Navn på forfatter
Marius Fagerli

Slutten på olje: Multiperiodiske porteføljevalg med likviditetsbegrensninger og ekstern kontantstrøm, samt CvaR risikokontroll og multivariat GARCH modell

The end of oil: Multiperiod portfolio decisions under liquidity constraints and external cashflow with CVaR risk control and multivariate GARCH model

MASTEROPPGAVE - Økonomi og administrasjon/siviløkonom
Trondheim, 26. Mai 2016

Hovedprofil: Finansiering og investering

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Konfidensiell til: -

Samarbeidsbedrift: -

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Abstract

This master thesis examines the modelling and solution of the allocation effect an external, correlated cashflow has on a two asset portfolio with liquidity constraints. This is interesting because research shows that a constant proportion portfolio normally has a good risk/return trade-off at 50/50 or 60/40. In practical portfolio management this is a popular allocation policy, e.g. in the Norwegian Government Pension Fund Global. If an external cashflow to and from the portfolio calls for another allocation to attain an optimum under CVaR risk control, the self-financing portfolio theory used in practical portfolio management would consequently be suboptimal under these conditions. In this thesis the optimal allocations are tested for four types of cashflow conditions; correlated, none, negative and positive, under full and low liquidity for identical scenarios. The objective function minimizes CVaR at a given expected return level for all cases by allocating between equity and fixed income securities in a 5-stage stochastic linear programming with recourse using Matlab. The optimization scenarios is modeled by means of a BEKK multivariate GARCH(1,0,1) model, adopted to provide characteristics of conditional variance in the price processes. The procedure used to incorporate liquidity and cashflow constraints are discussed in detail. Numerical results show that cashflow is actually irrelevant both in full and low liquidity conditions for all types of cashflow.

Keywords: Portfolio allocation, CVaR, Multistage Stochastic Linear Programming with recourse, BEKK MGARCH, liquidity, external cashflow.

Sammendrag

Denne masteroppgaven modellerer og undersøker løsningen en ekstern, korrelert kontantstrøm har på en portføljes allokering med to aktiva under likviditetsrestriksjoner. Dette er interessant fordi forskning viser at en portefølje med konstant andelsmessig eksponering mot aktivaene har best risiko/avkastningsforhold med 50/50- eller 60/40-vekter. I praktisk porteføljeforvaltning er dette ofte en populær allokeringsregel, f.eks. i Statens Pensjonsfond Utland. Dersom en ekstern kontantstrøm fremtvinger en annen allokering for å nå et optimum under CvaR risikokontroll, så vil det si at en allokering i henhold til selv-finansierende porteføljeteori i samme tilfellet ville være suboptimalt. Denne oppgaven tester optimale allokeringer for fire typer kontantstrømmer: korrelert, ingen, negativ og positiv, under komplett og lav likviditet i identiske scenarioer. Målfunksjonen minimerer CvaR for et gitt forventet avkastningsnivå i alle tilfeller ved å allokere mellom aksjer og obligasjoner i en 5-periodisk stokastisk lineærprogrammering med rekursjon i Matlab. Scenarioene i optimeringen er modellert ved hjelp av en BEKK multivariat GARCH modell, som er anvendt på grunn av sin evne til å modellere betinget varians i prisprosessene. Metoden for å implementere likviditets- og kontantstrømsrestriksjoner er diskutert i detalj. Resultatene viser at kontantstrømmen faktisk er irrelevant med både komplett og lav likviditet for alle typer kontantstrømmer.

Nøkkelord: Portføljeallokering, CVaR, multiperiodisk stokastisk lineærprogrammering med rekursjon, BEKK MGARCH, likviditet, ekstern kontantstrøm.

Acknowledgements

I am thankful to Dr. Denis Becker for providing guidance and insight into the art of optimization, and Dr. Are Oust for an inspiring topic and fruitful discussions on portfolio theory. I also want to thank my wife and daughter for being patient and supportive, and finally Hans Aaknes and Fredrik Sjetne who have been great discussion partners during the planning of this work.

Contents

1 Introduction

Since 2010 the target weights of the Norwegian Government Pension Fund Global (GPFG) has been 60 % equity, 35 % fixed income securities and 5 % real estate. A constant proportion portfolio allocation policy as such is meant to maintain a constant exposure of stocks proportional to the current portfolio wealth. This means selling off equity on rising stock prices, and buying equity on declining stock prices: A sort of "buy cheap, sell expensive"-policy. This is thoroughly studied in the field of finance and the 60/40-allocation has proven to be good risk/return allocation for self-financing portfolios. But, contrary with the self-financing portfolio, the GPFG has an external cashflow transferred to the portfolio several times a year, which originates from the petroleum activites in the North Sea. Also, it has a withdrawal once a year to cover the government fiscal budget spending. Additionally, 2016 is the first year in the GPFG's 26 year long history where the petroleum related cashflow is negative, indicating a possible change in the Norwegian prospects. This said Øystein Olsen in his Norges Bank Governor's yearly speech for 2016, and highlighting how the end of the prosperous oil income for the Norwegian government may be near. Further, he said that the Norwegian nation must prepare for persistently low oil prices and lower spending in years to come[1].

Certainly, such prospects should demand a sober strategy in order to reduce the risk of large losses, give a reasonable return and secure the wealth for future generations. Therefore, I wish to study whether or not the optimal allocation weights in a standard two asset portfolio is influenced by the introduction of a withdrawing or transferring cashflow from and to the portfolio. The assets are equity and bonds, whereas the cashflow is petroleum related. I employ an optimization method where the objective is to minimize risk (measured as conditional value-at-risk) at a given expected return level. This involves a BEKK-MGARCH model for conditional covariances in the scenario generations and portfolio optimization with 5-stage stochastic linear programming. The model includes recursion, which allows for intermediate rebalancings of the portfolio, that way creating an environment where one can study the constant proportion portfolio for different cashflow types. I have chosen this model because of its flexibility and ability to account for stochasticity. Such models base decisions on long term benefits and avoiding myopic short term choices, which in the long run could prove to be suboptimal. I have segmented the portfolios into a 1) portfolio with a correlated cashflow, 2)

[1] For more on the GPFG see appendix.

no cashflow (which serves as the self-financing portfolio), 3) negative cashflow and 4) positive cashflow. Additionally, I study the portfolio allocations of 1) to 4) in both a fully liquid and a low liquid market. Liquidity risk is a well studied phenomenon, and the introduction of illiquidity creates restrictions on the allocation possibilities due to the reduced ability to invest or divest in a certain asset. Within certain limits I expect that the portfolio can be allocated in a way that offsets the liquidity restrictions, still reaching the same risk/return as the full liquidity case. As a result, a consistent weight spread between the self-financing portfolio, and the positive and negative cashflow portfolios should emerge. If this is proves to be the case, a self-financing portfolio allocation would be suboptimal for cashflow portfolios. Then it is interesting to see how this affects the portfolio with a *correlated* cashflow (e.g. oil income correlated with equities), which should also be allocated differently as well.

Hypothesis H_1: Under the assumption of a perfect, fully liquid market a financial portfolio with cash in- and outflow will generally have the same average risk/return-optimal allocation weights as the theoretical self-financing portfolio.

Hypothesis H_2: Under the assumption of a low-liquid market, the average allocation of a financial portfolio with a strictly positive or negative cashflow will differ from the theoretical self-financing portfolio. Thus, a portfolio with cash in- and outflow that is correlated with equity will also differ in average optimal allocation compared to the theoretical self-financing portfolio.

Section 2 presents relevant theory, section 3; the method used to solve the problem, Section 4; the data used in the numerical results in section 5, and concluding remarks is given in section 6.

Apart from references stating otherwise, the content of this thesis is the result of the work, and therefore the sole responsibility, of the author.

2 Theory

2.1 Interdependent markets

In accord with advancing globalization and international markets, the price movements in economic factors and commodities has become increasingly more interdependent and correlated. In his ground breaking work on the subject, Hamilton J.D. (1983) studies the relationship between oil shocks and macroeconomic factors after the second World War. He discovered that oil prices to some degree altered the timing, magnitude and duration in a number of macroeconomic recessions in the period. After this study a vast amount of research has been done on the evolving interrelationships between oil prices impact and macroeconomic factors. E.g. Jones C. and Kaul G. (1996) was some of the first to test whether the reaction of international stock markets to oil shocks can be justified by current and future changes in real cash flows and/or changes in expected returns. They succeed to explain a rational reaction to oil price change in the U.S. and Canadian stock market, but were unable to fully model the same phenomenon in the Japanese and U.K. stock market, even though still indicating some relationship.

2.2 Crude oil and stock markets

In later years, an equivalent relationship between oil prices and asset classes has been investigated. The common findings so far is that there exist some sort of cross asset and cross commodity relationship, which exhibits dynamic, volatile and time varying correlation patterns (see i.e. Malik F. and Hammoudeh S. (2007), Filis G. Degiannakis S. and Floros C. (2011), Turhan M. I. et. al. (2014) and the references therein). Malik F. and Hammoudeh S. (2007) study the volatility and shock transmission mechanisms between U.S. equity, global market and three middle eastern countries by means of a multivariate BEKK-GARCH model. A thorough elaboration of this model will be conducted later in this thesis. Their findings show significant variance transmission in the US equity and global oil markets, and a spillover effect to other equity markets as well. Ewing and Thompsons (2007) found that crude oil prices are pro-cyclical and lag stock prices by 6 months (in Filis G. Degiannakis S. and Floros C. 2011). Filis G. Degiannakis S. and Floros C. (2011) further investigate the dynamic correlation with multivariate GARCH models, and find that time-varying correlations occur according to phases in the global business cycle and periods of world turmoil. Turhan M. I. et. al. (2014) reveal a trend in the period after the global 2008 financial crisis by means of Markov switching regressions. They find that there is an increasing positive correlation in both the stock-oil and bond-oil pairs, which previously has been oscillating around zero. They continue explaining this phenomenon by the shift in oil price mechanisms: Earlier it had been a mere result of supply and demand, but as financial markets has evolved, financial products written on oil have been used more for taking positions or speculating.

2.3 Crude oil and bond markets

A less studied subject is the relationship between oil and bond markets. It is anticipated that the connection in this case is channeled through inflationary expectations instead of traded financial derivatives (see e.g. Kilian and Lewis (2011) in Ciner C. Constantin G. and Lucey B. M. (2013)). Oppositely, Nicolau (2011, in Turhan M. I. et. al. 2014) studies cross-asset correlations between 1991 and 2009, and claims that there exists a negative correlation between bond and crude oil prices. Also, Ciner C. Constantin G. and Lucey B. M. (2013) examines the long-run dynamic relationship between crude oil futures traded on NYMEX, future prices of Brent oil and 10 year US and UK government bond prices, and the hedgeability of these in a financial portfolio. They find that there is a low correlation between oil and bond prices, with certain extraordinary breaks connected to world turmoil incidents.

2.4 Portfolio

The choice of an optimal portfolio of assets has been subject to substantial research over the last 60 years. The Markovitz (1952) mean-variance portfolio theory proposed the idea to allocate risky portfolio assets in such a way that return is maximized at a given risk-level along a portfolio frontier, implying a trade-off between risk and return. But, as time goes, the fluctuations due to market movements changes the allocations, and alters the initial risk-return profile. Perold A. F. and Sharpe W. F. (1988) examines a set of dynamic strategies dealing with the nesseccary response to trending and mean reverting markets. They find that dynamic constant-mix strategies, which maintains a constant exposure of stocks proportional to wealth, performes better than passive buy-and-hold strategies in reverting markets, but worse in upward trending markets. Oppositely, constant-proportion strategies buys stocks as they rise, and sell off as they decline, thus creating a floor, performing better than buy-and-hold strategies in upward trending markets, and worse in reverting markets. Many studies (Perold A. F. and Sharpe W. F. 1988, Harjoto M. A. and Jones F. J. 2006, Tokat Y. and Wicas N. 2007) use the 60-40 allocation as base. Jones and Wilson (2003) in Harjoto M. A. and Jones F. J. (2006) argue that investors, independent of their risk profile, should at least invest 40 % of their portfolio in equity, and further show that a 50-50 portfolio in stock and bonds has higher return and less risk compared to 100 % bonds.

2.5 Optimization

As future uncertainty is introduced into the portfolio decision, an optimization problem quickly becomes more complicated than the single stage problem. Stochastic linear programs (SLP) is a powerful toolbox in financial scenario modelling, first introduced by Dantzig G. B. (1955) and Beale E. M. L (1955). Ever since the early work of Mossin J. (1968), Merton R. C. (1969) and Samuelson P.A. (1969), being some of the first to successfully describe the multiperiod analysis of a portfolio, portfolio optimization has evolved far from the original single stage mean-variance optimization. They show how a sequence of portfolio decisions rely on the outcomes of the previous actions, at the same time accounting for the information of the future probability distribution. One of the increasingly more popular objectives in portfolio decision theory is to minimize risk defined as conditional Value-at-Risk at a given level (β-CVaR, hereby called CVaR), which was proposed by Rockafellar R.T. and Uryasev S. (2000). They show how this measure is superior in computational simplicity and shortfall risk controll compared to the simpler variance and Value-at-Risk (see e.g. Markovitz (1952) and Jorion P.H. (1996) respectively). CVaR is also known as mean excess loss or mean shortfall, stating for a given time horizon the mean expected shortfall under given confidence level. A rational investor would prefer low CVaR for high return, and in the specific case of SLP, CVaR is defined as the portfolio's worst outcome on average among all possible scenarios at the end of the horizon. Rockafellar R.T. and Uryasev S. (2002) show how VaR-centered problems in optimization is unstable entities while CVaR-centered problems is not, and further how CVaR can be formulated into an objective or constraint, or both. Also, optimizing CVaR means optimizing VaR because CVaR is always greater or equal to VaR (Rockafellar R.T. and Uryasev S. 2000, Uryasev S. 2000). Topaloglou N. Vladimirou H. and Zenios S. (2005) among others have developed elaborate multistage financial models using this objective under expected return constraints. Sakar C. T. and Köksalan M. (2013) found some interesting properties on the efficient frontiers with different CVaR probabilities (beta): There is a trade-off between CVaRs and returns. On one extreme with probability levels close to 100 %, the expected return levels will be close to maximum, whereas on the other extreme with probability levels close to 0 %, the expected return and CVaR are identical. Thus, with decreasing probability levels (betas), there is a positive correlation between risk and return.

2.6 Illiquidity in assets, correlation and portfolio choice

The portfolio choice litterature has over the years grown to a vast range of research. In their work on portfolio choice with illiquid assets, Ang A. Papanikolau D. and Westerfield M. (2013) present a model of optimal allocation to liquid and illiquid assets. They define illiquidity as "the difficulty in finding a counterparty with whom to trade, i.e. increasing bid-ask spread, where a transaction cannot be performed without suffering penalty through substantially lower asset price". They show that illiquidity leads to increased and state-dependent risk aversion, and reduces the allocation to both liquid and illiquid assets. Also, they show that annualized turnover varies heavily between assets classes. Public equity has over 100 % turnover annually, while OTC (Pinksheet) equities $\sim 35\ \%$ (Ang, Shtauber and Tetlock 2012), corporate bonds $25 - 35\ \%$ (Bao, Pan and Wang (2011)) and municipal bonds $< 10\ \%$ (Ang and Green (2011)) (all references in Ang A. Papanikolau D. and Westerfield M. 2013). A rational investor should prefer a high liquidity portfolio over a low one. In portfolio optimization under liquidity- and Conditional Value-at-Risk-constraints, Sakar C. T. and Köksalan M. (2013) employ a multi-period Stochastic Linear Programming on data from the Istanbul Stock Exchange. They demonstrate how the portfolio characteristics change at different levels of liquidity, CVaR and expected return. For low levels of liquidity the expected return is high, while the CVaR is low. They observe regions of efficient sollutions in the three criteria environment, and liquidity is considered as the factor that forces expected return and CVaR to their worst values. This means that a rational investor should be willing to sacrifice expected return and/or CVaR for good liquidity.

2.7 Multivariate dynamics

Also, understanding the comovements of financial returns in the portfolio is of great importance in order to successfully generate scenarios that represent the asset's price processes realistically. The SLP is not any better than it's input. It is therefore necessary to develop a model with the right statistical properties for the input in the scenario generation. This subject has therefore received a lot of attention in financial litterature (see e.g. Kaut (2003) for moment matching and otherwise Zivot E. (2008) for scenario generation with the references therein). Buraschi A. Porchia P. and Trojani F. (2010) study the importance of modeling the multivariate nature of second moments (variance/covariance) in optimal portfolio allocation, and say it is necessary to extend the portfolio scenario generation to model the volatility processes. The orgininal groundwork for this conditional variance modelling was orginially developed by

Engle R. F. (1982) in the famous ARCH model, and later the generalized ARCH (GARCH) model by Bollerslev T. (1986). They find that conditional covariances varies over time, and therefore is a significant determinant of the time-varying risk premia. As a result, the variance depends on the history of returns up to the estimation point, constantly changing at every point in time. So, the available information of past returns up to and including $t-1$ is non-stochastic, whereas the future outcomes is indeed stochastic. Also, the portfolio returns and diversification has been shown to depend on the covariance of the assets returns in a portfolio, meaning that some assets are positively or negatively correlated, while others are not. For this reason it is not only necessary to model the price processes depending on their own historic volatility, but also the covariance in the portfolio. Bollerslev, Engle, and Wooldridge (1998) in Silvennoinen A. and Teräsvirta T. (2009) demonstrate the multivariate VEC GARCH model, which allows both the variance and covariance to depend on the (historic) information set (e.g. in a return vector). A difficulty with these models is the estimation of a growing number of parameters. This often leads to infeasible sollutions, forcing simplifications and restrictions. Further development has been done to formulate more parsimonious models e.g. imposing positive definiteness like e.g. in the BEKK MGARCH case (Silvennoinen A. and Teräsvirta T. 2009). Chen (2005) captures the time-varying moments and conditional heteroskedasticity by implementing such a multivariate GARCH (MGARCH) framework in his study on multiperiod consumption and portfolio decision. By implementing the conditional variance-covariance matrix attained in the MGARCH model, it is possible to model the dynamics between several assets as time progresses.

