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Planning Gaits for Underactuated Compass-Biped Robot With Torso

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Problem Description

The project is aimed at planning gaits for a model of an underactuated 3-link biped robot with torso consistent with various physical constraints imposed on robot's dynamics. Specifically, steps in organization of numerical procedures for searching gaits, which represent periodic cycles in the state space the robot's hybrid dynamics, are comprehensively discussed. The arguments and their software implementation are based on a geometrical parameterization of such gaits through evolution of a scalar parameter introduced instead of time. Its appearance allows conveniently reformulating different properties and physical constraints inherent to expected behaviors of all degrees of freedom as properties of this scalar variable. The work investigates parameterization of system's solutions for an efficient and reliable organization of the gait's search.

Assignment given: 11. January 2016
Supervisor: Anton Shiriaev, ITK

Preface

This Master's thesis documents the work done during my final semester at NTNU, and concludes five years of studying Engineering Cybernetics.

I would first like to thank my supervisor Professor Anton Shiriaev for guiding my academic progress during the last two semesters. I have highly valued his patience and encouragements during our many meetings. Further, I would thank two of my colleagues during this project, Torleif Anstensrud and Leonid Paramonov for their invaluable help with software development and testing. I would also like to especially thank Damir Anicic for two semesters of fruitful collaboration, many interesting discussions and countless laughs. My closest family also deserve thanks for always being there for me and for providing unconditional support.

Lastly, I would like to thank my girl, Nora for five eventful years in Trondheim that went by ever so quickly. These have truly been *our* five years.

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Abstract

A structured approach to gait planning for a planar underactuated 3-link biped robot is presented. The method of virtual holonomic constraints (VHC) is employed to this end. It is shown how this method effectively reduces the dimensionality of the problem and allows for the biped's dynamics to be re-written in reduced form. The reduced dynamic system is then used to structure requirements for the existence of periodic gait-cycles in the robot's state-space. The gait planning task is then solved in a structured manner by organizing a numerical search that finds parameterizations of the VHCs that lead to nominal gait cycles for the biped.

Sammendrag

En strukturert tilnærming for planlegging av ganglag for en underaktuert tobent robot med overkropp er presentert. Dette er gjort ved å benytte såkalte vituelle geometriske begresninger for robotens dynamikk. Metoden gjør det mulig å skrive robtens dynamiske modell på redusert form. Denne reduserte formen har blitt brukt til å strukturere et sett med betingleser for lukkede periodiske løsninger i robotens tilstandsrom. Disse lukkede løsningene vil definere periodiske ganglag for roboten. Planlegging av slike ganglag blir gjort ved å strukturere et numerisk søk som finner parameterisenger for de virtuelle begresningene til det reduserte systemet som produserer periodiske ganglag for roboten.

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1 Introduction

This thesis-report describes the application of a certain methodology for developing walking behaviour in a 3-link bipedal robot with torso. Developing bipedal locomotion has been, and continues to be a challenge. A quick review of some existing literature reveals a considerable number of different walking machines possessing different capabilities and features. The problem has also attracted serious research interest from a wide range of researching fields. Nonetheless, the development of walking robots have not made progress to the point where such machines have found any widespread use commercially, or in our daily lives. This clearly motivates continued research interest in the field.

This chapter provides a brief introduction to the field of legged locomotion, focusing on bipedal robots. Some of the main motivating factors for pursuing research in this field is presented accompanied by a brief historical view on the topic.

Having presented some motivating factors and provided some historical context, the focus is shifted to the different strategies and methodologies that have been employed in the pursuit of bipedal locomotion in robots. More precisely the methodologies and strategies that have been applied in order to plan and subsequently control walking motions in bipedal robots. The strategy and methodology employed in this thesis can then be placed in some historical context and compared to other dominant and successful design paradigms or lack thereof.

The chapter then proceeds to establish the scope of this thesis, as well as the structure of the text.

1.1 Bipedal Robot Locomotion

1.1.1 Motivation and Brief Historical Review

Legged locomotion is found in both animals and humans. In the broadest sense locomotion is simply the ability to move through space. By looking at the movement of animals, it is evident that there are many different successful varieties of legged locomotion. There are numerous factors that motivate the development of robots capable of moving using legs. There are environments where other means of locomotion such as belts or wheels are unsuited or useless. The presence of uneven terrain, obstacles or structures such as ladders and staircases, are examples where legs become an obvious choice. Machines developed to operate using legs as a means of locomotion could be used in environments and tasks that require legged locomotion, but are hazardous or unsuited for humans. Search-and-rescue operations, military interventions, de-mining or exploration of foreign planets are examples of such tasks.

Further, the appeal of creating animal-like behaviour in machines, has undoubtedly been a key motivating factor for the development of many legged robots. In general one could argue that in the development of animal-like machines one can often find elegant solutions to certain problems by replicating mechanisms found in animals. Conversely the development of such machines could give insights into certain aspects of the original organism and its mechanisms. Although biomimetics [20] and the diversity and complexity of animal movement [6] are fascinating, there is put little emphasis on these fields in this thesis. A somewhat obvious, but important point is made in both ([6], pp. 104) and ([21], pp. 3). Mechanical robots and animals are indeed very different. Electrical actuators are not muscles, nerves are different from copper wires and animal brains are very different from computers.

Although the current state of technology clearly divides the mechanisms found in humans and animals from those we can replicate in robots, this does not imply that there is nothing to learn about human and animal locomotion from the development and analysis of legged robots. The fundamental dynamics of walking on legs can perhaps be better understood through such developments. Legged mechanical robots may serve in the role as simpler or introductory examples of legged walking mechanisms. In such machines some of the complexity of the legged locomotion observed in animals or humans are removed, while many of the fundamental characteristics and inherent difficulties remain intact. In [6], Dickinson et al. argues that walking on legs can be modelled through the dynamics of an inverted pendulum:

> *When animals walk, the body vaults up and over each stiff leg in an arc, analogous to an inverted pendulum.*

Bipedal robot locomotion constitutes a subclass of the legged robots that have been discussed thus far. However this subclass is the main focus in the following. The image of the humanoid bipedal robot is well known from popular culture. Consequently the appeal of creating human-like walking machines has undoubtedly been one of the key motivating factors for research and development of mechanical bipedal robots. However there is still a lot that is unknown about human bipedal locomotion. Developing and analysing control strategies for simpler yet representative bipedal mechanisms may provide insights into human locomotion control. Such insights could also aid the development of prosthetic devices. Mechanical walking robots exhibiting similar dynamics as those found in human walking could perhaps also serve as valuable test-beds for such prosthetics.

The first bipedal robots emerged as early as in the 1970s and since then a considerable number of prototypes has been developed. In [21] it is stated that over a hundred different bipedal robots have been designed and built. Within all these prototypes one finds a considerable variety. It is important to note that the different robots have been developed to exhibit many different features and accomplish different tasks and may therefore not be directly comparable. Some robots have been designed and built only to walk on two legs, while other projects have tried to develop more complex human-like robots. In the following two very different walking robot designs and corresponding strategies and methodologies for the creation of walking behaviour will presented. These two robots are chosen to show the span of the different developments that have been made in the field.

Humans are highly versatile, capable of doing diverse and complex tasks. Creating human-like or humanoid robots are therefore a daunting task. State of the art human-like robots include Honda Asimo [1] and recently Boston Dynamics' Atlas [2]. Asimo is capable of doing a variety of tasks, and is in this chapter used as the main example of a complex humanoid robot. Asimo show great versatility in its ability to move on two legs, being able to make turns when both running and walking. Asimo is also capable of performing a wide variety of other tasks. However the focus of this chapter or indeed this thesis is not humanoid robots. However, the elegant, stable and robust behaviour of human walking and running is one of the most striking and defining human characteristics. Although Asimo is versatile in both it's ability to locomote and perform other tasks, it's walking and running behaviours are not very human-like or efficient in terms of energetic economy [12].

A key point in the following discussion is therefore introduced. Although different bipedal robots have been designed with different features, abilities and criteria, they share the defining characteristic of walking on two legs. A.D. Kuo presents an interesting perspective in [12], where he investigates the trade-offs between economy and versatility in walking bipedal robots. Two important paradigms for the design of walking robots are also introduced here, namely the ZMP-paradigm and so-called dynamical walking. These two terms will be presented and discussed later in this chapter. For now it is suffice to say that they present two different very different philosophies for the development of walking robots that greatly affect both the mechanical design and the development of motion-planning and control strategies.

The human-like Honda Asimo is an example of the most complex and versatile bipedal robots to date. Asimo has a total of 57 DOFs, and is capable of much more than (simply) walking and running on two legs. On the other side of the spectrum one finds so-called passive dynamic walkers. These are deceptively simple and entirely passive walking mechanisms exhibiting very natural or human-like walking behaviour. The research into such simple but highly fascinating walking robots started with the seminal work of McGeer in the late 1980s. His 1990 paper [14] gained wide recognition and created a considerable research interest in passive dynamical walking. Passive dynamical walkers brings the inherent inverted pendulum dynamics of walking to the forefront. The remarkable simple design of passive dynamical walkers are capable of natural looking walking behaviours that is stable and robust, without any control actuation. Although these results are impressive and groundbreaking in their own right it has sparked an interest in the development of dynamical walking machines in general. In [12], the author elegantly describes so-called dynamic walking:

> *We define dynamic walking to refer specifically to machines designed*

to harness leg dynamics, using control more to shape and tune these dynamics than to impose prescribed kinematic motions.

The two examples of bipedal robots presented in the previous were chosen to tell the tale of two extremes. Two extremes in terms of complexity and ability to perform useful tasks. But also two extremes in philosophies for developing walking behaviour in machines. The completely passive gaits found in passive dynamic walker clearly demonstrates the inverted pendulum dynamics that is observed in human locomotion. More complex bipedal robots such as Asimo does not include this passive and inherently unstable dynamics in their walking gaits. The question one must ask is whether dynamical walking can be accomplished on flat ground, driven by actuators. The 3-link biped investigated in this thesis is a machine designed for the development of dynamical gaits on flat ground.

However, before the presentation of motion planning and control strategies for bipedal robots are presented, some useful terminology and definitions are presented.

1.1.2 Terminology and Definitions

The terminology and definitions will be similar to those found in ([21], Section 1.2), although not as comprehensive.

Bipedal robots are in general open kinematic chains. The legs and torso are then referred to as subchains. A bipedal robot has two legs possibly with knees and/or feet. Bipedal robots may or may not have a torso. The point where legs, and possibly, the torso is connected is referred to as the *hip* of the robot. Any biped is said to have feet, but this also includes point-feet.

Walking is defined through two distinct phases known as *single support* or *swing phase* and *double support phase*. In the single support or swing phase one of the leg's feet is on the ground while the other one is swinging in order to create the forward walking motion. The leg whose foot is in contact with the ground is refereed to as the *stance leg* and the other leg is referred to as the *swing leg*. The double support phase then describes the phase when both feet are in contact with the ground. Having established these two distinct phases, *steps* and consequently a *gait* can be defined.

A *step* is defined as the movement made between consecutive double support phases. The repetition of steps will constitute a walking *gait*. Note that one or two legs are always in contact with the ground during a walking gait, as opposed to running gaits. This thesis-report is only concerned with walking gaits. Further, robots that have so-called non-trivial feet will have steps that consist of different phases related to the feet's configuration such as heel-rise and toe-roll. However this thesis-report will not deviate in this direction and is only concerned with robots with (trivial) point-feet.

The dynamics of walking is defined through the two different phases and must therefore also consist of at least two parts. When a step is made the robot will start in the single support phase and then as the swing leg collides with ground the system will switch to the double support phase before a new step is made and the biped switches back to the single support phase. The collision of the swing leg with the ground is called an *impact*. A model of a biped robot must be *hybrid*, in order to create the switching between the two different phases. A *hybrid* model is one that captures both continuous dynamics as well as discrete switches.

1.1.3 The Inherent Dynamics and Difficulties of Walking

Having defined some key concepts and established some useful terminology, some important characteristics of walking can be presented.

Underactuation The bipedal locomotion of humans exhibits *underactuation*. This means that there are passive dynamics present in the walking motions. This implies that there are mechanical degrees of freedom that cannot be directly manipulated through the system's actuators. Underactuation is generally undesired in control systems. It means that there are parts of the systems state that cannot be directly actuated and subsequently controlled. Although undesired from a control perspective, there are several well-known examples of systems where underactuation is present. For instance are airplanes, bicycles and automobiles all underactuated systems. You cannot directly and independently affect both the position and orientation of such machines.

In a bipedal robot with point-feet underactuation is present. During the single support phase the angle between the stance leg and the ground cannot be directly controlled as there is no ankle and consequently no ankle-actuator. It is assumed that the end of the stance leg acts as a pivot point for the stance leg. The stance-leg then has the dynamics of an inverted pendulum. It is assumed that the foot is rigidly attached to ground and can rotate freely. There is no friction, i.e. the foot creates an ideal pivot point.

Limb Coordination Walking does not put any explicit requirements on the movement of the torso. However humans do coordinate the movements of legs, torso and arms when walking. This means that there are infinitely many different ways to coordinate legs, arms and torso. In terms of just simply walking no solution can be seen as more or less good. From a motion planning perspective, choosing such coordination may then become a highly nontrivial task.

1.2 Motion Planning and Control for Bipedal Robot Locomotion

Having put the problem of bipedal robot locomotion in some historical context, provided some motivating factors, and established some useful terminology it is time to look into potential solutions for creating walking behaviour in robots. First, some useful terms and concepts will be defined.

Gait Synthesis The task of combining the mechanical design of robots with strategies and methodologies for motion planning and control in order to create successful walking behaviours.

Motion-Planning As the name suggests motion planning is the task of planning motions in some way and then formulating the desired motions in some useful format. For bipedal robot locomotion any motions that result in walking behaviour could then be defined as desired motions. Time-dependent kinematic trajectories from the walking of humans could be one such example. However motions can also be found to be time-independent geometric relations between a systems coordinates, where motions are planned or rather shaped through these geometrical relations.

Nominal behaviour The desired behaviour of a robot is often referred to as *nominal behaviour*. The desired trajectories for a robot's coordinates (or state variables) and actuators are then referred to as *nominal trajectories*.

Control of walking motions Controlling walking motions could be defined as the task of tracking or alternatively stabilizing the nominal trajectories found when planning motions.

1.2.1 Brief Historical review

As already discussed there has been built a considerable amount of different prototypes of robots capable of bipedal locomotion. However successful strategies for creating efficient, stable and robust walking behaviours have not seen the same development. In this section, two different paradigms is presented, discussed and compared.

Again, the reader is reminded of [12] where the author, A.D. Kuo looks into trade-offs between energetic walking economy and versatility in bipedal robots. Some definitions are in order: *versatility* refers to a bipedal robot's ability to execute other tasks than walking, but also the *versatility* in terms of walking in itself, i.e. the ability to make turns or traverse rough terrain. Again looking to Hondas Asimo and so-called passive dynamic walkers presented earlier these trade-offs became very clear. The passive dynamical walker is not capable of anything other than passively walk down a flat slope under the influence of gravity. From the perspective of locomotion, it is clear that wheels would be a better alternative. However as already stated there is considerable motivation for understanding and developing walking bipedal robots. On the other extreme Asimo is highly versatile both in it walking behaviour and ability to accomplish other tasks. However these machines represent two very different approaches to gait-synthesis. Being two extremes these two machines are not interesting to directly compare, but they will be informative for presenting the two different paradigms for gait-synthesis.

Sustained Local Stability and the ZMP-paradigm The so-called ZMP-paradigm has historically been successful and many walking robots have employed this paradigm or some variation of the main idea. A detailed explanation and discussion of ZMP will not be included here. A more thorough treatment of ZMP is readily available in [12], [11], [21] and references therein. However a discussion of the implications of the ZMP-paradigm is worthwhile. A key concept in the ZMP-paradigm is *sustained local stability.* The employment of ZMP is taken to be an argument for stability in the walking behaviour of robots. The *sustained local stability* of walking in robots employing ZMP means that the walking robot must be locally stable everywhere along the nominal trajectories of the gait. Any nominal gait that does not violate the ZMP-criterion will then be locally stable everywhere along it's nominal trajectories. The continuum of locally stable points then create sustained local stability through the gait. However, this put severe constraints on the robots motions, as they must satisfy the ZMP-criterion.

