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Existence of solitary-wave solutions to a class of pseudodifferential evolution equations

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Abstract

In this thesis we present an original existence proof of solitary-wave solutions to a class of pseudodifferential evolution equations. We seek traveling wave solutions with constant velocity c of the form $u(x-ct)$ of

$$u_t + (n(u) - Lu)_x = 0 \quad \text{in } \mathbb{R},$$

through variational methods. By integrating over $\mathbb{R}$ with respect to the spatial variable, and assuming that the solution vanishes at infinity, we arrive at

$$-cu + n(u) - Lu = 0 \quad \text{in } \mathbb{R},$$

which is the standing point for our analysis. We prove existence of solutions to these equations by the technique previously employed by Albert [1] and Arnesen [4] amongst others. Here n is a nonlinear term, and compared to what has earlier been studied, it is now inhomogeneous and includes a higher order term. L is a Fourier multiplier operator of order $s \geq 0$. The higher order term included in the nonlinearity significantly changes the characteristics of the problem compared to what has previously been studied for this combination of equation and linear operator. We also introduce the principle of concentration compactness; the main ingredient in order to prove that we have compactness, despite working on an unbounded domain.

Sammendrag

I denne oppgaven presenterer vi et originalt eksistensbevis av solitære bølger til en klasse av pseudodifferensiale evolusjonslikninger. Vi søker reisende bølger med konstant hastighet c på formen $u(x-ct)$ av

$$u_t + (n(u) - Lu)_x = 0 \quad \text{i } \mathbb{R},$$

gjennom variasjonelle metoder. Ved å integrere opp med hensyn på romvariabelen, og ved å anta at løsningen forsvinner uendelig langt borte, ender vi opp med

$$-cu + n(u) - Lu = 0 \quad \text{i } \mathbb{R},$$

som er utgangspunktet for vår analyse. Vi beviser eksistens av løsninger til denne likningen ved en teknikk som tidligere har blitt brukt av blant andre Albert [1] og Arnesen [4]. Her er n et ikke-lineært ledd, og sammenliknet med hva som har blitt forsket på tidligere, er den nå inhomogen og inkluderer et høyere ordensledd. L er en Fourier multiplikator operator av orden $s \geq 0$. Det høyere ordensleddet som inngår i ikke-lineæritetsleddet endrer problemets karakter signifikant i forhold til hva som har blitt forsket på tidligere for denne kombinasjonen av likning og lineær operator. Vi introduserer også konsentrasjon kompakthetsprinsippet; et resultat som gjør det mulig for oss å bevise at vi fremdeles har kompakthet, til tross for at vi jobber på et ubundet domene.

Preface

This master's thesis is written in the last five months of my period at NTNU in order to complete my master's degree within the study program Physics and Mathematics. Prior to the thesis I had taken several courses in statistics and numerical analysis as well as a very basic introduction course in functional analysis. I encountered variational calculus for the first time in a course in optimization theory and quickly realized that this was something I wanted to pursue further. After taking a course in Partial Differential Equations, I realized I could combine this with the theory of variational calculus. I found out that Professor Mats Ehrnström had a project for a master student involving such theory, and thus began my journey into this field. I first wrote a project for Mr. Ehrnström worth 15 study points at NTNU, and after having done this there was no doubt; I wanted to write my master thesis about this theme. Having minimal to no knowledge of Sobolev theory and Fourier theory, I was encouraged to take Functional analysis and Fourier analysis while writing my project to prepare me for the master thesis. The proper research that was conducted during my master thesis was so fulfilling that I decided to accept a PhD offer at the University of Oslo.

Without the exceptional guidance of Professor Mats Ehrnström, my work in this thesis would not have been near what it became in the end. I would like to express my deep gratitude for his contribution. He made sure that every time I walked out of his office I was filled with motivation and excitement, and he also helped me with the many mathematical challenges I met along the way. In addition, I have to thank Mathias Arnesen, PhD student at NTNU, for offering me detailed insight into his article related to a similar theme as the one presented in this thesis.

Lastly, I would like to thank my father, Reidar Haugå, who has been one of the greatest inspirations of my life. May you rest in peace.

Contents

1 Introduction

History

"Understanding generalized solutions or weak solutions is fundamental, because many PDEs, especially nonlinear PDEs, do not in general possess smooth solutions."

- Lawrence C.Evans

A partial differential equation (PDE) does not always have solutions in the classical form, but it might allow solutions in a nonclassical way; distributional solutions. It was not until the 1930's [19], when Sergei Sobolev introduced the concept of distributions for the first time that this area within PDEs arose. When first introduced however, Sobolev simply called them functionals. Sobolev expanded the classical notion of a derivative, increasing the range of application of the techniques applied by Newton and Leibniz in the 17th century. In the 1940's, Laurent Schwartz gave a full description of the concept of distributions, a contribution to mathematics that Schwartz would receive the Fields medal for in 1950. Since its introduction, the theory of distributions has been used within the field of PDEs with great success. The Sobolev spaces, along with its embedding theorems, are invaluable tools in the search of nonclassical solutions of PDEs. The Sobolev spaces are natural homes for weak (and also classical) solutions of PDEs, and it is in these spaces we search for solutions in this thesis. The Sobolev spaces can be defined through some growth conditions on the Fourier transform, leading us to the notion of fractional derivatives. The concept itself, however, is not new. Leibniz [14] discussed the meaning of derivatives of order one half, in a note, centuries before Sobolev addressed the matter. For centuries the fractional derivatives were of purely theoretical interest. However, in the late 20th century, fractional PDEs modeling physical situations better than their predecessors, were introduced. There are now several textbooks written on the subject, for instance [20], and it is currently an area of active research both theoretically and with regards to applications.

The history of water waves is a long one and we will only be able to scratch the surface here. A very special kind of water waves are what we call solitons, localized solitary waves propagating with a constant velocity which can cross each other and emerge from the collision unchanged. In [18], John Scott Russell describes what he named the *Wave of Translation* in 1834 on the Union Canal near Edinburgh, Scottland:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on a horseback, and overtook it still rolling on at a rate of some eight or nine miles and hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance overview with that singular and beautiful phenomenon which I have called the Wave of Translation. "

The theory at the time, which was linear in character, could not describe such a phenomenon, and the discovery was met with skepticism. However, in the 1870s, Joseph Boussinesq, along with Lord Rayleigh, developed theory that did allow for Russell's discovery. In 1877 in [6], Boussinesq introduced the shallow-water wave equation, which we today know as the Korteweg-de Vries (KdV) equation:

$$u_t + uu_x + u_{xxx} = 0.$$

Diederik Korteweg and Gustav de Vries re-derived the equation in 1895 [13], whence the name. This equation, as opposed to earlier water wave theory, does indeed admit soliton solutions. However, the problem that solutions admitted by the KdV equation may not break, which is certainly a natural phenomenon for water waves, led to the introduction of the more general model known as the Whitham equation, named after Gerald Whitham. It takes the form

$$u_t + uu_x + (Lu)_x,$$

where L is an operator defined through the Fourier transform as $\widehat{Lf}(\xi) = \left(\frac{tanh\xi}{\xi}\right)^{1/2}\widehat{f}(\xi)$. It can be shown that in the limit $\xi \to 0^+$, we re-discover the KdV equation as an approximation. It can also be shown that a solution of the Whitham equation will, as opposed to the KdV equation, break if the slope of the initial profile is sufficiently large and negative at some point. Generalizing even further we obtain the equation

$$u_t + (n(u) - Lu)_x = 0,$$

which includes, for a generalization of the operator we are studying in this thesis, the Whitham equation with capillary effects and the generalized KdV equation.

In this thesis we apply the calculus of variations in order to prove the existence of solitary-wave solutions to the PDE under study. It is a method that deals with finding maxima or minima of functionals, operators that map from a function space to the space of real numbers. The interest lies in what is commonly referred to as extremal functions, functions yielding a zero rate of change of the functional under study. We might say that the calculus of variations started with the introduction of the brachistochrone curve problem raised by Johann Bernoulli in 1696 [5]. He introduced the problem as follows:

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The problem he posed was the following:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Johann Bernoulli was not the first to consider the brachistocrone problem. In 1638, Galileo had studied the problem in his famous work *Discourse on two new sciences* [10]. Galileo correctly deduced that the shortest path was not a straight line, but instead that an object would follow a curved path in order to minimize the time used to reach point B. However, he then stated that the arc would have the shape of a circle; an incorrect conclusion. Besides from Johann Bernoulli's solution, Newton, Jacob Bernoulli (Johann's brother), Leibniz, and de L'hopital also solved the problem. The May 1697 publication of Acta Eruditorium contained Leibniz's solution on page 205, Johann Bernoulli's solution on pages 206 to 211, Jacob Bernoulli's solution on pages 211 to 214, and a latin translation of Newton's solution on page 223. Somewhat surprisingly, the solution of L'Hopital was not published until nearly 300 years later in an appendix in [12].

The subject was first elaborated by Leohnard Euler, his contribution beginning in 1733. His elementa Calculi Variationum gave the science its name. Almost every famous mathematician has at some point dedicated some time to the field, for instance did Lagrange also contribute extensively during the 17th century. Perhaps the most important contributions of the 18th century is that of Weierstrass, and it may be asserted that he was the first to place the area on a firm and unquestionable foundation. In the 20th century, David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard amongst others made significant contributions. For instance did Hilbert and Zaremba introduce the method of direct variations around 1900, which is a general method of constructing a proof of the existence of a minimizer for a given functional. The method used in this thesis is heavily inspired by this.

Similar to finding maxima or minima of a function, one can find extrema of functionals where the functional derivative is equal to zero. Consider the functional

$$J(y) = \int_{x_1}^{x_2} L(x, y(x), y'(x))\, \mathrm{d}x,$$

where $x_1, x_2 \in \mathbb{R}$, $y(x)$ is a twice continuously differentiable function and $L(x, y(x), y'(x))$ is twice continuously differentiable with respect to its arguments. If u is a minima for $J(\cdot)$, then for all functions η, we have $J(u) \leq J(u + \epsilon\eta)$ for all $\epsilon \in \mathbb{R}$. By taking the derivative with respect to ϵ of $J(u + \epsilon\eta)$, evaluating it at $\epsilon = 0$ and setting it equal to zero, it can easily be shown that we arrive at the famous Euler-Lagrange equation:

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'} = 0.$$

However, the existence of such a solution must be established beforehand, a task that is often the most challenging part. The calculus of variations can be applied to many different settings besides from the brachistochrone problem such as isoperimetric problems, geodesics on surfaces, Plateau's problem and optimal control.

Background theory

The thesis starts by introducing the classical Sobolev spaces and the concept of weak derivatives. We also show that the classical Sobolev spaces are complete normed spaces (Banach spaces). We continue by introducing the space of distributions, the dual space of all compactly supported infinitely differentiable functions. Proving some basic results along the way helps us to better understand the complete theory of distributions, and, in particular, how distributions and functions are related. We introduce the Schwartz space and define the Fourier transform on it. We extend the definition to all square integrable functions and show that the Fourier transform is well defined on $L_2(\mathbb{R})$. Plancherel's identity is one of the most frequently used results in this paper and for this reason we give an original proof of this result . Through the Fourier transform we are also able to extend the classical Sobolev spaces to fractional Sobolev spaces of nonnegative order s. We also give a proof of the fact that functions in Sobolev spaces are bounded and continuous for sufficiently large values of s.

We go on by introducing the concept of weak convergence and prove weak lower semi-continuity of the Hilbert norm. An important result in our analysis will be weak convergence for some subsequence of bounded minimizing sequences, and for this purpose we present Banach-Alaoglu's theorem. The two most used theorems in this paper are the Sobolev embedding theorem and the Sobolev interpolation theorem, and for this reason we introduce them as well. The Sobolev embedding theorem ensures us that the functions we are working with in some fractional Sobolev space are also members of an appropriate Lebesgue space, under certain conditions. As we are working on unbounded domains, the Sobolev embedding theorem does not yield a compact embedding, and hence does not necessarily produce convergent subsequences. The main ingredient in dealing with this problem is the concentration compactness principle, whose credit is due to Pierre-Louis Lions [16].

The problem at hand

After having established all the theory needed, we approach a nonlinear equation of the form $u_t + (n(u) - Lu)_x = 0$. Here n consists of two parts, n_p and n_r. We have that $n_p(u) = c_p|u|^p$ with $c_p \neq 0$ or $n_p(u) = c_p u|u|^{p-1}$ with $c_p > 0$, and $n_r(u) = \mathcal{O}(|u|^{p+1+\gamma})$ for some $\gamma > 0$. The operator L is a Fourier multiplier operator defined through $\widehat{Lu} = \left(1 + |\xi|^2\right)^{s/2} \widehat{u}$. In work done by Arnesen [4] and Albert [1], only nonlinearities of the form $n(u) = n_p(u)$ have been studied previously. However, in [9], inhomogeneous nonlinearities including a higher order term is considered, but in that case the operator L is a smoothing operator. Besides, only small solutions are studied, whereas in this paper we find solutions without restrictions on the $L_2(\mathbb{R})$ norm. We break the proof down into a sequence of lemmas in order to get an appropriate overview of the proof. The concentration compactness principle states that either vanishing, dichotomy or compactness occurs for a subsequence of a sequence $\{\rho\}_n \subset L_1(\mathbb{R})$ that satisfies $\rho_n \geq 0$ with $\int_{\mathbb{R}} \rho \, \mathrm{d}x = \mu$ for a fixed $\mu > 0$ for all n. We prove that the minimal value of a translation of our functional is bounded from below and is less than zero. This is an important result in order to prove the subadditivity property of our functional, which again is needed to exclude dichotomy. Any minimizer of this translated functional will also be a minimizer to our original functional, which is trivially seen. With help from our lemmas, we use standard arguments to show that vanishing does not occur. Having excluded vanishing and dichotomy, we know from the concentration compactness principle that compactness occurs. We use this property to prove the existence of a minimizer to our minimization problem. We end the proof by showing that such a minimizer actually solves our equation (in the sense of distributions). Lastly we argue that solutions of our

PDE in fact inherit regularity from the equation itself. That is, we show that any solution u belonging to $H^{s/2}(\mathbb{R})$, also belongs to $H^{s}(\mathbb{R})$. By iteration, any solution found will be in $H^{\infty}(\mathbb{R})$.

2 Background theory

This chapter presents the theory needed in order to fully understand the content in the next chapter. We start by defining the weak derivative and the classical Sobolev spaces, and show that these are complete normed spaces. We go on by introducing the space of distributions and give some basic results for this space and the space of test functions. Furthermore, we define the Fourier transform on the Schwartz space and extend it to the space of Lebesgue p-integrable functions. The dual space of the Schwartz space, the space of tempered distributions, is also given some attention. Plancherel's identity is a frequently used result in this paper, and for this reason we give an original proof of this result. Through the Fourier transform we are able to define the fractional Sobolev spaces of nonnegative order s. We present and prove an embedding from the fractional Sobolev spaces into the space of bounded and continuous functions. Two of the most important theorems in this paper are then introduced; the Sobolev embedding theorem and the Sobolev interpolation theorem. Weak lower semi-continuity of the Hilbert norm is crucial in the final stages of our existence proof, and is therefore presented and proven. The theorem of Banach and Alaoglu is also given some attention due to its importance. Lastly, we introduce the principle of concentration compactness; the main ingredient in proving that compactness occurs.