3 Method

In accord with the above discussion, I formulate a multi-period portfolio allocation problem with asset correlated cash in- and outflow, which is methodologically solved with multi-period stochastic linear programming with recourse (MSLPR). It is modeled at a strategic level, where capital is allocated amongst two aggregated asset classes, stocks and bonds, and a cash flow in and out of the portfolio in the form of transfers or withdrawals under full and low liquidity. It models a discrete investment environment where the returns of the portfolio assets over time is generated in scenarios, and an outcome at any given state is contingent on the preceding outcomes before it. The objective is to minimize CVaR risk at a given return level in identical scenarios by allocating capital exclusively between equities and bonds after substracting or adding the external cashflow. The model is implemented in Matlab R2015b with the fconmin-function and Sheppard K. (2013) MFE Toolbox for the BEKK-MGARCH model on a Intel Core i5-4210U 1,7 GHz, 1 GB RAM computer running Windows 10. Total calculation time is approximately 70 seconds. For relevant Matlab code and schematic representation of the optimization model, see appendix 8.1 to 8.3.

3.1 Stochastic Programming Problem

3.1.1 Price process

The logarithmic return is defined as $\ln\left(\frac{y_t}{y_{t-1}}\right)$ where y is the asset price at any given moment t. The object is to model this return process for the scenario generation. In line with Bauwens L. Laurent S. and Rombouts J. (2006) a vector stochastic process $r_t \in \mathbb{R}^m$ (e.g. return of asset) of dimension $N \times 1$ has a vector of past information (see sigma field/filtered information), which can be denoted I_{t-1}. This means that only *past information* is available, and the future outcome is uncertain (stochastic). Hence, the return process for a given asset can be written as:

$$r_t = \mu_t(\theta) + \varepsilon_t \tag{1}$$

Where θ is defined as a finite, unknown parameter vector, $\mu_t(\theta)$ is the conditional mean vector, and ε_t the stochastic error term/conditional variance vector. The conditional covariance is modeled the following way:

$$\varepsilon_t = H_t^{\frac{1}{2}}(\theta) z_t \tag{2}$$

Where $H^{\frac{1}{2}}(\theta)$ is a $N \times N$ positive definite matrix, and z_t is a $N \times 1$ normal distributed random vector assumed to have first two moments:

$$\begin{gathered} E(z_t) = 0 \\ Var(z_t) = I_N \end{gathered} \tag{3}$$

Where the I_N is the identity matrix of order N (not to be confused with I_{t-1} – sigma field/filtered information), while $H_t^{\frac{1}{2}}$ is a positive definite matrix, obtained by the Cholesky fatorization of H_t. Further, H_t is in turn the conditional variance matrix of r_t. As stated, H_t is dependent on the unknown parameter vector θ. Given past information, the conditional variance matrix of r_t is calculated as follows:

$$
\begin{aligned}
Var(r_t|I_{t-1}) &= Var_{t-1}(r_t) = Var_{t-1}(\varepsilon_t) \\
&= H_t^{\frac{1}{2}} Var_{t-1}(z_t)(H_t^{\frac{1}{2}})' \\
&= H_t
\end{aligned}
\tag{4}
$$

Hence, $H_t^{\frac{1}{2}}$ is independent of $\mu(\theta)$. Therefore for simplicity it is assumed that μ equals zero, which is a reasonable assumtion given my primary interest in the asset covariance, which here can be obtained through the error term alone. Consequently, $r_t = \varepsilon_t$.

3.1.2 BEKK-MGARCH

A general formulation of H_t was originally proposed in the VEC-MGARCH model by Bollerslev T. Engle R.F. Wooldridge J.M. (1988). Even though it is very flexible, it has disadvantages and is computationally very demanding (Silvennoinen A. and Teräsvirta T. 2009). It also demands the positivity of H_t, which is hard to guarrantee in the VEC-model. An attractive property of the Baba-Engle-Kraft-Kroner (BEKK) MGARCH, which is a restrictive version of the VEC model, is that it has positive definite conditional covariance matrices by construction, thus obtaining a solution without imposing strong restrictions on the parameters (Bauwens L. Laurent S. and Rombouts J. 2006). Additionally, it makes for an easy incorporation into the scenario generation algorithms, as will be shown further down. The $BEKK(1,1)$ model is defined as:

$$H_t = C^{*\prime}C^* + \sum_{k=1}^{K} A_k^{*\prime}\,\varepsilon_{t-1}\varepsilon'_{t-1}A_k^* + \sum_{k=1}^{K} B_k^{*\prime}\,H_{t-1}B_k^* \tag{5}$$

Or as a *linear algebraic* expression:

$$\mathbf{H_t = C'C + A'\varepsilon_{t-1}\varepsilon'_{t-1}A + B'H_{t-1}B} \tag{6}$$

Where C, A and B are $N \times N$ matrices and C is lower triangular. The parameters in the BEKK model does not represent the impact of the lagged terms on the elements of H_t directly, so interpretation of the mentioned parameters is therefore difficult. Because numerical difficulties are so common in the estimation of BEKK models, it is typically assumed that the model is $BEKK(p, K, q)$, where $p = K = q = 1$ (Bauwens L. Laurent S. and Rombouts J. 2006). Here,

p is a positive, scalar integer representing the number of symmetric innovations, K is non-negative, scalar integer representing the number of asymmetric innovations, and q is a non-negative, scalar integer representing the number of conditional variance lags (Sheppard K. 2013). I further simplify by excluding the assumption of assymmetric returns, setting $K = 0$ and $BEKK(1,0,1)$. Consequently, the number of parameters to be estimated in the full $BEKK(1,0,1)$ model is $3N^2 - \sum_{i=1}^{N}(N_i - 1)$. The solution is maximized through a maximum likelihood function assuming a normally distributed error term:

$$L(\theta) = -Tln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}(\ln|H_{t-1}| + \varepsilon_{t-1}^{T} H_{t-1}^{-1} \varepsilon_{t-1} \qquad (7)$$

Where θ is the estimated parameter vector and T is the number of observations. The starting value H_1 is estimated as the mean H_t for a simulated path for all returns over a 20 year period chosen to meet the requirements of no residual autoregression or heteroskedasticity.

3.1.3 Scenario tree

An optimizing model involving some random parameters can be modelled as a stochastic programming problem. According to Sen S. and Higle J. (1999) a multistage stochastic linear programming model gives the opportunity for multistage decision-making, thus avoiding myopic choices. The random variables depend on the values that precede them. Thus, the multitude of successive stochastic outcomes can easily be depicted as a scenario "tree". In the figure below, a binomial scenario tree in 4 stages with 8 scenarios is depicted for illustrative purposes. The scenario tree consists of nodes $n \in P \ \forall \ t \in T$ and branches $b \in \{1,2\}$ where each node $p \in P \ \forall \ t \in T - 1$ represent points in which a decision $x_n \in X$ is possible, and branches relating the nodes with a probability $prob \in (0,1)$ in a consecutive manner. I have chosen a equiprobable non-recombining scenario model, meaning a single branch has a 50 % probability of occurring and an endnode will have $0{,}50^{T-1}$ probability of occurring. Stage $t = 1$ in node $n = 1$ is a startoff point from where every scenario originates. Each complete path from the first node to an end node is refered to as a scenario $J(n) \subset$ P, and each vertical set of nodes (i.e. 1 and 2, 3 to 7, 8 to 15 etc.) lies within a descrete time step t. Also for every parentnode p in $t < T$ there is two consective childnodes τ reliant on parentnode p. For example p_3 in n_3 governs $\tau_{3,1}$ in node n_6 and $\tau_{3,2}$ in node n_7 (see figure 3). Scenario $J(10)$ is highlighted for its only possible path ($n = p = 1 \rightarrow 2 \rightarrow \ 5 \rightarrow \ 10$). In a 4 stage equiprobable binomial scenario tree the probability is 0.50 for a single branch and $prob_{J(10)} 0{,}50^3 = 0.125$ for an endnode.

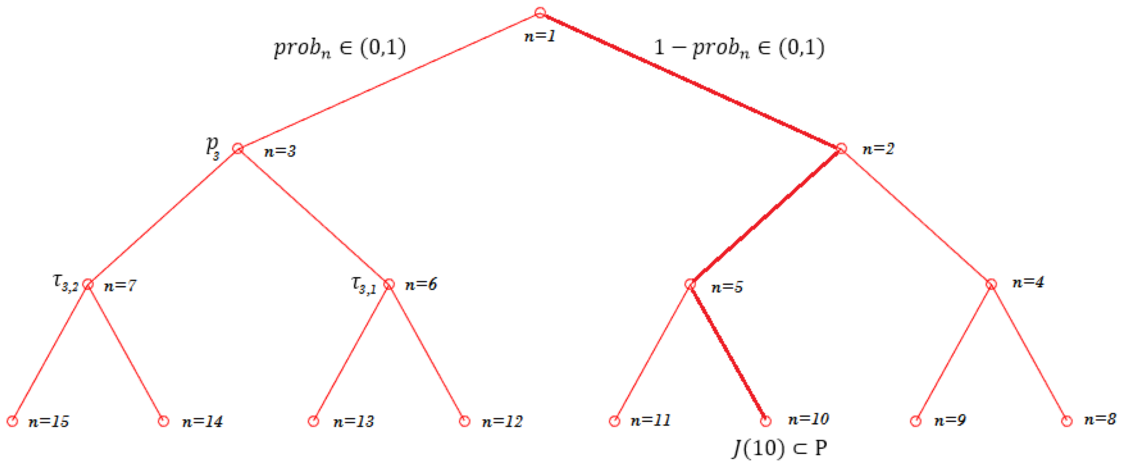


Figure 1: Depiction of how events may unfold through the binomial scenario tree in 4 stages $(T = 4)$ *and 8 scenarios.*

As shown above, $H_{t>1}$ in r_t relies on the two preceding variables H_{t-1} and ε_{t-1}. In the scenario generation I have stated this as the following algorithm, which connects children nodes to their preceding parent nodes:

$$H_{\tau_{n,b}} = C'C + A'\varepsilon_{p_n}\varepsilon'_{p_n}A + B'H_{p_n}B \tag{8}$$

In order to obtain the scenario generated returns, (1) and (2) is then applied. The generation of a scenario tree quickly becomes problematic due to the exponential growth in dimensionality. For a binomial scenario tree, every node has two branches, which gives $2^T - 1$ nodes and 2^{T-1} scenarios. It will be shown later that this can result in very large optimization programs for higher T's.

3.1.4 A Multistage SLP in general

According to Ruszczyński A. and Shapiro A. (2003) a general multistage SLP model with recourse can be stated as the following nested formulation (For the SLP-model I abuse notation, so it does not correspond to the MGARCH-model stated above):

$$\min_{\substack{A_{11}x_1 = b_1 \\ x_1 \geq 0}} c_1^T x_1 + \mathrm{E}\left[\min_{\substack{A_{21}x_1 + A_{22}x_2 = b_2 \\ x_2 \geq 0}} c_2^T x_2 + \mathrm{E}\left[\dots + \mathrm{E}\left[\min_{\substack{A_{T,T-1}x_{T-1} + A_{TT}x_T = b_T \\ x_T \geq 0}} c_T^T x_T\right]\right]\right] \tag{9}$$

Where the information structure is denoted in stages $t = 1, \dots, T$, where the minimization thorugh the corresponding decision vectors $x_1, \dots, x_T$ as a function of $\xi_{[1,t]} = (\xi_1, \dots, \xi_t) \rightarrow x_t = x_t(\xi_{[1,t]})$ is performed, representing a certain portfolio at time t. The vectors c_1, A_1 and b_1 are known, while some or all of the cost vectors $c_2, \dots, c_T$, matrices $A_{t,t-1}$ and A_{tt}, $t = 2, \dots, T$ and right hand side vectors $b_2, \dots, b_T$ are random. Thus the sequence of actions is as follows:

$$
\begin{gathered}
decision(x_1) \\
observation\ \xi_2 \coloneqq (c_2, A_{21}, A_{22}, b_2) \\
decision(x_2) \\
\vdots \\
observation\ \xi_T \coloneqq \left(c_T, A_{T,T-1}, A_{T,T}, b_T\right) \\
decision(x_T)
\end{gathered}
$$

In particular $\xi_{[1,t]}$ represents the information available up to time t. The important point is that every decision relies on the available information and the expectation of the following outcome, and not the outcome itself. In the first stage the problem is the following simple linear programming problem:

$$
\begin{gathered}
\min_{x_1} c_1^T x_1 + \mathrm{E}[\mathrm{Q}_2(x_1, \xi_2)] \\
s.t.\ A_{11} x_1 = b_1, \\
x_t \geq 0.
\end{gathered}
$$

Where the optimal value, called the "cost-go-function", is denoted $Q_t\left(x_{t-1}, \xi_{[1,t]}\right)$. For the follwowing general stages $t = 2, \dots, T-1$, the problem is stated:

$$
\begin{gathered}
\min_{x_t} c_t^T x_t + \mathrm{E}\left[\mathrm{Q}_{\mathrm{t+1}}\left(x_t, \xi_{[1,t+1]} \middle| \xi_{[1,t]}\right)\right] \\
s.t. A_{t,t-1} x_{t-1} + A_{t,t} x_t = b_t, \\
x_t \geq 0.
\end{gathered}
$$

For the last time period T the values of all problem data $\xi_{[1,T]}$ are allready known, and the values of earlier vectors $x_1, \dots, x_{T-1}$ have been decided. As a linear expression the last stage problem is therefore:

$$\min_{x_T} c_T^T x_T$$
$$s.t. A_{T,T-1}x_{T-1} + A_{TT}x_T = b_T,$$
$$x_T \geq 0.$$

3.1.5 The CVaR minimizing model

Next presented is the specific optimization model used in this thesis. As stated in Rockafellar R.T. and Uryasev S. (2000), let $f(x,r)$ be the loss associated with the decision vector x (subscripts is left out for notational ease). The underlying distribution of r is assumed to have the probability density $p(y)$. The probability of $f(x,r)$ not exceeding a threshold α is then given by:

$$\psi(x,\alpha) = \int_{f(x,r)\leq\alpha} p(r)dr \tag{10}$$

Where ψ is the cumulative distribution function for the loss associated with x. Any specified probability level β in (0,1) will be denoted by $\alpha_\beta(x)$ and $\emptyset_\beta(x)$ (VaR and CVaR respectively):

$$\alpha_\beta(x) = \min\{\alpha \in \mathbb{R}: \psi(x,\alpha) \geq \beta\} \tag{11}$$

and

$$\emptyset_\beta(x) = (1-\beta)^{-1} \int_{f(x,r)\leq\alpha_\beta(x)} f(x,r)p(r)dr \tag{12}$$

Hence, CVaR equals the cummulative losses $f(x,r)$, greater or equal to the VaR ($\alpha_\beta(x)$) divided by the confidence level $(1-\beta)$. The major contribution by Rockafellar R.T. and Uryasev S. (2000) was the reformulation that circumvented the demand for explicit calculation of VaR in the previous expression, thus yielding a profound simplification of the optimization calulation. So, a continous, nondecreasing expression for the conditional expectation of the loss associated with x relative to a loss being $\alpha_\beta(x)$ or greater, is stated as follows:

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{r\in\mathbb{R}^m} [f(x,r)-\alpha]^+ p(r)dr, \tag{13}$$

Where

$$[t]^{+} = \begin{cases} t \ when \ t > 0, \\ 0 \ when \ t \leq 0. \end{cases}$$

Consequently, they show that minimizing the CVaR ($\emptyset_{\beta}(x)$) over x is equivalent to minimizing $F_{\beta}(x, \alpha)$ over x, thus giving the opportunity for application of stochastic programming approaches to the minimization of CVaR. Minimizing the CVaR of the loss associated with $x \in X$ is equivalent to minimizing F_{β}:

$$\min_{x \in X} \emptyset_{\beta}(x) = \min_{(x,\alpha) \in X \times \mathbb{R}} F_{\beta}(x, \alpha) \tag{14}$$

By implementing this framework, it is not only the decision variable x which is optimized, but also the the quantile level α - Rockafellar R.T. and Uryasev S. (2000) prove how optimizing the CVaR expression also optimizes VaR, showing how a portfolio with low CVaR, must also have a low VaR as well. In order to solve the expression practically, they further formulate an approximation, which in turn can be directly implemented into stochastic programming:

$$\tilde{F}_{\beta}(x, \alpha) = \alpha + [J(1-\beta)]^{-1} \sum_{j=1}^{J} [-x^{T} r_k - \alpha]^{+} \tag{15}$$

Where $j \in J$ denotes the number of scenarios. Further I assume the vector x_i to be the weights for assets $k \in K$ in the portfolio, and constraining their sum to one, therefore not allowing for short-positions:

$$x \geq 0 \text{ with } x^{T} I = 1 \tag{16}$$

Where x is larger than zero and sums to one (I is an $N \times 1$ unity vector). The portfolio return is the returns of each asset weighted by the portfolio decisions x. Since the CVaR is a loss function, it is nessecary to specify this in the following negative manner:

$$f(x, r_k) = -x^{T} r_k \tag{17}$$

3.1.6 Linearization of the optimization problem

Because the max function $[-x^{t} r_k + \alpha]^{+}$ in (10) is not a linear expression, it is nessecary to rewrite the problem in order to optimize the problem with a linear solver e.g. with the *fconmin*–function in Matlab. Rockafellar R.T. and Uryasev S. (2000) introduces auxiliary variables u_j

for $j = 1, \dots, J$ in order to linearize the problem, showing that this is equivalent to minimizing (12):

$$\underset{(x,\alpha)}{\text{Min}}\ \tilde{F}_{\beta}(x,\alpha) = \alpha + J(1-\beta)^{-1}\sum_{j=1}^{J} u_j\,, \tag{18}$$

s.t. the linear constraints

$$\begin{gathered} x \geq 0 \text{ with } x^T I = 1 \\ \mu(x) \leq -R \\ u_j \geq 0, \\ x^T r_j + \alpha + u_j \geq 0, \\ \text{for } j = 1, \dots, J \end{gathered} \tag{19}$$

This states that when the loss is smaller than VaR: $x^T r_j + \alpha \geq 0 \rightarrow -x^T r_j - \alpha \leq 0$, and $u_j = \left[-x^T r_j - \alpha\right]^+ = 0$. When the loss is greater than or equal to VaR: $x^T r_j + \alpha \leq 0 \rightarrow -x^T r_j - \alpha \geq 0$ and therefore $u_k = \left[-x^T r_j - \alpha\right]^+ = -x^T r_j - \alpha$. Further, $\mu(x) = -x^T \boldsymbol{m}$ denotes the mean loss $\boldsymbol{m}$ associated with portfolio x for the end nodes in $t = T$, which imposes the requirement that the aggregated expected return on all admitted portfolios should at least be R, take the feasible set of portfolios $\forall x \in X$ satisfying (12) and (14), and thus make set X convex (Rockafellar R.T. and Uryasev S. 2000).

