ISBN 978-82-326-1784-5 (printed version)
ISBN 978-82-326-1785-2 (electronic version)
ISSN 1503-8181

Doctoral theses at NTNU, 2016:223

Karin Marie Jacobsen

Understanding module categories through triangulated categories using Auslander-Reiten theory

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of
Science and Technology
Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Mathematical Sciences

Doctoral theses at NTNU, 2016:223

NTNU

Karin Marie Jacobsen

Understanding module categories through triangulated categories using Auslander-Reiten theory

Thesis for the degree of Philosophiae Doctor

Trondheim, September 2016

Norwegian University of Science and Technology
Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the degree of Philosophiae Doctor

Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Mathematical Sciences

ISBN 978-82-326-1784-5 (printed version)
ISBN 978-82-326-1785-2 (electronic version)
ISSN 1503-8181

Doctoral theses at NTNU, 2016:223

Printed by Skipnes Kommunikasjon as

Acknowledgements

There are many people who have helped me get to the point where I have a thesis to hand in.

Aslak Bakke Buan has been a great advisor. He is friendly, knowledgeable and helpful, and an extraordinarily patient proof-reader. Thank you.

The remarkable Benedikte Grimeland has written two papers with me. She has a head for details that has saved us many times, and, luckily, a stubbornness to match my own. Thank you for taking me under your wing as a new PhD student, for sticking it out through endless revisions and errors in proofs, and for understanding that tea breaks and laughter are both essential. Thank you for sharing Steffen Oppermann with me, and thank you Steffen, for being my co-advisor in all but the name.

The Department of mathematical sciences has been a great place to work. From the great tech support on the third floor, to the conversations in the lunch room on the thirteenth, it has been a welcoming second home to me the last four years. In particular, I would like to thank Benedikte, Kristin, Yvonne and Øystein for inviting me to lunch in my first week and keeping me sane ever after. I may have taught you how to eat an elephant, but you helped me eat mine. Martin, thanks for putting up with me for nearly four years. Special thanks to Brynjulf for an e-mail correspondence in 2010 that kept me from giving up.

Finally, I would like to thank my friends, my family and in particular my parents Britt and Jon Erik for being there. Gunilla, thank you for being my ungodmother and my role model. And Petter, thank you for everything you do.

Introduction

This Ph.d. thesis consists of three papers:

(1) Benedikte Grimeland and Karin M. Jacobsen. *Abelian quotients of triangulated categories*. Journal of Algebra 439 (2015), 110–133.
(2) Benedikte Grimeland and Karin M. Jacobsen. *Realizing orbit categories as stable module categories - a complete classification*. Preprint, arXiv: 1508:02970v1.
(3) Karin M. Jacobsen. *Modules of finite projective dimension over a cluster-tilted algebra*. Manuscript.

All three papers concern the interaction between abelian and triangulated categories. We use triangulated categories to better understand the structure of module categories.

1. Background

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{D}^b(\mathcal{A})$ be the bounded derived category as defined by Verdier in [29]. We know that $\mathcal{D}^b(\mathcal{A})$ is triangulated [21], with suspension functor equal to the shift functor on complexes $X \mapsto X[1]$.

1.1. Orbit categories. In the last decade, some very interesting results have come from studying smaller categories obtained from the derived category. Given an automorphism on $\mathcal{D}^b(\mathcal{A})$, we can define an orbit category:

Definition 1. *Let be $\mathcal{C}$ an additive category and $F : \mathcal{C} \to \mathcal{C}$ an automorphism. The orbit category $\mathcal{C}/F$ has the same objects as $\mathcal{C}$, and its morphisms are given by* $\mathrm{Hom}_{\mathcal{C}/F}(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, F^n Y)$.

The orbit category is not necessarily triangulated. However, we know of a large class of functors that do give triangulated orbit categories:

Theorem 2 ([22]). *Suppose that $\mathcal{A}$ is an hereditary abelian k-category. Let F be an autoequivalence on $\mathcal{D}^b(\mathcal{A})$ that fulfills the following properties:*

(1) *For each indecomposable object $X \in \mathcal{D}^b(\mathcal{A})$, there are only finitely many $i \in \mathbb{Z}$ such that $F^i \in \mathcal{A}$.*
(2) *There exist an integer $N \geq 0$ such that each F-orbit of each indecomposable object of $\mathcal{D}^b(\mathcal{A})$ contains an object $U[n]$ where U is indecomposable in $\mathcal{A}$.*

Then $\mathcal{D}^b(\mathcal{A})/F$ is naturally a triangulated category and the functor $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})/F$ is a triangle functor. We call F an admissible functor.

Let τ be the AR-translation. Then the functor $F = \tau^{-1}[1]$ is admissible. We call the category $\mathcal{C}_{\mathcal{A}} = \mathcal{D}^b(\mathcal{A})/F$ the cluster category of $\mathcal{A}$ [13]. If $\mathcal{A} = \mathrm{mod}\,\Lambda$, we write $\mathcal{D}^b(\Lambda) := \mathcal{D}^b(\mathcal{A})$ and $\mathcal{C}_\Lambda := \mathcal{C}_{\mathcal{A}}$. The cluster categories on this form yield a categorification of the acyclic cluster algebras[15], but they also turn out to be interesting in its own right. It should be noted that in the cluster category we have $X[1] = \tau X$ by construction. Amiot has defined the cluster category for algebras of global dimension 2, and for quivers with potential [1].

1.2. Quotient categories and cluster-tilting objects. While the various triangulated categories are interesting to study, they are rather useless for our purposes without a way of getting back to the module category. One option is the Hom-functor: If X is an object in the triangulated category $\mathcal{C}$, then $\Gamma = \mathrm{End}_{\mathcal{C}}(X)$ is a ring. Moreover, for any object $Y \in \mathcal{C}$, the object $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a Γ-module, so $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is a functor to an abelian category.

We will also use the construction of a quotient category:

DEFINITION 3. *Let $\mathcal{C}$ be a category. An* ideal *$\mathcal{I}$ in $\mathcal{C}$ is a non-empty collection of morphisms such that for any composition of morphisms fg where $f \in I$ or $g \in I$, we have $fg \in I$. We set*

$$\mathcal{I}(X, Y) = \mathrm{Hom}(X, Y) \cap \mathcal{I}.$$

The quotient category $\mathcal{C}/\mathcal{I}$ has the same objects as $\mathcal{C}$, and has morphisms given by

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y).$$

The natural functor $\pi : \mathcal{C}/\mathcal{I}$ is called the quotient functor. It is universal with respect to sending the morphisms of $\mathcal{I}$ to zero.

The quotient functor is always full and dense. By universality we can show that any full and dense functor can be described as a quotient functor. For an object T, we define $\mathcal{I}_T$ to be the collection of morphisms factoring through T. We then write $\mathcal{C}/T$ for the quotient category.

The quotient category is particularly interesting when used in conjunction with cluster-tilting objects. We call $T \in \mathcal{C}$ a cluster-tilting object if it has no self-extensions, and is maximal with respect to that property:

DEFINITION 4 ([13]). *An object $T \in \mathcal{C}$ is called a* cluster-tilting object *if the following holds:*

$$\mathrm{add}\, T = \{X \in \mathcal{C} | \, \mathrm{Ext}_{\mathcal{C}}(T, X) = 0\} = \{Y \in \mathcal{C} | \, \mathrm{Ext}_{\mathcal{C}}(Y, T) = 0\}$$

We call $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{op}$ a cluster-tilted algebra. Buan, Marsh and Reiten showed in [11] that in the cluster category, the functor $\mathrm{Hom}(T, -)$ induces an equivalence $\mathrm{mod}\, \Gamma \cong \mathcal{C}/\tau T$.

Let Λ be a hereditary path algebra over an acyclic quiver Q. We say that the cluster-tilted algebra $\tilde{\Lambda} = \mathrm{End}_{\mathcal{C}_\Lambda}(T)$ is cluster-tilted of type Q.

Keller and Reiten generalised the results of [11] to 2-Calabi-Yau triangulated categories [23]. In particular, they show that the cluster-tilted algebra is Gorenstein.

Furthermore, in [26] Koenig and Zhu generalise the results of [11] to all triangulated categories, using cluster-tilting subcategories. These are subcategories $\mathcal{T}$ such that

$$\mathcal{T} = \{X \in \mathcal{C} | \, \mathrm{Ext}_{\mathcal{C}}(T, X) = 0\} = \{Y \in \mathcal{C} | \, \mathrm{Ext}_{\mathcal{C}}(Y, T) = 0\}.$$

If $\mathcal{C}$ is a triangulated category and $\mathcal{T}$ is a cluster-tilting subcategory of $\mathcal{C}$, then $\mathcal{C}/\mathcal{T}$ is abelian. Note that if T is a cluster-tilting object, then $\mathrm{add}\, T$ is a cluster-tilting subcategory.

1.3. Surface algebras. Let Λ be a gentle algebra [5], where the relations are from 3-cycles with radical square zero (this technical condition means that we are dealing with a quiver with potential, hence we can use Amiot cluster categories [1]). The cluster category $\mathcal{C}_\Lambda$ can be represented geometrically [10]. We give the construction for path algebras of of Dynkin type A, first described in [14], but it holds in larger generality as shown in [10].

Let S be a disc and let M be a set of at least four marked points on the boundary ∂S of S. We call (S, M) a marked surface. Consider the set of arcs between the marked points, up to homotopy. The arcs that are not homomorphic to arcs in $\partial S \setminus M$ are in

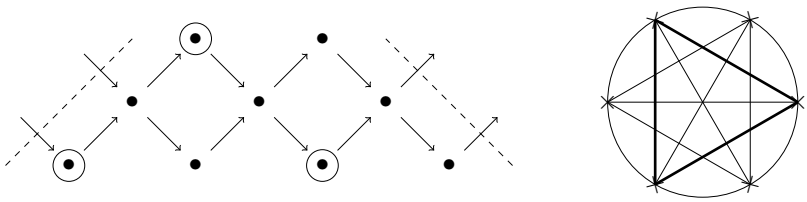


FIGURE 1. On the left: an AR-quiver for $\mathcal{C}_\Lambda$ of type $\mathbb{A}_3$ with the summands of a cluster-tilting object marked. On the right: the marked surface representation of the same cluster category, with the same cluster-tilting object bolded.

bijection to the indecomposable objects of $\mathcal{C}_\Lambda$. Together with a definition of irreducible morphisms, composition and mesh relations, this gives us a complete description of the cluster-category.

A maximal set of non-intersecting arcs in (S, M) is called a triangulation. When arcs intersect, the corresponding objects have non-trivial extensions. Hence a triangulation of (S, M) corresponds to a maximal object without self-extensions, or a cluster-tilting object. In [10] this correspondence is shown to be a bijection.

Let T be a cluster-tilting object and Δ its corresponding triangulation. The quotient $\mathcal{C}_\Lambda/T$ is equivalent to the module category of a cluster-tilted algebra $\tilde{\Lambda} \cong \operatorname{End}_{\mathcal{C}} \Lambda(T)$. If we set the arcs in Δ to correspond to zero objects, the marked surface gives us a representation of $\operatorname{mod} \tilde{\Lambda}$ [6][14].

2. Abelian quotients of triangulated categories.

Our first paper [16] concerns a simple question: When is a quotient category of a triangulated category abelian? As discussed in Section 1.2, we know sufficient conditions for a quotient categories to be abelian. We wanted to find necessary and sufficient conditions. We assume throughout this section that $\mathcal{C}$ is a Hom-finite, Krull-Schmidt triangulated category.

We start by considering $\mathcal{C}$ and an ideal $\mathcal{I}$. We show that if $\mathcal{C}/\mathcal{I}$ is abelian with a projective generator and the quotient functor is cohomological, then the quotient functor is representable.

Then the natural question is: When is a functor of the form $\operatorname{Hom}_{\mathcal{C}}(T, -)$ equivalent to a quotient functor? Before we give the theorem, we note that a right minimal morphism is a morphism f such that if $fg = f$, then g is an isomorphism.

THEOREM 5 ([16, Thm. 17]). *Fix some object $T \in \mathcal{C}$. Then $\operatorname{Hom}_{\mathcal{C}}(X, -)$ is a quotient functor, i. e. full and dense, if and only if the following two conditions are satisfied.*

a: *For all right minimal morphisms $T_1 \to T_0$ with $T_0, T_1 \in \operatorname{add} T$, all triangles*

$$T_1 \to T_0 \to X \xrightarrow{h} T_1[1]$$

satisfy $\operatorname{Hom}_{\mathcal{C}}(T, h) = 0$.

b: *For any object $X \in \mathcal{C}$ with $\operatorname{Hom}_{\mathcal{C}}(T, X) \neq 0$ there exists a triangle*

$$T_1 \to T_0 \to X \xrightarrow{h} T_1[1]$$

such that $T_1, T_0 \in \operatorname{add} T$, and $\operatorname{Hom}_{\mathcal{C}}(T, h) = 0$.

We also show that the AR-structure of the category is preserved under cohomological quotient functors.

It turns out that the objects that give full and dense functors are closely related to cluster-tilting objects:

THEOREM 6 ([16, Thm. 25]). *An object $T \in \mathcal{C}$ is a cluster-tilting object if and only if the following conditions are satisfied.*

a: *For all right minimal morphisms $T_1 \to T_0$ with $T_0, T_1 \in \operatorname{add} T$, all triangles*

$$T_1 \to T_0 \to X \xrightarrow{h} T_1[1]$$

satisfy $\operatorname{Hom}_{\mathcal{C}}(T, h) = 0$.

b*: *For any object $X \in \mathcal{C}$ there exists a triangle*

$$T_1 \to T_0 \to X \xrightarrow{h} T_1[1]$$

c: *If T' is a indecomposable summand of T, then $T[1] \notin \operatorname{add} T$.*

In particular, in the cluster category it is in practice only the cluster-tilting objects that give full and dense functors.

THEOREM 7 ([16, Thm. 27]). *Let $\mathcal{C}$ be the cluster category, and let T be an object in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \operatorname{mod} \operatorname{End}(T)^{op}$ is full and dense. Then either $\operatorname{End}(T) = k$ or T is a cluster-tilting object.*

3. Realizing orbit categories as stable module categories

It is well known that the stable module categories over self-injective algebras are triangulated, see e. g [18]. Holm and Jørgensen showed a triangulated equivalence between certain categories of higher cluster type and the stable module categories of certain self-injective algebras [19]. We extend their results in the second paper of the thesis.

By work of Riedtmann [27][28] and Bretcher, Läser and Riedtmann [9], we have a complete classification of the Auslander-Reiten-quivers (henceforth called AR-quivers) of representation-finite connected self-injective algebras. These AR-quivers are stable translation quivers of Dynkin tree type. Asashiba extended their work, by giving a classification of the representation-finite self-injective algebras up to derived equivalence. He assigns to each algebra a derived invariant called the *type* which is determined by the AR-quiver [3][4]. Asashiba also give representative algebras for each type.

Consider the orbit category $\mathcal{D}^b(\Lambda)/F$, where Λ is a path algebra over a Dynkin quiver. Provided that F is reasonably nice, the AR-quiver of $\mathcal{D}^b(\Lambda)/F$ is a stable translation quiver of Dynkin tree type. We find algebras Λ and admissible functors F so that we can match the AR-quiver of any stable module category of a representation-finite self-injective algebra. If the AR-quivers of two categories are identical, there is an additive equivalence between the categories. Building on that, we want to prove that the equivalence is a triangulated equivalence.

We use a theorem by Amiot [2] which helps us reduce the problem to showing equivalence of AR-quivers. We give the following corollary to the theorem, specializing it to our situation:

COROLLARY 8 ([17, Cor. 8]). *Let Λ be a representation-finite, self-injective, basic algebra such that $\underline{\operatorname{mod}}\Lambda$ is of standard type. Let Δ be a Dynkin diagram, and let $\Phi : \mathcal{D}^b(\operatorname{mod} k\Delta) \to \mathcal{D}^b(\operatorname{mod} k\Delta)$ be a functor such that $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ is triangulated.*

If the AR-quivers of $\underline{\operatorname{mod}}\Lambda$ and $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ are equivalent as translation quivers, then $\underline{\operatorname{mod}}\Lambda$ and $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ are equivalent as triangulated categories.

Using this corollary, we match up Asashiba's types with suitable functors for orbit categories. We give a full list of which orbit categories correspond to stable module categories over self-injective algebras.

4. Modules of finite projective dimension over a cluster-tilted algebra

In the final paper, we return to the subject of cluster categories and cluster-tilted algebras. Let Λ be a hereditary path algebra of Dynkin type, and let $\tilde{\Lambda} = \mathrm{End}_{\mathcal{C}_\Lambda}(T)$ be a cluster-tilted algebra of the same type as Λ.

We study the full subcategory $\mathcal{P}_{\leq 1}$ of $\mathrm{mod}\,\tilde{\Lambda}$ whose objects are the modules of projective dimension at most one. Reiten and Keller showed that cluster-tilted algebras have Gorenstein dimension one [23], so $\mathcal{P}_{\leq 1}$ contains all modules of finite projective dimension.

By Auslander and Smalø, the category $\mathcal{P}_{\leq 1}$ has AR-structure [7]. We can calculate the AR-translate $\tau_{\leq 1}$ in $\mathcal{P}_{\leq 1}$ using right approximations. We call $f : \mathfrak{r}_{\leq 1}X \to X$ a right minimal $\mathcal{P}_{\leq 1}$-approximation of X if f is right minimal, $\mathfrak{r}_{\leq 1}X \in \mathcal{P}_{\leq 1}$ and any morphism from $\mathcal{P}_{\leq 1}$ to X factors through f. If X is not projective then Kleiner and Perez showed that $\mathfrak{r}_{\leq 1}\tau X = I \oplus \tau_{\leq 1}X$, where I is an injective object [24][25].

The modules of infinite projective dimension have been given a very nice description by Beaudet, Brustle and Todorov in [8]. It turns out that an indecomposable module X has infinite projective dimension if and only if there is an endomorphism of T in $\mathcal{C}_\Lambda$ that factors through the preimage of X.

For type $\mathbb{A}$, we show that the description translates easily to the geometric representation:

THEOREM 9 ([20, Thm. 13]). *Let (S, M) be a marked surface, where S is a disc and $|M| \geq 4$. Let Δ be a triangulation of (S, M), and let $A(\Delta)$ be the cluster-tilted algebra corresponding to Δ. Let γ be an arc in (S, M) which is not in Δ, and let $N(\gamma)$ be the indecomposable module corresponding to γ. The following are equivalent:*

1. *The $A(\Delta)$-module $N(\gamma)$ has infinite projective dimension.*
2. *There is an internal triangle $\alpha\beta\delta$ of Δ, where α is a predecessor to γ and γ is a predecessor to β with respect to clockwise rotation at $\gamma(0)$. We say that γ is* trapped *by a triangle.*

The idea is illustrated in Figure 2.

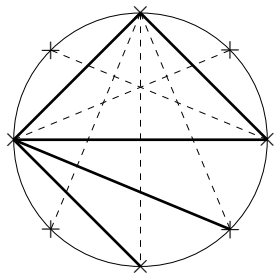


FIGURE 2. An indecomposable object of infinite projective dimension is represented by an arc (dashed lines) "trapped" by the triangulation (bold lines)

Using this description, we show that the number of non-isomorphic indecomposables in $\mathcal{P}_{\leq 1}$ is dependent only the number of nodes n and the number of directed three-cycles t in the quiver corresponding to Λ.

THEOREM 10 ([20, Thm. 16]). *Let Λ be a cluster-tilted algebra of type A_n. Let t be the number of three-cycles in the corresponding quiver. Then the number of indecomposable objects in $\mathcal{P}_{\leq 1}$ is*

$$|\mathcal{P}_{\leq 1}| = \frac{n(n+1)}{2} - nt + \frac{t(t-1)}{2}$$

Since the pair (n, t) is a derived invariant [12], this means that $|\mathcal{P}_{\leq 1}|$ is also a derived invariant.

We show how to calculate the right approximations from $\mathcal{P}_{\leq 1}$:

THEOREM 11 ([20, Thm. 18]). *Let $Z \in \operatorname{mod}\Lambda$ be indecomposable. Then*

$$\mathfrak{r}_{\leq 1}Z = \begin{cases} Z & \text{if } \operatorname{pd} Z \leq 1 \\ Y & \text{if } \operatorname{pd} Z = \infty, \exists \text{ irreducible } Y \to Z, \operatorname{pd} Y \leq 1 \\ \tau Z & \text{otherwise,} \end{cases}$$

where Y is unique up to isomorphism if it exists.

It follows that the approximation $\mathfrak{r}_{\leq 1}\tau X = I \oplus \tau_{\leq 1}X$ is either indecomposable or zero. In either case $I = 0$ and we get that

$$\tau_{\leq 1}X = \mathfrak{r}_{\leq 1}\tau X = \begin{cases} \tau X & \text{if } \operatorname{pd}\tau X \leq 1 \\ Y & \text{if } \operatorname{pd}\tau X = \infty, \exists \text{ irred. } Y \to \tau X, \operatorname{pd} Y \leq 1 \\ \tau^2 X & \text{otherwise} \end{cases}$$

where Y is unique up to isomorphism if it exists.

References

[1] C. Amiot. "Cluster categories for algebras of global dimension 2 and quivers with potential". In: *Ann. Inst. Fourier (Grenoble)* 59.6 (2009), pp. 2525–2590.

[2] C. Amiot. "On the structure of triangulated categories with finitely many indecomposables". In: *Bull. Soc. Math. France* 135.3 (2007), pp. 435–474.

[3] H. Asashiba. "On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras". In: *Algebr. Represent. Theory* 6.4 (2003), pp. 427–447.

[4] H. Asashiba. "The derived equivalence classification of representation-finite selfinjective algebras". In: *J. Algebra* 214.1 (1999), pp. 182–221.

[5] I. Assem and A. Skowroński. "Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$". In: *Math. Z.* 195.2 (1987), pp. 269–290.

[6] I. Assem et al. "Gentle algebras arising from surface triangulations". In: *Algebra Number Theory* 4.2 (2010), pp. 201–229.

[7] M. Auslander and S. O. Smalø. "Almost split sequences in subcategories". In: *J. Algebra* 69.2 (1981), pp. 426–454.

[8] L. Beaudet, T. Brüstle, and G. Todorov. "Projective dimension of modules over cluster-tilted algebras". In: *Algebr. Represent. Theory* 17.6 (2014), pp. 1797–1807.

[9] O. Bretscher, C. Läser, and C. Riedtmann. "Self-injective and simply connected algebras". In: *Manuscripta Math.* 36.3 (1981/82), pp. 253–307.

[10] T. Brüstle and J. Zhang. "On the cluster category of a marked surface without punctures". In: *Algebra Number Theory* 5.4 (2011), pp. 529–566.

[11] A. B. Buan, R. J. Marsh, and I. Reiten. "Cluster-tilted algebras". In: *Trans. Amer. Math. Soc.* 359.1 (2007), 323–332 (electronic).

[12] A. B. Buan and D. F. Vatne. "Derived equivalence classification for cluster-tilted algebras of type A_n". In: *J. Algebra* 319.7 (2008), pp. 2723–2738.

[13] A. B. Buan et al. "Tilting theory and cluster combinatorics". In: *Adv. Math.* 204.2 (2006), pp. 572–618.

[14] P. Caldero, F. Chapoton, and R. Schiffler. "Quivers with relations arising from clusters (A_n case)". In: *Trans. Amer. Math. Soc.* 358.3 (2006), pp. 1347–1364.

[15] S. Fomin and A. Zelevinsky. "Cluster algebras. I. Foundations". In: *J. Amer. Math. Soc.* 15.2 (2002), 497–529 (electronic).

[16] B. Grimeland and K. M. Jacobsen. "Abelian quotients of triangulated categories". In: *J. Algebra* 439 (2015), pp. 110–133.

[17] B. Grimeland and K. M. Jacobsen. "Realizing orbit categories as stable module categories - a complete classification". arXiv:1508.02970. 2015.

[18] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*. Vol. 119. Cambridge University Press, Cambridge, 1988.

[19] T. Holm and P. Jørgensen. "Realizing higher cluster categories of Dynkin type as stable module categories". In: *Q. J. Math.* 64.2 (2013), pp. 409–435.

[20] K. M. Jacobsen. "Modules of finite projective dimension over a cluster-tilted algebra". Unpublished manuscript.

[21] M. Kashiwara and P. Schapira. *Sheaves on manifolds*. Vol. 292. With a chapter in French by Christian Houzel. Springer-Verlag, Berlin, 1990.

[22] B. Keller. "On triangulated orbit categories". In: *Doc. Math.* 10 (2005), pp. 551–581.

[23] B. Keller and I. Reiten. "Cluster-tilted algebras are Gorenstein and stably Calabi-Yau". In: *Adv. Math.* 211.1 (2007), pp. 123–151.

[24] M. Kleiner. "Approximations and almost split sequences in homologically finite subcategories". In: *J. Algebra* 198.1 (1997), pp. 135–163.

[25] M. Kleiner and E. Perez. "Computation of almost split sequences with applications to relatively projective and prinjective modules". In: *Algebr. Represent. Theory* 6.3 (2003), pp. 251–284.

[26] S. Koenig and B. Zhu. "From triangulated categories to abelian categories - cluster tilting in a general framework". In: *Mathematische Zeitschrift* 258.1 (2008), pp. 143–160.

[27] C. Riedtmann. "Algebren, Darstellungsköcher, Überlagerungen und zurück". In: *Comment. Math. Helv.* 55.2 (1980), pp. 199–224.

[28] C. Riedtmann. "Representation-finite self-injective algebras of class A_n". In: *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*. Vol. 832. 1980, pp. 449–520.

[29] J.-L. Verdier. "Des catégories dérivées des catégories abéliennes". In: *Astérisque* 239 (1996). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis, xii+253 pp. (1997).

PAPER 1

Abelian quotients of triangulated categories

Benedikte Grimeland and Karin M. Jacobsen

Journal of Algebra 439 (2015), 110–133

ABELIAN QUOTIENTS OF TRIANGULATED CATEGORIES

BENEDIKTE GRIMELAND AND KARIN M. JACOBSEN

Abstract. We study abelian quotient categories $\mathcal{A} = \mathcal{T}/\mathcal{J}$, where $\mathcal{T}$ is a triangulated category and $\mathcal{J}$ is an ideal of $\mathcal{T}$. Under the assumption that the quotient functor is cohomological we show that it is representable and give an explicit description of the functor. We give technical criteria for when a representable functor is a quotient functor, and a criterion for when $\mathcal{J}$ gives rise to a cluster-tilting subcategory of $\mathcal{T}$. We show that the quotient functor preserves the AR-structure. As an application we show that if $\mathcal{T}$ is a finite 2-Calabi-Yau category, then with very few exceptions $\mathcal{J}$ is a cluster-tilting subcategory of $\mathcal{T}$.

1. Introduction

In the literature there are several known methods for forming a triangulated category given an abelian category. Given an abelian category $\mathcal{A}$ one can form the homotopy category $\mathcal{K}(\mathcal{A})$ and the derived category $\mathcal{D}(\mathcal{A})$, both of which are triangulated, along with their bounded versions. Orbit categories $\mathcal{D}^b(\mathcal{A})/F$ are known [10] to be triangulated when $\mathcal{A}$ is hereditary and F is a suitable autoequivalence. The stable module category of a selfinjective algebra is also triangulated.

