 0 \tag{9}$$

The comparison lemma (Khalil, 2002, Lemma 3.4) then gives that

$$V_\Delta(t,x) \le V_\Delta(t_0,x_0)e^{-\frac{k_{3_\Delta}}{k_{2_\Delta}}(t-t_0)} \quad \forall t \ge t_0 \ge 0 \tag{10}$$

Using (3) this gives that $\forall x_0 \in B_{\tilde{\Delta}}$

$$\begin{aligned}\left\|x(t;t_0,x_0,\theta^*)\right\| &\le \left(\frac{V_\Delta(t,x)}{k_{1_\Delta}}\right)^{\frac{1}{a}} \\ &\le \left(\frac{V_\Delta(t_0,x_0)e^{-\frac{k_{3_\Delta}}{k_{2_\Delta}}(t-t_0)}}{k_{1_\Delta}}\right)^{\frac{1}{a}} \\ &\le \left(\frac{k_{2_\Delta}\|x_0\|^a e^{-\frac{k_{3_\Delta}}{k_{2_\Delta}}(t-t_0)}}{k_{1_\Delta}}\right)^{\frac{1}{a}} \\ &= \left(\frac{k_{2_\Delta}}{k_{1_\Delta}}\right)^{\frac{1}{a}}\|x_0\|\, e^{-\frac{k_{3_\Delta}}{a k_{2_\Delta}}(t-t_0)} \end{aligned} \tag{11}$$

$\forall t \ge t_0 \ge 0$. In other words $B_{\tilde{\Delta}}$ is a subset of the region of attraction in which the system has exponential convergence. Since the subset of the region of attraction with exponential convergence, $B_{\tilde{\Delta}}$, can be made arbitrarily large by increasing Δ, the origin of System (1) is USGES. □

Remark 6 *The set $B_{\tilde{\Delta}}$ is a subset contained in the region of attraction $R_A = \{x \in \mathbb{R}^n$: $x(t;t_0,x_0,\theta^*)$ is defined for all $t \ge t_0 \ge 0$ and $\lim_{t\to\infty} x(t;t_0,x_0,\theta^*) = 0\}$. The subset $B_{\tilde{\Delta}}$ thus provides an estimate of the region of attraction, and in particular of the region of attraction in which the system has exponential convergence. Estimates based on Lyapunov functions, like $B_{\tilde{\Delta}}$, are often conservative, i.e. they may be much smaller than the actual region of attraction.*

Remark 7 *The proof also holds when the exponent a is parameterized by Δ, i.e. when there exists a positive constant a_Δ such that the conditions of Theorem 5 hold. However, since it does not seem feasible that the exponent will depend on the size of the estimate of the region of attraction, we state the theorem for the non-parameterized exponent a.*

Remark 8 *The proof shows that for the special case when conditions (3-4) are satisfied with positive constants k_1, k_2, k_3 that are uniform in Δ, then the overshoot and convergence rate (given in (11)) are independent of Δ, i.e. independent of the size of the subset of the region of attraction in which exponential convergence is shown.*

2.3 Example

We will now show a simple example of how Theorem 5 can be applied. Consider the following system

$$\dot{x} = -u(t)\arctan(\frac{x}{\theta}) \tag{12}$$

where the time-varying function $u(t)$ is continuously differentiable (C^1) in t, and $0 < u_{\min} \le u(t) \le u_{\max}$. This simple model can represent a system with a saturated P-controller, where the parameter $\theta > 0$ decides the slope of the saturation function. The dependency on a time-varying function $u(t)$ is motivated by marine control systems using LOS guidance, where the time-varying forward velocity $u(t)$ of the system affects the dynamics of the cross-track error (Fossen and Pettersen (2014)).