3.1.7 Extensions

As an extension to this I have incorporated the cash in- and outflow of the portfolio by customizing (18). This represents the total wealth to be allocated in the portfolio with the return orginating from development in asset prices on the left side, and the cashflow in or out on the right side. This means a relative increase or reduction in the total portfolio size by $c_f * 100$ % through a cashflow. I formulate this in the following expression:

$$x \geq 0 \text{ with } x^T I = 1 + c_f \tag{20}$$

Where c_f for $f \in \{1,2,3,4\}$: $c_1 = r_{t,oi}$ is the asset correlated cashflow (subscript: oil), $c_2 = 0$ is no cashflow (self-financing portfolio), $c_3 = r_{neg}$ negative, constant cashflow, and $c_4 = r_{pos}$ positive, constant cashflow. If for example $c_f = 0{,}10$ the portfolio will have a relative cash inflow of 10 % in a given state, whereas $c_f = -0{,}10$ results in a 10 % cash outflow in a given state all else being equal. Consequently, if $r_{t,eq}$ and $r_{t,oi}$ is perfectly correlated ($\rho = 1{,}0$ and

$r_{t,bo} = 0$), 5 % in both will give a portfolio wealth growth of 10 %, and oppositely 10 % wealth decrease if both is -5 %. The development of portfolio wealth W_t for this model is stated as the following:

$$W_{t-1}(1 + r_{t,W}) = W_t$$
$$\text{where } r_{t,W} = r_{t,eq}x_{t-1,eq} + r_{t,bo}x_{t-1,bo}$$

$r_{t,W}$ is the aggregated portfolio return between two stages, and subscript eq is equity and subscript bo is bonds. In (18) the original restriction in a self-financing portfolio is:

$$x_{t,eq} + x_{t,bo} = 1$$

Expressed in total wealth W_t, this is obviously:

$$W_t(x_{t,eq} + x_{t,bo}) = W_t \tag{21}$$

(In the following I exclude subscript t for simplicity). If cashflow is introduced in the self-financing portfolio, and c_f is proportional to W, the restriction in (21) would become:

$$W(x_{eq} + x_{bo} + c_f) = W + Wc_f \rightarrow$$
$$x_{eq} + x_{bo} + c_f = 1 + c_f$$

Where c_f is allocated between x_{eq} and x_{bo} in the following manner, giving the *relative weights* x^*:

$$x_{eq}(1 + c_f) + x_{bo}(1 + c_f) = 1 + c_f \rightarrow$$
$$x_{eq}^* + x_{bo}^* = 1 \tag{22}$$

And

$$x^* \geq 0 \text{ with } (x^*)^T I = 1$$

Relative weights is the allocation weights relative to whatever total size the portfolio has at any time[2]. Under hypothesis $H1$, the optimal portfolio weights $x^*_{c_{f\in\{1,3,4\}}}$ should equal the optimal

[2] Cashflow corrected portfolio weights makes it possible to compare allocations (a 10 % cashflow would give e.g. 66/44 allocation to a 60/40 allocated portfolio. The relative weights are still 60/40 ($= \frac{66}{66+44} = 60$ %).

self-financing portfolio weights $x^*_{c_{f=2}}$ because the cashflow will be distributed as not to disturb the pre-cashflow allocation weights (a 60/40 portfolio will thus distribute cashflow 60/40 as well). This is because the optimal allocation is optimal regardless of the portfolio size, and given no other restrictions and all else equal, any amount of assets can be bought and sold so the cashflow will only increase or decrease the total portfolio size. In other terms:

$$x^*_{c_{f\in\{1,3,4\}}} = x^*_{c_{f=2}}, W \in \mathbb{R}$$

For the second case under hypothesis $H2$ liquidity restrictions is introduced in order to ensure less liquid bond assets. As mentioned earlier, bonds generally have lower liquidity than equity, which will consequently lead to allocation of some or all of the cash in- and outflow to equity, depending on the liquidity rate γ. Inspired by the eariler work mentioned, I define a simple, deterministic turnover restriction where only a percentage γ of the bonds can be traded in any given stage. This is a liquidity restriction on *the percentage of assets transacted*, in opposition to e.g. Sakar C. T. and Köksalan M. (2013)s “number of transaction” restriction. Thus, bonds cannot be bought at a higher rate than γ_U and sold at a higher rate than γ_L, $0 < \gamma_U < 1$ and $-1 < \gamma_U < 0$ relative to the portfolio size. This allows for a slower allocation rate than what would else be optimal, and should therefore shift allocations. I express the liquidity restriction as follows:

$$x^T_{p_n,bo} - x^T_{\tau,bo} \leq \gamma_U \tag{23}$$

$$x^T_{p_n,bo} - x^T_{\tau,bo} \geq \gamma_L \tag{24}$$

Where $x^T_{p_n,bo}$ is the relative allocation weight in bonds for the parent node, and $x^T_{\tau,bo}$ is the same for the corresponding child node. If $\gamma_U = \gamma_L = |1|$; the bonds are fully liquid, while if $\gamma_U = \gamma_L = 0$; no capital can be allocated to or from bonds. For example if liquidity is 0,20, this would mean a maximum of 20 %-point turnover per allocation and thus take the bond allocations 5 periods to fully turn. If the liquidity is zero, this would result in a drifting portfolio where capital is initially allocated, but not rebalanced for $t > 0$. Under this restriction hypothesis $H2$ expects the allocation between the portfolios with cashflow $c_{f\in\{1,3,4\}}$ to differ from the self-financing portfolio (cashflow $c_{f=2} = 0$). In general when it is lower liquidity in the bonds than what the cashflow withdrawal demands (over one or several stages), an applicable proportion of capital must be readily available in equity. Hence, I expect the portfolio allocation means for $c_{f\in\{3,4\}}$ to be higher or lower, than the self-financing portfolio.

The size of this stichastic linear program depends on the number of stages, branches per node, number of assets and the restricions which the optimization problem is subject to. In this case the size of the decision vector x is $M = 1 \times N = 3(2^T - 1)$, the matrix A from (9) is $M[= 3(2^T - 1)] \times N[= 7 * 2^T - 10]$ and b from (9) is $M[= 6(2^T - 1) - 3] \times N[= 1]$. I have tried a different number of stages, but optimizations failed for $T > 5$. This may be due to the problem of dimensionality, which can be solved with a decomposition method. For example if $T = 4$ (recall the scenario tree) matrix A will be $M = 45$ and $N = 102$. If $T = 12$ the size of the optimization program will have grown to a staggering $M = 12285 \times N = 28662$! For more branches than two, the problem would grow even faster, making the use of scenario reduction techniques and decomposition methods unavoidable. Because of limitations in capacity and time, I have chosen to not forego such methods and rather restrict the SLP model to only binomial branches over 5 stages. With this choice the risk is that I model an incorrect distribution and lose statistical precision in the outcomes at each state. Additionally, more stages would serve as a larger foundation to draw the conclusion on. Even though a larger program would give a better foundation, a smaller program should still be able to show the expected patterns if they are present.

3.2 One-Way ANOVA

The output of the optimization model consists of four separate populations of allocation decisions (cashflow, none, negative and positive) between equity and bonds for two liquidity cases, so 31 decision points in 8 portfolios. Under hypothesis 1 the mean allocations in these are assumed to be similar, whereas for hypothesis 2 it assumed that an optimally allocated portfolio with one of the four cashflow types will have a systematic and significantly different allocation mean compared to a self-financing portfolio (e.g. the negative cashflow portfolio should *on average* be allocated higher/lower than the self- financing portfolio and similar for the positive portfolio). In other words: This means that at least two of the four portfolios should *on average* have a significantly unequal mean from the self-financing portfolio. Under the assumption of homogeneity in variances, the One-Way analysis of variance (ANOVA) is therefore a well suited test to measure the distance between means. If the ANOVA test supports hypothesis 2, that one or more means are unequal from the other, it would in turn be necessary to study the effect of how the correlated cashflow portfolio dictates the optimal allocation as well. It is assumed that:

$$E\left(x_{c_f}^*\right) = \mu_{CF} \text{ and } Var\left(x_{c_f}^*\right) = \sigma^2$$

Where $x_{c_f}^*$ is the relative allocation weights in a decision point, μ_{CF} is the mean of relative allocations for the portfolio and σ^2 is the variance of the portfolio. All allocation distributions are tested with Levene's test for equal variances. The ANOVA-test is performed once for the full liquidity case, and once for the low liquidity case, under the hypothesis:

$$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 \text{ and } H_1: at\ least\ two\ unequal$$

At significance level $a = 0{,}05$. The testobservator used to measure the within and between groups variance is given as:

$$F = \frac{n\frac{\sum(\mu_{CF} - \bar{\bar{x}})^2}{a-1}}{\frac{\sum\sum\left(x_{c_f}^* - \mu_{CF}\right)^2}{a(n-1)}}$$

Where $n\frac{\sum(\mu_{CF}-\bar{\bar{x}})^2}{a-1}$ is the variance between groups and $\sum(\mu_{CF} - \bar{\bar{x}})^2$ is the sum of squares (SS), $\frac{\sum\sum\left(x_{c_f}^*-\mu_{CF}\right)^2}{a(n-1)}$ is variance is within groups and $\sum\sum\left(x_{c_f}^* - \mu_{CF}\right)^2$ is the mean of squares (MS). $a - 1$ and $a(n-1)$ is the degrees of freedom for the two variances, respectively, and a is the number of factors, while n is the number of instances (nodes) in the population. $\bar{\bar{x}}$ is the group mean. If $F \geq F_{\alpha}^{a-1,a(n-1)}$; then the null hypothesis is rejected (Wonnacott T. and Wonnacott R. 1990).

4 Data

In order to model equity, the MSCI World Index is used as a proxy. This index captures large and mid cap representation across 23 "developed market" countries, with 1649 constistuents covering approximately 85 % of the free float-adjusted markets capitalization in each country. The four largest sector weights is financials (19 %), information technology (14 %), consumer discretionary (13 %) and health care (13 %) in United States (59 %), Japan (9 %), UK (7 %), France (4 %), Switzerland (3 %) and other (18 %). As a correlated cashflow, the ICE Brent crude oil index (oil) is applied as a proxy. It represents the average price of trading in the BFOE (Brent-Forties-Oseberg-Ekofisk fields located in the North sea) (Intercontinental Exchange 2016). The Brent oil price index is widely used as a reference for measuring crude oil price levels (MSCI Inc. 2016). As a proxy for bond investments, the Datastream US benchmark 10 year government bond index is used. This bond index was picked based on data availability. All indices is orginally stated in USD and daily observations, which is afterwards compunded into logarithmic monthly returns over a 20 year period between 11.03.96 – 11.03.16.

Table 1: Descpritive statistics for monthly data, 1996-2016[3]

	Oil	*Equity*	*Bond*
Mean	-0,139 %	0,269 %	-0,085 %
Standard Error	0,00449	0,00199	0,00091
Median	0,00690	0,00664	0,00030
Standard Deviation	0,06969	0,03093	0,01418
Sample Variance	0,00486	0,00096	0,00020
Excess kurtosis	3,20450	4,23243	1,30666
Skewness	-0,98054	-1,05617	-0,09425
Minimum	-0,32708	-0,17195	-0,04721
Maximum	0,19215	0,08740	0,04506
Jarque-Bera	141,73	224,69	17,50
Q	13,3070**	16,0153**	10,8658**
Q^2	19,2471**	8,4836**	25,7585**
A.Dickey Fuller	-12.1651**	-11.8066**	-12.4244**
Count	241	241	241
Oil	1		
Equity	**0,30****	1	
	(61,3 %)		
Bond	**-0,27****	**-0,40****	1
	(34,8 %)	*(30,4 %)*	

Top: Descriptive statistics for monthly observations of ICE Brent crude oil index (Oil), MSCI World Index (Equity) and Datastream US benchmark 10 year government bond index (Bond) over a 20 year period between 11.03.96 – 11.03.2016. Bottom: In bold: 20 year period correlation for oil, equity and bond. Significance on a α=0,01 level (**: p≤0,01). In parenthesis: Percentage of incidents where asset pairs are positively correlated in 12 month trailing correlation for the same period. As expected the oil and bond has on average had a negative

[3] See appendix for various graphical representation of data.

development during the sample peroid, whereas the stock market has had a positive trend. Oil is highly volatile (0,07) and stocks twice as volatile as bonds (0,031 vs. 0,014), all leptokurtic (> 0) and negatively skewed. This is also obvious in the minimum/maximum values, where the biggest monthly shifts for oil has been -33 % and +19 %, against -17/+9 and -5/+5 for stocks and bonds, respectively. The Jarque-Bera test[4] shows non-normal distrubutions ($JB > 5{,}99$) for all time series over the 241 month period. In the Ljung-Box test the Q-statistics with 1 lag show significant autocorrelation, which indicate that historic returns have a considerable impact on future returns. The Q^2-statistics of the McLeod A. I. and Li W. K. (1983) Ljung-Box test for heteroscedasticity with 1 lag and critical value $Q^2 = 3{,}8415$ indicates significant ARCH effects in the residuals of the returns for all three assets. The Augmented Dickey Fuller test for unit root rejects the null hypothesis ($ADF < -1.9421$) in all three cases, indicating stationarity in the asset returns. The total 20-year correlation (ρ) between oil and equity is as expected significantly positive (0,30). The equity/bond correlation is significantly negative (-0,40), and negatively correlated oil/bond-pair (-0,27) as well. As expected, the descriptive statistics strongly motivates the MGARCH model for conditional variance.