Dynamic Walking and Limit Cycles Dynamic walking does not impose any explicit criteria on the walking motions such as the ZMP-criterion. However in order to create dynamical walking gaits one assumes that there are passive dynamics present in the system. In other words the walking robot is underactuated as will be the case for bipedal robots with point-feet. This underactuation enables dynamical walking, but as discussed earlier underactuation pose challenges for motion-planning and control. These challenges calls for a radically different conceptual approach to the problems of motion planning and control than that of ZMP. Observing that gaits are periodic in their nature, one can restate the problem of gait-synthesis as: creating stable limit cycles for the bipedal robot. Limit cycles are periodic solutions of nonlinear differential equations. The hybrid dynamics of bipedal robots are nonlinear differential equations. This allows for a new approach to plan walking gaits. By defining gaits to by periodic solutions of the biped's dynamics, the search of gaits can be reformulated as the task of creating so-called *gait-cycles*.

Summing this up the main conceptual difference between ZMP-walking and dynamical walking become clear. A stabilized gait-cycle is stable as a whole. It is not locally stable, i.e. it does not make any sense to ask whether any point along the limit cycle is stable or unstable. This means that there is no stable points along the nominal trajectory constituting a nominal gait. As opposed to ZMP-walking where the continuum of locally stable points will create gaits that have sustained local stability.

Although there is a considerable conceptual divide between the two paradigms it is worthwhile to compare the two as they have both been employed to find solutions to the same problem: plan and control motions that will create walking behaviour in bipedal robots. Through applying the ZMP-paradigm the planning of motions indirectly also solves the problem of control as the ZMP-criterion will ensure sustained local stability. This makes this approach appealing and the employment of ZMP has created bipedal robots capable of stable walking behaviour. Robots employing ZMP as a strategy for walking have been capable of performing many other tasks as well, showing high versatility. However the ZMP-paradigm puts considerable constraints on the robots motions, and requires active torque control in knees and ankles to ensure that the ZMP-criterion is not violated.The dynamical walking paradigm does not set any such constraints, and have been shown to create walking behaviour with little active control. However the method requires the planning and stabilization of nonlinear hybrid limit cycles for the robot, a nontrivial and difficult task.

1.2.2 The Virtual Holonomic Constraints Approach and Limit Cycle Walking

The methodology of applying so-called virtual holonomic constraints has gained attention as a method for planning and stabilizing motions in systems where underactuation is present. The method has been applied to many different machines exhibiting underactuation of degree one. Examples include snake robots [16], pendubot [7], and most notably the 3-link compass biped with torso [8]. Many of the method's main concepts and implications are rather technical, and will be presented and treated in more detail in later chapters. However, some of the main advantages gained through applying this method for gait-synthesis in walking machines will be presented here. Although the method can be applied to systems with more than one degree of underactuation [10], this is not discussed here.

The main idea behind the methodology is to apply virtual holonomic constraints (VHCs) to an underactuated dynamical system. Virtual holonomic constraints are non-physical constraints that are satisfied through feedback control. By formulating the constraints as relations between the systems coordinates and a chosen *motion ganerator*. As opposed to physical constraints, these virtual constraints can be designed and redesigned to create desired motions for the system in which they are applied.

One very attractive property of the method is the reduction of dimensionality of motion-planning and control problems. For system's with one underactuated degree of freedom, the method will transform the dimension of the system dynamics from n DOF to 1. Being able to work with an one-dimensional system significantly decreases the computational complexity when solving motion planning and control problems for any system. By imposing chosen VHCs on the system, the state of the system can be uniquely defined through the motion generator.

The dynamical walking paradigm relies on the ability to plan and stabilize limit cycles for nonlinear hybrid dynamical systems. This is indeed a difficult task. In this context the VHC methodology has been suggested as a very attractive approach. Again, the reasons for this lies on more detailed technical than what will be discussed here. However the method indirectly simplify many of the difficulties inherent in the synthesis of dynamical walking gaits. By reducing the dimension of the problem to 1, the problem of *limb coordination* is indirectly dealt with. The movements of the torso and its DOFs does not define the act of walking as such. Any semi-erect torso is valid in terms of walking. By redefining the dynamics of the system through the imposing of VHCs, the nominal trajectory of the motion generator will define the motions of all links during a walking gait, and thereby also deal with limb coordination of legs and torso. The requirements of walking may then be formulated as constraints on these trajectories. Any feasible trajectories for the motion generating variable that satisfies these constraints will generate motions for both legs and torso in order to shape the motions of the walking gait-cycle.

Further the method also entails strategies for stabilizing and controlling the planned gait-cycles. In [18], Shiriaev and collaborators propose a general approach for orbital stabilization of underactuated nonlinear systems subject to virtual holonomic constraints. One of the most important implications of the method is the "timelessnes" of the nominal gait-cycles. The desired trajectories of a nominal gait-cycle is expressed in the motion generator variable as opposed to time. This means that a stabilizing controller must only ensure that the motions of the walking robot converge to the gait-cycle. This means that the controller does not need to catch up with desired trajectories in time.

1.3 Why 3-Link Biped With Torso ?

Having now provided some introductory background material, the focus can be shifted to the 3-link biped that is studied in this thesis. It is a planar biped equal to that found in [9] and [8]. The biped has stiff legs without knees and have point-feet. This make the robot underactuated in terms of it's stance leg. Actuators provide torque between the torso and the legs. The robot is therefore a machine that may exhibit dynamical walking gaits such as those found in passive dynamic walkers . However the 3-link biped is different from the dynamical walker in two important ways

- *active actuation:* as opposed the completely passive nature of the passive dynamical walkers, the 3-link biped studied in this thesis is capable of powering it's dynamic gaits through it's actuators. This enables the biped to walk on flat ground.
- *torso:* the torso of the biped introduces the problem of limb coordination. The motions of a gait cycle must also involve the coordination between torso and legs.

The 3-link biped is therefore interesting benchmark example for creating dynamical walking motions in a actively powered walking machine with torso. The biped is therefore a valuable example as the inherent difficulties of underactuation and limb coordination are held intact while the mechanical design is kept to minimum in terms of engineering complexity. The main concepts and insights gained through the study of this biped could therefore be valuable and applicable in the search for creating dynamical walking in more complex walking robots.

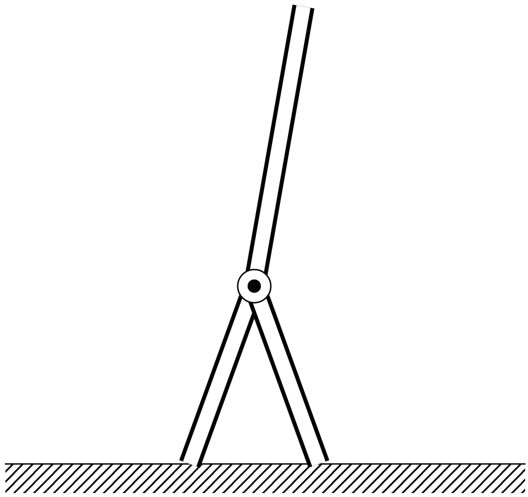


Figure 1.1: The 3-Link Biped Robot With Torso

1.4 Scope of this Text

Having presented and discussed some basic terminology and concepts, as well as provided some historical context the work documented in this thesis can be described. This thesis-report documents the application of the virtual holonomic constraints approach in order to plan dynamical gaits for the 3-link biped with torso visualized in 1.1. By describing a structured approach to motion-planning it is shown how one can formulate the task of gait-searching into optimization problems that can be solved in software.

A mechanical prototype of the 3-link biped was under development and not ready for experiments during the time of the project, and therefore no experiments have been made to verify the results documented. Further, there is no emphasis on electrical or mechanical engineering, instrumentation or system-identification aspects of prototype developments.

The method of applying virtual holonomic constraints for motion planning is not sufficient to create walking behaviour in an actual 3-link prototype. The nominal gaits and their trajectories are unstable and is not realizable through an open-loop control system. Closing the loop and applying a stabilizing controller is key to creating stable, and robust walking gaits in an actual machine. However the task of designing a stabilizing controller is not treated in this thesis. The emphasis lie entirely on planning realizable gaits for the 3-link biped.

1.5 Structure of Thesis Report

The main part of this thesis report is divided into chapters aiming to gradually document and explain the application of the vhc methodology employed to solve the task of gait planning for the biped. Chapter 2 contains theoretical foundations, and introduces concepts that will be central in the modelling of the biped and in the solving of the gait planning task. Chapter 3 develops the hybrid dynamical model used for the biped. Chapter 4 is concerned with the application of vhcs to the model developed in chapter 3, and the biped's subsequent reduced dynamics. In Chapter 5 and 6 it is shown how structure and solve tasks of gait planning and searching. Chapter 7 presents some numerical results found through employment of the gait planning approach presented in Chapter 5 and 6. In chapter 8 a general discussion is made on the work done in connection with this thesis, before a conclusion is presented in Chapter 9.

The code that has been developed is found in the attachment delivered together with this text.

2 | Theoretical Foundations

The 3-link biped with torso system exhibits both continuous and discrete dynamics. Planning walking gaits for this system using virtual holonomic constraints requires models of both the system's discrete and continuous dynamics and the formulation of the subsequent hybrid dynamical model. Walking gaits for the biped planned via VHCs will be periodic cycles in the systems state-space. Such periodic cycles in nonlinear systems is therefore also presented. The application of virtual holonomic constraints and the subsequent reduced dynamics of the biped are central to the work presented in this report, and is therefore described.

2.1 Euler-Lagrange Equations

The 3-DOF planar biped is a mechanical system. The continuous dynamics of mechanical systems can be formulated as a mathematical model. Such a model will form a necessary basis for motion planning and analysis. The procedure and the resulting equations presented here are known as Euler-Lagrange equations of the second kind. The Euler-Lagrange equations of the second kind describes a general procedure for deriving the dynamics of mechanical systems subject to so-called *holonomic constraints*, where the system can be described through a set of *generalized coordinates*.

The treatment found here is far from rigorous, for a more detailed treatment of Euler-Lagrange equations, the reader is referred to [19]

Degrees of Freedom, Holonomic Constraints and Generalized Coordinates A mechanical system consisting of rigid bodies will in general have 6 *degrees of freedom* (DOFs) per rigid body. Denoting the number of rigid bodies n, the number of coordinates needed to describe the configuration or kinematic state of the system is then $n \times 6$. However there may be constraints present that constrain the movement of the bodies. If all of these constraints are holonomic, the system is also deemed to be holonomic. Holonomic constraints are constraints that relate the positions of particles of rigid bodies within a system. A typical example is a system of two bodies connected rigidly by a massless rod of length L. Denoting position vectors of a particle in each body as r_a and r_b, these positions are related through the holonomic constraint

$$(r_a - r_b)^T (r_a - r_b) = L^2 \tag{2.1}$$

For a system of n rigid bodies, the presence of holonomic constraints could make the $n \times 6$ coordinates excessive. It may be possible to express the configuration of the system at any given time through an alternative set of coordinates known as *generalized coordinates.* Having chosen a set of generalized coordinates, a system's dynamics can be written in generalized coordinates as Euler-Lagrange equations of the second kind.

There is freedom in the choice of generalized coordinates for a holonomic system, however there are two requirements that must be met by the chosen coordinates in order to write the dynamics of the system as Euler-Lagrange equations. The generalized coordinates must satisfy the following two requirements:

1. *independent*: the set of generalized coordinates must be independent of each other, i.e. constraining the range of motion of one or more coordinate does not constrain the remaining coordinates in any way. All other coordinates will still have full range of motion.

2. *complete*: the set of chosen generalized coordinates must be able to completely describe the system at any given time.

If one can find such a set of generalized coordinates for a holonomic system, the Euler-Lagrange approach is applicable. The dynamics of the system can then be derived and written in generalized coordinates. In this text, the vector of n generalized coordinates and their time derivatives are denoted q and $\dot{q}$ respectively. The generalized force corresponding to the generalized coordinate indexed q_k is denoted τ_k. The Euler-Lagrange approach then allows for the dynamics of the system to be derived through it's kinetic and potential energy expressed in the n generalized coordinates q

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k \quad k \in [1, 2..., n] \tag{2.2}$$

where $\mathcal{L}$ denotes the Lagrangian

$$\mathcal{L} = \mathcal{K} - \mathcal{P} \tag{2.3}$$

Here $\mathcal{K}$ is the kinetic energy and $\mathcal{P}$ is the potential energy expressed in the chosen generalized coordinates. The Euler-Lagrange approach makes it remarkably straightforward to derive dynamic equations, without having to deal with the internal forces of the system. However these internal forces may be of interest, and if this is the case, the E-L equations of the first kind will deal with these forces explicitly. However in this text only the second kind will be used, as the internal constraint forces will not be analysed in any way.

Having derived the system's k differential equations from (2.2), these equations will define the motions $[q(t); \dot{q}(t)]$ of the system. For a mechanical systems, these equations can always re-written on the standard matrix form known from robotics

$$M(q) + C(q, \dot{q})\dot{q} + G(q) = Bu \tag{2.4}$$

This then completes the description of Euler-Lagrange equations. In Chapter 3, the method will be applied to the 3-link robot in order to model it's continuous single support phase

2.2 Hybrid System Definition

Hybrid systems are systems with switches, i.e. discrete update laws that makes the state of the system $[q(t); \dot{q}(t)]$, change instantaneously. The standard continuous robot equation (2.4) is incapable of capturing such dynamics. It must be augmented by an update law and conditions for updating or switching the state of the system in order to model a hybrid dynamic system. To this end some formalism, terminology and notation will be introduced. The notation will be similar to that found in [17].

The continuous dynamics expressed through (2.4) are only valid when there is no restriction on the movement of the mechanical system. Geometric surfaces created by other objects, will restrict the movement of the mechanical system. Collisions with the surfaces defined by such objects will change the state of the system. The continuous trajectories of (2.4) will be broken up by the discrete updates. Consequently a hybrid system will have trajectories $[q(t); \dot{q}(t)]$, consisting of continuous-in-time sub-arcs interconnected through discrete jumps in state-space.

The state-space of a hybrid system will have a collection of pairs of hypersurfaces denoted $\Gamma_-^{(i)}, \Gamma_+^{(i)}$. Associated with these hypersurfaces there will be mappings denoted F^i. The hypersurfaces and mappings will the make up triplets which are denoted:

$$\left\{\Gamma_-^i, \Gamma_+^i, F^{(i)}(\cdot)\right\}_{i=1}^{N_d}, \quad F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)} \tag{2.5}$$

The superscripts $-$ and $+$ denotes hypersurfaces immediately (in time) before and after a discrete switch. N_d denote the number of possible jumps in state-space. The corresponding mappings $F^{(i)}(\cdot)$ will update the state $[q(t); \dot{q}(t)]$, of the system

$$[q^-, \dot{q}^-] \equiv [q(T_h^-), \dot{q}(T_h^-)] \in \Gamma_- \to [q^+, \dot{q}^+] = [q(T_h^+); \dot{q}(T_h^+)] \in \Gamma_+$$

$$F([q(T_h^-), \dot{q}(T_h^-)]) \equiv [q(T_h^+), \dot{q}(T_h^+)]$$

at the time of switching T_h.

Having introduced switching surfaces $\Gamma_-^{(i)}, \Gamma_+^{(i)}$ and the update mapping $F^{(i)}(\cdot)$, the continuous (2.4) can be combined with the discrete dynamics defined by (2.5) to formulate the resulting hybrid system dynamics:

$$M(q) + C(q,\dot{q})\dot{q} + G(q) = Bu, \quad q^- \notin \Gamma_- \tag{2.6}$$

$$F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)}, \quad q^- \in \Gamma_- \tag{2.7}$$

Such hybrid systems will in later chapters be derived and formulated both for the original dynamics of the biped in q-space both also for the reduced dynamics of the system in θ-space[1]

2.3 Periodic Cycles for Hybrid Systems

Closed trajectories in the phase-plane of a nonlinear system are called *periodic orbits* or *limit cycles*. Planning and stabilizing such cycles is one way of creating oscillatory behaviour in a nonlinear system. Although nonlinear theory is not the focus of this thesis-report, some important concepts and definitions are needed in order to formulate and understand some key concepts and problems arising in the planning of dynamical walking gaits for the 3-link biped.

Limit Cycles For a general nonlinear system, $\dot{x} = f(x)$, a periodic solution is defined as $[x(t); \dot{x}(t)] = [x(t+T); \dot{x}(t+T)]$. Such a solution will then have period T. However this definition does not exclude equilibrium points as solutions, but these represent trivial and uninteresting periodic solutions. Nontrivial periodic solutions where T>0 are however more interesting and can be used in order to create nonlinear oscillators. In the system's phase-portrait such solutions are easily identified as *closed trajectories*. Such closed trajectories are referred to as *limit cycles*. Limit cycles can by both *stable* or *unstable*. A stable limit cycle is one where all other trajectories starting in some vicinity of of the limit cycle will converge to the limit cycle. For an unstable limit cycle other trajectories will not converge to the limit cycle. However if a solution of the system starts at the limit cycle it will stay on the limit cycle. Stable limit cycles will be robust in terms of disturbances while unstable ones will vanish in the presence of any disturbances.