Note that even though all the elaborated proofs are done by the author, they are pretty standard, and can be found in the literature (see for instance [11]). The exception is the proof of Plancherel's identity, which is original.

2.1 The space of test functions, $\mathscr{D}(\mathbb{R})$

We start this chapter by introducing the space of compactly supported smooth functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 2.1. (Compactly supported smooth functions).

$$\mathscr{D}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) : \ \mathrm{supp}(\phi) \text{ is compact in } \mathbb{R}\}. \tag{2.1}$$

Remark 2.1. The elements in this space are sometimes referred to as test functions.

One can also define the compactly supported smooth functions from a domain $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ similarly by changing $\mathbb{R}$ with Ω in (2.1). We need a way to define convergence in this space, and define it as follows:

Definition 2.2. (Convergence in $\mathscr{D}(\mathbb{R})$). We say that ϕ_j converges to ϕ in $\mathscr{D}(\mathbb{R})$, written $\phi_j \xrightarrow{\mathscr{D}} \phi$, if all the derivatives converge uniformly, that is,

$$\|\phi_j - \phi\|_{C^m(\mathbb{R})} \to 0 \quad j \to \infty \text{ for all } m \in \mathbb{N}, \tag{2.2}$$

and if there exists a compact domain $K \subset \mathbb{R}$ such that $\mathrm{supp}(\phi_j) \subset K$ for all $n \in \mathbb{N}$.

2.2 The weak derivative

We want to introduce the weak derivative for later use, but in order for it to make sense, we first need to define the locally p-integrable functions from $\mathbb{R}$ to $\mathbb{C}$.

Definition 2.3. ($L_p^{loc}(\mathbb{R})$ spaces). A function $f : \mathbb{R} \mapsto \mathbb{C}$ is in $L_p^{loc}(\mathbb{R})$ if for every compact subset $K \subset \mathbb{R}$,

$$\int_K |f|^p \, \mathrm{d}x < \infty.$$

Remark 2.2. Note that $L_p(\mathbb{R}) \subset L_p^{loc}(\mathbb{R})$.

For such functions we can define the weak derivative:

Definition 2.4. (Weak derivative of order α). We call v the weak derivative of order α of f, written $D^\alpha f$, if for every $\phi \in \mathscr{D}(\mathbb{R})$, we have:

$$\int_{\mathbb{R}} \phi v \, \mathrm{d}x = (-1)^\alpha \int_{\mathbb{R}} f D^\alpha \phi \, \mathrm{d}x. \tag{2.3}$$

Remark 2.3. In this paper we will only perform analysis in one dimension, but it is possible to extend most of the results and definitions here to higher dimensions $d > 1, d \in \mathbb{N}$. (α would then be introduced as a d-dimensional multi-index).

When the classical derivative of order α of f exists it coincides with the weak derivative. This justifies the way we define the weak derivative since, if v is sufficiently smooth, we can use integration by parts to move the derivatives over to ϕ and obtain (2.3). Also, we note at this point that the weak derivative is unique up to a set of measure zero.

2.3 The classical Sobolev spaces

We now have the necessary tools to define the classical Sobolev spaces.

Definition 2.5. (The classical Sobolev spaces). Let $k \in \mathbb{N}$. Then the spaces

$$W_p^k(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : D^\alpha f \in L_p(\mathbb{R}) \text{ for all } \alpha \in \mathbb{N}, \alpha \leq k\}$$

are what we call the classical Sobolev spaces.

The following theorem shows that the classical Sobolev spaces are Banach spaces:

Theorem 2.1. *(Completeness of $W_p^k(\mathbb{R})$). The classical Sobolev spaces become Banach spaces when equipped with the norm*

$$\|f\|_{W_p^k(\mathbb{R})} = \left(\sum_{\alpha \leq k} \|D^\alpha f\|_{L_p(\mathbb{R})}^p\right)^{\frac{1}{p}}. \tag{2.4}$$

Proof. First we prove that $\|\cdot\|_{W_p^k(\mathbb{R})}$ is indeed a norm and hence generates a normed space. We follow up with the proof of completeness. $\|\cdot\|_{W_p^k(\mathbb{R})}$ inherits the property $\|x\| = 0$ if and only if $x = 0$ almost everywhere from $\|\cdot\|_{L_p(\mathbb{R})}$. $\|\gamma x\|_{W_p^k(\mathbb{R})} = |\gamma|\|x\|_{W_p^k(\mathbb{R})}$, $\gamma \in \mathbb{C}$ also follows from this. Lastly, we need to show the triangle inequality. We raise both sides of (2.4) to the power p and get:

$$\begin{aligned}\|f+g\|^p_{W^k_p(\mathbb{R})} &= \sum_{\alpha\le k}\|D^\alpha(f+g)\|^p_{L_p(\mathbb{R})}\\ &= \sum_{\alpha\le k}\|D^\alpha f + D^\alpha g\|^p_{L_p(\mathbb{R})}\\ &\le \sum_{\alpha\le k}\|D^\alpha f\|^p_{L_p(\mathbb{R})} + \|D^\alpha g\|^p_{L_p(\mathbb{R})}\\ &= \|f\|^p_{W^k_p(\mathbb{R})} + \|g\|^p_{W^k_p(\mathbb{R})}\\ &\le \left(\|f\|_{W^k_p(\mathbb{R})} + \|g\|_{W^k_p(\mathbb{R})}\right)^p\end{aligned}$$

which completes the proof that $\|\cdot\|_{W^k_p(\mathbb{R})}$ is a norm. We have used the triangle inequality for $L_p(\mathbb{R})$ spaces in the second transition. The last inequality holds simply since norm evaluations are nonnegative. Next we prove completeness: Let $\{f_n\}_n$ be Cauchy in $W^k_p(\mathbb{R})$. This implies that $\{f_n\}_n$ is Cauchy in $L_p(\mathbb{R})$. By the completeness of the $L_p(\mathbb{R})$ spaces we know that $\{f_n\}_n$ attains its limit in $L_p(\mathbb{R})$, i.e. $f_n \to f,\ f \in L_p(\mathbb{R})$. Moreover, due to (2.4), we also have that $D^\alpha f_n$ is Cauchy in $L_p(\mathbb{R})$. In other words, there is a $f^\alpha \in L_p(\mathbb{R})$ such that $D^\alpha f_n \to f^\alpha$. It remains to prove that $D^\alpha f \in L_p(\mathbb{R})$, and it will be sufficient to show $f^\alpha = D^\alpha f$. To deduce this, choose $\phi \in \mathscr{D}(\mathbb{R})$ and let q be such that $\frac{1}{p}+\frac{1}{q}=1$. By Hölder's inequality we then get

$$\begin{aligned}&\left|\int_{\mathbb{R}}(f_n-f)D^\alpha\phi\ \mathrm{d}x\right|\\ &\le\left(\int_{\mathbb{R}}|f_n-f|^p\ \mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}}|D^\alpha\phi|^q\ \mathrm{d}x\right)^{\frac{1}{q}}\\ &=\|f_n-f\|_{L_p(\mathbb{R})}\|D^\alpha\phi\|_{L_q(\mathbb{R})}\to 0,\end{aligned}\tag{2.5}$$

since $f_n \to f$ in $L_p(\mathbb{R})$ and $\|D^\alpha\phi\|_{L_q(\mathbb{R})}$ is bounded due to $\phi \in \mathscr{D}(\mathbb{R})$. Furthermore, we have by similar arguments that

$$\begin{aligned}&\left|\int_{\mathbb{R}} (D^\alpha f_n - f^\alpha)\,\phi \, \mathrm{d}x\right| \\ &\leq \left(\int_{\mathbb{R}} |D^\alpha f_n - f^\alpha|^p \, \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |\phi|^q \, \mathrm{d}x\right)^{\frac{1}{q}} \to 0,\end{aligned} \tag{2.6}$$

as a consequence of $D^\alpha f_n \to f^\alpha$ in $L_p(\mathbb{R})$. This is precisely what we need, as we get by recalling the definition of the weak derivative in (2.3), that:

$$\begin{aligned}\int_{\mathbb{R}} f D^\alpha \phi \, \mathrm{d}x &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n (D^\alpha \phi) \, \mathrm{d}x \\ &= (-1)^\alpha \lim_{n\to\infty} \int_{\mathbb{R}} (D^\alpha f_n)\phi \, \mathrm{d}x \\ &= (-1)^\alpha \int_{\mathbb{R}} (f^\alpha)\phi \, \mathrm{d}x,\end{aligned}$$

implying that $D^\alpha f = f^\alpha$, which is what we wanted to show. We conclude that the classical Sobolev spaces equipped with the norm in (2.4) are Banach spaces.

The case when $p = 2$ is particularly interesting since the space $W_2^k(\mathbb{R})$, when equipped with an appropriate inner product, inherits the Hilbert space structure from $L_2(\mathbb{R})$. Defining the Sobolev spaces in the classical way, as we have done, only makes sense when $k \in \mathbb{N}$. In later chapters we work in fractional Sobolev spaces with real $k \geq 0$, demanding their own definition and attention. The introduction of the Fourier transform is vital in order to define and understand these spaces. We look into these matters further in section 2.9.

2.4 The dual space of $\mathscr{D}(\mathbb{R})$ and some basic results

Next we will introduce an important space in PDE theory, namely the dual space of $\mathscr{D}(\mathbb{R})$. This space is usually referred to as the space of distributions. We define the space as follows:

Definition 2.6. (The space of distributions, $\mathscr{D}'(\mathbb{R})$). We define $\mathscr{D}'(\mathbb{R})$ as the collection of all complex valued linear continuous functionals T over $\mathscr{D}(\mathbb{R})$, meaning,

$$T : \mathscr{D}(\mathbb{R}) \to \mathbb{C}, \quad T : \phi \mapsto T(\phi), \quad \phi \in \mathscr{D}(\mathbb{R}),$$

$$T(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 T(\phi_1) + \alpha_2 T(\phi_2) \quad \alpha_1, \alpha_2 \in \mathbb{C}, \quad \phi_1, \phi_2 \in \mathscr{D}(\mathbb{R}),$$

and

$$T(\phi_j) \to T(\phi) \text{ whenever } \phi_j \xrightarrow{\mathscr{D}} \phi \text{ as } j \to \infty.$$

Remark 2.4. The elements $T \in \mathscr{D}'(\mathbb{R})$ are called distributions.

We have for two distributions, T_1 and T_2, that they are equal if $T_1(\phi) = T_2(\phi)$ for all $\phi \in \mathscr{D}(\mathbb{R})$. Furthermore, we furnish $\mathscr{D}'(\mathbb{R})$ with the so called simple convergence topology. That is,

$$T_j \to T \quad \text{in } \mathscr{D}'(\mathbb{R}),\ T_j \in \mathscr{D}'(\mathbb{R}),\ j \in \mathbb{N},\ T \in \mathscr{D}'(\mathbb{R}),$$

means that

$$T_j(\phi) \to T(\phi) \quad \text{in } \mathbb{C} \text{ as } j \to \infty \text{ for any } \phi \in \mathscr{D}(\mathbb{R}).$$

At this point we make a short introduction to equivalence classes, where we assume the reader to be familiar with some basic measure theory. An important thing to emphasise is that the elements $L_p(\mathbb{R})$ spaces are not strictly speaking functions, but they are what we call equivalence classes. The difference between two functions in an equivalence class is contained in a set of measure zero, i.e.

$$[f] = \{g : |\{x \in \Omega : f(x) \neq g(x)\}| = 0\}.$$

We will search for solutions in a Sobolev space also consisting of equivalence classes. In words this means that we are searching for any representative in an equivalence class satisfying our equation. Throughout this paper we will usually refer to the elements as functions, although they are strictly speaking equivalence classes.

The results in this chapter is included mostly to gain understanding of the relationship between functions and distributions. A function $f \in L_p(\mathbb{R})$ can always be identified with a distribution, but a distribution $T \in \mathscr{D}'(\mathbb{R})$ may not always be identified with a function. In the next proposition, a somewhat surprising relationship between $\mathscr{D}(\Omega)$ and $L_p(\Omega)$ is addressed.

Proposition 2.1.

1. *Let Ω be an arbitrary domain in $\mathbb{R}$. Then the space $\mathscr{D}(\Omega)$ is dense in $L_p(\Omega)$ for $1 \leq p < \infty$.*

2. *Let $f \in L_1^{loc}(\Omega)$. If*

$$\int_\Omega f(x)\phi(x)\,\mathrm{d}x = 0 \quad \textit{for all } \phi \in \mathscr{D}(\Omega),$$

then $[f] = 0$.

Proof. Consult [11, p. 28-30] for a thorough proof of the above.

Definition 2.7. (Regular distributions). A distribution $T \in \mathscr{D}'(\Omega)$ is said to be *regular* if there exists an $f \in L_1^{loc}(\Omega)$ such that $T = T_f$, where

$$T_f(\phi) = \int_\Omega f(x)\phi(x)\,\mathrm{d}x \ \text{ for all } \phi \in \mathscr{D}(\Omega). \tag{2.7}$$

Remark 2.5. For some $g \in [f] \in L_1^{loc}(\Omega)$ it follows directly that $T_f = T_g$, since f and g only differ on a set of measure zero.

Remark 2.6. We see that for any f satisfying (2.7), there exists a unique corresponding distribution $T_f \in \mathscr{D}'(\mathbb{R})$. In this sense, any such f can be viewed as a distribution as well as a function.

There also exists nonregular distributions, an example being the Dirac delta distribution.

Proposition 2.2. *The Dirac delta distribution, defined as $\delta_a\phi = \phi(a)$, is a non-regular distribution.*

Proof. We check that δ_a fulfills the requirements of a distribution. We have

$$\delta_a : \mathscr{D}(\Omega) \to \mathbb{C} \quad \text{since } \phi(a) \in \mathbb{C},$$

$$\delta_a(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1\phi_1(a) + \alpha_2\phi_2(a) = \alpha_1\delta_a\phi_1 + \alpha_2\delta_a\phi_2,$$

and

$$\delta_a(\phi_j) \to \delta_a(\phi) \quad \text{as } j \to \infty \text{ since } \phi_j(a) \to \phi(a) \text{ whenever } \phi_j \xrightarrow{\mathscr{D}} \phi.$$

So the Dirac delta is indeed a distribution, but it cannot be regular, since if $\delta_a = \phi(a) = 0$, then that would give us

$$\int_\Omega f(x)\phi(x)dx = 0 \quad \text{for all} \quad \phi \in \mathscr{D}(\Omega),$$

yielding $f = 0$. We conclude that, in this case, Definition 2.7 cannot be satisfied for any nontrivial f.

Remark 2.7. Nonregular distributions are frequently referred to as singular distributions.

2.5 The Schwartz space and the Fourier transform

After having introduced the space $\mathscr{D}(\mathbb{R})$ and the space of distributions, $\mathscr{D}'(\mathbb{R})$, we are now looking for appropriate spaces on which to define the Fourier transform. The Fourier transform is one of the most powerful instruments in the theory of distributions and function spaces. For the purpose of the Fourier transform, $\mathscr{D}(\Omega)$ is too small, and $\mathscr{D}'(\Omega)$ is simply too large. When asking for something appropriate in between one arrives at $\mathscr{S}(\mathbb{R})$ and its dual $\mathscr{S}'(\mathbb{R})$.