We use the C^1 Lyapunov function candidate $V(x) = \frac{1}{2}x^2$. Condition (3) is satisfied with $k_1 = k_2 = \frac{1}{2}$ and $a = 2$. Furthermore

$$\begin{aligned}\dot{V}(t,x) &= -u(t)\frac{\arctan(\frac{x}{\theta})}{x}x^2 \\ &\le -u_{\min}\frac{\arctan(\frac{x}{\theta})}{x}x^2\end{aligned} \tag{13}$$

For any given Δ, we thus have that $\forall x \in B_\Delta$

$$\dot{V} \leq -u_{\min}\frac{\arctan(\frac{\Delta}{\theta})}{\Delta}x^2 \tag{14}$$

i.e. Condition (4) is satisfied with $k_{3_\Delta} = u_{\min}\frac{\arctan(\frac{\Delta}{\theta})}{\Delta}$. By Theorem 5 the origin $x = 0$ is thus USGES. Moreover, from Eq. (11) in the proof we see that the overshoot is independent on Δ. The guaranteed convergence rate, however, will depend on the desired size of the subset $B_{\tilde{\Delta}}$ of the region of attraction in which exponential convergence is shown, since $\frac{k_{3_\Delta}}{ak_2} = u_{\min}\frac{\arctan(\frac{\Delta}{\theta})}{\Delta}$ decreases when $\tilde{\Delta} = \sqrt{\alpha}\Delta$ increases. Also, note that we cannot conclude from (Khalil, 2002, Th. 4.10) that the origin is GES, since it is not possible to find a positive constant k_3 that is independent of the size of the estimated region of attraction, $B_{\tilde{\Delta}}$. In particular there exists no lower bound k_3 on the gain function $u_{\min}\frac{\arctan(\frac{x}{\theta})}{x}$ $\forall x \in \mathbb{R}^n$ since the function converges to zero as $|x| \longrightarrow \infty$. This is an inherent property of (12) because of the saturation.

2.4 USGES of cascaded systems

As noted in Remark 3, the Lyapunov sufficient conditions in Theorem 5 can be utilized in analysing USGES of cascaded systems. In particular, the cascaded systems theory result for USGES systems in Loria and Panteley (2004) Proposition 2.3 may thus be restated as follows:

Consider the cascaded parameterized nonlinear time-varying system

$$\dot{x}_1 = f_1(t, x_1, \theta) + g(t, x, \theta)x_2 \tag{15}$$
$$\dot{x}_2 = f_2(t, x_2, \theta) \tag{16}$$

where $t \in \mathbb{R}_{\geq 0}$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $x = \mathrm{col}\,[x_1, x_2]$, and $\theta \in \Theta \subset \mathbb{R}^m$ is a constant parameter. The functions $f_1(t, x_1, \theta)$, $f_2(t, x_2, \theta)$ and $g(t, x, \theta)$ are continuous in t and continuous and locally Lipschitz in x_1, x_2 and x, respectively, for all $\theta \in \Theta$. Furthermore, $f_1(t, x_1, \theta)$ is C^1 in t and x_1 for all $\theta \in \Theta$. The origin $x = 0$ is an equilibrium point of (15-16).

Proposition 9 *Let each of the systems*

$$\dot{x}_1 = f_1(t, x_1, \theta) \tag{17}$$

and

$$\dot{x}_2 = f_2(t, x_2, \theta) \tag{18}$$

be uniformly globally asymptotically stable (UGAS) and satisfy the conditions of Theorem 5. Furthermore, let the following assumptions be satisfied:
Assumption 1*: There exist constants* $c_1, c_2, \eta > 0$ *and a positive definite, radially unbounded Lyapunov function* $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_1} \longrightarrow \mathbb{R}_{\geq 0}$ *of* (17) *such that* $\dot{V}_{(17)}(t, x_1) \leq 0$ *and*

$$\left\|\frac{\partial V}{\partial x_1}\right\| \|x_1\| \leq c_1 V(t, x_1) \quad \forall \|x_1\| \geq \eta \tag{19}$$

$$\left\|\frac{\partial V}{\partial x_1}\right\| \leq c_2 \qquad \forall \|x_1\| \leq \eta \tag{20}$$

Assumption 2*: There exist two continuous functions* $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*, such that* $g(t, x, \theta)$ *satisfies*

$$\left\|g(t, x, \theta)\right\| \leq \alpha_1(\|x_2\|) + \alpha_2(\|x_2\|)\|x_1\| \tag{21}$$

Then the origin of (15-16) is USGES and UGAS.