Limit Cycles for a Hybrid System A considerable amount of literature is concerned with the analysis of continuous nonlinear systems and their solutions. However as Grizzle et al. points out in [9], nonlinear hybrid systems have not investigated in the same way. However formulating a periodic solution for the hybrid dynamic system defined in (2.6)-(2.7) is straightforward. Only periodic cycles with one jump are considered here. Any such nontrivial closed trajectories of a hybrid system will consist of repeated continuous sub-arcs and instantaneous

[1]the terms *reduced dynamics* and *θ-space* will be defined later in this chapter

jumps in its state-space. Designing and stabilizing such cycles through feedback will induce periodic stable behaviour in the system.

Looking back at (2.6)-(2.7), a closed trajectory or limit cycle can be formulated for the trajectories of a hybrid system. Recurring periodic behaviour is the repetition of limit cycles. For any given cycle, the starting time of the continuous sub-arc is denoted T_0. In the end of the continuous sub-arc the switch is engaged, this moment in time is denoted T_h. The period of a given periodic orbit is denoted $T_\star$. A limit cycle for the hybrid system can be defined as any periodic trajectory $[q_\star(t); \dot{q}_\star(t)]$, that satisfies:

$$[q_\star(t), \dot{q}_\star(t)] = [q_\star(t + T_\star), \dot{q}_\star(t + T_\star)] \ \ \forall t, \ , \ T_\star < 0 \tag{2.8}$$

$$\begin{aligned} &[q(T_0+); \dot{q}(T_0+)] \in \Gamma_+, \ [q(T_h-); \dot{q}(T_h-)] \in \Gamma_-, \\ &F([q(T_h-); \dot{q}(T_h-)]) = [q(T_0+); \dot{q}(T_0+)] \end{aligned} \tag{2.9}$$

The consecutive repetition of such hybrid sub-cycles indexed i can be formulated using the switching hypersurface Γ_+ and written on the form

$$[q_i; \dot{q}_i] \in \Gamma^+ = [q_{i+1}; \dot{q}_{i+1}] \in \Gamma_+ \tag{2.10}$$

This will be a repetitive closed cycle in the phase-portraits of the system's coordinates. These closed cycles will then have continuous-in-time sub-arcs that will be combined by a discrete jumps at $[q_\star(iT_h); \dot{q}_\star(iT_h)]$ to form closed orbits.

The concept of a recurring periodic orbit, and conditions for the existence of these closed trajectories in state-space will be central in the formulation and searching of gait-cycles in chapters 5 and 6.

2.4 Virtual Holonomic Constraints and Reduced Dynamics

The main idea of the VHC approach lies in rewriting the dynamics of an underactuated system expressed in generalized coordinates q into a new *virtually constrained system* written in a new variable θ. This new variable, θ is referred to as the *motion generating variable* or simply *motion generator*. The dynamics of the virtually constrained system will be referred to as *reduced dynamics*. The virtual holonomic constraints will effectively reduce the dimension of the system from the original n generalized coordinates to the single θ-variable. Thereby reducing the dimensionality of the system. The original system written in q and the reduced dynamical system written in θ will have trajectories written in these variables respectively. This motivates the introduction of the terms q-space and θ-space referring to the state-spaces of the two systems.

In this section the methodology is briefly presented. Some important implications of the method for planning motions will be discussed, however the treatment found here is far from comprehensive. For a more rigorous introduction the reader is referred to [18],[13],[5] and references therein.

Consider an underactuated Euler-Lagrange system with n generalized coordinates $q = [q_1, q_2, ..., q_n]$ and $(n-1)$ actuators. Such a system will have underactuation of degree one, i.e. the system has one more degree of freedom than control actuators. The reduced dynamics of such an system can be derived, by imposing virtual holonomic constraints on the system. These virtual constraints are assumed to be held invariant by feedback control.

The virtual holonomic constraints is a set of mathematical relations between the original generalized coordinates and the motion generator variable θ. The θ-variable can be chosen to represent one of the n generalized coordinates by choosing $\theta = q_i$, or the θ-variable can be completely *abstract*, i.e. it will not have any physical meaning as such.

$$q_1 = \phi_1(\theta),\ q_2 = \phi_2(\theta),\ .....,\ q_n = \phi_n(\theta) \tag{2.11}$$

By substituting these relations into the original system dynamics, one will obtain the reduced dynamics written in θ.

Some convenient vector notation is introduced

$$q(t) = \Phi(\theta) = \begin{bmatrix} \phi_1(\theta) \\ \phi_2(\theta) \\ \vdots \\ \phi_n(\theta) \end{bmatrix},\ \ \dot{q} = \Phi'(\theta)\dot{\theta},\ \ \ddot{q} = \Phi''(\theta)\dot{\theta}^2 + \Phi'(\theta)\ddot{\theta} \tag{2.12}$$

by substituting these relations into (2.4) the virtually constrained system written in θ-space can formulated

$$M(q) \to M(\Phi(\theta)) \tag{2.13}$$
$$C(q, \dot{q}) \to C(\Phi(\theta), \Phi'(\theta)\dot{\theta}) \tag{2.14}$$
$$G(q) \to G(\Phi(\theta)) \tag{2.15}$$
$$B(q) \to B(\Phi(\theta)) \tag{2.16}$$

$$M(\Phi(\theta))[\Phi''(\theta)\dot{\theta}^2 + \Phi'(\theta)\ddot{\theta}] + C(\Phi(\theta), \Phi'(\theta)\dot{\theta})[\Phi'\dot{\theta}] + G(\Phi\theta) = B(\Phi((\theta)) \tag{2.17}$$

Both the q-space dynamics and reduced dynamics (2.17) can be written out into n differential equations. Having imposed the relations (2.11) on the system, the n differential equations in θ-space can be always written on the form:

$$\begin{aligned}
&\alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) = b_1 u \\
&\alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) = b_2 u \\
&\vdots \\
&\alpha_n(\theta)\ddot{\theta} + \beta_n(\theta)\dot{\theta}^n + \gamma_n(\theta) = b_n u
\end{aligned} \tag{2.18}$$

For a E-L system with underactuation of degree one, it is always possible to combine these equations into a new equation where the resulting input u is annihilated. The resulting equation then express a system that, in the absence of input u, is *free*. This equations is given the subscript 0

$$\alpha_0(\theta)\ddot{\theta} + \beta_0(\theta)\dot{\theta}^2 + \gamma_0(\theta) = 0 \tag{2.19}$$

This free differential equation is of central importance in the VHC methodology, and will in this text be referred to as the virtually constrained system's *zero-dynamics* or *motion-generating equation*. Again, assuming that the VHCs (2.11) are kept invariant, the solutions $[\theta(t); \dot{\theta}(t)]$ of (2.19) will define the motions of the system and the corresponding nominal control input u. These solutions will in this respect *generate* motions for all n generalized coordinates, and inversely also the $(n-1)$ control inputs. These generated trajectories will be referred to as *nominal trajectories*. Further this system also has the useful and important property that there exists a preserved quantity for it's solutions $[\theta(t); \dot{\theta}(t)]$.

Conserved quantity For a virtually constrained Euler-Lagrange system's zero-dynamics (2.19) it has been shown and proved in ([18], Theorem 1), that there exists a general integral of motion. This quantity, denoted $I(\theta(0), \dot{\theta}(0), \theta(t), \dot{\theta}(t))$, preserves it's zero-value along any well-defined solution $[\theta(t); \dot{\theta}(t)]$ of (2.19). The expression can be written on two alternate forms.

In [17] the expression is written on the form:

$$I = \dot{\theta}^2(t) - e^{\left\{-\int_{\theta(0)}^{\theta(t)} \frac{2\beta(\tau)}{\alpha(\tau)} d\tau\right\}} \dot{\theta}^2(0) + \int_{\theta(0)}^{\theta(t)} e^{\left\{\int_{\theta(t)}^{s} \frac{2\beta(\tau)}{\alpha(\tau)}\right\}} \frac{2\gamma(s)}{\alpha(s)} ds = 0 \tag{2.20}$$

In [18] it is written on the form:

$$I = \dot{\theta}^2(t) - \psi(\theta(0), \theta(t)) \left[\dot{\theta}(0)^2 - \int_{\theta(0)}^{\theta(t)} \psi(s, \theta(0)) \frac{2\gamma(s)}{\alpha(s)} ds \right] = 0 \tag{2.21}$$

$$\psi(\theta_0, \theta_1) = e^{\left\{ -\int_{\theta_0}^{\theta_1} -\frac{2\beta(\tau)}{\alpha(\tau)} d\tau \right\}} \tag{2.22}$$

This property of the zero-dynamics of a virtually constrained system will prove to be very useful when searching for walking gaits for the 3-link biped. It allows for more efficient computations of $\dot{\theta}(t)$, which would else be found through a full integration of the zero-dynamics.

2.4.1 Motion Planning for Underactuated Euler-Lagrange Systems

It is clear that the solutions of zero-dynamics that generate nominal trajectories, are shaped through the choice of VHCs. This implies that desired motions for the full state of the system $[q(t); \dot{q}(t)]$ can be planned through the choice of VHCs (2.11) and analysis of the systems zero-dynamics (2.19). This is the main idea behind using virtual holonomic constraints as a means of planning and analysing motions for underactuated systems. The reduction of dimensionality of the motion planning significantly reduces computational complexity, but it also has some additional and important implications.

Time-less representation of nominal trajectories The motions planned using virtual holonomic constraints does not explicitly depend on time. Rather the nominal trajectories of the system become functions of θ through (2.11). This implies that the representation of nominal trajectories is effectively *time-less*. The nominal trajectories can be represented using a monotonically increasing θ, instead of time.

Link coordination Consider a general multi-link robotic system. Only the movements of some links may be essential for some desired behaviour. This means that there only exist explicit requirements for some of the system's trajectories $[q_i, ...q_n; \dot{q}_i, ..., \dot{q}_n]$. Having an endless set of feasible motions for the rest of the links, may complicate the task of formulating requirements for their nominal trajectories. The VHC-approach avoids this problem as the combination of (2.19), (2.11), will generate nominal trajectories for all links. If necessary and sufficient conditions for the desired motions can be written in θ-space, then any feasible motions meeting these conditions through solutions of (2.19) will also coordinate the motions of all links in the robotic system in order to achieve desired motions.

The application of virtual holonomic constraints is therefore a powerful tool when planning motions for underactuated systems. However the task of planning desired motions, is still challenging. Depending on the system and the motions one wish to plan, the virtual holonomic constraint functions cannot be any. Choosing these mathematical relations may be a highly non-trivial task.

Typically the derivation of VHCs that will create desired motions is done through trial-and-error or through the structuring of numerical searches in order to arrive at VHCs that will generate motions exhibiting some desired characteristics. It is therefore useful to formulate different classes of virtual holonomic constraints and subsequently alter these VHCs through parameterization. Such classes will then refer to the mathematical structure of the VHCs. The search of VHCs that will create the desired nominal trajectories can be done first through choosing the general structure of the VHCs ϕ_i, and then adjusting their parametrizations P_i to create desired nominal trajectories. The VHCs imposed on the system will then be denoted

$$q_1 = \phi_1(\theta, P_1),\ q_2 = \phi_2(\theta, P_2),\ .....,\ q_n = \phi_n(\theta, P_n) \tag{2.23}$$

where the parameter vectors P_i will numerically shape these functions, and subsequently the nominal trajectories of the system's coordinates.

The method of virtual holonomic constraints have found wide use, and its application to continuous underactuated E-L systems is well represented in the literature [13],[5], [18], [15]. However, planning of gaits for the 3-link biped requires the VHC-methodology to be applied to a hybrid dynamical system.

2.4.2 Motion Planning for Hybrid Systems

This section will focus on the planning of periodic motions in hybrid systems using virtual holonomic constraints. As discussed earlier in this chapter, periodic motions in hybrid systems are referred to as periodic orbits, limit cycles, or simply periodic cycles. By planning and stabilizing such periodic cycles, one can create periodic behaviour in a nonlinear system. Consider a hybrid system as the one formulated in (2.6)-(2.7). Having applied VHCs and derived reduced dynamics for the continuous dynamics of such a system, the update laws and mappings that define the system's discrete dynamics must also be taken into account when planning periodic cycles. As discussed earlier, the periodic cycles consist of continuous sub-arcs interconnected by jumps in the system's state-space. To create periodic cycles the continuous sub-arcs can be shaped such that the state of the system is returned back to the starting conditions for the continuous sub-arc after the discrete switch. The formulation of necessary conditions that guarantee the existence of such cycles is therefore required.

In [17], the VHC-methodology is applied to a hybrid system (2.6)-(2.7) as defined earlier in this chapter. Theorem 1 herein states necessary conditions for the existence of hybrid cycles and is re-stated here

Theorem 1 *Given the controlled impulsive mechanical system, where the continuous time dynamics (2.6) is with n degrees of freedom and is of underactuation one, and where the hypersurfaces* Γ_- *and* Γ_+ *and the mapping* $F : \Gamma_- \to \Gamma_+$ *define the discrete dynamics. Suppose that for some control signal* $u = u(t)_\star = u(t + T_h)$, *the impulsive mechanical system has a periodic solution*

$$q = q_\star(t) = q_\star(t + T_h), \quad \forall t, \quad T_h > 0 \tag{2.24}$$

with only one jump, i.e. $[q_\star(T_h+); \dot{q}_\star(T_h+)] \in \Gamma_+$, $[q_\star(T_h-); \dot{q}_\star(T_h-)] \in \Gamma_-$, $F([q_\star(T_h-); \dot{q}_\star(T_h-)]) = [q_\star(T_h+); \dot{q}_\star(T_h+)$. *Suppose the continuous-in-time arc of (2.24) admits a re-parameterization* $q_\star = \phi(\theta_\star(t))$ *defined by (2.11) with* C^2*-functions* $\phi_1(\cdot), ..., \phi_n(\cdot)$. *Compute the dynamics of (2.6), when these relations are kept invariant, i.e. compute the coefficients of the second order system (2.19). Then, by necessity, the algebraic equations*

$$I(\theta_\star(0), \dot{q}_\star(0), \theta_\star(T_h), \dot{q}_\star(T_h)) \;=\; 0 \tag{2.25}$$

$$F\left(\left[\begin{array}{c} q_- \\ \dot{q}_- \end{array}\right]\right)\Bigg|_{\substack{q_- = \phi(\theta_\star(T_h)) \\ \dot{q}_- = \phi'(\theta_\star(T_h))\dot{\theta}_\star(T_h)}} = \left[\begin{array}{c} q_+ \\ \dot{q}_+ \end{array}\right]\Bigg|_{\substack{q_+ = \phi(\theta_\star(0)) \\ \dot{q}_+ = \phi'(\theta_\star(0))\dot{\theta}_\star(0)}} \tag{2.26}$$

hold. Here $I(\cdot)$ *is an integral of (2.19) and can be taken as (2.20).*

In addition to Theorem 1, the steps defined in ([17], pp 2883) is used as a template for planning gait-cycles for the 3-link biped in later chapters.

3 Modelling

In this chapter the hybrid q-space dynamics of the planar 3-link biped with torso are modelled. The resulting hybrid dynamical model will be used throughout the rest of this thesis-report. The chapter begins with describing the 3-link biped system, stating assumptions and simplifications as well as introducing some useful terminology. Kinematics and dynamics for the robot is derived using two different sets of generalized coordinates. The continuous and discrete parts of the model is in the last section combined to form the full hybrid dynamic model.

3.1 Biped model

The 3-link biped with torso is a planar nonlinear mechanical system subject to impacts with the environment. A prototype of such a robot has been developed and built, however this chapter does not include any discussions on the engineering aspects of this prototype as such.

The simplified model of a 3-link biped with torso is visualized in Figure 3.1. Shown here is the biped with one foot in the ground during single support phase. l_i denotes lengths to the links center of mass (CoM), L_i denotes the total effective length of the links.

The complete model of this robot includes both continuous Euler-Lagrange dynamics as well as impact-effects modelled as an discrete impact-map updating the state of the system.

Some definitions and useful terminology is stated

- *single support phase*: the phase when the robot has on foot in the ground. Also referred to the *swing phase*
- *double support phase*: the phase when the robot has both feet on the ground.
- *stance leg*: is the leg whose foot is on the ground during single support phase. It is numbered *link 1*, given blue colour.
- *swing leg*: the freely swinging leg in the single support phase. It is numbered *link 3*, and given red colour.

