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A Convergent Crank-Nicolson Galerkin Scheme for the Benjamin-Ono Equation

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Abstract

In this thesis we prove the convergence of a Crank–Nicolson type Galerkin finite element scheme for the initial value problem associated to the Benjamin–Ono equation. The proof is based on a recent result for a similar numerical method for the Korteweg–de Vries equation, and utilises a commutator estimate related to a local smoothing effect to bound the $H^{\frac{1}{2}}$-norm of the approximations locally. This enables us to show that the scheme converges strongly in $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))$ to a weak solution of the equation for initial data in L^2 and some $T > 0$. Finally we illustrate the convergence with some numerical examples.

Samandrag

I denne oppgåva viser me at ein Crank–Nicolson-variant av eit Galerkin endeleg-element-skjema konvergerer for initialverdiproblemet tilknytta Benjamin–Ono-likninga. Beviset byggjer på eit nyleg resultat for ein tilsvarande numerisk metode for Korteweg–de Vries-likninga, og nyttar eit kommutatorestimat knytta til ein lokal utjamningseffekt for å avgrensa $H^{\frac{1}{2}}$-norma av dei tilnærma løysingane lokalt. Dette gjer oss i stand til å visa at skjemaet konvergerer sterkt i $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}))$ til ei svak løysing av likninga for initialdata i L^2 og ei tid $T > 0$. Til slutt illustrerer me konvergensen med nokre numeriske eksempel.

Preface

This thesis is the conclusion of my master's degree in Industrial Mathematics within the Applied Physics and Mathematics study programme at the Norwegian University of Science and Technology (NTNU). It is a continuation of my specialisation project where I studied properties of the Benjamin–Ono equation and implemented a finite element scheme to solve its Cauchy problem based on a method originally developed for the Korteweg–de Vries equation.

In my work I have put to use much of what I have learned during the course of my degree, be it techniques for which it was clear from the beginning would be useful, such as the finite element method, or theory which suddenly proved itself instrumental as I progressed with the convergence results, such as Fourier analysis. I have also learned much about mathematical topics of which I knew little or nothing prior to this work; e.g. dispersive and completely integrable partial differential equations which has been an underlying theme throughout the work, as well as fractional Sobolev spaces which emerged a central part of the convergence analysis.

Finally, I would like to thank my supervisor Helge Holden, professor at the Department of Mathematical Sciences at NTNU, for valuable ideas and excellent supervision during both my specialisation project and thesis.

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Contents

Chapter 1

Introduction

The objective of this master's thesis is to prove the convergence of a Crank–Nicolson type Galerkin finite element scheme for the Benjamin–Ono equation. This numerical method has previously been applied to the Korteweg–de Vries equation and proved convergent for initial data in L^2 by Dutta and Risebro [12]. Their work can be seen as a generalisation to higher order approximations in time of the work of Dutta, Koley and Risebro [11] where the implicit Euler method is used as temporal discretisation for the Galerkin scheme, and it is observed that the higher order Crank–Nicolson scheme gives significantly better approximations in practice than its implicit Euler counterpart.

1.1 The Benjamin–Ono equation

The Benjamin–Ono (BO) equation was first derived by Benjamin [3] as a governing equation for internal waves in stratified fluids of great depth. Later, said equation was derived by Ono [27] who additionally showed that the equation admitted several conserved quantities, among them mass and momentum. Benjamin's application of the equation is in fact a special case, as it serves as a generic model for weakly nonlinear long waves where only the lowest-order effects of nonlinearity and non-local dispersion appear, as mentioned in [23].

The BO equation is a one-dimensional nonlinear partial differential equation of integro-differential type and its non-dimensional initial value problem reads

$$\begin{cases} u_t + uu_x - \mathcal{H}u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathcal{H}$ denotes the Hilbert transform, which is a non-local operator defined as

$$\mathcal{H}u(x,\cdot) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(x-y,\cdot)}{y}\,dy.$$

Here p.v. denotes the Cauchy principal value of the integral.

One may also define the $2L$-periodic version of (1.1),

$$\begin{cases} u_t + uu_x - \tilde{\mathcal{H}}u_{xx} = 0, & (x,t) \in \mathbb{T} \times (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{1.2}$$

where $\mathbb{T} := \mathbb{R}/(2L\mathbb{Z})$, the mapping of the real line onto the torus of period $2L$, and $\tilde{\mathcal{H}}$ denotes the periodic Hilbert transform

$$\tilde{\mathcal{H}}u(x,\cdot) := \text{p.v.}\frac{1}{2L}\int_{-L}^{L} u(x-y,\cdot)\cot\left(\frac{\pi}{2L}y\right)dy.$$

In recent years the BO equation has been studied extensively due to its many interesting properties. Much work has been done on the existence of solutions to the equation, and there have been several improvements of the condition on lower regularity for the initial value problem (1.1) to be well-posed. Regarding well-posedness of (1.1) in $H^s(\mathbb{R})$, as the lower bound for s has been improved, the methods of proof have developed from relatively straightforward general procedures to more intricate methods utilising specific properties of the equation itself to remedy the problems associated with low-regularity initial data. Iório [18], using only hyperbolic energy methods, proved local well-posedness for $s > \frac{3}{2}$ and global well-posedness for $s \geq 2$ by utilising the conserved quantities of the equation. By deriving a local smoothing effect due to the dispersivity of the equation and combining this with compactness methods in the form of parabolic regularisation, Ponce [28] lowered the global bound to $s \geq \frac{3}{2}$. Improving upon Ponce's method by means of Strichartz estimates, Koch and Tzvetkov [22], and Kenig and Koenig [21] respectively reached $s > \frac{5}{4}$ and $s > \frac{9}{8}$ for local solutions. Tao [30] made a significant improvement when he proved global well-posedness for $s \geq 1$ by introducing a gauge transform resembling the well-known Cole-Hopf transform for the viscous Burgers equation to remove the most problematic terms involving the derivative. This transform was further refined by Burq and Plancheon [5] and by Ionescu and Kenig [17] to achieve respectively $s > \frac{1}{4}$ and $s \geq 0$ for the local well-posedness.

Regarding the well-posedness in the periodic case (1.2), Molinet has proven this globally for $s \geq 0$ using Tao's gauge transform and Strichartz estimates [25] and that the bound $L^2(\mathbb{T})$ is sharp [26].

Also within the field of numerical analysis there has been done work on the BO equation, as its non-local nature and many conserved quantities present an interesting challenge to approximate numerically. Dutta, Holden, Koley and Risebro have proved the convergence of a finite difference scheme for the BO equation in both the full line (1.1) and periodic (1.2) cases for sufficiently regular initial data [9]. The same authors have also analysed operator splitting of Godunov and Strang type applied to the BO equation and shown convergence given sufficiently regular initial data [10].

1.2 Outline of thesis

This thesis is organised as follows. In Chapter 2 we first present mathematical concepts and results which will be of importance in the convergence analysis, and we highlight

some relevant properties shared by the Benjamin–Ono and Korteweg–de Vries equations. Chapter 3 starts by comparing our proof with the approach in the case of the Korteweg–de Vries equation, before presenting the commutator estimates which play a key role in the analysis and making a priori estimates for the semi-discretised equation. In Chapter 4 we define the fully discrete scheme and prove that it is solvable for each time step, while Chapter 5 contains the actual convergence proof. Chapter 6 includes some numerical examples for the scheme, before we draw conclusions in Chapter 7.

Chapter 2

Background theory

2.1 L^p spaces

L^p-spaces, also known as Lebesgue spaces, are function spaces which satisfy the following properties

$$L^p(U) := \{v : U \to \mathbb{R} \text{ or } \mathbb{C} \mid v \text{ Lebesgue measurable}, \|v\|_{L^p(U)} < \infty\},$$

where

$$\|v\|_{L^p(U)} := \begin{cases} \left(\int\limits_U |v|^p \, dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}\limits_{x \in U} |v| := \inf\{M \geq 0 \mid |v(x)| \leq M \text{ for a.e. } x \in U\}, & p = \infty. \end{cases}$$

Here $U \subseteq \Omega$, where Ω is the ambient space. When $p = 2$ the norm is induced by the inner product

$$\langle u, v \rangle_{L^2(U)} := \int_U u\bar{v} \, dx,$$

where $\bar{v}$ is the complex conjugate of v. Hence, $\langle v, v \rangle = \|v\|^2_{L^p(U)}$. For a definition of Lebesgue measurable functions, see e.g. [24]. These function spaces are complete with respect to their norms, and therefore Banach spaces. In the case $p = 2$, $L^2(U)$ is a complete inner-product space, and thus a Hilbert space.