With the introduction of cluster algebras [7] and cluster-tilting theory [5], cluster-tilting subcategories (or maximal 1-orthogonal subcategories) have been defined, see [8]. In [11], Koenig and Zhu show that the quotient of any triangulated category by a cluster-tilting subcategory is abelian. However not all triangulated categories contain a cluster-tilting subcategory, but they may still admit an abelian quotient (for an example, see [11]). It is also known that for the cluster categories of coherent sheaves on weighted projective lines it is possible to obtain an abelian quotient by factoring out morphisms, without any objects being sent to zero [3] .

Consider the orbit category $\mathcal{D}^b(kQ)/\Sigma$, where Q is a Dynkin diagram and Σ is the suspension functor. This category has the same (finite) number of isomorphism classes of indecomposable objects as $\operatorname{mod} kQ$, but has a greater number of irreducible morphisms. This motivates us to find out if we can factor out an ideal to obtain an abelian category, possibly without sending any non-zero objects to zero. Both of the examples mentioned will be revisited in detail in Section 4.

Factoring out an ideal from the cluster category of a hereditary algebra has been studied [4]. All known abelian quotients of these cluster categories arise from factoring out cluster-tilting subcategories. We show that in the finite case these are in fact all possible abelian quotient categories.

In Section 2 we define some notation and show that in the finite case, if an abelian quotient category exists, it has enough projectives.

In Section 3 we study a quotient functor from a triangulated category to an abelian category with projective generator. We show that it is representable and naturally equivalent to an explicitly described functor.

Section 4 contains the main result:

THEOREM 1. $\mathrm{Hom}_{\mathcal{T}}(T,-)$ *is a quotient functor from a triangulated category* $\mathcal{T}$ *if and only if the following two criteria are satisfied*

- **a:** *For all right minimal morphisms* $T_1 \to T_0$, *where* $T_0, T_1 \in \operatorname{add} T$, *all triangles* $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ *satisfy* $\mathrm{Hom}_{\mathcal{T}}(T,h) = 0$.
- **b:** *For all indecomposable* T*-supported objects* X *there exists a triangle* $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ *with* $T_1, T_0 \in \operatorname{add} T$ *and* $\mathrm{Hom}_{\mathcal{T}}(T,h) = 0$.

In Section 5 we show that if it exists, the AR-structure is preserved by the quotient functor.

In Section 6 we discuss the special case of triangulated categories with Calabi-Yau dimension 2. We show

THEOREM 2. *Let* $\mathcal{T}$ *be a 2-CY connected triangulated category with finitely many isomorphism classes of indecomposable objects. If* T *is an object in* $\mathcal{T}$ *such that* $\mathrm{Hom}_{\mathcal{T}}(T,-) : \mathcal{T} \to \operatorname{mod}\Gamma$ *is full and dense, then* T *is either Schurian or a 2-cluster-tilting object in* $\mathcal{T}$.

We would like to thank Professor Steffen Oppermann for many helpful discussions during the work on this paper.

2. Background

SETUP. k is a field and $\mathcal{T}$ is a Hom-finite Krull-Schmidt triangulated k-category. Σ is the suspension functor of $\mathcal{T}$.

By $\mathcal{J}$ we denote an ideal of $\mathcal{T}$. The quotient category $\mathcal{T}/\mathcal{J}$ has the same objects as $\mathcal{T}$, and has morphisms $\mathrm{Hom}_{\mathcal{T}/\mathcal{J}}(X,Y) = \mathrm{Hom}_{\mathcal{T}}(X,Y)/\mathcal{J}(X,Y)$.

By construction the projection functor $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{J}$ is full and dense. We also assume it to be cohomological.

Note that the properties of being Hom-finite and a k-category carries over from $\mathcal{T}$ to the quotient $\mathcal{T}/\mathcal{J}$. The property of being a Krull-Schmidt category is also inherited by the quotient category. The proof is a slightly adapted version of the proof found in [9], taking into account we do not assume that the ideal $\mathcal{J}$ always contains objects.

LEMMA 3. *Let* $\mathcal{T}$ *be a triangulated Krull-Schmidt* k*-category, and let* $\mathcal{J}$ *be an ideal in* $\mathcal{T}$. *Then the quotient category* $\mathcal{T}/\mathcal{J}$ *is also a Krull-Schmidt category.*

PROOF. Let $\overline{X}$ be an indecomposable non-zero object of $\mathcal{T}/\mathcal{J}$. Then the preimage X in $\mathcal{T}$ can be decomposed into a finite direct sum of indecomposable objects: $X = \oplus_{i=1}^{n} X_i$. Let $e_i : X \xrightarrow{\rho_i} X_i \xrightarrow{\iota_i} X$ be the canonical morphisms in $\mathcal{T}$ for $i \in \{1,\dots,n\}$, and denote by $\overline{e_i}$ the image of e_i in $\mathcal{T}/\mathcal{J}$.

Since $\overline{X}$ is indecomposable, all except one of the e_i has to be such that $\overline{e_i} = 0$. Therefore we may assume that $\overline{e_1} \neq 0$ and $\overline{e_i} = 0$ for $i \in \{2,\dots,n\}$. Note that $\overline{\rho_i}$ is an epimorphism, since $\rho_i \circ \iota_i = 1_{X_i}$ so that $\overline{\rho_i} \circ \overline{\iota_i} = \overline{\rho_i \circ \iota_i} = \overline{1_{X_i}} = 1_{\overline{X_i}}$. Therefore $\overline{\iota_i \circ \rho_i} = 0$ means that $\iota_i \in \mathcal{J}$. However, since $\iota_i \in \mathcal{J}$ and $\overline{\iota_i \circ \rho_i} = 0$ this means that we also have $\rho_i \in \mathcal{J}$. Then $1_{\overline{X_i}} = 0$ and so $\overline{X_i} = 0$. Looking at the endomorphism ring of $\overline{X}$ we then have that

$$\mathrm{End}_{\mathcal{T}/\mathcal{J}}(\overline{X}) = \mathrm{End}_{\mathcal{T}/\mathcal{J}}(\overline{X_i})$$

which is a local ring. □

For more details we refer the reader to [9], Section 2 and 3.

Let $\mathcal{A}$ be an abelian Hom-finite Krull-Schmidt k-category with finitely many isomorphism classes of indecomposable objects. We call a projective object P in $\mathcal{A}$ a projective generator if for any object X in $\mathcal{A}$ there is an epimorphism $P^n \twoheadrightarrow X$ for some $n \in \mathbb{N}$.

Our first aim is to establish that $\mathcal{A}$ has a projective generator. We need this when we study abelian quotients $\mathcal{A} = \mathcal{T}/\mathcal{J}$ in later sections. The existence of the projective generator was shown for the case when all objects of $\mathcal{A}$ has finite length by Deligne in [6]. We do not know that we have finite length yet, but we do know that $\mathcal{A}$ is a Krull-Schmidt category. Therefore we give a different proof. We will use the Harada-Sai lemma for the proof, so we need to recall the standard definition of length in a category.

DEFINITION 4. *An object X in an abelian category $\mathcal{A}$ has finite length if there exists a finite chain of subobjects*

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_{n-1} \subsetneq X_n = X$$

such that each quotient X_{i+1}/X_i is a simple object.

We also define a different measure on the indecomposable objects of $\mathcal{A}$. This will help us show that every object in A has finite length. By $\operatorname{Ind}\mathcal{A}$ we denote the set isomorphism classes of indecomposable objects in $\mathcal{A}$.

DEFINITION 5. *Let X be an indecomposable object in $\mathcal{A}$. We define*

$$\hat{l}(X) = \sum_{I \in \operatorname{Ind}\mathcal{A}} \dim_k \operatorname{Hom}_{\mathcal{A}}(I, X).$$

Since $\mathcal{A}$ is Hom-finite and there are finitely many isomorphism classes of indecomposables, $\hat{l}(X)$ must be a finite number.

LEMMA 6. *Let X and Y be objects in $\mathcal{A}$. If there exists a proper monomorphism $i : X \to Y$ then $\hat{l}(X) < \hat{l}(Y)$.*

PROOF. Assume that $i : X \to Y$ is a proper monomorphism. For any I, the induced morphism $\operatorname{Hom}_{\mathcal{A}}(I, X) \to \operatorname{Hom}_{\mathcal{A}}(I, Y)$ is an inclusion. Therefore $\dim \operatorname{Hom}_{\mathcal{A}}(I, X) \leq \dim \operatorname{Hom}_{\mathcal{A}}(I, Y)$ for all I, and $\hat{l}(X) \leq \hat{l}(Y)$.

The identity 1_Y cannot factor through i, as i is assumed not to split. Therefore there is at least one indecomposable summand Y' of Y such that $1_{Y'} : Y' \hookrightarrow Y$ does not factor through $i : X \to Y$. Therefore $\dim \operatorname{Hom}_{\mathcal{A}}(Y', X) < \dim \operatorname{Hom}_{\mathcal{A}}(Y', Y)$, and we must have $\hat{l}(X) < \hat{l}(Y)$. □

THEOREM 7. *Let X be an object in an abelian Krull-Schmidt* Hom*-finite k-category with finitely many indecomposable objects. Then X has finite length.*

PROOF. Consider a finite chain of subobjects of X

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_{n-1} \subsetneq X_n = X$$

where not all quotients X_{i+1}/X_i are necessarily simple objects. Choose a non-simple quotient X_{j+1}/X_j. Let Z be a nonzero, proper subobject of X_{j+1}/X_j. Consider the following commutative diagram with short exact rows.

$$\begin{array}{ccccc} X_j & \hookrightarrow & Y & \twoheadrightarrow & Z \\ \| & & \downarrow & & \downarrow \\ X_j & \hookrightarrow & X_{j+1} & \twoheadrightarrow & X_{j+1}/X_j \end{array}$$

The object Y is the pullback of $Z \to X_{j+1}/X_j$ and $X_{j+1} \to X_{j+1}/X_j$. We see that Y is such that $X_j \subseteq Y \subseteq X_{j+1}$. We now need to show the inclusions to be proper inclusions.

If $X_j = Y$, then $Z \to X_{j+1}/X_j$ is an epimorphism, contradicting the choice of Z. If $Y = X_{j+1}$, the exact sequence in the top row of the diagram would force $Z = 0$, which again contradicts the choice of Z.

Hence we find a refined finite chain

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_j \subsetneq Y \subsetneq X_{j+1} \subsetneq \ldots \subsetneq X_{n-1} \subsetneq X_n = X.$$

If the quotients of this chain are still not simple, the process can be repeated. However, since $\hat{l}(X)$ is finite and $\hat{l}(X_i) < \hat{l}(X_{i+1})$, we can only do a finite number of iterations of the process before reaching a chain where all the quotients are simple. Thus any object has a finite composition series and also finite length. □

Our aim now is to show that there are enough projectives in the category $\mathcal{A}$. In order to achieve this the following lemma will be useful:

LEMMA 8. *(Harada-Sai)*
Consider a chain of length 2^n of non-isomorphisms f_i between indecomposable objects of maximal length n:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots X_{2^n-1} \xrightarrow{f_{2^n}} X_{2^n}.$$

Then the composition $f_{2^n} f_{2^n-1} \cdots f_2 f_1$ is zero.

A proof of this lemma, which is also valid in abelian categories can for example be found in [16].

For each indecomposable object X of $\mathcal{A}$ we will show that there is a projective object with an epimorphism to X by iterating a certain process, which will build a tree with X as the root-node. Let N be the maximal length of any indecomposable object of $\mathcal{A}$. Then by the Harada-Sai lemma, any chain of 2^N non-isomorphisms in $\mathcal{A}$ is zero.

THEOREM 9. *An abelian category $\mathcal{A}$ with finitely many isomorphism classes of indecomposable objects has enough projectives.*

PROOF. Let X be an indecomposable, non-zero object of $\mathcal{A}$ which is not projective. Let X^+ be an object in $\mathcal{A}$ with an epimorphism f^+ to X which is not split. If X^+ is projective, we are done. If X^+ is not projective, decompose X^+ into a finite sum of indecomposable objects:

$$X^+ = \bigoplus_{i=1}^{m} X_i$$

with morphisms $f_i : X_i \to X$. Note that each morphism $f_i \in \mathrm{Rad}(X_i, X)$.

Now, consider each summand X_i. If X_i is projective, we take no further action. Otherwise, we can again find an object X_i^+ of $\mathcal{A}$ with a non-split epimorphism $f_i^+ : X_i^+ \to X_i$. The objects X_i^+ can be decomposed into finite sums of indecomposable objects again, and we iterate the process.

This iteration process builds a directed tree, where each node is an indecomposable object of $\mathcal{A}$, and each edge is a radical morphisms. We continue the iteration process until all leaf nodes of the tree are either projective indecomposable objects, or at a branch of length 2^N. The sum of the compositions of morphisms along all paths from the leaf nodes to the root nodes is an epimorphism by construction. Consider this morphism:

$$\bigoplus_{\text{leaf nodes}} X_{\text{leaf}} \xrightarrow{g} X.$$

If there are no projective leaf nodes in the tree, g is a composition of 2^N radical morphisms, and thus zero. Since it is an epimorphism, $X = 0$, which is a contradiction of the initial assumptions.

Let P be the sum of all projective leaf nodes occurring in the tree, we now know $P \neq 0$. Consider the inclusion i from P to the sum of the leaf nodes. It is easy to see that $g \circ i$ is an epimorphism from a projective object to X. □

3. The quotient functor is representable

In the remaining sections we assume that $\mathcal{T}/\mathcal{J}$ is an abelian category denoted by $\mathcal{A}$. As established in the previous section $\mathcal{A}$ has a projective generator when it is a finite category. For the rest of the article, neither $\mathcal{A}$ nor $\mathcal{T}$ are required to be finite. However, we require $\mathcal{A}$ to have a projective generator P.

In this section we define T as the minimal preimage of P in $\mathcal{T}$. We first show that the functor $\mathrm{Hom}_{\mathcal{T}}(T, -)$ takes the ideal $\mathcal{J}$ to zero in the module category $\mathrm{mod}\,\mathrm{End}_{\mathcal{T}}(T)^{^{op}}$. Second, we show that the k-algebras $\Gamma \coloneqq \mathrm{End}_{\mathcal{T}}(T)^{^{op}}$ and $\Lambda \coloneqq \mathrm{End}_{\mathcal{A}}(P)^{^{op}}$ are isomorphic. Finally, the main result of the section is proved, namely that the quotient functor π is naturally isomorphic to $\mathrm{Hom}_{\mathcal{T}}(T, -)$.

Definition 10. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{J}$ an ideal in $\mathcal{T}$ such that $\mathcal{A} = \mathcal{T}/\mathcal{J}$ is an abelian category with a basic projective generator P. Let π be the quotient functor from $\mathcal{T}$ to $\mathcal{A}$, and assume π is cohomological. We define the minimal preimage of P in $\mathcal{T}$ to be the basic object $T \in \mathcal{T}$ such that:*

- $\pi(T) = P$
- *for all indecomposable summands T' of T we have $\pi(T') \neq 0$.*

The following lemma will prove useful in the remaining sections.

Lemma 11. *Let $f : X \to Y$ be a morphism in $\mathcal{T}$. Then*

1. *if Y is indecomposable and $\pi(Y) \neq 0$, then f is a split epimorphism if and only if $\pi(f)$ is a split epimorphism.*
2. *if X is indecomposable and $\pi(X) \neq 0$, then f is a split monomorphism if and only if $\pi(f)$ is a split monomorphism.*

Proof. We only prove the first statement, as the second is dual.

If f is a split epimorphism, then there exists a morphism $g : Y \to X$ such that $fg = 1_Y$. But then $\pi(f)\pi(g) = \pi(1_Y) = 1_\pi(Y)$, so $\pi(f)$ is a split epimorphism.

If $\pi(f)$ is a split epimorphism, there exists a morphism $g' : \pi(Y) \to \pi(X)$ such that $\pi(f)g' = 1_{\pi(Y)}$. Since π is a full functor, there exists a morphism $g : Y \to X$ with $\pi(g) = g'$. Since $fg \in \mathrm{End}_{\mathcal{T}}(Y)$, which is a local ring, fg is either nilpotent or an isomorphism. As $\pi(fg) = \pi(f)g' = 1_{\pi(Y)}$, it clearly cannot be nilpotent. Hence fg is an isomorphism, and thus f is split epimorphism. □

For the rest of this section we fix P as the projective generator of $\mathcal{A}$, and we fix T to be the minimal preimage of P in $\mathcal{T}$. From now on we will denote a morphism $\mathrm{Hom}_{\mathcal{T}}(T, f)$ by $\overline{f}$.

Lemma 12. *Let $f \in \mathcal{J}$. Then* $\mathrm{Hom}_{\mathcal{T}}(T, f) = \bar{f} = 0$.

Proof. Assume that we have $\bar{f} \neq 0$, where $f : X \to Y$. For at least one indecomposable $T' \in \mathrm{add}\,T$, there exists at least one map $g : T' \to X$ such that the composition $T' \xrightarrow{g} X \xrightarrow{f} Y$ is non-zero. Since $f \in \mathcal{J}$, we also have that $fg \in \mathcal{J}$. The morphism $T' \xrightarrow{fg} Y$ can be completed to the triangle $Z \xrightarrow{h} T' \xrightarrow{fg} Y \to \Sigma Z$.

Since π is cohomological, we get the following exact sequence in $\mathcal{A}$:

$$\pi(Z) \xrightarrow{\pi(h)} \pi(T') \xrightarrow{\pi(fg)=0} \pi(Y).$$

Since $fg \in \mathcal{J}$ we have $\pi(fg) = 0$ in $\mathcal{A}$, so $\pi(h)$ is an epimorphism. Since T' is an indecomposable summand of T we have that $\pi(T')$ is an indecomposable projective in $\mathcal{A}$, and hence $\pi(h)$ is split epi. By Lemma 11, h is split epi, giving a morphism u such that $1_{T'} = hu$.

From the distinguished triangle $Z \xrightarrow{h} T' \xrightarrow{fg} Y \to \Sigma Z$ we see that the composition $fgh = 0$. By composing with u, we get that $0 = fghu = fg$, which is a contradiction. Hence $\bar{f} = 0$. □

LEMMA 13. *Let* $\Gamma = \mathrm{End}_{\mathcal{T}}(T)^{op}$ *and* $\Lambda = \mathrm{End}_{\mathcal{A}}(P)^{op}$. *Then* Γ *and* Λ *are isomorphic as* k*-algebras.*

PROOF. We know that π is a full and dense k-functor between $\mathcal{T}$ and $\mathcal{A}$. Hence it induces an algebra epimorphism

$$\pi : \Gamma \to \mathrm{End}_{\mathcal{A}}(\pi(T))^{op} = \Lambda.$$

It remains to show that π is a monomorphism as well.

Let $f \in \Gamma = \mathrm{End}_{\mathcal{T}}(T)^{op}$ be such that $\tilde{\pi}(f) = 0$ in $\mathcal{A}$. Then f is in the ideal $\mathcal{J}$, and hence $\mathrm{Hom}_{\mathcal{T}}(T, f) = 0$ by Lemma 12. This means that for any $g \in \Gamma$ we must have $gf = 0$. In particular, $f = 1_T f = 0$. Hence $\ker(\tilde{\pi}) = 0$ and $\Gamma \cong \Lambda$ as k-algebras. □

From [13] it is known that since $\mathcal{A}$ has a projective generator P, there is an equivalence of categories $\mathcal{A} \cong \operatorname{mod} \Lambda = \operatorname{mod} \mathrm{End}_{\mathcal{A}}(P)^{op}$. From the previous result we now know that $\operatorname{mod} \Gamma \cong \operatorname{mod} \Lambda$. That is, we have two functors π and $\mathrm{Hom}_{\mathcal{T}}(T, -)$ such that $\pi, \mathrm{Hom}_{\mathcal{T}}(T, -) : \mathcal{T} \to \operatorname{mod} \Gamma$. Next we show that these two functors are naturally isomorphic.

THEOREM 14. *Let* $\pi : \mathcal{T} \to \mathcal{A}$ *be a quotient functor from a triangulated category to an abelian category. Let* P *be the projective generator of* $\mathcal{A}$*, and let* T *be its minimal preimage in* $\mathcal{T}$.

Then π *is naturally isomorphic to* $\mathrm{Hom}_{\mathcal{T}}(T, -)$.

PROOF. By [13], the equivalence of categories between $\mathcal{A}$ and $\operatorname{mod} \Lambda$ is given by $\mathrm{Hom}_{\mathcal{A}}(P, -)$.

Let X be an object in $\mathcal{T}$. Consider the map

$$\eta : \mathrm{Hom}_{\mathcal{T}}(T, X) \to \mathrm{Hom}_{\mathcal{A}}(P, \pi(X))$$

induced by π (recall that $P = \pi(T)$). This is an epimorphism, since π is a full functor. Let $g \in \mathrm{Hom}_{\mathcal{T}}(T, X)$. If $\pi(g) = 0$, then $g \in \mathcal{J}$, so $\mathrm{Hom}_{\mathcal{T}}(T, g) = 0$. Hence $g = 0$, and η is an isomorphism.

For any object X in $\mathcal{T}$ we thus know that

$$\pi(X) \cong \mathrm{Hom}_{\mathcal{A}}(P, \pi(X)) \cong \mathrm{Hom}_{\mathcal{C}}(T, X).$$

We will show that this is a natural transformation.

Let $f : X \to Y$ be a morphism in $\mathcal{T}$. Consider the following diagram:

$$\begin{array}{ccccc}
\pi(X) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(P,-)} & \mathrm{Hom}_{\mathcal{A}}(P,\pi(X)) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{T}}(T,X) \\
\downarrow{\scriptstyle \pi(f)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{A}}(P,\pi(f))} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{T}}(T,f)} \\
\pi(Y) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(P,-)} & \mathrm{Hom}_{\mathcal{A}}(P,\pi(Y)) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{T}}(T,Y)
\end{array}$$

The first square commutes because $\mathrm{Hom}_{\mathcal{A}}(P,\pi(f)) = \mathrm{Hom}_{\mathcal{A}}(P,-)\circ\pi(f)$. The second square commutes by the functoriality of π. Hence we have defined a natural transformation. Since the map for each object is an isomorphism, it is also a natural isomorphism. □

4. When is $\mathrm{Hom}_{\mathcal{T}}(T,-)$ a quotient functor?

In the previous section, we showed that the quotient functor is representable. This poses the question of when representable functors are quotient functors.

In this section we give our main result. It concerns technical conditions on an object T that are equivalent to $\mathrm{Hom}_{\mathcal{T}}(T,-)$ being a quotient functor (i.e. full and dense).

We start by giving two useful lemmas. The first is well known, and the proof in [2, ch. II. 2] extends our case. For more details see e.g. [12]. Recall that $\Gamma = \mathrm{End}_{\mathcal{T}}(T)^{op}$.

Lemma 15. *Let T be an arbitrary object in $\mathcal{T}$. $\mathrm{Hom}_{\mathcal{T}}(T,-)$ induces an equivalence*

$$\mathrm{add}\, T \cong \mathrm{proj}\, \Gamma$$

The second lemma is an extension of the first. Recall that we write $\mathrm{Hom}_{\mathcal{T}}(T,f) = \overline{f}$.

Lemma 16. *Let T be an arbitrary object in $\mathcal{T}$. Let $T_0 \in \mathrm{add}\, T$, let $X, Y \in \mathcal{T}$ and let $T_0 \xrightarrow{f} X$, $T_0 \xrightarrow{g} Y$ and $X \xrightarrow{h} Y$ be morphisms.*

If $\overline{g} = \overline{h}\circ\overline{f}$, then $g = hf$.

Proof. We have assumed that $\mathcal{T}$ is Krull-Schmidt. Let

$$T_0 = \bigoplus_{j=0}^{n} T_0^j$$

be the decomposition of T_0 where the T_0^j are all indecomposable. We rewrite $f = [f_0 \cdots f_n]$ and $g = [g_0 \cdots g_n]$ with respect to this composition. Note that if for each j we have $g_j = hf_j$, then $g = hf$, so fix one j.

Since $\overline{g} = \overline{hf}$, and $\mathrm{Hom}_{\mathcal{T}}(T,-)$ distributes over direct sums, we know that $\overline{g}_j = \overline{hf}_j$. Since T_0^j is an indecomposable element in $\mathrm{add}\, T$, T_0^j must be a summand of T. Let $i : T_0^j \to T$ and $p : T \to T_0^j$ be the direct sum injection and projection respectively. We then construct the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{T}}(T,T_0^j) & \xrightarrow{\overline{f}_j} & \mathrm{Hom}_{\mathcal{T}}(T,X) & \xrightarrow{\overline{h}} & \mathrm{Hom}_{\mathcal{T}}(T,Y) \\
\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{T}}(i,T_0)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{T}}(i,X)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{T}}(i,Y)} \\
\mathrm{Hom}_{\mathcal{T}}(T_0^j,T_0^j) & \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(T_0^j,f_j)} & \mathrm{Hom}_{\mathcal{T}}(T_0^j,X) & \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(T_0^j,h)} & \mathrm{Hom}_{\mathcal{T}}(T_0^j,Y)
\end{array}$$

We know that $p \in \mathrm{Hom}_{\mathcal{T}}(T, T_0^j)$. By chasing p through the diagram, we get that

$$\begin{aligned} g_j &= g_j p i \\ &= [\mathrm{Hom}_{\mathcal{T}}(i, Y)\overline{g}_j](p) = [\mathrm{Hom}_{\mathcal{T}}(i, Y)\overline{hf}_j](p) \\ &= [\mathrm{Hom}_{\mathcal{T}}(T_0^j, h)\,\mathrm{Hom}_{\mathcal{T}}(T_0^j, f_i)\,\mathrm{Hom}_{\mathcal{T}}(i, T_0)](p) \\ &= h f_j p i = h f_j \end{aligned}$$

□

An object X is called *T-supported* if $\mathrm{Hom}_{\mathcal{T}}(T, X) \neq 0$.

We are now ready to prove one of our main theorems.

THEOREM 17. *$\mathrm{Hom}_{\mathcal{T}}(T, -)$ is a quotient functor, i.e. full and dense, if and only if the following two conditions are satisfied*

- **a:** *For all right minimal morphisms $T_1 \to T_0$, where $T_0, T_1 \in \operatorname{add} T$, all triangles $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ satisfy $\mathrm{Hom}_{\mathcal{T}}(T, h) = 0$.*
- **b:** *For all indecomposable, T-supported objects X there exists a triangle $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ with $T_1, T_0 \in \operatorname{add} T$ and $\mathrm{Hom}_{\mathcal{T}}(T, h) = 0$.*

PROOF. Assume first that **a** and **b** hold. We will first show that this means that $\mathrm{Hom}_{\mathcal{T}}(T, -)$ is dense, and then that it is full.

a implies dense: Let X be an arbitrary object in $\operatorname{mod}\Gamma$. We need to find an object Y in $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(T, Y) \cong X$. We have the following minimal projective presentation of X

$$\mathrm{Hom}_{\mathcal{T}}(T, T_1) \xrightarrow{\overline{f}} \mathrm{Hom}_{\mathcal{T}}(T, T_0) \xrightarrow{g} X \to 0,$$

where $T_1, T_0 \in \operatorname{add} T$, by the equivalence $\operatorname{proj}\Gamma \cong \operatorname{add} T$.