Definition 2.8. (Schwartz space). We define the following space as the Schwartz space:

$$\mathscr{S}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) : \|\phi\|_{k,l} < \infty \text{ for all } k \in \mathbb{N}_0, l \in \mathbb{N}_0\},$$

where

$$\|\phi\|_{k,l} = \sup_{x\in\mathbb{R}} \left(1 + |x|^2\right)^{\frac{k}{2}} \sum_{|\alpha|\le l} |D^\alpha \phi(x)|.$$

Convergence in this space is defined as follows: A sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}(\mathbb{R})$ is said to converge in $\mathscr{S}(\mathbb{R})$ to $\phi \in \mathscr{S}(\mathbb{R})$, written $\phi_j \xrightarrow{\mathscr{S}} \phi$, if

$$\|\phi_j - \phi\|_{k,l} \to 0 \quad \text{as } \; j \to \infty \text{ for all } k \in \mathbb{N}_0, l \in \mathbb{N}_0.$$

Remark 2.8. Elements of $\mathscr{S}(\mathbb{R})$ are often referred to as *rapidly decreasing functions*. The name can be justified by noticing that in the case when $l = 0$, we have $|\phi(x)| \le c_k(1 + |x|^k)^{-1}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Similarly for all derivatives $D^\alpha \phi(x)$, $\alpha \in \mathbb{N}$.

Notice that by the definition of $\mathscr{D}(\mathbb{R})$ we have that $\mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$, and that $\phi_j \xrightarrow{\mathscr{D}} \phi$ implies $\phi_j \xrightarrow{\mathscr{S}} \phi$. However, there are functions in $\mathscr{S}(\mathbb{R})$ which do not belong to $\mathscr{D}(\mathbb{R})$, the most famous example being $\phi(x) = e^{-x^2}, x \in \mathbb{R}$.

We are now ready to define the one-dimensional Fourier transform on the Schwartz space.

Definition 2.9. Let $f \in \mathscr{S}(\mathbb{R})$. We define the Fourier transform of $f, \mathscr{F}\{f\}(\xi)$, as:

$$\mathscr{F}\{f\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \mathrm{d}x \quad \xi \in \mathbb{R}. \tag{2.8}$$

Similarly we define the *inverse Fourier transform* for $f \in \mathscr{S}(\mathbb{R})$ as:

$$\mathscr{F}^{-1}\{f\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \,\mathrm{d}x \quad \xi \in \mathbb{R}. \tag{2.9}$$

Remark 2.9. Since $f \in \mathscr{S}(\mathbb{R})$, both (2.8) and (2.9) make sense.

Remark 2.10. Note also that $\mathscr{F}^{-1}\{f\}(\cdot) = \mathscr{F}\{f\}(-\cdot)$.

In this thesis we often write $\widehat{f}(\xi)$ instead of $\mathscr{F}\{f\}(\xi)$. From these definitions we can also state and prove the following theorem:

Theorem 2.2. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Then $\mathscr{F}\phi \in \mathscr{S}(\mathbb{R})$ and $\mathscr{F}^{-1}\phi \in \mathscr{S}(\mathbb{R})$. Furthermore, $x^\alpha \phi \in \mathscr{S}(\mathbb{R})$ and $D^\alpha \phi \in \mathscr{S}(\mathbb{R})$ for $\alpha \in \mathbb{N}_0$. Also,*

$$D^\alpha(\mathscr{F}\phi)(\xi) = (-i)^\alpha \mathscr{F}(x^\alpha \phi(x))(\xi), \quad \alpha \in \mathbb{N}_0, \xi \in \mathbb{R}, \tag{2.10}$$

and

$$\xi^\alpha(\mathscr{F}\phi)(\xi) = (-i)^\alpha \mathscr{F}(D^\alpha \phi)(\xi), \quad \alpha \in \mathbb{N}_0, \xi \in \mathbb{R}. \tag{2.11}$$

Proof. $x^\alpha \phi \in \mathscr{S}(\mathbb{R})$ and $D^\alpha \phi \in \mathscr{S}(\mathbb{R})$ follows immediately from Definition 2.8. Hence both (2.10) and (2.11) make sense. By Lebesgue's dominated convergence theorem and the mean value theorem, we also have

$$\frac{d}{d\xi}(\mathscr{F}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-ix) e^{-ix\xi} \phi(x) \,\mathrm{d}x,$$

which by iteration yields (2.10). As for (2.11), notice that

$$\mathscr{F}(\frac{d}{dx}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \frac{d}{dx}\phi(x) \,\mathrm{d}x = i\xi(\mathscr{F}\phi)(\xi).$$

Since $\phi \in \mathscr{S}(\mathbb{R})$, iterated integration by parts gives (2.11). This completes the proof.

The Fourier inversion theorem

This section aims to justify the way we defined the Fourier transform and its inverse on $\mathscr{S}(\mathbb{R})$. The following theorem is vital for us in order to carry out our analysis in later sections:

Theorem 2.3. *Let $\phi \in \mathscr{S}(\mathbb{R})$. Then*

$$\phi = \mathscr{F}^{-1}\mathscr{F}\phi = \mathscr{F}\mathscr{F}^{-1}\phi.$$

Furthermore, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ map $\mathscr{S}(\mathbb{R})$ one-to-one onto itself,

$$\mathscr{F}\mathscr{S}(\mathbb{R}) = \mathscr{S}(\mathbb{R}) \text{ and } \mathscr{F}^{-1}\mathscr{S}(\mathbb{R}) = \mathscr{S}(\mathbb{R}).$$

Proof. Proof of this can be found in [11, p. 42-43].

2.6 The space of tempered distributions, $\mathscr{S}'(\mathbb{R})$

Having already introduced the space of all linear continuous functionals over $\mathscr{D}(\mathbb{R})$, namely $\mathscr{D}'(\mathbb{R})$, we shall now do the same for $\mathscr{S}(\mathbb{R})$.

Definition 2.10. Let $\mathscr{S}(\mathbb{R})$ be as in Definition 2.8. Then $\mathscr{S}'(\mathbb{R})$ is the collection of all complex valued linear continuous functionals T over $\mathscr{S}(\mathbb{R})$:

$$T : \mathscr{S}(\mathbb{R}) \to \mathbb{C}, \quad T : \phi \mapsto T(\phi), \quad \phi \in \mathscr{S}(\mathbb{R}),$$

$$T(\lambda_1\phi_1 + \lambda_2\phi_2) = \lambda_1 T(\phi_1) + \lambda_2 T(\phi_2), \quad \lambda_1, \lambda_2 \in \mathbb{C};\ \phi_1, \phi_2 \in \mathscr{S}(\mathbb{R}),$$

and

$$T(\phi_j) \to T(\phi) \quad \text{for } j \to \infty \text{ whenever } \phi_j \xrightarrow{\mathscr{S}(\mathbb{R})} \phi.$$

Remark 2.11. The elements in $\mathscr{S}'(\mathbb{R})$ are called *tempered distributions* or *slowly increasing distributions*.

We look at $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ as a dual pairing of locally convex spaces. We have that

$$T_1 = T_2 \quad \text{in } \mathscr{S}'(\mathbb{R}) \;\text{ means that } T_1(\phi) = T_2(\phi) \text{ for all } \phi \in \mathscr{S}(\mathbb{R}).$$

As with $\mathscr{D}'(\mathbb{R})$, it is sufficient for us to furnish $\mathscr{S}'(\mathbb{R})$ with simple convergence topology:

$$T_j \to T \quad \text{in } \mathscr{S}'(\mathbb{R}), T_j \in \mathscr{S}'(\mathbb{R}),\ j \in \mathbb{N},\ T \in \mathscr{S}'(\mathbb{R}),$$

means that

$$T_j(\phi) \to T(\phi) \quad \text{in } \mathbb{C} \text{ if } j \to \infty \text{ for any } \phi \in \mathscr{S}(\mathbb{R}).$$

At this point it is natural to pose the question; which $f \in L_1^{loc}(\mathbb{R})$ generates a regular distribution that is also a tempered distribution? The answer is provided in the following proposition:

Proposition 2.3. *Let $1 \leq p \leq \infty$. Then*

$$L_p(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}) \tag{2.12}$$

in the interpretation

$$T_f(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x, \quad \phi \in \mathscr{S}(\mathbb{R}).$$

Proof. Let q be so that $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder's inequality, since $\phi \in \mathscr{S}(\mathbb{R})$, we have

$$\left| \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x \right| \leq \|f\|_{L_p(\mathbb{R})}\|\phi\|_{L_q(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R})}\|\phi\|_{k,0} \tag{2.13}$$

for some $k \in \mathbb{N} \geq k(p,n)$. This yields (2.12).

Remark 2.12. Note that (2.13) holds for any $\phi \in \mathscr{S}(\mathbb{R})$. This means that for any $f \in L_p(\mathbb{R})$, there exists a unique $T = T_f \in \mathscr{S}'(\mathbb{R})$ such that f can be identified with T_f. This is what we mean when we say that $L_p(\mathbb{R})$ is a subspace of $\mathscr{S}'(\mathbb{R})$. However, it is important to point out that the nature of the elements residing in these spaces is different.

2.7 The Fourier transform in $\mathscr{S}'(\mathbb{R})$ and in $L_p(\mathbb{R})$

We have already introduced the Fourier transform on $\mathscr{S}(\mathbb{R})$ and we now wish to extend it to $\mathscr{S}'(\mathbb{R})$. In the proper sense, by (2.12), one can consider $\mathscr{S}(\mathbb{R})$ as a subset of $\mathscr{S}'(\mathbb{R})$.

Definition 2.11. Let $T \in \mathscr{S}'(\mathbb{R})$. Then the Fourier transform $\mathscr{F}T$ and the inverse Fourier transform $\mathscr{F}^{-1}T$ are given by

$$(\mathscr{F}T)(\phi) = T(\mathscr{F}\phi) \quad \text{and} \quad (\mathscr{F}^{-1}T)(\phi) = T(\mathscr{F}^{-1}\phi), \quad \phi \in \mathscr{S}(\mathbb{R}). \tag{2.14}$$

Remark 2.13. We will not show it here, but it can be proven that $\mathscr{F}T \in \mathscr{S}'(\mathbb{R})$, and similarly $\mathscr{F}^{-1}T \in \mathscr{S}'(\mathbb{R})$, whenever $T \in \mathscr{S}'(\mathbb{R})$. From this we deduce that $\mathscr{F}$ and $\mathscr{F}^{-1}$ extend the Fourier transform and its inverse from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}'(\mathbb{R})$, respectively. Consult [11, p.47-48] for more discussion related to these matters.

It can also be shown that theorem 2.3 holds for $\mathscr{S}'(\mathbb{R})$ replaced with $\mathscr{S}(\mathbb{R})$. We now wish to define the Fourier transform on the space of Lebesgue p-integrable functions. By recalling (2.12), we have that any $f \in L_p(\mathbb{R})$ can be interpreted as a regular distribution belonging to $\mathscr{S}'(\mathbb{R})$. Consequently we also have $\mathscr{F}f \in \mathscr{S}'(\mathbb{R})$. The question regarding regularity of the distribution, however, remains to be answered.

Theorem 2.4. *Let $n \in \mathbb{N}$. We then have that:*

1. *For $f \in L_p(\mathbb{R})$ with $1 \leq p \leq 2$, $\mathscr{F}f \in \mathscr{S}'(\mathbb{R})$ is regular.*
2. *If $f \in L_1(\mathbb{R})$, then*

$$(\mathscr{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,\mathrm{d}x, \quad \textit{for all } f \in L_1(\mathbb{R}).$$

3. *The restrictions of $\mathscr{F}$ and $\mathscr{F}^{-1}$, respectively, to $L_2(\mathbb{R})$, generate unitary operators in $L_2(\mathbb{R})$. Furthermore*

$$\mathscr{F}\mathscr{F}^{-1} = \mathscr{F}^{-1}\mathscr{F} = id \quad (\textit{identity in } L_2(\mathbb{R})).$$

Proof. One may consult [11] for a proof of the above. We will address the first part of the third point in the next section.

Remark 2.14. It is worth pointing out that for $2 < p \leq \infty$, there does exist functions $f \in L_p(\mathbb{R})$ such that $\mathscr{F} f$ is not regular. The simplest case is when $p = \infty$. The Fourier transform of a constant function $f(x) = c \neq 0$ equals $c'\delta$ (see [11]) with $c' \neq 0$. But according to Proposition 2.2, δ is not a regular distribution, and hence neither is the Fourier transform of the constant function.

2.8 Plancherel's identity

The following relation is one of the most used results in this paper, and we therefore give an original proof of the result.

Proposition 2.4. *The Fourier transform is an isometry on $L_2(\mathbb{R})$: Let $f, g \in L_2(\mathbb{R})$. We then have*

$$\langle f, g \rangle_{L_2(\mathbb{R})} = \langle \mathscr{F}\{f\}, \mathscr{F}\{g\} \rangle_{L_2(\mathbb{R})}.$$

Remark 2.15. This is often referred to as Plancherel's identity. Also, the result obviously holds for the inverse Fourier transform as well.

In order to prove this we need some additional results.

Proposition 2.5. *We have the following integral relation:*

$$\int_{-\infty}^{\infty} \frac{\sin(Nx)}{x} \, \mathrm{d}x = \operatorname{sgn}(N)\pi \quad N \in \mathbb{R}. \tag{2.15}$$

Proof. Let $f = \chi_{(-N,N)} \in L_2(\mathbb{R})$, where $\chi_{(-N,N)}$ is the characteristic function. The Fourier transform of f is

$$\mathscr{F}\{f\}(\xi) = \int_{-\infty}^{\infty} \chi_{(-N,N)}(x) e^{-i\xi x} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-i\xi x} \, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \frac{\sin(N\xi)}{\xi}.$$

We have by the Fourier inversion theorem that

$$f(x) = \mathscr{F}^{-1}\{\mathscr{F}\{f(x)\}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin(N\xi)}{\xi} e^{i\xi x} \, \mathrm{d}\xi,$$

which gives us

$$f(0) = 1 = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\sin(N\xi)}{\xi}\,\mathrm{d}\xi. \tag{2.16}$$

Due to the odd nature of the sine function, we have that the value of the integral in (2.16) will depend on the sign of N. By introducing the sign function the proof is complete.

Proposition 2.6. *(Riemann-Lebesgue). Let $f \in L_1(\mathbb{R})$. Then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)e^{inx}dx = 0 \quad n \in \mathbb{Z}.$$

Proof. Assume first that f is a compactly supported smooth function. We then have by integration by parts that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)e^{inx}dx = \lim_{n\to\infty}\left(\frac{1}{in}f(x)e^{inx}\Big|_{-\infty}^{\infty}\right) - \lim_{n\to\infty}\left(\frac{1}{in}\int_{-\infty}^{\infty} f'(x)e^{inx}dx\right) \to 0.$$

Since $f \in L_1(\mathbb{R})$, due to Proposition 2.1, it may be approximated in the $L_1(\mathbb{R})$ norm by a compactly supported smooth function. The result then follows.

Remark 2.16. Proposition 2.6 will also hold when substituting the complex exponential function with the sine or cosine function (due to the Euler identity).