One also defines the spaces of locally p-integrable functions in U as

$$L^p_{\text{loc}}(U) := \{v : U \to \mathbb{R} \text{ or } \mathbb{C} \mid v \in L^p(K),\ K \subset U, K \text{ compact}\}.$$

2.2 Weak derivatives

The notion of weak derivatives is essential in the study of partial differential equations. Requiring solutions to be smooth enough to be differentiable in the classical sense, also known as strongly differentiable, severely restricts the amount of feasible solutions and

makes the search for them much harder. This is why one instead often looks for solutions which are less smooth and only permit so-called weak derivatives. The definitions presented here will follow [13].

Let α be a multi-index. Then the differential operator D^α is defined as

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}.$$

Denote the space of infinitely differentiable functions $\varphi : U \to \mathbb{R}$ with compact support in U by $C_c^\infty(U)$. The functions belonging to this space are often called test functions.

Definition 2.1. *Let $u, v \in L^1_{loc}(U)$ while α is a multi-index. Then v is defined as the α-th weak partial derivative of u,*

$$D^\alpha u = v,$$

given that

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U v \varphi \, dx, \quad \varphi \in C_c^\infty(U),$$

is satisfied.

Remark 2.1. Note that if u is differentiable in the classical sense, the weak derivative coincides with the strong derivative.

Define also the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \mid \|x^\alpha D^\beta \varphi\|_{L^\infty(\mathbb{R}^n)} < \infty, \, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}_0^n\},$$

as presented in [23]. Here $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Weak derivatives and the Schwartz space are part of the theory of distributions which will not be elaborated on here.

2.3 The Fourier transform

Here we present some theory on Fourier transforms which will be of use in later chapters.

Definition 2.2. *For a function u satisfying $u, \hat{u} \in L^1(\mathbb{R}^n)$, the Fourier transform and its inverse are respectively defined as*

$$\mathcal{F}[u](\xi) = \hat{u}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-i\langle x, \xi \rangle} \, dx, \tag{2.1}$$

and

$$\mathcal{F}^{-1}[\hat{u}](x) = u(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\langle x, \xi \rangle} \, d\xi. \tag{2.2}$$

Remark 2.2. When a function belongs exclusively to $L^1(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ there are some technicalities regarding these definitions which will be omitted here. For the rest of the chapter functions are assumed to belong to $L^2(\mathbb{R}^n)$.

One of the nice relations between a function and its Fourier transform is the Plancherel identity

$$\langle u, v\rangle_{L^2(\mathbb{R}^n)} = \langle \hat{u}, \hat{v}\rangle_{L^2(\mathbb{R}^n)}, \quad u, v \in L^2(\mathbb{R}^n). \tag{2.3}$$

Another interesting property of the Fourier transform is revealed when it is applied to derivatives of functions. Assume for simplicity that $D^\beta u \in L^2(\mathbb{R}^n)$ for $0 \le |\beta| \le |\alpha|$ to avoid complications regarding convergence. Using integration by parts it is easy to verify the identity

$$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi). \tag{2.4}$$

Thus the Fourier transform turns differentiation into multiplication with a polynomial. This property is very useful for linear differential equations as they can be restated as algebraic equations involving the Fourier transform of the solution, which are often more tractable than the original equations. The above property may also be used to show the following relation. Given $f \in C^N(\mathbb{R}^n)$ and $D^\alpha f \in L^1(\mathbb{R}^n)$ for $0 \le |\alpha| \le N$ we have

$$|\hat{f}(\xi)| \le \frac{C}{(1+|\xi|)^N}, \tag{2.5}$$

for some suitable C depending on $\|D^\alpha f\|_{L^1(\mathbb{R}^n)}$, n and N. The above estimate is easily obtained using (2.4), the inequality

$$\sum_{0\le|\alpha|\le N} |\widehat{D^\alpha f}| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{0\le|\alpha|\le N} \|D^\alpha f\|_{L^1(\mathbb{R}^n)}$$

which follows from the definition (2.1), and the inequality

$$C_N^{-1}(1+|\xi|)^N \le \sum_{0\le|\alpha|\le N} |\xi^\alpha| \le C_N(1+|\xi|)^N,$$

where C_N only depends on n and N.

There is also a simple relation between the Fourier transform of a convolution of two functions u and v, defined as

$$u * v(x) = \int_{\mathbb{R}^n} u(y)v(x-y)\,dy.$$

Assume $u, v \in L^2(\mathbb{R}^n)$, then we have

$$\widehat{u * v}(\xi) = (2\pi)^{\frac{n}{2}} \hat{u}(\xi)\hat{v}(\xi), \tag{2.6}$$

and

$$\widehat{uv}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \hat{u} * \hat{v}(\xi). \tag{2.7}$$

For convolutions we have Young's inequality: Let $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Then for $u \in L^p(\mathbb{R}^n)$ and $v \in L^r(\mathbb{R}^n)$ we have

$$\|u * v\|_{L^q(\mathbb{R}^n)} \leq \|v\|_{L^r(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}. \tag{2.8}$$

Furthermore, using the Fourier transform one may also define the so-called homogeneous fractional derivative of order $\beta \in \mathbb{R}^n$ as

$$D_x^\beta u(x) = (-\Delta)^{\beta/2} u(x) := \mathcal{F}^{-1}[|\xi|^\beta \hat{u}(\xi)](x). \tag{2.9}$$

For proofs of the above properties and further information regarding Fourier analysis the reader is referred to [16].

2.4 Sobolev spaces

With the concept of weak derivatives readily defined, one may move on to the definition of Sobolev spaces, which are very useful in the study of partial differential equations.

Definition 2.3. *For fixed k and p such that $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0$, the Sobolev space $W^{k,p}(U)$ is defined as*

$$W^{k,p}(U) := \{u \in L^1_{\text{loc}}(U) \,:\, D^\alpha u \text{ exists and belongs to } L^p(U),\ |\alpha| \leq k\}.$$

The spaces $W^{k,p}$ are assigned the standard norm

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum\limits_{|\alpha| \leq k} \int_U |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sum\limits_{|\alpha| \leq k} \operatorname*{ess\,sup}_{x \in U} |D^\alpha u|, & p = \infty, \end{cases}$$

and the seminorm

$$|u|_{W^{k,p}(U)} := \begin{cases} \left(\sum\limits_{|\alpha| = k} \int_U |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sum\limits_{|\alpha| = k} \operatorname*{ess\,sup}_{x \in U} |D^\alpha u|, & p = \infty. \end{cases}$$

Remark 2.3. It is customary to use the notation $H^k(U) := W^{k,2}(U)$, as the Sobolev spaces with index $p = 2$ are Hilbert spaces.

Utilising the Fourier transform one may characterise $H^k(\mathbb{R}^n)$ by restrictions on the Fourier transforms of the functions belonging to the space. That is, a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{\frac{k}{2}}\hat{u} \in L^2(\mathbb{R}^n)$. One is then able to define another norm for $H^s(\mathbb{R}^n)$,

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|^2)^{\frac{s}{2}}\hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{2.10}$$

This norm is equivalent to the previous Sobolev norm for integer s, but it is also well defined for non-integer s. Therefore, given $s \in \mathbb{R}$ and u such that $\|u\|_{H^s(\mathbb{R}^n)} < \infty$, u is said to belong to the *fractional Sobolev space* $H^s(\mathbb{R}^n)$. Additionally, for $s > 0$ the space $H^{-s}(\mathbb{R}^n)$ is said to be the dual space of $H^s(\mathbb{R}^n)$. In particular, u belongs to $H^{\frac{1}{2}}(\mathbb{R})$ when both $\|u\|_{L^2(\mathbb{R})}$ and $\left\|D_x^{\frac{1}{2}}u\right\|_{L^2(\mathbb{R})}$ are finite, as can be seen from

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R})} &= \int_{\mathbb{R}} \sqrt{1+|\xi|^2}\,|\hat{u}|^2\,d\xi \\
&\leq \int_{\mathbb{R}} (1+|\xi|)\,|\hat{u}|^2\,d\xi \\
&= \int_{\mathbb{R}} |\hat{u}|^2\,d\xi + \int_{\mathbb{R}} \left(|\xi|^{\frac{1}{2}}\,|\hat{u}|\right)^2 d\xi \\
&= \|u\|^2_{L^2(\mathbb{R})} + \left\|D_x^{\frac{1}{2}}u\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

For further information on the Sobolev spaces with integer k see [13]. However, as the matter of fractional Sobolev spaces is only very briefly mentioned in the above introductory text, we give without proof a useful result presented in Proposition 3.1 in the more advanced text [23].

Lemma 2.1 (Sobolev interpolation). *If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1-\theta)s_2, 0 \leq \theta \leq 1$, then*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq \|u\|^{\theta}_{H^{s_1}(\mathbb{R}^n)}\|u\|^{1-\theta}_{H^{s_2}(\mathbb{R}^n)}. \tag{2.11}$$

The above interpolation inequality makes us able to use two fractional Sobolev norms to bound an intermediate Sobolev norm.

