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A new anisotropic failure criterion for transversely isotropic solids

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SUMMARY

A coordinate-free formulation of a failure criterion for transversely isotropic solids is proposed. In the three-dimensional stress space the criterion is represented by an elliptic paraboloid. The anisotropic form of the proposed criterion is based on generalization of the second invariant of the deviatoric stress and of the mean stress obtained through the introduction of a unique fourth-order tensor. For isotropic conditions, the criterion reduces to the Mises–Schleicher failure condition. It is shown that the criterion satisfactorily predicts the strength anisotropy of transversely isotropic rocks subjected to an axisymmetric stress state. The procedure for the identification of the parameters of the criterion from a few simple laboratory tests is outlined. © 1998 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Most rocks are anisotropic. We can distinguish between an intrinsic anisotropy and an anisotropy induced by the stress–strain history. The intrinsic anisotropy is the result of the process of rock formation, and of various environmental factors (such as diagenesis, metamorphosis or weathering). At the macroscopic level, the directionality of the mechanical properties is related to the existence of well-defined rock fabric elements, such as bedding, layering, foliation and lamination planes, or the existence of linear structures. The symmetries most frequently encountered are: transverse isotropy and orthotropy. Other geomaterials, such as sands or normally consolidated clays, or rock salt, may be essentially isotropic under zero effective stress and become anisotropic due to the deformation process. To determine the type or degree of anisotropy of rocks, dynamic methods can be used. Measurements of the travel times of seismic waves propagating in various directions provide the matrix of elastic coefficients. Afterwards, uniaxial and triaxial tests can be performed in the directions that are necessary for the determination of the constitutive equation. Anisotropy is responsible for a formidable complication of both theoretical and experimental aspects of rock modelling. The

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number of parameters to be experimentally determined is very large. Since the failure defines the boundary of the constitutive domain, the strength characteristics reflect the anisotropy of the material, which is a combination of both the structural anisotropy and the anisotropy induced by the irreversible deformation. Therefore, the first step in modelling rock anisotropy consists of describing the short-term failure surface. The failure surface is defined as the locus of points that separate the stress state, that can be reached in a given material, from the states of stress that cannot be attained. The term failure is used to describe either the stress state at which macrofracture occurs, or the peak stress attained during ductile deformation. To determine this surface, laboratory experiments of different types may be performed to obtain stress points that produce failure. However, the failure surface is not unique. It may depend on various factors, such as size and shape of the specimen, loading history, environmental conditions (e.g., temperature, humidity, etc.), porosity and homogeneity of the material.

Experimental studies, on rock exhibiting planar anisotropy subjected to an axisymmetric state of stress, have shown that the strength varies with the orientation angle β[1—10] (β is the angle between the strata's planes and the direction of maximum compression). The minimum strength occurs for β around 30°, (for most rocks $\beta \in (30^\circ, 45^\circ)$) while the maximum compressive strength occurs at an orientation angle β of 0° or 90° depending upon the rock type. In some cases, the curve between compression strength and orientation is concave upwards over its whole range (e.g., schists, slates, shales, phyllites); in other cases the curve tends to have flat 'shoulders' near the extreme orientations (as in the case of jointed rocks), or to be of undulatory type.[11] Three types of failure have been observed: shear both across and along the bedding or cleavage planes; slip along the bedding planes; and formation of kink boards.[12] Theories for describing the continuous variation of strength with orientation for transversely isotropic rocks that fail in shear have been proposed by several authors (e.g. Jaeger,[5] McLamore and Gray,[7] Ramamurthy[11]). Starting from submicroscopic conditions of brittle fracture, several criteria have been derived from the Griffith crack model in which it is assumed that there are two populations of microcracks: long cracks parallel to the planar isotropy, and short cracks randomly orientated (e.g. Hoek,[13] Walsh and Brace,[14] Barron[15]). However, all these theories require a wide range of tests and a large amount of curve fitting. The numerical implementation is generally difficult, these theories cannot be applicable to truly 3-D stress states. A more general approach was proposed by Goldenblat and Kopnov.[16] These authors suggested the use of strength tensors and formulated an anisitropic criterion in the following form:

$$(F_i\sigma_i)^\alpha + (F_{ij}\sigma_i\sigma_j)^\beta + (F_{ijk}\sigma_i\sigma_j\sigma_k)^\gamma + \cdots = 1, \tag{1}$$

where the contracted notation is used, and $i, j, k = 1, 2, \ldots, 6$ ($\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$, $\sigma_3 = \sigma_{33}$, $\sigma_4 = \sigma_{23}$, $\sigma_5 = \sigma_{13}$, $\sigma_6 = \sigma_{12}$). They investigated the special case: $\alpha = 1, \beta = \frac{1}{2}, \gamma = -\infty$. A failure criterion that ignores strength tensors of higher order than two are proposed by Tsai and Wu.[17] Though proposed and investigated in the context of fibre-reinforced composites, Tsai and Wu's criterion is widely used in engineering for different types of anisotropic materials. For geological materials a widely used criterion is Pariseau's criterion.[18] To take into account the possibility of unequal tensile and compressive strengths, and to describe the influence of the hydrostatic stress, Pariseau extended Hill's criterion[19] by including a linear term in σ_{11}, σ_{22} and σ_{33}. However, these criteria are not expressed in co-ordinate-free form, i.e., in terms of invariants. The application of these criteria to general stress conditions (i.e., for any orientation of the principal stress axes with respect to the co-ordinate system associated with the specific structural material symmetry) is rather difficult. A general theory of the flow and fracture of anisotropic solids was developed by Boehler and Sawczuk[20,21] and Boehler[22] in the framework of the theory of invariance. Specific forms of failure criteria were proposed for rocks (e.g., diatomite[8]) and for composites.[23] Generalizations of Coulomb–Navier and von Mises isotropic failure criteria to anisotropic conditions (orthotropic and transversely isotropic media) can be found in Boehler.[24] Nova and Sacchi[25] proposed a generalized failure condition for orthortropic solids that was subsequently applied to describe the failure of transversely isotropic rocks in compression (see Nova[26]). Theocaris[27–29] proposed an elliptic paraboloid failure criterion also presenting the differential strength effect. The criterion was applied to a great number of transversely isotropic materials, such as fibre-reinforced composites, cellular solids and brittle foams.