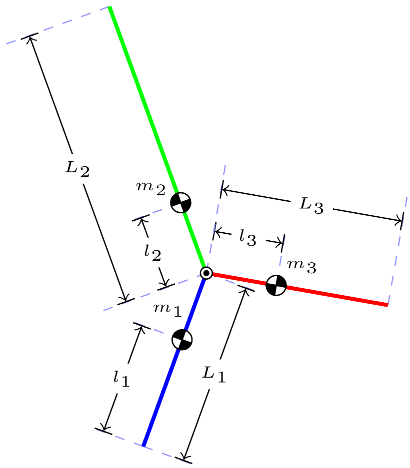


Figure 3.1: The 3-Link Biped with lengths L_i and l_i during single support phase

- *torso*: the torso is connected to the two legs in the hip joint. It is numbered *link 2*, and given green colour

- *impacts*: refers to the swing leg colliding with the ground and the subsequent switching from single support phase to double support phase

- *heel strike*: when the swing leg impacts the ground before the step is complete. This is solved through knees in many robots and humans. For this model it is assumed that the effective length of the swing leg is shortened in order to avoid this

The 3-link biped model that will be developed here is a mathematical approximation of the real-world dynamics found in the prototype system. There are a number of simplifications and assumptions that are made in order to formulate this model. Some key assumptions and simplifications are:

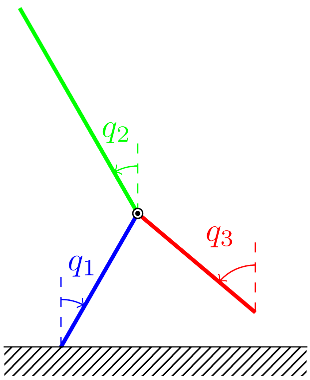
- *rigid links*: the system's three links are assumed to be rigid bodies with (non-zero) inertia about their CoM

- *stance leg is an ideal pivot*: the model assumes that the stance leg of the robot is rigidly attached to the ground, i.e. there is no slipping or sliding. The end of the legs are point-feet the stance legs foot acts like an ideal pivot during the single support phase.

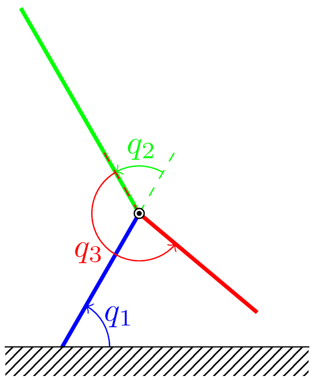
- *instantaneous impacts*: the impacts experienced by the system is assumed to be discrete in time, i.e. the impact event happens during an infinitesimally short period of time.

3.2 Kinematics

The kinematics of the robot have been developed using two different sets of generalized coordinates. In the figure below the two conventions are pictured. One of them defines the generalized coordinates of the system to be *absolute* angles between the vertical and the 3 links. The other convention is similar to the standard convention used for robotic manipulators and define the generalized coordinates to be *relative* angles between the links.



(a) absolute angle convention



(b) relative angle convention

Figure 3.2: Absolute and relative angle conventions

Holonomic Constraints and Degrees of Freedom The biped is subject to the holonomic constraints, and it is therefore possible to express the systems kinematics through three generalized coordinates as opposed to the 3×6 excessive coordinates needed to express the kinematics of a general system with 3 rigid bodies in 3-dimensional space. This is due to the following constraints

- *planar system*: the biped has no movement in the z^0 direction (out of the page)
- *stance leg*: the stance leg is rigidly attached to the ground
- *swing leg*: the swing leg is rigidly attached to the hip joint
- *torso*: the torso is rigidly attached to the hip joint

The constraints imposed through the planar conditions removes $3 \times 3 = 9$ degrees of freedom. Each of the constraints for the legs further remove the possibility of independent movement in the x and y - directions for each rigid body. Subtracting 18-9-6 $= 3$ show that the system can be expressed through 3 generalized coordinates. The only unrestricted and independent degree of freedom for all three links are their rotation about the inertial z^0-axis.

Again looking back at Figure 3.2, conventions for the angles of rotation about z^0-axis have been chosen. From there it is straightforward to derive the system's kinematic relations using both conventions. In both cases the position and velocities for the link's centers of mass, the hip and the robot's swing-foot (end of swing-leg) are derived. With i indexing the links, the position and velocity vectors of the system are denoted

1. position and velocity of links' CoM are denoted: p_i, v_i

2. position and velocity of the swing-foot is denoted: p_{if}, v_{if}

3.2.1 Absolute Angle Convention

$$p_1 = \begin{bmatrix} -l_1 sin(q_1) \\ l_1 cos(q_1) \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -l_1 cos(q_1)\dot{q}_1 \\ -l_1 sin(q_1)\dot{q}_1 \\ 0 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} -L_1 sin(q_1) - l_2 sin(q_2) \\ L_1 cos(q_1) + l_2 cos(q_2) \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -L_1 cos(q_1)\dot{q}_1 - l_2 cos(q_2)\dot{q}_2 \\ -L_1 sin(q_1)\dot{q}_1 - l_2 sin(q_2)\dot{q}_2 \\ 0 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} -L_1 sin(q_1) + l_3 sin(q_3) \\ L_1 cos(q_1) - l_3 cos(q_3) \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -L_1 cos(q_1)\dot{q}_1 + l_3 cos(q_3)\dot{q}_3 \\ -L_1 sin(q_1)\dot{q}_1 + l_3 sin(q_3)\dot{q}_3 \\ 0 \end{bmatrix}$$

$$p_{3f} = \begin{bmatrix} -L_1 sin(q_1) + L_3 sin(q_3) \\ L_1 cos(q_1) - L_3 cos(q_3) \\ 0 \end{bmatrix}$$

$$v_{3f} = \begin{bmatrix} -L_1 cos(q_1)\dot{q}_1 + L_3 cos(q_3)\dot{q}_3 \\ -L_1 sin(q_1)\dot{q}_1 + L_3 sin(q_3)\dot{q}_3 \\ 0 \end{bmatrix}$$

3.2.2 Relative Angle Convention

$$p_1 = \begin{bmatrix} l_1 cos(q_1) \\ l_1 sin(q_1) \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -l_1 sin(q_1)\dot{q}_1 \\ l_1 cos(q_1)\dot{q}_1 \\ 0 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} L_1 cos(q_1) + l_2 cos(q_1 + q_2) \\ L_1 sin(q_1) + l_2 sin(q_1 + q_2) \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 - l_2 sin(q_1 + q_2)(i\dot{q}_1 + \dot{q}_2) \\ L_1 cos(q_1)\dot{q}_1 + l_2 cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \\ 0 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} L_1 cos(q_1) + l_3 cos(q_1 + q_2 + q_3) \\ L_1 sin(q_1) + l_3 sin(q_1 + q_2 + q_3) \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 + l_3 sin(q_1 + q_2 + q_3)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ L_1 cos(q_1)\dot{q}_1 + l_3 cos(q_1 + q_2 + q_3)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ 0 \end{bmatrix}$$

$$p_{3f} = \begin{bmatrix} L_1 cos(q_1) + L_3 cos(q_1 + q_2 + q_3) \\ L_1 sin(q_1) + L_3 sin(q_1 + q_2 + q_3) \\ 0 \end{bmatrix}$$

$$v_{3f} = \begin{bmatrix} -L_1 sin(q_1)\dot{q}_1 + L_3 sin(q_1 + q_2 + q_3)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ \\ L_1 cos(q_1)\dot{q}_1 + L_3 cos(q_1 + q_2 + q_3)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ 0 \end{bmatrix}$$

3.3 Euler-Lagrange Equations

Having established the kinematics of the system, it it straightforward to express the kinetic and potential energy of the system in order to derive the Lagrangian and find the system's dynamics during the single support phase.

The Euler-Lagrange method for deriving dynamics are applicable for the biped as the system only has holonomic mechanical constraints and it has been shown that the system can described through the choice of three generalized coordinates. To derive the Euler-Lagrange equations for the system, the first step is to derive the kinetic and potential energy for the system.

Kinetic energy:

$$\mathcal{K} = \frac{1}{2}[m_1 v_1^T v_1 + m_2 v_2^T v_2 + m_3 v_3^T v_3] + \frac{1}{2}[J_1 \dot{q}_1^2 + J_2 \dot{q}_2^2 + J_3 \dot{q}_3^2] \tag{3.1}$$

Potential energy:

$$\mathcal{P} = mgp_1[1] + mgp_2[1] + mgp_3[1] \tag{3.2}$$

Having formulated the kinematic and potential energy the left hand side of the Euler-Lagrange equations 2.2 can be derived. For convenience and correspondence to the software implementation a small reformulation of the left-hand side is made. Since this is a mechanical system the potential energy is not a function of $\dot{q}$. (2.2) can therefore be written as

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial \mathcal{K}}{\partial q_k} + \frac{\partial \mathcal{P}}{\partial q_k} = \tau_k \tag{3.3}$$

Having showed how the left hand side(LHS) of the dynamic equations are found through the Euler-Lagrange approach, the attention is for now shifted to the right hand side (RHS) of (3.3).

Generalized Torques On the right hand side of the E-L equations τ_k is assumed to be *generalized forces for the system* that directly correspond to the generalized coordinates. This implies that the generalized forces will directly act on the generalized coordinates. The 3-link biped has two actuators. u_2 produces torques between torso and stance leg, and u_3 acts between torso and swing leg. For the mechanical prototype of a 3-link biped that is modelled here the relative angle convention for generalized coordinates is such that the actuators of the biped act directly along the coordinates q_2 and q_3. This makes it straightforward to state the RHS of the E-L equations:

$$\tau_k = Bu = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{3.4}$$

However, in the absolute angle convention the actuators does not directly align with the generalized coordinates. This requires another structure for the B-matrix. It is possible to use slightly different, but equally correct B-matrices depending on the physical and electrical set-up of the prototype. The B-matrix that will be used in this thesis-report is:

$$\tau_k = Bu = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{3.5}$$

This then completes the formulation of the continuous dynamics for the biped. However, these dynamics does not include the discrete state updates that will occur when the biped's legs impacts the ground. To this end an impact map must be formulated and derived.

3.4 Impact Map

When the swing leg impacts the ground, it is assumed that the impact happens during an infinitesimally short period of time. Further, during the impact the geometrical configuration of the biped does not change, but the velocities of the links changes instantaneously. However the hybrid model derived here entails the re-labelling of the legs at impact. This is because the physical legs of the robot interchangeably act as stance leg and swing leg from step to step. This means that the entire state $[q;\dot{q}]$ is instantaneously changed when an impact occurs. This is then formulated mathematically as a discrete-in-time *impact map*.

In the following T_h denotes the time of impact, further the following shorthand notations are introduced

$$[q^-;\dot{q}^-] \equiv [q(T_h^-);\dot{q}(T_h^-)]$$
$$[q^+;\dot{q}^+] \equiv [q(T_h^+);\dot{q}(T_h^+)]$$

Assuming that the impact occurs during an infinitesimally short period of time, - and $+$ denotes the system's configuration and velocities immediately prior to and after impact respectively.

An impact map can then be formulated as

$$[q^+;\dot{q}^+] = F([q^-;\dot{q}^-]) \tag{3.6}$$

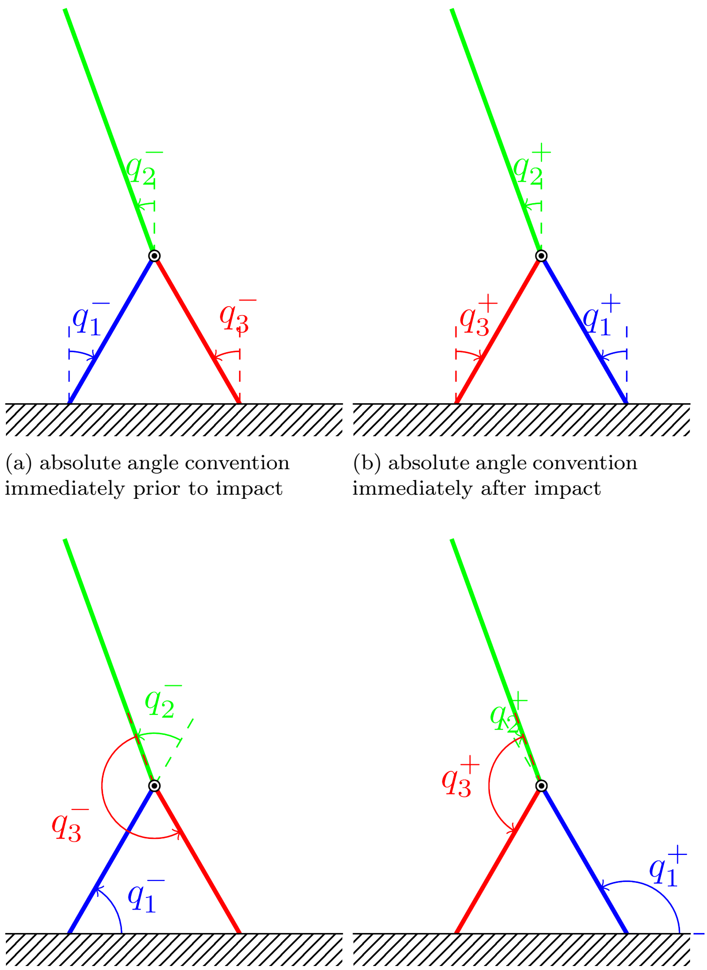
where

$$[q^+;\dot{q}^+] \in \Gamma_+,\ [q^-;\dot{q}^-] \in \Gamma_- \tag{3.7}$$

Updating q at impact For the biped the update of the coordinates only involve re-labelling of the legs. The update law for q can therefore be formulated directly from geometrical considerations. See Fig 3.3.

Absolute coordinates:

$$q_1^+ = -q_1^- \tag{3.8}$$
$$q_2^+ = q_2- \tag{3.9}$$
$$q_3^+ = -q_3^- \tag{3.10}$$



(a) absolute angle convention immediately prior to impact

(b) absolute angle convention immediately after impact

(c) relative angle convention immediately prior to impact

(d) relative angle convention immediately after impact

Figure 3.3: Relative and absolute angle conventions immediately prior to and after swing leg impacts the ground

Relative coordinates:

$$q_1^+ = -q_1^- + \pi \tag{3.11}$$
$$q_2^+ = 2q_1^- + q_2^- - \pi \tag{3.12}$$
$$q_3^+ = 2q_1^- + q_3^- - \pi \tag{3.13}$$

3.4.1 Conservation of Angular Momentum

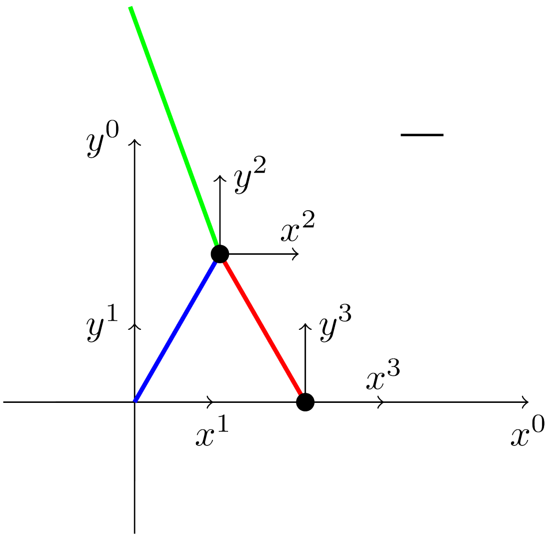
The update laws for the coordinates q (3.8)-(3.13), can be directly extracted from Fig 3.3. The derivatives of the coordinates $\dot{q}$ are also subject to instantaneous updates at impact. Although collisions in E-L systems are nontrivial and there exists a wealth of literature on the topic, in the biped model developed here the update of angular velocities $\dot{q}$ will rely on the assumption that the angular momentum of the biped is conserved at impact. The task is to derive an update rule for mapping $\dot{q}^- \rightarrow \dot{q}^+$. This means that there are three unknowns to solve for namely, $\dot{q}_1^+, \dot{q}_3^+, \dot{q}_3^+$. It is therefore clear that the mapping will consist of three equations. By observing the fact that the internal forces at swing leg/stance leg and hip cancel each other out, i.e. does no work, during the impact, it is clear that the angular momentum of the robots links about these point will be conserved. It is possible to derive three equations relating $\dot{q}^-$ and $\dot{q}^+$ in this way and then solve for $\dot{q}^+$.