Here, $\overline{f}$ is the composition of the monomorphism $\operatorname{Ker} g \to \mathrm{Hom}_{\mathcal{T}}(T, T_0)$ and the projective cover $\mathrm{Hom}_{\mathcal{T}}(T, T_1) \to \operatorname{Ker} g$. The former is right minimal because it is a monomorphism. The latter is right minimal because it is a projective cover (see e.g. [2, thm I.4.1]). Hence $\overline{f}$ is right minimal.

The morphism $f : T_1 \to T_0$ in $\mathcal{T}$ is right minimal by virtue of the equivalence $\operatorname{proj}\Gamma \cong \operatorname{add} T$. We can complete f to an distinguished triangle

$$T_1 \xrightarrow{f} T_0 \to Y \to \Sigma T_1.$$

Applying $\mathrm{Hom}_{\mathcal{T}}(T, -)$ and using **a** we get the following exact sequence:

$$\mathrm{Hom}_{\mathcal{T}}(T, T_1) \xrightarrow{\overline{f}} \mathrm{Hom}_{\mathcal{T}}(T, T_0) \to \mathrm{Hom}_{\mathcal{T}}(T, Y) \to 0.$$

It follows from the uniqueness of cokernels that $X \cong \mathrm{Hom}_{\mathcal{T}}(T, Y)$, and thus $\mathrm{Hom}_{\mathcal{T}}(T, -)$ is dense.

b implies full: Let X and Y be two objects in $\mathcal{T}$. Let $f : \mathrm{Hom}_{\mathcal{T}}(T, X) \to \mathrm{Hom}_{\mathcal{T}}(T, Y)$ be an arbitrary morphism in $\operatorname{mod}\Gamma$. We need to find a morphism $f' : X \to Y$ in $\mathcal{T}$ such that $\overline{f'} = f$. Since the functor $\mathrm{Hom}_{\mathcal{T}}(T, -)$ distributes over direct sums, we assume without loss of generality that X and Y are indecomposable.

If X or Y is not T-supported, then obviously $f = 0$, so $0 : X \to Y$ maps to f. In the following we therefore assume $\mathrm{Hom}_{\mathcal{T}}(T, X) \neq 0$ and $\mathrm{Hom}_{\mathcal{T}}(T, Y) \neq 0$

Using property **b**, we define the following exact triangles:

$$\begin{array}{ccccccc} T_1 & \longrightarrow & T_0 & \xrightarrow{g} & X & \longrightarrow & \Sigma T_1 \\ T_1' & \longrightarrow & T_0' & \xrightarrow{g'} & Y & \longrightarrow & \Sigma T_1' \end{array}$$

By applying the functor $\mathrm{Hom}_{\mathcal{T}}(T,-)$ and using the comparison theorem for projective resolutions, we get the following diagram in $\operatorname{mod}\Gamma$ with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{T}}(T,T_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(T,T_0) & \xrightarrow{\overline{g}} & \mathrm{Hom}_{\mathcal{T}}(T,X) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \overline{u}} & & \downarrow{\scriptstyle f} & & \\
\mathrm{Hom}_{\mathcal{T}}(T,T_1') & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(T,T_0') & \xrightarrow{\overline{g'}} & \mathrm{Hom}_{\mathcal{T}}(T,Y) & \longrightarrow & 0
\end{array}$$

By the equivalence between $\operatorname{add} T$ and the projective objects in $\operatorname{mod}\Gamma$, we can lift the left commutative square in the diagram back to $\mathcal{T}$. The commutative square in $\mathcal{T}$ can be completed to a morphism of triangles.

$$\begin{array}{ccccccc}
T_1 & \longrightarrow & T_0 & \xrightarrow{g} & X & \longrightarrow & \Sigma T_1 \\
\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f'} & & \downarrow \\
T_1' & \longrightarrow & T_0' & \xrightarrow{g'} & Y & \longrightarrow & \Sigma T_1'
\end{array}$$

Applying $\mathrm{Hom}_{\mathcal{T}}(T,-)$ once again, we get the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{T}}(T,T_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(T,T_0) & \xrightarrow{\overline{g}} & \mathrm{Hom}_{\mathcal{T}}(T,X) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \overline{u}} & & \downarrow{\scriptstyle \overline{f'}} & & \\
\mathrm{Hom}_{\mathcal{T}}(T,T_1') & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(T,T_0') & \xrightarrow{\overline{g'}} & \mathrm{Hom}_{\mathcal{T}}(T,Y) & \longrightarrow & 0
\end{array}$$

Since

$$f\overline{g} = \overline{g'}\overline{u} = \overline{f'}\overline{g}$$

and $\overline{g}$ is an epimorphism, it follows that $f = \overline{f'}$, and we have shown $\mathrm{Hom}_{\mathcal{T}}(T,-)$ to be dense, thus finishing the first implication.

Next, assume that $\mathrm{Hom}_{\mathcal{T}}(T,-)$ is a full and dense functor.

Full and dense implies a: Let $f : T_1 \to T_0$ be a right minimal morphism between objects in $\operatorname{add} T$. Complete the morphism to the following triangle:

$$T_1 \xrightarrow{f} T_0 \xrightarrow{g} X \xrightarrow{h} \Sigma T_1$$

We want to show that $\mathrm{Hom}_{\mathcal{T}}(T,h) = 0$. Use $\mathrm{Hom}_{\mathcal{T}}(T,-)$ on this triangle to obtain the diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{T}}(T,T_1) & \xrightarrow{\overline{f}} & \mathrm{Hom}_{\mathcal{T}}(T,T_0) & \xrightarrow{\overline{g}} & \mathrm{Hom}_{\mathcal{T}}(T,X) & \xrightarrow{\overline{h}} & \mathrm{Hom}_{\mathcal{T}}(T,\Sigma T_1) \\
 & & & {\scriptstyle \overline{u}}\searrow \quad \nearrow{\scriptstyle \overline{v}} & & & \\
 & & & \mathrm{Hom}_{\mathcal{T}}(T,Y) & & &
\end{array}$$

where $\mathrm{Hom}_{\mathcal{T}}(T,Y) = \operatorname{Im}\overline{g}$. The image and the maps all have preimages in $\mathcal{T}$, since $\mathrm{Hom}_{\mathcal{T}}(T,-)$ is full and dense. We assume (without loss of generality) that all summands of Y are T-supported. We want to show that Y is a direct summand of X, and we start by showing that $\mathrm{Hom}_{\mathcal{T}}(T,Y)$ is a summand of $\mathrm{Hom}_{\mathcal{T}}(T,X)$.

By $\overline{u} \circ \overline{f} = 0$ and Lemma 16, we get that $uf = 0$.

Using $\mathrm{Hom}_{\mathcal{T}}(-,Y)$ on the triangle, we get the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(g,Y)} \mathrm{Hom}_{\mathcal{T}}(T_0,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(f,Y)} \mathrm{Hom}_{\mathcal{T}}(T_1,Y)$$

Starting with $u \in \mathrm{Hom}_{\mathcal{T}}(T_0,Y)$ we get that $\mathrm{Hom}_{\mathcal{T}}(f,Y)(u) = uf = 0$, so u must be in the image of $\mathrm{Hom}_{\mathcal{T}}(g,Y)$. Thus there exists some $w \in \mathrm{Hom}_{\mathcal{T}}(X,Y)$ with $wg = u$.

Since

$$\overline{w} \circ \overline{v} \circ \overline{u} = \overline{w} \circ \overline{g} = \overline{u},$$

and $\overline{u}$ is an epimorphism, we must have

$$\overline{w} \circ \overline{v} = 1_{\mathrm{Hom}_{\mathcal{T}}(T,Y)}.$$

Thus $\mathrm{Hom}_{\mathcal{T}}(T,Y)$ is a direct summand of $\mathrm{Hom}_{\mathcal{T}}(T,X)$.

In order to show that Y is a direct summand of X, we first show that u is a left minimal morphism. Use the functor $\mathrm{Hom}_{\mathcal{T}}(-,Y)$ to obtain

$$\mathrm{Hom}_{\mathcal{T}}(Y,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(u,Y)} \mathrm{Hom}_{\mathcal{T}}(T_0,Y)$$

in $\operatorname{mod}\operatorname{End}_{\mathcal{T}} Y$. By [2, thm I.2.2], there exists a decomposition

$$\mathrm{Hom}_{\mathcal{T}}(Y,Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y_1,Y) \oplus \mathrm{Hom}_{\mathcal{T}}(Y_2,Y)$$

such that $u^* = \mathrm{Hom}_{\mathcal{T}}(u,Y)|_{\mathrm{Hom}_{\mathcal{T}}(Y_1,Y)}$ is a right minimal morphism and $\mathrm{Hom}_{\mathcal{T}}(u,Y)|_{\mathrm{Hom}_{\mathcal{T}}(Y_2,Y)} = 0$. The preimages in $\mathcal{T}$ exist by the equivalence between $\operatorname{add} Y$ and $\operatorname{proj}\operatorname{End}_{\mathcal{T}} Y$.

By the dual of Lemma 15, there exists a preimage u_1 of u^* such that $u_1 = u|_{Y_1}$, and $u = \left(\begin{smallmatrix} u_1 \\ 0 \end{smallmatrix}\right)$. Suppose that for a morphism $x : Y_1 \to Y_1$, we have $xu_1 = u_1$. Then $\mathrm{Hom}_{\mathcal{T}}(u_1,Y)\,\mathrm{Hom}_{\mathcal{T}}(x,Y) = \mathrm{Hom}_{\mathcal{T}}(u_1,Y)$. We have that $u^* = \mathrm{Hom}_{\mathcal{T}}(u_1,Y)$ is right minimal; thus $\mathrm{Hom}_{\mathcal{T}}(x,Y)$ is an isomorphism. By (the dual of) Lemma 15, x is also an isomorphism. Consequently, u_1 is left minimal.

We know that $\overline{u} = \left(\begin{smallmatrix} \overline{u_1} \\ 0 \end{smallmatrix}\right)$. Since $\overline{u}$ is an epimorphism, Y_2 cannot be T-supported. By choice of Y we have $Y_2 = 0$. Therefore $u = u_1$ is left minimal.

We have $wvu = wg = u$. Thus wv is an isomorphism and Y is a direct summand of X.

We rewrite the original triangle to

$$T_1 \xrightarrow{f} T_0 \xrightarrow{\left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right)} Y \oplus R \xrightarrow{\left(\begin{smallmatrix} h_Y & h_R \end{smallmatrix}\right)} \Sigma T_1 \tag{1}$$

where $h_Y = h|_Y$ and $h_R = h|_R$. The next step is to show that R is a direct summand of ΣT_1. Using $\mathrm{Hom}_{\mathcal{T}}(-,R)$ on the triangle, we get the following exact sequence:

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma T_1,R) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(\left(\begin{smallmatrix} h_Y & h_R \end{smallmatrix}\right),R)} \mathrm{Hom}_{\mathcal{T}}(Y \oplus R,R) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}\left(\left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right),R\right)} \mathrm{Hom}_{\mathcal{T}}(T_0,R)$$

The projection $p_R : Y \oplus R \to R$ is contained in $\mathrm{Hom}_{\mathcal{T}}(Y \oplus R,R)$. It is obviously in the kernel of $\mathrm{Hom}_{\mathcal{T}}\left(\left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right),R\right)$, so it must be in the image of $\mathrm{Hom}_{\mathcal{T}}\left(\left(\begin{smallmatrix} h_Y & h_R \end{smallmatrix}\right),R\right)$. Thus h_R is a split monomorphism, and we have $\Sigma T_1 = R \oplus S$. The one-sided inverse of h_R we denote as z.

Let $y \in \mathrm{End}_{\mathcal{T}}(\Sigma T_1)$ be such that $(\Sigma f) \circ y = \Sigma f$. Since Σ is an autoequivalence we have $f \circ \Sigma y = f$. Since f is right minimal, Σy is an isomorphism. But that means that y is an isomorphism as well. Thus Σf is right minimal, and so is $\overline{\Sigma f}$.

Now consider

$$\begin{aligned}\overline{\Sigma f} &= \overline{\Sigma f} \circ 1_{\mathrm{Hom}_{\mathcal{T}}(T,\Sigma T)} \\ &= \overline{\Sigma f} \circ \begin{pmatrix} \overline{h_R} \circ \overline{z} & 0 \\ 0 & 1_{\mathrm{Hom}_{\mathcal{T}}(T,S)} \end{pmatrix} = \overline{\Sigma f} \circ \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathrm{Hom}_{\mathcal{T}}(T,S)} \end{pmatrix}\end{aligned}$$

Consequently $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_{\mathrm{Hom}_{\mathcal{T}}(T,S)} \end{smallmatrix}\right)$ is an isomorphism. Thus R is not T-supported.

It follows that $\overline{h} = 0$, and **a** holds.

Full and dense implies b: Let X be a T-supported indecomposable object in $\mathcal{T}$. Consider its minimal projective presentation in $\operatorname{mod}\Gamma$:

$$\mathrm{Hom}_{\mathcal{T}}(T,T_1) \xrightarrow{\overline{f}} \mathrm{Hom}_{\mathcal{T}}(T,T_0) \xrightarrow{\overline{g}} \mathrm{Hom}_{\mathcal{T}}(T,X)$$

We know that $\overline{f}$ is right minimal, and by Lemma 15, so is f. Complete $T_1 \xrightarrow{f} T_0$ to a triangle in $\mathcal{T}$

$$T_1 \xrightarrow{f} T_0 \xrightarrow{u} Y \xrightarrow{v} \Sigma T_1,$$

where $\overline{v} = 0$ by condition **a**.

We use $\mathrm{Hom}_{\mathcal{T}}(T,-)$ on the triangle to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{T}}(T,T_1) & \xrightarrow{\overline{f}} & \mathrm{Hom}_{\mathcal{T}}(T,T_0) & \xrightarrow{\overline{u}} & \mathrm{Hom}_{\mathcal{T}}(T,Y) & \xrightarrow{\overline{v}=0} & \mathrm{Hom}_{\mathcal{T}}(T,\Sigma T_1) \\ \| & & \| & & & & \\ \mathrm{Hom}_{\mathcal{T}}(T,T_1) & \xrightarrow{\overline{f}} & \mathrm{Hom}_{\mathcal{T}}(T,T_0) & \xrightarrow{\overline{g}} & \mathrm{Hom}_{\mathcal{T}}(T,X) & \xrightarrow{0} & 0 \end{array}$$

By uniqueness of cokernels we must have $\mathrm{Hom}_{\mathcal{T}}(T,X) \cong \mathrm{Hom}_{\mathcal{T}}(T,Y)$. Thus there are maps $x : X \to Y$ and $y : Y \to X$ such that

$$\overline{x} \circ \overline{y} = 1_{\mathrm{Hom}_{\mathcal{T}}(T,Y)} \text{ and } \overline{y} \circ \overline{x} = 1_{\mathrm{Hom}_{\mathcal{T}}(T,X)}.$$

Once again, the preimages of the isomorphisms exist due to $\mathrm{Hom}_{\mathcal{T}}(T,-)$ being full.

We have that $yx \in \mathrm{End}_{\mathcal{T}}(X)$, which is a local ring since X was assumed to be indecomposable. Since $\overline{yx}$ is an isomorphism, yx clearly cannot be nilpotent, so it must be an isomorphism. Thus X is a direct summand of Y, and we write $Y = X \oplus R$.

We are now in the same situation as diagram (1), and we can use the same argument to show that $\Sigma T_1 = R \oplus S$. By [14, Lemma 1.2.4],

$$T_1 \xrightarrow{f} T_0 \to X \to \Sigma T_1$$

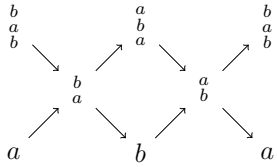
is a distinguished triangle. It fulfills the requirements of **b**. □

Example 18. We revisit the last example presented in [11]. This is an example of an abelian quotient of a triangulated category with no cluster-tilting subcategories, hence not covered by the theory developed in [11]. Let $A = kQ/I$ be the self-injective algebra given by the quiver

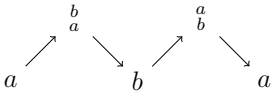
$$Q : a \underset{\beta}{\overset{\alpha}{\rightleftarrows}} b$$

and the relations $\alpha\beta\alpha,\ \beta\alpha\beta$.

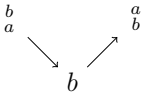
The AR-quiver of $\operatorname{mod} A$ is



where the first and the last columns are identified. The stable module category $\underline{\mathrm{mod}}A$ is triangulated with suspension functor Ω^{-1}, the cozysygy. Its AR-quiver is:



As explained in detail in [11] this triangulated category does not have any cluster-tilting subcategories. An abelian quotient can be formed, by factoring out $\mathrm{add}(a)$. This abelian category has the following AR-quiver:



The projective generator of this category is $\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b$. The preimage of the projective generator is $\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b$ considered as an object in $\underline{\mathrm{mod}}A$. The functor $\mathrm{Hom}_{\underline{\mathrm{mod}}A}(\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b, -)$ gives rise to the an abelian category with the same AR-quiver as $\underline{\mathrm{mod}}A/\operatorname{add}(a)$.

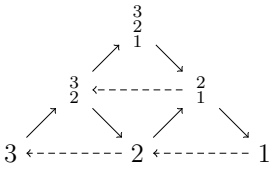
There is only one right minimal morphism between indecomposable objects of $\mathrm{add}(\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b)$, namely $\begin{smallmatrix} b \\ a \end{smallmatrix} \to b$. The triangle

$$\begin{smallmatrix} b \\ a \end{smallmatrix} \to b \to \begin{smallmatrix} a \\ b \end{smallmatrix} \to \Omega^{-1} \begin{smallmatrix} b \\ a \end{smallmatrix}$$

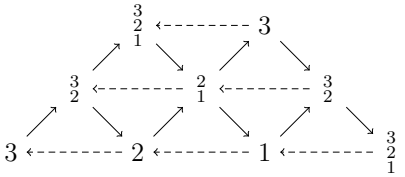
shows that condition **a** is fulfilled.

The triangle also shows that condition **b** is fulfilled for $\begin{smallmatrix} a \\ b \end{smallmatrix}$. For $\begin{smallmatrix} b \\ a \end{smallmatrix}$ and b, condition **b** is fulfilled by the completion of the identity morphism to a triangle. Hence Theorem 17 implies that $\mathrm{Hom}_{\underline{\mathrm{mod}}A}(\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b, -)$ is full and dense.

Example 19. We revisit the class of examples mentioned in the introduction. We will study $\mathcal{D}^b(kQ)/\Sigma$ where kQ is a Dynkin diagram. Specifically we consider the quiver A_3 with orientation $1 \to 2 \to 3$. The AR-quiver of $\mathrm{mod}\, kA_3$ is:



The AR-quiver of the triangulated category $\mathcal{T} = \mathcal{D}^b(kA_3)/\Sigma$ is:



where we include some objects twice to indicate which objects are identified.

This category does not have any cluster-tilting subcategories, so it is not possible to attain an abelian quotient by the method used in [11]. However it is in some sense already close to being an abelian category; the difference is just two irreducible maps! We let $T = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 3$. Applying the functor $\mathrm{Hom}_{\mathcal{T}}(\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 3, -)$ we return to an abelian category equivalent to the module category. The functor is easily seen to be full and dense directly.

There are two right minimal morphisms between indecomposable summands of T, namely $3 \to \begin{smallmatrix}3\\2\end{smallmatrix}$ and $\begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix}$. Thus the triangles

$$3 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 2 \to 3$$

and

$$\begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to 1 \to \begin{smallmatrix}3\\2\end{smallmatrix}$$

show that part **a** of Theorem 17 is satisfied.

The objects 3, $\begin{smallmatrix}3\\2\end{smallmatrix}$ and $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$ are in $\operatorname{add} T$, so the completions of the identity maps on these objects fulfills condition **b** of Theorem 17. The above triangles fulfill condition **b** for the objects 1 and 2. The only object that remains is $\begin{smallmatrix}2\\1\end{smallmatrix}$, and in this case the triangle

$$3 \to \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 3$$

satisfies condition **b**. Hence by Theorem 17 the functor $\operatorname{Hom}_{\mathcal{T}}(\begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3, -)$ is full and dense.

5. AR-structure in the abelian quotient

In this section we show that the AR-structure of $\mathcal{T}$ is preserved as much as one can hope for in the abelian quotient. Let

$$\Delta : \tau X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} \Sigma\tau X$$

be an AR-triangle in $\mathcal{T}$. Assume that none of the objects in Δ are sent to zero, that τX is not sent to an injective and that X is not sent to a projective. Then we show that Δ is sent to an AR-sequence in $\mathcal{A}$.

Before proceeding we need to define two new notions. We call a category $\mathcal{T}$ locally finite if for each indecomposable object X of $\mathcal{T}$ there are only finitely many isomorphism classes of indecomposable objects Y such that $\operatorname{Hom}_{\mathcal{T}}(X, Y) \neq 0$. A Serre functor $\mathbb{S}$ is an autoequivalence $\mathbb{S} : \mathcal{T} \to \mathcal{T}$ such that for each object X in $\mathcal{T}$ there is an isomorphism

$$D \operatorname{Hom}_{\mathcal{T}}(X, -) \cong \operatorname{Hom}_{\mathcal{T}}(-, \mathbb{S}X)$$

where D is the duality $\operatorname{Hom}_k(-, k)$.

The following theorem is due to [15].

THEOREM 20. *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt k-category. Then $\mathcal{T}$ has AR-triangles if and only if $\mathcal{T}$ has a Serre functor $\mathbb{S}$.*

Many triangulated categories have a Serre functor. For example Amiot showed in [1] that any locally finite Krull-Schmidt triangulated k-category has a Serre functor.

We assume in the following that $\mathcal{T}$ has AR-triangles, and we proceed to study the AR-structure in the abelian quotient category. As before we let $T \in \mathcal{T}$ be an object such that $\operatorname{Hom}_{\mathcal{T}}(T, -) : \mathcal{T} \to \operatorname{mod} \Gamma$ is full and dense, where $\Gamma = \operatorname{End}(T)^{op}$.

LEMMA 21. *Let*

$$\Delta : \tau X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} \Sigma\tau X$$

be an AR-triangle in $\mathcal{T}$. Then $\overline{h} = 0$ if and only if $X \notin \operatorname{add} T$

PROOF. Assume first that $\overline{h} = 0$. Then $\overline{g}$ is an epimorphism. If $X \in \operatorname{add} T$, Then $\operatorname{Hom}_{\mathcal{T}}(T, X)$ is projective, and the epimorphism $\overline{g}$ is split. By Lemma 11, g is also a split epimorphism. However this leads to $h = 0$ which is a contradiction to Δ being an AR-triangle, hence $X \notin \operatorname{add} T$.

Now assume that $X \notin \operatorname{add} T$. Then $\operatorname{Hom}_{\mathcal{T}}(T, X)$ is not projective. If X is not T-supported then clearly $\overline{h} = 0$. If on the other hand X is T-supported then there exists a non-split epimorphism

$$\operatorname{Hom}_{\mathcal{T}}(T, A) \xrightarrow{\overline{u}} \operatorname{Hom}_{\mathcal{T}}(T, X),$$

giving rise to a morphism $A \xrightarrow{u} X$ which is not a split epimorphism. Since g is an almost split morphism, there exists a morphism $v : A \to Y$ such that $u = g \circ v$. We have $\overline{u} = \overline{g} \circ \overline{v}$, where $\overline{u}$ is an epimorphism. Hence $\overline{g}$ is an epimorphism and so $\overline{h} = 0$.

□

LEMMA 22. *If $\Sigma^{-1}X \in \operatorname{add} T$ then $\operatorname{Hom}_{\mathcal{T}}(T, \tau X)$ is injective and nonzero.*

PROOF. We have

$$\operatorname{Hom}_{\mathcal{T}}(T, \tau X) \cong \operatorname{Hom}_{\mathcal{T}}(T, \mathbb{S}\Sigma^{-1}X) \cong D \operatorname{Hom}_{\mathcal{T}}(\Sigma^{-1}X, T).$$

Hence if $\Sigma^{-1}X \in \operatorname{add} T$ then $\operatorname{Hom}_{\mathcal{T}}(\Sigma^{-1}X, T) \in \operatorname{proj} \Gamma^{op}$. It follows that $D \operatorname{Hom}_{\mathcal{T}}(\Sigma^{-1}X, T)$ is injective in $\operatorname{mod} \Gamma$. □

LEMMA 23. *Let Δ be the AR-triangle*

$$\Delta : \tau X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} \Sigma\tau X.$$

Assume that X and τX are both T-supported, with $\Sigma^{-1}X \notin \operatorname{add} T$, and $X \notin \operatorname{add} T$. Then the functor $\operatorname{Hom}_{\mathcal{T}}(T, -)$ takes the AR-triangle Δ to the following AR-sequence in $\operatorname{mod} \Gamma$:

$$0 \to \operatorname{Hom}_{\mathcal{T}}(T, \tau X) \xrightarrow{\overline{f}} \operatorname{Hom}_{\mathcal{T}}(T, Y) \xrightarrow{\overline{g}} \operatorname{Hom}_{\mathcal{T}}(T, X) \to 0 \tag{2}$$

PROOF. From Lemma 21 it is clear that $\operatorname{Hom}_{\mathcal{T}}(T, -)$ takes Δ to the short exact sequence (2). Note that $\operatorname{Hom}_{\mathcal{T}}(T, \tau X)$ and $\operatorname{Hom}_{\mathcal{T}}(T, X)$ are indecomposable since X and τX are indecomposable. Since $\operatorname{Hom}_{\mathcal{T}}(T, \tau X)$ is indecomposable it its enough to show that $\overline{g}$ is right almost split [2, thm V.1.14].

If $\overline{g}$ is a split epimorphism then by Lemma 11 we have that g is a split epimorphism, which contradicts the fact that Δ is an AR-triangle. Hence $\overline{g}$ is not a split epimorphism.

Assume that $u : W \to X$ is a morphism such that $\overline{u}$ is not a split epimorphism. By Lemma 11, u is not a split epimorphism.

Since u is not a split epimorphism and g is right almost split, there is a morphism $v : A \to Y$ such that $u = g \circ v$. Applying $\operatorname{Hom}_{\mathcal{T}}(T, -)$ to this we obtain exactly what we want, which is a morphism $\overline{v} : \operatorname{Hom}_{\mathcal{T}}(T, A) \to \operatorname{Hom}_{\mathcal{T}}(T, Y)$ such that $\overline{u} = \overline{g} \circ \overline{v}$. □

6. Cluster-tilting objects and the 2-Calabi-Yau case

In this section we will work under the additional assumption that $\mathcal{T}$ is a 2-Calabi-Yau category. We give two notable results. First we show for which objects T applying the functor $\operatorname{Hom}_{\mathcal{T}}(T, -)$ coincides with the cluster-tilting case studied in [11]. Then we apply this result, to show that in many finite categories the only possible way to obtain an abelian quotient is with the previously known method from [11].