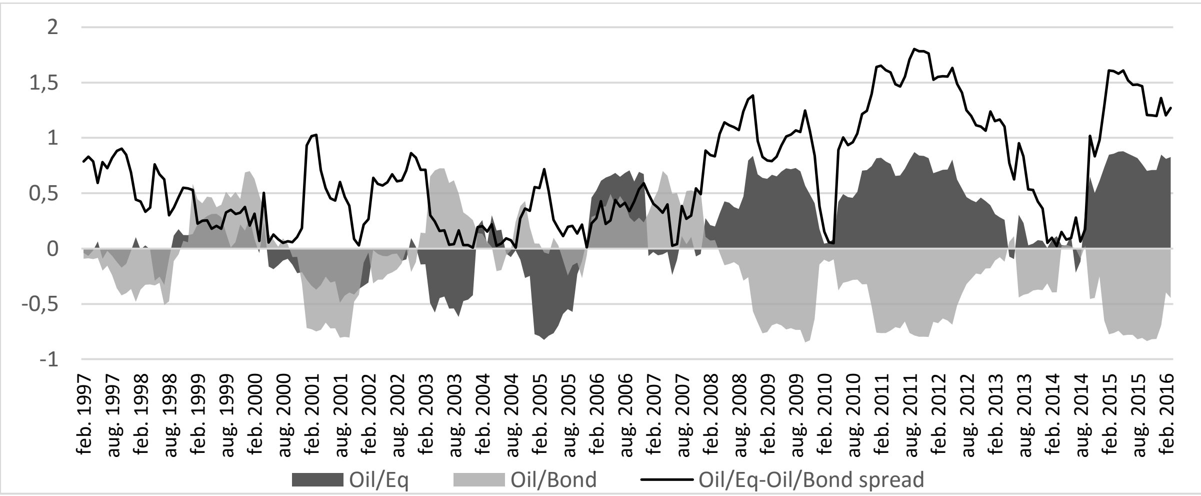


Figure 2: 12 month trailing correlation and absolute spread

Figure 4 shows a 12 month trailing correlation for oil/equity (dark) and oil/bond (light) in the 20 year period. The absolute spread (black line) indicates periods where equity and bond has similar correlation (spread $\cong 0$) or dissimilar correlation (spread > 0). The oil/equity correlation is positive 61 % of the time, whereas for bond/oil and bond/equity it is only positive

[4] For expressions of the statistical tests Jarque-Bera for normality, Ljung-Box Q for autocorrelation and ARCH-effects, and Augmented Dickey Fuller for unit root, see appendix.

around a third of the time (35 % and 30 %, respectively). Figure 4 display a 12 month trailing correlation for the oil/equity (dark) and oil/bond (light) pairs over 230 months, feb 1997-march 2016. I have chosen a 12 month window because it provides a clearer, smoothened presentation of the correlation dynamics, compared to the more detailed 6 month window proposed by Filis G. Degiannakis S. and Floros C. (2011). Additionally the spread $s = |\rho_{oil/eqity} - \rho_{oil/bonds}|$ is the absolute value of the distance between the correlation pairs in every point, indicating events of large shifts in the correlations. When the spread is large, the correlations is at two extremes, while when the spread is close to zero, both equity and bonds have similar correlation with oil. This is interesting because it indicates instances where the correlations move against one another in large shifts. Some important patterns arise: A sudden stabilizing in correlations after 2008 is obvious and as found in Turhan M. I. et. al. (2014). Both correlation pairs has fluctuated between negative and positive from 1996 to 2008 and then suddenly stabilizes in the period after 2008. In line with the findings of the authors the oil/equity correlation here is positive after 2008, but seems to be opposite (negative) of the findings for the oil/bond correlation after 2008, stabilizing below zero. In the period prior to the 2008 financial crisis, the oil/equity correlation is positive 35 %, while in the post-period, the oil/equity correlation is positive ($\rho > 0$) 96 % of the time. Not only is it positive, but highly correlated ($\rho \geq 0{,}60$) in 50 % of the time. The oil/bond the correlation is positive in 47 % of the cases pre-2008, against only 9 % post-2008 (with $\rho > 0{,}1$ zero percent of the time).

5 Numerical results

The results from the model described in section 3 will be presented in the following. I report two groups of empirical results: Optimal portfolio allocation in a perfectly liquid and a low liquid market for the four types of cashflow described earlier. The first group shows the CVaR optimal allocations in accordance with the cashflow types under the assumption that all assets can be fully bought and sold in one period, whereas the other group enforce a different and slower turnover rates for the same bond assets compared to the full liquidity case. All portfolios is modeled on the same generated scenarios. To test if the portfolio allocation means are significantly different from one another, an ANOVA test is performed for both liquidity conditions.

5.1 Scenario generation

The scenario tree returns is generated so that the child node return and conditional covariance matrix is reliant on the previous parent node output. The start-off point parameters for the $(1,1)$-order, trivariate, full, symmetric conditional covariance matrix at $t = 0$ is estimated as the mean of all simulated covariances from the $3 \times 3 \times 241$ conditional covariance matrix H obtained from estimating the BEKK(1,0,1) MGARCH-model[5]. The estimated A, B and C-vector has the following parameters:

$$A = \begin{bmatrix} 0{,}5742 & 0{,}0007 & -0{,}0005 \\ 0{,}0001 & 0{,}5742 & 0{,}0003 \\ -0{,}0003 & -0{,}0002 & 0{,}5738 \end{bmatrix}, B = \begin{bmatrix} 0{,}3178 & 0{,}0005 & 0{,}0002 \\ -0{,}0003 & 0{,}318 & 0{,}0002 \\ 0{,}0009 & -0{,}0001 & 0{,}3174 \end{bmatrix},$$

$$C = \begin{bmatrix} 0{,}0525 & 0 & 0 \\ 0{,}0069 & 0{,}0223 & 0 \\ -0{,}0029 & -0{,}0036 & 0{,}0096 \end{bmatrix}$$

The log likelihood at the optimum is $L = 1568{,}4$. I consider a $T = 5$ period problem with five decision stages in $n = 31$ nodes and $J = 16$ scenarios. The scenario tree is selected so that the correlation ρ is $\geq 0{,}15$, $\leq -0{,}15$ and ≤ 0 for the oil/equity-, equity/bond- and oil/bond-pairs, respectively. This is to ensure that the overall correlation is similar to the dataset's correlation characteristics described in section 4. For the scenarios generated, the complete tree by nodes and returns $r_{t,oi}, r_{t,eq}$ and $r_{t,bo}$, respectively is presented in figure 5. The same return data is presented nodewise in table 2, along with a visual representation of the cummulative returns in figure 6. Lastly, the descriptive statistics for the scenario generated returns is shown in table 3. The optimization in all 8 cases has this scenario tree as a basis.

[5] See the BEKK-model under "Method". Local optimum found within the default limits.

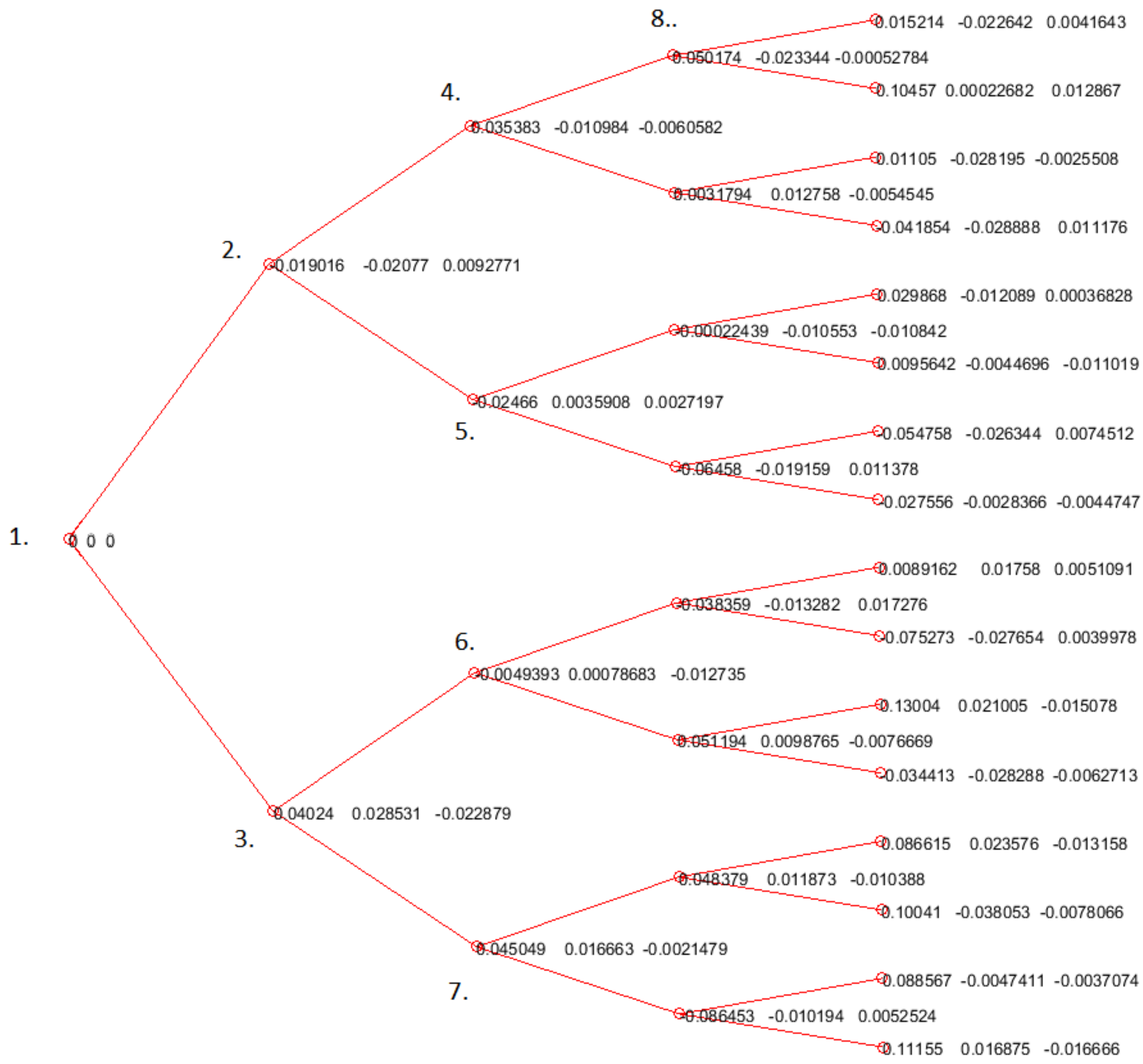


Figure 3: Taxonomic scenario tree with return for oil, equity and bonds per node. See table 2 for return data.

Table 2: Generated return data for $r_{t,oi}, r_{t,eq}$ and $r_{t,bo}$ from node 1 to 31:

	Oil	*Equity*	*Bonds*
1	0	0	0
2	-0,019	-0,0208	0,0093
3	0,0402	0,0285	-0,0229
4	0,0354	-0,011	-0,0061
5	-0,0247	0,0036	0,0027
6	-0,0049	0,0008	-0,0127
7	0,045	0,0167	-0,0021
8	0,0502	-0,0233	-0,0005
9	0,0032	0,0128	-0,0055
10	-0,0002	-0,0106	-0,0108
11	-0,0646	-0,0192	0,0114
12	-0,0384	-0,0133	0,0173
13	0,0512	0,0099	-0,0077
14	0,0484	0,0119	-0,0104
15	-0,0865	-0,0102	0,0053
16	0,0152	-0,0226	0,0042
17	0,1046	0,0002	0,0129
18	0,0111	-0,0282	-0,0026
19	-0,0419	-0,0289	0,0112
20	0,0299	-0,0121	0,0004
21	0,0096	-0,0045	-0,011
22	-0,0548	-0,0263	0,0075
23	-0,0276	-0,0028	-0,0045
24	0,0089	0,0176	0,0051
25	-0,0753	-0,0277	0,004
26	0,13	0,021	-0,0151
27	-0,0344	-0,0283	-0,0063
28	0,0866	0,0236	-0,0132
29	0,1004	-0,0381	-0,0078
30	0,0886	-0,0047	-0,0037
31	0,1116	0,0169	-0,0167

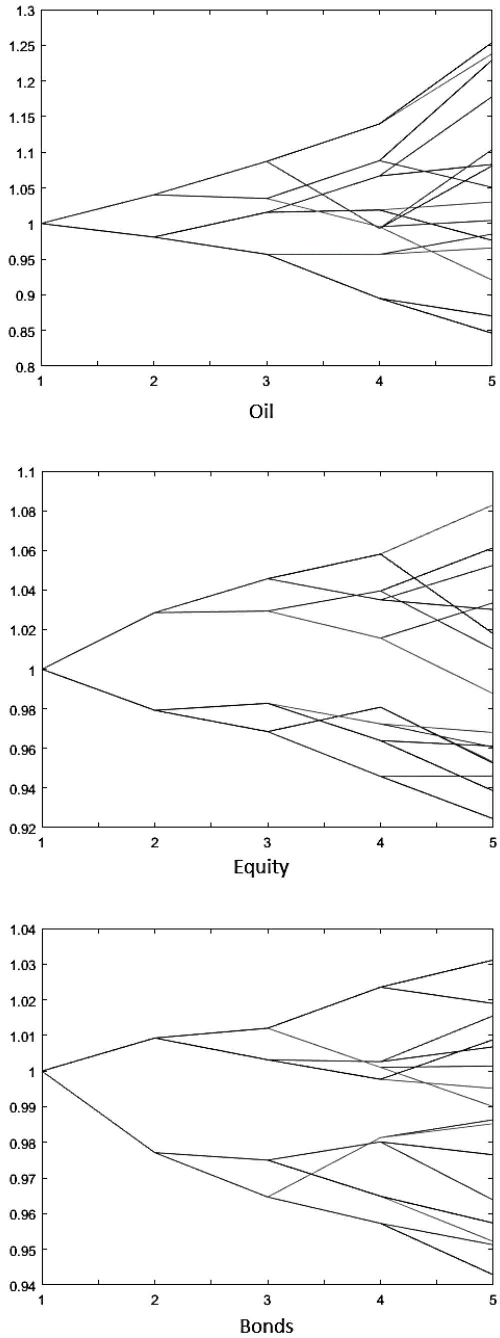


Figure 4: Binomial scenario generated cummulative returns over $T = 5$ stages for three assets: oil, equity and bonds. Total percentage along the vertical axis and stages T along the horisontal axis.

The generated returns in every child node is reliant on the preceding parent node, giving endnode scenarios:

$$J(15) = \{1,2,4,8,16\}, J(16) = \{1,2,4,8,17\}, \dots, J(31) = \{1,3,7,15,31\}.$$

From figure 5 it is apparent that oil has the largest delvelopment with max/min total return over the five stage period between 1,25 and 0,85, while equity and bonds have 1,08/0,925, and 1,03/0,943, respectively. The expected cummulative return in stage 5 is 5,11 % for oil, -0,75 % for equity and -1,35 % for bonds. Further, the 90 %-CVaR is 13,94 % for oil, 6,71 % for equity and 5,21 % for bonds[6]. See appendix 8.16 for corresponding tables.

Table 3: Desciptive statistics for scenario generated returns for $r_{t,oi}, r_{t,eq}$ and $r_{t,bo}$.

	Oil	*Equity*	*Bonds*
Mean	1,606 %	-0,545 %	-0,220 %
Standard Error	0,01027	0,00329	0,00173
Median	0,00960	-0,00470	-0,00260
Standard Deviation	0,05717	0,01834	0,00965
Sample Variance	0,00327	0,00034	0,00009
Excess kurtosis	-0,63976	-1,06626	-0,50764
Skewness	0,21412	0,13417	0,03004
Minimum	-0,08650	-0,03810	-0,02290
Maximum	0,13000	0,02850	0,01730
Count	31	31	31
Oil	1		
Equity	0,459**	1	
Bond	-0,525**	-0,519**	1

*Top: Descriptive statistics for scenario generated returns for oil, equity and bonds over five stages (31 nodes and 16 scenarios). Bottom: Overall correlation for oil, equity and bond. Significance on a $\alpha = 0{,}01$ level (**: $p \leq 0{,}01$).*

In table 3, the mean and standard deviation is 0,016/0,057 for oil, -0,005/0,018 for equity and -0,002/0,01 for bonds. All is leptokurtic and positively skewed. Oil is positively correlated with equity ($\rho = 0{,}46$) and negatively correlated with bonds ($\rho = -0{,}52$). Naturally, equity is

[6] Here 10 %-linear interpolation has been applied between the two lowest cummulative outcomes of the 16 scenarios within each asset class (representing a cummulative probability outcome of 6,25 % and 12,50 % respectively).

negatively correlated with bonds ($\rho = -0{,}53$). Compared to the original data, the scenarios generated in the model shows similar first three moments and correlation, but less extreme shortfall due to the normal distribution assumption and they are not negatively skewed. This can be partly explained by the low number of branches in the scenario tree, which can fail to give a good approximation of the underlying distribution of the simulated return processes. Further, it could be argued that the scenarios should be modeled to also account for extreme values and negative 4. moments, which would give a better understanding of the specific cases for oil, equity and bond dynamics. But for the sake of optimal allocation between the different portfolio types with cashflow and liquidity restrictions, the model should in any case yield the same conclusion, regardless of these differences in statistical attributes and distributions.

5.2 Optimal portfolio allocations

In this section the optimal allocations for the different portfolios is presented. The expected return for all scenarios over the 5 stage period is arbitrarily set to $R = 2{,}15\ \%$[7]. The CVaR beta is set to $\beta = 0{,}90$, meaning the objective is to minimize the 10 % worst shortfall cases in the endnode scenarios. This is also equal to a 90 % likelihood that the return will stay above a certain cut off point, i.e. Value-at-Risk, which is implicitly calculated in the linear program (recall Rockafellar R.T. and Uryasev S. (2000) and Rockafellar R.T. and Uryasev S. (2002)). The process of optimal allocation is in the following termed "the investor" for the sake of intuitive description: The optimization process is naïve and without expectations, meaning "the investor" allocates optimally for all future stages without knowing for certain the outcome in the stages ahead. Further, the portfolio contains two assets where "the investor" freely determines the allocation of wealth according to given restrictions. Also, the portfolio consumption and investments is exogenous, meaning "the investor" must withdraw or transfer a given amount at any point in time and simultaneously allocate the portfolio optimally for future outcomes (except in the first node 1 where only the optimal allocation is performed). No cash holdings is allowed, so between stages, the whole portfolio is invested in either equity or bonds. The portfolios has four individual cases of cashflows: 1) a cashflow which is positively correlated with equity, 2) no cashflow, 3) a constant -10 % negative cashflow and 4) a constant +10 % cashflow. These cashflows results in an increase, decrease or no change in the total portfolio wealth size at every stage, beginning in the first node. The second hypothesis expects

[7] This is an expected return level where the SLP-model obtained a local minimum with all constraints satisfied for all 8 portfolios simultaneously. Generally one or more failed to obtain a minimum for other return levels.

that liquidity restrictions on the bond assets should result in larger allocation differences between the portfolios to reach the same risk/return profile as in the full liquidity case (the minimum CVaR at the given expected return R). The decision vector x in table 4 and 7 has 62 points where an allocation decision is made (allocation to either equity or bond over 31 nodes), where the last 32 is endnodes. The VaR, in the 63rd position in the decision vector is calculated as part of the optimization, and the auxiliary real variables u in the last 30 positions of the decision vector serve as slack variables which should be close to zero in an optimal point. In all cases the 2nd no-cashflow case serves as the theoretical self-financing portfolio, and thus the theoretical optimal allocation.