The main idea is to write equations for conservation of angular momentum for all three links about the foot of swing leg/stance leg, and for two of the links about the hip during impact. These equations will roughly be on the form

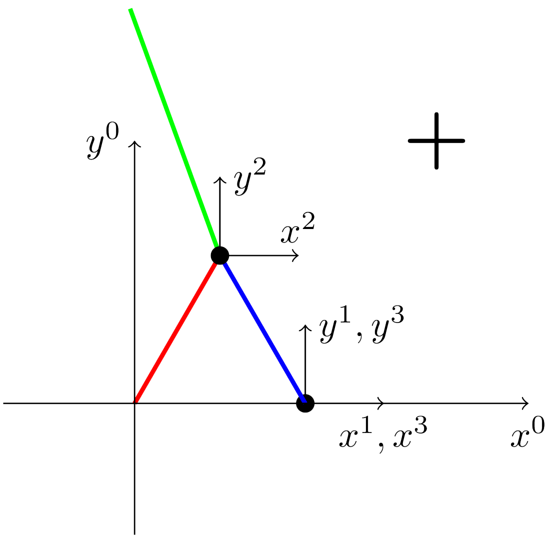
$$r(q^-) \times v(q^-, \dot{q}^+) + J\dot{q}^+ = r(q^-) \times v(q^-, \dot{q}^-) + J\dot{q}^-$$

Formulating these three equations is somewhat involved, and some necessary notation must be introduced. Fig 3.4 shows the coordinate frames that will be used in the map formulation.

- *frame 0* is the inertial frame
- *frame 1* is the frame corresponding to origo for the kinematic relations and E-L equations
- *frame 2* is attached to the hip of the robot
- *frame 3* is attached to swing leg prior to impact, and stance leg after impact and re-labelling
- $v_{b/c}^a$ denotes velocity vector of the point b with respect to c expressed in coordinate frame a
- $r_{b/c}^a$ denotes position vector of the point b with respect to c expressed in coordinate frame a



(a) Biped immediately prior to impact



(b) Biped immediately after impact

Figure 3.4: The coordinate frames used in derivation of impact map

Using the introduced vector notation and Fig 3.4, the three equations expressing conservation of angular momentum can be formulated. The left hand side (LHS) expresses angular momentum immediately prior to impact corresponding to Fig 3.4b. The right hand side (RHS) expresses angular momentum immediately after impact corresponding to Fig 3.4a. By setting LHS = RHS from the assumption of conserved angular momentum, the three equations that is need to find the update law for $\dot{q}^- \rightarrow \dot{q}^+$ stated

Conservation of angular momentum of torso and swing leg/stance leg about the hip

$$\begin{aligned} &p^1_{3/o2}(q^+) \times m_3 v^1_{3/o1}(q^+, \dot{q}^+) + J_3 \dot{q}_3^+ = \\ &p^1_{1/o3}(q^-) \times m_1 v^1_{1/o1}(q^-, \dot{q}^-) + J_3 \dot{q}_3^- \end{aligned} \tag{3.14}$$

$$\begin{aligned} &p^1_{2/o2}(q^+) \times m_2 v^1_{2/o1}(q^+, \dot{q}^+) + J_2 \dot{q}_2^+ = \\ &p^1_{2/o2}(q_-) \times m_2 v^1_{2/o1}(q_-, \dot{q}^-) + J_2 \dot{q}_2^- \end{aligned} \tag{3.15}$$

Conservation of all links about the foot of swing leg/stance leg

$$\begin{aligned} &p^1_{1/o1}(q^+) \times m_1 v^1_{1/o1}(q^+, \dot{q}^+) + \\ &p^1_{2/o1}(q^+) \times m_2 v^1_{2/o1}(q^+, \dot{q}^+) + \\ &p^1_{3/o1}(q^+) \times m_2 v^1_{2/o1}(q^+, \dot{q}^+) + \\ &J_1 \dot{q}_1^+ + J_2 \dot{q}_2^+ + J_3 \dot{q}_3^+ \\ &= \\ &p^1_{1/o3}(q^-) \times m_1 v^1_{1/o1}(q^-, \dot{q}^-) + \\ &p^1_{2/o3}(q^-) \times m_2 v^1_{2/o1}(q^-, \dot{q}^-) + \\ &p^1_{3/o3}(q^-) \times m_2 v^1_{2/o1}(q^-, \dot{q}^-) + \\ &J_1 \dot{q}_1^- + J_2 \dot{q}_2^- + J_3 \dot{q}_3^- \end{aligned} \tag{3.16}$$

By substituting in for q^+ as defined in (3.8)-(3.10) and (3.11)-(3.13) it is possible to re-write these three equations on vector form as

$$A(q^-)\dot{q}^+ = B(q^-)\dot{q}^- \tag{3.17}$$

The velocity vector immediately after impact, $\dot{q}^+$, can then be solved for

$$\dot{q}^+ = [A^- B]\dot{q}^- \tag{3.18}$$

By combining this with the updates for the q-coordinate vector, (3.8)-(3.10) and (3.11)-(3.13), the necessary equations that will constitute an impact map (3.6) have been derived for the biped using the notion of conservation of angular momentum.

3.5 Hybrid Model Formulation

Having derived both the continuous and discrete-in-time parts of the biped's dynamics, it is possible to formulate the system's hybrid dynamics in the format (2.6)-(2.7).

However, the dynamics is not complete before conditions that will initiate an update through the impact map F have been defined. For the 3-link biped this is indeed a simple and straightforward task. The biped's dynamics will switch when the swing leg impacts the ground. This will make the biped enter double support phase, i.e. both feet are on the ground, and this is thus also the condition that will initiate the switch $F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)}, \quad q^- \in \Gamma_-$. The switching condition, both feet on the ground, then defines a hypersurface dependent only on the biped's configuration defined by q. This hypersurface is denoted S(q), and defines all configurations where both legs of the biped is on the ground.

$$\forall\ [q^-; \dot{q}^-] \in \Gamma_-$$
$$q^- \in S(q)$$

Although this formulation does not exclude heel strikes or upside-down-configurations from initiating impacts these can be excluded in software in an ad-hoc manner by applying simple if/else statements in the implementation of of the impact-condition S. In the prototype this model is derived from, heel strike is avoided mechanically by effectively shortening the effective length of the leg by a retracting mechanism in the swing leg during single support phase. Finally the hybrid system for the 3-link biped can be written as:

$$M(q) + C(q,\dot{q})\dot{q} + G(q) = B(q)u, \quad q \notin S \tag{3.19}$$
$$F(q^-, \dot{q}^-) : \Gamma_- \to \Gamma_+, \quad q^- \in S \tag{3.20}$$

4 Reduced Dynamics

Having developed the hybrid model for the system in the generalized coordinates $q = [q_1, q_2, q_3]$, it is now possible derive the biped's reduced dynamics as discussed in section 2.4. The biped will be subject to three virtual holonomic constrains, that will be assumed to be held invariant by feedback control

$$
\begin{gathered}
q_1 = \phi_1(\theta), \;\; q_2 = \phi_2(\theta), \; q_3 = \phi_3(\theta) \\
q = \Phi(\theta) = \begin{bmatrix} \phi_1(\theta) \\ \phi_2(\theta) \\ \phi_3(\theta) \end{bmatrix}
\end{gathered} \tag{4.1}
$$

$$
\begin{gathered}
\dot{q}_1 = \phi_1'(\theta)\dot{\theta} \; \dot{q}_2 = \phi_2'(\theta)\dot{\theta} \; \dot{q}_3 = \phi_3'(\theta)\dot{\theta} \\
\dot{q} = \Phi'(\theta)\dot{\theta} = \begin{bmatrix} \phi_1'(\theta) \\ \phi_2'(\theta) \\ \phi_3'(\theta) \end{bmatrix} \dot{\theta}
\end{gathered} \tag{4.2}
$$

The relations (4.1), will rewrite the biped's *q-space* dynamics into a new reduced dynamic system. Subsequently the biped's hybrid dynamics (3.19)-(3.20) can be rewritten in the motion generator variable θ. The reduced hybrid dynamics written in θ will be used for planning and searching feasible gaits for the biped. With the background material from section 2.4, it is straightforward to substitute the relations (4.1) into the q-space equations (3.19)-(3.20) and derive the reduced hybrid dynamics in θ-space.

4.1 Continuous Reduced Dynamics

The continuous E-L equations written in q-space (2.4) is transformed into reduced dynamics written in theta on the α, β, γ-form (2.18). The 3-link biped's continuous dynamics consist of three second order differential equations that through substitutions defined in (4.1) are re-written in to three new second order differential equations in θ

$$\begin{aligned}\alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) &= \tau_1 \\ \alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) &= \tau_2 \\ \alpha_3(\theta)\ddot{\theta} + \beta_3(\theta)\dot{\theta}^2 + \gamma_3(\theta) &= \tau_3\end{aligned} \tag{4.3}$$

These three equations can be combined to one equation where the right hand side of the equations become zero. This is then the virtually constrained biped's zero-dynamics

$$\alpha_0(\theta)\ddot{\theta} + \beta_0(\theta)\dot{\theta}^2 + \gamma_0(\theta) = 0 \tag{4.4}$$

The solutions $[\theta(t), \dot{\theta}(t)]$ of (4.4) will define the motions of the biped's links during the continuous-in-time single-support phase. This equation is therefore of essential importance to the solving of motion-planning and gait searching tasks for the biped discussed in later chapters. The two different conventions for generalized coordinates will as already discussed change the structure of the right hand side of the continuous equations, and subsequently the equations (4.3), will be different for the two conventions

Relative Coordinates

$$\alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) = 0 \tag{4.5}$$

$$\alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) = u_2 \tag{4.6}$$

$$\alpha_3(\theta)\ddot{\theta} + \beta_3(\theta)\dot{\theta}^2 + \gamma_3(\theta) = u_3 \tag{4.7}$$

zero-dynamics will the simply be

$$[\alpha_1]\ddot{\theta} + [\beta_1]\dot{\theta}^2 + [\gamma_1] = 0 \tag{4.8}$$

Absolute Coordinates

$$\alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) = u_2 \tag{4.9}$$

$$\alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) = -u_2 - u_3 \tag{4.10}$$

$$\alpha_3(\theta)\ddot{\theta} + \beta_3(\theta)\dot{\theta}^2 + \gamma_3(\theta) = u_3 \tag{4.11}$$

zero-dynamics will be

$$[\alpha_1 + \alpha_2 + \alpha_3]\ddot{\theta} + [\beta_1 + \beta_2 + \beta_3]\dot{\theta}^2 + [\gamma_1 + \gamma_2 + \gamma_3] = 0 \tag{4.12}$$

Further, the biped's continuous θ-dynamics will have the conserved integral quantity $I(\theta(0), \dot{\theta}(0), \theta(t), \dot{\theta}(t))$ (2.20). This quantity will be very useful when planning motions and searching for gaits for the biped. Having derived the continuous reduced dynamics, these must be augmented by discrete reduced dynamics.

4.2 Discrete Reduced Dynamics

Similar to the development of the continuous reduced dynamics, the transformation of the biped's discrete dynamics will be done through the substitution of (4.1) into the equations defining the discrete-in-time impact dynamics of the biped (3.20). This will entail the re-writing of the impact map $F(q^-, \dot{q}^-)$ into it's θ-space equivalent $\mathcal{F}(\theta^-, \dot{\theta}^-)$.

Updating $\theta^- \to \theta^+$ The gaits that will be planned and searched in this thesis-report will be periodic, implying

$$\Phi(\theta(t)) = \Phi(\theta(t + T_h)) \tag{4.13}$$

T_h will then be time it takes for the biped to make one step. The time at the start of a step is denoted T_0, and the time in the end of step is denoted θ_T. The following shorthand notation is introduced:

$$\theta_0 \equiv \theta(T_0) \quad \theta_T \equiv \theta(T_h)$$

The update of θ in theta-space is trivial and is formulated

$$\theta_T \to \theta_0 \tag{4.14}$$

where θ_0 and θ_T will be defined by the choice of motion-generator.

Updating $\dot{\theta}^- \to \dot{\theta}^+$ Having formulated the update $\theta^- \to \theta^+$, it is possible to rewrite the equations for conservation of angular momentum (3.14)-(3.16), (3.17) in *theta*-space. From these equations two different, but equally valid approaches can be taken. However, they both rely on the substitution of relations (4.1),(4.2). The equations(3.14)-(3.16), (3.17) are linear with respect to the angular velocities $\dot{q}_i$. Having $\dot{q}_i = \phi_i'(\theta)\dot{\theta}$, it is clear that re-writing these equations into θ-space will create equations linear in theta. This can be done in two ways

Either one could re-write (3.17) into

$$Q(\theta^-)\Phi'(\theta)\dot{\theta}^+ = P(\theta^-)\dot{\theta}^- \tag{4.15}$$

and then solve for $\Phi'(\theta)\dot{\theta}^+$, and subsequently solve for $\dot{\theta}^+$

Or, alternatively re-write (3.14)-(3.16) on the form

$$lhs_1(\theta^-)\dot{\theta}^+ = rhs_1(\theta^-)\dot{\theta}^- \tag{4.16}$$

$$lhs_2(\theta^-)\dot{\theta}^+ = rhs_2(\theta^-)\dot{\theta}^- \tag{4.17}$$

$$lhs_3(\theta^-)\dot{\theta}^+ = rhs_3(\theta^-)\dot{\theta}^- \tag{4.18}$$

and solve for $\dot{\theta}^+$

$$\dot{\theta}^+ = (rhs_1(\theta^-)/lhs_1(\theta^-))\dot{\theta}^- \tag{4.19}$$
$$\dot{\theta}^+ = (rhs_2(\theta^-)/lhs_2(\theta^-))\dot{\theta}^- \tag{4.20}$$
$$\dot{\theta}^+ = (rhs_3(\theta^-)/lhs_3(\theta^-))\dot{\theta}^- \tag{4.21}$$

The approach (4.19)-(4.21) is the one that will be used in the following, and the one that have been used in the software implementation. Regardless of approach, the equations resulting from the substitution of (4.1)-(4.2) into (3.14)-(3.16) will be overdetermined, i.e. there will be three equations and only one unknown $\dot{\theta}^+$. The three solutions for $\dot{q}^+$ expressed in (4.19)-(4.21), can be re-written:

$$\dot{\theta}^+ = k_1(q^-)\dot{\theta}^- \tag{4.22}$$
$$\dot{\theta}^+ = k_2(q^-)\dot{\theta}^- \tag{4.23}$$
$$\dot{\theta}^+ = k_3(q^-)\dot{\theta}^- \tag{4.24}$$

The quantities k_1, k_2, k_3 will in the following be referred to as the θ-impact map's *impact factors*. For a particular solution $\dot{\theta}^+_\star$ to make sense then the three impact factors must be equal, i.e. $k_1 = k_2 = k_3$.

4.3 Hybrid Reduced Dynamics

Having formulated the reduced continuous and discrete dynamics in θ-space, these can be combined to formulate the hybrid dynamics. The update condition will be denoted $\mathcal{S}(\theta)$, and the reduced hybrid dynamics is then denoted[1]

$$\alpha_i(\theta)\ddot{\theta} + \beta_i(\theta)\dot{\theta}^2 + \gamma_i(\theta) = \tau_i, \quad i = 0, .., 3, \quad \theta \notin \mathcal{S} \tag{4.25}$$
$$\mathcal{F}(\theta^-, \dot{\theta}^-) : \gamma_- \to \gamma_+, \ \theta^- \in \mathcal{S} \tag{4.26}$$

This then completes the formulation of the biped's reduced dynamics. This hybrid system will be used in order to plan and search for walking gaits for the biped in the next chapters.

[1]Note that the γ-symbol is used both in α, β, γ-equations as well as for the switching surface

5 Gait Planning

Having derived the necessary models in the previous chapters, the task of planning motions that will constitute walking gaits for the 3-link biped can be presented and discussed. The main contribution of this chapter is to derive and structure the necessary and almost sufficient requirements for the planning of motions that will create walking behaviour in the 3-link biped.

Planning gaits for the 3-link biped, involves the task of defining desired characteristics for nominal gait-cycles, and then deriving the necessary conditions that must be met in order to guarantee their existence. In the following chapter these conditions will take on the role as constraints in a numerical search routine. The terms constraint, requirements and conditions will be used interchangeably throughout this section. The constraints are written in a format that corresponds to equality and inequality constraints in numerical optimization problems. This also makes the constraints written here correspond directly to those found in the software implementation `gait_search_equations.py`.