The present paper focuses on the development of a new macroscopic failure criterion that can describe accurately the strength characteristics of transversely isotropic materials. For isotropic conditions, the criterion reduces to the Mises–Schleicher criterion. The proposed anisotropic Mises–Schleicher criterion (AMS), is written in a general form. The adequacy of the AMS criterion is demonstrated by applying it to the various transversely isotropic rocks, using experimental results taken from the literature.

2. FORMULATION OF THE FAILURE CRITERION

Anisotropic behaviour is detected if there exists a rotation of the applied stresses that results in a different strain-rate history or, equivalently, if a different stress response is obtained from a rotation of the applied displacements. Materials symmetries are described by those orthogonal transformations $\mathbf{Q}$ that produce no such difference in response. The set of all these orthogonal tensors $\mathbf{Q}$ forms the material symmetry group.[30] Transverse isotropy is usually characterized by the group g of rotations about a preferred direction denoted, say, by the unit vector S_1:

$$g = \{\mathbf{Q} \in O(3) | \mathbf{Q}(S_1) = S_1 \text{ or } \mathbf{Q}(S_1) = -S_1\}, \tag{2}$$

or equivalently,

$$\begin{gathered} g = \{\mathbf{Q} \in O(3) | \mathbf{Q}\mathbf{M}\mathbf{Q}^{\mathrm{T}} = \mathbf{M}\}, \\ \mathbf{M} = S_1 \otimes S_1 \cdot \end{gathered} \tag{3}$$

We emphasize that the anisotropy characterized by second-order tensor $\mathbf{M}$ is the initial material symmetry, if the undeformed state is taken as the reference configuration. The directional material properties impose definite restrictions on the form of the constitutive equations. Accordingly, an appropriate framework for modelling the mechanical behaviour of anisotropic solids is offered by the theory of invariance.[31–33] It was proved that any scalar, vector or second-order tensor-valued anisotropic function of vectors and second-order tensors, can be expressible as an isotropic function of the original arguments, and of the structural tensors as additional arguments.[34] Therefore, any scalar property, such as the failure function, say, $f(\sigma)$ can be represented relative to its symmetry group by an isotropic function f_1 of σ, and the structural tensors. For a transversely isotropic material it follows that $f_1 = f_1(\sigma, \mathbf{M})$. The requirement that $f_1(\sigma, \mathbf{M})$ is isotropic implies that f_1 is a function of the five independent invariants:

$$I_1 = \mathrm{tr}\ \sigma, \qquad J_2 = \frac{1}{2}\mathrm{tr}\ (\sigma')^2, \qquad I_3 = \mathrm{tr}\ (\sigma)^3, \qquad I_4 = \mathrm{tr}\ \mathbf{M}\sigma, \qquad I_5 = \mathrm{tr}\ \mathbf{M}\sigma^2, \tag{4}$$

where prime stands for deviator, and 'tr' denotes the trace operator. Thus, failure occurs when the loading conditions satisfy a relationship of the following type:

$$f_1(I_1, J_2, I_3, I_4, I_5) = 1, \tag{5}$$

When modelling rock behaviour, special consideration of the effects of hydrostatic pressure on deformation and strength is required.

To describe these particular characteristics, we propose the following criterion:[35,36]

$$\frac{3}{2}\operatorname{tr}(\Sigma') - \frac{m}{3} tr\, \Sigma - 1 = 0, \tag{6}$$

where m is a material constant, and Σ' is the deviator of the second-order tensor Σ, defined by:

$$\Sigma_{ij} = B_{ijkl}\sigma_{kl}. \tag{7}$$

The anisotropy is introduced by means of the fourth-order tensor $\mathbf{B}$ that satisfies the usual symmetry conditions:

$$B_{ijkl} = B_{jikl} = B_{klij} = B_{ijlk}. \tag{8}$$

$\mathbf{B}$ is supposed to be constant: it does not depend on time or other environmental conditions. The idea to introduce a fourth-order tensor to describe the strength anisotropy is not new (see Goldenblat and Kopnov,[16] Boehler and Sawzuck,[20,21] among others). However, in contrast to the other existing criteria, the only restriction we impose on $\mathbf{B}$ is to be invariant under any orthogonal transformation belonging to the symmetry group g. Rather than making simplified assumptions concerning the form of $\mathbf{B}$, as done by Boehler[22] all the five components of this tensor are considered as independent strength parameters. Thus, in the structural system (S_1, S_2, S_3) the truncated matrix of $\mathbf{B}$ is:

$$\mathbf{B} = \begin{bmatrix} a & b & b & 0 & 0 & 0 \\ b & d & e & 0 & 0 & 0 \\ b & e & d & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d-e}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{c}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c}{2} \end{bmatrix} \tag{9}$$

where a, b, c, d and e are material constants and (S_2, S_3) defines the symmetry plane. Using (7) and (9), we can express the invariants of Σ in terms of the stress invariants, and of the mixed invariants of σ and $\mathbf{M}$:

$$\begin{aligned} \operatorname{tr}\Sigma &= (a+b-d-e)\operatorname{tr}\mathbf{M}\sigma + (b+d+e)\operatorname{tr}\sigma \\ \operatorname{tr}(\Sigma')^2 &= AA(\operatorname{tr}\mathbf{M}\sigma)^2 + BB(\operatorname{tr}\sigma)^2 + CC(\operatorname{tr}\sigma)(\operatorname{tr}\mathbf{M}\sigma) \\ &\quad + DD(\operatorname{tr}\mathbf{M}\sigma^2) + EE\operatorname{tr}\sigma^2, \end{aligned} \tag{10}$$