Remark 10 *Note that Conditions* (3)-(4) *of Theorem 5 only need to be satisfied* $\forall x_i \in B_{\Delta_i}$, $i \in \{1, 2\}$*, while Assumptions 1-2 need to be satisfied* $\forall x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$*, respectively.*

Remark 11 *In Proposition 2.3/Theorem 2.1 of Loria and Panteley (2004) there is also a third assumption:*
Assumption 3*: There exists a class* $\mathcal{K}$ *function* $\alpha(\cdot)$ *such that, for all* $t_0 \geq 0$*, the trajectories of the system* (18) *satisfy*

$$\int_{t_0}^{\infty} \left\|x_2(t; t_0, x_2(t_0), \theta^*)\right\| dt \leq \alpha(\left\|x_2(t_0)\right\|) \tag{22}$$

Note that this assumption is satisfied from the conditions in Proposition 9. In particular, since System (18) *is USGES it is also uniformly locally exponentially stable (ULES). Furthermore,* (18) *is UGAS by assumption. The properties UGAS + ULES imply that Assumption 3 is satisfied:*

Since the system is ULES, there exist positive constants c, k, λ*, independent on* t_0*, such that* $\forall x_2(t_0) \in B_c$

$$\left\|x_2(t; t_0, x_2(t_0), \theta^*)\right\| \leq k\left\|x_2(t_0)\right\| e^{-\lambda(t-t_0)} \quad \forall t \geq t_0 \geq 0$$

Since the system is UGAS, there exists a class $\mathcal{KL}$ *function* β *such that* $\forall x_2(t_0) \in \mathbb{R}^n$

$$\left\|x_2(t; t_0, x_2(t_0), \theta^*)\right\| \leq \beta(\left\|x_2(t_0)\right\|, t - t_0) \quad \forall t \geq t_0 \geq 0$$

By the UGAS property, we know that $\exists T > 0$ *such that at* $t = t_0 + T$ *the solution enters the neighbourhood of the origin where the convergence is exponential. Consequently,*

$$\begin{aligned}\int_{t_0}^{\infty} \left\|x_2(t; t_0, x_2(t_0), \theta^*)\right\| dt &\leq \int_{t_0}^{t_0+T} \beta(\left\|x_2(t_0)\right\|, t - t_0)dt \\ &+ \int_{t_0+T}^{\infty} k\left\|x_2(t_0 + T)\right\| e^{-\lambda(t-(t_0+T))}dt \\ &\leq T\beta(\left\|x_2(t_0)\right\|, 0) + \frac{k}{\lambda}\beta(\left\|x_2(t_0)\right\|, T)\end{aligned}$$

The right hand side is a class $\mathcal{K}$ function $\alpha(\|x_2(t_0)\|$, i.e. Assumption 3 is satisfied.

2.5 Example continued

To illustrate how Proposition 9 can be used, we build further on the simple example in Section 2.3. Consider the following system

$$\dot{x}_1 = -u(t)\arctan(\frac{x_1}{\theta_1}) + x_1x_2 \quad (23)$$

$$\dot{x}_2 = -(1+2e^{-t})\theta_2x_2 \quad (24)$$

where θ_1 and θ_2 are constant and positive parameters. We first consider System (17), i.e.

$$\dot{x}_1 = -u(t)\arctan(\frac{x_1}{\theta_1}) \quad (25)$$

The C^1 Lyapunov function candidate $V_1(x_1) = \frac{1}{2}x_1^2$ is positive definite, decrescent and radially unbounded on $\mathbb{R}$. Furthermore, $\dot{V}(x_1) \le -W(x_1)$ where $W(x_1) = u_{\min}\frac{\arctan(\frac{x_1}{\theta_1})}{x_1}x_1^2$ is a continuous positive definite function on $\mathbb{R}$. The origin of (25) is thus UGAS, and we have shown in Section 2.3 that the system satisfies the conditions of Theorem 5.
Assumption 1 is clearly satisfied with $V_1(x_1)$:

$$\left\|\frac{\partial V_1}{\partial x_1}\right\|\|x_1\| = \|x_1\|^2 = 2V_1(x_1) \quad \forall\|x_1\| \quad (26)$$

$$\left\|\frac{\partial V_1}{\partial x_1}\right\| = \|x_1\| \le \eta \quad \forall\|x_1\| \le \eta \quad (27)$$

i.e. with $c_1 = 2$, and $c_2 = \eta$ for any choice of $\eta > 0$.