As stated in earlier chapters, a set of virtual holonomic constraints, $[\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)]$ imposed on the biped will shape the solutions $[\theta(t); \dot{\theta}(t)]$ of the reduced zero-dynamics. These solutions will subsequently also shape the trajectories $[q(t), \dot{q}(t)]$ and $[u(t)]$ that define the biped's movements. However, there is a considerable amount of conditions that must be met through a chosen set of VHCs in order to guarantee the existence of a walking cycle. By formulating these conditions as mathematical equality and inequality constraints, it is possible to constrain the gait search-space. By employing these constraints to constrain the search space a feasible numerical optimization problem can be organized to numerically search for gait cycles.

5.1 Defining Walking Gaits

Although walking gaits have been loosely defined and discussed in earlier chapters, there is need for more detailed definitions and additional terminology. Walking in the broadest sense, set very few constraints for the walking motions, it is simply the consecutive execution of steps. However this definition is of no use for solving the task of planning periodic motions that will produce walking gaits. This definition lead to an infinite number of different valid motions, and consequently lead to an infinitely large search-space for gait searching. This would indeed make it very difficult to find any gaits at all.

Gait planning involves the reduction of the gait search space, and then serves as a prerequisite for the gait search that will be the topic of chapter 6. The gait planning done here will rely on a set of necessary requirements that must be explicitly stated. These requirements will constrain the set of *feasible motions*, i.e. motions that will produce gaits. Constraining the set of feasible motions, will then effectively reduce the search-space and make it possible to search and find gaits using numerical optimization routines.

A key requirement made for the walking gaits that will be planned in this text, is symmetry. This thesis only deal with the planning of *symmetric walking gaits*. A *symmetric gait* has the property that each step of it's recurring gait-cycle will be identical. This implies that the starting and ending configurations will be identical in the geometrical sense, i.e. not considering the labelling of swing and stance leg. This is visualized in Fig 5.1

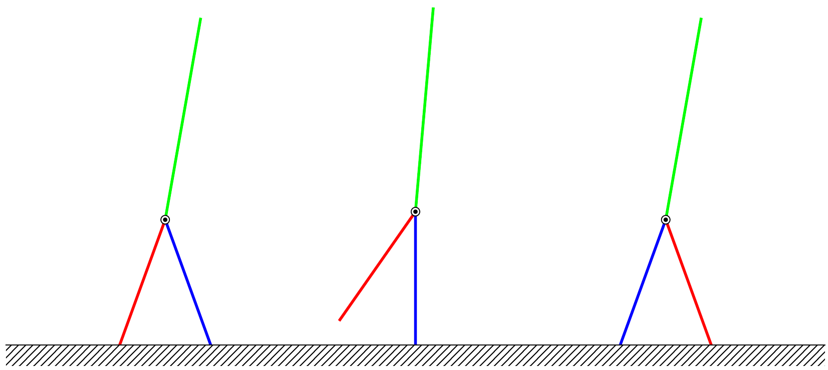


Figure 5.1: One step of a symmetric recurring gait-cycle

For symmetric gait cycles all steps will be identical, and planning motions for symmetric gaits is really the planning of motions during a single step. The repetition of identical steps will then constitute the planned nominal gait-cycle. A step starts immediately after an impact and ends after the next impact have occurred and a new step is initiated. A step in a symmetric gait will be such that when the swing leg impacts the ground at $[q(T^-), \dot{q}(T^-)]$, the update laws will transform the system back into the starting state $[q(T^+), \dot{q}(T^+)] \equiv [q(0); \dot{q}(0)]$. Such behaviour is within the class of periodic cycles for nonlinear systems that were discussed in chapter 2.4. Further, Theorem 1 [17], re-stated in 2.4.2 will be applicable for the gait-cycles of the 3-link biped.

Before the different requirements for symmetric gaits are derived and structured in this section, the planning of gaits can in general be summarized as

The task of creating motions that during the course of a step will take *the biped from a set of desired starting conditions to a set of desired ending conditions in a well behaved way.*

Although this formulation is rather general and informal, it does define the gait planning task in a neat way. By reviewing Theorem 1, the condition I = 0, is simply guaranteeing that the step's continuous-in-time sub-arc in θ-space is well defined. The next condition then guarantees that the update law $F(\cdot)$ will switch the system from desired ending conditions defined as $[q^-, \dot{q}^-] \in \Gamma_-$ back into the step's desired starting conditions defined $[q^+, \dot{q}^+] \in \Gamma_+$.

5.2 Parameterization of Reduced Dynamics and Reduction of Gait Search Space

As briefly discussed in section 2.4.1, complicated motion planning tasks require the parameterization of VHCs. The structure of a given VHC can be a type of mathematical expressions of choice. Such functions can be shaped by their numerical parameterization. This is the approach that is taken for planning and searching gaits for the biped. The virtual holonomic constraints can be written on parameterized form

$$q_1 = \phi_1(\theta, P_1) \;\; q_2 = \phi_2(\theta, P_2), \;\; q_3 = \phi_3(\theta, P_3) \quad \forall\, \theta \in [\theta_0, \theta_T] \tag{5.1}$$

Here ϕ_i refers to the VHC function/structure and the vectors P_i refer to their numerical parameterization. The reduced dynamics of the system will then be differential equations written on the form

$$\alpha_k(\theta, P)\ddot{\theta} + \beta_k(\theta, P)\dot{\theta}^2 + \gamma_k(\theta, P) = b_k u_k \tag{5.2}$$

Where P refers to the parameterization vector $[P_1, P_2, P_3]$. For convenience the reduced dynamics of the biped from in section 4.1 is re-stated here on parameterized form

Relative Coordinates

$$\begin{aligned} \alpha_1(\theta, P)\ddot{\theta} + \beta_1(\theta, P)\dot{\theta}^2 + \gamma_1(\theta, P) &= 0 \\ \alpha_2(\theta, P)\ddot{\theta} + \beta_2(\theta, P)\dot{\theta}^2 + \gamma_2(\theta, P) &= u_2 \\ \alpha_3(\theta, P)\ddot{\theta} + \beta_3(\theta, P)\dot{\theta}^2 + \gamma_3(\theta, P) &= u_3 \end{aligned} \tag{5.3}$$

Absolute Coordinates

$$\begin{aligned} \alpha_1(\theta, P)\ddot{\theta} + \beta_1(\theta, P)\dot{\theta}^2 + \gamma_1(\theta, P) &= u_2 \\ \alpha_2(\theta, P)\ddot{\theta} + \beta_2(\theta, P)\dot{\theta}^2 + \gamma_2(\theta, P) &= -u_2 - u_3 \\ \alpha_3(\theta, P)\ddot{\theta} + \beta_3(\theta, P)\dot{\theta}^2 + \gamma_3(\theta, P) &= u_3 \end{aligned} \tag{5.4}$$

Regardless of convention, the zero dynamics are then denoted

$$\alpha_0(\theta, P)\ddot{\theta} + \beta_0(\theta, P)\dot{\theta}^2 + \gamma_0(\theta, P) = 0 \tag{5.5}$$

Further, for one step the motion generator θ, will be monotonically increasing between $[\theta_0, \theta_T]$, where 0, and T denotes the start and end of a step respectively.

Gait planning and searching then involves the shaping of solutions $[\theta_\star, \dot{\theta}_\star]$ of (5.5) through numerical parameterization. These solutions will then shape the gait-cycles nominal trajectories

$$[q_\star(t), \dot{q}_\star(t)], \quad u_\star(t) \tag{5.6}$$

Having chosen some mathematical structure for $[\phi_1, \phi_2, \phi_3]$,the numerical parameterization vectors P_1, P_2, P_3 will shape the biped's motions and its corresponding nominal actuation . The searching of gaits then becomes the task of numerically determining the vectors P_1, P_2, P_3 such that the motions of the system defined by (5.6) will constitute a walking gait. By determining necessary requirements that will ensure the existence of a gait-cycle, it is possible to find walking gaits by meeting these requirements. In the following, such requirements and their corresponding constraints will be formulated. The constraints, will then constrain the gait search-space through requirements arising from considerations of different aspects of the desired walking behaviour for the biped.

Throughout the rest of this chapter the following notation will be used

- *a step starts at the time t=0*, and the state of the system at this time in θ-space and q-space is written in shorthand notation as $[q_0, \dot{q}_0]$ and $[\theta_0, \dot{\theta}_0]$ respectively
- *a step ends at the time t=T*, and the state of the system at this time in θ-space and q-space is written in shorthand notation as $[q_T, \dot{q}_T]$ and $[\theta_T, \dot{\theta}_T]$ respectively
- $\phi_i^+(\theta) \equiv \phi_i(\theta^+)$, where $\phi_i^+(\cdot)$ is defined by the biped's impact map.

5.3 Necessary Conditions for Double Support Phase

During the instantaneous double support phase there will be several conditions that must be met. The double support phase include both ending and starting conditions for the biped since $[q(T^+), \dot{q}(T^+)] = [q(0), \dot{q}(0)]$. Requirements for starting and ending kinematic states include conditions that arise from geometrical considerations, the impact map, and nominal actuator trajectories.

5.3.1 Kinematic Considerations

The kinematics of the biped's configuration during double support phase must meet five requirements:

1. *symmetry*: The start and end configurations $[q(0), q(T)]$ must be symmetric, i.e. start and end configurations must be geometrically identical as visualized in Fig 5.1.

2. *both feet on the ground*: Another constraint for the bipeds starting and ending configurations is simply that the feet of the biped must be on the ground in the start and the end of the step.
3. *upright posture*: the steps of the biped's gaits must be such that the biped has a upright posture at the start and end of a step. This puts constraints on $q(0)$ and $q(T)$.
4. *swing leg lift off at t=0*: Immediately after impact the swing leg of the robot should lift off from the ground, i.e. the y-component of the swing-foot should be positive
5. *swing leg downstroke at t=T*: Immediately prior to impact the swing leg of the robot should be moving down into the ground, i.e. the y-component of the swing-foot should be negative

Symmetry The requirement of symmetry can be formulated as the following equality constraint:

$$\begin{aligned}&(\phi_1(\theta_0, P_1) - \phi_1^+(\theta_T, P_1))^2 + \\ &(\phi_2(\theta_0, P_2) - \phi_2^+(\theta_T, P_2))^2 + \\ &(\phi_3(\theta_0, P_3) - \phi_3^+(\theta_T, P_3))^2 = 0\end{aligned} \tag{5.7}$$

Both feet on the ground Requiring that both feet are indeed on the ground at the start and end of a gait can be formulated as the following equality constraints

Absolute angle convention:

$$\phi_1(\theta_{0/T}, P_1) + \phi_3(\theta_{0/T}, P_3) = 0 \tag{5.8}$$

Relative angle convention:

$$2\phi_1(\theta_{0/T}, P_1) + \phi_2(\theta_{0/T}, P_2) + \phi_3(\theta_{0/T}, P_3) - 2\pi = 0 \tag{5.9}$$

$$\tag{5.10}$$

Upright Posture To avoid gaits where the biped is essentially walking upside-down, constraints can be made for the starting configuration and then by symmetry this will also apply for the end configurations. These requirements will be constraints on ϕ_1 and ϕ_2, as ϕ_3 will be constrained to comply through the requirement that both feet must be on the ground. For both angle conventions such constraints will be in the format of max/min constraints that can be adjusted when setting up a gait-search

$$\begin{aligned}&\phi_i(\theta_0, P_i) < q_{i,max} \\ &\phi_i(\theta_0, P_i) > q_{i,min} \\ &i \in [1, 2]\end{aligned} \tag{5.11}$$

Swing leg lift-off The velocity of the swing-foot were expressed in chapter 3 and denoted v_{3f}. The y-component of this can be rewritten into θ-space on the form $\varphi(\theta, P)\dot{\theta}$. Assuming that $\theta(t)$ is monotonically increasing, i.e. $\dot{\theta} > 0$ the requirement of swing-leg lift-off can be formulated as the inequality constraint:

$$\varphi(\theta_0, P) > 0 \tag{5.12}$$

Swing leg downstroke The velocity of the swing-foot were expressed in chapter 3 and denoted v_{3f}. The y-component of this can be rewritten into θ-space on the form $\varphi(\theta, P)\dot{\theta}$. Assuming that $\theta(t)$ is monotonically increasing, i.e. $\dot{\theta} > 0$ the requirement of swing-leg downstroke can be formulated as the inequality constraint:

$$\varphi(\theta_T, P) < 0 \tag{5.13}$$

5.3.2 Impact Map Considerations

It is the combination of continuous-in-time sub-arcs and jumps that will create gait-cycles. The set of chosen VHCs and their parameterizations will shape both the sub-arcs and the impact maps. Remember from earlier that for any impact map in θ-space to be valid the three resulting impact factors $k_1(q^-), k_2(q^-), k_3(q^-))$ must equal for the impact map to be valid. This gives rise to the following equality constraint:

$$(k_1(\theta^-) - k_2(\theta^-))^2 + (k_1(\theta^-) - k_3(\theta^-))^2 = 0 \tag{5.14}$$

5.3.3 Actuation Considerations

The actuation of a gait cannot be any. An actuator cannot produce torque-trajectories that have discrete-in-time jumps. It is therefore required that the actuator-trajectories for a nominal gait avoids this behaviour during the impact with the ground. The biped has two actuators denoted u_2 and u_3. The reduced dynamics of the biped (5.3) and (5.4) define the nominal trajectories of u_2 and u_3. Using the zero-dynamics equation (5.5) combined with (5.3) or (5.4), the actuator trajectories can be expressed through θ and $\dot{\theta}$.

Further, because of the re-labelling of legs during double support phase, nominal torque-trajectories that is continuous through impact must be such that:

$$\begin{aligned} u_3(\theta_0, \dot{\theta}_0, P) &= u_2(\theta_T, \dot{\theta}_T, P) \\ u_2(\theta_0, \dot{\theta}_0, P) &= u_3(\theta_T, \dot{\theta}_T, P) \end{aligned} \tag{5.15}$$

This can be formulated as the equality constraint:

$$(u_3(\theta_0, \dot{\theta}_0, P) - u_2(\theta_T, \dot{\theta}_T, P))^2 + (u_2(\theta_0, \dot{\theta}_0, P) - u_3(\theta_T, \dot{\theta}_T, P))^2 = 0 \tag{5.16}$$

An efficient approach to calculations of $\dot{\theta}_0$ and $\dot{\theta}_T$, is included later in this section in (5.20),(5.24).

5.4 Necessary Conditions for Single Support Phase

Having stated necessary conditions for a walking gaits double support phase, the necessary conditions that will ensure a well-defined and realizable single support phase must be stated. The single support phase is entirely defined by the bipeds continuous dynamics. Requirements for this phase will arise from the VHC methodology itself and from the biped's actuators.

5.4.1 General Requirements of the VHC methodology

Consider the zero-dynamics of the biped

$$\alpha_0(\theta, P)\ddot{\theta} + \beta_0(\theta, P)\dot{\theta}^2 + \gamma_0(\theta, P) = 0 \tag{5.17}$$

α separated from zero It is clear that this equation has a singularity for $\alpha(\theta, P) = 0$. During a gait $\alpha(\theta, P)$ must therefore be separated from zero for all $\theta \in [\theta_0, \theta_T]$. This can be stated as inequality constraints

$$\begin{aligned} &\alpha(\theta, P) > 0 \quad \forall\theta \in [\theta_0, \theta_T] \\ &\text{or} \\ &\alpha(\theta, P) < 0 \quad \forall\theta \in [\theta_0, \theta_T] \end{aligned} \tag{5.18}$$

Monotonically increasing motion generator Another requirement is that θ is monotonically increasing from θ_0 to θ_T. Without this requirement, the mappings between q-space and θ-space would not be one-to-one. This gives rise to the inequality constraint:

$$\dot{\theta} > 0 \quad \forall \quad \theta \in [\theta_0, \theta_T] \tag{5.19}$$

This constraint requires the motion generators velocity $\dot{\theta}$ to be calculated at points along the trajectory $[\theta(\theta), \dot{\theta}(t)]$. This must be in the search loop, and a full integration of the reduced zero-dynamics is therefore too inefficient. However the conserved quantity and it's equation on the form (2.21) can be utilised to find a more efficient solution. The equation can be rewritten as

$$I = \dot{\theta}(t)^2 - \rho_1\dot{\theta}(0) + \rho_2 \tag{5.20}$$

where

$$\rho_1 = \psi(\theta(0), \theta(t)) = e^{\left\{-\int_{\theta_0}^{\theta_1} -\frac{2\beta(\tau)}{\alpha(\tau)} d\tau\right\}} \tag{5.21}$$

$$\rho_2 = \psi(\theta(0), \theta(t)) \int_{\theta(0)}^{\theta(t)} \psi(s, \theta(0)) \frac{2\gamma(s)}{\alpha(s)} ds \tag{5.22}$$

By solving the integral quantities ρ_1 and ρ_2 in a cumulative fashion along the vector $[\theta(0), ..., \theta(t)]$, it is possible to find the corresponding vector $[\dot{\theta(0)}, ..., \dot{\theta}(t)$. $\theta(0)$ can be found by assuming that the impact factors will converge to one value on any valid gait-cycle, i.e.

$$\theta(0)^2 = [\theta_T^+]^2 = k[\theta_T^-]^2 \tag{5.23}$$

combining this with (5.20), the starting velocity of the step can be expressed:

$$\theta(0)^2 = \frac{k^2 \rho_2}{\rho_1 k^2 - 1} \dot{\theta_T}^2 \tag{5.24}$$

This then allows for the computation of the velocity vector for θ. The cumulative integral form is such that the resolution of the $[\theta(0), ..., \theta(t)$ can be chosen accordingly to the precision demands of the functions in which it is employed. Software implementation of this can found in `gait_search_equations.py`.