Proposition 2.7. *(Dirac's Delta integral). In the sense of distributions we have*

$$\frac{1}{2\pi}\int\limits_{\infty}^{\infty} e^{ix\xi}\,\mathrm{d}x = \delta(\xi). \tag{2.17}$$

Proof. Recall the definition of Dirac's delta given in Proposition 2.2. We then have that

$$\delta_0\phi = \int\limits_{-\infty}^{\infty} \phi(\xi)\int\limits_{-\infty}^{\infty} e^{ix\xi}\,\mathrm{d}x\mathrm{d}\xi \quad \phi \in \mathscr{D}(\mathbb{R}).$$

We perform the inner integration from $-N$ to N, $N \in \mathbb{R}$, and take the limit $N \to \infty$;

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(\xi) \int_{-\infty}^{\infty} e^{ix\xi} \, \mathrm{d}x\mathrm{d}\xi &= \int_{-\infty}^{\infty} \phi(\xi) \lim_{N\to\infty} \int_{-N}^{N} e^{ix\xi} \, \mathrm{d}x\mathrm{d}\xi \\
&= \lim_{N\to\infty} \int_{-\infty}^{\infty} \phi(\xi) 2\frac{\sin(N\xi)}{\xi} \, \mathrm{d}\xi \\
&= \lim_{N\to\infty} \int_{-\infty}^{\infty} (\phi(\xi) + \phi(-\xi)) \frac{\sin(N\xi)}{\xi} \, \mathrm{d}\xi \\
&= \lim_{N\to\infty} \int_{-\infty}^{\infty} 2\phi(0) \frac{\sin(N\xi)}{\xi} \, \mathrm{d}\xi \\
&+ \lim_{N\to\infty} \int_{-\infty}^{\infty} (\phi(\xi) + \phi(-\xi) - 2\phi(0)) \frac{\sin(N\xi)}{\xi} \, \mathrm{d}\xi. \qquad (2.18)
\end{aligned}
$$

Notice that due to Taylor's theorem there exists $C \in \mathbb{R}$ such that $\left|\phi(\xi) + \phi(-\xi) - 2\phi(0)\right| \leq C\xi^2$. This means, by proposition 2.6, that this term in 2.18 will vanish as $N \to \infty$. Since N is positive, the first term will according to proposition 2.5 equal $2\phi(0)\pi$. In total we get:

$$
\int_{-\infty}^{\infty} \phi(\xi) \int_{-\infty}^{\infty} e^{ix\xi} \, \mathrm{d}x\mathrm{d}\xi = 2\pi\phi(0),
$$

which completes the proof of proposition 2.7.

We now have the necessary tools to prove proposition 2.4:

Proof of proposition 2.4. Let $f, g \in L_2(\mathbb{R})$, and let $\mathscr{F}f, \mathscr{F}g \in L_2(\mathbb{R})$ be their respective Fourier transforms as defined in (2.8). Let ξ_1 and ξ_2 be the frequency parameters for the two transforms of f and g, respectively. Then, by recalling Theorem 2.3, we have:

$$
\int_{\infty}^{\infty} f\overline{g} \, \mathrm{d}x = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} (\mathscr{F}f)(\xi_1) e^{i\xi_1 x} \, \mathrm{d}\xi_1 \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \overline{(\mathscr{F}g)(\xi_2) e^{i\xi_2 x}} \, \mathrm{d}\xi_2 \mathrm{d}x,
$$

where the overline denotes complex conjugation. By Fubini's theorem we get

$$\int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} (\mathscr{F} f)(\xi_1) e^{i\xi_1 x}\, \mathrm{d}\xi_1 \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \overline{(\mathscr{F} g)(\xi_2) e^{i\xi_2 x}}\, \mathrm{d}\xi_2 \mathrm{d}x \tag{2.19}$$

$$= \frac{1}{2\pi} \int_{\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty} (\mathscr{F} f)(\xi_1)\overline{(\mathscr{F} g)(\xi_2)} e^{ix(\xi_1-\xi_2)}\, \mathrm{d}x\mathrm{d}\xi_1\mathrm{d}\xi_2. \tag{2.20}$$

By recalling Proposition 2.7 we then obtain:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\infty}^{\infty}\int_{\infty}^{\infty}\int_{\infty}^{\infty} (\mathscr{F} f)(\xi_1)\overline{(\mathscr{F} g)(\xi_2)} e^{ix(\xi_1-\xi_2)}\, \mathrm{d}x\mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&= \int_{\infty}^{\infty}\int_{\infty}^{\infty} (\mathscr{F} f)(\xi_1)\overline{(\mathscr{F} g)(\xi_2)} \delta(\xi_1 - \xi_2)\, \mathrm{d}\xi_1\mathrm{d}\xi_2 \\
&= \int_{\infty}^{\infty} (\mathscr{F} f)(\xi)\overline{(\mathscr{F} g)(\xi)}\, \mathrm{d}\xi,
\end{aligned}$$

after relabeling. This completes the proof of proposition 2.4.

2.9 The fractional Sobolev spaces, $H^s(\mathbb{R})$

It is finally time to introduce the fractional Sobolev spaces, which will be our solution spaces in chapter 3. Recalling (2.4), we have of particular interest the case when $p = 2$ and $k \in \mathbb{N}_0$. When equipped with the inner product

$$\langle f, g\rangle_{H^k(\mathbb{R})} = \sum_{\alpha \leq k} \langle D^\alpha f, D^\alpha g\rangle_{L_2(\mathbb{R})} = \sum_{\alpha \leq k} \int_{\mathbb{R}} (D^\alpha f)\overline{(D^\alpha g)}\, \mathrm{d}x, \tag{2.21}$$

the spaces $W_2^k(\mathbb{R})$ become Hilbert spaces. This is quite obvious since the inner product in (2.21) inherits the properties of the inner product of $L_2(\mathbb{R})$.

We want to characterize the spaces $W_2^k(\mathbb{R})$ in terms of the Fourier transform, and in order to do so we introduce weighted L_2 spaces:

Definition 2.12. Let $n \in \mathbb{N}$ and let ω be a continuous positive function in $\mathbb{R}$. Then

$$L_2(\mathbb{R},\omega) = \{f \in L_1^{loc}(\mathbb{R}) : \omega f \in L_2(\mathbb{R})\}. \tag{2.22}$$

When furnished with the inner product

$$\langle f, g\rangle_{L_2(\mathbb{R},\omega)} = \int_{\mathbb{R}} \omega(x) f(x) \overline{\omega(x) g(x)}\, \mathrm{d}x = \langle \omega f, \omega g\rangle_{L_2(\mathbb{R})}, \tag{2.23}$$

$L_2(\mathbb{R},\omega)$ becomes a Hilbert space. We also notice that $f \mapsto \omega f$ maps $L_2(\mathbb{R},\omega)$ unitarily onto $L_2(\mathbb{R})$. In this paper we will work with a nonlocal operator involving the weights

$$\omega_s(\xi) = (1 + |\xi|^2)^{\frac{s}{2}}, \quad s \geq 0,\ \xi \in \mathbb{R}. \tag{2.24}$$

Consequently, the spaces $L_2(\mathbb{R},\omega_s)$ are of particular interest.

Proposition 2.8. *Let $L_2(\mathbb{R},\omega_s)$ be given by (2.22) and (2.24). Then $L_2(\mathbb{R},\omega_s)$ together with the inner product in (2.23) is a Hilbert space. Furthermore,*

$$\mathscr{S}(\mathbb{R}) \subset L_2(\mathbb{R},\omega_s) \subset \mathscr{S}'(\mathbb{R}), \tag{2.25}$$

when interpreted in the sense of definition 2.7.

Proof. We refer to [11, p. 60] for the proof.

Having already defined the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ on $\mathscr{S}'(\mathbb{R})$, we can restrict $\mathscr{F}$ and $\mathscr{F}^{-1}$ to $W_2^k(\mathbb{R})$ and $L_2(\mathbb{R},\omega_k)$. Next we present an important theorem that will be crucial in the development of the fractional Sobolev spaces.

Theorem 2.5. *Let $k \in \mathbb{N}_0$. The Fourier transform $\mathscr{F}$, and its inverse $\mathscr{F}^{-1}$, generate unitary maps of $W_2^k(\mathbb{R})$ onto $L_2(\mathbb{R},\omega_k)$, and of $L_2(\mathbb{R},\omega_k)$ onto $W_2^k(\mathbb{R})$,*

$$\mathscr{F} W_2^k(\mathbb{R}) = \mathscr{F}^{-1} W_2^k(\mathbb{R}) = L_2(\mathbb{R},\omega_k). \tag{2.26}$$

Proof. Let $f \in W_2^k(\mathbb{R})$. From (2.21) and Proposition 2.4 we have

$$\begin{aligned}
\|f\|^2_{W_2^k(\mathbb{R})} &= \sum_{\alpha\le k} \|D^\alpha f\|^2_{L_2(\mathbb{R})} \\
&= \sum_{\alpha\le k} \|\mathscr{F}(D^\alpha f)\|^2_{L_2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left(\sum_{\alpha\le k} |\xi^\alpha|^2\right) |(\mathscr{F} f)(\xi)|^2 \,\mathrm{d}\xi,
\end{aligned}$$

and since $\sum_{\alpha\le k} |\xi^\alpha|^2 \sim |\omega_k^2(\xi)|$, we obtain with respect to this equivalent norm, denoted by $\|\cdot\|_{L_2(\mathbb{R},\omega_k)_*}$, that

$$\|f\|_{W_2^k(\mathbb{R})} = \|\mathscr{F} f\|_{L_2(\mathbb{R},\omega_k)_*}. \tag{2.27}$$

Hence $\mathscr{F}$ is an isometric map from $W_2^k(\mathbb{R})$ to $L_2(\mathbb{R},\omega_k)$. Conversely, let $g \in L_2(\mathbb{R},\omega_k)$ and $f = \mathscr{F}^{-1} g$. By the counterparts for $\mathscr{F}^{-1}$ of (2.10), we have:

$$D^\alpha f = i^\alpha \mathscr{F}^{-1}(x^\alpha g) \in L_2(\mathbb{R}), \quad \alpha \le k,$$

proving that $f \in W_2^k(\mathbb{R})$. Hence $\mathscr{F}$ maps $W_p^k(\mathbb{R})$ unitarily onto $L_2(\mathbb{R},\omega_k)$.

Remark 2.17. We can quite obviously rewrite (2.26) as

$$W_2^k(\mathbb{R}) = \mathscr{F} L_2(\mathbb{R},\omega_k) = \mathscr{F}^{-1} L_2(\mathbb{R},\omega_k).$$

This means that the space $W_2^k(\mathbb{R})$ can be defined as the Fourier image of $L_2(\mathbb{R},\omega_k)$. But by doing so there is no need to demand $k \in \mathbb{N}$. The relation in (2.25) will hold for any $s \ge 0$. The resulting spaces W_2^s are so useful and powerful that they require their own notation and definition.

Definition 2.13. Let $s \ge 0$. We then define the fractional Sobolev spaces as

$$H^s(\mathbb{R}) = \{f \in \mathscr{S}'(\mathbb{R}) : \int_{\mathbb{R}} (1+|\xi|^2)^s |\mathscr{F}(f)(\xi)|^2 \,\mathrm{d}\xi < \infty\}. \tag{2.28}$$

Remark 2.18. It follows immediately from our discussion (Theorem 2.5 and (2.27)) that

$$H^s(\mathbb{R}) = W_2^k(\mathbb{R}) \quad \text{whenever } k \in \mathbb{N}_0.$$

Remark 2.19. One can substitute $\mathscr{F}$ with $\mathscr{F}^{-1}$ in (2.28). Either way, $H^s(\mathbb{R})$ extends the classical Sobolev spaces $W_2^k(\mathbb{R})$ from $k \in \mathbb{N}$ to $s \ge 0$.

We equip the fractional Sobolev spaces with the following inner product:

$$\langle f, g\rangle_{H^s(\mathbb{R})} = \langle (1+|\xi|^2)^{\frac{s}{2}} \mathscr{F} f, (1+|\xi|^2)^{\frac{s}{2}} \mathscr{F} g\rangle_{L_2(\mathbb{R})}. \tag{2.29}$$

The inner product in (2.29) naturally induces a norm on $H^s(\mathbb{R})$:

$$\|f\|_{H^s(\mathbb{R})} = \|(1+|\xi|)^{\frac{s}{2}} f\|_{L_2(\mathbb{R})}.$$

Proposition 2.9. *The fractional Sobolev spaces defined in (2.28), equipped with the inner product in (2.29), are Hilbert spaces. Furthermore,*

$$\mathscr{S}(\mathbb{R}) \subset H^s(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R}). \tag{2.30}$$

Proof. By definition,

$$f \mapsto \omega_s \mathscr{F} f : \ H^s(\mathbb{R}) \to L_2(\mathbb{R}) \tag{2.31}$$

generates an isometric map into $L_2(\mathbb{R})$. If one chooses $f = \mathscr{F}^{-1}(\omega_{-s} g) \in \mathscr{S}'(\mathbb{R})$ for any given $g \in L_2(\mathbb{R})$, it follows that (2.31) is a unitary map onto $L_2(\mathbb{R})$. This yields that $H^s(\mathbb{R})$ is a Hilbert space. (2.31) also maps both $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ onto itself. Since $L_2(\mathbb{R}) \subset H^s(\mathbb{R})$ and $\mathscr{S}(\mathbb{R}) \subset L_2(\mathbb{R})$, it follows that $\mathscr{S}(\mathbb{R}) \subset H^s(\mathbb{R})$. Although we will not show it here (see [11]), it can be shown that if $T \in \mathscr{S}'(\mathbb{R})$, then also $D^\alpha T \in \mathscr{S}'(\mathbb{R})$. The right hand side of (2.30) follows immediately from this.

2.10 Embedding in $BC(\mathbb{R})$

We are interested in under which conditions one can find a representative function $f \in BC(\mathbb{R})$ of the equivalence class $[f] \in H^s(\mathbb{R})$. We will need the following lemma:

Lemma 2.1. *(Approximation by smooth functions). The Schwartz space $\mathscr{S}(\mathbb{R})$, and thus also $C^\infty(\mathbb{R})$, is dense in $H^s(\mathbb{R})$.*

Proof. Consult [3] for a proof of the above.

Having established the above result we are now able to prove the following:

Theorem 2.6. *For $s > 1/2$ one has*

$$H^s(\mathbb{R}) \hookrightarrow BC(\mathbb{R}),$$

which means that for such s, we can in each equivalence class $[f] \in H^s(\mathbb{R})$ find a representative function $f \in BC(\mathbb{R})$. In fact, one has

$$\|f\|_{BC(\mathbb{R})} \leq \|[f]\|_{H^s(\mathbb{R})}.$$

Proof. Since $\mathscr{S}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$, it is sufficient to prove that there exists some constant $C > 0$, such that for all $\phi \in \mathscr{S}(\mathbb{R})$, we have

$$|\phi(x)| \leq C\|\phi\|_{H^s(\mathbb{R})}, \quad \text{for all } x \in \mathbb{R}.$$

Making use of the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
|\phi(x)| &= |(\mathscr{F}^{-1}\mathscr{F}\phi)(x)| \\
&= \frac{1}{\sqrt{2\pi}}|\int_{\mathbb{R}} e^{i\xi x}(\mathscr{F}\phi)(\xi)\,\mathrm{d}x| \\
&\leq c\int_{\mathbb{R}} (1+|\xi|^2)^{s/2}|\mathscr{F}\phi(\xi)|(1+|\xi|^2)^{-s/2}\,\mathrm{d}x \\
&\leq c\left(\int_{\mathbb{R}} (1+|\xi|^2)^{s}|\mathscr{F}\phi(\xi)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \frac{1}{(1+|\xi|^2)^s}\,\mathrm{d}x\right)^{\frac{1}{2}} = C\|\phi\|_{H^s(\mathbb{R})}.
\end{aligned}$$

Note that the last integral converges since $s > 1/2$.