where AA, BB, CC, DD and EE are algebraic combinations of the coefficients a, b, c, d, e and m. Therefore, the AMS criterion (with $\mathbf{B}$ defined by (9)) is a specific form of the general criterion (5). Also, the first invariant of Σ is linear in σ and thus can be thought as a generalization to transversely isotropic conditions of the mean stress (or I_1). The second invariant of the transformed tensor deviator is a quadratic homogeneous function of σ, and reduces to J_2 for $a = d = c = 1$, $b = e = 0$. Let us show that for isotropic conditions, the AMS criterion reduces to a Mises–Schleicher paraboloid surface. Indeed, if σ_T and σ_C denote the tensile and compressive strengths respectively, the Mises–Schleicher criterion may be expressed in the form (e.g., see Lubliner[37]):

$$3J_2 + (\sigma_C - \sigma_T)I_1 - \sigma_T\sigma_C = 0, \tag{11}$$

where $I_1 = \sigma_{kk}, J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij}$. For isotropic conditions, **B** is of the form:

$$B_{ijkl} = b\delta_{ij}\delta_{kl} + \left(\frac{a-b}{2}\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{12}$$

Consequently, if we replace in the expression (7) of Σ, **B** by its expression (12), the AMS criterion becomes:

$$\frac{3(a-b)^2}{2}\mathrm{tr}(\sigma')^2 - \frac{m(a+2b)}{3}\mathrm{tr}(\sigma) - 1 = 0, \tag{13}$$

From uniaxial tensile and compression tests, we get:

$$(a-b)^2 = \frac{1}{\sigma_C\sigma_T}, \tag{14}$$

$$-\frac{m(a+2b)}{3} = \frac{1}{\sigma_C} - \frac{1}{\sigma_T}. \tag{15}$$

Thus, (13) coincides with (11).

In the structural system (S_1, S_2, S_3), the AMS criterion is expressed by:

$$\begin{aligned} a_1\sigma_{11} + a_2(\sigma_{22} + \sigma_{33}) + A_{11}\sigma_{11}^2 + A_{22}(\sigma_{22}^2 + \sigma_{33}^2) + 2A_{12}\sigma_{11}(\sigma_{22} + \sigma_{33}) \\ + 2A_{23}\sigma_{22}\sigma_{33} + A_{44}\sigma_{23}^2 + A_{55}(\sigma_{12}^2 + \sigma_{13}^2) = 1. \end{aligned} \tag{16}$$

where the coefficients in (16) are given by:

$$\begin{aligned} a_1 &= \frac{-m(a+2b)}{3} \\ a_2 &= \frac{-m(b+e+d)}{3} \\ A_{11} &= (a-b)^2 \\ A_{22} &= \frac{(b-e)^2 + (d-e)^2 + (b-d)^2}{2} \\ A_{44} &= 3(d-e)^2 \\ A_{55} &= 3c^2 \\ A_{23} &= A_{22} - \frac{A_{44}}{2} \\ A_{12}^2 &= \frac{A_{11}(A_{22} + A_{23})}{2} \end{aligned} \tag{17}$$

The physical interpretation of the parameters of the criterion may be revealed from simple laboratory tests. From shear tests in the (S_2, S_3) plane and in the (S_1, S_2) plane respectively, we get

$$\begin{aligned} A_{44} &= 1/Q^2 \\ A_{55} &= 1/R^2 \end{aligned} \tag{18}$$

where Q denotes the pure shear strength in the (S_2, S_3) plane, whereas R is the corresponding one in the (S_1, S_2) plane. From uniaxial tests along the S_1 and the S_2 axis, respectively:

$$\begin{aligned} a_1 &= 1/X_C - 1/X_T \\ a_2 &= 1/Y_C - 1/Y_T \\ A_{11} &= 1/(X_T X_C) \\ A_{22} &= 1/(Y_T Y_C). \end{aligned} \tag{19}$$

Here, and throughout the text, the compressive stresses are taken positive. X_C and $(-X_T)$ are the uniaxial compressive and tensile strengths along S_1 while Y_C and $(-Y_T)$ are the uniaxial compressive and tensile strengths along S_2. Relations (17) imply that for the AMS criterion the interaction coefficients A_{12} and A_{23} are interrelated with the diagonal components A_{11}, A_{22} and A_{44}, so they are directly defined in terms of the basic engineering strengths of the material. This is a significant advantage of the AMS criterion over most existing failure criteria. As an example, in the Tsai and Wu criterion,[17] the determination of the off-diagonal coefficient F_{12} (i.e., the coefficient of $\sigma_{11}\sigma_{22}$ in the expression of the criterion in the structural coordinate system) has been found to be very sensitive and dependent on the nature of the particular test used for its determination. To overcome the difficulties related to the optimal experimental evaluation of F_{12}, several definitions have been proposed (see Tsai and Hahn,[38] Cowin,[39] Wu and Stackhurski[40]). However, estimating the value of F_{12} is still a debated question (Labossière and Neale[41]).