2.5 The Hilbert transform

Prior to defining the Hilbert transform one considers the related concept of Cauchy principal values. Let $f : \mathbb{R} \to \mathbb{R}$ have a singularity at a point x_0 such that the integral of f over the real line is undefined. Then one may define the Cauchy principal value, denoted p.v., of the integral as the following symmetric limit

$$\text{p.v.} \int_{\mathbb{R}} f(x)\,dx = \lim_{\varepsilon \to 0} \int_{|x-x_0| \geq \varepsilon} f(x)\,dx,$$

which may exist even if the integral does not.

Definition 2.4. *The Hilbert transform of $\varphi \in \mathcal{S}(\mathbb{R})$ is defined as*

$$\mathcal{H}\varphi(x) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\varphi(x-y)}{y}dy = \frac{1}{\pi}\text{p.v.}\left(\frac{1}{x} * \varphi(x)\right). \tag{2.12}$$

The above definition can be stated in terms of the truncated Hilbert transform

$$\mathcal{H}^{(\varepsilon)}\varphi(x) := \frac{1}{\pi}\int_{|y|\geq\varepsilon} \frac{\varphi(x-y)}{y}dy, \tag{2.13}$$

as

$$\mathcal{H}\varphi(x) = \lim_{\varepsilon\to 0}\mathcal{H}^{(\varepsilon)}\varphi(x).$$

Remark 2.4. Note that the Hilbert transform is well defined at a point $x \in \mathbb{R}$ for all integrable functions f on $\mathbb{R}$ which satisfy the following Hölder condition in a neighbourhood of x, $\exists\, C_x > 0, \varepsilon_x > 0$ such that

$$|f(x) - f(y)| \leq C_x\,|x-y|^{\varepsilon_x}\,,$$

when $|x-y| < \delta_x$. From this it is clear that the Hilbert transform is well defined for all integrable, Hölder-continuous functions. Notice also that the truncated transform is well defined for $f \in L^p, 1 \leq p < \infty$. This is because $\frac{1}{x}$ belongs to $L^{p'}$ for $|x| \geq \varepsilon$, where p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$, and so the claim follows from Hölder's inequality. From this and limiting arguments one may deduce that $\mathcal{H}f$ is defined for $f \in L^p$.

It is clear from the definition of the transform that it is a linear operator. One can also verify that the transform commutes with differentiation, that is

$$\frac{d\mathcal{H}\varphi(x)}{dx} = \mathcal{H}\frac{d\varphi(x)}{dx}. \tag{2.14}$$

Complex contour integration yields the following characterisation of the Fourier transform of $\mathcal{H}(\varphi)$,

$$\widehat{\mathcal{H}\varphi}(\xi) = -i\,\text{sgn}(\xi)\hat{\varphi}(\xi). \tag{2.15}$$

The above identity allows the extension of the Hilbert transform to an isometry in $L^2(\mathbb{R})$, as the Plancherel identity (2.3) shows that

$$\|\mathcal{H}\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}. \tag{2.16}$$

Combining this property with the interchanging of $\mathcal{H}$ and differentiation, it is clear that the Hilbert transform also is isometric in the Sobolev spaces $H^k(\mathbb{R}), k \in \mathbb{N}_0$.

The relation

$$\mathcal{H}(\mathcal{H}\varphi) = -\varphi, \tag{2.17}$$

is readily obtained from (2.15) by applying the inverse Fourier transform to the identity

$$\mathcal{F}[\mathcal{H}(\mathcal{H}\varphi)](\xi) = -i\,\mathrm{sgn}(\xi)\widehat{\mathcal{H}\varphi}(\xi) = (-i\,\mathrm{sgn}(\xi))^2\,\hat{\varphi}(\xi) = -\hat{\varphi}(\xi).$$

Another important property of the Hilbert transform is orthogonality or antisymmetry. Assume $f \in L^p$ for $1 < p < \infty$ and $g \in L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This implies

$$\int_{\mathbb{R}} \mathcal{H}f(x)g(x)\,dx = -\int_{\mathbb{R}} f(x)\mathcal{H}g(x)\,dx. \tag{2.18}$$

In particular this holds for $f, g \in L^2$. Choosing $f = g$ gives $\langle f, \mathcal{H}f\rangle_{L^2} = 0$, hence f is orthogonal to its Hilbert transform.

For a $2L$-periodic function f one can derive its periodic Hilbert transform

$$\tilde{\mathcal{H}}f(x) := \text{p.v.}\frac{1}{2L}\int_{-L}^{L} f(x-y)\cot\left(\frac{\pi}{2L}y\right)\,dy \tag{2.19}$$

from the Hilbert transform on the real line by a series expansion. The derivation follows [31]. Start with the truncated expression

$$\begin{aligned}
\int_{\varepsilon\le|y|\le(2N+1)L} \frac{f(x-y)}{y}\,dy &= \int_{\varepsilon\le|y|\le L} \frac{f(x-y)}{y}\,dy + \sum_{\substack{k=-N\\k\neq 0}}^{N} \int_{(2k-1)L}^{(2k+1)L} \frac{f(x-y)}{y}\,dy \\
&= \int_{\varepsilon\le|y|\le L} \frac{f(x-y)}{y}\,dy + \sum_{\substack{k=-N\\k\neq 0}}^{N} \int_{-L}^{L} \frac{f(x-y+2kL)}{y-2kL}\,dy \\
&= \int_{\varepsilon\le|y|\le L} \frac{f(x-y)}{y}\,dy + \int_{-L}^{L} \sum_{\substack{k=-N\\k\neq 0}}^{N} \frac{f(x-y)}{y-2kL}\,dy,
\end{aligned}$$

where the last equality is due to the periodicity of f. Because of cancellation one may write

$$\sum_{\substack{k=-N\\k\neq 0}}^{N} \frac{1}{y-2kL} = \sum_{\substack{k=-N\\k\neq 0}}^{N} \left(\frac{1}{y-2kL} + \frac{1}{2kL}\right).$$

From this and the following representation which can be derived by a change of variables in the identity given on page 189 in [2],

$$\frac{\pi}{2L}\cot\left(\frac{\pi}{2L}y\right) = \frac{1}{y} + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \left(\frac{1}{y-2kL} + \frac{1}{2kL}\right), \quad |y| \le L,$$

one obtains

$$\lim_{\substack{\varepsilon\to 0\\N\to\infty}} \frac{1}{\pi}\int_{\varepsilon\le|y|\le(2N+1)L} \frac{f(x-y)}{y}\,dy = \text{p.v.}\frac{1}{2L}\int_{-L}^{L} f(x-y)\cot\left(\frac{\pi}{2L}y\right)\,dy,$$

which is exactly (2.19). For proofs and further properties of the Hilbert transform the reader is referred to [16].

2.6 Some useful results for embedded spaces

For normed spaces X and Y such that $X \subseteq Y$, we say that X is continuously embedded in Y, denoted $X \hookrightarrow Y$, if there is some constant C such that $\|u\|_Y \leq C\|u\|_X$ for every $u \in X$. Also, we say that X is compactly embedded in Y, denoted $X \hookrightarrow\hookrightarrow Y$, if it is continuously embedded in Y, and every bounded sequence in X has a convergent subsequence in Y. The following lemma is a classical result within the theory of partial differential equations.

Lemma 2.2 (Aubin–Simon). *Let X, B, Y be Banach spaces such that $X \subseteq B \subseteq Y$, $X \hookrightarrow\hookrightarrow B$ and $B \hookrightarrow Y$. Let $T > 0$ and $\{u_n\}_{n\in\mathbb{N}}$ a sequence of functions such that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;X)$ and $\{\partial_t u_n\}_{n\in\mathbb{N}}$ is bounded in $L^q(0,T;Y)$ for any $1 \leq p, q \leq \infty$. Then there exists $u \in L^p(0,T;B)$ such that, up to a subsequence,*

$$u_n \to u \text{ strongly in } L^p(0,T;B). \tag{2.20}$$

Proof. The reader is referred to [29]. □

In our convergence analysis we will use the fact that for the compact interval $[-R, R]$ we have $H^{\frac{1}{2}}([-R,R]) \hookrightarrow\hookrightarrow L^2([-R,R]) \hookrightarrow H^{-2}([-R,R])$, and then apply Lemma 2.2.

We now turn to a result related to fractional Sobolev spaces and their embeddings. The following is a restatement of Theorem 6.5 in [7] without proof.

Lemma 2.3 (Fractional Sobolev embedding). *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|^p_{L^{p^*}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy, \tag{2.21}$$

where $p^* := \frac{np}{n-sp}$ is the so-called "fractional critical exponent". Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^*]$.