5.2.1 Full liquitity

Setting $\gamma_U = \gamma_L = |1|$ results in the bonds being fully liquid and can be 100 % transacted at any stage. The output in table 4 shows that in general the optimal allocation for all cases is a 50% equity and 50% bonds-allocation in accord with Jones and Wilson (2003) in Harjoto M. A. and Jones F. J. (2006). The 90 % CVaR for the given scenarios are 1) 2,95 %, 2) 2,87 %, 3) 2,58 % and 4) 3,15 % (which are all larger than VaR, as expected). Further, $CVaR_3 < CVaR_2 < CVaR_1 < CVaR_4$. All auxiliary variables u equals zero, except u_{19} which is slightly violated. Notice how all the portfolios (i.e. x1 eq + x1 bo) sums to one when corrected for their respective positive or negative cashflow. The exception is portfolio 2 (no cashflow), which sums to one directly due to the absence of cashflow, and serve as the theoretical self-financing portfolio.

Table 4: Output: Optimal allocation (decision vector), full liquidity, four cashflow cases. The portfolios sum to one when corrected for the respective cashflow. See figure 7, 8 and appendix 8.14 for the relative allocation to equity.

Decision variable (x)	1.Corr. CF	2.No CF	3.Neg. CF	4.Pos. CF
CVaR	**0,0295**	**0,0287**	**0,0258**	**0,0315**
x1 eq	0,50	0,50	0,50	0,50
x1 bo	0,50	0,50	0,50	0,50
x2 eq	0,49	0,50	0,45	0,55
x2 bo	0,49	0,50	0,45	0,55
x3 eq	0,52	0,50	0,45	0,55
x3 bo	0,52	0,50	0,45	0,55
x4 eq	0,52	0,50	0,45	0,55
x4 bo	0,52	0,50	0,45	0,55
x5 eq	0,23	0,14	0,15	0,05
x5 bo	0,75	0,86	0,75	1,05

x6 eq	0,50	0,50	0,45	0,55
x6 bo	0,50	0,50	0,45	0,55
x7 eq	0,10	0,12	0,10	0,13
x7 bo	0,95	0,88	0,80	0,97
x8 eq	0,53	0,50	0,45	0,55
x8 bo	0,53	0,50	0,45	0,55
x9 eq	0,00	0,00	0,00	0,00
x9 bo	1,00	1,00	0,90	1,10
x10 eq	0,50	0,50	0,45	0,55
x10 bo	0,50	0,50	0,45	0,55
x11 eq	0,13	0,07	0,06	0,08
x11 bo	0,81	0,93	0,84	1,02
x12 eq	0,00	0,01	0,00	0,00
x12 bo	0,96	0,99	0,90	1,10
x13 eq	0,00	0,00	0,00	0,00
x13 bo	1,05	1,00	0,90	1,10
x14 eq	0,52	0,50	0,45	0,55
x14 bo	0,52	0,50	0,45	0,55
x15 eq	0,46	0,50	0,45	0,55
x15 bo	0,46	0,50	0,45	0,55
x16 eq	0,51	0,50	0,45	0,55
x16 bo	0,51	0,50	0,45	0,55
x17 eq	0,55	0,50	0,45	0,55
x17 bo	0,55	0,50	0,45	0,55
x18 eq	0,51	0,50	0,45	0,55
x18 bo	0,51	0,50	0,45	0,55
x19 eq	0,48	0,50	0,45	0,55
x19 bo	0,48	0,50	0,45	0,55
x20 eq	0,51	0,50	0,45	0,55
x20 bo	0,51	0,50	0,45	0,55
x21 eq	0,50	0,50	0,45	0,55
x21 bo	0,50	0,50	0,45	0,55
x22 eq	0,47	0,50	0,45	0,55
x22 bo	0,47	0,50	0,45	0,55
x23 eq	0,49	0,50	0,45	0,55
x23 bo	0,49	0,50	0,45	0,55
x24 eq	0,50	0,50	0,45	0,55
x24 bo	0,50	0,50	0,45	0,55
x25 eq	0,46	0,50	0,45	0,55
x25 bo	0,46	0,50	0,45	0,55
x26 eq	0,57	0,50	0,45	0,55
x26 bo	0,57	0,50	0,45	0,55
x27 eq	0,48	0,50	0,45	0,55
x27 bo	0,48	0,50	0,45	0,55
x28 eq	0,54	0,50	0,45	0,55
x28 bo	0,54	0,50	0,45	0,55
x29 eq	0,55	0,50	0,45	0,55
x29 bo	0,55	0,50	0,45	0,55
x30 eq	0,54	0,50	0,45	0,55
x30 bo	0,54	0,50	0,45	0,55
x31 eq	0,56	0,50	0,45	0,55
x31 bo	0,56	0,50	0,45	0,55
VaR	**0,0290**	**0,0283**	**0,0255**	**0,0311**
u2-31	0,0000	0,0000	0,0000	0,0000
u19	0,0000	0,0006	0,0005	0,0007

In figure 7 and 8 the allocations in table 4 is corrected for cashflow and show the relative allocation to equity for all four cases[8]. Since all allocation in both figures seems close to identical, this gives an indication that the cashflow is generally irrelevant for the allocation policy under a fully liquid market. Mostly the optimal allocation is 50/50, except in node 5, 7, 9, 11 and 13 where the majority part of the capital is in bonds. A special case is in node 5 and 11, which is the only two points where the allocation differs between the four portfolios. In node 5 the allocation in equity is highest for the correlated cashflow portfolio (23 %), followed by the no cashflow/negative cashflow portfolio in the middle (15 % and 13 %, respectively), and the positive cashflow portfolio with the lowest allocation to equity (5 %). In node 11 the correlated portfolio is allocated 14 % in equity, compared to 6-8 % in the three other cases.

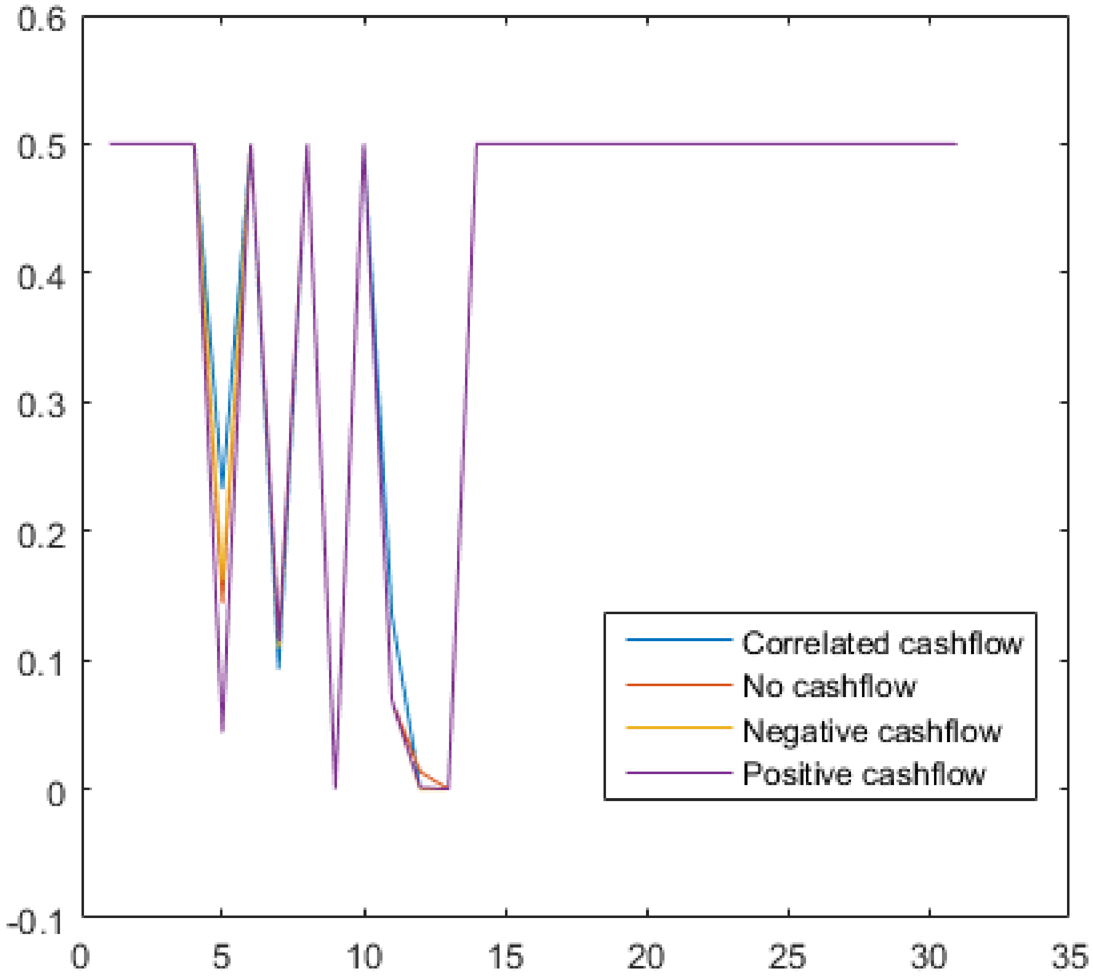


Figure 5: Cashflow corrected allocation weights (v.axis) in equity for four cashflow types per nodes (h.axis). Except for node 5 and 11 allocation policy is not affected by cashflow type under full liquidity.

[8] See appendix for corresponding table.

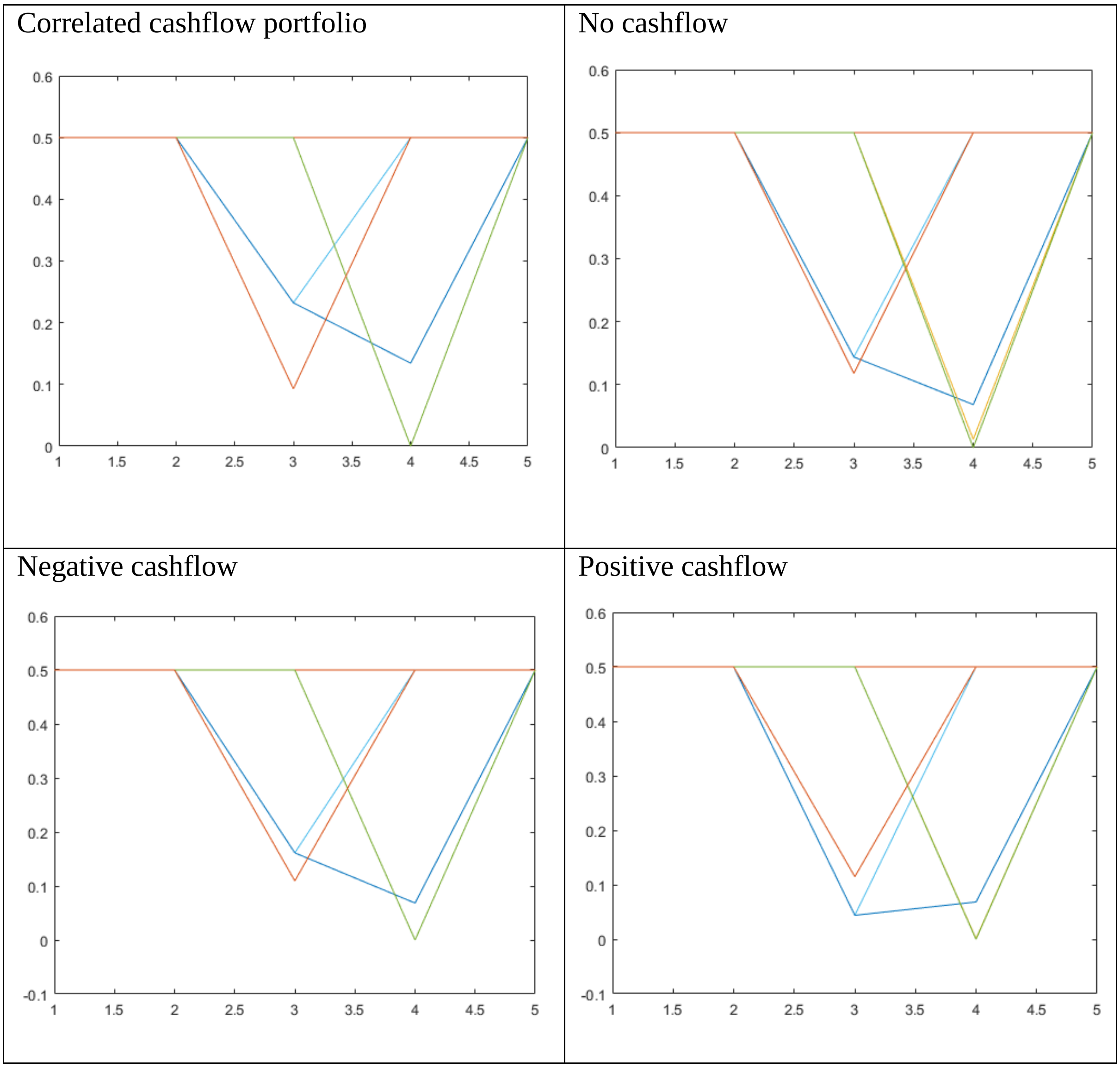


Figure 6: Cashflow corrected allocation weights (v.axis) in equity for four cashflow types after scenarios (h.axis), full liquidity.

5.2.2 Results from One-Way ANOVA[9]

The mean allocation for the four cashflow portfolios lie in the interval 1) 41,48 % to 4) 41,06 %, which the ANOVA test shows is unsignificantly different. The F-value (0,009) is below the critical level (2,68) and the p-value (0,999) is larger than the critical significance level 0,05. Therefore hypothesis H_1 is not rejected. Cashflow is irrelevant for the optimal allocation policy under a fully liquid market.

[9] Levene's test show equal variance in populations: $p = 0{,}98 > 0{,}05$. It is therefore appropriate to proceed with ANOVA.

Table 5: Relative allocation means computed from cashflow corrected decision outputs, full liquidity.

	1. Corr. CF	2. No CF	3. Neg. CF	4. Pos. CF
Mean allocation	41,81%	41,43%	41,42%	41,06%

Table 6: One-Way ANOVA for full liquidity case.

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0,000864	3	0,000288	0,008864	0,998849	2,680168
Within Groups	3,899777	120	0,032498			
Total	3,900641	123				

5.2.3 Low liquitity

On the other hand, setting $\gamma_U = \gamma_L = |0{,}20|$ results in the bonds being low liquid and thus cannot be fully sold before after five periods. That CVaR and VaR here is equal to the fully liquid case shows that the optimal scenarios are obtainable with different allocation weights. Again all portfolios within a respective node sums to one, corrected for the cashflow type. Interestingly the start-off-point for the low liquid case is not the 50/50 allocation seen under full liquidity above. Here, only the negative cashflow portfolio starts out at 50/50, while the no cashflow starts out at 47/53 and the positive cashflow portfolio at 22/78. Notice how every change in bond allocation is below 20 %-points, i.e. scenario $J(16) = \{1{,}2{,}4{,}8{,}16\}$ has $x_{\{1,2,4,8,16\},CF=2} = \{53\%, 0{,}73\%, 91\%, 71\%, 53\%\}$. Similarily to the fully liquid output, the u is slightly violated in node 19 and also in node 27.

Table 7: Outout: Optimal allocation (decision vector), low liquidity, four cashflow cases. The portfolios sum to one when corrected for the respective cashflow. See figure 9, 10 and appendix 8.15 for the relative allocation to equity.