For a better understanding of the characteristics of the failure surface it is required to represent it in the three-dimensional space of the principal stresses $\sigma_1, \sigma_2, \sigma_3$. The case of one of the principal stress axes, the σ_2-axis say, coinciding with the corresponding structural axis S_2, whereas the two other stress axes rotate about S_2, will be analysed. This allows the description of failure when one of the principal stress directions is parallel to the strike of the planes of symmetry (see Figure 1). The AMS criterion is expressed in the principal stress system (X_1, X_2, X_3) by:

$$a'_1\sigma_1 + a'_2\sigma_2 + a'_3\sigma_3 + A'_{11}\sigma_1^2 + A'_{22}\sigma_2^2 + A'_{33}\sigma_3^2 + 2A'_{12}\sigma_1\sigma_2 + 2A'_{13}\sigma_3\sigma_1 + 2A'_{23}\sigma_2\sigma_3 = 1. \tag{20}$$

The expressions of the new coefficients A'_{ij} and a'_i, in terms of the coefficients A_{ij}, a_i and the angle θ are given in the Appendix. To determine the shape of the failure surface (20), the following quantities

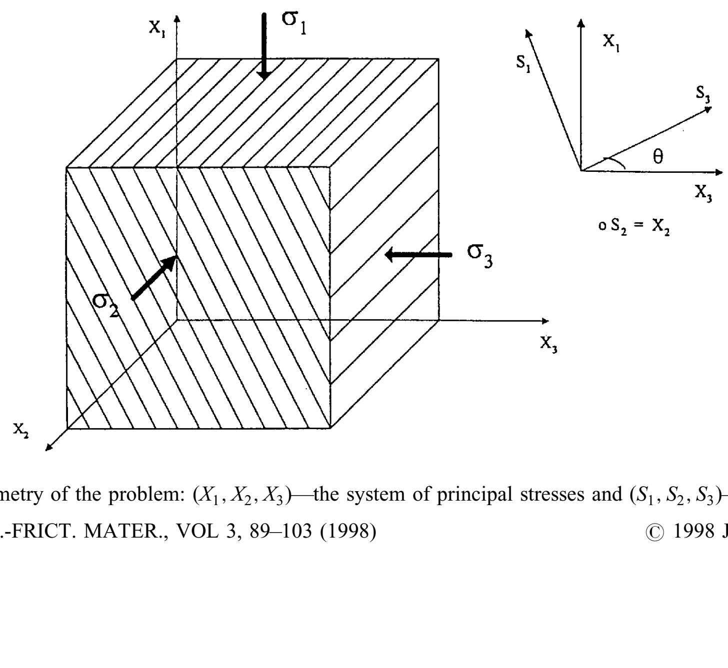

Figure 1. Geometry of the problem: (X_1, X_2, X_3)—the system of principal stresses and (S_1, S_2, S_3)—the structural system

need to be evaluated: (i) determinant of D_4, and (ii) determinant of D_3, where the matrices D_4 and D_3 are defined by:

$$D_4 = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & \frac{a'_1}{2} \\ A'_{12} & A'_{22} & A'_{23} & \frac{a'_2}{2} \\ A'_{13} & A'_{23} & A'_{33} & \frac{a'_3}{2} \\ \frac{a'_1}{2} & \frac{a'_2}{2} & \frac{a'_3}{2} & -1 \end{bmatrix} \tag{21}$$

$$D_3 = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{12} & A'_{22} & A'_{23} \\ A'_{13} & A'_{23} & A'_{33} \end{bmatrix} \tag{22}$$

For $A_{12} < 0$, using (41) we get

$$\det D_3 = \frac{A_{55}A_{44}}{16}[(\sqrt{A_{11}} - \sqrt{del})\sin 2\theta]^2 \tag{23}$$

and

$$\det D_4 = \frac{(E + FA_1 + GA_1^2)\sin(2\theta)^2 - A_{44}(2A_2\sqrt{A_{11}} + A_1\sqrt{del})^2}{16} \tag{24}$$

where:

$$\begin{aligned} E &= -A_{55}A_{44}[A_2^2 + (\sqrt{A_{11}} - \sqrt{del})^2] + A_{11}A_2^2(-A_{55} + 4A_{44}) \\ F &= A_2[(4A_{44} - A_{55})\sqrt{A_{11}del} - A_{55}A_{44}] \\ G &= A_{44}del - A_{22}A_{55} \\ del &= 4A_{22} - A_{44}. \end{aligned} \tag{25}$$

For $A_{12} > 0$:

$$\det D_3 = \frac{A_{55}A_{44}}{16}[(\sqrt{A_{11}} + \sqrt{del})\sin 2\theta]^2, \tag{26}$$

$$\det D_4 = \frac{(M + NA_1 + GA_1^2)\sin(2\theta)^2 - A_{44}(-2A_2\sqrt{A_{11}} + A_1\sqrt{del})^2}{16} \tag{27}$$

where

$$\begin{aligned} M &= -A_{55}A_{44}[A_2^2 + (\sqrt{A_{11}} + \sqrt{del})^2] + A_{11}A_2^2(-A_{55} + 4A_{44}) \\ N &= -A_2[(4A_{44} - A_{55})\sqrt{A_{11}del} + A_{55}A_{44}]. \end{aligned} \tag{28}$$

From (23) to (28) it follows that for $\theta = 0°$ and $\theta = 90°$: $\det D_3 = 0$ and $\det D_4 < 0$. Therefore, for these orientations, the failure surface is an *elliptic paraboloid*. For the failure surface to be of the same type for any orientation, $\det D_3$ must be equal to zero for any θ. From (23) and (25) it follows that the following conditions must be fulfilled:

$$\begin{aligned} A_{12} &< 0 \\ A_{11} &= del \end{aligned} \tag{29}$$

If conditions (29) hold, equation (24) becomes:

$$\det D_4 = -\frac{(A_1 + 2A_2)^2[A_{44}A_{11}(\cos 2\theta)^2 + A_{22}A_{55}(\sin 2\theta)^2]}{16}. \tag{30}$$

Hence, $\det D_4 < 0$, for any θ. In conclusion, if the conditions (29) between the engineering strengths hold, for any orientation θ the AMS failure surface is an elliptic paraboloid in the three-dimensional space of the principal stresses.