5.4.2 Actuator Requirements

Having stated the necessity for continuous torque-trajectories earlier, this is augmented by an additional requirement. The actuators can only deliver a finite amount of torque u_{lim}. This can be formulated into the inequality constraint

$$|max(u(\theta, \dot{\theta}))| < u_{lim} \quad \forall \theta \in [\theta_0, \theta_T] \tag{5.25}$$

There may be more constraints imposed by a real-world actuator, such as max power consumption, bandwidth etc. but such requirements are not included in this thesis, but could obviously be added through the formulation of corresponding constraints.

6 Gait Search

To find gaits it is necessary to organize a numerical gait search. This involves structuring the constraints found in a numerical optimization routine. Having chosen three VHC-functions

$$q_1 = \phi_1(\theta, P_1) \;\; q_2 = \phi_2(\theta, P_2), \;\; q_3 = \phi_3(\theta, P_3) \quad \forall\, \theta \in [\theta_0, \theta_T] \tag{6.1}$$

the task of a numerical gait search is to find a parametrization $P = [P_1, P_2, P_3]$ that will meet all the necessary requirements as stated in the previous chapter (5.7) - (5.25). Further, in this chapter there will be made a distinction between so-called *feasible gaits* and *realizable gaits*. It is also useful to introduce the notion of a so-called *optimal gaits*

- *feasible gaits* are periodic gait-cycles in the phase-portraits of the . However they may or not be *realizable* due to the torque demands required to create the periodic cycle.

- *realizable gaits* are gait cycles that are realized through nominal (open-loop) toque trajectories that are realizable for the biped's actuators. They then satisfy the requirements stated in (5.16), (5.25).

Finding gait-cycles for the 3-link biped is not an easy task. The biped's dynamics are highly nonlinear. Further there is considerable computational complexity involved for several of the necessary constraints derived in chapter 5. However a step-by-step approach in gait-searching proved to be successful, this was done by first finding a parametrizations for feasible gaits, and then using this as a starting point for further optimization of the gait cycle into a realizable one. The search algorithm that was used was a sequential quadratic programming routine available via the Scipy [3] software package for Python.

6.1 Organizing Numerical Gait Search

6.1.1 Choice of Motion Generator Variable

Within the VHC paradigm, one is free to choose motion generator variable, θ. It can be chosen to represent a physical quantity, or can conversely also be chosen to be an entirely *abstract* quantity. An *abstract* motion generator will not directly correspond to any physical quantity, but the relations defined by (6.1), will define the mapping between $\theta(t)$ and the biped's physical configuration $q(t)$.

The choice of θ can be motivated in several ways. Choosing θ such that it directly represents a physical quantity can be useful with respect to implementation of walking gaits in a physical prototype. A physical quantity can either be directly measured or found via an observer. However such a choice may negatively affect the tasks of motion planning and gait searching. For gait searching a non-physical and completely abstract θ may be a more convenient choice.

For the 3-link biped, the passive q_1 coordinate is a reasonable candidate for motion generator among the generalized coordinates. However such a choice has the effect of severally restricting the gait search space, by not allowing the choice of mathematical structure and parameterization for $\phi_1(\theta, P1)$. Further, this also implies that q_1 must be monotonically increasing during the steps of the gait. This has the effect of limiting the range of feasible and realizable gait-cycles that could be found through a numerical search. As the focus of the project is the planning of gaits, the choice was made to use a completely abstract motion generator for planning and searching gaits. This allows the free choice of VHC-structure for $[\phi_1, \phi_2, \phi_3]$.

6.1.2 Searching a Feasible Gait-cycle

A feasible gait cycle is one where the nominal trajectories $[q_\star(t), \dot{q}_\star(t)], u_\star(t)$ meet all requirements for a walking gait, except those that arise from the biped's actuators, i.e. $u_\star(t)$ cannot be reproduced by the actuators.

Stripping away the actuator requirements, there is requirements from the VHC-method itself, impact map and kinematics that need to be accounted for when organizing the numerical search. The numerical optimization routine need an objective function to be optimized. For the task of finding feasible gait cycles the reduced impact map constraint served as the quantity to minimize, having a zero global minimum function value. Further it was found that it was beneficiary to choose max and min of the impact map factors (4.22),(4.23),(4.24), that define the relation between $i\dot{\theta}$ at the beginning and end of each step. The three different impact factors will converge to one k-value for a feasible gait.

$$\theta_0 = \theta_T^+ = k\theta_T^- \tag{6.2}$$

This is done by creating the inequality constraint $k_m in < k < k_m ax$. By adding this constraint to a set of the constraints derived in chapter 5, a gait search can be organized

$$\mathbf{min}\ (k_1(\theta^-) - k_2(\theta^-))^2 + (k_1(\theta^-) - k_3(\theta^-))^2 = 0 \tag{6.3}$$

subject to

symmetry

$$\begin{aligned}&(\phi_1(\theta_0, P_1) - \phi_1^+(\theta_T, P_1))^2 + \\ &(\phi_2(\theta_0, P_2) - \phi_2^+(\theta_T, P_2))^2 + \\ &(\phi_3(\theta_0, P_3) - \phi_3^+(\theta_T, P_3))^2 = 0\end{aligned} \tag{6.4}$$

both feet on the ground

absolute angle convention:

$$\phi_1(\theta_{0/T}, P_1) + \phi_3(\theta_{0/T}, P_3) = 0 \tag{6.5}$$

relative angle convention:

$$2\phi_1(\theta_{0/T}, P_1) + \phi_2(\theta_{0/T}, P_2) + \phi_3(\theta_{0/T}, P_3) - 2\pi = 0 \tag{6.6}$$

upright posture

$$\begin{aligned}&\phi_i(\theta_0, P_i) < q_{i,max} \\ &\phi_i(\theta_0, P_i) > q_{i,min} \\ &i \in [1, 2]\end{aligned} \tag{6.7}$$

swing leg lift-off

$$\varphi(\theta_0, P) > 0 \tag{6.8}$$

swing leg downstroke

$$\varphi(\theta_T, P) < 0 \tag{6.9}$$

impact factors

$$k_{min} < k_i < k_{max} \quad i \in [1, 2, 3] \tag{6.10}$$

θ monotonically increasing

$$\dot{\theta} > 0 \quad \forall \quad \theta \in [\theta_0, \theta_T] \tag{6.11}$$

$\alpha(\theta)$ separated from zero

$$\alpha(\theta, P) > 0 \quad \forall \theta \in [\theta_0, \theta_T] \tag{6.12}$$

$$\text{or} \tag{6.13}$$

$$\alpha(\theta, P) < 0 \quad \forall \theta \in [\theta_0, \theta_T] \tag{6.14}$$

This search structure is found implemented in the software script `random_kickoff_search.py`. It was made more effective by being wrapped in a while loop which for each iteration randomly chooses a new initial guess P_0 for the VHC-parameterization. This proved to be an effective strategy for finding feasible gait cycles using both angle conventions.

6.1.3 Searching a Realizable Gait-cycle

To search and find a realizable gait-cycle for the biped, actuation constraints must be added to the search routine. By using the VHCs and corresponding parametrizations of realizable gait-cycles, these can be re-shaped to meet the actuation constraints of the biped. This entails that torques must be continuous, i.e. the nominal torque trajectories must be continuous-in-time. Further the max peak torque values that were deemed realizable was set to 1 Nm. By structuring a search with these requirements it was possible to find realizable gait-cycles for the biped. The most effective structure was to be that were the torque continuity constraint took the role as objective function, and the max peak constraint was added as an inequality constraint. This search structure was formatted as

$$\mathbf{min}\ (u_3(\theta_0, \dot{\theta}_0, P) - u_2(\theta_T, \dot{\theta}_T, P))^2 + (u_2(\theta_0, \dot{\theta}_0, P) - u_3(\theta_T, \dot{\theta}_T, P))^2 = 0 \tag{6.15}$$

subject to

symmetry

$$\begin{aligned} &(\phi_1(\theta_0, P_1) - \phi_1^+(\theta_T, P_1))^2 + \\ &(\phi_2(\theta_0, P_2) - \phi_2^+(\theta_T, P_2))^2 + \\ &(\phi_3(\theta_0, P_3) - \phi_3^+(\theta_T, P_3))^2 = 0 \end{aligned} \tag{6.16}$$

both feet on the ground

absolute angle convention:

$$\phi_1(\theta_{0/T}, P_1) + \phi_3(\theta_{0/T}, P_3) = 0 \tag{6.17}$$

relative angle convention:

$$2\phi_1(\theta_{0/T}, P_1) + \phi_2(\theta_{0/T}, P_2) + \phi_3(\theta_{0/T}, P_3) - 2\pi = 0 \tag{6.18}$$

upright posture

$$\begin{aligned} &\phi_i(\theta_0, P_i) < q_{i,max} \\ &\phi_i(\theta_0, P_i) > q_{i,min} \\ &i \in [1, 2] \end{aligned} \tag{6.19}$$

swing leg lift-off

$$\varphi(\theta_0, P) > 0 \tag{6.20}$$

swing leg downstroke

$$\varphi(\theta_T, P) < 0 \tag{6.21}$$

impact factors

$$(k_1(\theta^-) - k_2(\theta^-))^2 + (k_1(\theta^-) - k_3(\theta^-))^2 = 0 \tag{6.22}$$

$$k_{min} < k_i < k_{max} \;\; i \;\in [1, 2, 3] \tag{6.23}$$

θ **monotonically increasing**

$$\dot{\theta} > 0 \quad \forall \quad \theta \in [\theta_0, \theta_T] \tag{6.24}$$

$\alpha(\theta)$ **separated from zero**

$$\alpha(\theta, P) > 0 \quad \forall \theta \in [\theta_0, \theta_T] \tag{6.25}$$

$$\text{or} \tag{6.26}$$

$$\alpha(\theta, P) < 0 \quad \forall \theta \in [\theta_0, \theta_T] \tag{6.27}$$

realizable peak torque values

$$|max(u(\theta, \dot{\theta}))| < u_{lim} \quad \forall \theta \in [\theta_0, \theta_T] \tag{6.28}$$

The implementation of this search routine can be found in `realizable_gait_search.py`

7 | Numerical Results

The organization of gait searches as presented in the previous made it possible to find realizable gait-cycles for the three link biped. A selection of the gaits that were found is included here. The chapter visualize the trajectories of both so-called feasible gait-cycles and realizable ones as defined in the previous chapter. Both relative and absolute angles were used when searching gaits, an this section show gaits in both set of coordinates. Realizable gaits were however only found using absolute coordinates. The VHC structure that have been employed to find gaits are polynomials of 5th order, and the VHCs are written:

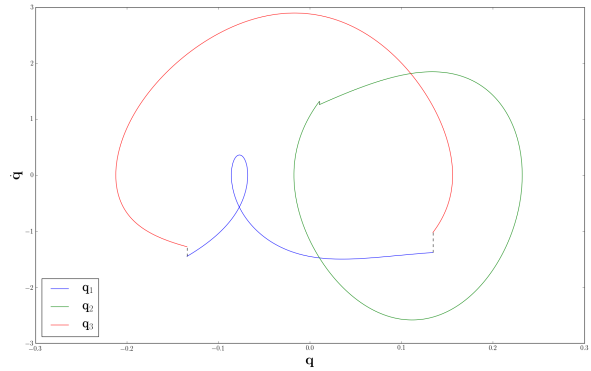
$$
\begin{aligned}
\phi_1(\theta, P_1) &= p_{10} + p_{11}\theta + p_{12}\theta^2 + p_{13}^3 + p_{14}\theta^4 + p_{15}\theta^5 \\
\phi_2(\theta, P_2) &= p_{20} + p_{21}\theta + p_{22}\theta^2 + p_{23}^3 + p_{24}\theta^4 + p_{25}\theta^5 \\
\phi_3(\theta, P_3) &= p_{30} + p_{31}\theta + p_{32}\theta^2 + p_{33}^3 + p_{34}\theta^4 + p_{35}\theta^5
\end{aligned}
\tag{7.1}
$$

The set of physical parameters that were used, were based on CAD calculations for the prototype, and they can be found in `CAD_params_ILeg.xml`.

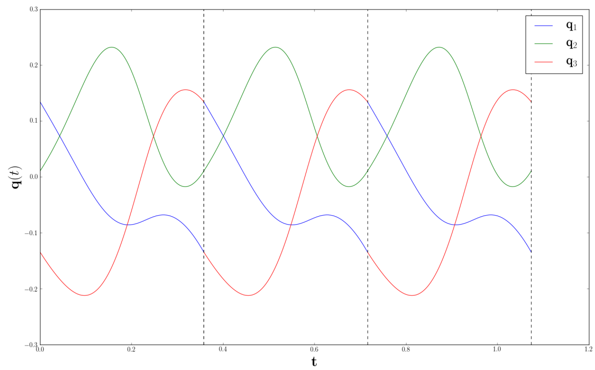
Feasible Gait Cycles The gait cycles that were found by running `random_kickoff_search.py` were not realizable because of their torque properties, but nonetheless they are periodic gait-cycles. The plots of two such cycles are included here one for each set of generalized coordinates in Fig 7.1 and Fig 7.2.

Realizable Gait Cycles By running the script `realizable_gait_search.py` several realizable gait-cycles were found using absolute coordinates. Realizable max peak values for the biped was taken to be |1|Nm. Two of the relaizable gaits that were found are included shown in Fig 7.3 and Fig 7.4.

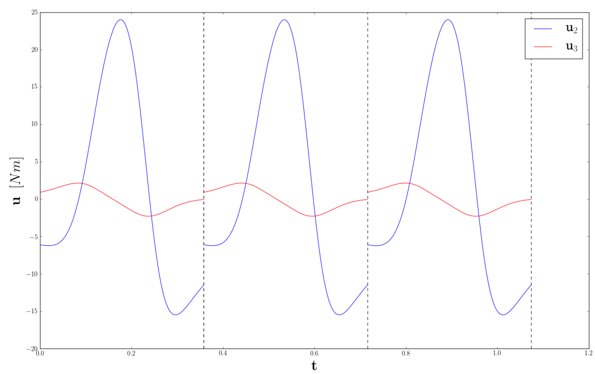
The vhc parameterizations for each gait and hi-res plots can be found in the folder `thesis_report_gaits` delivered with this text.