We then consider System (18) which is

$$\dot{x}_2 = -(1+2e^{-t})\theta_2x_2 \quad (28)$$

The C^1 Lyapunov function candidate $V_2(x_2) = \frac{1}{2}x_2^2$, for which

$$\dot{V}_2 = -(1+2e^{-t})\theta_2x_2^2 \le -\theta_2x_2^2 \quad (29)$$

clearly satisfies the conditions of Theorem 5. Indeed, by (Khalil, 2002, Theorem 4.10) the origin of (29) is UGES, which implies both UGAS and USGES.

Finally, it remains to investigate Assumption 2, i.e. the assumption that the interconnection term $g(t,x,\theta)$ has linear growth in x_1:

$$\|g(t,x,\theta)\| = \|x_1x_2\| = \|x_1\|\|x_2\| \quad (30)$$

i.e. Assumption 2 is satisfied with $\alpha_1 = 0$ and $\alpha_2(r) = r$. By Proposition 9 the origin of the cascaded system (23-24) is thus USGES and UGAS.

3 Robustness to nonvanishing perturbations

In this section we discuss the robustness properties that USGES may provide with respect to nonvanishing perturbations, and we use the simple example from Section 2.3 to illustrate this result as well.

Lemma 9.2 of Khalil (2002) shows why UGES is such a powerful property with respect to robustness to nonvanishing perturbations. We will now investigate the robustness properties that USGES can provide. Consider the system

$$\dot{x} = f(t,x,\theta) + g(t,x,\theta) \quad (31)$$

which is a perturbation of the nominal system (1), and where the perturbation $g(t,x,\theta)$ is locally Lipschitz in x and piecewise continuous in t for all $\theta \in \Theta$.

Lemma 12 *Assume that the conditions of Theorem 5 are satisfied and that there exists a positive constant k_{4_Δ} and a constant $0 < c < 1$ such that $\forall x \in B_\Delta$*

$$\left\|\frac{\partial V_\Delta}{\partial x}\right\| \le k_{4_\Delta}\|x\|^{a-1} \quad (32)$$

$$\|g(t,x,\theta)\| \le \delta < \frac{k_{3_\Delta}}{k_{4_\Delta}}\left(\frac{k_{1_\Delta}}{k_{2_\Delta}}\right)^{1/a}\Delta c \quad (33)$$

$\forall t \ge t_0 \ge 0$. Then $\forall x_0 \in B_{\hat{\Delta}}$ there is a $T \ge 0$ (dependent on x_0 and μ) such that the solution $x(t;t_0,x_0,\theta^*)$ of the perturbed system (31) satisfies

$$\|x(t;t_0,x_0,\theta^*)\| \le \left(\frac{k_{2_\Delta}}{k_{1_\Delta}}\right)^{\frac{1}{a}}\|x_0\|e^{-\frac{(1-c)k_{3_\Delta}}{ak_{2_\Delta}}(t-t_0)} \quad (34)$$

$\forall\, t_0 \le t \le t_0 + T$ and

$$\|x(t;t_0,x_0,\theta^*)\| \le \frac{k_{4_\Delta}}{k_{3_\Delta}}\left(\frac{k_{2_\Delta}}{k_{1_\Delta}}\right)^{1/a}\frac{\delta}{c} \quad \forall t \ge t_0+T. \quad (35)$$