2.11 Sobolev embedding and interpolation theorems

We will now introduce the two most frequently used theorems in this paper; embedding and interpolation on Sobolev spaces. The Sobolev embedding theorem yields criteria for when functions in Sobolev spaces are also members of some appropriate Lebesgue space. This is an invaluable tool for us in order to see when the functionals we are working with make sense. The interpolation is invaluable in this thesis since it yields inequalities that makes our lemmas possible to prove.

Theorem 2.7. *(Sobolev embedding theorem). If $k > l$ and $1 \leq p < q < \infty$ are two real numbers such that $(k-l)p < 1$ and*

$$p \leq q \leq \frac{p}{1-kp},$$

then

$$W^{k,p}(\mathbb{R}) \hookrightarrow W^{l,q}(\mathbb{R}).$$

More precisely, under the given conditions, there exists a constant $C \in \mathbb{R}$ such that for all $\phi \in W^{k,p}(\mathbb{R})$,

$$\|\phi\|_{W^{l,q}(\mathbb{R})} \leq C\|\phi\|_{W^{k,p}(\mathbb{R})}.$$

Proof. Consult [7] for proof of the above.

Remark 2.20. The theorem is valid if one replaces $\mathbb{R}$ with a compact domain $\Omega \subset \mathbb{R}$ as well. In this case the embedding is compact.

Theorem 2.8. *(Sobolev interpolation theorem). Let*

$$-\infty < s_1 < s < s_2 < \infty \quad \text{and} \quad s = (1-\theta)s_1 + \theta s_2$$

with $0 < \theta < 1$. Then there is a positive constant c such that for any $\epsilon > 0$, and any $f \in H^{s_2}(\mathbb{R})$, the following inequalities hold:

$$\|f\|_{H^s(\mathbb{R})} \leq \|f\|^{1-\theta}_{H^{s_1}(\mathbb{R})}\|f\|^{\theta}_{H^{s_2}(\mathbb{R})} \leq \epsilon\|f\|_{H^{s_2}(\mathbb{R})} + c\epsilon^{-\frac{\theta}{1-\theta}}\|f\|_{H^{s_1}(\mathbb{R})}.$$

Proof. ([11]) From the definition of the norm on $H^s(\mathbb{R})$ and Hölder's inequality we get

$$\begin{aligned}
\|f\|_{H^s(\mathbb{R})} &= \left(\int_{\mathbb{R}} (1+|\xi|^2)^s|\widehat{f}|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \int_{\mathbb{R}} (1+|\xi|^2)^{\theta s_2}|\widehat{f}|^2(1+|\xi|^2)^{(1-\theta)s_1}|\widehat{f}|^2 \, \mathrm{d}x \\
&\leq \left(\epsilon^{-\frac{1}{1-\theta}}\|f\|_{H^{s_1}(\mathbb{R})}\right)^{1-\theta}\left(\epsilon^{\frac{1}{\theta}}\|f\|_{H^{s_2}(\mathbb{R})}\right)^{\theta} &(2.32)\\
&\leq c'\epsilon^{-\frac{1}{1-\theta}}\|f\|_{H^{s_1}(\mathbb{R})} + c'\epsilon^{\frac{1}{\theta}}\|f\|_{H^{s_2}(\mathbb{R})}. &(2.33)
\end{aligned}$$

In the last transition we have made use of Young's inequality. We get the first inequality in (2.8) by setting $\epsilon = 1$ in (2.32), and the second inequality with $\epsilon' = c'\epsilon^{\frac{1}{\theta}}$ in (2.33).

Remark 2.21. As in the embedding theorem, the result also holds on compact domains as well as the whole real line.

2.12 Weak lower semi-continuity of the Hilbert norm

When proving the existence of a minimizer to a functional through variational methods, an important result is that the Hilbert norm is weakly lower semi-continuous. In our case the result is vital in the last stages of our existence proof in the next chapter. We first define the concept of weak convergence:

Definition 2.14. A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to x in H if

$$\langle x_n, y\rangle \to \langle x, y\rangle \quad \text{for all } y \in H.$$

Proposition 2.10. *Let K be convex and norm-closed, and suppose that $x_j \to x$ weakly. Then $x \in K$.*

Proof. Suppose that $x \notin K$. Then there exists $r > 0$ such that $B_r \cap K = \emptyset$, where B_r is the ball of radius r. Then there exists $l \in X'$, where X' indicates the dual space of X, such that $l \neq 0$ and $Re\ l|_K \leq Re\ l|_{B_r}$. Then using y such that $Re\ l(y) = 1$ (pick arbitrarily and scale appropriately), we have that

$$Rel|_K \leq Re\ l(x - \epsilon y) = Re\ l(x) - \epsilon$$

for $\epsilon < r$. However, this implies that $Re\ l(x_j) \leq Re\ l(x) - \epsilon$. Letting $j \to \infty$ we get $Re\ l(x) \leq Re\ l(x) - \epsilon$, a contradiction.

Corollary 2.1. *If $f : X \to \mathbb{R}$ is convex and continuous, and $x_j \to x$ weakly, then*

$$f(x) \leq \liminf_j f(x_j)$$

Proof. Suppose that $c = \liminf_{j\to\infty} f(x_j)$. Consider the set $K = \{x : f(x) \leq c + \epsilon\}$. This is a norm-closed, convex set by assumption. By the definition of liminf, we can find a subsequence x'_j such that $f(x'_j) < c + \epsilon$. Since $x'_j \to x$ weakly and K is closed and convex, Proposition 2.10 shows that $x \in K$ so that $f(x) \leq c + \epsilon$. Since ϵ is arbitrary, the result is proven.

Remark 2.22. Since the norm mapping $x \to \|x\|$ is convex and continuous, we get immediately from Corollary 2.1 that

$$\|x\| \leq \liminf_j \|x_j\|,$$

which is exactly what we set out to show.

Remark 2.23. Note that we have simultaneously showed weak lower semicontinuity of the Banach norm, and not only for the Hilbert norm.

2.13 Banach-Alaoglu's theorem

The biggest challenge when doing variational calculus is to prove that the infimum value of our functional is attained for some element in our solution space. This minimizer will be found through a minimizing sequence, and hence convergence properties are of utmost importance. As already mentioned, we will be searching for solutions in Sobolev spaces. Boundedness of a sequence does not necessarily yield strong convergence of that sequence, but it does guarantee the existence of a weakly convergent subsequence whenever we are working in reflexive spaces (which the Sobolev spaces are). We introduce the theorem of Banach and Alouglu, proved for separable normed vector spaces in 1932 by Banach, and in the general case by Alaoglu in 1940.

Definition 2.15. A subset K of a Banach space is *weakly sequentially compact (w.s.c)* if for any sequence $\{x_n\}_n \subset K$ there is a weakly convergent subsequence with limit in K.

Theorem 2.9. *(Banach Alaoglu's theorem). Let X be a reflexive Banach space (i.e. $X = X''$) and let $B = \{x \in X : \|x\| \leq 1\}$ be its closed unit ball. Then B is w.s.c.*

Proof. See section 3.15 in [17].

2.14 The concentration compactness principle

In this paper we analyze a class of partial differential equations on an unbounded domain. When working on compact domains, the Sobolev embedding theorem guarantees the existence of convergent subsequences. However, on unbounded domains the compactness property is not enjoyed, and hence we must make use of a new technique; the concentration compactness principle. Even though a sequence $\{x_n\}_n$ does not have a subsequence converging to some element x, it may have a subsequence converging to some nonconstant sequence $\{y_n\}_n$. This new sequence holds a very structured form; it "concentrates" to a point, "travels" off to infinity or is a superposition of several concentrating or traveling profiles of this form. More precisely we have the following:

Lemma 2.2. *(Concentration compactness). Let $\{\rho_n\}_n \subset L_1(\mathbb{R})$ be a sequence that satisfies*

$$\rho_n \geq 0 \quad \textit{a.e. on } \mathbb{R} \textit{ with}$$

$$\int_{\mathbb{R}} \rho_n \, \mathrm{d}x = \mu$$

for a fixed $\mu > 0$ and all $n \in \mathbb{N}$. Then there exists a subsequence $\{\rho_{n_k}\}_k$ that satisfies one of the three following properties:

1. *(Compactness). There exists a sequence $\{y_k\}_k \subset \mathbb{R}$ such that for every $\epsilon > 0$, there exists $r < \infty$ satisfying for all $k \in \mathbb{N}$:*

$$\int_{y_k - r}^{y_k + r} \rho_{n_k}(x) \, \mathrm{d}x \geq \mu - \epsilon.$$

2. *(Vanishing). For all $r < \infty$,*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \rho_{n_k} \, \mathrm{d}x = 0.$$

3. *(Dichotomy). There exists $\bar{\mu} \in (0, \mu)$ such that for every $\epsilon > 0$, there exists a natural number $k_0 \geq 1$ and two sequences of positive L^1 functions*

$\{\rho_k^{(1)}\}, \{\rho_k^{(2)}\}$ satisfying, for $k \geq k_0$:

$$\|\rho_{n_k} - \left(\rho_k^{(1)} + \rho_k^{(2)}\right)\|_{L^1} \leq \epsilon,$$
$$\Big|\int_{\mathbb{R}} \rho_k^{(1)}\, \mathrm{d}x - \bar{\mu}\Big| \leq \epsilon,$$
$$\Big|\int_{\mathbb{R}} \rho_k^{(2)}\, \mathrm{d}x - (\mu - \bar{\mu})\Big| \leq \epsilon$$

and

$$dist\left(supp(\rho_k^{(1)}), supp(\rho_k^{(2)})\right) \to \infty.$$

Proof. The original proof of this can be found in [15].

Remark 2.24. The condition $\int_{\mathbb{R}} \rho_n\, \mathrm{d}x = \mu$ can be replaced by $\int_{\mathbb{R}} \rho_n\, \mathrm{d}x = \mu_n$ where $\mu_n \to \mu$ (see [8]).

3 Existence of solitary-wave solutions

In this section we present an original existence proof of solitary-wave solutions to a class of pseudodifferential evolution equations. A similar class of equations have been studied by Albert and Arnesen in [1] and [4]. However, this paper discusses the admittance of solutions when the nonlinearity part is inhomogeneous and includes a higher order term. A similar situation is considered in [9], but here the linear operator is a smoothing operator. This chapter progresses as follows:

Section 3.1 presents the class of equations and the assumptions made. Compared to earlier literature, attention should be given to the introduction of the higher order term included in the nonlinearity part. The functionals we are working with are also introduced, and we make assumptions so that all the integrals make sense.

Section 3.2 starts by proving that the infimum of our functional is bounded from below as well as is less than zero. The higher order term in the nonlinearity part demands more care and attention in comparison to what is considered in [1] and [4], where the nonlinearity is homogeneous. Perhaps the most important (and most challenging) result is the subadditivity property of the infimum value of our functional. Also here the higher order term complicates things as we are dealing with two separate integrals instead of one as in the homogeneous situation. This leads us to analyze several different cases and requires us to prove that the result still holds, independent of the signs of the integrals. We also show that any minimizing sequence is bounded in $H^{s/2}(\mathbb{R})$ from above for all n, and that the minimizing sequence is strictly greater than zero measured in the $L_{p+1}(\mathbb{R})$ and $L_{p+1+\gamma}(\mathbb{R})$ norm for all sufficiently large n.

Section 3.3 deals with excluding vanishing. This is a straight forward task with the help of the lemmas from section 3.2.

Section 3.4 is the part where we exclude dichotomy. This is done by examining what the consequences will be if dichotomy were to occur. Due to the subadditivity property of our functional, we are able to show that such circumstances would yield a contradiction, and hence prove that dichotomy does not occur.

Section 3.5 is where the main result is proven; the existence of a minimizer to our variational problem. Having excluded vanishing and dichotomy, we know from the concentration compactness principle that compactness occurs. We use this together with the Banach Alaoglu theorem to show strong convergence in $L_{p+1}(\mathbb{R})$ and $L_{p+1+\gamma}(\mathbb{R})$. Finally, due to the weak lower semi-continuity of the Hilbert norm, we are able to finish the existence proof.

Section 3.6 proves that the minimizer found actually solves the PDE under study. In this section we could have easily introduced, and applied, the Lagrange multipler principle, but we present an original approach in order to get a better understanding of the result.

Section 3.7 is included to show that solutions to our PDE inherits regularity from the equation itself. We are able to show that any solution belonging to $H^{s/2}(\mathbb{R})$, also belongs to $H^s(\mathbb{R})$. By iteration we can conclude that any solution found will be in $H^\infty(\mathbb{R})$, and hence the solutions we find are smooth.

3.1 Equation and assumptions

We study the family of pseudodifferential evolution equations of the form

$$u_t + (n(u))_x - (Lu)_x = 0 \quad \text{in } \mathbb{R}, \tag{3.1}$$

where both u and n are real valued functions, and L is a Fourier multiplier operator of order $s > 0$:

$$\widehat{Lu}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{u}(\xi). \tag{3.2}$$

We assume n to be made up of two parts:

$$n(x) = n_p(x) + n_r(x), \tag{3.3}$$

where n_p can be on one of two forms:

$$n_p(x) = \begin{cases} c_p|x|^p & \text{with } c_p \neq 0 \\ c_p x|x|^{p-1} & \text{with } c_p > 0, \end{cases}$$

and n_r is an order term:

$$n_r(x) = \mathcal{O}(|x|^{p+\gamma}) \quad \text{with } \gamma > 0. \tag{3.4}$$

The notation $\mathcal{O}(|x|^{p+\gamma})$ means that this term is of order $p+\gamma$, i.e. there exists a constant $C > 0$ such that $|\mathcal{O}(|x|^{p+\gamma})| \leq C|x|^{p+\gamma}$. Next we introduce the primitive of n, denoted by N:

$$N(x) = N_p(x) + N_r(x) \tag{3.5}$$

where

$$N_p(x) = \int_0^x n_p(s)\,\mathrm{d}s,$$

and

$$N_r(x) = \int_0^x n_r(s)\,\mathrm{d}s.$$

We will also assume $p \in (1, 2s+1-\gamma)$. Furthermore, we assume traveling wave solutions of the form $u = u(x-ct)$ with wave speed c. Making a change of variable, assuming that the solutions vanishes at infinity and integrating up with respect to the spatial variable yields

$$-cu + n(u) - Lu = 0 \quad \text{in } \mathbb{R}, \tag{3.6}$$

which will be the standing point for our analysis in what follows. With inspiration from [1] and [4] we search for minimizers of

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}} uLu\,\mathrm{d}x - \int_{\mathbb{R}} N(u)\,\mathrm{d}x, \tag{3.7}$$

with respect to the constraint

$$\mathcal{Q}(u) = \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x = q \neq 0. \tag{3.8}$$

This section deals with proving the existence of minimizers of

$$I_q := \inf\{\mathcal{E}(\omega) \;:\; \omega \in H^{s/2}(\mathbb{R}) \text{ and } \mathcal{Q}(\omega) = q\}. \tag{3.9}$$