We start by defining a cluster-tilting object.

DEFINITION 24. *An object T in a triangulated category $\mathcal{T}$ is called a* cluster-tilting object *if*

$$\operatorname{add}(T) = \{X \mid \operatorname{Hom}_{\mathcal{T}}(T, \Sigma X) = 0\} = \{X \mid \operatorname{Hom}_{\mathcal{T}}(X, \Sigma T) = 0\}.$$

Cluster-tilting objects turn out to be very closely related to the objects where T is such that $\operatorname{Hom}_{\mathcal{T}}(T, -)$ is a full and dense functor. In Section 4 we showed that $\operatorname{Hom}_{\mathcal{T}}(T, -)$ is full and dense if and only if condition **a** and **b** were satisfied. We consider the following, stronger, version of condition **b**:

b*: For all indecomposable objects X there exists a triangle

$$T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$$

with $T_1, T_0 \in \operatorname{add} T$ and $\operatorname{Hom}_{\mathcal{T}}(T, h) = 0$.

The difference from **b** is that we require existence of such a triangle not only for objects X that are T-supported, but for all objects.

THEOREM 25. *An object T in $\mathcal{T}$ is a cluster-tilting object if and only if **a** from Theorem 17 and **b*** are satisfied and furthermore:*

c: *if T' is an indecomposable summand of T, then $\Sigma T' \notin \operatorname{add} T$*

PROOF. Suppose **a**, **b*** and **c** holds, we need to show that T is cluster-tilting.

Assume $T' \in \operatorname{add} T$ is indecomposable. We need to show that $\Sigma T'$ is not T-supported, i.e $\operatorname{Hom}_{\mathcal{T}}(T, \Sigma T') = 0$. By **b*** a distinguished triangle

$$T_1 \to T_0 \to \Sigma T' \xrightarrow{f} \Sigma T_1$$

exists, where $T_1, T_0 \in \operatorname{add} T$ and $\overline{f} = 0$. We have that $f \neq 0$ by **c**. Since T' is indecomposable, $\Sigma^{-1} f : T' \to T_1$ is right minimal. By rotating the above triangle, we get the distinguished triangle

$$T' \xrightarrow{\Sigma^{-1} f} T_1 \to T_0 \xrightarrow{g} \Sigma T'$$

where $\overline{g} = 0$ by **a**. This means that the following sequence is exact:

$$\operatorname{Hom}_{\mathcal{T}}(T, T_0) \xrightarrow{\overline{g}=0} \operatorname{Hom}_{\mathcal{T}}(T, \Sigma T') \xrightarrow{\overline{f}=0} \operatorname{Hom}_{\mathcal{T}}(T, \Sigma T_1).$$

Consequently $\operatorname{Hom}_{\mathcal{T}}(T, \Sigma T') = 0$.

Conversely, suppose X is such that $\operatorname{Hom}_{\mathcal{T}}(T, \Sigma X) = 0$. We need to show that X is in $\operatorname{add} T$. By **b*** there exists $T_0, T_1 \in \operatorname{add} T$ such that following triangle is distinguished:

$$T_1 \to T_0 \xrightarrow{0} \Sigma X \to \Sigma T_1.$$

The zero follows from the assumption on X.

Using the axioms for triangulated categories we see that the following triangle is also distinguished:

$$X \to T_1 \to T_0 \xrightarrow{0} \Sigma X.$$

By [14], this is a split triangle; thus $T_1 \cong X \oplus T_0$ and $X \in \operatorname{add} T$.

We now have shown that $\operatorname{add} T = \{X \mid \operatorname{Hom}_{\mathcal{T}}(T, \Sigma X) = 0\}$; the other equality in definition 24 can be shown similarly. Thus T must be cluster-tilting. Suppose now instead that T is a cluster-tilting object; we show **a**, **b*** and **c** in order.

Let $T_1 \to T_0$ be a right minimal morphism between objects in $\operatorname{add} T$, and complete this morphism to a triangle:

$$T_1 \to T_0 \to Y \xrightarrow{g} \Sigma T_1.$$

Since $\operatorname{Hom}_{\mathcal{T}}(T, \Sigma T_1) = 0$, we must also have $\overline{g} = 0$, and **a** holds.

Note that $\operatorname{add} T$ is a functorially finite subcategory; this is well known and not dependent on T being a cluster-tilting object. In particular $\operatorname{add} T$ is contravariantly finite. We now follow [11, thm 3.2] to show that condition **b*** holds.

Let X be an arbitrary object of $\mathcal{T}$. Let $f : T_0 \to X$ be a right $\operatorname{add} T$-approximation of X. We complete this to the triangle

$$Y \to T_0 \xrightarrow{f} X \xrightarrow{g} \Sigma Y.$$

Applying $\mathrm{Hom}_{\mathcal{T}}(T,-)$ we get the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(T,T_0) \xrightarrow{\overline{f}} \mathrm{Hom}_{\mathcal{T}}(T,X) \xrightarrow{\overline{g}} \mathrm{Hom}_{\mathcal{T}}(T,\Sigma Y) \to \mathrm{Hom}_{\mathcal{T}}(T,\Sigma T_0) \to \cdots$$

We have $\mathrm{Hom}_{\mathcal{T}}(T,\Sigma Y)=0$, since $\overline{f}$ is surjective and $\mathrm{Hom}_{\mathcal{T}}(T,\Sigma T_0)=0$. Consequently, $Y \in \operatorname{add} T$, and condition **b*** holds.

To show condition **c**, assume that T' is an indecomposable summand of T with $\Sigma T' \in \operatorname{add} T$. As $\mathrm{Hom}_{\mathcal{T}}(T,\Sigma T)=0$, we must have $\mathrm{Hom}_{\mathcal{T}}(T,\Sigma T')=0$. Since $\Sigma T' \in \operatorname{add} T$, this means that $\Sigma T'=0$. Hence $T'=0$. □

A triangulated category $\mathcal{T}$ with a Serre functor $\mathbb{S}$ is said to be d-Calabi-Yau if $\mathbb{S}=\Sigma^d$, and d is the smallest positive integer for which this holds. In particular 2-Calabi-Yau categories have been studied quite extensively. Examples of such categories include the classical cluster categories, $D^b(H)/\tau^-\Sigma$ where H is an hereditary algebra [5].

We will show that in finite 2-Calabi-Yau categories if $\mathrm{Hom}_{\mathcal{T}}(T,-)$ is full and dense, then in most cases T must be a cluster-tilting object. To do this we first need to study the structure of finite 2-Calabi-Yau categories.

LEMMA 26. *Let $\mathcal{T}$ be a connected 2-Calabi-Yau category with finitely many isomorphism classes of indecomposable objects. Let $T \in \operatorname{Ob}\mathcal{T}$ be a non-zero object such that condition **b** is satisfied. Then*

$$\operatorname{Ind}\mathcal{T} = \{X \in \operatorname{Ind}\mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(T,X) \neq 0\} \cup \{X \in \operatorname{Ind}\mathcal{T} \mid X \in \operatorname{add}\Sigma T\}.$$

PROOF. Let $\mathcal{D} = \{X \in \operatorname{Ind}\mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(T,X) \neq 0\} \cup \{X \in \operatorname{Ind}\mathcal{T} \mid X \in \operatorname{add}\Sigma T\}$. Note that $\mathcal{D}$ is non-empty, as all indecomposable summands of T are in $\mathcal{D}$.

We will show that for any object $X \in \mathcal{D}$ we can find a triangle

$$T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$$

such that $T_1, T_0 \in \operatorname{add} T$ and $\mathrm{Hom}_{\mathcal{T}}(T,h)=0$. If X is T-supported, this follows immediately from condition **b**. If not, then $X \in \operatorname{add}\Sigma T$, so $\Sigma^{-1}X \in \operatorname{add} T$. The following split triangle fulfills the conditions.

$$\Sigma^{-1}X \to 0 \to X \to X$$

Assume that $X \in \mathcal{D}$ and $Y \in \operatorname{Ind}\mathcal{T}$. Let $f : X \to Y$ be a non-zero morphism. We will show that $Y \in \mathcal{D}$. By the above we can form the following diagram:

$$\begin{array}{ccccccc} T_1 & \longrightarrow & T_0 & \xrightarrow{g} & X & \xrightarrow{h} & \Sigma T_1 \\ & & & & f\downarrow & & \\ & & & & Y & & \end{array}$$

If $gf \neq 0$, then Y is T-supported. Hence $Y \in \mathcal{D}$, and we are done.

If $gf=0$, then by the weak cokernel property of triangulated categories, there exists a morphism $f' : \Sigma T_1 \to Y$ such that $f'h=f$. Hence $\Sigma^{-1}Y$ is T-supported. By condition **b** we can form a distinguished triangle

$$T_1' \to T_0' \to \Sigma^{-1}Y \to \Sigma T_1',$$

with $T_1', T_0 \in \operatorname{add} T$. Hence the triangle

$$\Sigma T_1' \to \Sigma T_0' \to Y \xrightarrow{h'} \Sigma^2 T_1'$$

is distinguished. If $h'=0$, then the triangle splits, and Y is a summand of $\Sigma T_0'$. Hence $Y \in \operatorname{add}\Sigma T$.

If $h' \neq 0$, then

$$0 \neq \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^2 T) = \operatorname{Hom}_{\mathcal{T}}(Y, \mathbb{S}T) \cong D \operatorname{Hom}_{\mathcal{T}}(T, Y).$$

Hence Y is T-supported.

In [1], the author shows that any connected triangulated category with finitely many indecomposables has an AR-quiver of the form $\mathbb{Z}\Delta/G$, where Δ is a Dynkin diagram and G is a group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. By Corollary 6.3.3 in [1] $\mathcal{T}$ is an orbit category. Since, by the above, any τ-orbit of the AR-quiver of $\mathcal{T}$ contains an element of $\mathcal{D}$, we see that $\operatorname{Ind}\mathcal{T} = \mathcal{D}$. □

A consequence of this lemma is that for the connected 2-Calabi-Yau case **b*** is implied by **b**. We are now ready to prove the final theorem. Recall that an object X in a triangulated category $\mathcal{T}$ such that $\operatorname{End}_{\mathcal{T}}(X)^{op} \cong k$ is called a Schurian object.

THEOREM 27. *Let $\mathcal{T}$ be a 2-CY connected triangulated category with finitely many isomorphism classes of indecomposable objects. If T is an object in $\mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(T, -) : \mathcal{T} \to \operatorname{mod}\Gamma$ is full and dense, then T is either Schurian or a 2-cluster-tilting object in $\mathcal{T}$.*

PROOF. Condition **a** is satisfied, since $\operatorname{Hom}_{\mathcal{T}}(T, -)$ is full and dense. By the above, so is condition **b***. If T satisfies **c**, then by Lemma 25 T is a cluster-tilting object.

Assume that T does not satisfy condition **c**. We will show that $\operatorname{mod}\Gamma = \operatorname{mod} k$.

Let T' be an indecomposable summand of T such that $T', \Sigma T' \in \operatorname{add} T$. Since $\mathcal{T}$ is a 2-CY triangulated category, we have $\Sigma T' \cong \tau T'$. Therefore we have the following AR-triangle

$$\Delta : \tau T' \xrightarrow{f} E \xrightarrow{g} T' \xrightarrow{h} \Sigma\tau T'.$$

Applying $\operatorname{Hom}_{\mathcal{T}}(T, -)$ to the above AR-triangle yields the following long exact sequence:

$$\ldots \to \operatorname{Hom}_{\mathcal{T}}(T, \tau T') \xrightarrow{\overline{f}} \operatorname{Hom}_{\mathcal{T}}(T, E) \xrightarrow{\overline{g}} \operatorname{Hom}_{\mathcal{T}}(T, T') \to \ldots$$

where by the proof of Lemma 23 the map $\overline{g}$ is right almost split.

Since T' in $\operatorname{add} T$, we have that $\operatorname{Hom}_{\mathcal{T}}(T, T')$ is projective. Hence there exists a right almost split monomorphism $r : \operatorname{Rad}_{\Gamma}\operatorname{Hom}(T, T') \to \operatorname{Hom}_{\mathcal{T}}(T, T')$. Since $\overline{g}$ and r are both right almost split, there are morphisms

$$a : \operatorname{Rad}_{\Gamma}(T, T') \to \operatorname{Hom}_{\mathcal{T}}(T, E) \text{ and } a' : \operatorname{Hom}_{\mathcal{T}}(T, E) \to \operatorname{Rad}_{\Gamma}(T, T')$$

such that $\overline{g}a = r$ and $ra' = \overline{g}$. Hence

$$ra'a = \overline{g}a = r.$$

Since r is a monomorphism, $a'a = 1_{\operatorname{Rad}_{\Gamma}(T,T')}$. Thus $\operatorname{Rad}_{\Gamma}(T, T')$ must be a direct summand of $\operatorname{Hom}_{\mathcal{T}}(T, E)$. We rewrite in terms of this direct summand:

$$\operatorname{Hom}_{\mathcal{T}}(T, E) = \operatorname{Rad}_{\Gamma}(T, T') \oplus \operatorname{Hom}_{\mathcal{T}}(T, U)$$

for some object U in $\mathcal{T}$. We rewrite the morphism $\overline{g} = \begin{pmatrix} r & u \end{pmatrix}$

We have

$$\begin{pmatrix} r & u \end{pmatrix} = \begin{pmatrix} r & u \end{pmatrix} \circ a \circ a' = \begin{pmatrix} r & 0 \end{pmatrix}.$$

Hence $u = 0$.

Let $R \in \mathcal{T}$ be the preimage of $\operatorname{Rad}_{\Gamma}(T, T')$, i.e. $\operatorname{Hom}_{\mathcal{T}}(T,R) = \operatorname{Rad}_{\Gamma}(T, T')$. Let g' be the preimage of r, so that $r = \overline{g'}$. Since $\operatorname{Hom}_{\mathcal{T}}(T, R)$ is a summand of $\operatorname{Hom}_{\mathcal{T}}(T, E)$ it is clear that R is a summand of E, that is $E = R \oplus V$ where $\operatorname{Hom}_{\mathcal{T}}(T, V) = \operatorname{Hom}_{\mathcal{T}}(T, U)$. Hence Δ can be written as

$$\tau T' \xrightarrow{\binom{f_R}{f_V}} R \oplus V \xrightarrow{(g' \; 0)} T' \to \Sigma\tau T'$$

which, by applying $\operatorname{Hom}_{\mathcal{T}}(T,-)$, is sent to the long exact sequence

$$\cdots \to \operatorname{Hom}_{\mathcal{T}}(T,\tau T') \xrightarrow{\binom{\overline{f_R}}{\overline{f_V}}} \begin{matrix}\operatorname{Hom}_{\mathcal{T}}(T,R)\\ \oplus \\ \operatorname{Hom}_{\mathcal{T}}(T,V)\end{matrix} \xrightarrow{(\overline{g'}\ 0)} \operatorname{Hom}_{\mathcal{T}}(T,T') \to \cdots$$

We know that $\overline{g'}$ is a monomorphism. Exactness of the sequence gives us that $\overline{g'} \circ \operatorname{Hom}_{\mathcal{T}}(T,f_R) = 0$ and hence $\operatorname{Hom}_{\mathcal{T}}(T,f_R) = 0$. Due to Lemma 16 we also have $f_R = 0$.

We have

$$\begin{pmatrix} 0 & 0 \\ 0 & 1_V \end{pmatrix} \begin{pmatrix} 0 \\ f_V \end{pmatrix} = \begin{pmatrix} 0 \\ f_V \end{pmatrix}.$$

Since $f = \left(\begin{smallmatrix} 0 \\ f_V \end{smallmatrix}\right)$ is left minimal, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_V \end{smallmatrix}\right)$ must be an automorphism on $R \oplus V$, so $R = 0$. Then $\operatorname{Rad}\operatorname{Hom}_{\mathcal{T}}(T,T') = \operatorname{Hom}_{\mathcal{T}}(T,R) = 0$, and $\operatorname{Hom}_{\mathcal{T}}(T,T')$ is a simple projective.

Consider Δ under $\operatorname{Hom}_{\mathcal{T}}(T,-)$:

$$\operatorname{Hom}_{\mathcal{T}}(T,\Sigma^{-1}T') \to \operatorname{Hom}_{\mathcal{T}}(T,\tau T) \to \operatorname{Hom}_{\mathcal{T}}(T,E) \xrightarrow{0} \operatorname{Hom}_{\mathcal{T}}(T,T')$$

The first morphism cannot be zero, as $\tau T \ncong E$. By Lemma 21, we must have $\Sigma^{-1}T' \in \operatorname{add} T$. By induction, for any $n \in \mathbb{N}$, we have $\Sigma^{-n}T' \in \operatorname{add} T$. In particular $\Sigma^{-2}T' = \Sigma^{-1}\tau^{-}T' \in \operatorname{add} T$. Hence, by Lemma 22, $\operatorname{Hom}_{\mathcal{T}}(T,T')$ is injective. Thus $\operatorname{Hom}_{\mathcal{T}}(T,T')$ is simple, projective and injective as a $\operatorname{End}_{\mathcal{T}}(T)^{^{op}}$-module.

We assumed $\mathcal{T}$, and thus also $\operatorname{mod}\Gamma$, to be a connected category. However, if $\operatorname{Hom}_{\mathcal{T}}(T,T')$ is a simple, projective and injective module, it must be the only indecomposable object in its connected component. It follows that $\operatorname{mod}\Gamma = \operatorname{mod} k$. □

References

[1] C. Amiot. "On the structure of triangulated categories with finitely many indecomposables". In: *Bull. Soc. Math. France* 135.3 (2007), pp. 435–474.

[2] M. Auslander, I. Reiten, and S. O. Smalø. *Representation Theory of Artin Algebras*. Cambridge University Press, 1995.

[3] M. Barot, D. Kussin, and H. Lenzing. "The cluster category of a canonical algebra". In: *Trans. Amer. Math. Soc.* 362.8 (2010), pp. 4313–4330.

[4] A. B. Buan, R. J. Marsh, and I. Reiten. "Cluster-tilted algebras". In: *Trans. Amer. Math. Soc.* 359.1 (2007), 323–332 (electronic).

[5] A. B. Buan et al. "Tilting theory and cluster combinatorics". In: *Adv. Math.* 204.2 (2006), pp. 572–618.

[6] P. Deligne. "Catégories tannakiennes". In: *The Grothendieck Festschrift, Vol. II*. Vol. 87. 1990, pp. 111–195.

[7] S. Fomin and A. Zelevinsky. "Cluster algebras. I. Foundations". In: *J. Amer. Math. Soc.* 15.2 (2002), 497–529 (electronic).

[8] O. Iyama. "Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories". In: *Proceedings of the 37th Symposium on Ring Theory and Representation Theory*. 2005, pp. 24–30.

[9] M. Jun-Ichi. "Derived categories with applications to representations of algebras". In: (2000).

[10] B. Keller. "On triangulated orbit categories". In: *Doc. Math.* 10 (2005), pp. 551–581.

[11] S. Koenig and B. Zhu. "From triangulated categories to abelian categories - cluster tilting in a general framework". In: *Mathematische Zeitschrift* 258.1 (2008), pp. 143–160.

[12] H. Krause. "Krull-Remak-Schmidt categories and projective covers".

[13] B. Mitchell. "The Full Imbedding Theorem". In: *American Journal of Mathematics* 86 (1964).

[14] A. Neeman. *Triangulated Categories*. 148. Princeton University Press, 2001.

[15] I. Reiten and M. Van den Bergh. "Noetherian hereditary abelian categories satisfying Serre duality". In: *J. Amer. Math. Soc.* 15.2 (2002), pp. 295–366.
[16] L. H. Rowen. *Ring Theory, Volume I.* Academic Press, 1988.

PAPER 2

Realizing orbit categories as stable module categories - a complete classification

Benedikte Grimeland and Karin M. Jacobsen

Preprint

REALIZING ORBIT CATEGORIES AS STABLE MODULE CATEGORIES - A COMPLETE CLASSIFICATION

BENEDIKTE GRIMELAND AND KARIN M. JACOBSEN

Abstract. We classify all triangulated orbit categories of path-algebras of Dynkin diagrams that are triangle equivalent to a stable module category of a representation-finite self-injective standard algebra. For each triangulated orbit category T we give an explicit description of a representation-finite self-injective standard algebra with stable module category triangle equivalent to T.

1. Introduction

Let k be an algebraically closed field. In this paper we will focus on two types of triangulated categories with finitely many isomorphism classes of indecomposable objects: triangulated orbit categories of path algebras of Dynkin quivers of type $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$, and stable module categories of representation-finite self-injective algebras of Dynkin tree type.

It is well-known that the stable module category of a self-injective algebra is a triangulated category. Riedtmann showed in [15] that all connected stable components of the AR-quiver of a representation-finite algebra are of Dynkin tree type. In two subsequent papers by Riedtmann [16] and Bretschner, Läser and Riedtmann [5], a complete classification of all representation-finite self-injective algebras of Dynkin type is given in terms of their quivers with relations. Continuing their work, Asashiba gives an invariant under derived equivalence for representation-finite self-injective algebras, based on the shape of the AR-quiver [2][3], called the type of the algebra. Algebras of one type are stably equivalent, as well as derived equivalent. He also determines which types contain standard algebras.

Triangulated orbit categories have been well studied, see e.g. [6], [7] and [13]. The orbit category of a triangulated category is not necessarily triangulated itself. However Keller showed that the orbit category $\mathcal{D}^b(H)/F$ is triangulated for H a hereditary algebra and with certain restrictions on the functor F [13]. In the case where $F = \tau^{-1}[m-1]$ for $m \in \mathbb{N}$, the orbit category $\mathcal{D}^b(H)/F$ is known as the m-cluster category $\mathcal{C}_m(H)$. The Calabi-Yau dimension of $\mathcal{C}_m(H)$ is m.

Keller and Reiten proved in [14] that any algebraic triangulated category of Calabi-Yau dimension m that contains an $(m-1)$-cluster-tilting object T with a hereditary endomorphism algebra H such that $\mathrm{Hom}(T, \Sigma^{-i}T) = 0$ for $i = 0, \ldots, m-2$ is triangle equivalent to the m-cluster category $\mathcal{C}_m(H)$.

More recently in [8], Dugas was able to determine the Calabi-Yau dimension to some of the stable module categories of representation-finite self-injective algebras.

The theorem of Keller and Reiten, combined with the Calabi-Yau dimensions calculated by Dugas, was used by Holm and Jørgensen [11] to classify which stable module categories of self-injective algebras are triangle equivalent to an m-cluster category.

We classify all triangulated orbit categories of path-algebras of Dynkin diagrams that are triangle equivalent to the stable module category of a representation-finite standard

self-injective algebra. This is done by showing that all self-injective algebras of standard type are triangle equivalent to orbit-categories (but not necessarily m-cluster categories), using a theorem by Amiot [1, thm 7.0.5]. Amiot's theorem reduces the problem from finding triangle equivalences to finding isomorphisms between translation quivers. In the last section we sum up the results, giving a complete overview of all the orbit categories that are possible to realize as a stable module category.

The following theorem sums up our results.

THEOREM 1. *Let Δ be a Dynkin diagram and let Φ be an autoequivalence such that $\mathcal{D}^b(k\Delta)/\Phi$ is triangulated. Let Λ be a self-injective algebra. The orbit category $\mathcal{C} = \mathcal{D}^b(k\Delta)/\Phi$ is triangle equivalent to $\underline{\mathrm{mod}}\Lambda$ precisely in the cases described in Table 1.*

2. Translation quivers and Automorphism groups

Translation quivers can be seen as an abstraction of the properties of AR-quivers. They are central in Riedtmann's classification of all self-injective algebra of Dynkin type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$. Background on translation quivers can be found in [10], [4].

DEFINITION 2. *We define a quiver $Q = (Q_0, Q_1, s, t)$ to consist of a set of vertices Q_0, a set of arrows Q_1, a source map s and a tail/sink map t.*

x^- and x^+: *For a vertex $x \in Q_0$ we denote by x^- the set of direct predecessors of x in Q, and by x^+ the set of direct successors of x in Q.*

Locally finite quiver: *A quiver Q is called locally finite if for each $x \in Q_0$ the sets x^- and x^+ are finite.*

Translation quiver: *Let θ be an injective map from a subset of Q_0 to Q_0. The pair (Q, θ) is called a translation quiver if the following is satisfied:*

1. *Q has no loops and no multiple arrows*
2. *For $x \in Q_0$ such that $\theta(x)$ is defined, we have that $x^- = \theta(x)^+$*

The map θ is called the translation of the translation quiver (Q, θ).