Note that the above lemma uses a different characterisation of the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ where the norm is defined by the above iterated integral. For $p = 2$, which is the relevant case for us, this characterisation is equivalent to the previously defined $H^s(\mathbb{R}^n)$ spaces characterised by the Fourier transform, as pointed out in Proposition 3.4 in [7].

2.7 Dispersive PDEs

A partial differential equation (PDE) is labelled dispersive if its wave solutions disperse, or spread out, in space as time passes. Consider as an example the linear Airy equation

$$u_t + u_{xxx} = 0, \tag{2.22}$$

and its wave-component solutions of the form $u(x,t) = Ae^{i(kx-\omega t)}$. These are plane waves with amplitude A, wave number k, and angular frequency ω. By rewriting the expression as $Ae^{ik(x-\frac{\omega}{k}t)}$ it is clear that these waves propagate to the right on $\mathbb{R}$ with phase velocity $c_p = \frac{\omega}{k}$. Insert these solutions in (2.22) to obtain

$$-i(\omega + k^3)Ae^{i(kx-\omega t)} = 0,$$

which in turn implies

$$\omega(k) = -k^3. \tag{2.23}$$

This last equation is called the dispersion relation for (2.22) and governs the motion of its plane wave solutions. In general, the dispersion relation expresses the correspondence between ω and k. In many cases, like above, it is possible to write the angular frequency as an explicit function of the wave number, $\omega = \omega(k)$. From (2.23) it is clear that the phase velocity is $c_p(k) = -k^2$. One also defines the group velocity of the plane waves to be $c_g = \frac{d\omega}{dk}$, which in the current case becomes $c_g(k) = -3k^2$. For this example we see that both velocities vary with k, hence solutions of different wave numbers propagate with different velocities. Consequently one may define dispersivity in terms of the dispersion relation: an equation is called dispersive if the group velocity of its wave solutions is non-constant. Here one also requires that ω is a real function of k to avoid changes in the amplitude of the plane waves.

2.8 Properties of BO and KdV

2.8.1 Comparison of linear terms

The Benjamin–Ono equation (1.1) has several interesting properties. Firstly it closely resembles the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{2.24}$$

where the distinguishing feature between the two is seen to be the linear highest order derivative term. In (1.1) said term has been constructed by applying the linear, non-local, operator $\mathcal{H}$ to $-u_{xx}$. This similarity is also seen from applying the Fourier transform to the terms discussed, which for (2.24) yields $-i\xi^3\hat{u}(\xi)$. Using the relation (2.15) one obtains the expression $-i\,\mathrm{sgn}(\xi)\xi^2\hat{u}(\xi)$ for the term in (1.1), which differs from the above expression by a factor $|\xi|$. On the other hand, applying the Fourier transform to $-u_{xx}$ gives $\xi^2\hat{u}$, which differs from the BO term by the factor $-i\,\mathrm{sgn}(\xi)$. Consequently one may regard $-\mathcal{H}(u_{xx})$ as an intermediate stage between $-u_{xx}$ and u_{xxx}. This could suggest that this term introduces a form of fractional derivative in the analysis of the BO equation, and we will in the following see that this is indeed the case.

2.8.2 Conserved quantities

From the preceding statement one could expect that the two equations would share certain properties, and this has been shown to hold true. One such property is complete integrability. The Benjamin–Ono equation is, at least in the formal sense, completely integrable [1]. Thus it admits infinitely many conserved or time-invariant quantities, including mass, momentum and energy defined respectively as

$$m(u) := \int_{\mathbb{R}} u\,dx, \tag{2.25}$$

$$I(u) := \int_{\mathbb{R}} u^2\,dx, \tag{2.26}$$

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} |D_x^{\frac{1}{2}} u|^2 dx - \frac{1}{6}\int_{\mathbb{R}} u^3\,dx. \tag{2.27}$$

This also holds true for KdV when $D_x^{\frac{1}{2}} u$ is replaced by u_x in the energy expression. The Korteweg–de Vries equation has also been shown to be completely integrable by means of the inverse scattering transform [8]. Here we see that the (homogeneous) fractional derivative $D_x^{\frac{1}{2}}$ appears in the energy expression for BO, as predicted earlier.

2.8.3 Dispersivity

Both the Korteweg–de Vries and the Benjamin–Ono equation are dispersive. Nonlinear equations such as these are regarded as dispersive if their respective linearised counterparts are dispersive. To show this for (2.24) one simply omits the nonlinear term to obtain (2.22), which was shown to be dispersive in the previous section. To motivate the fact that (1.1) is dispersive, consider its linearised counterpart,

$$u_t(x,t) = \mathcal{H}u_{xx}(x,t).$$

From the Fourier transform and the relation (2.15) one obtains

$$\hat{u}_t(k,t) = i\,\mathrm{sgn}(k)k^2\hat{u}(k,t),$$

where one has transformed from the spatial variable x to the wave number k. This can be written by the definition of the Fourier transform as

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u_t(x,t)e^{-ikx}\,dx = i\,\mathrm{sgn}(k)k^2\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x,t)e^{-ikx}\,dx,$$

which is equivalent to

$$\int_{\mathbb{R}} u_t(x,t)e^{-ikx}\,dx = \int_{\mathbb{R}} i\,\mathrm{sgn}(k)k^2 u(x,t)e^{-ikx}\,dx.$$

Inserting the ansatz $u(x,t) = e^{i(kx-\omega t)}$ one obtains

$$\int_{\mathbb{R}} -i\omega u(x,t)e^{-ikx}\,dx = \int_{\mathbb{R}} i\,\mathrm{sgn}(k)k^2 u(x,t)e^{-ikx}\,dx,$$

which implies that

$$\omega(k) = -\operatorname{sgn}(k)k^2$$

holds almost everywhere. From the above dispersion relation it is clear that the group velocity is a non-constant function of k and thus the Benjamin–Ono equation is dispersive.

Another interesting feature shared by Benjamin–Ono and Korteweg–de Vries is that they exhibit infinite speed of propagation for initial data [19]. That is, for any $t > 0$, the initial data u_0 will have affected the solution $u(\cdot, t)$. This is also a well-known property of the heat equation.

2.8.4 Soliton solutions

The existence of soliton solutions is another interesting property the Benjamin–Ono equation shares with KdV. To understand the concept of solitons one first introduces the solitary wave solution, which is a travelling wave of permanent form propagating at constant speed. For linear equations the shape of these solutions may be arbitrary, and is not of great interest. Consider for example the one-dimensional transport equation

$$u_t + cu_x = 0, \quad c \in \mathbb{R}. \tag{2.28}$$

It is easily verified that any function $u(x - ct)$, where u is differentiable represents a solution of (2.28), and thus the shape of u is of no importance. On the other hand, for a nonlinear dispersive equation such solutions are not trivial, but exist as a consequence of a delicate balance between nonlinearity and dispersion. Consider the inviscid Burgers equation,

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = u_0, & x \in \mathbb{R}. \end{cases}$$

The nonlinear term uu_x forces the wave-like solutions to steepen and in finite time become multivalued. This phenomenon is often called "breaking" of the wave, and it inhibits the possibility of a steady wave profile.

Dispersion will also tend to change the shape of solutions over time, but in a sense opposite to the nonlinear effect. As solution components with different wave numbers propagate with different velocities, an initial travelling wave will spread out as the high-speed components leave the rest behind. When nonlinear terms such as uu_x tend to "pull" the wave profile together, while the dispersive part of the equation tends to "pull" the solution apart, and these effects balance, the result is exactly a solitary wave.

A soliton solution, as informally described on page 15 in [8], is a solution of a nonlinear equation which represents a wave of permanent form, which is localised in the sense that it decays or approaches a constant at infinity, and which can interact strongly with other solitons and still retain its identity. A more mathematically precise definition relies upon the inverse scattering transform and is also included in [8], but this is outside the scope of this thesis. From the first two properties of a soliton it is clear that it is a solitary travelling wave, and therefore it is the third property that distinguishes solitons from ordinary solitary waves. When two travelling waves collide they will in general not

proceed unaffected by this interaction, as changes in amplitude, speed, or phase may occur, or the waves can even merge to a new single wave. The prominent feature of solitons is that when they collide, the solitons will proceed seemingly unaffected, as if the collision never happened. One could from this be misled to think that the interaction is linear and the superposition principle could be applied to these waves. Yet there is a nonlinear feature involved in the interaction; after colliding the solitons will be slightly phase-shifted compared to how the waves would have propagated on their own, without any interaction. This phenomenon is remarkable, as no energy is lost in the interaction, and the sole sign of nonlinearity at play is the phase shift. Based on this particle-like behaviour, Zabusky and Kruskal [32] coined the term soliton for this type of wave when it was observed in their numerical solutions of the KdV equation.

2.8.5 Local smoothing effects

The KdV and BO equations also share a form of local smoothing effect for their solutions which will play a key role in our convergence analysis. That is, given initial data in some space for their Cauchy problems, the solutions will locally belong to a more regular, or smoother, space.