As one can see in Lemma 1 in [2], if u solves (3.1) with initial conditions

$u(x,0) = \psi(x)$ for all $x \in \mathbb{R}$ where $\psi \in H^r(\mathbb{R}), r \geq s/2$, then the functionals in (3.7) and (3.8) are time independent. Moreover, we have that our main results are contained within the following theorem:

Theorem 3.1. *(Existence of solitary-wave solutions). Assume L is of the form as in* (3.2) *and n is as in* (3.3)*. Then, if $p \in (1, 2s+1-\gamma)$, the set of minimizers of I_q, D_q, is nonempty for any $q > 0$, and every element of D_q is a solution to* (3.6) *with the wave speed c being the Lagrange multiplier in this constrained variational problem. Moreover, if $\{u_n\}_n \subset H^{s/2}(\mathbb{R})$ is a minimizing sequence of I_q under the previous conditions, then there exists a sequence $\{y_n\}_n \subset \mathbb{R}$ such that a subsequence $\{u_n(\cdot + y_n)\}_n$ converges in $H^{s/2}(\mathbb{R})$ to an element of D_q. Furthermore, $D_q \subset H^s(\mathbb{R})$.*

Note that in order for the integrals in (3.7) to make sense we must require that both $u \in L_{p+1}(\mathbb{R})$ and $u \in L_{p+1+\gamma}(\mathbb{R})$ whenever $u \in H^{s/2}(\mathbb{R})$. In other words we would like to have the embeddings $H^{s/2}(\mathbb{R}) \hookrightarrow L_{p+1}(\mathbb{R})$ and $H^{s/2}(\mathbb{R}) \hookrightarrow L_{p+1+\gamma}(\mathbb{R})$. Applying Theorem 2.7 with $k = s/2$, $l = 0$, $p = 2$, $q = p+1$ and $n = 1$ we get that this is satisfied if

$$1 < p \leq \frac{1+s}{1-s},$$

where $p \in (1, \frac{1+s}{1-s})$ should be interpreted as $p \in (1, \infty)$ whenever $s \geq 1$. Doing the same with $q = p+1+\gamma$ yields $1 < p+\gamma \leq \frac{1+s}{1-s}$. However, the assumption $p \in (1, 2s+1-\gamma)$ assures us that this is always satisfied since $\frac{1+s}{1-s} = 1 + \frac{2s}{1-s}$, which will always be greater than $2s+1$. Hence our more strict assumption $p \in (1, 2s+1-\gamma)$ yields directly that the integrals in (3.7) and (3.8) make sense.

3.2 Boundedness of infimum and subadditivity of functional

To get a better overview in this section the results are broken down into a sequence of lemmas. Since $\widehat{Lu}(0) \neq 0$, which has been important in the proofs in the papers of Albert [1] and Mathias [4], we prove the lemmas for a translation of I_q, $\overline{I_q} = I_q - \frac{1}{2}q$. It is obvious that any minimizer of $\overline{I_q}$, if it exists, is also a minimizer of I_q.

Lemma 3.1. *For all $q > 0$, one has*

$$-\infty < \overline{I_q} < 0.$$

Proof. We note that by Proposition 2.4 the first integral in 3.7 is nonnegative;

$$\frac{1}{2}\int_{\mathbb{R}} uLu \, \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \widehat{u}\widehat{Lu} \, \mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} (1+|\xi|^2)^{\frac{s}{2}} |\widehat{u}(\xi)|^2 \, \mathrm{d}x \geq 0.$$

Thus in order to prove $\overline{I_q} < 0$, it is sufficient to prove that there exists a $\phi \in H^{s/2}(\mathbb{R})$ with $\mathcal{Q}(\phi) = \frac{1}{2}\int_{\mathbb{R}} \phi^2 \, \mathrm{d}x = q > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}} \phi L\phi \, \mathrm{d}x - \frac{1}{2}q < \int_{\mathbb{R}} N(\phi) \, \mathrm{d}x. \tag{3.10}$$

We can obtain $\int_{\mathbb{R}} N_p(\phi) \, \mathrm{d}x > 0$ by taking ϕ to be nonnegative if $c_p > 0$ and nonpositive if $c_p < 0$. Now for any $t > 0$, define

$$\phi_t(x) = \sqrt{t}\phi(tx).$$

Then $\mathcal{Q}(\phi_t) = \frac{1}{2}\int_{\mathbb{R}} t^2\phi(tx)^2 \, \mathrm{d}x = q$ and

$$\int_{\mathbb{R}} N_p(\phi) \, \mathrm{d}x = \frac{t^{(p-1)/2}}{p+1} \|\phi\|_{L_{p+1}(\mathbb{R})}^{p+1}. \tag{3.11}$$

Before proceeding we note that we have, for $\phi \in H^{s/2}(\mathbb{R})$, that $\widehat{\phi(tx)}(\xi) = \frac{1}{t}\hat{\phi}(\frac{\xi}{t})$. This gives us:

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} \phi_t L\phi_t \, \mathrm{d}x - \frac{1}{2}q &= \frac{1}{2}\int_{\mathbb{R}} \widehat{\phi_t}\widehat{L\phi_t} \, \mathrm{d}\xi - \frac{1}{2}q \\
&= \frac{1}{2}t\int_{\mathbb{R}} \widehat{\phi(tx)}\widehat{L\phi(tx)} \, \mathrm{d}\xi - \frac{1}{2}q \\
&= \frac{1}{2}\frac{1}{t}\int_{\mathbb{R}} (1+|\xi|^2)^{\frac{s}{2}} |\widehat{\phi(\xi t)}|^2 \, \mathrm{d}\xi - \frac{1}{2}q \\
&= \frac{1}{2}\int_{\mathbb{R}} (1+|t\xi|^2)^{\frac{s}{2}} |\widehat{\phi(\xi)}|^2 \, \mathrm{d}\xi - \frac{1}{2}q \\
&\leq C' t^s \|\phi\|_{H^{s/2}(\mathbb{R})}^2,
\end{aligned} \tag{3.12}$$

for some constant $C' > 0$ which is independent of t. We are seeking the inequality

$$\frac{t^{\left(\frac{p-1}{2}\right)}}{p+1}\|\phi\|^{p+1}_{L_{p+1}(\mathbb{R})} + t^{\left(\frac{p-1+\gamma}{2}\right)}\int_{\mathbb{R}} N_r(\phi)\,\mathrm{d}x > C't^s\|\phi\|^2_{H^{s/2}(\mathbb{R})} \tag{3.13}$$

which is satisfied, since if $\int_{\mathbb{R}} N_r(\phi)\,\mathrm{d}x \le 0$, we get

$$\frac{t^{\left(\frac{p-1}{2}\right)}}{p+1}\|\phi\|^{p+1}_{L_{p+1}(\mathbb{R})} > C't^s\|\phi\|^2_{H^{s/2}(\mathbb{R})}. \tag{3.14}$$

Due to the assumption $(p-1)/2 < s$, the expression on the right hand side of (3.14) goes to zero faster as $t \longrightarrow 0^+$ than the left hand side. Furthermore, if $\int_{\mathbb{R}} N_r(\phi)\,\mathrm{d}x > 0$, the inequality in (3.13) still holds as $t \longrightarrow 0^+$ since $(p-1)/2 < (p-1+\gamma)/2 < s$. This implies $\overline{I_q} < 0$. In order to prove $\overline{I_q} > -\infty$, we make effective use of Theorem 2.7 and Theorem 2.8 to obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}} N(\phi)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} N_p(\phi) + N_r(\phi)\,\mathrm{d}x\right| &\le \int_{\mathbb{R}} |N_p(\phi)|\,\mathrm{d}x + \int_{\mathbb{R}} |N_r(\phi)|\,\mathrm{d}x \\
&\le K_1\|\phi\|^{p+1}_{H^{(p-1)/(2(p+1))}(\mathbb{R})} + K_2\|\phi\|^{p+1+\gamma}_{H^{(p+\gamma-1)/(2(p+1+\gamma))}(\mathbb{R})} \\
&\le K_1 q^{\frac{((p+1)s-p+1)}{s}}\|\phi\|^{\frac{(p-1)}{s}}_{H^{s/2}(\mathbb{R})} + K_2 q^{\frac{((p+1+\gamma)s-p+1-\gamma)}{s}}\|\phi\|^{(p+\gamma-1)/s}_{H^{s/2}(\mathbb{R})}.
\end{aligned}$$

From this we have

$$
\begin{aligned}
\mathcal{E}(\phi) - \frac{1}{2}q &= \frac{1}{2}\int_{\mathbb{R}} \phi L\phi \,\mathrm{d}x - \int_{\mathbb{R}} N(\phi)\,\mathrm{d}x - \frac{1}{2}q \\
&\geq \frac{1}{2}\int_{\mathbb{R}} \phi L\phi \,\mathrm{d}x - \int_{\mathbb{R}} |N_p(\phi)| + |N_r(\phi)|\,\mathrm{d}x - \frac{1}{2}q \\
&\geq \frac{1}{2}\|\phi\|^2_{H^{s/2}(\mathbb{R})} - \frac{c_p}{p+1}\|\phi\|^{p+1}_{L_{p+1}(\mathbb{R})} - C\|\phi\|^{p+1+\gamma}_{L_{p+1+\gamma}(\mathbb{R})} - \frac{1}{2}q \\
&\geq \frac{1}{2}\|\phi\|^2_{H^{s/2}(\mathbb{R})} - K_1\|\phi\|^{p+1}_{H^{(p-1)/(2(p+1))}(\mathbb{R})} - K_2\|\phi\|^{p+1+\gamma}_{H^{(p+\gamma-1)/(2(p+1+\gamma))}(\mathbb{R})} - \frac{1}{2}q \\
&\geq \frac{1}{2}\|\phi\|^2_{H^{s/2}(\mathbb{R})} - K_1 q^{\frac{((p+1)s-p+1)}{s}}\|\phi\|^{\frac{(p-1)}{s}}_{H^{s/2}(\mathbb{R})} - K_2 q^{\frac{((p+1+\gamma)s-p+1-\gamma)}{s}}\|\phi\|^{\frac{(p+\gamma-1)}{s}}_{H^{s/2}(\mathbb{R})} - \frac{1}{2}q,
\end{aligned}
$$

where $K_1, K_2 > 0$ depend only on the Sobolev embedding constants. Since $\frac{p-1}{s} < \frac{p+\gamma-1}{s} < 2$, the growth of the term with negative sign is bounded by the growth of the positive term. It follows that $\overline{I_q} > -\infty$.

Although the proof above goes in the same fashion as in [1] and [4], we see that we must always analyze the different scenarios depending on whether $\int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x$ is greater than, less than or equal to zero. The method of finding such a ϕ and scaling it is widely used with success in for instance [4], [1] and [9]. The next step is to show that there are bounds on any minimizing sequence for $\overline{I_q}$. In the case when the nonlinearity, n, only consists of n_p, it would be sufficient to show $\|u_n\|_{L_{p+1}(\mathbb{R})} \geq \delta > 0$ for sufficiently large values of n. However, the higher order term requires a slightly different bound as seen in the lemma below:

Lemma 3.2. *If $\{u_n\}_n$ is a minimizing sequence for $\overline{I_q}$, then there exists constants $K > 0$ and $\delta > 0$ such that*

1. $\|u_n\|_{H^{s/2}(\mathbb{R})} \leq K$ *for all n,*

2. $\int_{\mathbb{R}} |N(u_n)|\,\mathrm{d}x \geq \delta$ *for all sufficiently large n.*

Proof. From the definition of the norm on $H^{s/2}(\mathbb{R})$, Theorem 2.7 and Theorem

2.8 we have

$$\begin{aligned}
\frac{1}{2}\|u_n\|^2_{H^{s/2}(\mathbb{R})} &= \mathcal{E}(u_n) + \int_{\mathbb{R}} N(u_n)\,\mathrm{d}x \\
&\leq \sup_n \ \mathcal{E}(u_n) + \frac{c_p}{p+1}\|u_n\|^{p+1}_{L_{p+1}(\mathbb{R})} + C\|u_n\|^{p+1+\gamma}_{L_{p+1+\gamma}(\mathbb{R})} \\
&\leq K' + K_1\|u_n\|^{p+1}_{H^{(p-1)/(2(p+1))}(\mathbb{R})} + K_2\|u_n\|^{p+1+\gamma}_{H^{(p+\gamma-1)/(2(p+1+\gamma))}(\mathbb{R})} \\
&\leq K' + K_1 q^{\frac{((p+1)s-p+1)}{s}}\|u_n\|^{\frac{(p-1)}{s}}_{H^{s/2}(\mathbb{R})} + K_2 q^{\frac{((p+1+\gamma)s-p+1-\gamma)}{s}}\|u_n\|^{\frac{(p+\gamma-1)}{s}}_{H^{s/2}(\mathbb{R})}.
\end{aligned}$$

We have by assumption that $\frac{p-1}{2} < \frac{p+\gamma-1}{2} < 2$, and hence we have bounded the square of the norm by a smaller power guaranteeing the existence of a bound K. In order to prove the second statement we argue by contradiction. If no such δ exists, then we would have

$$\liminf_{n\to\infty} \int_{\mathbb{R}} N(u_n)\,\mathrm{d}x \leq 0.$$

This would imply

$$\overline{I_q} = \liminf_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}} u_n L u_n\,\mathrm{d}x - \int_{\mathbb{R}} N(u_n)\,\mathrm{d}x\right) - \frac{1}{2}q \geq \liminf_{n\to\infty}\left(-\int_{\mathbb{R}} N(u_n)\,\mathrm{d}x\right) \geq 0,$$

which contradicts Lemma 3.1.

Remark 3.1. From $\int_{\mathbb{R}} |N(u_n)|\,\mathrm{d}x \geq \delta > 0$ we have immediately that $\|u_n\|_{L_{p+1}(\mathbb{R})} \geq \delta' > 0$ as well for some $\delta' \in \mathbb{R}_+$. To see this, assume to the contrary that $\|u_n\|_{L_{p+1}(\mathbb{R})} = 0$. This implies $u_n = 0$ almost everywhere which again would imply $\|u_n\|_{L_{p+1+\gamma}(\mathbb{R})} = 0$. However, this would again mean that $\int_{\mathbb{R}} |N(u_n)|\,\mathrm{d}x = 0$, a contradiction of Lemma 3.2.

The next lemma is essential to excluding dichotomy later and was not proved with ease, the reason being the several different scenarios that needed to be considered. We make use of the fact that $\left(\liminf_{n\to\infty}\int_{\mathbb{R}} N(u_n)\,\mathrm{d}x\right) > 0$, but as we do not know the signs of the two parts, $\left(\liminf_{n\to\infty} \int\limits_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x\right)$ and $\left(\liminf_{n\to\infty} \int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x\right)$, separately, this initially led to some concern for the author. We solve this issue by splitting up the different scenarios and show that the lemma below will hold for all these cases.