Stable translation quiver: *A translation quiver (Q,θ) is called stable if $\theta : Q_0 \to Q_0$ is a bijection.*

TABLE 1. The cases, up to triangulated equivalence, where $\mathcal{C} = \mathcal{D}^b(k\Delta)/\Phi$ is triangle equivalent to $\underline{\mathrm{mod}}\Lambda$

$\mathcal{C}$		Λ	*Sec.*
$\mathcal{D}^b(\mathbb{A}_r)/\tau^w$	$r \geq 1, w \geq 1$	*Nakayama alg.* $N_{w,r+1}$	*6.1*
$\mathcal{D}^b(\mathbb{A}_r)/\tau^w\phi$	$r = 2l+1, l \geq 1$ $w = rv,\ r \geq 1$	*Möbius alg.* $M_{l,v}$	*6.2*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w$	$r \geq 4, w = s(2r-3), s \geq 1$	$D_{n,s,1}$	*7.1*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w\phi$	$r \geq 4, w = s(2r-3), s \geq 1$	$D_{n,s,2}$	*7.2*
$\mathcal{D}^b(\mathbb{D}_4)/\tau^{5w}\rho$	$w \geq 1$	$D_{4,s,3}$	*7.3*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w$	$r = 3m, m \geq 2$ $w = s(2r-3)/3, s \geq 1, 3 \nmid s$	$D_{3m,\frac{s}{3},1}$	*7.4*
$\mathcal{D}^b(\mathbb{E}_r)/\tau^w$	$r = 6$ *and* $w = 11s$ $r = 7$ *and* $w = 17s\ \ s \geq 1$ $r = 8$ *and* $w = 29s$	$E_{r,s,1}$	*8.1*
$\mathcal{D}^b(\mathbb{E}_6)/\tau^w\phi$	$w = 11s, s \geq 1$	$E_{6,s,2}$	*8.2*

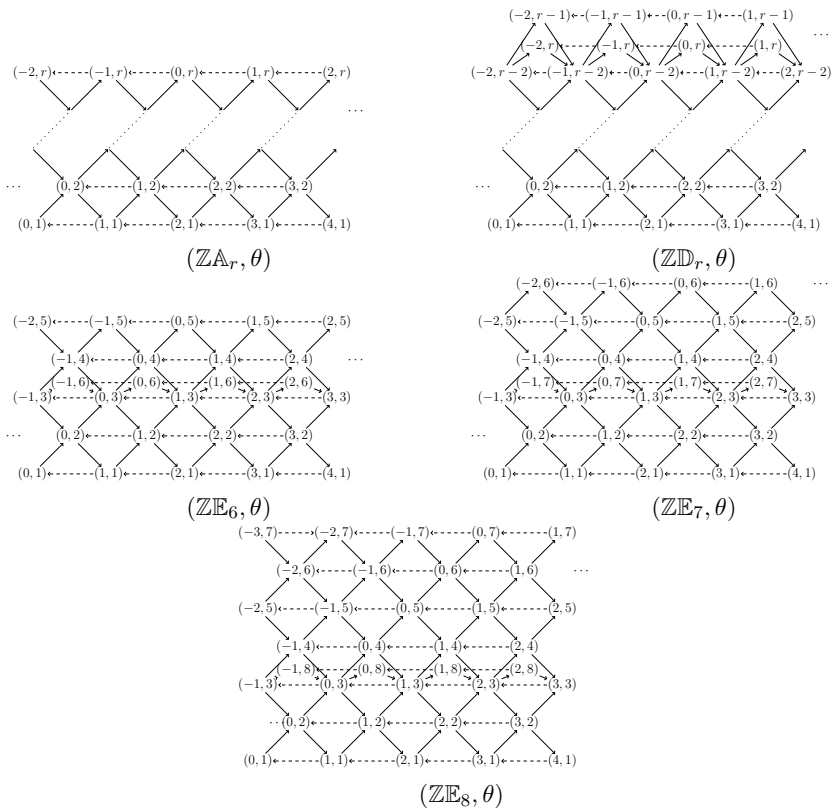


FIGURE 1. Translation quivers of Dynkin diagrams

Morphism of translation quivers: *Given two translation quivers* (Q, θ) *and* (Q', θ')*, a morphism* $f : (Q, \theta) \to (Q', \theta')$ *is a pair of maps* $f_0 : Q_0 \to Q'_0$ *and* $f_1 : Q_1 \to Q'_1$ *such that*

- *if* $\alpha \in Q_1$*, and* $\alpha : x \to y$ *then* $f_1(\alpha) \in Q'_1$ *is the arrow* $f_1(\alpha) : f_0(x) \to f_0(y)$*.*
- *for all vertices* $x \in Q$ *where* θ *is defined we have* $f_0(\theta(x)) = \theta'(f_0(x))$*.*

Our focus will be on translation quivers of the form $(\mathbb{Z}\Delta, \theta)$ for Δ of Dynkin type $\mathbb{A}, \mathbb{D}$ and $\mathbb{E}$. We use the following orientation on the Dynkin diagrams:

$\mathbb{A}_r$: $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow r-1 \longrightarrow r$

$\mathbb{D}_r$: $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow r-2 \begin{array}{l} \nearrow r \\ \searrow r-1 \end{array}$

$\mathbb{E}_r$: $\begin{array}{ccccccccccc} & & & & & & r & & & & \\ & & & & & & \uparrow & & & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & \cdots & \longrightarrow & r-2 & \longrightarrow & r-1 \end{array}$

The corresponding stable translation quivers $(\mathbb{ZA}_r, \theta)$, $(\mathbb{ZD}_r, \theta)$, $(\mathbb{ZE}_6, \theta)$ $(\mathbb{ZE}_7, \theta)$ and $(\mathbb{ZE}_8, \theta)$ with $\theta(p, q) = (p-1, q)$ are shown in Figure 1.

TABLE 2. The definition of the automorphism S in translation quivers of Dynkin type

Translation quiver	Automorphism S
$(\mathbb{Z}\mathbb{A}_n, \theta)$	$S(p,q) = (p+q, n+1-q)$
$(\mathbb{Z}\mathbb{D}_n, \theta)$ n even	$S = \theta^{-(n+1)}$
$(\mathbb{Z}\mathbb{D}_n, \theta)$ n odd	$S = \theta^{-(n+1)}\phi$, where ϕ is the automorphism on $(\mathbb{Z}\mathbb{D}_n, \theta)$ which exchanges the vertices (x,r) and $(x,r-1)$ for $x \in \mathbb{Z}$
$(\mathbb{Z}\mathbb{E}_6, \theta)$	$S = \phi\theta^{-6}$, where ϕ is the automorphism on $(\mathbb{Z}\mathbb{E}_6, \theta)$ exchanging $(x,5)$ with $(x+2,1)$ and $(y,4)$ with $(y+1,2)$ for $x, y \in \mathbb{Z}$
$(\mathbb{Z}\mathbb{E}_7, \theta)$	$S = \theta^{-9}$
$(\mathbb{Z}\mathbb{E}_8, \theta)$	$S = \theta^{-15}$

The set of automorphisms on a translation quiver (Q, θ) forms a group A. A *group of automorphisms* of (Q, θ) is a subgroup of A.

DEFINITION 3. *Let G be a group of automorphisms of a translation quiver (Q, θ). The group G is called admissible if each orbit of G intersects the set $\{x\} \cup x^+$ in at most one point, and intersects the set $\{x\} \cup x^-$ in at most one point for each $x \in Q_0$.*

Given a (stable) translation quiver (Q, θ) and an admissible group G of automorphisms of (Q, θ), one can form the (stable) translation quiver $(Q, \theta)/G$, where $(Q/G)_0 = Q_0/G$ and $(Q/G)_1 = Q_1/G$. The maps s, t and θ are induced by the corresponding maps of (Q, θ) [15]. For the stable translation quivers given by $\mathbb{Z}\Delta$, where Δ is a Dynkin diagram, all admissible automorphism groups are known [15][1].

Some examples of automorphisms on $(\mathbb{Z}\Delta, \theta)$ are given in Table 2. The action of S as given in the table is the same as that of the suspension functor on $\mathcal{D}^b(k\Delta)$; see also [1, sec. 2].

3. Orbit Categories

Throughout the rest of this paper we will assume k to be an algebraically closed field.

DEFINITION 4. *Given an additive category $\mathcal{A}$ and an automorphism $F : \mathcal{A} \to \mathcal{A}$, the orbit category $\mathcal{A}/F$ is given as the category with the same objects as $\mathcal{A}$ and morphisms given by* $\mathrm{Hom}_{\mathcal{A}/F}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, F^n Y)$.

Certain orbit categories of triangulated categories were shown by Keller in [13] to be triangulated:

THEOREM 5 ([13]). *Let $\mathcal{H}$ be a hereditary abelian k-category such that there is a triangle equivalence*

$$\mathcal{D}^b(\mathrm{mod}\, k\Delta) \cong \mathcal{D}^b(\mathcal{H}).$$

If F is an autoequivalence on $\mathcal{D}^b(\mathcal{H})$ such that

- *for each indecomposable object U of $\mathcal{H}$ there are only finitely many objects F^iU that lie in $\mathcal{H}$ for $i \in \mathbb{Z}$.*

- *there exist some integer $N \geq 0$ such that the F-orbit of each indecomposable object of $\mathcal{D}^b(H)$ contains an object $U[n]$ for some $0 \leq n \leq N$ and some indecomposable object U of $\mathcal{H}$.*

Then the orbit category $\mathcal{O}_F(\mathcal{H}) := \mathcal{D}^b(\mathcal{H})/F$ is naturally a triangulated category, and the projection functor $\pi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{O}_F(\mathcal{H})$ is a triangle functor.

We now let Δ be a Dynkin diagram, and consider the category $\mathcal{D}^b(k\Delta)$. The AR-translation τ and the suspension functor $[1]$ satisfies the requirements on F. In many cases, as we will see, so will the composition $\tau^m[n]$.

The AR-quiver of $\mathcal{D}^b(k\Delta)$ is equivalent as a translation quiver to $(\mathbb{Z}\Delta, \theta)$. The action of τ and $[1]$ on the AR-quiver of $\mathcal{D}^b(k\Delta)$ are equivalent to the action of respectively θ and S on $(\mathbb{Z}\Delta, \theta)$. Hence, if $\tau^m[n]$ satisfies the requirements on F, the AR-quiver of $\mathcal{D}^b(\Delta)/\tau^m[n]$ is isomorphic as a translation quiver to $(\mathbb{Z}\Delta, \theta)/(\theta^m S^n)$.

In $\mathcal{D}^b(k\Delta)$ we know that $[2] = \tau^{-h}$ where h is the Coxeter number of Δ see [9][12]. The Coxeter number is known to be $n+1$ for $\mathbb{A}_n$, $2n-2$ for $\mathbb{D}_n$, 12 for $\mathbb{E}_6$, 18 for $\mathbb{E}_7$ and 30 for $\mathbb{E}_8$.

4. Amiot's theorem

A very important tool we will use is a theorem by Amiot [1, theorem 7.0.5]. We first need to give a definition of two special classes of triangulated categories.

DEFINITION 6. *A triangulated category $\mathcal{T}$ is called*

algebraic: *if it is triangle equivalent to the stable category of a Frobenius category.*
standard: *if it equivalent as a k-linear category to the mesh category $k\Gamma$ where Γ is the AR-quiver of $\mathcal{T}$*

THEOREM 7. *[1, p. 7.0.5] Let $\mathcal{T}$ be a finite triangulated category which is algebraic and standard. Then there exists a Dynkin diagram Δ of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, and an auto-equivalence Φ on $\mathcal{D}^b(\operatorname{mod} k\Delta)$ such that $\mathcal{T}$ is triangle equivalent to the orbit category $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$.*

We specialize the theorem to deal with the cases we will use:

COROLLARY 8. *Let Λ be a representation-finite, self-injective, basic algebra such that $\underline{\operatorname{mod}}\Lambda$ is of standard type. Let Δ be a Dynkin diagram, and let $\Phi : \mathcal{D}^b(\operatorname{mod} k\Delta) \to \mathcal{D}^b(\operatorname{mod} k\Delta)$ be a functor such that $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ is triangulated.*

If the AR-quivers of $\underline{\operatorname{mod}}\Lambda$ and $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ are equivalent as translation quivers, then $\underline{\operatorname{mod}}\Lambda$ and $\mathcal{D}^b(\operatorname{mod} k\Delta)/\Phi$ are equivalent as triangulated categories.

PROOF. Obviously, $\underline{\operatorname{mod}}\Lambda$ is a finite standard triangulated category. It is algebraic, because Λ is self-injective and basic, and hence Frobenius. By the proof of Theorem 7 in [1], the equivalence follows. □

5. Self-injective Representation-finite Algebras

Our aim is to use Claire Amiot's theorem to show that many orbit categories of hereditary algebras (more than known before) are actually realizable as stable module categories of self-injective algebras. In order to apply the theorem on the stable module categories of self-injective algebras, we need to know that the categories are algebraic and standard. It is clear that they are algebraic, as any representation-finite self-injective algebra is Frobenius.

Asashiba has in his paper [3] defined an invariant under derived and stable equivalence, called the *type* of the representation-finite self-injective algebra. Furthermore, he shows

that any two standard (resp. non-standard) representation-finite self-injective algebras have the same type if and only if they are derived equivalent, and also if and only if they are stably equivalent. In the appendix to [2] a list of algebras, in terms of quivers with relations, is given for each type defined in [3]. In Sections 6, 7 and 8, we make use of the explicit representatives for each type, and give the details of equivalent orbit categories and stable module categories of self-injective algebras.

We give a brief summary of the classification of Asashiba.

DEFINITION 9. *[3] Let Δ be a Dynkin diagram type $\mathbb{A}, \mathbb{D}, \mathbb{E}_6, \mathbb{E}_7$ or $\mathbb{E}_8$. We define the type of a representation-finite self-injective algebra Λ to be the triple $(\Delta(\Lambda), f(\Lambda), t(\Lambda))$. The parameters are defined as follows:*

$\Delta(\Lambda)$**:** *the tree type of Λ (for this definition, we write $\Delta = \Delta(\Lambda)$).*

Let m_Δ be the Loewy length of the mesh category $k\mathbb{Z}\Delta$. From [5] we know that $m_{\mathbb{A}_n} = n$, $m_{\mathbb{D}_n} = 2n-3$, $m_{\mathbb{E}_6} = 11$, $m_{\mathbb{E}_7} = 17$ and $m_{\mathbb{E}_8} = 29$. The AR-quiver of the stable module category of Λ is known [15] to be of the form $\mathbb{Z}\Delta/\langle\phi\tau^{-r}\rangle$ for some automorphism ϕ with a fixed vertex.

$f(\Lambda)$**:** *the frequency of Λ is given by $f(\Lambda) := r/m_\Delta$.*

$t(\Lambda)$**:** *the torsion order $t(\Lambda)$ is the order of ϕ.*

Using this notation, Asashiba gives a list of all the possible types for a standard representation-finite self-injective algebra.

THEOREM 10. *[3] The set of types of standard representation-finite self-injective algebras is the disjoint union of the following sets:*

- $\left\{(\mathbb{A}_n, \frac{s}{n}, 1) | n, s \in \mathbb{N}\right\}$
- $\{(\mathbb{A}_{2p+1}, s, 2) | p, s \in \mathbb{N}\}$
- $\{(\mathbb{D}_n, s, 1) | n, s \in \mathbb{N}, n \geq 4\}$
- $\left\{(\mathbb{D}_{3m}, \frac{s}{3}, 1) | m, s \in \mathbb{N}, m \geq 2, 3 \nmid s\right\}$
- $\{(\mathbb{D}_n, s, 2) | n, s \in \mathbb{N}, n \geq 4\}$
- $\{(\mathbb{D}_4, s, 3) | s \in \mathbb{N}\}$
- $\{(\mathbb{E}_n, s, 1) | n = 6, 7, 8, s \in \mathbb{N}\}$
- $\{(\mathbb{E}_6, s, 2) | s \in \mathbb{N}\}$

6. Type $\mathbb{A}$

There are two standard types of representation-finite self-injective algebras that have AR-quivers of the form $\mathbb{Z}\mathbb{A}_n/G$, up to stable equivalence. The representatives given for these two standard types by [2] and also by [16] are the Nakayama algebras, with AR-quivers of cylindrical shape, and the Möbius algebras, which have AR-quivers shaped like a Möbius band.

For the Nakayama algebras, the stable module categories will be equivalent to orbit categories using functors that are some power of the AR-translation τ. For Möbius algebras we need a "flip functor" to get the Möbius shape of the quiver:

DEFINITION 11. *Let $n = 2l+1$ with $l \in \mathbb{N}$. The flip functor φ on $\mathcal{D}^b(\mathbb{A}_n)$ is given by $\varphi = \tau^{l+1}[1]$.*

6.1. Self-injective Nakayama algebras.

DEFINITION 12. *A self-injective Nakayama algebra is a path algebra $N_{v,r} = Q_v/I_r$, for $v \geq 1, r \geq 2$, where Q_v is the quiver in Figure 2 and I_r is the ideal generated by paths of length r.*

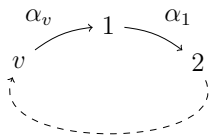


FIGURE 2. Quiver of a self-injective Nakayama algebra $N_{v,r}$

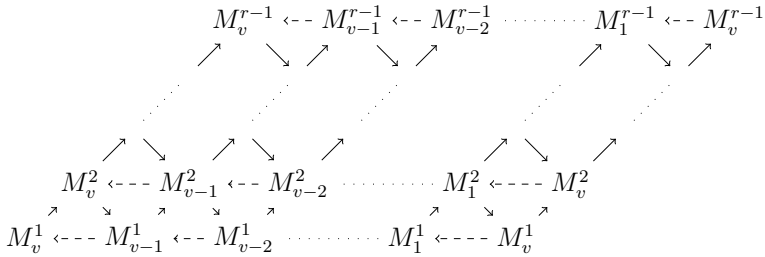


FIGURE 3. AR-quiver of $\underline{\mathrm{mod}}N_{v,r}$. The leftmost and rightmost diagonal are identified.

These algebras are self-injective, and the stable module category $\underline{\mathrm{mod}}N_{v,r}$ is triangulated. The AR-quiver of $\underline{\mathrm{mod}}N_{v,r}$ has been described by Riedtmann in [16]. As a translation quiver it is of the form $\mathbb{Z}\mathbb{A}_{r-1}/(\theta^v)$. In the notation of Asashiba this is of type $(A_n, \frac{v}{r}, 1)$.

If we denote the indecomposable modules over $N_{v,r}$ by M_n^l, where n is the socle of the module, and l is the (Loewy) length of the module, the AR-quiver of $\underline{\mathrm{mod}}N_{v,r}$ is shown in Figure 3.

PROPOSITION 13. *The categories $\underline{\mathrm{mod}}N_{v,r}$ and $\mathcal{D}^b(\mathbb{A}_{r-1})/\tau^v$ are triangle equivalent for $r \geq 2$ and $v \in \mathbb{N} \setminus \{0\}$.*

PROOF. For $v \neq 0$, the functor τ^v fulfils the conditions in Theorem 5, so $\mathcal{D}^b(\mathbb{A}_{r-1})/\tau^v$ is triangulated. The algebra $N_{v,r}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

6.2. Möbius algebras.

DEFINITION 14. *Let $l, v \geq 1$. The Möbius algebra $M_{l,v}$ is the path algebra kQ/I, where Q is the quiver in Figure 4 and I is generated by the relations:*

(1) $\alpha_l^i \cdots \alpha_0^i = \beta_l^i \cdots \beta_0^i$ *for* $i \in \{1, \ldots, v\}$
(2) $\beta_0^{i+1}\alpha_l^i = 0$ *and* $\alpha_0^{i+1}\beta_l^i = 0$ *for* $i \in \{1, \ldots, v-1\}$
(3) $\alpha_0^1\alpha_l^v = 0$ *and* $\beta_0^1\beta_l^v = 0$
(4) *paths of length* $l+2$ *are equal to zero*

EXAMPLE 15. Let $l = 1$ and $v = 2$. The algebra $M_{1,2}$ is given by the quiver in Figure 5 with relations

$$\begin{aligned} \alpha_1^0\alpha_0^0 &= \beta_1^0\beta_1^0 & \alpha_1^1\alpha_0^1 &= \beta_1^1\beta_1^1 \\ \beta_0^1\alpha_0^0 &= 0 & \alpha_0^1\beta_1^0 &= 0 \\ \alpha_0^0\alpha_1^1 &= 0 & \beta_0^0\beta_1^1 &= 0. \end{aligned}$$

The AR-quiver of this algebra is shown in Figure 6. We see that $\underline{\mathrm{mod}}M_{1,2}$ is triangle equivalent to $\mathcal{D}^b\mathbb{A}_3/\phi\tau^6$.

Riedtmann[16] showed that in general the AR-quiver of the stable module category of a Möbius algebra $M_{l,v}$ is of the form $\mathbb{Z}\mathbb{A}_{2l+1}/(\theta^{(2l+1)v}\phi)$, where $\phi = \theta^{\frac{2l+2}{2}}S$ and S is as in Table 2. It is the asymmetry of relations (2) and (3) in I that gives rise to the "Möbius" twist.

In Asashiba's notation these algebras are of type $(A_{2l+1}, v, 2)$.

PROPOSITION 16. *Let $l, v \geq 1$ and let $n = 2l + 1$. The categories $\underline{\mathrm{mod}}M_{l,v}$ and $D^b(\mathbb{A}_n)/\tau^{nv}\varphi$ are equivalent as triangulated categories.*

PROOF. Since $nv \geq 1$, we know that $\tau^{nv}\varphi$ fulfils the requirements on F in Theorem 5. Hence $D^b(\mathbb{A}_{2l+1})/\tau^{nv}\varphi$ is triangulated. The algebra $M_{l,v}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

7. Type $\mathbb{D}$

We will now look in detail at the classes of self-injective algebras that have AR-quivers of the form $\mathbb{Z}\mathbb{D}_n/G$. For this purpose we will make use of the detailed list of representatives of the standard types of representation-finite self-injective algebras provided as an

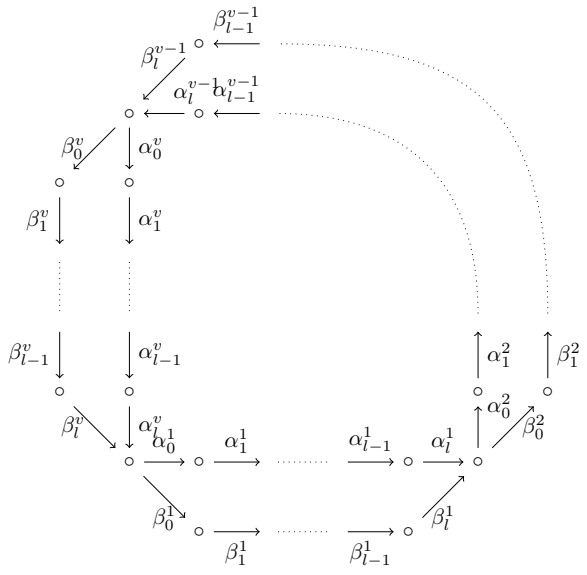


FIGURE 4. Quiver of the Möbuis algebra $M_{l,v}$

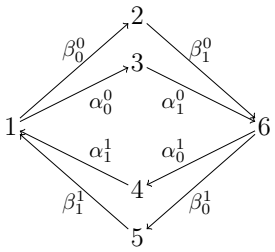


FIGURE 5. The quiver of $M_{1,2}$

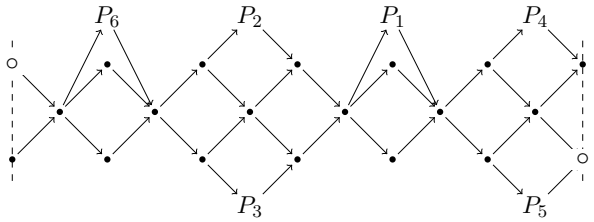


FIGURE 6. The AR-quiver of the algebra $M_{1,2}$. The identical objects on either side are identified.

appendix to [2]. There are, as indicated by Theorem 10, four cases to consider that are standard. Three of these share the same quiver but have different sets of relations, the last type has an entirely different quiver.

We will now define some functors that will be useful in later subsections.

DEFINITION 17. *We call an indecomposable object X in $\mathcal{D}^b(\mathbb{D}_n)$ an α-object if X is a summand of the middle term in exactly one AR-triangle, and the middle term of this AR-triangle has 3 indecomposable summands. All objects that are not α-objects are called β-objects.*

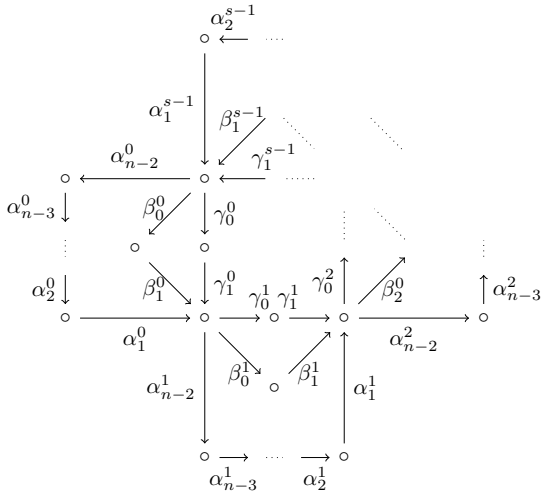


FIGURE 7. $(\mathbb{D}_n, s)$

DEFINITION 18. *We define the flip functor* $\varphi : \mathcal{D}^b(\mathbb{D}_n) \to \mathcal{D}^b(\mathbb{D}_n)$ *for* $n > 4$*, in the following way :*

$$\varphi(X) = \begin{cases} X \text{ if } X \text{ is a } \beta\text{-object} \\ \text{the other } \alpha\text{-object in the middle term containing } X \\ \quad \text{if } X \text{ is an } \alpha\text{-object.} \end{cases}$$

For $n = 4$ *choose two of the* τ*-orbits containing* α*-objects in* $\mathcal{D}^b(\mathbb{D}_4)$*. We define the objects of these two* τ*-orbits to be* α^**. We then define* φ *by:*

$$\varphi(X) = \begin{cases} X \text{ if } X \text{ is not an } \alpha^*\text{-object} \\ \text{the other } \alpha^*\text{-object in the middle term containing } X \\ \quad \text{if } X \text{ is an } \alpha^*\text{-object.} \end{cases}$$

DEFINITION 19. *We define the rotation functor* $\rho : \mathcal{D}^b(\mathbb{D}_4) \to \mathcal{D}^b(\mathbb{D}_4)$*. Enumerate the* τ*-orbits containing* α*-objects in* $\mathcal{D}^b(\mathbb{D}_4)$ *by* $1, 2$ *and* 3*. Let* σ *be a permutation of order* 3 *on the set* $\{1, 2, 3\}$*. We then define*

$$\rho(X) = \begin{cases} X \text{ if } X \text{ is a } \beta\text{-object} \\ \text{the } \alpha\text{-object in } \tau\text{-orbit } \sigma(i), \text{ in the middle term containing } X \\ \quad \text{if } X \text{ is an } \alpha\text{-object in } \tau\text{-orbit nr } i. \end{cases}$$

7.1. Type $(\mathbb{D}_n, s, 1)$.

DEFINITION 20. *The representative of self-injective algebras of type* $(\mathbb{D}_n, s, 1)$ *is given by the path algebra* $D_{n,s,1} := kQ/I$ *where* Q *is the quiver of Figure 7 and the ideal* I *is generated by the following set of relations:*

(1) $\alpha_1^i\alpha_2^i\cdots\alpha_{n-2}^i = \beta_1^i\beta_0^i = \gamma_1^i\gamma_0^i$ *for all* $i \in \{0, \ldots, s-1\}$
(2) *For all* $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\begin{aligned} &\beta_0^{i+1}\alpha_1^i = 0, && \gamma_0^{i+1}\alpha_1^i = 0, \\ &\alpha_{n-2}^{i+1}\beta_1^i = 0, && \alpha_{n-2}^{i+1}\gamma_1^i = 0, \\ &\gamma_0^{i+1}\beta_1^i = 0, && \beta_0^{i+1}\gamma_1^i = 0; \end{aligned}$$

(3) $\alpha_{j-n+2}^{i+1}\cdots\alpha_j^i = 0$ *for all* $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$ *and for all* $j \in \{1, \ldots, n-2\} = \mathbb{Z}/\langle n-2\rangle$.

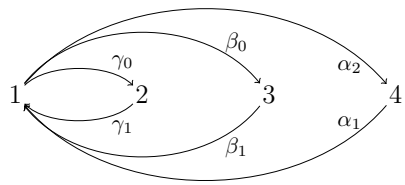


FIGURE 8. The quiver of algebras $D_{4,1,1}, D_{4,1,2}$ and $D_{4,1,3}$.

EXAMPLE 21. Let $n = 4$ and $s = 1$. The algebra $D_{4,1,1}$ is given by the quiver in Figure 8 with relations:

$$\alpha_1\alpha_2 = \beta_1\beta_0 = \gamma_1\gamma_0$$

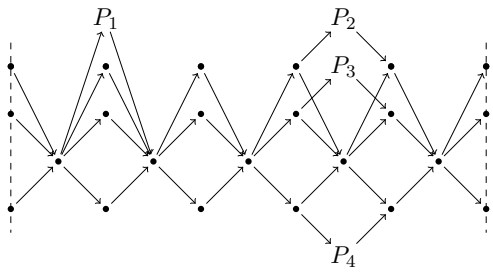


FIGURE 9. $D_{4,1,1}$

$$\alpha_2\beta_1 = 0\,, \qquad \beta_0\alpha_1 = 0\,, \qquad \gamma_0\alpha_1 = 0,$$
$$\alpha_2\gamma_1 = 0\,, \qquad \beta_0\gamma_1 = 0\,, \qquad \gamma_0\beta_1 = 0,$$

and all paths of length 3 are 0. Note that the relations in point 2 makes it impossible to compose arrows from different loops, this leads to an AR-quiver which has cylinder shape. The AR-quiver of this algebra is shown in Figure 9. In this case $\underline{\mathrm{mod}}D_{4,1,1}$ is triangle equivalent to $\mathcal{D}^b(\mathbb{D}_4)/\tau^5$.