3. INTERSECTION OF THE AMS CRITERION BY THE PLANE $\sigma_1 = \sigma_2$

Because rock specimens are often submitted to a triaxial stress state in which two of the principal stresses are equal, it is worthwhile analysing the intersection of the AMS criterion with the usual triaxial plane. The equation expressing the intersection of the AMS criterion with the plane $(\sigma_3, \sqrt{2}\sigma_1 = \sqrt{2}\sigma_2)$ is readily defined by putting into equation (20), $\sigma_1 = \sigma_2$:

$$\alpha x^2 + \beta\sigma_3^2 + 2\delta x\sigma_3 + 2\gamma x + 2\eta\sigma_3 - 1 = 0 \tag{31}$$

where:

$$\begin{aligned} x &= \sqrt{2}\sigma_1 \\ \alpha &= \frac{1}{2}(A'_{11} + A'_{22} + 2A'_{12}) \\ \beta &= A'_{33} \\ \delta &= \frac{1}{\sqrt{2}}(A'_{13} + A'_{23}) \\ \gamma &= \frac{1}{2\sqrt{2}}(a'_1 + a'_2) \\ \eta &= \frac{1}{2}a'_3. \end{aligned} \tag{32}$$

To decide the nature of the conic (31), the following quantities need to be evaluated:

$$\Delta = \begin{bmatrix} \alpha & \delta & \gamma \\ \delta & \beta & \eta \\ \gamma & \eta & -1 \end{bmatrix}, \qquad J = \begin{bmatrix} \alpha & \delta \\ \delta & \beta \end{bmatrix}, I = \alpha + \beta \tag{33}$$

For $A_{12} < 0$ and $A_{11} = del$ (i.e., if conditions (29) hold):

$$\begin{aligned} J &= 0 \\ \Delta &= -\frac{1}{32}(A_1 + 2A_2)^2\{[3(\cos\theta)^2 - 2]^2 del + A_{44}(\cos 2\theta)^4 + 4A_{55}(\cos\theta)^2(\sin\theta)^2\}. \end{aligned} \tag{34}$$

Hence, for any orientation θ, the intersection is a parabola. Thus, the failure locus is 'open' on the compressive side, showing that the axial pressure may be increased without limit, if the confining pressure is increased proportionally. However, the failure curve is 'closed' on the tensile side. Thus, failure can occur under tensile hydrostatic pressure. The hydrostatic tensile strength is:

$$p = \frac{1}{2(1/Y_C - 1/Y_T) + (1/X_C - 1/X_T)}. \tag{35}$$

In a plane-stress test, on the other hand, the failure curve is 'closed' and the strength can only achieve a finite magnitude. As an example, in Figure 2 is shown the intersection of the failure surface with the

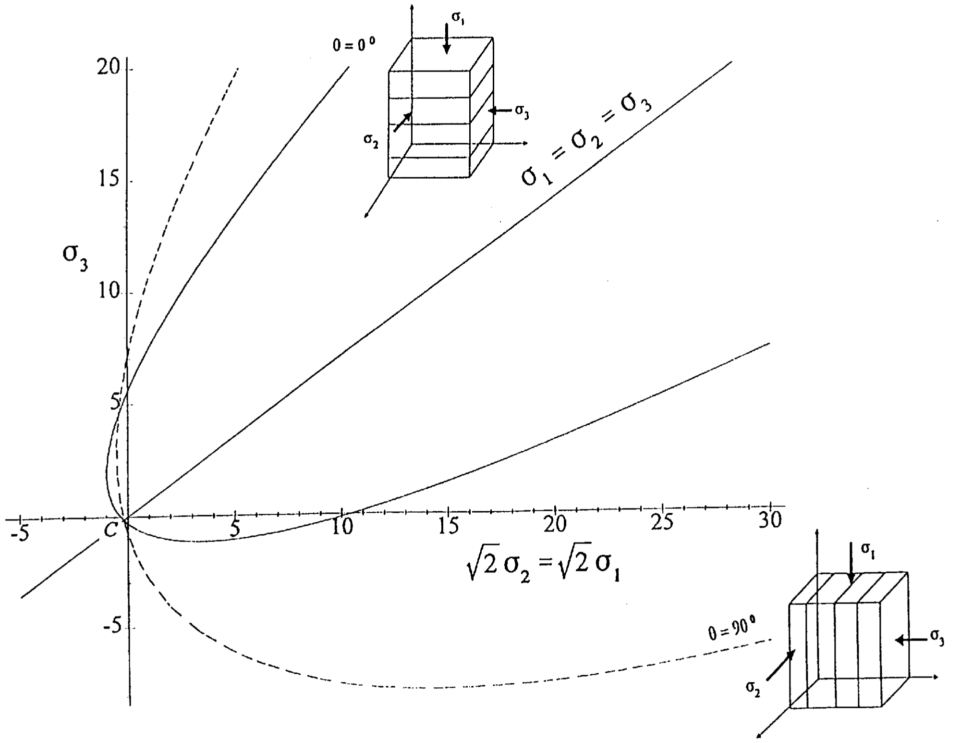

Figure 2. Intersection of the AMS criterion with the triaxial plane (σ_3, $\sqrt{2}\sigma_1 = \sqrt{2}\sigma_2$) for $\theta = 0°$ and $\theta = 90°$ (data after Allirot and Bohler[8])

plane $\sigma_1 = \sigma_2$, for $\theta = 0°$ and $\theta = 90°$, for a *diatomite* (data after Allirot and Boehler[8]). The intersections of the parabola corresponding to $\theta = 0°$ with the σ_3 axis are at the uniaxial compressive strength (Y_C), and the uniaxial tensile strength ($-Y_T$), respectively. Point *C* represents the limiting loading condition for a hydrostatic tensile stress state. The intersections with the ($\sigma_1 = \sigma_2, \sigma_3 = 0$) axis represent the biaxial failure strengths in compression and tension, respectively. Similarly, for $\theta = 90°$, the intersections with the σ_3 axis are the uniaxial compressive strength (X_C) and the uniaxial tensile strength ($-X_T$), respectively. The parabola passes through the same point *C*, expressing that hydrostatic strength does not depend on the orientation of the applied loading with respect to the structural axis of the material. The biaxial compressive strength for $\theta = 90°$ is of 51 MPa and it is not represented in Figure 2. From (35) it follows that if the condition $2X_T < Y_C$ is fulfilled, as is the case for most rocks, then $|p| < Y_T$. Similarly, if condition $2Y_T < Y_C$ is satisfied then: $|p| < X_T$. Thus, the hydrostatic absolute value of the hydrostatic tensile strength $|p|$ is lower than Y_T and/or X_T. This is in contrast to most existing criteria that postulate that failure occurs when the major principal stress is equal to the uniaxial tensile strength. The use of tension cut-offs on the failure surface produces numerical instabilities when used in computer codes. The application to boundary-value problems is rather difficult, since these criteria are defined by several expressions in the three-dimensional stress space. For a multiaxial tensile stress however, it can be shown that crack propagation which ultimately leads to failure may occur if none of the principal stresses reaches the uniaxial strength.[42] Also, it is reasonable to believe that a smooth failure curve in the neighbourhood of *C* (as in the case of the AMS criterion) is physically sound. The AMS criterion captures these experimental observed features and thus, can describe quite well failure under both tensile and compressible stress states.