Proof: The proof follows along the lines of the proof of Lemma 9.2 of Khalil (2002). In particular, the derivative of V_Δ along the trajectories of (31) satisfies

$$\dot{V}_\Delta = \frac{\partial V_\Delta}{\partial t} + \frac{\partial V_\Delta}{\partial x}f(t,x,\theta) + \frac{\partial V_\Delta}{\partial x}g(t,x,\theta) \quad (36)$$

$$\le -k_{3_\Delta}\|x\|^a + \left\|\frac{\partial V_\Delta}{\partial x}\right\|\|g(t,x,\theta)\| \quad (37)$$

$$\le -k_{3_\Delta}\|x\|^a + k_{4_\Delta}\delta\|x\|^{a-1} \quad (38)$$

$$= -(1-c)k_{3_\Delta}\|x\|^a - ck_{3_\Delta}\|x\|^a + k_{4_\Delta}\delta\|x\|^{a-1} \quad (39)$$

$$\le -(1-c)k_{3_\Delta}\|x\|^a \qquad \|x\| \ge \frac{k_{4_\Delta}\delta}{k_{3_\Delta}c} \quad (40)$$

Applying Theorem 4.18 of Khalil (2002) with $\alpha_1(r) = k_{1_\Delta} r^a$, $\alpha_2(r) = k_{2_\Delta} r^a$, $\mu = \frac{k_{4_\Delta}\delta}{k_{3_\Delta} c}$ and $W(x) = (1-c)k_{3_\Delta} r^a$ completes the proof. In particular, the condition

$$\mu < \alpha_2^{-1}(\alpha_1(\Delta)) \tag{41}$$

gives condition (33). Furthermore, from the proof of Theorem 4.18 we have that for all $t \in [t_0, t_0+T]$

$$\dot{V}_\Delta \leq -\frac{(1-c)k_{3_\Delta}}{k_{2_\Delta}} V_\Delta \tag{42}$$

which by the comparison lemma gives that

$$V_\Delta(t,x) \leq V_\Delta(t_0,x_0) e^{-\frac{(1-c)k_{3_\Delta}}{k_{2_\Delta}}(t-t_0)} \quad \forall\, t_0 \leq t \leq t_0+T$$

which again gives that the class $\mathcal{KL}$ function of Theorem 4.18 is given by $\beta(r,s) = \left(\frac{k_{2_\Delta}}{k_{1_\Delta}}\right)^{\frac{1}{a}} r e^{-\frac{(1-c)k_{3_\Delta}}{a k_{2_\Delta}} s}$. □

Remark 13 *Note that when the system is UGES then Lemma 9.2 of Khalil (2002) shows that for all uniformly bounded disturbances, irrespective of their magnitude, the solution of the perturbed system will be uniformly bounded. This robustness property is not given for UGAS systems, and it is interesting to see that it can neither be concluded in general for USGES systems from Lemma 12. In particular, note that the right-hand side of* (33)*, which gives the upper bound of the perturbation term, does not necessarily converge to* ∞ *as* $\Delta \to \infty$*, since the parameters* k_{i_Δ}*,* $i = 1,\ldots,4$ *depend on* Δ*. We therefore need further information about how* k_{i_Δ}*,* $i = 1,\ldots,4$ *depend on* Δ *in order to conclude whether the upper limit of the perturbation can be made arbitrarily large by increasing* Δ*.*

Let us continue using the simple system (12) to illustrate the theory. In particular, consider

$$\dot{x} = -u(t)\arctan(\frac{x}{\theta}) + g(t,x,\theta) \tag{43}$$

where the perturbation $g(t,x,\theta)$ is locally Lipschitz in x and piecewise continuous in t. We have shown that the Lyapunov function candidate $V(x) = \frac{1}{2}x^2$ satisfies the conditions of Theorem 5 with $k_1 = k_2 = \frac{1}{2}$, $k_{3_\Delta} = u_{\min}\frac{\arctan(\frac{\Delta}{\theta})}{\Delta}$ and $a = 2$. Furthermore,

$$\left\| \frac{\partial V}{\partial x} \right\| = \|x\| \tag{44}$$

i.e. (32) is satisfied with $k_4 = 1$. For the perturbation term, the condition given by Lemma 12 is

$$\begin{aligned} \|g(t,x,\theta)\| \leq \delta &< \frac{k_{3_\Delta}}{k_{4_\Delta}}\left(\frac{k_{1_\Delta}}{k_{2_\Delta}}\right)^{1/a}\Delta c \\ &= u_{\min}\arctan(\frac{\Delta}{\theta})c \\ &< \frac{\pi}{2}u_{\min} \end{aligned} \tag{45}$$

i.e. the upper bound δ on the perturbation is smaller than $u_{\min}\pi/2$. We can thus only conclude that the boundedness properties (34-35) hold for uniformly bounded perturbations of sufficiently small magnitude.