In general the AR-quiver of the stable module category of algebras of type $(\mathbb{D}_n, s, 1)$ is of the form $\mathbb{Z}\mathbb{D}_n/\theta^{s(h-1)}$, where h is the Coxeter number for $\mathbb{D}_n$.

PROPOSITION 22. *Let $n \geq 4$ and $n, s \in \mathbb{N}$. The categories $\underline{\mathrm{mod}}D_{n,s,1}$ and $\mathcal{D}^b(\mathbb{D}_n)/\tau^{s(h-1)}$ are equivalent as triangulated categories.*

PROOF. Since $s(h-1) > 0$ the functor $\tau^{s(h-1)}$ satisfies the conditions of Theorem 5, hence the category $\mathcal{D}^b(\mathbb{D}_n)/\tau^{s(h-1)}$ is triangulated. The algebra $D_{n,s,1}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

7.2. Type $(\mathbb{D}_n, s, 2)$.

DEFINITION 23. *The representative of self-injective algebras of type $(\mathbb{D}_n, s, 2)$ is given by the path algebra $D_{n,s,2} := kQ/I$ where Q is the quiver of Figure 7 and the ideal I is generated by the following set of relations:*

(1) $\alpha_1^i\alpha_2^i\cdots\alpha_{n-2}^i = \beta_1^i\beta_0^i = \gamma_1^i\gamma_0^i$ *for all* $i \in \{0, \ldots, s-1\}$
(2) *for all* $i \in \{0, \ldots, s-1\} = \mathbb{Z}/\langle s\rangle$,

$$\beta_0^{i+1}\alpha_1^i = 0 \qquad \gamma_0^{i+1}\alpha_1^i = 0,$$
$$\alpha_{n-2}^{i+1}\beta_1^i = 0 \qquad \alpha_{n-2}^{i+1}\gamma_1^i = 0,$$

and for all $i \in \{0, \ldots, s-2\}$,

$$\gamma_0^{i+1}\beta_1^i = 0 \qquad \beta_0^{i+1}\gamma_1^i = 0,$$
$$\beta_0^0\beta_1^{s-1} = 0, \qquad \gamma_0^0\gamma_1^{s-1} = 0;$$

(3) *α-paths of length $n-1$ are zero, and for all* $i \in \{0, \ldots, s-2\}$,

$$\beta_0^{i+1}\beta_1^i\beta_0^i = 0, \qquad \gamma_0^{i+1}\gamma_1^i\gamma_0^i = 0,$$
$$\beta_1^{i+1}\beta_0^{i+1}\beta_1^i = 0, \qquad \gamma_1^{i+1}\gamma_0^{i+1}\gamma_1^i = 0, \text{ and}$$
$$\gamma_0^0\beta_1^{s-1}\beta_0^{s-1} = 0, \qquad \beta_0^0\gamma_1^{s-1}\gamma_0^{s-1} = 0,$$
$$\gamma_1^0\gamma_0^0\beta_1^{s-1} = 0, \qquad \beta_1^0\beta_0^0\gamma_1^{s-1} = 0.$$

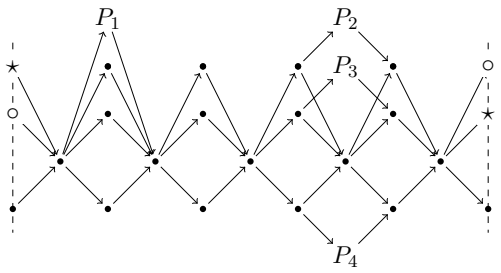


FIGURE 10. $\operatorname{mod} D_{4,1,2}$. The quiver is glued together by identifying the matching symbols on either side.

EXAMPLE 24. Let $n = 4$ and $s = 1$. The algebra $D_{4,1,2}$ is given by the quiver in Figure 8 with relations:

$$\alpha_1\alpha_2 = \beta_1\beta_0 = \gamma_1\gamma_0$$

$$\begin{array}{lll} \alpha_2\beta_1 = 0\,, & \beta_0\alpha_1 = 0\,, & \gamma_0\alpha_1 = 0, \\ \alpha_2\gamma_1 = 0\,, & \beta_0\beta_1 = 0\,, & \gamma_0\gamma_1 = 0, \end{array}$$

and all paths of length 3 are 0. The AR-quiver of this algebra is shown in Figure 10. This time the zero relations in point 2 glues together two of the τ-orbits of $\mathbb{Z}\mathbb{D}_4$. In this case $\underline{\operatorname{mod}} D_{4,1,2}$ is triangle equivalent to $\mathcal{D}^b(\mathbb{D}_4)/\tau^5\varphi$.

In general the AR-quiver of the stable module category of algebras of type $(\mathbb{D}_n, s, 2)$ is of the form $\mathbb{Z}\mathbb{D}_n/\theta^{s(h-1)}\phi$, where h is the Coxeter number for $\mathbb{D}_n$, and ϕ is the automorphism described in Table 2 for $n > 4$ and for $n = 4$ it is an automorphism of order 2.

PROPOSITION 25. *Let $n \leq 4$ and $s, n \in \mathbb{N}$. The categories $\underline{\operatorname{mod}} D_{n,s,2}$ and $\mathcal{D}^b(\mathbb{D}_n)/\tau^{s(h-1)}\varphi$ are equivalent as triangulated categories.*

PROOF. Since $s(h-1) > 0$ the functor $\tau^{s(h-1)}\varphi$ satisfies the conditions given in Theorem 5. Hence the category $\mathcal{D}^b(\mathbb{D}_n)/\tau^{s(h-1)}\varphi$ is triangulated. The algebra $D_{n,s,2}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

7.3. Type $(\mathbb{D}_4, s, 3)$.

DEFINITION 26. *The representative of self-injective algebras of type $(\mathbb{D}_4, s, 3)$ is given by the path algebra $D_{4,s,3} := kQ/I$ where Q is the quiver of Figure 7 and the ideal I is generated by the following set of relations:*

(1) *The same relations as for $(\mathbb{D}_4, s, 1)$, part 1.*
(2) *For all $i \in \{0, \ldots, s-2\}$*

$$\begin{array}{ll} \beta_0^{i+1}\alpha_1^i = 0, & \gamma_0^{i+1}\alpha_1^i = 0, \\ \alpha_0^{i+1}\beta_1^i = 0, & \gamma_0^{i+1}\beta_1^i = 0, \\ \alpha_0^{i+1}\gamma_1^i = 0, & \beta_0^{i+1}\gamma_1^i = 0, \ \textit{and} \\ \alpha_0^0\alpha_1^{s-1} = 0, & \gamma_0^0\alpha_1^{s-1} = 0, \\ \alpha_0^0\beta_1^{s-1} = 0, & \beta_0^0\beta_1^{s-1} = 0, \\ \beta_0^0\gamma_1^{s-1} = 0, & \gamma_0^0\gamma_1^{s-1} = 0; \end{array}$$

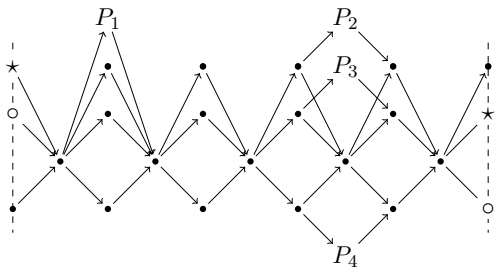


FIGURE 11. $\operatorname{mod} D_{4,1,3}$. The quiver is glued together by identifying the matching symbols on either side.

(3) *all paths of length* 3 *are zero.*

EXAMPLE 27. Let $n = 4$ and $s = 1$. The algebra $D_{4,1,3}$ is given by the quiver in Figure 8 with relations:

$$\alpha_1\alpha_2 = \beta_1\beta_0 = \gamma_1\gamma_0$$

$$\begin{aligned} &\alpha_0\alpha_1 = 0\,, && \alpha_0\beta_1 = 0\,, && \beta_0\gamma_1 = 0,\\ &\gamma_0\alpha_1 = 0\,, && \beta_0\beta_1 = 0\,, && \gamma_0\gamma_1 = 0, \end{aligned}$$

and all paths of length 3 are 0. The AR-quiver of this algebra is shown in Figure 11. As the figure shows, three of the τ-orbits of $\mathbb{Z}\mathbb{D}_4$ are glued together, this is due to the zero relations of length two. In this case $\underline{\operatorname{mod}} D_{4,1,3}$ is triangle equivalent to $\mathcal{D}^b(\mathbb{D}_4)/\tau^5\rho$.

In general the AR-quiver of the stable module category of algebras of type $(\mathbb{D}_4, s, 3)$ is of the form $\mathbb{Z}\mathbb{D}_n/\theta^{5s}\phi$, where ϕ is the automorphism of order 3 described in Table 2.

PROPOSITION 28. *Let $n = 4$ and $s \in \mathbb{N}$. The categories $\underline{\operatorname{mod}} D_{4,s,3}$ and $\mathcal{D}^b(\mathbb{D}_4)/\tau^{5s}\rho$ are equivalent as triangulated categories.*

PROOF. Since $5s > 0$, the functor $\tau^{5s}\rho$ satisfies the conditions given in Theorem 5. Hence the category $\mathcal{D}^b(\mathbb{D}_4)/\tau^{5s}\rho$ is triangulated. The algebra $D_{n,s,3}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

7.4. Type $(\mathbb{D}_{3m}, \frac{s}{3}, 1)$. This is the only type of tree type $\mathbb{D}$ where the frequency is not an integer. If $3|s$, then the type is already described, in Section 7.1; hence we require that s is not divisible by 3.

DEFINITION 29. *Let $m \geq 2$ and $s \geq 1$ with $3 \nmid s$. The representative of self-injective algebras of type $(\mathbb{D}_{3m}, \frac{s}{3}, 1)$ is given by the path algebra $D_{3m,\frac{s}{3},1} := kQ/I$ where Q is the quiver of Figure 12 and the ideal I is generated by the following set of relations:*

(1) $\alpha_m^i \cdots \alpha_2^i\alpha_1^i = \beta_{i+1}\beta_i$ *for all* $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$;
(2) $\alpha_1^{i+2}\alpha_m^i = 0$ *for all* $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$;
(3) $\alpha_j^{i+3} \cdots \alpha_1^{i+3}\beta_{i+2}\alpha_m^i \cdots \alpha_j^i = 0$ *for all* $i \in \{1, \ldots, s\} = \mathbb{Z}/\langle s\rangle$ *and for all* $j \in \{1, \ldots, m\}$

EXAMPLE 30. Let $m = 2$ and $s = 1$. The algebra $D_{6,\frac{1}{3},1}$ is given by the quiver in Figure 13 with relations:

$$\begin{aligned} \beta^2 &= \alpha_2\alpha_1 & \alpha_1\alpha_2 &= 0\\ \alpha_1\beta\alpha_2\alpha_1 &= 0 & \alpha_2\alpha_1\beta\alpha_2 &= 0. \end{aligned}$$

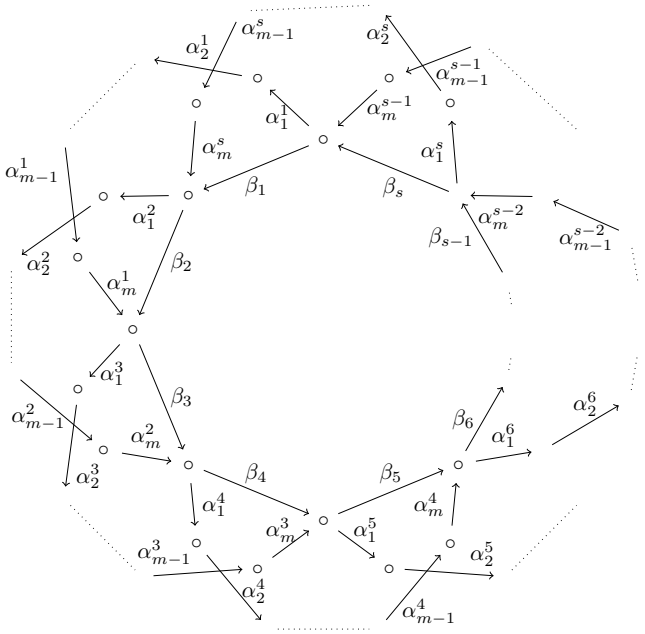


FIGURE 12. $(\mathbb{D}_{3m}, \frac{s}{3})$

The AR-quiver of this algebra is shown in Figure 14. In this case $\underline{\mathrm{mod}} D_{6,\frac{1}{3},1}$ is triangle equivalent to $\mathcal{D}^b(\mathbb{D}_6)/\tau^3$.

In general the AR-quiver of the stable module category of algebras of type $(\mathbb{D}_{3m}, \frac{s}{3}, 1)$ is of the form $\mathbb{Z}\mathbb{D}_{3m}/\theta^{s(h-1)/3}$, where h is the Coxeter number for $\mathbb{D}_{3m}$, and ϕ is the automorphism described in Table 2. (Note that since $h-1 = 2n-3 = 6m-3$ we have that $s(h-1)/3$ is a natural number).

PROPOSITION 31. *Let $m \geq 2$ and $s \geq 1$ with $3 \nmid s$. The categories $\underline{\mathrm{mod}} D_{3m,\frac{s}{3},1}$ and $\mathcal{D}^b(\mathbb{D}_{3m})/\tau^{s(h-1)/3}$ are equivalent as triangulated categories.*

PROOF. Since $s(h-1)/3 > 0$, the functor $\tau^{s(h-1)/3}$ satisfies the conditions of Theorem 5, hence the category $\mathcal{D}^b(\mathbb{D}_{3m})/\tau^{s(h-1)/3}$ is triangulated. The algebra $D_{3m,\frac{s}{3},1}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

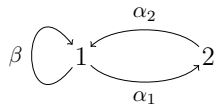


FIGURE 13. Quiver of the path algebra $D_{6,\frac{1}{3},1}$

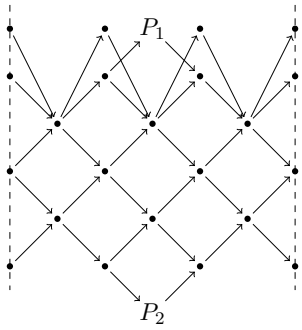


FIGURE 14. $D_{6,\frac{1}{3},1}$

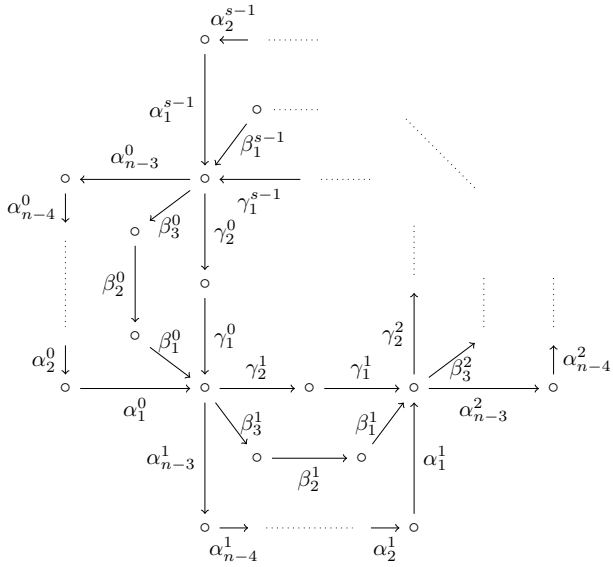


FIGURE 15. Type $(\mathbb{E}_n, s)$

8. Type $\mathbb{E}$

We now look at self-injective algebras with AR-quivers of the form $\mathbb{Z}\mathbb{E}_n/G$. These algebras are all standard [3], and they are divided into two main groups; those with a cylindrical AR-quiver, and those with a Möbius-shaped AR-quiver. In Asashiba's notation, the former are of type $(E_n, s, 1)$, while the latter are of type $(\mathbb{E}_6, s, 2)$, see [3]. For the first group, the stable module categories will be equivalent to orbit categories using functors that are some power of the AR-translation τ. For the latter, however, we need a "flip functor" to get the Möbius shape of the quiver.

DEFINITION 32. *The flip functor φ on $\mathcal{D}^b(\mathbb{E}_6)$ is given by $\varphi = \tau^6[1]$.*

We follow the classification due to Asashiba for the rest of the section. Note that the representative algebras all share the quiver given in Figure 15; however the relations are different.

8.1. Type $(\mathbb{E}_n, s, 1)$.

DEFINITION 33. *The representative of self-injective algebras of type* $(\mathbb{E}_n, s, 1)$ *is given by the path algebra* $E_{n,s,1} := kQ/I$ *where* Q *is the quiver of Figure 15 and the ideal* I *is generated by the following set of relations:*

(1) $\alpha_1^i\alpha_2^i\cdots\alpha_{n-3}^i = \beta_1^i\beta_2^i\beta_3^i = \gamma_1^i\gamma_2^i$ *for all* $i \in \{0,\dots,s-1\}$*;*
(2) *For all* $i \in \{0,\dots,s-1\} = \mathbb{Z}/\langle s\rangle$*,*

$$\begin{aligned} \beta_3^{i+1}\alpha_1^i &= 0, & \gamma_2^{i+1}\alpha_1^i &= 0,\\ \alpha_{n-3}^{i+1}\beta_1^i &= 0, & \gamma_2^{i+1}\beta_1^i &= 0,\\ \alpha_{n-3}^{i+1}\gamma_1^i &= 0, & \beta_3^{i+1}\gamma_1^i &= 0, \text{ and} \end{aligned}$$

(3) α*-paths of length* $n-2$ *are equal to 0,* β*-paths of length* 4 *are equal to 0 and* γ*-paths of length* 3 *are equal to 0.*

EXAMPLE 34. Let $n = 6$ and $s = 1$. The algebra $E_{6,1,1}$ is given by the quiver in Figure 16, together with the relations

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2$$

$$\begin{aligned} \alpha_3\beta_1 &= 0 & \alpha_3\gamma_1 &= 0 & \beta_3\alpha_1 &= 0\\ \beta_3\gamma_1 &= 0 & \gamma_2\alpha_1 &= 0 & \gamma_2\beta_1 &= 0\\ \alpha_2\alpha_3\alpha_1\alpha_2 &= 0 & \beta_2\beta_3\beta_1\beta_2 &= 0. \end{aligned}$$

The AR-quiver of the module category over this algebra is given in Figure 17. It turns out that $\underline{\mathrm{mod}}E_{6,1,1}$ is triangulated equivalent to $\mathcal{D}^b(k\mathbb{E}_6)/\tau^{11}$.

In general, the AR-quiver of the stable module categories of self-injective algebras of type $(\mathbb{E}_n, s, 1)$ is isomorphic to $\mathbb{Z}\mathbb{E}_n/\theta^{t_n s}$, where $t_6 = 11$, $t_7 = 17$ and $t_8 = 29$.

PROPOSITION 35. *Let* $n = 6, 7, 8$ *and* $s \geq 1$*. The categories* $\underline{\mathrm{mod}}E_{n,s,1}$ *and* $\mathcal{D}^b(k\mathbb{E}_n)/\tau^{t_n s}$ *are triangle equivalent.*

PROOF. Since $t_n s > 0$, the functor $\tau^{t_n s}$ satisfies the conditions of Theorem 5, hence $\mathcal{D}^b(k\mathbb{E}_n)/\tau^{t_n s}$ is a triangulated category. The algebra $E_{n,s,1}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

8.2. Type $(\mathbb{E}_6, s, 2)$.

DEFINITION 36. *The representative of self-injective algebras of type* $(\mathbb{E}_6, s, 2)$ *is given by the path algebra* $E_{6,s,2} := kQ/I$ *where* Q *is the quiver of Figure 15 and the ideal* I *is generated by the following set of relations:*

(1) $\alpha_1^i\alpha_2^i\cdots\alpha_{n-3}^i = \beta_1^i\beta_2^i\beta_3^i = \gamma_1^i\gamma_2^i$ *for all* $i \in \{0,\dots,s-1\}$*;*

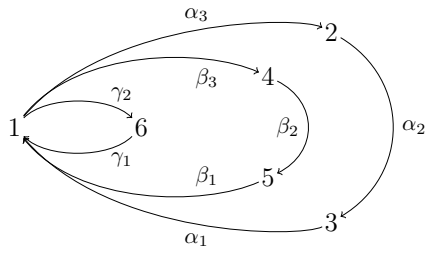


FIGURE 16. Quiver of $E_{6,1,n}$ for $n = 1, 2$

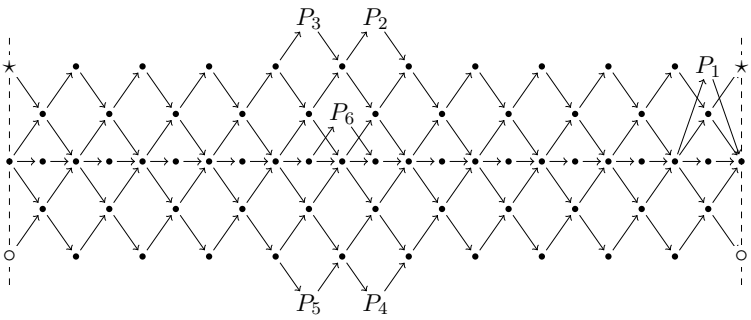


FIGURE 17. AR-quiver of mod $E_{6,1,1}$. The quiver is glued together by identifying the matching symbols on either side.

(2) *For all* $i \in \{0, \dots, s-1\} = \mathbb{Z}/\langle s \rangle$,

$$\alpha_3^{i+1}\gamma_1^i = 0, \qquad \beta_3^{i+1}\gamma_1^i = 0,$$
$$\gamma_2^{i+1}\alpha_1^i = 0, \qquad \gamma_2^{i+1}\beta_1^i = 0,$$

and for all $i \in \{0, \dots, s-2\}$,

$$\beta_3^{i+1}\alpha_1^i = 0, \qquad \alpha_3^{i+1}\beta_1^i = 0,$$
$$\alpha_3^0\alpha_1^{s-1} = 0, \qquad \beta_3^0\beta_1^{s-1} = 0, \text{ and}$$

(3) γ*-paths of length 3 are equal to 0, and for all* $i \in \{0, \dots, s-2\}$ *and for all* $j \in \{1, 2, 3\} = \mathbb{Z}/\langle 3 \rangle$,

$$\alpha_{j-3}^{i+1} \cdots \alpha_j^i = 0, \qquad \beta_{j-3}^{i+1} \cdots \beta_j^i = 0,$$
$$\beta_{j-3}^0 \cdots \beta_3^0\alpha_1^{s-1} \cdots \alpha_j^{s-1} = 0, \qquad \alpha_{j-3}^0 \cdots \alpha_3^0\beta_1^{s-1} \cdots \beta_j^{s-1} = 0.$$

EXAMPLE 37. Let $n = 6$ and $s = 1$. The algebra $E_{6,1,2}$ is given by the quiver in Figure 16, together with the relations

$$\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = \gamma_1\gamma_2$$

$$\alpha_3\alpha_1 = 0 \qquad \alpha_3\gamma_1 = 0 \qquad \beta_3\beta_1 = 0$$
$$\beta_3\gamma_1 = 0 \qquad \gamma_2\alpha_1 = 0 \qquad \gamma_2\beta_1 = 0$$
$$\alpha_2\alpha_3\beta_1\beta_2 = 0 \qquad \beta_2\beta_3\alpha_1\alpha_2 = 0$$

The AR-quiver of the module category over this algebra is given in Figure 18. It turns out that $\underline{\mathrm{mod}}E_{6,1,2}$ is triangulated equivalent to $\mathcal{D}^b(k\mathbb{E}_6)/\tau^{11}\varphi$.

In general, the AR-quiver of the stable module categories of self-injective algebras of type $(\mathbb{E}_6, s, 2)$ is isomorphic to $\mathbb{Z}\mathbb{E}_6/\theta^{11s}\phi$, where ϕ is described in Table 2.

PROPOSITION 38. *Let* $s \geq 1$. *The categories* $\underline{\mathrm{mod}}E_{6,s,2}$ *and* $\mathcal{D}^b(k\mathbb{E}_6)/\tau^{11s}\varphi$ *are triangle equivalent.*

PROOF. Since $11s > 0$, the functor $\tau^{11s}\varphi$ satisfies the conditions of Theorem 5, hence $\mathcal{D}^b(k\mathbb{E}_6)/\tau^{11s}\varphi$ is a triangulated category. The algebra $E_{6,s,2}$ is a representation-finite, self-injective, basic algebra, whose stable module category is standard by Theorem 10. The conclusion follows from Corollary 8. □

9. Summary

From the results of Section 6, 7 and 8 it is clear that all but one of the representation-finite self-injective standard algebras are stably triangle equivalent to an orbit category of the form $\mathcal{D}^b(k\Delta_r)/\tau^w\varphi^i$ where $i \in \{0,1\}$ and φ is the functor described in Definition 11 for type $\mathbb{A}$, Definition 18 and 19 for $\mathbb{D}$ and Definition 32 for type $\mathbb{E}_6$. However not all triangulated orbit categories of the form $\mathcal{D}^b(k\Delta_r)/F$ are equivalent to a stable module category of a representation finite self-injective algebra. We therefore sum up our findings in a table below, aiming at a way to easily look up if a certain orbit category is in fact equivalent or not to a stable module category of a self-injective algebra.

Recall that given a functor of the form $F = \tau^m[n]$ on $\mathcal{D}^b(k\Delta_r)$, it can be expressed on the form $F = \tau^w\varphi^i$ using the Coxeter relation for Δ_r, and the above-mentioned definitions of φ.

The following theorem sums up our results.

THEOREM 39. *Let Δ be a Dynkin diagram and let Φ be an autoequivalence such that $\mathcal{D}^b(k\Delta)/\Phi$ is triangulated. Let Λ be a self-injective algebra. The orbit category $\mathcal{C} = \mathcal{D}^b(k\Delta)/\Phi$ is triangle equivalent to $\underline{\mathrm{mod}}\Lambda$ precisely in the cases described in Table 3.*

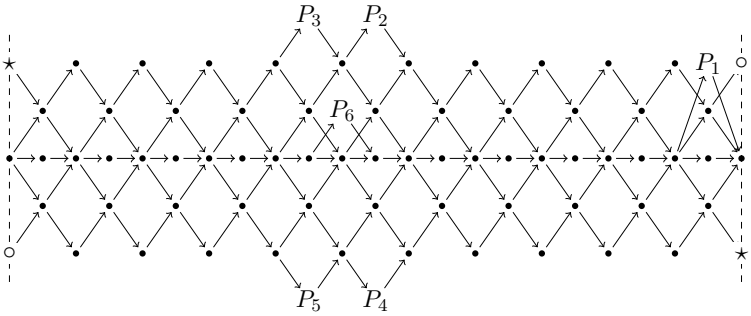


FIGURE 18. AR-quiver of mod $E_{6,1,2}$. The quiver is glued together by identifying the matching symbols on either side.