Decision variable (x)	1.Corr. CF	2.No CF	3.Neg CF	4.Pos CF
CVaR	**0,0295**	**0,0287**	**0,0258**	**0,0315**
x1 eq	0,43	0,47	0,50	0,22
x1 bo	0,57	0,53	0,50	0,78
x2 eq	0,21	0,27	0,26	0,22
x2 bo	0,77	0,73	0,64	0,88
x3 eq	0,37	0,34	0,37	0,12
x3 bo	0,67	0,66	0,53	0,98
x4 eq	0,19	0,09	0,07	0,09
x4 bo	0,85	0,91	0,83	1,01
x5 eq	0,22	0,09	0,14	0,07
x5 bo	0,76	0,91	0,76	1,03

x6 eq	0,12	0,16	0,20	0,20
x6 bo	0,87	0,84	0,70	0,90
x7 eq	0,21	0,20	0,20	0,16
x7 bo	0,84	0,80	0,70	0,94
x8 eq	0,33	0,29	0,21	0,29
x8 bo	0,72	0,71	0,69	0,81
x9 eq	0,00	0,00	0,00	0,00
x9 bo	1,00	1,00	0,90	1,10
x10 eq	0,42	0,29	0,34	0,25
x10 bo	0,58	0,71	0,56	0,85
x11 eq	0,15	0,07	0,00	0,08
x11 bo	0,78	0,93	0,90	1,02
x12 eq	0,00	0,01	0,01	0,01
x12 bo	0,96	0,99	0,89	1,09
x13 eq	0,00	0,00	0,00	0,00
x13 bo	1,05	1,00	0,90	1,10
x14 eq	0,31	0,38	0,38	0,36
x14 bo	0,74	0,62	0,52	0,74
x15 eq	0,19	0,38	0,38	0,36
x15 bo	0,72	0,62	0,52	0,74
x16 eq	0,47	0,48	0,41	0,49
x16 bo	0,54	0,53	0,49	0,61
x17 eq	0,55	0,48	0,41	0,49
x17 bo	0,55	0,53	0,49	0,61
x18 eq	0,00	0,05	0,01	0,00
x18 bo	1,01	0,95	0,89	1,10
x19 eq	0,15	0,05	0,01	0,00
x19 bo	0,81	0,95	0,89	1,10
x20 eq	0,51	0,49	0,45	0,44
x20 bo	0,51	0,51	0,45	0,66
x21 eq	0,50	0,49	0,45	0,44
x21 bo	0,50	0,51	0,45	0,66
x22 eq	0,30	0,27	0,09	0,28
x22 bo	0,64	0,73	0,81	0,82
x23 eq	0,31	0,27	0,09	0,28
x23 bo	0,66	0,73	0,81	0,82
x24 eq	0,12	0,08	0,15	0,11
x24 bo	0,89	0,92	0,75	0,99
x25 eq	0,05	0,08	0,15	0,11
x25 bo	0,87	0,92	0,75	0,99
x26 eq	0,28	0,20	0,19	0,20
x26 bo	0,85	0,80	0,71	0,90
x27 eq	0,11	0,20	0,19	0,20
x27 bo	0,85	0,80	0,71	0,90
x28 eq	0,54	0,50	0,45	0,52
x28 bo	0,54	0,50	0,45	0,58
x29 eq	0,55	0,50	0,45	0,52
x29 bo	0,55	0,50	0,45	0,58
x30 eq	0,54	0,50	0,45	0,52
x30 bo	0,54	0,50	0,45	0,58
x31 eq	0,56	0,50	0,45	0,52
x31 bo	0,56	0,50	0,45	0,58
VaR	**0,0290**	**0,0283**	**0,0255**	**0,0311**
u2-u31	0,0000	0,0000	0,0000	0,0000
u19	0,0000	0,0006	0,0005	0,0007
u27	0,0008	0,0000	0,0000	0,0000

As expected, figure 9 and 10 reveal a difference in allocations from the fully liquid case. But it does not seem to be a consistent pattern between the negative-, positive- and no cashflow portfolios. H_2 expects one of the cashflow portfolios to be consistently above, and the other consistently below the self-financing portfolio. Instead, they vary above or below without any apparent orderliness. In figure 9 one can see that the optimal allocation goes from below the 50/50 allocation to lower levels between the startnode and endnodes, but as the uncertainty deteriorates, it moves back toward the 50/50-allocation. This is the same pattern as in the fully liquid case, but which had the luxury of allocating instantly instead of incrementally toward the optimal allocation.

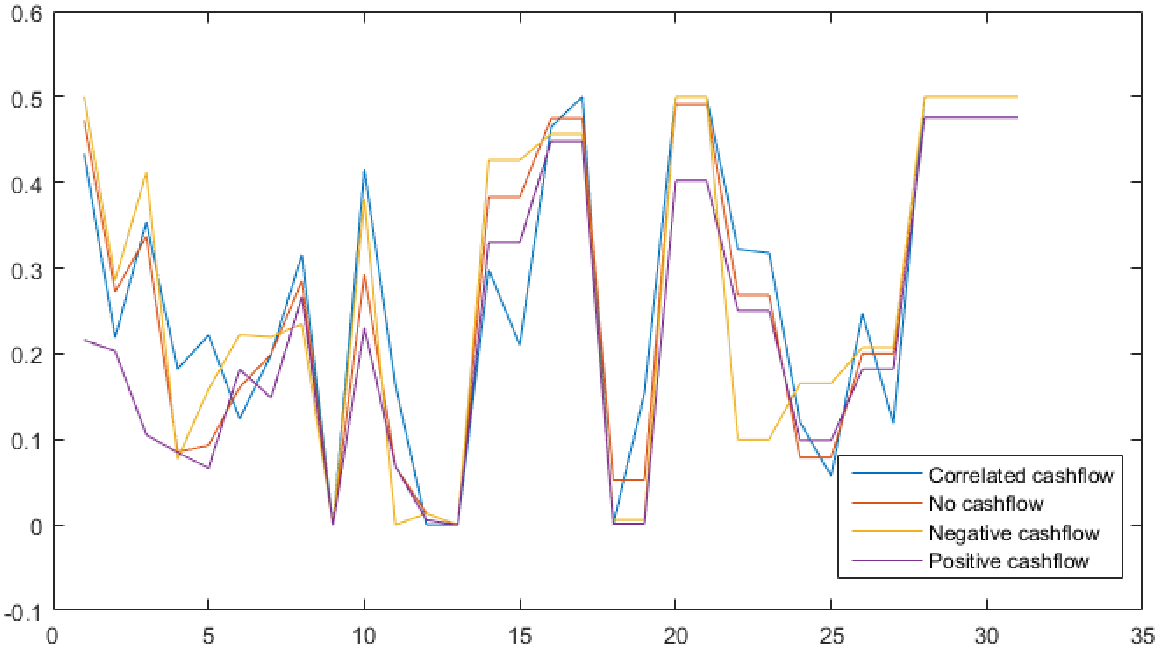
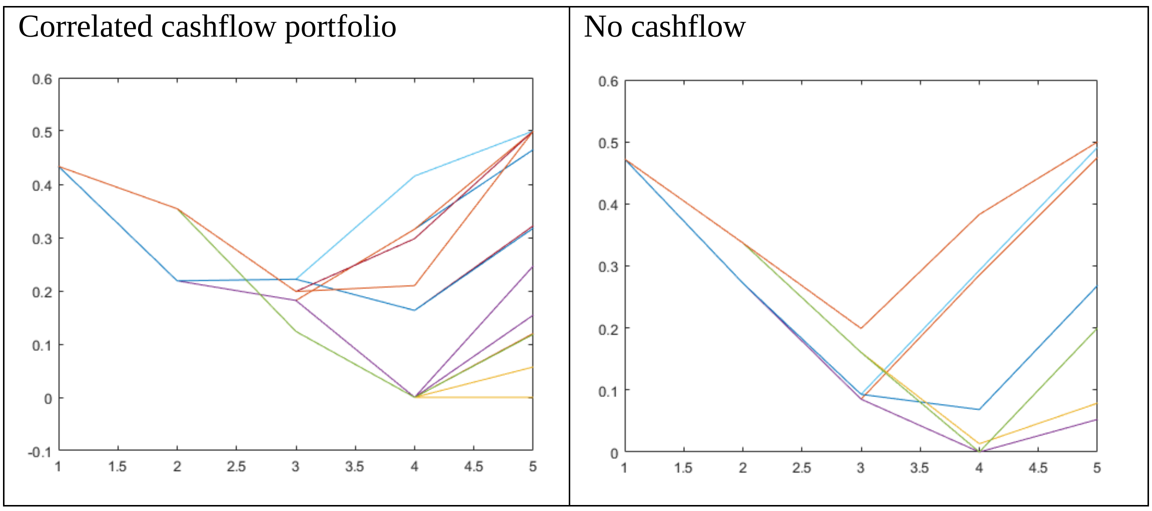


Figure 7: Cashflow corrected allocation weights (v.axis) in equity for four cashflow types after nodes (h.axis), low liquidity.



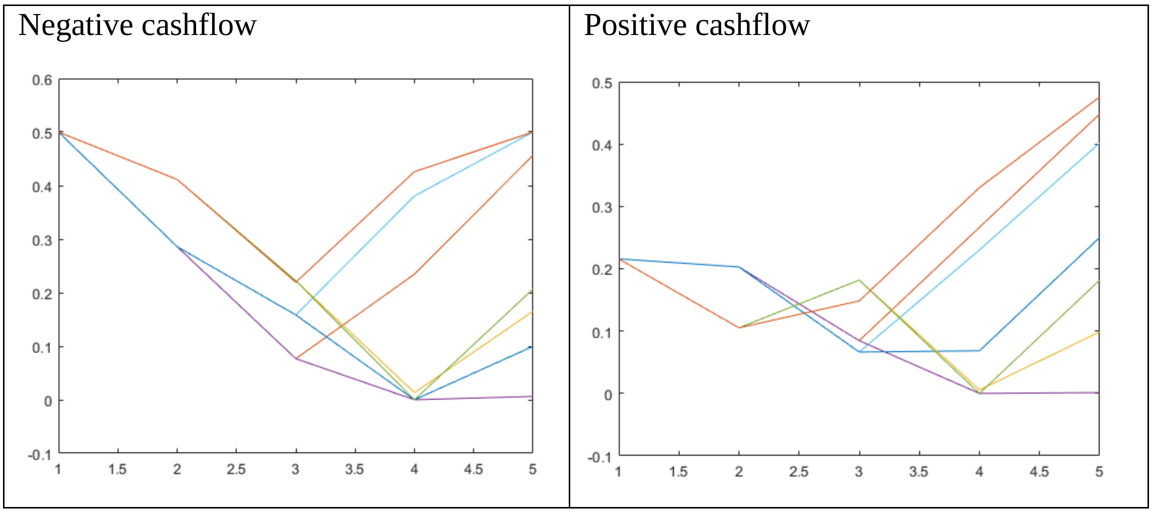


Figure 8: Cashflow corrected allocation weights (v.axis) in equity for four cashflow types after scenarios (h.axis), low liquidity.

5.2.4 Results from One-Way ANOVA[10]

In the low liquidity case, the allocation means for the four cashflow portfolios is lower that the fully liquid case, and show larger spreads between the means: 27,21 % for correlated cashflow, 26,38 % for the self-financing portfolio, 26,53 % for the negative cashflow and 22,27 % for the positive cashflow portfolio. The F-value (0,495) is below the critical level (2,68) and the p-value (0,686) is larger than the critical significance level 0,05. Therefore hypothesis H_2 is rejected. Hence, the mean spreads is not sufficiently large, and therefore also the cashflow in the low liquidity case is irrelevant for the optimal allocation policy.

Table 8: Relative allocation mean computed from cashflow corrected decision outputs, low liquidity.

	1. Corr. CF	2. No CF	3. Neg. CF	4. Pos. CF
Mean allocation	27,21%	26,38%	26,53%	22,27%

Table 9: One-Way ANOVA for low liquidity case.

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	0,047048	3	0,015683	0,495005	0,686426	2,680168
Within Groups	3,801823	120	0,031682			

[10] Levene's test show equal variance in populations: $p = 0{,}54 > 0{,}05$. It is therefore appropriate to proceed with ANOVA.

Total	3,848871	123

6 Conclusion

This article studies whether or not a positive or negative cashflow has an impact on the optimal allocation policy for a two asset portfolio under the restrictions of full and low liquid bonds compared to a self-financing portfolio. Cashflow here is defined as the transfer or withdrawal of capital to or from the portfolio, which therefore increase or reduce the total portfolio size. The optimal allocation is generated as a scenario tree with 16 scenarios over 5 stages through BEKK MGARCH conditional variance modelling and 5-stage stochastic linear programming with recourse. The optimal allocations is defined as the percentage portfolio weights in a given portfolio type for every decision point in time that reduces the downside risk over the period. In other words: The objective is to minimize shortfall risk defined as conditional value-at-risk (CVaR) for the whole scenariotree at a given expected return level through optimal allocation between equity and bonds exclusively (no cash holdings). Earlier studies have shown that allocating correctly gives a risk/return relationship which is optimal for a given portfolio. Thus, there is a tradeoff between expected return and CVaR. Also, studies have shown that liquidity will introduce a third element which should to be considered in the optimal allocation policy of a portfolio. The optimizations show that optimal allocation in the fully liquid case is for most cases a 50 % equity and 50 % bonds arrangement, whereas in the low liquid case a substantially lower equity allocation and with higher variability. As expected, the ANOVA test for the first case of fully liquid bond assets supports the hypothesis that cashflow is irrelevant for optimal allocation (be it correlated, negative or positive cashflow). Moreover, in the second case of low liquidity, the conclusion is the same as in the first case: Cashflow is also here irrelevant for the optimal allocation. The ANOVA test shows that there are larger differences between the means in the second case, but not enough to be significant. Allocating according to the optimal self-financing portfolio will therefore not be suboptimal when cashflows are introduced into the portfolio.

Further work should include scenario reduction techniques and decomposition of the main optimization problem into smaller subproblems. This will lead to a model which will have better statistical precision in scenario generation and the possibility to include more stages. Also, inclusion of allocation thresholds (which demands allocation only after a certain divergence from the target allocation), transaction costs and taxation could yield further insight.

Under these conditions it would be interesting to investigate the efficient frontier for different cashflows over a range of returns, restrictions and liquidities to uncover any effect on the self-financing portfolio allocation.

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8 Appendix

8.1 Matlab code for binomial scenario tree generation with BEKK MGARCH from MFE Toolbox

```
%BEKK MGARCH(1,0,1) parameter estimation
[PARAMETERS,LL,HT,VCV,SCORES]=bekk(r,[],1,0,1,'full',[],[]);

%Correlation criteria
correl=zeros(3,3);
while correl(2,1)<=0.15||correl(2,3)>=-0.15||correl(3,1)>=0

%Parent matrix generator
nodes=2^T-1;                    %node count
endnodes=nodes-(2^(T-1)-1)
    counta=1;
        for cp=1:nodes-endnodes
                ps(2*cp:2*cp+1)=counta;
                counta=counta+1;
        end

[rowp colp]=size(ps);

%Tree values
r_t=[0 0 0];
y_t(:,:,1)=[1 1 1];
[C,A,G,B] = bekk_parameter_transform(PARAMETERS,1,0,1,3,'full');

%HT=zeros(3,3,nodes);
HT(:,:,1)=[mean(HT(1,1,:)) mean(HT(1,2,:)) mean(HT(1,3,:));
           mean(HT(2,1,:)) mean(HT(2,2,:)) mean(HT(2,3,:));
           mean(HT(3,1,:)) mean(HT(3,2,:)) mean(HT(3,3,:))];
varepsilon=zeros(3,1,nodes);
for k=2:nodes

HT(:,:,k)=C+A'*varepsilon(:,:,ps(k))*varepsilon(:,:,ps(k))'*A+B'*HT(:,:,ps(k))*B;
    varepsilon(:,:,k)=chol(HT(:,:,k))*normrnd(0,1,1,3)';
    r_t(k,:)=varepsilon(:,:,k)';
    y_t(:,:,k)=y_t(:,:,ps(k)).*(1+r_t(k,:));
end
correl=corr(r_t);
end

HT=HT(:,:,1:31);
```

8.2 Matlab code for MSLPR model with liquidity and cashflow contraints

```
CF=1;    %1=corrCF 2=noCF 3=negCF 4=posCF
beta=0.90;
assets=2;
nodes=2^T-1;
endnodes=nodes-(2^(T-1)-1);
decision=nodes*(assets+1);
sizeAb=nodes*(2+assets)-1+2*(nodes-1);
firstu=assets*nodes+2;
numberofu=assets*(nodes-endnodes);
var=firstu-1;

Aeq=zeros(nodes,decision);
beq=zeros(nodes,1);         %Cashflow entry/exit
A=zeros(sizeAb,decision);
b=zeros(sizeAb,1);
lb=zeros(1,decision);
ub=inf(1,decision);
p=1/endnodes;

y1=r_t(:,1:2); %sim.input
y2=zeros(numberofu+1,2); %var+u_k
y=[y1; y2];

%Aeq_________________________________________________________________
%(11) Sum x to 1
g=0;
for i=1:nodes
   Aeq(i,i+g:i+g+1)=1;
   g=g+1;
end

%beq (external portfolio cashflow)___________________________________
beq(:,1)=1+r_t(:,1);
beq(:,2)=ones(nodes,1);   %zero cashflow
beq(1,3)=1;
beq(2:nodes,3)=0.9;       %nocorr neg CF
beq(1,4)=1;
beq(2:nodes,4)=1.1;       %nocorr pos CF
beq=beq(:,CF);

%A___________________________________________________________________
%(15) mu(x)<= -R
    g=numberofu;
    for i=1:endnodes
        A(1,i+g:i+g+1)=y(i+nodes-endnodes,1:2)*p;
        g=g+1;
    end

%u_k >= 0
    A(2:nodes,firstu:decision)=diag(-ones(numberofu,1));

%x^Ty+a+u_k >= 0
g=0;
```