The criterion representation in the octahedral (π) plane for different orientations θ, as well as the representation in the plane $(\sqrt{J_2}, I_1)$ are given in Reference 43.

4. COMPARISON WITH EXPERIMENTAL DATA

The significance of any failure criterion ultimately lies in its ability to describe the behaviour of real materials. In this section we will apply the AMS criterion to several transversely isotropic rocks. The coefficients of the AMS criterion as functions of the engineering strengths are given by equations (18) and (19). The additional condition $1/Q^2 = 4/(Y_T Y_C) - 1/X_C$ (see equation (29)) implies that the shear strength in the isotropy plane (S_2, S_3) is defined directly in terms of the uniaxial strengths of the material. Thus, for the determination of the parameters of the AMS criterion, only two types of test need to be performed:

(a) uniaxial compression and uniaxial tensile tests in the S_1 and S_2 direction, respectively;
(b) shear test in the (S_1, S_2) plane.

Since, for rocks, shear tests are very difficult to perform and to interpret, the parameter c can be estimated by least-squares fit using the compression strengths at a given confining pressure for the orientations $\theta = 0°$, $\theta = 90°$, and at least another intermediate orientation. Indeed, for triaxial compression under confining pressure p_c the AMS criterion writes:

$$H_1\sigma_a^2 + H_2\sigma_a + H_3 = 0 \tag{36}$$

$$H_1 = \left(\frac{2u}{X_C} + \frac{v}{Y_C} + 3c^2\right)(\cos\theta)^4 + \left(-\frac{u}{X_C} - 2\frac{v}{Y_C} - 3c^2\right)(\cos\theta)^2 + \frac{v}{Y_C} \tag{37}$$

$$\begin{aligned} H_2 = &\left[\left(\frac{-4u}{X_C} - 2\frac{v}{Y_C} + 6c^2\right)(\cos\theta)^4 + \left(2\frac{u}{X_C} + 4\frac{v}{Y_C} - 6c^2\right)(\cos\theta)^2 - \frac{2v}{Y_C}\right]p_c \\ &+ \left(\frac{1}{X_C} - \frac{1}{Y_C} - \frac{1}{Y_C} - u + v\right)(\cos\theta)^2 + \frac{1}{Y_C} - v \end{aligned} \tag{38}$$

$$\begin{aligned} H_3 = &\left[\left(\frac{2u}{X_C} + \frac{v}{Y_C} - 3c^2\right)(\cos\theta)^4 + \left(-\frac{u}{X_C} - \frac{2v}{Y_C} + 3c^2\right)(\cos\theta)^2 + \frac{v}{Y_C}\right]p_c^2 \\ &+ \left[\frac{1}{X_C} - u - \left(\frac{1}{X_C} - u - \frac{1}{Y_C} + v\right)(\cos\theta)^2 + \frac{1}{Y_C} - v\right]p_c - 1. \end{aligned} \tag{39}$$

In equations (37) to (39), σ_a denotes the applied axial stress, and the following notation has been used: $u = 1/X_T$, $v = 1/Y_T$.

Consider first, the experimental data on Tournemire *shale* obtained in the Lille Mechanics laboratory by Niandou.[10] The rock is an upper Toarcian massive shale. At the macroscopical level, the rock is characterized by a well-defined stratified structure. Ultrasonic measurements carried out on cubical specimens have shown that this rock exhibits intrinsic transverse isotropy (see Cuxac[44]). This type of anisotropy is conserved up to a high value of the differential stress level (i.e., the difference between the axial stress and the applied confining pressure) as shown by the compression test results.[10] Five replications of each test were performed. For Tournemire shale, the mean arithmetic value of X_C is 48 MPa, whereas $Y_C = 50$ MPa. No tensile test results were available. We assumed that: $X_T = 3.92$ MPa, and $Y_T = 4.1$ MPa, the estimate being based on tensile strength results obtained for oily shales by other researchers.[45] For the estimation of the parameter c the test results at $p_c = 50$ MPa for the orientations $\theta = 0°, 30°, 45°, 60°$ and $90°$ were used. The numerical values obtained for the coefficients are: $a = 1.205\,\text{MPa}^{-1}$, $b = 1.13529\,\text{MPa}^{-1}$, $c = 0.098\,\text{MPa}^{-1}$, $d = 3.517\,\text{MPa}^{-1}$, $e = -1.325\,\text{MPa}^{-1}$ and $m = 4.613\,\text{MPa}.^{-1}$ The Tsai and Wu criterion was also applied to Tournemire shale. The same set of data points were used for the estimation of the