TABLE 3. The cases, up to triangulated equivalence, where $\mathcal{C} = \mathcal{D}^b(k\Delta)/\Phi$ is triangle equivalent to $\underline{\mathrm{mod}}\Lambda$.

$\mathcal{C}$		Λ	*Sec.*
$\mathcal{D}^b(\mathbb{A}_r)/\tau^w$	$r \geq 1, w \geq 1$	*Nakayama alg.* $N_{w,r+1}$	*6.1*
$\mathcal{D}^b(\mathbb{A}_r)/\tau^w\phi$	$r = 2l+1, l \geq 1$ $w = rv,\ r \geq 1$	*Möbius alg.* $M_{l,v}$	*6.2*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w$	$r \geq 4, w = s(2r-3), s \geq 1$	$D_{n,s,1}$	*7.1*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w\phi$	$r \geq 4, w = s(2r-3), s \geq 1$	$D_{n,s,2}$	*7.2*
$\mathcal{D}^b(\mathbb{D}_4)/\tau^{5w}\rho$	$w \geq 1$	$D_{4,s,3}$	*7.3*
$\mathcal{D}^b(\mathbb{D}_r)/\tau^w$	$r = 3m, m \geq 2$ $w = s(2r-3)/3, s \geq 1, 3 \nmid s$	$D_{3m,\frac{s}{3},1}$	*7.4*
$\mathcal{D}^b(\mathbb{E}_r)/\tau^w$	$r = 6$ *and* $w = 11s$ $r = 7$ *and* $w = 17s\ \ s \geq 1$ $r = 8$ *and* $w = 29s$	$E_{r,s,1}$	*8.1*
$\mathcal{D}^b(\mathbb{E}_6)/\tau^w\phi$	$w = 11s, s \geq 1$	$E_{6,s,2}$	*8.2*

Acknowledgements. We would like to thank Steffen Oppermann and Aslak Bakke Buan for helpful comments and feedback.

References

[1] C. Amiot. "On the structure of triangulated categories with finitely many indecomposables". In: *Bull. Soc. Math. France* 135.3 (2007), pp. 435–474.

[2] H. Asashiba. "On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras". In: *Algebr. Represent. Theory* 6.4 (2003), pp. 427–447.

[3] H. Asashiba. "The derived equivalence classification of representation-finite selfinjective algebras". In: *J. Algebra* 214.1 (1999), pp. 182–221.

[4] M. Auslander, I. Reiten, and S. O. Smalø. *Representation Theory of Artin Algebras*. Cambridge University Press, 1995.

[5] O. Bretscher, C. Läser, and C. Riedtmann. "Self-injective and simply connected algebras". In: *Manuscripta Math.* 36.3 (1981/82), pp. 253–307.

[6] A. B. Buan et al. "Tilting theory and cluster combinatorics". In: *Adv. Math.* 204.2 (2006), pp. 572–618.

[7] P. Caldero, F. Chapoton, and R. Schiffler. "Quivers with relations arising from clusters (A_n case)". In: *Trans. Amer. Math. Soc.* 358.3 (2006), pp. 1347–1364.

[8] A. Dugas. "Resolutions of mesh algebras: periodicity and Calabi-Yau dimensions". In: *Math. Z.* 271.3-4 (2012), pp. 1151–1184.

[9] P. Gabriel. "Auslander-Reiten sequences and representation-finite algebras". In: *Representation theory, I (Proc. Workshop, Carleton Univ., Ottawa, Ont., 1979)*. Vol. 831. 1980, pp. 1–71.

[10] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*. Vol. 119. Cambridge University Press, Cambridge, 1988.

[11] T. Holm and P. Jørgensen. "Realizing higher cluster categories of Dynkin type as stable module categories". In: *Q. J. Math.* 64.2 (2013), pp. 409–435.

[12] B. Keller. "Cluster algebras, quiver representations and triangulated categories". In: *Triangulated categories*. Vol. 375. 2010, pp. 76–160.

[13] B. Keller. "On triangulated orbit categories". In: *Doc. Math.* 10 (2005), pp. 551–581.

[14] B. Keller and I. Reiten. "Cluster-tilted algebras are Gorenstein and stably Calabi-Yau". In: *Adv. Math.* 211.1 (2007), pp. 123–151.

[15] C. Riedtmann. "Algebren, Darstellungsköcher, Überlagerungen und zurück". In: *Comment. Math. Helv.* 55.2 (1980), pp. 199–224.

[16] C. Riedtmann. "Representation-finite self-injective algebras of class A_n". In: *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*. Vol. 832. 1980, pp. 449–520.

PAPER 3

Modules of finite projective dimension over a cluster-tilted algebra

Karin M. Jacobsen

Manuscript

MODULES OF FINITE PROJECTIVE DIMENSION OVER A CLUSTER-TILTED ALGEBRA

KARIN M. JACOBSEN

Abstract. We study the category of modules of finite projective dimension $\mathcal{P}_{\leq 1}$ over a gentle cluster-tilted algebra using triangulations of marked surfaces. For Dynkin type A, we give the number of irreducible objects in $\mathcal{P}_{\leq 1}$ and calculate the AR-translation.

1. Introduction

Let Λ be a finite dimensional algebra, and let $\mathcal{D}^b(\Lambda)$ be the bounded derived category over $\operatorname{mod}\Lambda$. The cluster category, as introduced in [9] by Buan, Marsh, Reineke, Reiten and Todorov, has been widely studied the last decade. It is given by the orbit category $\mathcal{C} = \mathcal{D}^b(\Lambda)/\tau^{-1}[1]$, where Λ is a hereditary algebra.

In particular, the class of cluster-tilted algebras [7] is very interesting. These are the algebras of the form $\operatorname{End}(T)^{op}$, where T is a cluster-tilting (i.e. maximal rigid) object in the cluster category. Cluster-tilted algebras are not necessarily hereditary but their module categories still have the nice structure of the original, hereditary algebra.

The technique of using marked surfaces as way to represent the cluster category of type A was introduced in [10]. Marked surfaces were used by Assem, Brüstle, Charbonneau-Jodoin and Plamondon to represent module categories of gentle algebras, including cluster-tilted algebras of type A, in [3]. For algebras of finite type, the geometric descriptions give complete descriptions of the cluster category and the module category.

In this paper we let Λ be a cluster-tilted algebra, and consider the full subcategory $\mathcal{P}_{\leq 1}$ of $\operatorname{mod}\Lambda$ containing all modules of projective dimension at most one. It was shown in [12] that cluster-tilted algebras have Gorenstein dimension one, so in the cases we study, the category $\mathcal{P}_{\leq 1}$ contains all modules of finite projective dimension. In Section 2 we recount results by Auslander and Smalø given in [4] that shows that for any finitely generated Artin algebra, the subcategory $\mathcal{P}_{\leq 1}$ has AR-structure. We state additional results by Kleiner and Perez that calculates the AR-translation explicitly.

In Section 3, we give some results by Keller and Reiten on the structure of $\mathcal{P}_{\leq 1}$. We also recount a very useful theorem given in [5] by Beaudet, Brüstle and Todorov, which describes all modules of infinite projective dimension.

We state the basic facts about surface algebras in Section 4, including a conjecture on the proper translation of the theorem in [5] to the world of surface algebras. In Section 5 we show that the conjecture holds for Dynkin type A and use it to give an invariant for type A. We also calculate the AR-translation in $\mathcal{P}_{\leq 1}$ for type A.

1.1. Notation. k is an algebraically closed field. We assume all algebras are k-algebras and that all categories are k-categories.

For a category $\mathcal{C}$, we let $|\mathcal{C}|$ denote the number of isomorphism classes of indecomposable objects.

Let $\mathcal{U}$ be a subcategory $\mathcal{C}$, and let X be an object in $\mathcal{C}$. A minimal right $\mathcal{U}$-approximation of X is a right minimal morphism $\mathfrak{t}_{\mathcal{U}}X \xrightarrow{g} X$ with $\mathfrak{t}_{\mathcal{U}}X \in \mathcal{U}$, such that any

morphism from an object in $\mathcal{U}$ to X factors through g. Dually, we denote by $X \to \mathfrak{l}_{\mathcal{U}}X$ a minimal left $\mathcal{U}$-approximation of X.

For a Λ-module X, we let $\operatorname{pd} X$ and $\operatorname{id} X$ denote the projective and injective dimension of X respectively.

For T a cluster-tilting object, we let $\Lambda_T = \operatorname{End}(T)^{op} = kQ_T/I_T$.

2. Almost split sequences in subcategories

In [4], Auslander and Smalø studied the existence of almost split sequences (or AR-sequences) in subcategories of the module category. For convenience we state some central results. First we define the analogues to projectives and injectives for a subcategory.

DEFINITION 1 ([4]). *Let $\mathcal{U}$ be a subcategory, and let X be an object in $\mathcal{U}$.*
X is called Ext-projective *if $\operatorname{Ext}^1_{\mathcal{U}}(X, U) = 0$ for all $U \in \mathcal{U}$.*
X is called Ext-injective *if $\operatorname{Ext}^1_{\mathcal{U}}(U, X) = 0$ for all $U \in \mathcal{U}$.*

We give the definition of almost split sequences in subcategories.

DEFINITION 2 ([4]). *Let Λ be a k-algebra. Let $\mathcal{U}$ be a subcategory of* $\operatorname{mod}\Lambda$ *that is closed under extensions. We say that $\mathcal{U}$* has almost split sequences *if*

1. *For every indecomposable object $U \in \mathcal{U}$, there exists a right almost split morphism $V \to U$ and a left almost split morphism $U \to W$ in $\mathcal{U}$.*
2. *For each indecomposable non-*Ext*-projective object $W \in \mathcal{U}$ there exists an exact sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ such that f is left almost split and g is right almost split in $\mathcal{U}$.*
3. *For each indecomposable non-*Ext*-injective object $U \in \mathcal{U}$ there exists an exact sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ such that f is left almost split and g is right almost split in $\mathcal{U}$.*

They also give an existence theorem for the almost split sequences.

THEOREM 3 ([4, Thm. 2.4]). *Any functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions has almost split sequences.*

In this paper, we will study the following full subcategory of $\operatorname{mod}\Lambda$:

$$\mathcal{P}_{\leq 1} := \{M \in \operatorname{mod}\Lambda \mid \operatorname{pd} M \leq 1\}.$$

COROLLARY 4. *If Λ has Gorenstein dimension 1, then $\mathcal{P}_{\leq 1}$ has almost split sequences.*

PROOF. If Λ has Gorenstein dimension 1, then the injective envelope of Λ has projective dimension 1. In [11], the authors showed that this means that $\mathcal{P}_{\leq 1}$ is functorially finite. It is closed under extensions by the horseshoe lemma. □

Kleiner [13], and later Kleiner and Perez [14] gave a method for calculating the AR-translation in subcategories with almost split sequences.

THEOREM 5 ([14]). *Let $\mathcal{U}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$. *Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be an almost split sequence in $\mathcal{U}$. Let τ be the AR-translation in $\operatorname{mod}\Lambda$. *Then $\mathfrak{r}_{\mathcal{U}}\tau W \cong U \oplus I$, where I is* Ext*-injective.*

3. $\mathcal{P}_{\leq 1}$ for cluster-tilted algebras

Let $\mathcal{C}$ be a cluster category as defined in [9], and let T be a cluster-tilting object in $\mathcal{C}$. We denote by $\tau_{\mathcal{C}}$ the Auslander-Reiten translation on $\mathcal{C}$. Let Λ be given by $\Lambda = \operatorname{End}(T)^{op}$; we call Λ a cluster-tilted algebra. As we will study the subcategory $\mathcal{P}_{\leq 1}$ of Λ-modules of finite projective dimension, we first give a description of the projectives and injectives in $\operatorname{mod}\Lambda$.

LEMMA 6 ([7]). *Let X be an object in* $\operatorname{mod}\Lambda$.
X is projective if and only if $X \cong \operatorname{Hom}_{\mathcal{C}}(T, T')$ for some $T' \in \operatorname{add} T \subseteq \mathcal{C}$.
X is injective if and only if $X \cong \operatorname{Hom}_{\mathcal{C}}(T, \tau_{\mathcal{C}}^2 T')$ for some $T' \in \operatorname{add} T \subseteq \mathcal{C}$.

We will need the following statements, which were shown in [12], though not all stated explicitly. Statement (3) was also shown in [15]. We include the proofs for the convenience of the reader.

LEMMA 7 ([12]). *Let X be an object in* $\operatorname{mod}\Lambda$, *where Λ is a cluster-tilted algebra. Then the following holds:*

(1) *if X is injective, then* $\operatorname{pd} X \leq 1$;
(1*) *if X is projective, then* $\operatorname{id} X \leq 1$;
(2) *if* $\operatorname{pd} X < \infty$, *then* $\operatorname{id} X \leq 1$;
(2*) *if* $\operatorname{id} X < \infty$, *then* $\operatorname{pd} X \leq 1$;
(3) *either* $\operatorname{pd} X = \operatorname{id} X = \infty$, *or* $\operatorname{pd} X \leq 1$ *and* $\operatorname{id} X \leq 1$;
(4) *the Gorenstein dimension of Λ is 1.*

PROOF.

(1) Suppose X is injective. Choose a minimal projective presentation of X, say $P_1 \to P_0 \to X \to 0$. Using Lemma 6, we see rewrite this as

$$\operatorname{Hom}_{\mathcal{C}}(T, T_1) \to \operatorname{Hom}_{\mathcal{C}}(T, T_0) \to \operatorname{Hom}_{\mathcal{C}}(T, \tau^2 T'),$$

where T_1, T_0 and T' are in $\operatorname{add} T$.

Using the correspondence $\operatorname{add} T \cong \operatorname{proj}\Lambda$, together with the knowledge that $\operatorname{Hom}_{\mathcal{C}}(T, -)$ is full and dense, we see that there is a distinguished triangle

$$\tau T' \xrightarrow{f} T_1 \to T_0 \to \tau^2 T'.$$

We apply $\operatorname{Hom}_{\mathcal{C}}(T, -)$ to this triangle and then use that $\operatorname{Hom}_{\mathcal{C}}(T, \tau T) = \operatorname{Ext}^1(T, T) = 0$. We get that

$$0 \to P_1 \to P_0 \to X \to 0$$

is an exact sequence, hence $\operatorname{pd} X \leq 1$.

(1*) Dual of (1).

(2) Denote by ΩX the kernel of some epimorphism $P_1 \to X$, where P_1 is projective. Let $\Omega^{m+1}(X) = \Omega(\Omega^m(X))$. Suppose that $\operatorname{pd} X = n < \infty$. Then $\Omega^n(X)$ must be projective, and hence of injective dimension at most one. We will show that $\operatorname{id} X \leq 1$ by induction on p. Consider the short exact sequence

$$0 \to \Omega^{n-p} X \to P_{n-p} \to \Omega^{n-(p-1)} X \to 0,$$

where we assume that $\mathrm{id}\Omega^{n-p}X \leq 1$ and that P_{n-p} is projective, so that $\mathrm{id}P_{n-p} \leq 1$. For an arbitrary object $A \in \mathrm{mod}\,\Lambda$, we get the long exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}(A, \Omega^{n-p}X) \to \mathrm{Hom}(A, P_{n-p}) \to \mathrm{Hom}(A, \Omega^{n-p-1}X) \to \\ \to \mathrm{Ext}^1(A, \Omega^{n-p}X) \to \mathrm{Ext}^1(A, P_{n-p}) \to \mathrm{Ext}^1(A, \Omega^{n-p-1}X) \to \\ \to \mathrm{Ext}^2(A, \Omega^{n-p}X) \to \mathrm{Ext}^2(A, P_{n-p}) \to \mathrm{Ext}^2(A, \Omega^{n-p-1}X) \to \cdots\end{aligned}$$

Since $\mathrm{Ext}^{i+1}(A, \Omega^{n-p}X) = 0 = \mathrm{Ext}^i(A, P_{n-p})$ for $i > 1$ it follows that we have $\mathrm{Ext}^i(A, \Omega^{n-p-1}X) = 0$ for $i > 1$. Hence $\mathrm{id}\,\Omega^{n-p-1}X \leq 1$. By induction, we get that $\mathrm{id}\,X \leq 1$.

(2*) Dual of (2).

(3) Follows from (2) and (2*).

(4) Restatement of (3).

□

As we start investigating the AR-structure of $\mathcal{P}_{\leq 1}$, we need to know its Ext-projectives and -injectives.

LEMMA 8. *Let X be an object in $\mathcal{P}_{\leq 1}$.*
X is Ext*-projective in $\mathcal{P}_{\leq 1}$ if and only if X is projective in* $\mathrm{mod}\,\Lambda$.
X is Ext*-injective in $\mathcal{P}_{\leq 1}$ if and only if X is injective in* $\mathrm{mod}\,\Lambda$.

PROOF. If X is projective (injective) in $\mathrm{mod}\,\Lambda$, then X is Ext-projective (Ext-injective) in $\mathrm{mod}\,\Lambda$ and also in $\mathcal{P}_{\leq 1}$.

Suppose that X is an element in $\mathcal{P}_{\leq 1}$. Then there exists an exact sequence

$$0 \to P_1 \to P_0 \to X \to 0.$$

If X is not projective, the sequence does not split. Both P_1 and P_0 are nonzero elements in $\mathcal{P}_{\leq 1}$, and hence $\mathrm{Ext}_{\mathcal{P}_{\leq 1}}(X, P_1) \neq 0$, so X cannot be Ext-projective.

Assume that $X \in \mathcal{P}_{\leq 1}$ is not injective. Then $\mathrm{id}\,X = 1$ by Lemma 7. There exists a non-split short exact sequence

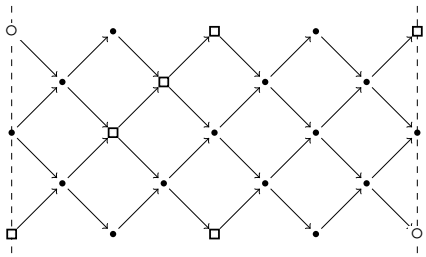
$$0 \to X \to I_0 \to I_1 \to 0,$$

which is also non-split in $\mathcal{P}_{\leq 1}$. Hence X is not Ext-injective. □

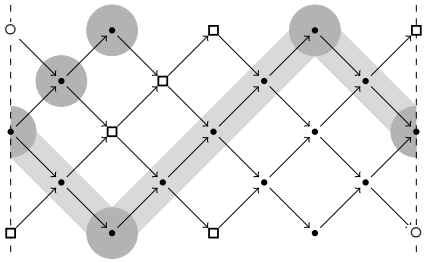
We now describe the objects of $\mathcal{C}$ that $\mathrm{Hom}_{\mathcal{T}}(T, -)$ send to $\mathcal{P}_{\leq 1}$.

THEOREM 9 ([5, Theorem 1.1]). *Let $X \in \mathcal{C}$. Then* $\mathrm{Hom}_{\mathcal{C}}(T, X)$ *has infinite projective dimension if and only if there exists a non-zero morphism $\tau T \to \tau T$ factoring through X.*

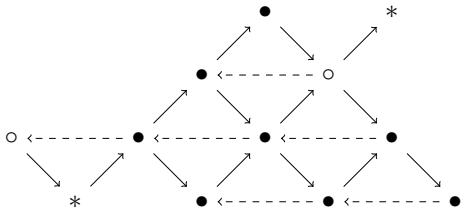
EXAMPLE 10. Let $\mathcal{C}$ be the cluster category over $\mathbb{A}_5$, and let T be the cluster-tilting object that is the direct sums of the objects marked with □ in the below figure.



We shade the summands of τT dark grey. The objects X such that $\operatorname{pd}\operatorname{Hom}(T, X) = \infty$ are shaded light grey.



The AR-quiver of $\mathcal{P}_{\leq 1}$ is:



The nodes marked $*$ and $\circ$ are identified.

4. Surface algebras

As shown in [3], surface algebras are useful when we study cluster-tilted algebras. In particular, they give a description of the AR-structure of the module category and cluster category of a gentle algebra [6]. In this section we recall the most relevant definitions and results from the two papers.

Let S be an oriented surface with boundary ∂S. Let M be a finite, nonempty set of points on ∂S such that there is at least one point from each connected component of ∂S in M. We call the pair (S, M) a *marked surface* (without punctures).

A *curve* in (S, M) is a continuous function $\gamma : [0, 1] \to S$ such that $\gamma(0), \gamma(1) \in M$. It is closed if $\gamma(0) = \gamma(1)$. Its inverse curve is γ^{-1} given by $\gamma^{-1}(t) = \gamma(1 - t)$.

An *arc* in (S, M) is a homotopy class of non-contractible curves in (S, M), subject to the equivalence relation $\gamma \sim \gamma^{-1}$. By abuse of notation we will denote the arc containing a curve γ simply by γ. If γ is homotopic to an arc on ∂S which only intersects M in its endpoints, we call it a *boundary arc*. Otherwise, we call it an *internal arc*.

A *triangulation* Δ of (S, M) is a maximal collection of non-intersecting arcs. A triangle in Δ is called an *internal triangle* if all the edges of the triangle are internal arcs. If α, β and γ are internal arcs forming a triangle, that triangle is denoted $\alpha\beta\gamma$.

The *quiver* Q_Δ *associated to a triangulation* Δ is given as follows.

Vertices: For each internal arc in Δ we associate one vertex in Q_Δ.

Arrows: For each triangle in Δ containing two internal arcs α and β, there is an arrow in $\mathcal{Q}_\Delta$ between the corresponding vertices $\alpha \to \beta$ if α is a predecessor to β with respect to a clockwise rotation at their joint end point in M.

The algebra associated to Δ is $A(\Delta) = k\mathcal{Q}_\Delta/I_\Delta$, where I_Δ is generated by all paths of length two which are part of a 3-cycle in $\mathcal{Q}_\Delta$. We call $A(\Delta)$ a *surface algebra*.

Surface algebras are gentle and finite-dimensional. Let Λ be a gentle path algebra where all the relations are from oriented 3-cycles with radical square zero. Then there exists a marked surface (S, M) with a triangulation Δ such that $\Lambda \cong A(\Delta)$. In particular,

let Λ be a cluster-tilted algebra of Dynkin type A_n, then we know that S is a disc and M contains $|M| = n + 3$ points [3].

We can also associate a cluster category to (S, M). Let Δ be a triangulation of (S, M). Let $\mathcal{C}_\Delta$ be the Amiot cluster category (see [1]) over (Q_Δ, W), where W is the potential corresponding to I_Δ above. We can show that the category $\mathcal{C}_\Delta$ is independent of the choice of triangulation Δ; hence we simply denote it by $\mathcal{C}_{(S,M)}$. The indecomposable objects of $\mathcal{C}_{(S,M)}$ are contained in two classes:

String objects: correspond to the internal arcs in (S, M). We set the boundary arcs to correspond to the zero object. For the arc γ we denote the corresponding string object by $C(\gamma)$

Band objects: indexed by $k^* \times \prod_1^*(S, M)/\sim$, where $k^* = k \setminus \{0\}$ and $\prod_1^*(S, M)/\sim$ is given by the nonzero elements of the fundamental group of (S, M) subject to the equivalence relation $\alpha \sim \alpha^{-1}$. We set (λ, b^0) to be equal to the zero object.

If Γ is a finite set of arcs, we set $\mathcal{C}(\Gamma) = \bigoplus_{\gamma\in\Gamma} C(\gamma)$.

The irreducible morphisms of $\mathcal{C}_{(S,M)}$ are most easily given by first defining two types of immediate successor objects:

- For any string object $C(\gamma)$, let ${}_sC(\gamma)$ and $C(\gamma)_e$ be the arcs obtained by moving respectively the start or end point of γ one step. Such a move is also called a elementary pivoting move. See Figure 1 for an illustration.

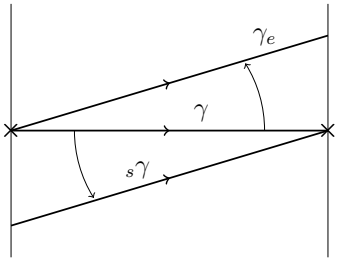


FIGURE 1. Elementary pivoting moves

- If (λ, b^n) is a band object, set $(\lambda, b^n)_e = (\lambda, b^{n-1})$ and ${}_s(\lambda, b^n) = (\lambda, b^{n+1})$.

Consider an indecomposable object $X \in \mathcal{C}_{(S,M)}$.

- If X_e is non-zero, there is an irreducible morphism $X \to X_e$.
- If ${}_sX$ is non-zero, there is an irreducible morphism $X \to {}_sX$.

All irreducible morphisms are of this form.

We see that ${}_e(X_s) = ({}_eX)_s = {}_eX_s$. The following commutativity relation holds on the irreducible morphisms:

$$X \to X_e \to {}_sX_e = X \to {}_sX \to {}_sX_e$$

If either X_e or ${}_sX$ is zero, this means that the other composition is zero.

The cluster category $\mathcal{C}_{(S,M)}$ is a triangulated category. The suspension functor is most easily described by its inverse: $X[-1] = {}_sX_e$. Moreover, the category $\mathcal{C}_{(S,M)}$ is 2-Calabi-Yau, and has AR-triangles:

$$X \to X_e \oplus {}_sX \to {}_sX_e \to X[1].$$

We can show that if S is a disc and $|M| \geq 4$, then $\mathcal{C}_{(S,M)}$ is equivalent to the cluster category of type $A_{|M|-3}$.

The triangulations Δ of (S, M) correspond to cluster-tilting objects in $\mathcal{C}_{(S,M)}$. For convenience, we let $C(\Delta) = \tau T_\Delta$, where T_Δ is a cluster-tilting object. Mutation on a Q_Δ corresponds to switching out an arc in Δ.

We know that the quotient category $\mathcal{C}/\tau T_\Delta$ is equivalent to $\operatorname{mod}\operatorname{End}(T)^{op}$ [7]. If we set the arcs of a triangulation Δ to correspond to zero objects, we obtain a description of the module category of $A(\Delta)$. Hence, from the geometric description of $\mathcal{C}_{(S,M)}$, we get a geometric description of $\operatorname{mod} A(\Delta)$, including a description of the AR-translation. For an internal arc γ which is not in Δ, we let $N(\gamma)$ denote the module corresponding to the arc γ. We see that $N(\gamma) \cong \operatorname{Hom}_{\mathcal{C}_{(S,M)}}(T_\Delta, C(\gamma))$.

We propose the following version of Theorem 9:

CONJECTURE 11. *Let (S, M) be a marked surface, and let Δ be a triangulation of (S, M). Let γ be an arc in (S, M) which is not in Δ. The following are equivalent:*

(1) *The $A(\Delta)$-module $N(\gamma)$ has infinite projective dimension.*
(2) *There is an internal triangle $\alpha\beta\delta$ in Δ where α is a predecessor to γ and γ is a predecessor to β with respect to clockwise rotation at $\gamma(0)$. We say that γ is* trapped by a triangle.