Lemma 3.3. *For all $q_1, q_2 > 0$, one has*

$$I_{q_1+q_2} < I_{q_1} + I_{q_2}. \tag{3.15}$$

Proof. First, we claim that for $t > 1$ and $q > 0$ it holds that

$$I_{tq} < tI_q.$$

In order to see this, let $\{u_n\}_n$ be a minimizing sequence of I_q and define the scaled sequence $\{\tilde{u}_n\}_n = \sqrt{t}\{u_n\}_n$ for all n such that $\mathcal{Q}(u_n) = tq$. Hence we have that $\mathcal{E}(\tilde{u}_n) \geq I_{tq}$ for all n. We then have for all n that

$$\begin{aligned}
I_{tq} &\leq \frac{1}{2}\int\limits_{\mathbb{R}} \tilde{u}_n L\tilde{u}_n\,\mathrm{d}x - \int\limits_{\mathbb{R}} N(\tilde{u}_n)\,\mathrm{d}x \\
&= t\mathcal{E}(u_n) + \int\limits_{\mathbb{R}} tN(u_n) - N(\tilde{u}_n)\,\mathrm{d}x \\
&= t\mathcal{E}(u_n) + (t - t^{\frac{p+1}{2}})\int\limits_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x + (t - t^{\frac{p+1+\gamma}{2}})\int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x.
\end{aligned}$$

At this point we need to analyze different scenarios depending on what signs the integrals $\int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x$ and $\int\limits_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x$ take. Recall that $t > 1$. If $\int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x \geq 0$ we have that

$$(t - t^{\frac{p+1+\gamma}{2}})\int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x < (t - t^{\frac{p+1}{2}})\int\limits_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x,$$

which gives us

$$\begin{aligned}
I_{tq} &\leq t\mathcal{E}(u_n) + (t - t^{\frac{p+1}{2}}) \int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x + (t - t^{\frac{p+1+\gamma}{2}}) \int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x \\
&< t\mathcal{E}(u_n) + (t - t^{\frac{p+1}{2}}) \int_{\mathbb{R}} N(u_n)\,\mathrm{d}x.
\end{aligned}$$

We now let $n \to \infty$ and get

$$I_{tq} < tI_q + (t - t^{\frac{p+1}{2}})\delta < tI_q.$$

On the other hand, if $\int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x < 0$ and $\int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x \leq 0$ we get

$$\begin{aligned}
I_{tq} &\leq t\mathcal{E}(u_n) + (t - t^{\frac{p+1}{2}}) \int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x + (t - t^{\frac{p+1+\gamma}{2}}) \int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x \\
&\leq \mathcal{E}(u_n) + (t - t^{\frac{p+1+\gamma}{2}}) \int_{\mathbb{R}} N(u_n)\,\mathrm{d}x,
\end{aligned}$$

and we can again take $\liminf$ and obtain $I_{tq} < tI_q$. The last case is when $\int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x < 0$ and $\int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x > 0$. But now $(t - t^{\frac{p+1}{2}})\liminf\left(\int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x\right) < 0$, yielding directly

$$\begin{aligned}
I_{tq} &\leq \liminf\left(\mathcal{E}(u_n) + (t - t^{\frac{p+1}{2}}) \int_{\mathbb{R}} N_p(u_n)\,\mathrm{d}x + (t - t^{\frac{p+1+\gamma}{2}}) \int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x\right) \\
&< tI_q + \liminf\left((t - t^{\frac{p+1+\gamma}{2}}) \int_{\mathbb{R}} N_r(u_n)\,\mathrm{d}x\right) < tI_q,
\end{aligned}$$

and we have covered all the different cases. Furthermore, assuming first $q_1 > q_2$, we get

$$I_{(q_1+q_2)} = I_{q_1(1+\frac{q_2}{q_1})} < (1 + \frac{q_2}{q_1})I_{q_1} = I_{q_1} + \frac{q_2}{q_1} I_{q_2\frac{q_1}{q_2}} < I_{q_1} + I_{q_2}.$$

The proof when $q_1 > q_2$ can be done the exact same way. The remaining case when $q_1 = q_2$ becomes

$$I_{q_1+q_2} = I_{2q_1} < 2I_{q_1} = I_{q_1} + I_{q_2},$$

which completes the proof.

Remark 3.2. By adding $\frac{1}{2}(q_1 + q_2)$ on both sides of (3.15), we get that the result holds for $\overline{I_q}$ as well.

Before proceeding we will also need the following result, taken from [1]:

Lemma 3.4. *Given $K > 0$ and $\delta > 0$, there exists $\eta = \eta(K, \delta) > 0$ such that if $v \in H^{s/2}(\mathbb{R})$ with $\|v\|_{H^{s/2}(\mathbb{R})} \leq K$ and $\|v\|_{L_{p+1}(\mathbb{R})} \geq \delta$, then*

$$\sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |v(x)|^{p+1} \, \mathrm{d}x \geq \eta.$$

Proof. Without loss of generality we may assume $\frac{s}{2} \leq 1$; if $\frac{s}{2} > 1$ then $\|v\|_{H^1(\mathbb{R})} \leq K$ and the analysis can be carried out for $H^1(\mathbb{R})$. Choose a smooth function $\zeta : \mathbb{R} \to [0,1]$ with support in $[-2,2]$ and satisfying $\sum_{j\in\mathbb{Z}} \zeta(x-j) = 1$ for all $x \in \mathbb{R}$, and define $\zeta_j(x) = \zeta(x-j)$ for $j \in \mathbb{Z}$. The map $T : H^r(\mathbb{R}) \to l_2(H^r(\mathbb{R}))$ defined by

$$Tv = \{\zeta_j v\}_{j\in\mathbb{Z}}$$

is bounded for $r = 0$ and $r = 1$. For $r = 0$,

$$\|Tv\|^2_{l_2(L_2(\mathbb{R}))} = \sum_{j\in\mathbb{Z}} \|\zeta_j v\|^2_{L_2(\mathbb{R})} \leq \Sigma_{j\in\mathbb{Z}} \int_{j-2}^{j+2} v^2 \, \mathrm{d}x = 4\|v\|^2_{L_2(\mathbb{R})}$$

and one can argue similarly when $r = 1$, recalling that ζ is a smooth function. By interpolation the map T is therefore also bounded for $r = \frac{s}{2}$. That is, there exists a constant C_0 such that for all $v \in H^{s/2}(\mathbb{R})$,

$$\sum_{j\in\mathbb{Z}} \|\zeta_j v\|^2_{H^{s/2}(\mathbb{R})} \leq C_0 \|v\|^2_{H^{s/2}(\mathbb{R})}. \tag{3.16}$$

Since $l_{p+1} \hookrightarrow l_1$, there exists a positive number C_1 such that $\sum_{j\in\mathbb{Z}} |\zeta(x-j)|^{p+1} \geq C_1$ for all $x \in \mathbb{R}$. We claim that for every $v \in H^{s/2}(\mathbb{R})$ that is not identically zero, there exists an integer j_0 such that

$$\|\zeta_{j_0} v\|^2_{H^{s/2}(\mathbb{R})} \leq \left(1 + C_2 \|v\|^{-p-1}_{L_{p+1}(\mathbb{R})}\right) \|\zeta_{j_0} v\|^{p+1}_{L_{p+1}(\mathbb{R})}$$

where $C_2 = C_0 K^2 / C_1$. To see this, assume to the contrary that

$$\|\zeta_j v\|^2_{H^{s/2}(\mathbb{R})} > \left(1 + C_2 \|v\|^{-p-1}_{L_{p+1}(\mathbb{R})}\right) \|\zeta_j v\|^{p+1}_{L_{p+1}(\mathbb{R})}$$

for every $j \in \mathbb{Z}$. Summing over j, we obtain

$$C_0 \|v\|^2_{H^{s/2}(\mathbb{R})} > \left(1 + C_2 \|v\|^{-p-1}_{L_{p+1}(\mathbb{R})}\right) \sum_{j \in \mathbb{Z}} \|\zeta_j v\|^{p+1}_{L_{p+1}(\mathbb{R})},$$

and by our choice of C_2 we get

$$C_0 K^2 > \left(1 + C_2 \|v\|^{-p-1}_{L_{p+1}(\mathbb{R})}\right) C_1 \|v\|^{p+1}_{L_{p+1}(\mathbb{R})} = C_1 \|v\|^{p+1}_{L_{p+1}(\mathbb{R})} + C_0 K^2$$

for $\|v\|_{H^{s/2}(\mathbb{R})} \leq K$, which is a contradiction. This proves the claim in (3.16). We observe that from (3.16) and the underlying assumptions of the lemma it follows that

$$\|\zeta_{j_0} v\|^2_{H^{s/2}(\mathbb{R})} \leq \left(1 + \frac{C_2}{\delta^{p+1}}\right) \|\zeta_{j_0} v\|^{p+1}_{L_{p+1}(\mathbb{R})}.$$

For $p \leq 1 + 2s - \gamma$, we have by the Sobolev embedding theorem that

$$\|\zeta_{j_0} v\|_{L_{p+1}(\mathbb{R})} \leq C \|\zeta_{j_0} v\|_{H^{s/2}(\mathbb{R})},$$

where C is independent of v. Combining the two inequalities above we get that

$$\|\zeta_{j_0} v\|_{L_{p+1}(\mathbb{R})} \geq \left[C^2 \left(1 + \frac{C_2}{\delta^3}\right)\right]^{\frac{1}{1-p}},$$

and since

$$\int_{j_0-2}^{j_0+2} |v|^{p+1} \, \mathrm{d}x \geq \|\zeta_{j_0} v\|^{p+1}_{L_{p+1}(\mathbb{R})},$$

the result follows with $\eta = \left[C^2 \left(1 + \frac{C_2}{\delta^3}\right)\right]^{\frac{p+1}{1-p}}$.

Having gathered the tools needed, we will now preclude vanishing and dichotomy, and hence prove that compactness occurs.

3.3 Excluding vanishing

We make the observation that any minimizing sequence $\{u_n\}_n$ of $\overline{I_q}$ satisfies the conditions for Lemma 2.2 with $\rho_n = \frac{1}{2}u_n^2$ and $\mu = q$. Combined with the previous section this gives us everything we need in order to exclude vanishing, as proven in the next lemma. The proof follows in the same manner as if the nonlinearity term was homogeneous.

Lemma 3.5. *Vanishing does not occur.*

Proof. From Lemma 3.4 and Lemma 3.2 we conclude that there exists a number $\eta > 0$ and a sequence $\{y_n\}_n \subset \mathbb{R}$ such that

$$\int_{y_n-2}^{y_n+2} |u_n|^{p+1}\, \mathrm{d}x \geq \eta$$

for all n. Hence by the use of Theorem 2.7 and Theorem 2.8 we obtain

$$\eta \leq C\|u_n\|_{H^{s/2}(\mathbb{R})}^{\frac{p-1}{s}} \left(\int_{y_n-2}^{y_n+2} |u_n|^2\, \mathrm{d}x \right)^{\frac{((p+1)s-p+1)}{s}} \leq CK^{\frac{p-1}{s}} \left(\int_{y_n-2}^{y_n+2} |u_n|^2\, \mathrm{d}x \right)^{\frac{((p+1)s-p+1)}{s}},$$

where C is the embedding constant and K is the constant from Lemma 3.2 part 1. Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \int_{y_n-2}^{y_n+2} |u_n|^2\, \mathrm{d}x \geq \frac{\eta^{\frac{s}{((p+1)s-p+1)}}}{CK^{\frac{p-1}{((p+1)s-p+1}}} > 0,$$

which excludes vanishing.

3.4 Excluding dichotomy

The most difficult part when showing that the compactness property holds is to prove that the phenomenon dichotomy does not occur. We start by proving a result that would follow if dichotomy were to happen, and then use the subadditivity property of our functional to arrive at a contradiction. Before doing so, we will need the following result from [4]:

Theorem 3.2. *Assume L to be on the form as in (3.2). Let $u \in H^{s/2}(\mathbb{R})$ and $\phi, \psi \in C^\infty(\mathbb{R})$ satisfy $0 \leq \phi \leq 1$, $0 \leq \psi \leq 1$ with*

$$\phi(x) = \begin{cases} 1, & if\ |x| < 1, \\ 0, & if\ |x| > 2, \end{cases}$$

and

$$\psi(x) = \begin{cases} 0, & if\ |x| < 1, \\ 1, & if\ |x| > 2. \end{cases}$$

Define $\phi_r(x) = \phi(x/r)$ and $\psi_r(x) = \psi(x/r)$ for all $x \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \phi_r u \left(L(\phi_r u) - \phi_r L u\right) \,\mathrm{d}x \to 0,$$

and

$$\int_{\mathbb{R}} \psi_r u \left(L(\psi_r u) - \psi_r L u\right) \,\mathrm{d}x \to 0$$

as $r \to \infty$.

Proof. Consult [4] for a detailed proof of the above.

Lemma 3.6. *Assume Dichotomy occurs. Then for each $\epsilon > 0$ there is a subsequence of $\{u_n\}_n$, still denoted $\{u_n\}_n$, a real number $\bar{q} \in (0, q)$, $N \in \mathbb{N}$ and two sequences $\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n \subset H^{s/2}(\mathbb{R})$ satisfying for all $n \geq N$:*

$$|\mathcal{Q}(u_n^{(1)}) - \bar{q}| < \epsilon, \tag{3.17}$$

$$|\mathcal{Q}(u_n^{(2)}) - (q - \bar{q})| < \epsilon, \tag{3.18}$$

$$\mathcal{E}(u_n) \geq \mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) - \epsilon. \tag{3.19}$$

Proof. We can, by assumption, for every $\epsilon > 0$, find a number $N \in \mathbb{N}$ and sequences $\{\rho_n^{(1)}\}$ and $\{\rho_n^{(2)}\}$ of positive functions satisfying the properties of Lemma 2.2 part 3, where $\rho_n = \frac{1}{2}u_n^2, \mu = q$ and $\bar{q} = \bar{\mu}$. We may assume that $\{\rho_n^{(1)}\}$ and $\{\rho_n^{(2)}\}$ satisfy

$$\begin{aligned} &\operatorname{supp} \rho_n^{(1)} \subset (y_n - R_n, y_n + R_n), \\ &\operatorname{supp} \rho_n^{(2)} \subset (-\infty, y_n - 2R_n) \cup (y_n + 2R_n, \infty), \end{aligned}$$

where $y_n \in \mathbb{R}$ and $R_n \to \infty$. Then

$$\frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} u_n^2 \, \mathrm{d}x \le \epsilon \tag{3.20}$$

for all $n \ge N$. Now choose ϕ, ψ as in Theorem 3.2 satisfying $\phi^2 + \psi^2 = 1$ in addition, and define $\phi_n(x) = \phi((x-y_n)/R_n)$, $\psi_n(x) = \psi((x-y_n)/R_n)$, $u_n^{(1)} = \phi_n u_n$ and $u_n^{(2)} = \psi_n u_n$. By the definitions of $u_n^{(1)}$ and $\rho_n^{(1)}$ we have

$$\begin{aligned}\left| \mathcal{Q}\left(u_n^{(1)}\right) - \int_{\mathbb{R}} \rho_n^{(1)} \, \mathrm{d}x \right| &= \int_{|x-y_n| \le R_n} |\frac{1}{2}u_n^2 - \rho_n^{(1)}| \, \mathrm{d}x + \frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} \phi_n^2 u_n^2 \, \mathrm{d}x \\ &\le \epsilon + \frac{1}{2}\int_{R_n \le |x-y_n| \le 2R_n} u_n^2 \, \mathrm{d}x \le 2\epsilon,\end{aligned}$$

for all $n \ge N$. The last inequality follows from (3.20). Since $\left|\int_{\mathbb{R}} \rho_n^{(1)} \, \mathrm{d}x - \bar{q}\right| \le \epsilon$, we get (3.17) by repeating the procedure with $\epsilon/3$. Comparing $\mathcal{Q}\left(u_n^{(2)}\right)$ to $\int_{\mathbb{R}} \rho_n^{(2)} \, \mathrm{d}x$, we obtain (3.18) with exactly the same arguments. In order to prove (3.19) we write

$$\begin{aligned}\mathcal{E}\left(u_n^{(1)}\right) + \mathcal{E}\left(u_n^{(2)}\right) = &\frac{1}{2}\Big[\int_{\mathbb{R}} \phi_n^2 u_n L u_n \, \mathrm{d}x + \int_{\mathbb{R}} \phi_n u_n \left(L(\phi_n u_n) - \phi_n L u_n\right) \, \mathrm{d}x\Big] \\ &+ \frac{1}{2}\Big[\int_{\mathbb{R}} \psi_n^2 u_n L u_n \, \mathrm{d}x + \int_{\mathbb{R}} \psi_n u_n \left(L(\psi_n u_n) - \psi_n L u_n\right) \, \mathrm{d}x\Big] \\ &- \int_{\mathbb{R}} (\phi_n^2 + \psi_n^2) N(u_n) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}} \Big[(\phi_n^2 - \phi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1})\Big] N(u_n) \, \mathrm{d}x.\end{aligned}$$