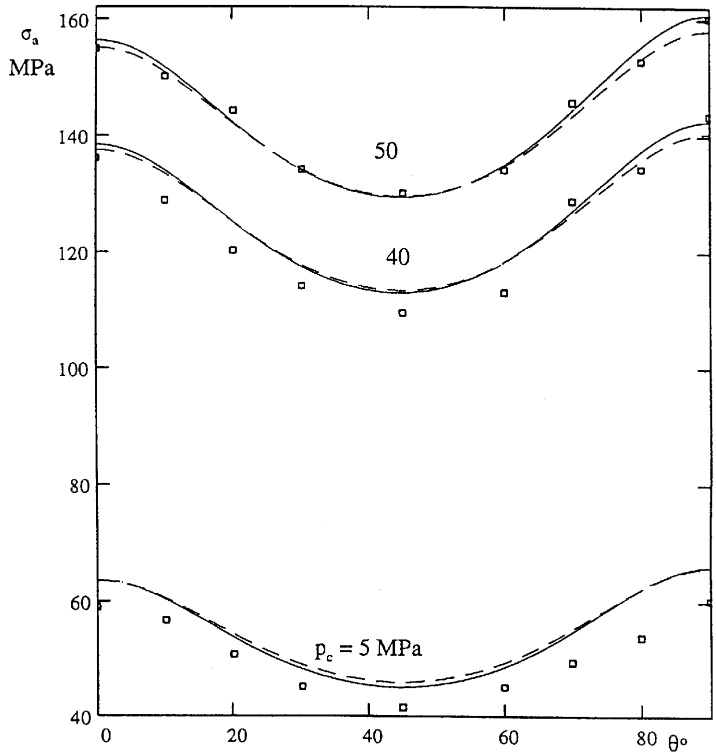

Figure 3. Comparison between the experimental results on Tournemire shale, the theoretical predictions of the AMS criterion and the prediction of the Tsai and Wu criterion[17] (data after Niandou[10])

parameters of the criterion. Figure 3 shows the variation of the peak axial stress σ_a with the orientation θ for several confining pressures. The interrupted lines are the theoretical predictions of the Tsai and Wu criterion,[17] the solid lines correspond to the predictions of the AMS criterion, while the experimental points are represented by symbols. The experimental results show that for each confining pressure the minimum strength is found between $\theta = 45^\circ$ and $\theta = 50^\circ$ while two maximum values of the strengths are obtained for $\theta = 0^\circ$ and $\theta = 90^\circ$. As a general remark, the strength anisotropy is decreasing as the confining pressure is increasing. For the AMS criterion the comparison with the data is successful on the whole. The influence of the confining pressure on the strength characteristics is well described although only the test results for $p_c = 50$ MPa were used for the determination of the parameter c. Figure 3 also shows a good agreement between the experimental data and the theoretical predictions of the Tsai and Wu criterion. Still, the better fit obtained with the Tsai and Wu criterion may be due to a larger number of material parameters.

The AMS criterion was also applied to Austin *slate*, using the experimental data obtained by McLamore and Gray.[7] The rock did not present discernible bedding planes but cleavage was well developed. No uniaxial test results were available. We assumed that: $X_T = 22.776$ MPa and $Y_T = 34.75$ MPa. By extrapolating the data shown in Figure 4 towards $\theta = 0^\circ$ and $\theta = 90^\circ$, we have found that: $X_C = 262.01$ MPa and $Y_C = 275.8$ MPa. To evaluate the coefficient c, we used the confined compression strengths at $p_c = 137.9$ MPa ($p_c = 20{,}000$ psi), for θ ranging from 0° to 90° at 10° intervals, and Equation (36). By least-square fit we obtained: $c = 0.022\ \mathrm{MPa}^{-1}$, while by making use of equations (18) and (19), we estimated: $a = 0.0321\ \mathrm{MPa}^{-1}$, $b = 0.0192\ \mathrm{MPa}^{-1}$, $d = 18.58\ \mathrm{MPa}^{-1}$, $m = 2.941\ \mathrm{MPa}^{-1}$ and $e = -18.13\ \mathrm{MPa}^{-1}$. In Figure 4 the calculated curves are plotted together

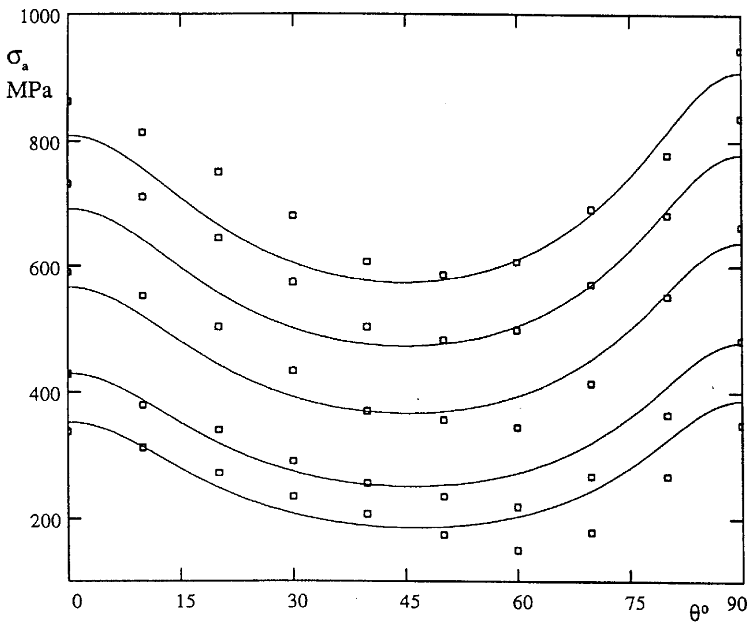

Figure 4. Comparison between theoretical and experimental results on Austin slate (data after MacLabmore and Gray[7])

with the experimental results. The AMS criterion describes well the effect of planar anisotropy on the strength of rock over the entire range of confining pressures. For lower confining pressures the experimental results are better matched for $\theta \in (0^\circ, \theta_C)$, θ_C corresponding to the minimum value of the fracture strength, while for higher confining pressures a better agreement is obtained for θ higher than θ_C.