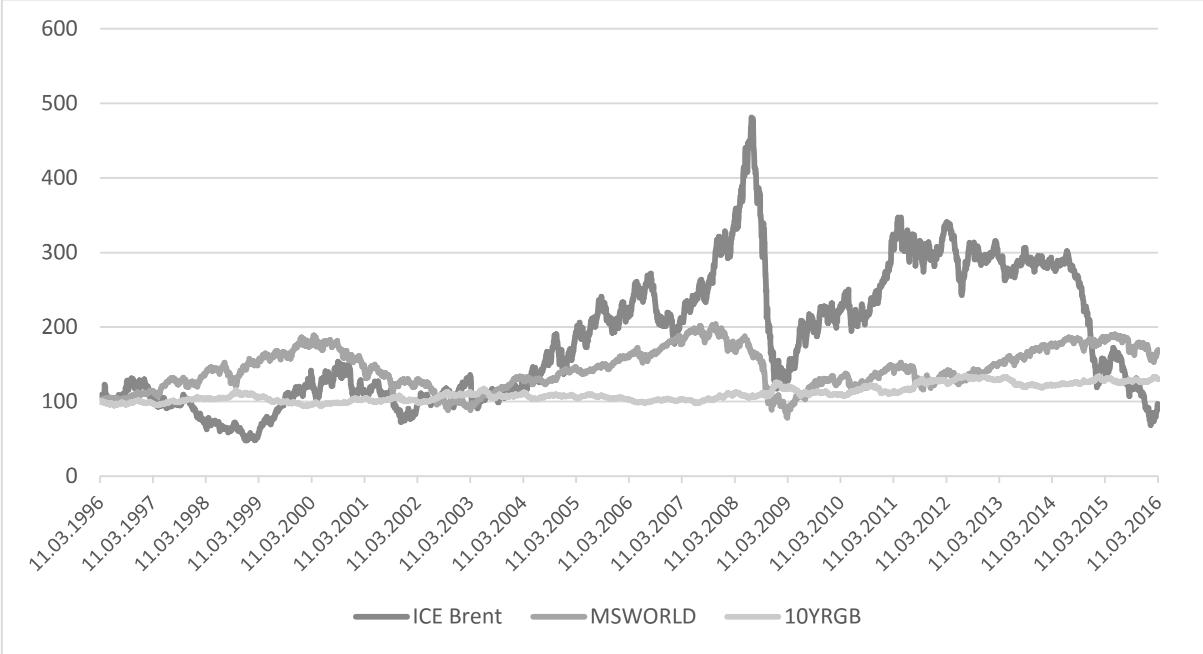
```
for i=1:nodes-endnodes
        xy(i+g,2*i-1)=-y(i+g+1,1);
        xy(i+g,2*i)=-y(i+g+1,2);

        xy(i+g+1,2*i-1)=-y(i+g+2,1);
        xy(i+g+1,2*i)=-y(i+g+2,2);
        g=g+1;
end

start=numberofu+2;
endst=start+assets*(nodes-endnodes)-1;
    A(start:endst,1:assets*(nodes-endnodes))=xy;
    A(start:endst,var)=-1;
    A(start:endst,firstu:decision)=diag(-ones(numberofu,1));

%lower/upper bound x>=0
start=endst+1;
endst=start+assets*nodes-1;
    A(start:endst,1:assets*nodes)=-diag(ones(1,assets*nodes));

%lbub liquidity_________________________________________________________
start=endst+1;
lbub=zeros(2*nodes-2,decision);
for i=1:nodes
    even(i)=2*i;
end

h=0;
en=1;
for g=1:2
    if h==1
        en=-1;
    end

    for i=2:nodes
    lbub(i-1+(nodes-1)*h,even(ps(i)))=en;
    end

    for i=2:nodes
    lbub(i-1+(nodes-1)*h,even(i))=-en;
    end
    h=1;
end

start=endst+1;
A(start:sizeAb,1:decision)=lbub;

%b_____________________________________________________________________
    b=zeros(sizeAb,1);
    R=0.0215;
    b(1,1)=-R;
    b(start:sizeAb-nodes,1)=1;
    b(sizeAb-nodes+1:sizeAb,1)=1;
```

```
%objective function____________________________________________________
f = @(w)w(var)+1/(endnodes*(1-beta))*sum(w(firstu:decision))

options =
optimset('Display','iter','MaxIter',5000,'MaxFunEvals',20000,'TolFun',1.0e-
08,'TolX',1.0e-08,'TolCon',0.4e-01,'Algorithm','sqp');

%solver____________________________________
w0=zeros(1,decision);
[w(CF,:), fval(CF)] = fmincon(f,w0,A,b,Aeq,beq,[],[],[],options);
```

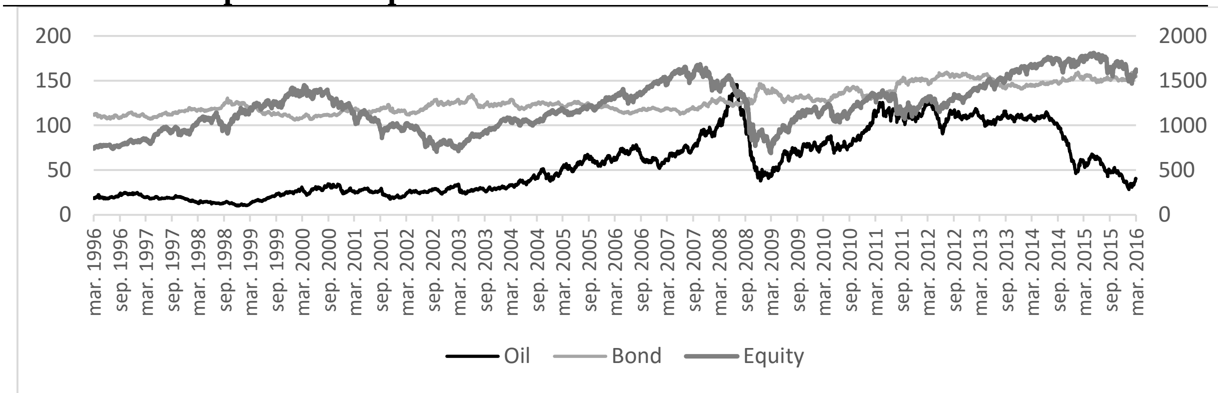
8.3 Explicit (schematic) 3 stage stochastic linear program with liquidity and cashflow constraints

Asset-liability management un			Stage 0		Stage 1				Stage 2																
node			1		2		3		4		5		6		7			2	3	4	5	6	7		
Decision	x^t		we1	wb1	we2	wb2	we3	wb3	we4	wb4	we5	wb5	we6	wb6	we7	wb7	VaR	u2	u3	u4	u5	u6	u7		
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21		
	sum x=1	Aeq	1	1																				=	1+c
					1	1																		=	1+c
							1	1																=	1+c
(11)									1	1														=	1+c
											1	1												=	1+c
													1	1										=	1+c
															1	1								=	1+c
(15)	mu(x) =< -A								p*re4	p*rb4	p*re5	p*rb5	p*re6	p*rb6	p*re7	p*rb7								<=	-R
	u_k >= 0	A																-1						<=	0
																			-1					<=	0
																				-1				<=	0
																					-1			<=	0
																						-1		<=	0
																							-1	<=	0
	x^Ty+a+u_k		-re2	-rb2													-1	-1						<=	0
			-re3	-rb3													-1		-1					<=	0
					-re4	-rb4											-1			-1				<=	0
					-re5	-rb5											-1				-1			<=	0
							-re6	-rb6									-1					-1		<=	0
							-re7	-rb7									-1						-1	<=	0
	x >= 0		-1																					<=	0
				-1																				<=	0
					-1																			<=	0
						-1																		<=	0
							-1																	<=	0
								-1																<=	0
									-1															<=	0
										-1														<=	0
											-1													<=	0
												-1												<=	0
													-1											<=	0
														-1										<=	0
															-1									<=	0
																-1								<=	0
	liquidity			1		-1																		<=	g1
				1				-1																<=	g1
						1				-1														<=	g1
						1						-1												<=	g1
								1						-1										<=	g1
								1								-1								<=	g1
				-1		1																		<=	g2
				-1				1																<=	g2
						-1				1														<=	g2
						-1						1												<=	g2
								-1						1										<=	g2
								-1								1								<=	g2

8.4 Price index for assets ICE Brent (oil), MSWORLD (equity) and 10YRGB (bond)

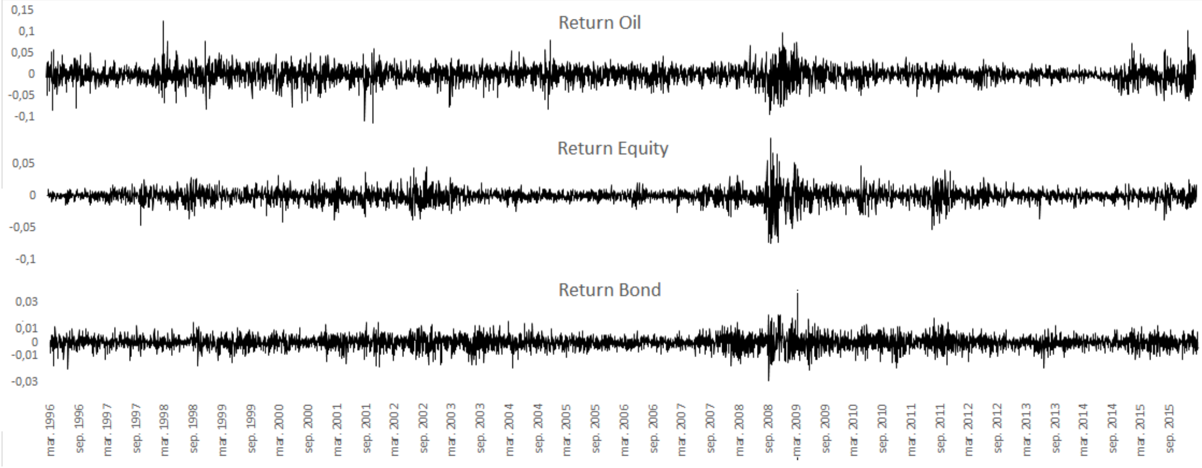


8.5 Absolute price development

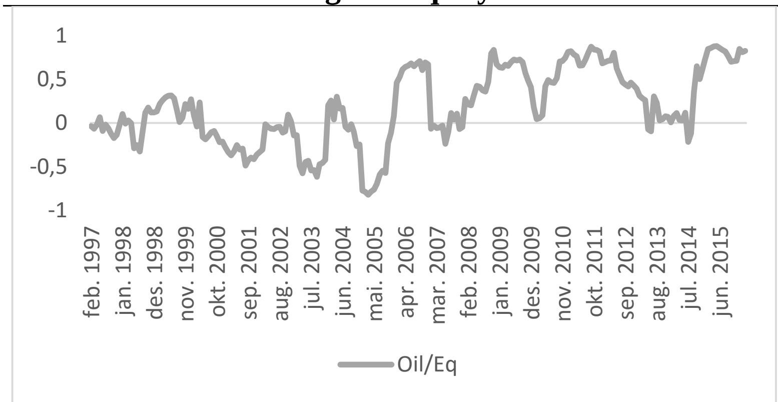


Prices from march 1996 to march 2016 for ICE Brent crude oil index (Oil), Datastream US benchmark 10 year government bond index (Bond) (thin grey) – left axis, and MSCI World Index (Equity) (thick grey) – right axis .

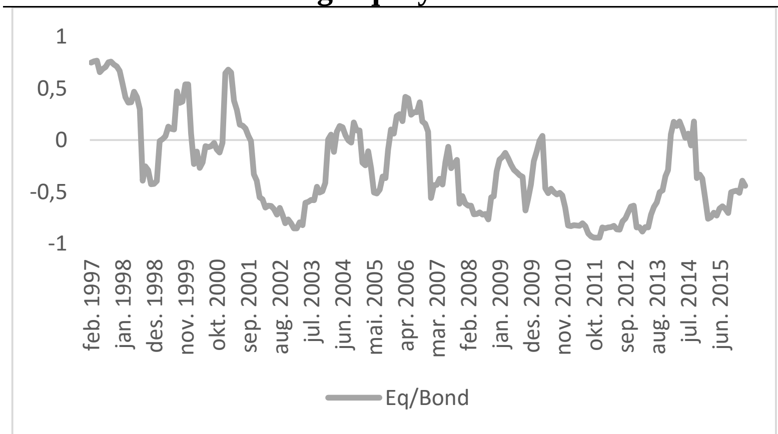
8.6 Daily returns for for assets ICE Brent (oil), MSWORLD (equity) and 10YRGB (bond)



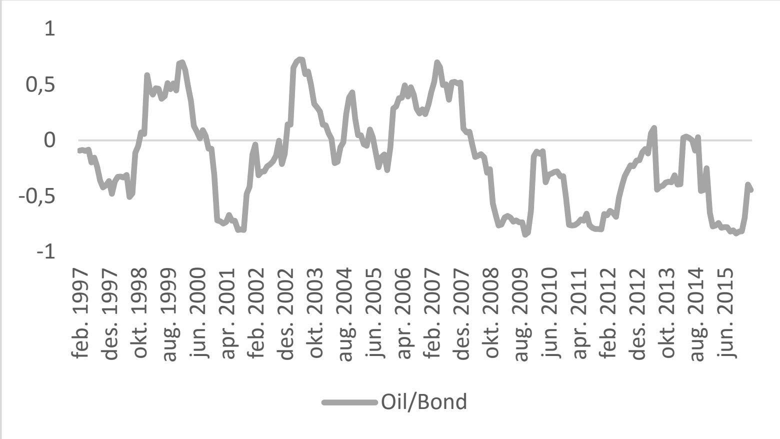
8.7 12 month trailing Oil/Equity correlation



8.8 12 month trailing Equity/Bond correlation



8.9 12 month trailing Oil/Bond correlation



8.10 Jarque-Bera test for normality

As a goodness-of-fit test for normality, the Jarque-Bera is widely used and has gained great acceptance among econometricians. It is a function of the measures of skewness S and excess kurtosis K computed from a sample of residuals. Under normality $S = 0$ and $K = 3$, respectively (Thadewald T and Büning H. 2007). The JB test is defined as follows:

$$JB = \frac{n}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \tag{25}$$

Where JB is asymptotically chi-squared distributed with two degrees of freedom. Thus, the hypothesis H_0 of normality is rejected at level α if $JB \geq \chi^2_{1-\alpha,2}$. At level $\alpha = 0{,}05$ the critical point $\chi^2_{0{,}95,2} = 5{,}99$.

8.11 Ljung-Box Q-test for autocorrelation and heteroscedasticity

The Ljung-Box Q-test has a null hypothesis saying residuals shows no autocorrelations for a fixed number of lags L versus autocorrelated residuals for the same number of lags, i.e. autocorrelation coefficient $\rho(k)$, $k = 1, \dots, L$, is nonzero. Under the null hypothesis the asymptotic distribution of the Q-statistics is χ^2_l with L degrees of freedom.

$$Q = N(N+2)\sum_{k=1}^{L}\left(\frac{\hat{\rho}_k^2(\varepsilon_t)}{N-k}\right)$$

Where N is the size of the samle, L is the number of autocorrelation lags, and $\rho(k)$ is the null hypothesis (Box G. E. P Jenkins G. M. and Reinsel G. C. 2008, The MathWorks Inc. 2016 a).

Similarily McLeod A. I. and Li W. K. (1983) proposed a formal test for heteroscedasticity based on the Ljung-Box test, checking for ARCH-effects. In this case the test operator is simply the residuals squared (McLeod A. I. and Li W. K. 1983, Wang W. et. al. 2005, The MathWorks Inc. 2016 a):

$$Q^2 = N(N+2)\sum_{k=1}^{L}\left(\frac{\hat{\rho}_k^2(\varepsilon_t^2)}{N-k}\right)$$

Where $\hat{\rho}_k^2$ is the squared sample autocorrelations of the residual series $\varepsilon_t = y - \hat{y}$ for model estimates or $\varepsilon_t = y - \bar{y}$ for sample average at lag k.

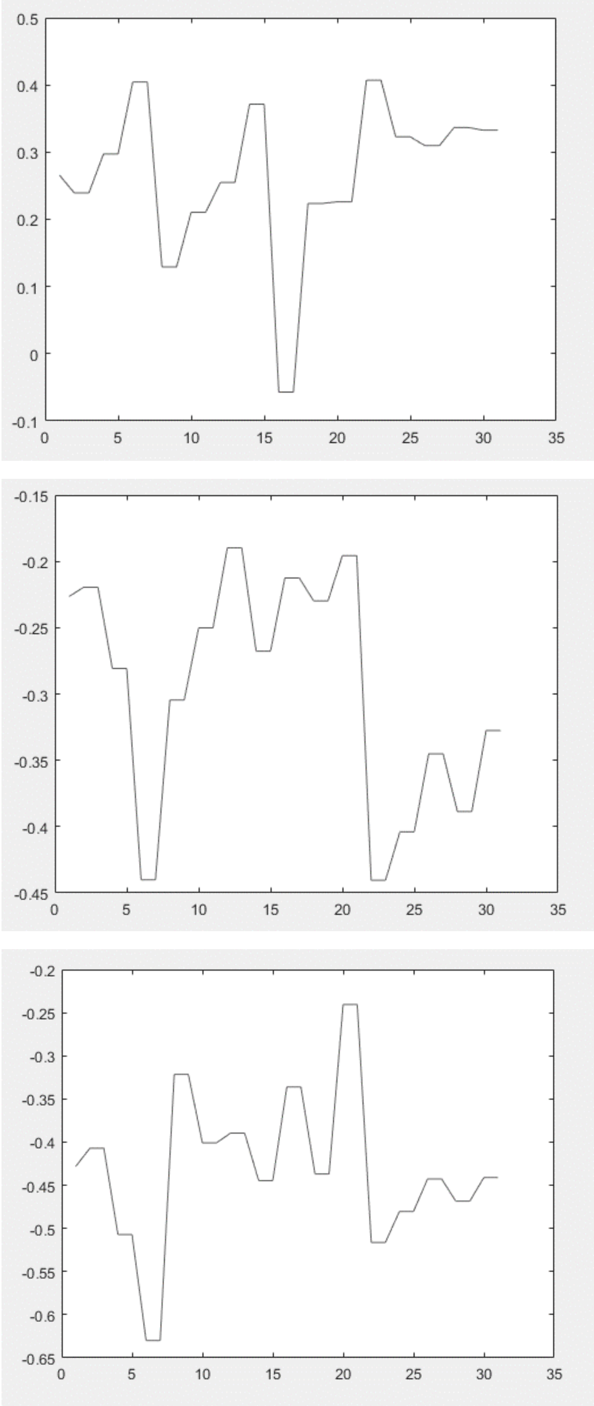
8.12 Augmented Dickey Fuller test for unit root

The Augmented Dickey Fuller test for unit root controlling for a unknown order $ARMA(p,q)$-model. The ADF tests the null hypothesis $\rho = 1$ that a sample y_t is $I(1)$ (unit root of one) against the alternative hypothesis $\rho = 0$, $I(0)$ (unit root of zero). The following model is considered:

$$y_t = \rho y_{t-1} + \varepsilon_t$$
$$\varepsilon_t = \alpha\varepsilon_{t-1} + e_t + \beta e_{t-1}$$

Where it is assumed that $|\alpha| < 1$, $|\beta| < 1$ and $y_0 = 0$ and e_t is a sequence of normal i.i.d random variables (Said E. S. and Dickey D. A. 1984).