It follows from Theorem 3.2 that by taking n sufficiently large enough (such that $\int_{\mathbb{R}} \phi_n u_n \left(L(\phi_n u_n) - \phi_n L u_n\right) \, \mathrm{d}x + \int_{\mathbb{R}} \psi_n u_n \left(L(\phi_n u_n) - \psi_n L u_n\right) \, \mathrm{d}x < \epsilon$), we get

$$\mathcal{E}\left(u_n^{(1)}\right) + \mathcal{E}\left(u_n^{(2)}\right) \le \mathcal{E}(u_n) + \epsilon + \int_{\mathbb{R}} \Big[(\phi_n^2 - \phi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1})\Big] N(u_n) \, \mathrm{d}x.$$

For $|x-y_n| \notin (R_n, 2R_n)$ we have, by our choice of ϕ and ψ, that $\phi_n^2 = \phi_n^{p+1}$ and $\psi_n^2 = \psi_n^{p+1}$. Thus we have

$$\begin{aligned}
&\left| \int\limits_{\mathbb{R}} \left[(\phi_n^2 - \phi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1}) \right] N(u_n)\, \mathrm{d}x \right| \\
&\leq \int_{R_n \leq |x-y_n| \leq 2R_n} \left[(\phi_n^2 - \phi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1}) \right] |N(u_n)|\, \mathrm{d}x.
\end{aligned}$$

We also have that $0 \leq \phi_n \leq 1$, yielding $\phi_n^{p+1} \leq \phi_n^2$. The same holds for ψ. This gives us

$$\begin{aligned}
&\int_{R_n \leq |x-y_n| \leq 2R_n} \left[(\phi_n^2 - \phi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1}) \right] |N(u_n)|\, \mathrm{d}x \\
&\leq \int_{R_n \leq |x-y_n| \leq 2R_n} 2|N(u_n)|\, \mathrm{d}x \\
&\leq \int_{R_n \leq |x-y_n| \leq 2R_n} 2|N_p(u_n)|\, \mathrm{d}x + \int_{R_n \leq |x-y_n| \leq 2R_n} 2|N_r(u_n)|\, \mathrm{d}x \\
&\leq 2K_1 \epsilon^{((p+1)s-p+1)/s} K^{(p-1)/s} + 2K_2 \epsilon^{((p+1+\gamma)s-p+1-\gamma)/s} K^{(p+\gamma-1)/s}.
\end{aligned}$$

Here we have again used Theorem 2.7 and 2.8 on compact domains combined with the boundedness of u_n in $H^{s/2}(\mathbb{R})$ and (3.20). Since $\frac{p-1}{s} < \frac{p-1+\gamma}{s} < 2$ and $p > 1$, we have that $((p+1)s - p + 1)/s = p + 1 - \frac{(p-1)}{s} > 0$ and $((p+1+\gamma)s - p + 1 - \gamma)/s = p + 1 + \gamma - \frac{(p+\gamma-1)}{s} > 0$. This proves that there is an N such that (3.19) holds for all $n \geq N$.

Also here the proof goes in the same fashion as in [4], but we have corrected some minor mistakes. The higher order term does change the proof, but not to an extent where it yielded any major problems. This lemma can now be used to get a contradiction with Lemma 3.3, as illustrated in the following corollary:

Corollary 3.1. *Dichotomy does not occur.*

Proof. Assume to the contrary that Dichotomy occurs for a minimizing sequence $\{u_n\}_n$. Then by Lemma 3.6 there is for every $\epsilon > 0$, a subsequence of $\{u_n\}_n$, still denoted $\{u_n\}_n$, a number $\bar{q} \in (0,q)$, $N \in \mathbb{N}$ and two sequences $\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n \subset H^{s/2}(\mathbb{R})$ such that (3.17), (3.18) and (3.19) are satisfied for all $n \geq N$. Then we also have

$$\begin{aligned}
\overline{I_q} &= \liminf_{n\to\infty}\left(\mathcal{E}(u_n) - \frac{1}{2}q\right) \\
&\geq \liminf_{n\to\infty}\left(\mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) - \frac{1}{2}q - \epsilon\right) \\
&\geq \overline{I_{\bar{q}+r_1}} + \overline{I_{(q-\bar{q})+r_2}} - \epsilon,
\end{aligned}$$

where $r_1, r_2 \in (-\epsilon, \epsilon)$. Letting $\epsilon \to 0^+$, we get a contradiction with Lemma 3.3. This proves that dichotomy does not occur. We are now finally ready to present the main existence result of this section.

3.5 Existence of minimizer

Having excluded vanishing and dichotomy we know that compactness occurs. This makes it possible for us to prove the existence of a minimizer to our variational problem.

Lemma 3.7. *Let $\{u_n\}_n$ be a minimizing sequence for $\overline{I_q}$. Then there exists a sequence $\{y_n\}_n \subset \mathbb{R}$ such that the sequence $\{\tilde{u}_n\}_n$, defined by $\{\tilde{u}_n\}_n = u_n(x+y_n)$, has a subsequence that converges in $H^{s/2}(\mathbb{R})$ to a minimizer of $\overline{I_q}$. In particular, the set of minimizers is nonempty.*

Proof. Let $\{u_n\}_n$ be a minimizing sequence for $\overline{I_q}$. By Corollary 3.1 and Lemma 3.4 we know that compactness occurs. That is, there is a subsequence of $\{u_n\}_n$, again denoted $\{u_n\}_n$, and a sequence $\{y_n\}_n \subset \mathbb{R}$ such that for every $\epsilon > 0$, there exists $0 < r < \infty$ satisfying for all $n \in \mathbb{N}$:

$$\frac{1}{2}\int_{|x-y_n|\leq r} u_n^2 \,\mathrm{d}x \geq q - \epsilon. \tag{3.21}$$

This implies that for every $k \in \mathbb{N}$ we can find $r_k \in \mathbb{R}_+$ so that

$$\frac{1}{2}\int_{|x|\leq r_k} \tilde{u}_n^2 \,\mathrm{d}x \geq q - \frac{1}{k}. \tag{3.22}$$

Furthermore, by Lemma 3.2 and Theorem 2.7, for every $k \in \mathbb{N}$, there is a subsequence of $\{\tilde{u}_n\}_n$, denoted $\{\tilde{u}_{n,k}\}_n$, and a function $\omega_k \in L_2\left(\left[-r_k, r_k\right]\right)$ such that $\{\tilde{u}_{n,k}\}_n \to \omega_k$ in $L_2\Big(\Big[-r_k, r_k\Big]\Big)$. From the inequalities in (3.21) and (3.22) we can deduce that $\mathcal{Q}(\omega_k) \geq q - \frac{1}{k}$.

A Cantor diagonalization argument on the sequences $\{\tilde{u}_{k,n}\}_n$ yields a subsequence of $\{\tilde{u}_n\}_n$, still denoted $\{\tilde{u}_n\}_n$, that converges strongly in $L_2(\mathbb{R})$ to some function $\omega \in L_2(\mathbb{R})$. Thus we have that

$$\mathcal{Q}(\omega) = q. \tag{3.23}$$

As the sequence $\{\tilde{u}_n\}_n$ is bounded we have weak convergence to some element $w \in H^{s/2}(\mathbb{R})$ by the Banach Alaoglu theorem. Weak convergence to ψ in $H^{s/2}(\mathbb{R})$ imply weak convergence to ψ in $L_2(\mathbb{R})$. Since limits are unique we have that $w = \omega$. Also, the weak $H^{s/2}(\mathbb{R})$ convergence and strong $L_2(\mathbb{R})$ convergence imply convergence in $L_{p+1+\gamma}(\mathbb{R})$:

$$\begin{aligned}
&\|\tilde{u}_n - \omega\|_{L_{p+1+\gamma}(\mathbb{R})} \leq C\|\tilde{u}_n - \omega\|_{H^{(p+\gamma+1)/(2(p+1+\gamma))}(\mathbb{R})} \\
&\leq C\|\tilde{u}_n - \omega\|_{L_2(\mathbb{R})}^{\frac{((p+1+\gamma)s-p+1-\gamma)}{s(p+1+\gamma)}} \|\tilde{u}_n - \omega\|_{H^{s/2}(\mathbb{R})}^{(p-1+\gamma)/(s(p+1+\gamma))} \\
&\leq C'\|\tilde{u}_n - \omega\|_{L_2(\mathbb{R})}^{\frac{((p+1+\gamma)s-p+1-\gamma)}{s(p+1+\gamma)}},
\end{aligned}$$

where we have made use of embedding and interpolation results by Sobolev. The same result can be shown similarly for the convergence in $L_{p+1}(\mathbb{R})$. Strong convergence in $L_{p+1+\gamma}(\mathbb{R})$ and $L_{p+1}(\mathbb{R})$ imply

$$\lim_{n\to\infty} \int_{\mathbb{R}} N(\omega) - N(u_n)\, \mathrm{d}x = 0.$$

By Corollary 2.1 we also have that

$$\liminf \|\omega\|_{H^{s/2}(\mathbb{R})} \leq \|u_n\|_{H^{s/2}(\mathbb{R})}.$$

These two results combined yields

$$\mathcal{E}(\omega) - \frac{1}{2}q \leq \liminf \mathcal{E}(u_n) - \frac{1}{2}q = \overline{I_q}. \tag{3.24}$$

Hence by (3.23), (3.24) and the definition of $\overline{I_q}$ we have that $\mathcal{E}(\omega) = \overline{I_q}$. This proves that $\tilde{u}_n \to \omega$, where ω is a minimizer of I_q and belongs to $H^{s/2}(\mathbb{R})$. We conclude that the proof is complete.

It now only remains to prove that a minimizer of I_q actually solves 3.6 (in the sense of distributions).

3.6 Minimizer found solves the PDE under study

In order to prove the result in this section, normal procedure would be to apply the method of Lagrange multipliers. However, the original proof given below provides a deeper understanding of the result, which is why we have chosen to take this more troublesome approach.

Lemma 3.8. *Any minimizer of I_q is a solution to 3.6 with wave speed c.*

Proof. Assume that ω minimizes

$$\mathcal{E}(\omega) = \int_{\mathbb{R}} \omega L \omega \, \mathrm{d}x - \int_{\mathbb{R}} N(\omega) \, \mathrm{d}x$$

with respect to the constraint

$$\int_{\mathbb{R}} \omega^2 \, \mathrm{d}x = q.$$

Now pick an arbitrary parametric curve $\phi(t) \in H^{s/2}(\mathbb{R})$, $t \in (-\epsilon, \epsilon)$ with $\phi(0) = \omega$. Since ω is assumed to minimize $\mathcal{E}(\cdot)$, we have

$$\frac{d}{dt}\Big|_{t=0} \left(\frac{1}{2} \int_{\mathbb{R}} \phi L \phi \, \mathrm{d}x - \int_{\mathbb{R}} N(\phi) \, \mathrm{d}x \right) = 0$$

yielding

$$\int_{\mathbb{R}} \dot{\phi}(0) L \phi(0) \, \mathrm{d}x - \left(\int_{\mathbb{R}} \dot{\phi}(0)(n_p(\phi) + n_r(\phi)) \, \mathrm{d}x \right) = 0,$$

where $\dot{\phi}(0)$ denotes the derivative of ϕ with respect to t at $t = 0$. In total we have that

$$\langle \dot{\phi}(0), L\phi(0) - (n_p + n_r) \rangle_{L_2(\mathbb{R})} = 0.$$

We also have that

$$\int_{\mathbb{R}} \omega^2 \, \mathrm{d}x = q \neq 0,$$

giving us

$$\frac{d}{dt}\Big|_{t=0}\left(\int_{\mathbb{R}} \phi(t)^2 \, \mathrm{d}x\right) = \int_{\mathbb{R}} 2\dot{\phi}(0)\phi(0) \, \mathrm{d}x = 0.$$

This gives us yet another orthogonality relation in $L_2(\mathbb{R})$:

$$\langle \dot{\phi}(0), \phi(0)\rangle_{L_2(\mathbb{R})} = 0.$$

From these arguments, and since $\phi(0) = \omega$, we can conclude that there exists $\lambda \in \mathbb{R}$ such that

$$L\omega - n(\omega) + \lambda\omega = 0$$

in the sense of distributions. The proof is complete by noticing that $\lambda = c$ gives (3.6).

3.7 Solutions inherit regularity from the equation itself

Having proven that there exists solutions to (3.6), it is possible to show that the solutions actually inherit regularity from the equation itself. Notice that the inverse of L can be written explicitly as $L^{-1}u = \mathscr{F}^{-1}\{(1+|\xi|^2)^{-s/2}\hat{u}(\xi)\}$ such that $L^{-1}Lu = u$. By rewriting (3.6) we get:

$$u = L^{-1}\left(n(u) - cu\right) \tag{3.25}$$

Also, since L defines a mapping from $H^r(\mathbb{R})$ to $H^{r-s}(\mathbb{R})$, i.e.

$$\|Lu\|_{H^{m-s}(\mathbb{R})} = \|u\|_{H^m(\mathbb{R})},$$

we have that L^{-1} defines a map from $H^r(\mathbb{R})$ to $H^{r+s}(\mathbb{R})$. We also have that if $u \in H^r(\mathbb{R})$, then also $n(u) \in H^r(\mathbb{R})$, since if $r > 1/2$, we have

$$|\widehat{u^p}| = \left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x} u^p \, \mathrm{d}x\right| \leq |\sup_x u^{p-1}| \left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x} u \, \mathrm{d}x\right|,$$

which yields

$$\|u^p\|_{H^r(\mathbb{R})} = \int_{\mathbb{R}} (1+|\xi|^2)^r |\widehat{u^p}|^2 \, \mathrm{d}x \leq \left(\sup_x |\widehat{u^{p-1}}|^2\right)\|u\|_{H^r(\mathbb{R})} < \infty.$$

From the above and (3.25) we conclude that if a solution u is in $H^r(\mathbb{R})$, then u is also in $H^{r+s}(\mathbb{R})$. By iteration we have that $u \in H^\infty(\mathbb{R})$. Note that we start the iteration with $r > 1/2$, since we then have the embedding $H^r(\mathbb{R}) \hookrightarrow BC(\mathbb{R})$.

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