Kato [20] first established the following local smoothing effect for the KdV equation (2.24); given $s > \frac{3}{2}$, $0 < T < \infty$ and initial data $u_0 \in C(0, T; H^s(\mathbb{R}))$, then

$$u \in L^2(0, T; H^{s+1}(-R, R))$$

for any $0 < R < \infty$. Ponce [28] proved a similar result for the BO equation (1.1); given $s \geq \frac{3}{2}$ and initial data $u_0 \in H^s(\mathbb{R})$, then

$$u \in L^2_{\text{loc}}(\mathbb{R}; H^{s+\frac{1}{2}}_{\text{loc}}(\mathbb{R})).$$

Note that the gain in regularity is greater for KdV than for BO. This is connected to the dispersivity of the equations; as the dispersive effect of BO is weaker than for KdV, the associated smoothing effect is weaker. We also emphasise that the gain in regularity for BO is fractional as opposed to the integer gain for KdV, and once more we see the appearance of fractional derivatives in the context of BO.

The above results have later been refined and play an important role in the existence proof for solutions with less regular initial data, that is $u_0 \in L^2(\mathbb{R})$, and for the BO equation this is treated by Ginibre and Velo [15].

Chapter 3

Remarks and preliminary estimates

We start with some remarks on notation. From here on we will use $D^\beta u$ to denote the fractional derivative of order $\beta \in \mathbb{R}$ of u with respect to x and D^β to denote the fractional derivative operator of order $\beta \in \mathbb{R}$ with respect to x. On the other hand, ∂_x^k will denote the standard differentiation operator of order k with respect to x, and the usual subscript notation u_x, u_{xx}, etc. will represent standard derivatives of u. Furthermore, $\varphi^{(k)}$ will both denote the k-th derivative of φ and the operator corresponding to multiplying with said derivative.

Additionally, C will represent various positive constants which exact value is of no importance to the arguments. Likewise, $C(R)$ will denote various positive constants which depend on the parameter R, and so on.

3.1 Comparison to the approach in the case of KdV

In our analysis of the convergence properties of the numerical scheme we will follow the work of Dutta and Risebro [12], where the main ingredient of the convergence proof is that the solutions of the scheme is shown to exhibit the same local smoothing effect as the solutions of the KdV equation itself, namely initial data in L^2 gives solutions in H^1_{loc}. To show this the authors use a smooth, non-decreasing cut-off function φ and integration by parts to derive the following identity

$$\int_{\mathbb{R}} w_x(\varphi w)_{xx}\,dx = \frac{3}{2}\int_{\mathbb{R}} w_x^2\varphi_x\,dx - \frac{1}{2}\int_{\mathbb{R}} w^2\varphi_{xxx}\,dx,$$

where the positivity of the first term on the right hand side is used to bound the solutions locally in H^1. This technique was first introduced by Kato [20]. The above relation is in fact a consequence of the commutator identity

$$[-\partial_x^3, \varphi] = -3\partial_x\varphi_x\partial_x - \varphi_{xxx},$$

where for two operators P and Q, one defines the commutator $[P, Q] := PQ - QP$. Such commutator identities have been generalised for the hierarchy of generalised Benjamin–Ono equations by Ginibre and Velo [14], [15], and consequently our analysis will rely

heavily on their results. Most importantly we will use their commutator expansions to extract a positive term involving the L^2-norm of the half-derivative $D^{\frac{1}{2}}$ of the solution from the term in the weak formulation emerging from $\mathcal{H}u_{xx}$ in (1.1),

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}w_x(\varphi w)_x\,dx &= \int_{\mathbb{R}} \varphi w\mathcal{H}(-\partial_{xx})w\,dx \\ &= \frac{1}{2}\langle w, \varphi\mathcal{H}(-\partial_{xx})w\rangle - \frac{1}{2}\int_{\mathbb{R}} w\mathcal{H}(-\partial_{xx})\varphi w\,dx \\ &= \frac{1}{2}\langle w, \varphi\mathcal{H}(-\partial_{xx})w\rangle - \frac{1}{2}\langle w, \mathcal{H}(-\partial_{xx})\varphi w\rangle \\ &= \frac{1}{2}\langle w, -[\mathcal{H}(-\partial_{xx}), \varphi]w\rangle\,.\end{aligned}$$

Here we may insert the commutator identity found on page 227 in [14] for the ordinary Benjamin–Ono equation corresponding to the parameter $\mu = \frac{1}{2}$,

$$-[\mathcal{H}(-\partial_{xx}), \varphi] = 2D^{\frac{1}{2}}\varphi_x D^{\frac{1}{2}} + R_{\frac{1}{2}}(\varphi),$$

where $R_{\frac{1}{2}}(\varphi)$ is some remainder operator which is bounded in L^2. Note the sign change on the left hand side, due to that [14] uses a different convention for $\mathcal{H}$ where its Fourier multiplier is $+i\,\mathrm{sgn}(\xi)$ instead of $-i\,\mathrm{sgn}(\xi)$. This makes us able to bound the L^2-norm of the approximate solution's half-derivative locally by the L^2-norm of the initial data.

3.2 Commutator estimates

Here we present estimates of remainder operators which will be used in the analysis of the numerical scheme. We start with the commutator identity

$$-[\mathcal{H}D^{2\mu+1}, h] = (2\mu+1)D^{\mu}h'D^{\mu} + R_{\mu}(h), \tag{3.1}$$

where h is the operator corresponding to multiplying with some function h. Choosing $\mu = \frac{1}{2}$ gives the relevant identity for the BO equation, which can be seen from the symbolic notation $D = (-\partial_{xx})^{\frac{1}{2}}$. We will now show that R_μ is bounded in L^2 given that $\hat{h}$ is sufficiently regular, but first we need a supporting lemma which is found as Lemma 3 in [14].

Lemma 3.1. *The following inequalities hold:*

$$2\left|\sinh((2\mu+1)t) - (2\mu+1)\sinh(t)\right| \le \left|\sinh(t)\right|^{2\mu+1} \tag{3.2}$$

for $0 \le \mu \le 1$ *and all* $t \in \mathbb{R}$*, and*

$$-4\mu\left(\cosh(t)\right)^{2\mu+1} \le 2\left(\cosh((2\mu+1)t) - (2\mu+1)\cosh(t)\right) \le \left(\cosh(t)\right)^{2\mu+1} \tag{3.3}$$

for all $\mu \ge 0$ *and all* $t \in \mathbb{R}$.

Proof. As sinh is an odd function it is sufficient to consider $t > 0$. Define

$$y(t) = (\sinh(t))^{-(2\mu+1)} \left(\sinh((2\mu+1)t) - (2\mu+1)\sinh(t)\right),$$

from which we observe that $y(t) \geq 0$. Also

$$y'(t) = (2\mu+1)\left(\sinh(t)\right)^{-2(\mu+1)} \left(\mu \sinh(2t) - \sinh(2\mu t)\right) \geq 0$$

for $t \geq 0$ and $\mu \leq 1$, so that $y(t)$ is increasing in t for $t \geq 0$. Observe that for large t we have

$$y(t) \approx 2^{2\mu} \frac{e^{(2\mu+1)t} - (2\mu+1)e^{t}}{e^{(2\mu+1)t}} = 2^{2\mu}\left[1 - (2\mu+1)e^{-2\mu t}\right] \xrightarrow{t\to\infty} 2^{2\mu},$$

so that $y(t)$ is saturated for $t \to \infty$. From the preceding facts we conclude that (3.2) holds.

Similarly we define

$$z(t) = (\cosh(t))^{-(2\mu+1)} \left(\cosh((2\mu+1)t) - (2\mu+1)\cosh(t)\right),$$

and we observe that $z(0) = -2\mu$. Note that

$$z'(t) = (2\mu+1)\left(\cosh(t)\right)^{-2(\mu+1)} \left(\mu \sinh(2t) - \sinh(2\mu t)\right)$$

has the same sign as t. For large $|t|$ we have

$$z(t) \approx 2^{2\mu} \frac{e^{(2\mu+1)|t|} - (2\mu+1)e^{|t|}}{e^{(2\mu+1)|t|}} = 2^{2\mu}\left[1 - (2\mu+1)e^{-2\mu|t|}\right] \xrightarrow{|t|\to\infty} 2^{2\mu},$$

and so $z(t)$ is saturated for $|t| \to \infty$. From the preceding properties it is clear that (3.3) holds. □

We are now ready to prove a lemma concerning the boundedness of $R_\mu(h)$ in L^2 based on a result in [14], but here with a more detailed proof.