4 Robustness to vanishing perturbations

For completeness, in this section we outline the robustness properties that USGES provides with respect to vanishing perturbations. Consider the system

$$\dot{x} = f(t,x,\theta) + g(t,x,\theta) \tag{46}$$

which is a perturbation of the nominal system (1), and where the perturbation $g(t,x,\theta)$ is locally Lipschitz in x and piecewise continuous in t for all $\theta \in \Theta$, and $g(t,0,\theta) = 0$.

Lemma 14 *Assume that the conditions of Theorem 5 are satisfied and that there exist a positive constant* k_{4_Δ} *and a nonnegative constant* γ_Δ*, satisfying* $0 \leq \gamma_\Delta < \frac{k_{3_\Delta}}{k_{4_\Delta}}$*, such that* $\forall x \in B_\Delta$

$$\left\| \frac{\partial V_\Delta}{\partial x} \right\| \leq k_{4_\Delta}\|x\|^{a-1} \tag{47}$$

$$\|g(t,x,\theta)\| \leq \gamma_\Delta \|x\| \tag{48}$$

$\forall t \geq t_0 \geq 0$*. Then the origin of system* (46) *is USGES on* Θ*.*

Proof: The proof follows along the lines of the proof of Theorem 5. In particular, from (8) and (46) it follows that $\forall x_0 \in B_{\tilde{\Delta}}$

$$\dot{V}_\Delta \leq -k_{3_\Delta}\|x\|^a + \left\| \frac{\partial V_\Delta}{\partial x} \right\| \|g(t,x,\theta)\| \tag{49}$$

$$\leq -k_{3_\Delta}\|x\|^a + k_{4_\Delta}\gamma_\Delta\|x\|^a \tag{50}$$

$$= -(k_{3_\Delta} - \gamma_\Delta k_{4_\Delta})\|x\|^a \tag{51}$$

$\forall t \geq t_0 \geq 0$, where $(k_{3_\Delta} - \gamma_\Delta k_{4_\Delta}) > 0$. It follows that $\forall x_0 \in B_{\tilde{\Delta}}$, V_Δ satisfies the differential inequality

$$\dot{V}_\Delta \leq -\frac{k_{3_\Delta} - \gamma_\Delta k_{4_\Delta}}{k_{2_\Delta}} V_\Delta \quad \forall t \geq t_0 \geq 0 \tag{52}$$

and the comparison lemma together with (3) gives that

$$\left\|x(t;t_0,x_0,\theta^*)\right\| \leq \left(\frac{k_{2_\Delta}}{k_{1_\Delta}}\right)^{\frac{1}{a}} \|x_0\| \, e^{-\frac{k_{3_\Delta}-\gamma_\Delta k_{4_\Delta}}{a k_{2_\Delta}}(t-t_0)} \tag{53}$$

$\forall t \geq t_0 \geq 0$. Since the subset of the region of attraction with exponential convergence, $B_{\tilde{\Delta}}$, can be made arbitrarily large by increasing Δ, the origin of System (46) is USGES. □

5 Conclusions

In this paper we have developed Lyapunov sufficient conditions for uniform semiglobal exponential stability (USGES) of parameterized nonlinear time-varying systems. Furthermore, we have investigated the robustness that USGES may provide with respect to perturbations. For vanishing perturbations of sufficiently small gain, it is seen that the USGES property is retained. When it comes to nonvanishing perturbations, it is shown that USGES provides robustness to uniformly bounded disturbances. It is interesting to note, however, that the strong robustness properties that UGES provides to perturbations of arbitrary magnitude cannot readily be concluded for USGES systems. USGES thus guarantees both stability and good performance in the sense of exponential convergence in a region of attraction that can be arbitrarily enlarged, while robustness to external disturbances of arbitrary magnitude must be considered for each particular case, for instance using the analysis tools provided in this paper.

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