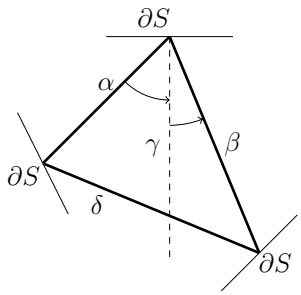


FIGURE 2. An arc γ trapped by an internal triangle $\alpha\beta\delta$.

Suppose condition (2) holds. There must be an arrow between the vertices corresponding to α and β in $\mathcal{Q}_\Delta$. The arrow corresponds to a non-zero morphism $C(\alpha) \to C(\beta)$ in the cluster category. That morphism factors through $C(\gamma)$, see Figure 2. By Theorem 9, the module $N(\gamma)$ has infinite projective dimension. Thus, condition (2) implies condition (1).

In Section 5 we show that the reverse implication, and thus the whole conjecture, holds in Dynkin type A.

5. $\mathcal{P}_{\leq 1}$ in type A

We will show that Conjecture 11 holds in type A. Thus we have a very nice description of the modules of infinite projective dimension: they are the ones corresponding to arcs trapped by some internal triangle of Δ. In this section we assume that S is a disc and $|M| \geq 4$. Hence $C_{(S,M)}$ will be the cluster category of type A_n, where $n = |M| - 3$.

The derived category of an algebra of type A_n is isomorphic as a translation quiver to $\mathbb{Z}A_n$. Let ϕ be the graph automorphism on $\mathbb{Z}A_n$ corresponding to the functor $\tau^{-1}[1]$. The AR-quiver of the cluster category is equivalent to the stable translation quiver $\mathbb{Z}A_n/\phi$. This quiver has a Möbius band shape [9], see Figure 3.

For an indecomposable object X in the cluster category $\mathcal{C}$, the Hom-hammock of X is the set of objects Y such that $\operatorname{Hom}_{\mathcal{C}}(X, Y) \neq 0$. Similarly, the Ext-hammock is the set of objects Y such that $\operatorname{Ext}_{\mathcal{C}}(X, Y) \neq 0$.

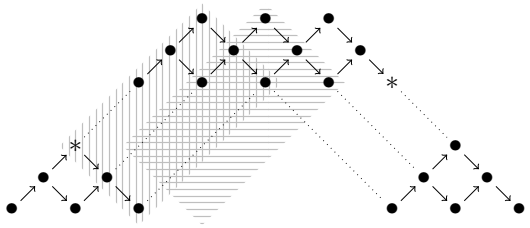


FIGURE 3. The AR-quiver of the cluster category of an algebra of type A_n. The first row and the last row are identified, with the $*$ indicating the identification of nodes. The Hom-hammock of $*$ is shaded with vertical lines. The Ext-hammock of $*$ is shaded with horisontal lines.

Let T_1 and T_2 be two indecomposable summands of a cluster-tilting object T, and suppose there is a non-zero morphism $T_1 \to T_2$. Then T_2 must lie in the Hom-hammock of T_1. However it cannot lie in the Ext-hammock of T_1, as they are both part of a cluster-tilting object. As we see in Figure 3 we are left with very few options for where in the AR-quiver T_2 may lie with respect to T_1.

We have an analogous result for triangulations on (S, M):

LEMMA 12. *Let α and β be internal arcs in the triangulation Δ in (S, M). If there is a non-zero morphism $f : C(\alpha) \to C(\beta)$, then α and β share a vertex, and f corresponds to rotation about this vertex.*

PROOF. Assume that α and β do not share a vertex. Since Δ is a triangulation, the arcs α and β cannot intersect. We are in the situation of Figure 4. We number the points of M anti-clockwise along ∂S by 1 to $|M|$. Let the a_1 and a_2 denote the vertices of α, and let b_1 and b_2 denote the vertices of β.

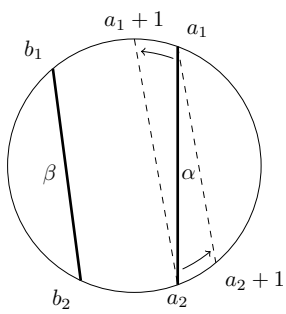


FIGURE 4. The situation in the proof of Lemma 12 if α and β do not share a vertex. The dashed arcs show the irreducible morphisms (elementary pivoting moves) from α.

We assume without loss of generality that f is a composition of finitely many irreducible maps [2, Cor. 5.5], corresponding to a sequence of elementary pivoting moves on α. We need to move the two vertices of α to the vertices of β without going through a boundary component. Due to the mesh relations, we can assume that we do all moves for a_1, and then all the moves for a_2, see Figure 5. In the figure we also see that since f is non-zero, we can assume without loss of generality that we first map a_1 to b_2, and then a_2 to a_1. However, this map also factors through (for example) the arc from a_1 to

$a_1 + 1$, which is a boundary arc. Boundary arcs represent the zero object, so f must be zero, giving us a contradiction.

It follows that α and β must share at least one vertex. By a similar argument, we see that f actually must correspond to a series of elementary pivoting moves about this vertex. □

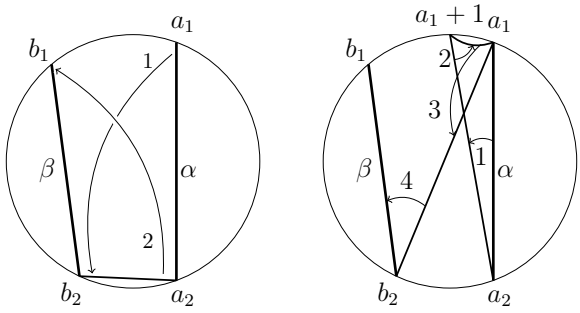


FIGURE 5. The two decompositions of $f : C(\alpha) \to C(\beta)$ discussed in Lemma 12 as seen on the marked surface.

We are now ready to prove Conjecture 11 in type A.

THEOREM 13. *Let (S, M) be a marked surface, where S is a disc and $|M| \geq 4$. Let Δ be a triangulation of (S, M). Let γ be an arc in (S, M) which is not in Δ. The following are equivalent:*

(1) *The $A(\Delta)$-module $N(\gamma)$ has infinite projective dimension.*
(2) *There is an internal triangle $\alpha\beta\delta$ of Δ, where α is a predecessor to γ and γ is a predecessor to β with respect to clockwise rotation at $\gamma(0)$. We say that γ is* trapped *by a triangle.*

PROOF. As we discussed after stating Conjecture 11, if γ is trapped by the triangle $\alpha\beta\delta$, then there is a non-zero morphism $C(\alpha) \to C(\beta)$ which factors through $C(\gamma)$. By Theorem 9, the module $N(\gamma)$ has infinite projective dimension.

Suppose $C(\gamma)$ has infinite projective dimension. We use Theorem 9 and the correspondence between the cluster category and arcs on the surface. We find that there are arcs α and β in Δ, and a non-zero morphism $f : C(\alpha) \to C(\beta)$ which factors through $C(\gamma)$. By Lemma 12, the arcs α and β share a vertex x, and f corresponds to a rotation of α about x.

By [2, Cor. 5.5] we can assume without loss of generality that any non-isomorphism is a finite composition of irreducible morphisms. In the case of f it corresponds to a series of elementary pivoting moves about x.

Let $g : C(\alpha) \to C(\gamma)$ and $h : C(\gamma) \to C(\beta)$ be such that $f = hg$. The maps g and h can also be assumed to be a finite composition of irreducible morphisms. Suppose that the corresponding elementary pivoting move of one of the morphisms is not a move about x. By applying the commutativity relation on $\mathcal{C}_{(S,M)}$ a finite number of times, we get that f factors through $\tau^{-1}C(\alpha) = {}_sC(\alpha)_e$; see Figure 6.

This means $\tau^{-1}C(\alpha)$ has infinite projective dimension, but it is supposed to be projective in the module category. Hence we reach a contradiction.

It follows that g and h must be composed of irreducible morphisms corresponding to elementary pivoting moves about x. Hence γ, α and β share x as a vertex, and moving

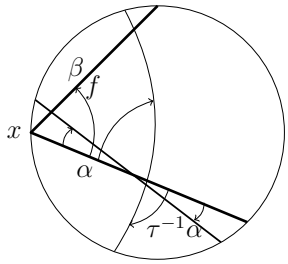


FIGURE 6. Let g be a composition of irreducible morphisms, starting in $C(\alpha)$. If at least one elementary pivoting move of g is not about x, then g factors through $\tau^{-1}\alpha$.

anticlockwise about x we can go from α to γ and then to β. That α and β must be part of an internal triangle $\alpha\beta\delta$ follows from maximality of the triangulation. □

5.1. Number of indecomposables. For a cluster-tilted algebra Λ of Dynkin type A, we want to investigate $|\mathcal{P}_{\leq 1}|$, the number of indecomposable objects in $\mathcal{P}_{\leq 1}$ up to isomorphism.

Let Q be a quiver with n vertices. In [8], Buan and Vatne showed that Q is the quiver of a cluster-tilted algebra of type A_n if and only if the following are satisfied:

- All non-trivial cycles are of length three.
- Any vertex has at most four neighbours.
- If a vertex has four neighbours, then two of the arrows belong to one three-cycle and the two other arrows belong to another three-cycle.
- If a vertex has three neighbours, then two of the arrows belong to a three-cycle, and the last arrow does not belong to a three-cycle.

They also showed that that two cluster-tilted algebras of type A_n are derived equivalent if and only if their quivers have the same number of three-cycles.

We will show that $|\mathcal{P}_{\leq 1}|$ depends only on the number of vertices and three-cycles in the quiver of Λ; this means that $|\mathcal{P}_{\leq 1}|$ is invariant under derived equivalence.

Let $\mathcal{C}_{A_n}$ be the cluster category of type A_n. Then $|\mathcal{C}_{A_n}| = \frac{1}{2}n(n+3)$, as seen in [9]. If we factor out a cluster-tilting object T, then we factor out n non-isomorphic objects; hence $|\operatorname{mod}\Lambda| = \frac{1}{2}n(n+1)$. We denote by $\mathcal{S}$ the full subcategory of objects of infinite projective dimension. Cluster-tilted algebras have Gorenstein dimension one. Thus an object is either in $\mathcal{P}_{\leq 1}$ or it has infinite projective dimension, so $|\mathcal{S}| + |\mathcal{P}_{\leq 1}| = |\operatorname{mod}\Lambda|$.

By Theorem 13, finding $|\mathcal{S}|$ reduces to finding the number of arcs trapped by a triangle. Remember that when representing an algebra of type A_n by a marked surface (S, M), we let S be a disc and $|M| = n + 3$.

LEMMA 14. *Let (S, M) be a marked surface, where S is a disc. Let Δ be a triangulation of (S, M). One internal triangle of Δ traps exactly $|M| - 3$ arcs.*

PROOF. Each vertex x of the internal triangle will trap arcs from x to the vertices that lie between the other two vertices of the triangle. So for each marked point which is not a vertex in the triangle, one arc is trapped by the triangle. Thus $|M| - 3$ arcs are trapped by the triangle. Figure 7 gives a visualisation of the proof. □

Naively, we might then conclude that if Δ contains t triangles, then $|\mathcal{S}| = t(|M| - 3) = tn$. However, we may have counted some arcs more than once, as one arc can be trapped by two internal triangles (one at each vertex). Obviously, an arc cannot be trapped by more

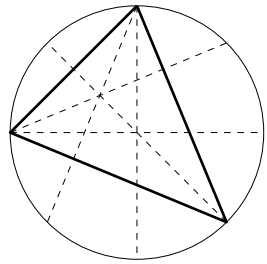
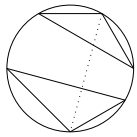
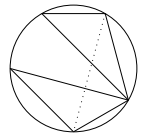


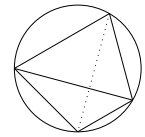
FIGURE 7. Arcs trapped by a triangle



(A) no shared vertices



(B) 1 shared vertex



(C) 2 shared vertices

FIGURE 8. The three cases in the proof of Lemma 15. The dashed line represents the trapped arc.

than two triangles, nor can it be trapped twice by one triangle, so if we find out how many arcs one pair of triangles can trap, we know how many arcs we have counted twice.

LEMMA 15. *For each pair of Δ-triangles, exactly one arc is trapped by both Δ-triangles.*

PROOF. Consider a pair of internal triangles. As seen in Figure 8, the two triangles can share 0, 1 or 2 vertices. In all cases, it is obvious that only one arc can be trapped by both triangles. □

THEOREM 16. *Let Λ be a cluster-tilted algebra of type A_n. Let t be the number of three-cycles in the corresponding quiver. Then the number of indecomposable objects in $\mathcal{P}_{\leq 1}$ is*

$$|\mathcal{P}_{\leq 1}| = \frac{n(n+1)}{2} - nt + \frac{t(t-1)}{2}$$

PROOF. As we mentioned before, the number of indecomposable modules in $\operatorname{mod}\Lambda$ is $\frac{1}{2}n(n+1)$. Each of the t triangles traps n arcs. Each pair of triangles trap one common arc, and t triangles form $\frac{1}{2}t(t-1)$ distinct pairs of triangles. Hence $|\mathcal{S}| = nt - \frac{1}{2}t(t-1)$, and the result follows. □

EXAMPLE 17. We consider the cluster-tilting object T given in Example 10. We consider it as a triangulation of (S, M) with $|M| = 8$; see Figure 9.

With reference to the notation in Theorem 16 we have $n = 5$ and $t = 1$, hence the size of $\mathcal{P}_{\leq 1}$ is 10, as we also saw in Example 10.

5.2. AR-translation. Let $\tau_{\leq 1}$ denote the AR-translation in $\mathcal{P}_{\leq 1}$, and let $\mathfrak{r}_{\leq 1}X \to X$ be the minimal right $\mathcal{P}_{\leq 1}$-approximation of X.

From by Theorem 5 we know that $\mathfrak{r}_{\leq 1}\tau X = \tau_{\leq 1}X \oplus I$, where I is Ext-injective. This means that if we can calculate the right $\mathcal{P}_{\leq 1}$-approximation, then we can calculate the AR-translation. For the sake of completeness, we also show how to compute the minimal left $\mathcal{P}_{\leq 1}$-approximation $X \to \mathfrak{l}_{\leq 1}X$.

THEOREM 18. *Let $X \in \operatorname{mod}\Lambda$ be indecomposable. Then*

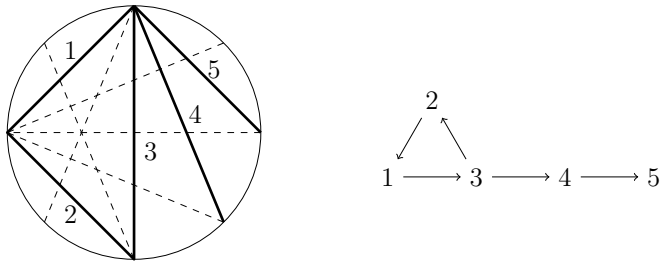


FIGURE 9. A cluster-tilting object T in $\mathcal{C}(\mathbb{A}_5)$ considered as a triangulation of (S, M). The dashed arcs correspond to indecomposables of infinite projective dimension in $\operatorname{mod}\operatorname{End}(T)^{op}$. On the right, the quiver Q_T corresponding to T.

$$\mathfrak{r}_{\leq 1}X = \begin{cases} X & \text{if } \operatorname{pd} X \leq 1 \\ Y & \text{if } \operatorname{pd} X = \infty, \exists \text{ irreducible } Y \to X, \operatorname{pd} Y \leq 1 \\ \tau X & \text{otherwise,} \end{cases}$$

where Y is unique up to isomorphism if it exists. Dually,

$$\mathfrak{l}_{\leq 1}X = \begin{cases} X & \text{if } \operatorname{pd} X \leq 1 \\ Z & \text{if } \operatorname{pd} X = \infty, \exists \text{ irreducible } X \to Z, \operatorname{pd} Z \leq 1 \\ \tau^{-1}X & \text{otherwise,} \end{cases}$$

where Z is unique up to isomorphism if it exists.

PROOF. Let γ be the arc corresponding to X (so that $N(\gamma) = X$) and let α and β be its immediate predecessors. We illustrate the setup in Figure 10. Furthermore, let T be the cluster-tilting object such that $\operatorname{End}(T)^{op} = \Lambda$ and let Δ be the triangulation of (S, M) corresponding to T.

If X has finite projective dimension, it is obviously its own right $\mathcal{P}_{\leq 1}$-approximation, and we are done.

If X has infinite projective dimension, then γ is trapped by a triangle. As we see from Figure 10, at least one of α or β must either be trapped by that same triangle, or be an arc in the triangle. In other words, there are three possibilities:

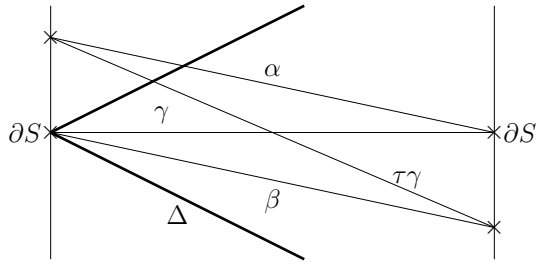


FIGURE 10. Illustration for the proof of Theorem 18; at least one of α or β must be trapped by or part of the triangle trapping γ.

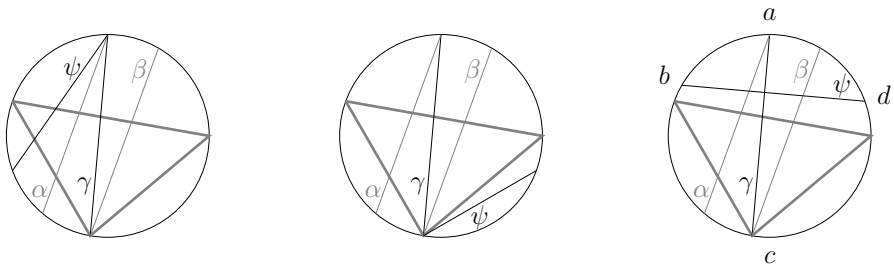
Case 1: $\operatorname{pd} N(\alpha) \leq 1$ and $\operatorname{pd} N(\beta) = \infty$.
Case 2: $\operatorname{pd} N(\alpha) = \infty$ and $\operatorname{pd} N(\beta) \leq 1$.

Case 3: $\operatorname{pd} N(\alpha) = \operatorname{pd} N(\beta) = \infty$.

In Case 1 we see that α is not trapped by or part of a triangle, but β is. We will show that $N(\alpha) \to N(\gamma)$ is the minimal right approximation of X. Minimality follows from irreducibility. It remains to show that an arbitrary morphism $f : N(\psi) \to N(\gamma)$ with $N(\psi) \in \mathcal{P}_{\leq 1}$ factor through $N(\alpha)$. If f is zero, we are done, so we assume that f is non-zero. We also assume without loss of generality that it is a composition of irreducible morphisms.

Consider the arcs ψ and α. Either the arcs intersect in the interior of the disc S, or they do not.

Suppose that ψ and γ have no intersections except possibly on their end points. Then the set $\{\gamma, \psi\}$ can be extended to a triangulation Δ' of (S, M). Since f is non-zero, we have by Lemma 12, that ψ shares a vertex with γ, and lie before γ with respect to anti-clockwise rotation about this vertex. Thus it must also share a vertex with α or β.



(A) Predecessor to α (B) Predecessor to β (C) Crossing γ

FIGURE 11. Three possible placements of ψ used in the proof of Theorem 18, Case 1.

If ψ shares a vertex with α, then f clearly factors through $N(\alpha)$, as illustrated in Figure 11a.

Suppose ψ shares a vertex with β. Let δ be the arc in the triangle trapping β and γ which lies before β going anti-clockwise about their shared vertex; see Figure 11b. Then the morphism $C(\psi) \to C(\gamma)$, which is the preimage of f in the cluster category, factors through $C(\delta)$. Hence $f = 0$ which is a contradiction.

On the other hand, suppose ψ and γ do intersect in the interior of S. Let a, c be the vertices that γ share with α and β respectively. Let b, d be the vertices of ψ, labelled such that when moving anti-clockwise around ∂S we encounter the vertices in the order a, b, c, d; see Figure 11c. Then f corresponds, up to mesh relation, to first a rotation about b, and then a rotation about a. Hence f factors through $N(\alpha)$.

It follows that for Case 1 the right minimal $\mathcal{P}_{\leq 1}$-approximation of $N(\gamma) = X$ is $N(\alpha) \to X$.

In Case 2, we can use a symmetric proof to the one for Case 1 to show that the right minimal $\mathcal{P}_{\leq 1}$-approximation of $N(\gamma) = X$ is $N(\beta) \to X$.

In Case 3, both α and β are trapped by or part of some triangle. We cannot have that $\tau\gamma \in \Delta$, as in that case $N(\gamma)$ is projective, and thus has finite projective dimension.

Let $\delta \in \Delta$ be an arc in the triangle trapping γ on the vertex shared with α, such that δ lies after γ when going anticlockwise about the vertex; see Figure 12. Any arc following $\tau\gamma$ when going anticlockwise about the vertex shared with β must cross δ. Hence $\tau\gamma$ cannot be trapped by a triangle on the vertex shared with β. Symmetrically,

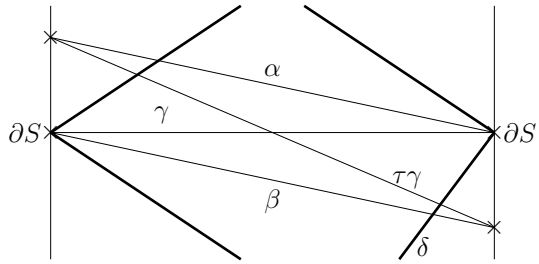


FIGURE 12. Illustration for the proof of Theorem 18. If both α and β are trapped by or part of the triangle, then $\tau\gamma$ is not trapped by or part of the triangle.

it cannot be trapped by a triangle on the vertex shared with α. It follows that $N(\tau\gamma)$ has finite projective dimension. We will show that $N(\tau\gamma) \to N(\gamma)$ is the right minimal $\mathcal{P}_{\leq 1}$-approximation of $N(\gamma) = X$.

Let $N(\psi)$ be a module of finite projective dimension, and let $f : N(\psi) \to N(\gamma)$ be a morphism. We will show that this morphism factors through $N(\tau\gamma)$. By repeating the proof for Case 1 we see that f factors through both $N(\alpha)$ and $N(\beta)$. By the mesh relations this means that it factors through $N(\tau\gamma)$ as well. It follows that in Case 3 the right minimal $\mathcal{P}_{\leq 1}$-approximation of $N(\gamma) = X$ is $N(\tau\gamma) \to N(\gamma)$.

The calculation of the left approximation is dual. □

COROLLARY 19.

$$\tau_{\leq 1}X = \mathfrak{r}_{\leq 1}\tau X = \begin{cases} \tau X & \text{if } \operatorname{pd}\tau X \leq 1 \\ Y & \text{if } \operatorname{pd}\tau X = \infty, \exists \text{ irred. } Y \to \tau X, \operatorname{pd} Y \leq 1 \\ \tau^2 X & \text{otherwise} \end{cases}$$

where Y is unique up to isomorphism if it exists.

PROOF. We know from Theorem 5 that $\mathfrak{r}_{\leq 1}\tau X = \tau_{\leq 1}X \oplus I$, where I is injective. Theorem 18 shows that right $\mathcal{P}_{\leq 1}$-approximations of indecomposables are again indecomposable. Hence $I = 0$, and the corollary follows. □

EXAMPLE 20. We still consider the cluster-tilting object T given in Example 10. We consider it as a triangulation of a marked surface as in Example 17. Let γ be the arc marked in Figure 13. Then we see that $\tau N(\gamma) = N(\tau\gamma)$ has infinite projective dimension. Using Corollary 19, we first look at the immediate predecessors of $\tau\gamma$, denoted α and β. We see that α is trapped by a triangle, but β is not. Hence $\tau_{\leq 1}N(\gamma) = N(\beta)$.

Acknowledgements. The author would like to thank Aslak Bakke Buan for many helpful discussions.

References

[1] C. Amiot. "Cluster categories for algebras of global dimension 2 and quivers with potential". In: *Ann. Inst. Fourier (Grenoble)* 59.6 (2009), pp. 2525–2590.

[2] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1.* Vol. 65. Techniques of representation theory. Cambridge University Press, Cambridge, 2006.

[3] I. Assem et al. "Gentle algebras arising from surface triangulations". In: *Algebra Number Theory* 4.2 (2010), pp. 201–229.

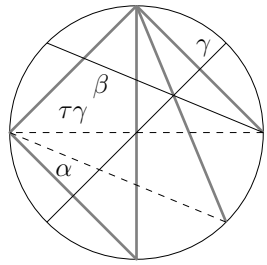


FIGURE 13. Illustration for Example 20. Arcs in the chosen triangulation are drawn with bold lines, arcs trapped in the triangle are dashed, and arcs corresponding to finite dimensional modules are thin lines.

[4] M. Auslander and S. O. Smalø. "Almost split sequences in subcategories". In: *J. Algebra* 69.2 (1981), pp. 426–454.

[5] L. Beaudet, T. Brüstle, and G. Todorov. "Projective dimension of modules over cluster-tilted algebras". In: *Algebr. Represent. Theory* 17.6 (2014), pp. 1797–1807.

[6] T. Brüstle and J. Zhang. "On the cluster category of a marked surface without punctures". In: *Algebra Number Theory* 5.4 (2011), pp. 529–566.

[7] A. B. Buan, R. J. Marsh, and I. Reiten. "Cluster-tilted algebras". In: *Trans. Amer. Math. Soc.* 359.1 (2007), 323–332 (electronic).

[8] A. B. Buan and D. F. Vatne. "Derived equivalence classification for cluster-tilted algebras of type A_n". In: *J. Algebra* 319.7 (2008), pp. 2723–2738.

[9] A. B. Buan et al. "Tilting theory and cluster combinatorics". In: *Adv. Math.* 204.2 (2006), pp. 572–618.

[10] P. Caldero, F. Chapoton, and R. Schiffler. "Quivers with relations arising from clusters (A_n case)". In: *Trans. Amer. Math. Soc.* 358.3 (2006), pp. 1347–1364.

[11] K. Igusa, S. O. Smalø, and G. Todorov. "Finite projectivity and contravariant finiteness". In: *Proc. Amer. Math. Soc.* 109.4 (1990), pp. 937–941.

[12] B. Keller and I. Reiten. "Cluster-tilted algebras are Gorenstein and stably Calabi-Yau". In: *Adv. Math.* 211.1 (2007), pp. 123–151.

[13] M. Kleiner. "Approximations and almost split sequences in homologically finite subcategories". In: *J. Algebra* 198.1 (1997), pp. 135–163.

[14] M. Kleiner and E. Perez. "Computation of almost split sequences with applications to relatively projective and prinjective modules". In: *Algebr. Represent. Theory* 6.3 (2003), pp. 251–284.

[15] D. Smith. "On tilting modules over cluster-tilted algebras". In: *Illinois J. Math.* 52.4 (2008), pp. 1223–1247.