Lemma 3.2. *Let $R_\mu(h)$ be defined as in* (3.1) *with $0 \leq \mu \leq 1$. Then*

$$|||R_\mu(h)||| \leq \frac{1}{\sqrt{2\pi}} \left\| \widehat{D^{2\mu} h'} \right\|_{L^1(\mathbb{R})}, \tag{3.4}$$

where $|||\cdot|||$ denotes the operator norm in $L^2(\mathbb{R})$.

Proof. Start by applying the operator to a function $u \in L^2(\mathbb{R})$ and taking the Fourier

transform

$$
\begin{aligned}
\widehat{R_\mu(h)u} &= -\mathcal{F}\left[\mathcal{H}D^{2\mu+1}hu\right] + \mathcal{F}\left[h\mathcal{H}D^{2\mu+1}u\right] - (2\mu+1)\mathcal{F}\left[D^\mu h' D^\mu u\right] \\
&= i\,\mathrm{sgn}(\xi)|\xi|^{2\mu+1}\frac{1}{\sqrt{2\pi}}\hat{h} * \hat{u} + \frac{1}{\sqrt{2\pi}}\hat{h} * (-i\,\mathrm{sgn}(\xi)|\xi|^{2\mu+1}\hat{u}) \\
&\quad - (2\mu+1)|\xi|^\mu \frac{1}{\sqrt{2\pi}}(i\xi\hat{h}) * (|\xi|^\mu \hat{u}) \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')\left[\mathrm{sgn}(\xi)|\xi|^{2\mu+1} - \mathrm{sgn}(\xi')|\xi'|^{2\mu+1}\right]\hat{u}(\xi')\,d\xi' \\
&\quad - \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')\left[(2\mu+1)|\xi|^\mu(\xi-\xi')|\xi'|^\mu\right]\hat{u}(\xi')\,d\xi' \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')\left[\mathrm{sgn}(\xi)|\xi|^{2\mu+1} - \mathrm{sgn}(\xi')|\xi'|^{2\mu+1}\right]\hat{u}(\xi')\,d\xi' \\
&\quad - \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')(2\mu+1)\left[\mathrm{sgn}(\xi)|\xi|^{\mu+1}|\xi'|^\mu - \mathrm{sgn}(\xi')|\xi|^\mu|\xi'|^{\mu+1}\right]\hat{u}(\xi')\,d\xi' \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi') \\
&\quad \times \left[\mathrm{sgn}(\xi)|\xi|^{\mu+\frac{1}{2}}|\xi'|^{\mu+\frac{1}{2}}e^{(2\mu+1)t} - \mathrm{sgn}(\xi')|\xi|^{\mu+\frac{1}{2}}|\xi'|^{\mu+\frac{1}{2}}e^{-(2\mu+1)t}\right]\hat{u}(\xi')\,d\xi' \\
&\quad - \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')(2\mu+1) \\
&\quad \times \left[\mathrm{sgn}(\xi)|\xi|^{\mu+\frac{1}{2}}|\xi'|^{\mu+\frac{1}{2}}e^{t} - \mathrm{sgn}(\xi')|\xi|^{\mu+\frac{1}{2}}|\xi'|^{\mu+\frac{1}{2}}e^{-t}\right]\hat{u}(\xi')\,d\xi' \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')|\xi\xi'|^{\mu+\frac{1}{2}} \\
&\quad \times \left[(\mathrm{sgn}(\xi)+\mathrm{sgn}(\xi'))(\sinh((2\mu+1)t) - (2\mu+1)\sinh(t))\right]\hat{u}(\xi')\,d\xi' \\
&\quad + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} i\hat{h}(\xi-\xi')|\xi\xi'|^{\mu+\frac{1}{2}} \\
&\quad \times \left[(\mathrm{sgn}(\xi)-\mathrm{sgn}(\xi'))(\cosh((2\mu+1)t) - (2\mu+1)\cosh(t))\right]\hat{u}(\xi')\,d\xi',
\end{aligned}
$$

where we have used that one may write $|\xi| = |\xi'|e^{2t}$ for some $t \in \mathbb{R}$. Note that

$$
|\,\mathrm{sgn}(\xi) \pm \mathrm{sgn}(\xi')|
$$

is equal to 0 or 2 almost everywhere. This gives that

$$
\frac{1}{2}|\,\mathrm{sgn}(\xi) \pm \mathrm{sgn}(\xi')| = |\frac{1}{2}(\mathrm{sgn}(\xi) \pm \mathrm{sgn}(\xi'))|^{2\mu+1}
$$

almost everywhere.

Combining the above with Lemma 3.1 we obtain

$$
\begin{aligned}
\widehat{R_\mu(h)u} \leq & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\xi\xi'\right|^{\frac{2\mu+1}{2}} \\
& \times \left[\left|\operatorname{sgn}(\xi)+\operatorname{sgn}(\xi')\right| \frac{1}{2} \left|2\sinh(t)\right|^{2\mu+1}\right] \hat{u}(\xi')\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\xi\xi'\right|^{\frac{2\mu+1}{2}} \\
& \times \left[\left|\operatorname{sgn}(\xi)-\operatorname{sgn}(\xi')\right| \frac{1}{2} \left(2\cosh(t)\right)^{2\mu+1}\right] \left|\hat{u}(\xi')\right|\, d\xi' \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|\xi\xi'\right|^{\frac{1}{2}} \left|\operatorname{sgn}(\xi)+\operatorname{sgn}(\xi')\right| \left|e^t-e^{-t}\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|\xi\xi'\right|^{\frac{1}{2}} \left|\operatorname{sgn}(\xi)-\operatorname{sgn}(\xi')\right| \left|e^t+e^{-t}\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|\operatorname{sgn}(\xi)+\operatorname{sgn}(\xi')\right| \left||\xi|-|\xi'|\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|\operatorname{sgn}(\xi)-\operatorname{sgn}(\xi')\right| \left||\xi|+|\xi'|\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \\
& \times \left[\frac{1}{2} \left|\xi+\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\xi-\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\xi'-\xi'\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \\
& \times \left[\frac{1}{2} \left|\xi-\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\xi+\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\xi'-\xi'\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|1+\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| \left|\xi-\xi'\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left[\frac{1}{2} \left|1-\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| \left|\xi-\xi'\right|\right]^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
= & \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \frac{1}{2} \left|1+\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| \left|\xi-\xi'\right|^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi' \\
& + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \frac{1}{2} \left|1-\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| \left|\xi-\xi'\right|^{2\mu+1} \left|\hat{u}(\xi')\right|\, d\xi',
\end{aligned}
$$

where in the last equality we have used that that $\frac{1}{2}|1 \pm \operatorname{sgn}(\xi)\operatorname{sgn}(\xi')|$ must be equal to 0 or 1 almost everywhere. This also implies

$$
\left|1+\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| + \left|1-\operatorname{sgn}(\xi)\operatorname{sgn}(\xi')\right| = 2
$$

almost everywhere. Inserting the above relation in the estimate gives

$$\begin{aligned}\widehat{R_\mu(h)u} &\leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\xi-\xi'\right|^{2\mu+1} \left|\hat{u}(\xi')\right| \, d\xi' \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\widehat{D^{2\mu}h'}(\xi-\xi')\right| \left|\hat{u}(\xi')\right| \, d\xi' \\ &= \frac{1}{\sqrt{2\pi}} \left|\widehat{D^{2\mu}h'}\right| * |\hat{u}| \, (\xi).\end{aligned}$$

Using Plancherel's identity (2.3) and Young's inequality (2.8) we arrive at the following estimate

$$\begin{aligned}\|R_\mu(h)u\|_{L^2(\mathbb{R})} &= \left\|\widehat{R_\mu(h)u}\right\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{\sqrt{2\pi}} \left\|\left|\widehat{D^{2\mu}h'}\right| * |\hat{u}|\right\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{\sqrt{2\pi}} \left\|\widehat{D^{2\mu}h'}\right\|_{L^1(\mathbb{R})} \|\hat{u}\|_{L^2(\mathbb{R})} \\ &= \frac{1}{\sqrt{2\pi}} \left\|\widehat{D^{2\mu}h'}\right\|_{L^1(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} \, ,\end{aligned}$$

from which we conclude that (3.4) holds. □

We also want to show that a similar estimate holds for the remainder operator S_μ in

$$[D^\mu, h] = S_\mu. \tag{3.5}$$

This is proved in Proposition 2.1 in [15], but we here present an alternative, and perhaps simpler proof for the case relevant to us, based on the approach in the preceding lemma.

Lemma 3.3. *Suppose S_μ is defined as in* (3.5) *and* $0 \leq \mu \leq 1$. *Then*

$$|||S_\mu||| \leq \frac{1}{\sqrt{2\pi}} \left\|\widehat{D^\mu h}\right\|_{L^1(\mathbb{R})} , \tag{3.6}$$

where $|||\cdot|||$ denotes the operator norm in $L^2(\mathbb{R})$.