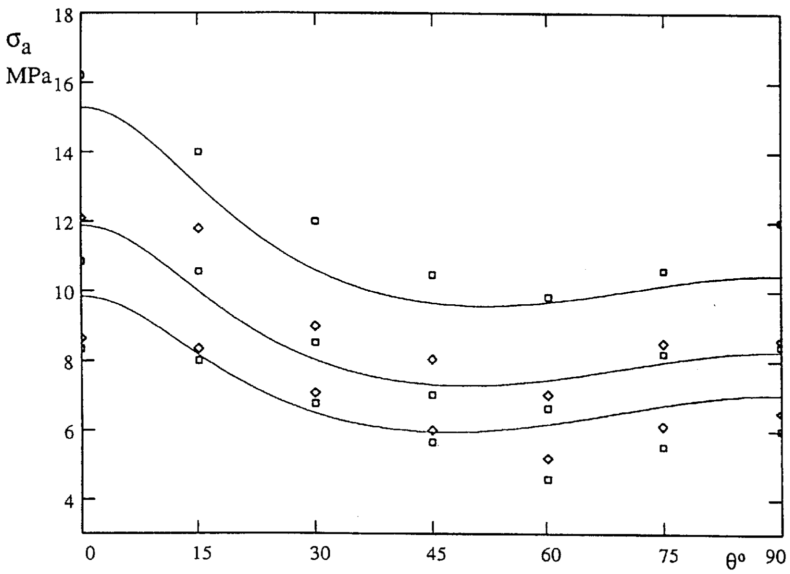

Figure 5. Comparison between theoretical and experimental results on diatomite (data after Allirot and Boehler[8])

The theoretical predictions for a *diatomite* (data after Allirot and Boehler[8]) are plotted in Figure 5. The experimental data points for $p_c = 0.5$ MPa, $p_c = 1$ MPa and $p_c = 2$ MPa are represented by symbols. For the confining pressure $p_c = 0.5$ MPa and $p_c = 1$ MPa, two replicates of each test were performed. For the identification of the parameter c, the mean arithmetic values of the confined strengths at $p_c = 0.5$ MPa were used. Following the procedure outlined we obtained: $c = 0.022\,\mathrm{MPa}^{-1}$, while by making use of equations (18) and (19), we estimated: $a = 0.126\,\mathrm{MPa}^{-1}$, $b = 0.28\,\mathrm{MPa}^{-1}$, $d = 0.3\,\mathrm{MPa}^{-1}$, $e = -0.693\,\mathrm{MPa}^{-1}$, $m = 11.357\,\mathrm{MPa}^{-1}$. The criterion reproduces the trend shown by the experimental data.

5. CONCLUSIONS AND FINAL REMARKS

A coordinate-free formulation of a failure criterion for transversely isotropic solids was proposed. For isotropic conditions, the criterion reduces to the Mises–Schleicher criterion. The anisotropic form of the AMS criterion is based on a generalization of the second invariant of the deviatoric stress and of the mean stress respectively, using a unique fourth-order tensor. The components of this tensor are directly expressible in terms of the basic engineering strengths of the material. For hydrostatic stress states, the generalized second invariant of the deviatoric stress is different from zero. Thus, it can describe the experimentally observed distortion of anisotropic materials under hydrostatic stress. The additional condition $1/Q^2 = 4/(Y_T Y_C) - 1/X_C$ (see equation (29)) that defines the shear strength in the isotropy plane (S_2, S_3) in terms of the uniaxial strengths of the material, ensures that the failure surface is an elliptic paraboloid for any orientation. The intersection of the AMS criterion with the usual triaxial plane reveals some interesting features of the criterion. It was shown that the AMS criterion models satisfactorily the failure characteristics under tensile stresses, thus no cut-offs are necessary. Though essentially phenomenological, the AMS criterion can accurately describe the observed failure characteristics of transversely isotropic rocks. The AMS criterion was used to model the strength anisotropy of different types of rock in triaxial compression. The procedure for the identification of the parameters of the criterion from a few simple laboratory tests was outlined. The comparison between the theoretical predictions and the data is reasonably good. The AMS criterion is further used as a short-term failure criterion in an elastic/viscoplastic constitutive model for initially transversely isotropic intact rocks.[46]

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APPENDIX

Assume that the applied principal stresses coordinate system (X_1, X_2, X_3) is such that $X_2 \| S_2$, and S_1 is obtained from X_1 by rotation about S_2 with the angle θ, in the coordinate system, the AMS criterion is expressed by:

$$a'_1\sigma_1 + a'_2\sigma_2 + a'_3\sigma_3 + A'_{11}\sigma_1^2 + A'_{22}\sigma_2^2 + A'_{33}\sigma_3^2 + 2A'_{12}\sigma_1\sigma_2 + 2A'_{13}\sigma_3\sigma_1 + 2A'_{23}\sigma_2\sigma_3 = 1. \tag{40}$$

The coefficients A'_{ij} and a'_i are expressed in terms of the coefficients A_{ij}, a_i and the angle θ by

$$\begin{aligned}
A'_{11} &= A_{11}\cos^4\theta + A_{22}\sin^4\theta + (2A_{12} + A_{55})\sin^2\theta\cos^2\theta \\
A'_{22} &= A_{22} \\
A'_{33} &= A_{11}\sin^4\theta + A_{22}\cos^4\theta + (2A_{12} + A_{55})\sin^2\theta\cos^2\theta \\
A'_{12} &= (A_{23}\sin^2\theta + A_{12}\cos^2\theta) \\
A'_{13} &= [(A_{11} + A_{22} - A_{55} - 2A_{12})\sin^2\theta\cos^2\theta + A_{12}] \\
A'_{23} &= (A_{12}\sin^2\theta + A_{23}\cos^2\theta) \\
a'_1 &= a_1\cos^2\theta + a_2\sin^2\theta \\
a'_2 &= a_2 \\
a'_3 &= a_1\sin^2\theta + a_2\cos^2\theta.
\end{aligned} \tag{41}$$

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