8.13 Correlations for scenario generated returns per nodes (oil/eq, oil/bonds and eq/bonds, respectively)



8.14 Relavtive equity allocation for full liquidity case (100 %)

Corr. CF	No CF	Neg. CF	Pos. CF
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,23	0,14	0,16	0,04
0,50	0,50	0,50	0,50
0,09	0,12	0,11	0,11
0,50	0,50	0,50	0,50
0,00	0,00	0,00	0,00
0,50	0,50	0,50	0,50
0,13	0,07	0,07	0,07
0,00	0,01	0,00	0,00
0,00	0,00	0,00	0,00
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50
0,50	0,50	0,50	0,50

8.15 Relavtive equity allocation for low liquidity case (20 %)

Corr CF	No CF	Neg CF	Pos CF
0,43	0,47	0,50	0,22
0,22	0,27	0,29	0,20
0,35	0,34	0,41	0,11
0,18	0,09	0,08	0,08
0,22	0,09	0,16	0,07
0,12	0,16	0,22	0,18
0,20	0,20	0,22	0,15
0,32	0,29	0,23	0,27
0,00	0,00	0,00	0,00
0,42	0,29	0,38	0,23
0,16	0,07	0,00	0,07
0,00	0,01	0,01	0,01
0,00	0,00	0,00	0,00
0,30	0,38	0,43	0,33
0,21	0,38	0,43	0,33
0,47	0,48	0,46	0,45
0,50	0,48	0,46	0,45
0,00	0,05	0,01	0,00
0,15	0,05	0,01	0,00
0,50	0,49	0,50	0,40
0,50	0,49	0,50	0,40
0,32	0,27	0,10	0,25
0,32	0,27	0,10	0,25
0,12	0,08	0,17	0,10
0,06	0,08	0,17	0,10
0,25	0,20	0,21	0,18
0,12	0,20	0,21	0,18
0,50	0,50	0,50	0,48
0,50	0,50	0,50	0,48
0,50	0,50	0,50	0,48
0,50	0,50	0,50	0,48

8.16 Cummulative return for oil, equity and bonds for $T = 5$.

Scenario (oil)

1	1	0,981	1,0157	1,0667	1,0829
2	1	0,981	1,0157	1,0667	1,1782
3	1	0,981	1,0157	1,0189	1,0302
4	1	0,981	1,0157	1,0189	0,9763
5	1	0,981	0,9568	0,9566	0,9851
6	1	0,981	0,9568	0,9566	0,9657
7	1	0,981	0,9568	0,895	0,846
8	1	0,981	0,9568	0,895	0,8703
9	1	1,0402	1,0351	0,9954	1,0043
10	1	1,0402	1,0351	0,9954	0,9205
11	1	1,0402	1,0351	1,0881	1,2296
12	1	1,0402	1,0351	1,0881	1,0506
13	1	1,0402	1,0871	1,1397	1,2384
14	1	1,0402	1,0871	1,1397	1,2541
15	1	1,0402	1,0871	0,9931	1,0811
16	1	1,0402	1,0871	0,9931	1,1039

Scenario (equity)

1	1	0,9792	0,9685	0,9459	0,9245
2	1	0,9792	0,9685	0,9459	0,9461
3	1	0,9792	0,9685	0,9808	0,9532
4	1	0,9792	0,9685	0,9808	0,9525
5	1	0,9792	0,9827	0,9724	0,9606
6	1	0,9792	0,9827	0,9724	0,968
7	1	0,9792	0,9827	0,9639	0,9385
8	1	0,9792	0,9827	0,9639	0,9612
9	1	1,0285	1,0293	1,0157	1,0335
10	1	1,0285	1,0293	1,0157	0,9876
11	1	1,0285	1,0293	1,0395	1,0613
12	1	1,0285	1,0293	1,0395	1,0101

13	1	1,0285	1,0457	1,0581	1,083
14	1	1,0285	1,0457	1,0581	1,0178
15	1	1,0285	1,0457	1,035	1,0301
16	1	1,0285	1,0457	1,035	1,0525

Scenario (bonds)

1	1	1,0093	1,0032	1,0026	1,0068
2	1	1,0093	1,0032	1,0026	1,0155
3	1	1,0093	1,0032	0,9977	0,9951
4	1	1,0093	1,0032	0,9977	1,0088
5	1	1,0093	1,012	1,001	1,0014
6	1	1,0093	1,012	1,001	0,99
7	1	1,0093	1,012	1,0235	1,0312
8	1	1,0093	1,012	1,0235	1,019
9	1	0,9771	0,9647	0,9813	0,9864
10	1	0,9771	0,9647	0,9813	0,9853
11	1	0,9771	0,9647	0,9573	0,9428
12	1	0,9771	0,9647	0,9573	0,9513
13	1	0,9771	0,975	0,9649	0,9522
14	1	0,9771	0,975	0,9649	0,9574
15	1	0,9771	0,975	0,9801	0,9765
16	1	0,9771	0,975	0,9801	0,9638

8.17 Inspiration: The Government Pension Fund Global

The Norwegian economy has for nearly three decades had prominent wealth increase due to the petroleum driven boom. The establishment of the Government Pension Fund Global (hereby-called GPFG) in 1990 was the political intent to support long term sustainable spending of the Norwegian government's petroleum income, as well as preserving the expenditures for the present and future Norwegian nation. The «pension» part of the name is actually quite misleading, as it holds no obligation to future pension payments, so the name itself is politically motivated, and not actual. The formal GPFG management responsibility lies with the Department of Finance, but the operative management is delegated to the Central Bank of Norway (hereby called Norges Bank). The daily operations responsibility is further delegated to the Norwegian Bank Investment Management (NBIM). The entire GPFG is invested in foreign currency to avoid destabilizing the Norwegian economy. During the spring of 2001 the fiscal rule for government fiscal budget spending was set to 4 % of the GPFG capital, which was assumed to represent expected real return of the fund (Norwegian Bank Investment Management 2010). In 2015, one ninth over the fiscal budget will come from the GPFG (Norwegian Petroleum Directorate and Norwegian Ministry of Petroleum and Energy 2016).

Transfers to the GPFG

Capital is normally transferred to the fund monthly with the exception of December, where the Ministry of Finance informs Norges Bank of how much is to be transferred. Thereafter the portfolio managers are informed of the inflow and decide which securities they wish to purchase. A central trading team executes the transactions (Norwegian Bank Investment Management 2010).

The petroleum related income is a combination of petroleum activity taxation, the governmental ownership in Statoil, and the State's Direct Financial Interest (SDFI), providing the government the opportunity for direct ownership in the Norwegian continental shelf (Store norske leksikon 2016). As shown in the figure below, in the 1970's the majority income was taxes from operations, and the Norwegian government only had ownership interests in production licenses through Statoil. In 1985, SDFI system was established to increase petroleum related income from the Norwegian continental shelf, thus yielding an important income source in the years to come (Norwegian Petroleum Directorate and Norwegian Ministry of Petroleum and Energy 2016).

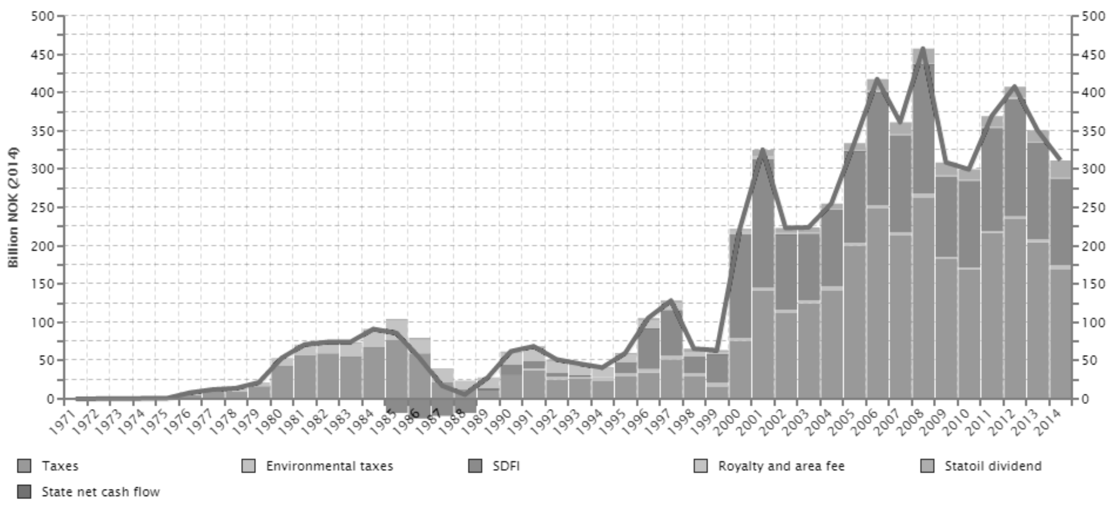


Figure: The net government cash flow from petroleum activities, 1973-2014. Source: (Norwegian Petroleum Directorate and Norwegian Ministry of Petroleum and Energy 2016)

The history of a rebalancing regime

The rebalancing history of the GPFG starts in the year of 1998 upon the introduction of equities as a complementary asset class in the fund. Until that moment, the transfers from petroleum activity had been part of the Norges Bank FXs reserves that was in its entirety invested in government bonds. For the period 1998 to 2001, the fund, in conjunctions with contribution transfers from petroleum activities, was rebalanced back every quarter to six strategic weights, and regional allocations to 50 % European, 30 % American and 20 % Asian. The performance of the GPFG is measured against a benchmark portfolio with the intended strategic attributes. In order to bring the actual index closer to the strategic weights, it was in 2001 decided to allocate on a monthly basis (Norwegian Bank Investment Management 2012).

In a letter to the Department of Finance from the then Norges Bank Governor Svein Gjedrem, dated 10. February 2006, it is argued by the GPFG's very long-term investment horizon, the fund should accept higher risk in order to increase return through a 50 or 60 % allocation to the equity portfolio. The argument is based on a comparison of the return the fund would have achieved had it allocated 40, 50 and 60 % over a 15-year period. Also an allocation comparison of the largest global pension funds was used to show that 60 % bonds is almost a 50 % higher allocation share than what these funds practice, even though they probably had shorter duration and a more well defined liability than the GPFG (Norwegian Bank Investment Management 2006). In 2007-2009 the equity portfolio was increased from 40 to 60 % (Norwegian Bank Investment Management 2012).

Around 65 % of petroleum related transfers to GPFG in 1996-2010 was allocated to the equity portfolio, while the remainder was channeled to the fixed income portfolio. The chart below shows the yearly inflow allocation to the equity and fixed income portfolio not adjusted for inflation. In the first years, the transfers was mostly allocated to the fixed income portfolio, while for 2007 and onwards the parts of the fixed income portfolio was reallocated to equity in order to meet the 20 % increase in the equity portfolio (Norwegian Bank Investment Management 2010).

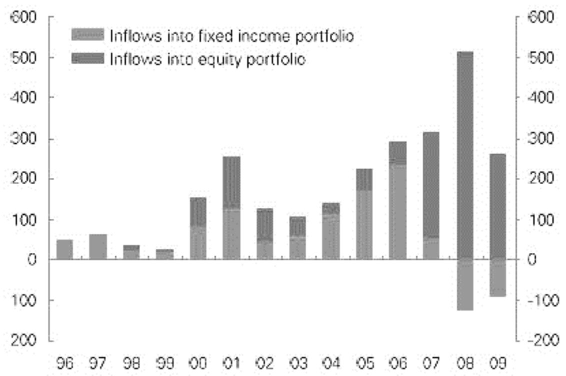


Figure: Annual inflows into the fund by asset class (BNOK), 1996-2009. Source: (Norwegian Bank Investment Management 2010)

In 2010 the fund announced that it intended to acquire 25 % of the Regent Street in London, England, thus including real estate as a another factor for diversification of risk in the GPFG portfolio. In the years to come, the CPFG acquired real estate property in Paris, Berlin, München, Frankfurt, Washington D.C., New York, Boston, San Fransisco and several other cities in Europe and United States, in various types from prime yield office buildings, to retail and storage facilites and many more (Norwegian Bank Investment Management a 2016).

The mandate

Today the investment strategy is to invest in a wide range of countries, companies and assets. In september 2015 the portfolio allocation weights consisted of 59,7 % equity investments, 37,3 % fixed-income investments and in 3,0 % real estate investments (Norwegian Bank Investment Management a 2016). The “Management Mandate” given to Norges Bank by the Ministry of Finance is the official guidelines, which the GPFG must act accordingly to. On investment of capital and the management objective includes statements like: “The Bank shall seek to achieve the highest possible return after costs measured in the investment portfolio's currency basket". Further, the strategic benchmark portfolio shall consist of 60 % equity, 40 % fixed income subtracted by the amount invested in real estate with a long term goal of 5 %. If the equity share in the actual benchmark portfolio deviates more than 4 % from the strategic benchmark portfolio, rebalancing back to strategic weights is enforced. The responsible management principles is that investments should have a long term horizon, involve good corporate and sustainable environmental, economical and social considerations. The equity portfolio may be invested in listed companies on regulated and recognised market places, or unlisted companies with the intention to become listed, whereas the bond portfolio may be invested in tradable bonds, depository receipts, and other tradable debt instruments, and the real estate portfolio in most kinds of real estate vehicles concerning land and/or the building on top of it. The portfolio may not be invested in Norwegian securities denominated in NOK or enterprises with head office in Norway, fixed income in countries under UN sanctions or similar, and real estate in the form of infrastructure (Norwegian Bank Investment Management c 2016).

Dark prospects

Since the establishment, the GPFG has been transferred NOK 3,5 trillion almost doubling the portfolio capital, which at the time of writing amounts to a total of NOK 6,9 trillion. Looking ahead, it is likely that the GPFG will further reduce its allocation to bonds, and increase its allocation to equity in order to position itself as a long-term investor (The Central Bank of Norway a 2016).

In the Norges Bank Governor's yearly speech for 2016, Øystein Olsen stated that the end of the prosperous oil income for the Norwegian government may be near. The Norwegian nation must prepare for persistently low oil prices and lower spending in years to come. He said that already at the time of presentation for the 2015 government budget plan, the prospects were that net savings from oil related income and spending would become negative. With the current oil price, the Norges Bank estimates the present value of the future oil related income to around a quarter of the current size of the GPFG. As shown in the figure below for most part of the first decade of 2000, net contributions have been highly positive, but with a NOK 200 billion downturn during the financial crisis 2008-2009. From 2012 and forward the net contributions have decreased, ending with the historically first negative net transaction in 2016. Thus 2015, he says, may be the last year of positive contributions to the GPFG (The Central Bank of Norway a 2016).

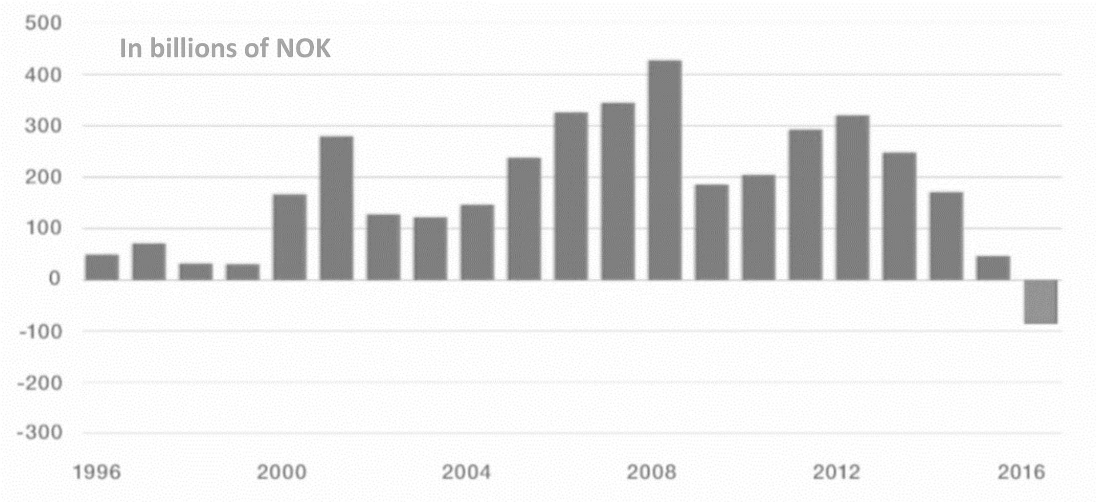


Figure: Net yearly contribution in billions of cash flow in GPFG, 1996-2016. Source: (The Central Bank of Norway b 2016)