Proof. We proceed as in the proof of the preceding lemma,

$$\begin{aligned}\widehat{S_\mu u}(\xi) &= \widehat{D^\mu h u}(\xi) - \widehat{h D^\mu u}(\xi) \\ &= \frac{1}{\sqrt{2\pi}} |\xi|^\mu \hat{h} * \hat{u}(\xi) - \frac{1}{\sqrt{2\pi}} \hat{h} * \left(|\xi|^\mu \hat{u}\right)(\xi) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \hat{h}(\xi-\xi') \left[|\xi|^\mu - |\xi'|^\mu\right] \hat{u}(\xi') \, d\xi' \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \hat{h}(\xi-\xi') |\xi\xi'|^{\frac{\mu}{2}} \left[2\sinh(\mu t)\right] \hat{u}(\xi') \, d\xi',\end{aligned}$$

where $|\xi| = |\xi'|e^{2t}$.

Now we want to show that

$$2\,|\sinh(\mu t)| \leq (2\,|\sinh(t)|)^{\mu} \tag{3.7}$$

for all $t \in \mathbb{R}$ and $0 \leq \mu \leq 1$. As sinh is odd it is sufficient to consider $t > 0$. We define

$$y(t) = (\sinh(t))^{-\mu} \sinh(\mu t),$$

for which L'Hôpital's rule yields $y(0) = 0$ when $0 \leq \mu < 1$. We see that

$$y'(t) = \mu\,(\sinh(t))^{-(\mu+1)} \sinh\left((1-\mu)t\right) \geq 0$$

for $t \geq 0$ and $0 \leq \mu \leq 1$, which shows that $y(t)$ is increasing in t for $t \geq 0$. Note also that for large t we have

$$y(t) \approx 2^{\mu-1}\frac{e^{\mu t}}{e^{\mu t}} = 2^{\mu-1},$$

which shows that $y(t)$ is saturated as $t \to \infty$. Together this implies that (3.7) holds.

Then we may continue the estimate of the operator S_μ with

$$\begin{aligned}
\widehat{S_\mu u}(\xi) &\leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\xi\xi'\right|^{\frac{\mu}{2}} |2\sinh(t)|^{\mu} \left|\hat{u}(\xi')\right| \, d\xi' \\
&\leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\left|\xi\xi'\right|^{\frac{1}{2}} (e^{t} - e^{-t})\right|^{\mu} \left|\hat{u}(\xi')\right| \, d\xi' \\
&\leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left||\xi| - \left|\xi'\right|\right|^{\mu} \left|\hat{u}(\xi')\right| \, d\xi' \\
&\leq \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\hat{h}(\xi-\xi')\right| \left|\xi - \xi'\right|^{\mu} \left|\hat{u}(\xi')\right| \, d\xi' \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left|\widehat{D^{\mu}h}(\xi-\xi')\right| \left|\hat{u}(\xi')\right| \, d\xi' \\
&= \frac{1}{\sqrt{2\pi}} \left|\widehat{D^{\mu}h}\right| * |\hat{u}|\,(\xi),
\end{aligned}$$

where in the last inequality we have used the reverse triangle inequality. Combining the above estimate with Plancherel's identity (2.3) and Young's inequality (2.8) yields

$$\begin{aligned}
\|S_\mu u\|_{L^2(\mathbb{R})} &= \left\|\widehat{S_\mu u}\right\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{\sqrt{2\pi}} \left\|\left|\widehat{D^{\mu}h}\right| * |\hat{u}|\right\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{\sqrt{2\pi}} \left\|\widehat{D^{\mu}h}\right\|_{L^1(\mathbb{R})} \|\hat{u}\|_{L^2(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}} \left\|\widehat{D^{\mu}h}\right\|_{L^1(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}\,,
\end{aligned}$$

which proves (3.6). □

3.3 A priori estimates

We momentarily define a weak solution to the Benjamin–Ono equation (1.1) to be a function $u(x,t)$ such that $u \in C^1\left([0,\infty); H^2(\mathbb{R})\right)$ which for all $v \in H^2(\mathbb{R})$ satisfies

$$\langle u_t, v\rangle + \langle u u_x, v\rangle + \langle \mathcal{H} u_x, v_x\rangle = 0, \tag{3.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard L^2-inner product. This formulation is easily obtained by multiplying the equation with a test function v and integrating by parts, utilising the fact that $v \in H^2(\mathbb{R})$ to obtain the vanishing of the function at infinity. We now discretise the equation in time using a Crank–Nicolson method. Let Δt be the time step size, $u^n \approx u(\cdot, n\Delta t)$ and $u^{n+\frac{1}{2}} = (u^{n+1} + u^n)/2$. For a given u^0, define u^n to be the solution of

$$\left\langle u^{n+1}, v\right\rangle + \Delta t \left\langle u^{n+\frac{1}{2}} \left(u^{n+\frac{1}{2}}\right)_x, v\right\rangle + \Delta t \left\langle \mathcal{H}\left(u^{n+\frac{1}{2}}\right)_x, v_x\right\rangle = \left\langle u^n, v\right\rangle, \tag{3.9}$$

for all $v \in H^2(\mathbb{R})$ and $n \geq 0$. Assuming that the above equation has a unique solution u^{n+1} we may choose $v = u^{n+1} + u^n = 2u^{n+\frac{1}{2}}$ to show that this discretisation preserves the L^2-norm of the solution,

$$\begin{aligned}\left\langle u^{n+1}, 2u^{n+\frac{1}{2}}\right\rangle &+ 2\Delta t \left\langle u^{n+\frac{1}{2}} \left(u^{n+\frac{1}{2}}\right)_x, u^{n+\frac{1}{2}}\right\rangle \\ &+ 2\Delta t \left\langle \mathcal{H}\left(u^{n+\frac{1}{2}}\right)_x, \left(u^{n+\frac{1}{2}}\right)_x\right\rangle = \left\langle u^n, 2u^{n+\frac{1}{2}}\right\rangle,\end{aligned}$$

which is equivalent to

$$\|u^{n+1}\|^2_{L^2(\mathbb{R})} + 2\Delta t \left\langle u^{n+\frac{1}{2}} \left(u^{n+\frac{1}{2}}\right)_x, u^{n+\frac{1}{2}}\right\rangle = \|u^n\|^2_{L^2(\mathbb{R})},$$

where one has used the orthogonality of the Hilbert transform. Observe that integration by parts yields

$$\int_{\mathbb{R}} v v_x v \, dx = -\int_{\mathbb{R}} \left(v^2\right)_x v \, dx = -2\int_{\mathbb{R}} v v_x v \, dx,$$

so it is clear that this term must be zero. The former equation then gives

$$\|u^{n+1}\|_{L^2(\mathbb{R})} = \|u^n\|_{L^2(\mathbb{R})} = \|u^0\|_{L^2(\mathbb{R})}. \tag{3.10}$$

Note that the above relation combined with Minkowski's inequality gives the estimate $\|u^{n+\frac{1}{2}}\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\left(\|u^{n+1}\|_{L^2(\mathbb{R})} + \|u^n\|_{L^2(\mathbb{R})}\right) = \|u^0\|_{L^2(\mathbb{R})}$.

Now we would like to follow [12] in using the local smoothing effect of the equation to obtain an a priori bound for a more regular norm than $L^2(\mathbb{R}) = H^0(\mathbb{R})$. For the KdV equation one is then able to bound the solutions in $H^1_{\text{loc}}(\mathbb{R})$ by $\left\|u^0\right\|_{L^2(\mathbb{R})}$, but as the smoothing effect of the BO equation is weaker we can expect to obtain at most a local bound of the $H^{\frac{1}{2}}$-norm.

Let us first define a smooth cut-off function $\varphi \in C^\infty(\mathbb{R})$ satisfying:

(a) $1 \leq \varphi(x) \leq 2 + 2R$,

(b) $\varphi'(x) = 1$ for $|x| < R$,

(c) $\varphi'(x) = 0$ for $|x| \geq R + 1$, and

(d) $0 \leq \varphi'(x) \leq 1$ for all x.