(a) Phase portraits of the biped's links during one step

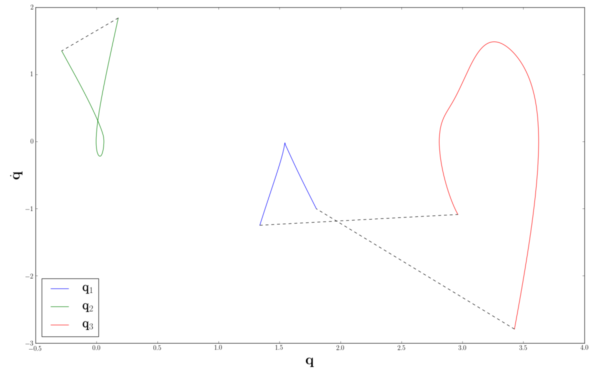


(b) The time-trajectories of the biped's links during the gait cycle

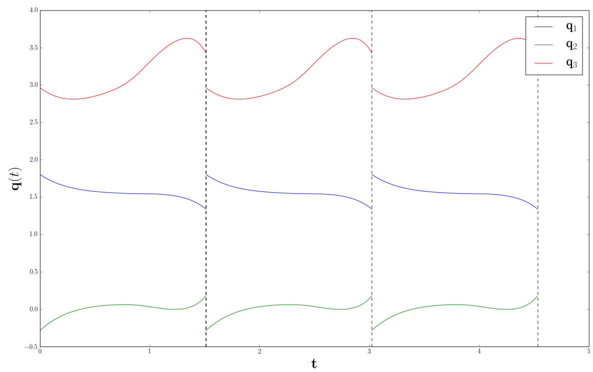


(c) The time trajectories of the biped's actuator torques during the gait cyle

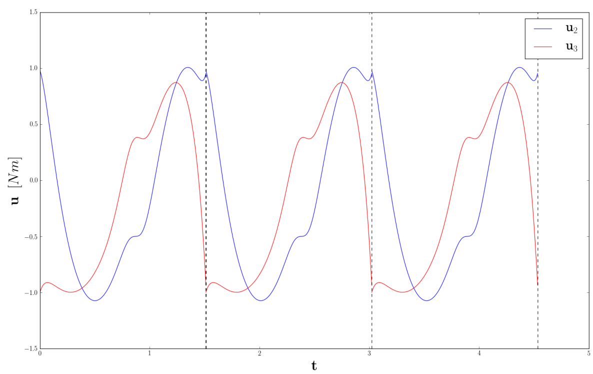
Figure 7.1: Feasible Gait Cycle in Absolute Coordinates



(a) Phase portraits of the biped's links during one step

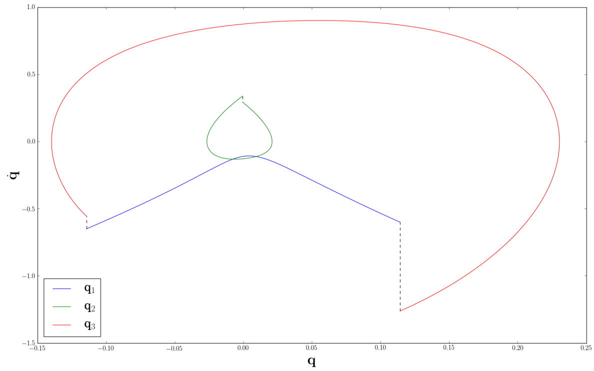


(b) The time-trajectories of the biped's links during the gait cycle

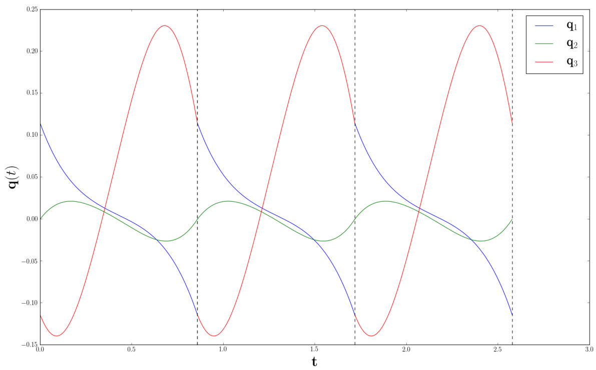


(c) The time trajectories of the biped's actuator torques during the gait cyle

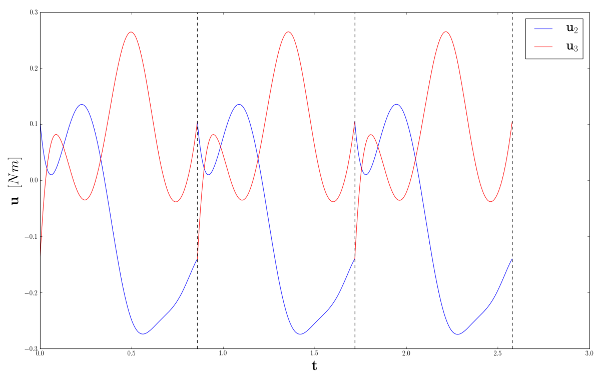
Figure 7.2: Feasible Gait Cycle in Relative Coordinates



(a) Phase portraits of the biped's links during one step

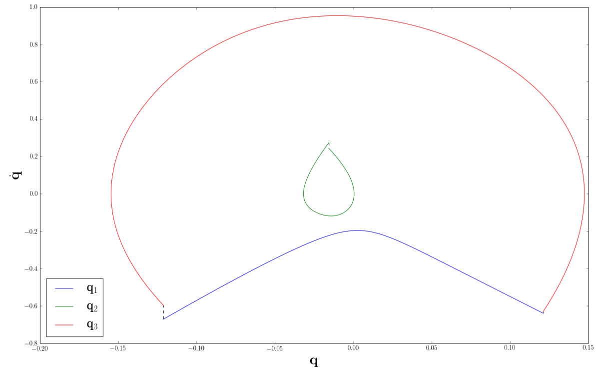


(b) The time-trajectories of the biped's links during the gait cycle

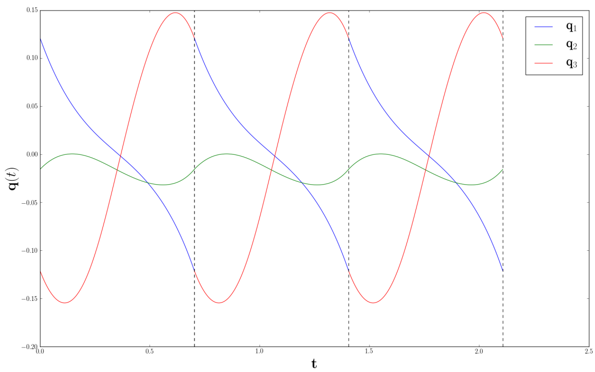


(c) The time trajectories of the biped's actuator torques during the gait cyle

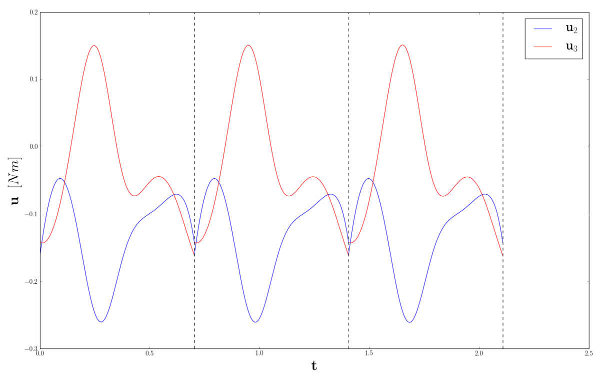
Figure 7.3: Realizable Gait Cycle 1 in Absolute Coordinates



(a) Phase portraits of the biped's links during one step



(b) The time-trajectories of the biped's links during the gait cycle



(c) The time trajectories of the biped's actuator torques during the gait cyle

Figure 7.4: Realizable Gait Cycle 2 in Absolute Coordinates

8 Discussion

The previous chapters have shown the application of virtual holonomic constraints for a planar 3-link biped with point-feet. It has been shown how the methodology can be used to structure the planning and searching for gait-cycles. A brief discussion of the results achieved and the methodology applied is included here before the thesis is concluded.

8.1 The Virtual Holonomic Constraints Approach

The application of virtual holonomic constraints to plan gaits for the biped have been documented in the previous chapters. However gait planning is only one step one the way in terms of creating actual walking behaviour in a real-world biped prototype. A number of engineering aspects and challenges have not been accounted for in this report. Further it is also important to point out that the nominal trajectories $[q_\star(t), \dot{q}_\star(t), u_\star(t)]$ found here are in general unstable motions, and needs the aid of a stabilizing controller to be realized in an actual walking biped. However the method show great promise as a powerful approach to motion-planning for mechanical hybrid system such as the 3-link biped

Gait Stabilization When planning nominal gait-cycles, it is assumed that the chosen VHCs are held invariant. The open-loop input of the nominal $u_\star(t)$ would not create a walking gait in a physical prototype. In general all trajectories found in gait-planning as described in this report will be unstable. To stabilize the gaits, a controller must be used to close the loop. Within the VHC-paradigm this requires the derivation of so-called *transverse coordinates* and *transverse linearization* for the system. These are not the topics of this thesis, but it is important to point out that planning gaits using VHCs, makes these method's applicable for stabilizing the planned trajectories. In [8], the derivation of transverse linearization and subsequent stability analysis for a planar 3-link biped is treated. Further, it is important to note that the choice of motion-generator may not be entirely independent of controller structure and measurement devices. The motion generator could ideally be chosen to comply with measurements devices, state-observers and controllers. A completely abstract motion generator is not directly measurable, and nominal trajectories expressed in such a variable may not be the most convenient. However this could be solved in some ad-hoc manner such as the use of a look-up table.

Gait Planning and Prototyping One interesting use of motion planning is in the prototype stage of a project. The existence of gait-cycles will be dependent on the physical parameters of a machine. In this thesis-report a set of physical parameters have been used, and VHC-parameterizations have been found to comply with these physical parameters in order to create realizable walking gaits. However, these roles can easily be reversed. In a prototyping stage of a walking biped, gait planning and searching can give insight into the walking capabilities of a hypothetical prototype. This can then be used to set reasonable specifications for physical parameters such as masses and lengths of links as well as specifications in terms of needed actuation. This clearly demonstrates the use of gait planning as a standalone tool for proving the feasibility of a walking machine.

Planning More Complicated Walking Motions The gaits that have been discussed in this thesis are very simple. They are periodic and symmetric walking motions with fixed gait-speed and step-lengths and would only make the biped capable of walking in a steady gait. Scenarios such as obstacle avoidance can be used to motivate the development of more complex walking motions. If the dynamical walking paradigm could have an impact in more complex and versatile walking robots, it must enable such robots to accomplish more than just simply walking in a steady gait. Humans are capable of changing speed and step length of their gaits in a seamless fashion. A natural next step in terms of gait planning could be to plan transitions between different gait-cycles. This is undoubtedly a challenging task. However the development of many different simple walking cycles such as those that have been shown for the biped in this report could prove to be valuable. A multitude of such motions would form a library of simple walking gaits. Algorithms could use this library as a starting point for planning more complex walking motions. Typically the combination of different step-lengths would be required for obstacle avoidance. This could possibly be achieved by making interconnections between different simple walking gaits.

Gait Searching The gaits that have been found in connection with these thesis was optimized in terms of their peak torque requirements. Unfortunately time did not allow for experimentation with other performance indexes. However this thesis report documents a structured approach to organizing numerical optimization routines for optimizing gaits. By formulating their mathematical objective functions, other gait-properties could indeed be optimized using the developed framework. Interesting performance indexes could include gait-speed, energy consumption, or a combination of these. Further the software implementation that have been developed in connection with this thesis does allow for other VHC-structures to be employed in gait planning and searching. However time did not allow for experimentation with different structures. The VHC approach is very attractive from a gait planning and gait searching perspective as it makes it possible to organize these difficult tasks sufficiently while still being versatile and allowing the design of different motion patterns. For the 3-link biped, different classes of VHCs and the definition of interesting performance indexes could create a variety of different walking gaits.

8.2 Software Implementation

The main body of work documented in this thesis-report is the development of a useful software framework for gait planning using virtual holonomic constraints. The software implementation written in the Python programming language was developed to be dynamic and convenient in use.

Design Choices Two important design choices was made early in the projects development.

- two different sets of generalized coordinates are supported
- the code is able to handle VHCs in a symbolic format and consequently derive q-space and θ-space dynamics for use in gait planning and analysis.

The motivation behind supporting two sets of generalized coordinates namely absolute and relative were taken on the basis that gait-cycles represented in relative coordinates could be beneficiary as they more directly correspond to the prototype's measurements and actuators. The absolute coordinates are more easy to use when planning gaits, and were supported for convenience. In addition, other members of the project had developed models with relative and absolute coordinates respectively. By supporting both conventions it was possible to directly compare and verify numerical results among a total of three different implementations during code debugging.

More importantly the code was developed to support symbolic VHCs. The models derived for the biped quickly become mathematically complex, and the use of symbolic mathematical software such as Maple become a necessity. However deriving mathematical expressions in a software such as Maple and then copy-pasting these expressions when need was deemed to be a very in-elegant solution. Instead considerable work was put into developing a framework for the biped that automated this process. This was done through extensive use of the SymPy package [4]. This allowed for any type of mathematically defined VHCs to be used for gait planning and searching without having to change any other parts of the code. This also enables the opportunity for combining different VHC-structures, and organizing search procedures that automatically try different structures or combinations of these in search for gaits with different characteristics.

9 Conclusion

The aim of this thesis was to study gait planning of a planar 3-link biped using virtual holonomic constraints. The 3-link biped is an interesting example for testing method's for the creation of dynamic walking gaits in underactuated bipedal robots. It's three links and two actuators make it an underactuated system of degree one. Historically, motion planning for underactuated systems have been a challenge. However the method of using virtual holonomic constraints have proven to be successful for planning motions in a considerable range of such systems.

This thesis show the application of said method to the hybrid underactuated 3-link biped system. The aim was to plan desired motions for dynamical walking gaits for this machine. The dynamical walking gaits of the biped have motions that constitute closed periodic cycles in the biped's state space. The task of gait planning is to create such cycles. Several such cycles was found for the 3-link biped with surprisingly low actuator torque requirements.

The biped was modelled using a combination of continuous Euler Lagrange equations and an impact map derived assuming conservation angular momentum. These were combined to model the hybrid dynamics exhibited of the biped during walking. By imposing virtual holonomic constraints, the biped's dynamics was reduced, and the subsequent reduced dynamic model was used to develop a structures for planning gaits. By employing the reduced dynamics it was shown how a structured approach to gait planning could be formulated for the biped by formulating necessary conditions for the existence of gait-cycles. These requirements was then used to structure a numerical gait search that successfully found parametrizations of the reduced dynamics such that the set of nominal trajectories $[u_\star(t), \dot{q}_\star(t), u_\star]$, induced walking in the biped.

A unified software framework was developed for the task. The framework is dynamical in terms of VHC-structure and allows for two different sets of coordinates to be used. This enables the user to easily use both coordinate conventions as well as different classes of VHCs when planning and searching gaits. The framework proved to be useful and capable of searching and finding gait cycles.

The results found and documented in this thesis further proves the VHC-paradigm to be a very exciting methodology for developing dynamical walking gaits in bipedal robots. It elegantly reduces the dimensionality of the gait planning problem. By doing so the method also indirectly deal with two inherent difficulties found in dynamics walking motions, namely limb coordination and underactuation. Further the method creates nominal motions that are completely time-independent allowing for time-less stabilizing controller structures. The VHC methodology is therefore a viable candidate in the future development of walking robots.

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A | Python Source Code

The attached folder delivered with this thesis-report contains source code developed during the project. The folder is divided in to subfolders listed here containing different parts of the software implementation.

A.1 `q_space_dynamics`

Includes files for modelling the biped's dynamics in generalized coordinates. It contains the following files

`biped_data.py` Handles the physical parameters of the biped. Originally written by Leonid Paramonov and reused in this project with permission.

`biped_dynamics_v1.py` Derives the mathematical dynamical model of the biped as Euler-Lagrange equations.

`biped_utils.py` Implements useful utilities for simulating differential equations as those found for the biped. Originally written by Leonid Paramonov and reused in this project with permission.

`biped_v1.py` Holds functions for simulating the biped's Euler-Lagrange equations.

`impact_map.py` Derives the impact map for the biped in generalized coordinates.

A.2 `theta_space_dynamics`

`abg.py` Derives the reduced dynamics of biped on α, β, γ-form.

`motion_generator.py` Holds functions for generating trajectories for the biped's generalized coordinates and actuators $[q(t), \dot{q}(t), u(t)]$ from motion generator trajectories $[\theta, \dot{\theta}]$.

`theta_impact_map_v2.py` Derives impact map for the biped's reduced dynamics.

`vhc.py` Handles virtual holonomic constraints, and implements related utility functions.

A.3 kinematics

`q_space_kin.py` Derive kinematic relations written in generalized coordinates

`theta_space_kin.py` Derive kinematic relations re-written into θ-space.

A.4 gait_search

`gait_data.py` Structure and save a gait's VHC parameterization as well as the gait's kinematic start and end conditions to an xml file.

`gait_data.py` Structure and save a gait's VHC parameterization as well as the gait's kinematic start and end conditions.

`gait_search_equations.py` Derives and holds functions for use in gait searching.

`random_kickoff_search.py` Gait searching script with randomized initial guess for the optimization routine.

`realizable_gait_search.py` Gait searching script constraining the search space to find realizable gaits.

`gait_run.py` Run through simulations for one step of a given gait and makes corresponding plots.

`gait_run_ms.py` Multi-step gait simulation that produces gait-plots.

A.5 thesis_report_gaits

Includes xml-files and hi-res plots of the gaits found in the Numerical Results chapter.

A.6 model_parameters_xml

Contains the physical parameters for the biped used to search gaits.