Now, by the definition of φ, $v = \varphi u^{n+\frac{1}{2}}$ is also an admissible test function in $H^2(\mathbb{R})$, and we will write $w := u^{n+\frac{1}{2}}$ to save space. Inserting this in (3.9) and using the identity

$$\langle ww_x, w\varphi \rangle = -\frac{1}{3} \int_{\mathbb{R}} w^3 \varphi_x \, dx$$

which is easily attained from integration by parts, we have

$$\frac{1}{2} \left\| u^{n+1} \sqrt{\varphi} \right\|^2_{L^2(\mathbb{R})} + \Delta t \int_{\mathbb{R}} \mathcal{H} w_x (\varphi w)_x \, dx - \frac{\Delta t}{3} \int_{\mathbb{R}} w^3 \varphi_x \, dx = \frac{1}{2} \left\| u^n \sqrt{\varphi} \right\|^2_{L^2(\mathbb{R})} . \tag{3.11}$$

As previously mentioned, the second term on the left hand side can be expressed as $\frac{1}{2} \langle w, -[\mathcal{H}(-\partial_{xx}), \varphi] w \rangle$, which according to (3.1) can be written as

$$\left\langle w, (D^{\frac{1}{2}} \varphi_x D^{\frac{1}{2}} + \frac{1}{2} R_{\frac{1}{2}}(\varphi)) w \right\rangle = \left\langle D^{\frac{1}{2}} w, \varphi_x D^{\frac{1}{2}} w \right\rangle + \frac{1}{2} \left\langle w, R_{\frac{1}{2}}(\varphi)) w \right\rangle .$$

Note that there is no sign change when one redistributes the fractional derivative, as opposed to the usual integration by parts, which can be seen by considering the inner product in Fourier transformed variables,

$$\begin{aligned}
\left\langle w, D^{\frac{1}{2}} \varphi_x D^{\frac{1}{2}} w \right\rangle &= \int_{\mathbb{R}} \overline{\hat{w}} \mathcal{F} \left(D^{\frac{1}{2}} \varphi_x D^{\frac{1}{2}} w \right) \, d\xi \\
&= \int_{\mathbb{R}} \overline{\hat{w}} |\xi|^{\frac{1}{2}} \mathcal{F} \left(\varphi_x D^{\frac{1}{2}} w \right) \, d\xi \\
&= \int_{\mathbb{R}} \overline{|\xi|^{\frac{1}{2}} \hat{w}} \mathcal{F} \left(\varphi_x D^{\frac{1}{2}} w \right) \, d\xi \\
&= \int_{\mathbb{R}} \overline{\mathcal{F} \left(D^{\frac{1}{2}} w \right)} \mathcal{F} \left(\varphi_x D^{\frac{1}{2}} w \right) \, d\xi \\
&= \left\langle D^{\frac{1}{2}} w, \varphi_x D^{\frac{1}{2}} w \right\rangle .
\end{aligned}$$

Here the Plancherel identity (2.3) has been used in the first and last equalities.

For the remainder operator, Lemma 3.2 with $\mu = \frac{1}{2}$ gives the estimate

$$|||R_{\frac{1}{2}}(\varphi)||| \leq \frac{1}{\sqrt{2\pi}} \left\| \widehat{D^1 \varphi_x} \right\|_{L^1(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \left\| \widehat{\varphi_{xx}} \right\|_{L^1(\mathbb{R})} ,$$

where $|||\cdot|||$ denotes the operator norm in $L^2(\mathbb{R})$. To show that this expression is bounded we use the fact that $\varphi_{xx} \in C_c^\infty(\mathbb{R})$, and particularly $\varphi_{xx} \in C_c^2(\mathbb{R})$ so that $\varphi^{(2+k)} \in L^1(\mathbb{R})$

for $k = 0, 1, 2$. According to (2.5) we then have $|\widehat{\varphi_{xx}}(\xi)| \leq \frac{C}{(1+|\xi|)^2}$, and thus $\|\widehat{\varphi_{xx}}\|_{L^1(\mathbb{R})} \leq 2C$. Then it follows that

$$\frac{1}{2}\left\langle w, R_{\frac{1}{2}}(\varphi)w\right\rangle \geq -\frac{C}{\sqrt{2\pi}}\|w\|^2_{L^2(\mathbb{R})} = -\widetilde{C}\|w\|^2_{L^2(\mathbb{R})}.$$

Next we want to estimate the term stemming from the nonlinearity in terms of $\int_{\mathbb{R}} |D^{\frac{1}{2}}w|^2\varphi_x\,dx$,

$$\begin{aligned}
\int_{\mathbb{R}} w^3\varphi_x\,dx &\leq \left(\int_{\mathbb{R}} w^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} w^4\varphi_x^2\,dx\right)^{\frac{1}{2}} \\
&= \|w\|_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|^2_{L^4(\mathbb{R})} \\
&\leq C\,\|w\|_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|^2_{H^{\frac{1}{4}}(\mathbb{R})} \\
&\leq C\,\|w\|_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|_{H^{\frac{1}{2}}(\mathbb{R})} \\
&\leq \frac{1}{2}\,\|w\sqrt{\varphi_x}\|^2_{H^{\frac{1}{2}}(\mathbb{R})} + \frac{C^2}{2}\,\|w\|^2_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{1}{2}\left\|D^{\frac{1}{2}}(w\sqrt{\varphi_x})\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\,\|w\sqrt{\varphi_x}\|^2_{L^2(\mathbb{R})} + \frac{C^2}{2}\,\|w\|^2_{L^2(\mathbb{R})}\,\|w\sqrt{\varphi_x}\|^2_{L^2(\mathbb{R})} \\
&\leq \frac{1}{2}\left\|D^{\frac{1}{2}}(w\sqrt{\varphi_x})\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left(1 + C^2\,\|w\|^2_{L^2(\mathbb{R})}\right)\|w\|^2_{L^2(\mathbb{R})}\,.
\end{aligned}$$

The second inequality above is an application of Lemma 2.3 with $n = 1$, $p = 2$, $s = \frac{1}{4}$, and $q = p^* = 4$, while the third inequality comes from Lemma 2.1 with $s = \frac{1}{4}, s_1 = 0, s_2 = \frac{1}{2}$, and $\theta = \frac{1}{2}$. For the first term in the last line we use the commutator identity (3.5) with $h = \sqrt{\varphi_x}$,

$$D^{\frac{1}{2}}\sqrt{\varphi_x} = \sqrt{\varphi_x}D^{\frac{1}{2}} + S_{\frac{1}{2}}.$$

We may assume $\sqrt{\varphi_x} \in C_c^\infty(\mathbb{R})$. If this was not the case we could have started by defining a non-negative function $h \in C_c^\infty(\mathbb{R})$ such that $h(x) = 1$ for $|x| < R$ and $h(x) = 0$ for $|x| \geq R + 1$. Then $h^2 \in C_c^\infty(\mathbb{R})$ with the same properties as stated above for h and by defining $\varphi(x) := 1 + \int_{-\infty}^x h(x)^2\,dx$ we obtain a function with the same properties as the original φ, and where $\sqrt{\varphi_x}$ is smooth and compactly supported by definition.

According to Lemma 3.3 we may estimate the L^2 operator norm of $S_{\frac{1}{2}}$ as

$$\begin{aligned}
|||S_{\frac{1}{2}}||| &\leq \frac{1}{\sqrt{2\pi}}\left\|\widehat{D^{\frac{1}{2}}h}\right\|_{L^1(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}}\left\||\xi|^{\frac{1}{2}}\hat{h}\right\|_{L^1(\mathbb{R})} \\
&\leq \frac{1}{\sqrt{2\pi}}\left\|(1+|\xi|)\hat{h}\right\|_{L^1(\mathbb{R})} \\
&= \frac{1}{\sqrt{2\pi}}\left(\|\hat{h}\|_{L^1(\mathbb{R})} + \|\widehat{h'}\|_{L^1(\mathbb{R})}\right).
\end{aligned}$$

Noting that $h \in C_c^\infty(\mathbb{R})$ and in particular $C_c^3(\mathbb{R})$, and using (2.5) we have the estimate

$$|||S_{\frac{1}{2}}||| \leq \frac{4C}{\sqrt{2\pi}} =: C_S.$$

Thus, taking the L^2-norm we obtain

$$\begin{aligned}\left\|D^{\frac{1}{2}}(w\sqrt{\varphi_x})\right\|_{L^2(\mathbb{R})} &\leq \left\|\sqrt{\varphi_x}D^{\frac{1}{2}}w\right\|_{L^2(\mathbb{R})} + \left\|S_{\frac{1}{2}}w\right\|_{L^2(\mathbb{R})} \\ &\leq \left\|\sqrt{\varphi_x}D^{\frac{1}{2}}w\right\|_{L^2(\mathbb{R})} + C_S\left\|w\right\|_{L^2(\mathbb{R})},\end{aligned}$$

for some constant C_S depending on φ_x.

Inserting the above estimates in (3.11) we obtain

$$\begin{aligned}\frac{1}{2}\left\|u^{n+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} &+ \Delta t\left\|\sqrt{\varphi_x}D^{\frac{1}{2}}w\right\|^2_{L^2(\mathbb{R})} - \Delta t\widetilde{C}\|w\|^2_{L^2(\mathbb{R})} \\ &\leq \frac{1}{2}\left\|u^n\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} + \frac{\Delta t}{3}\left\|\sqrt{\varphi_x}D^{\frac{1}{2}}w\right\|^2_{L^2(\mathbb{R})} \\ &\quad + \frac{\Delta t}{3}C_S^2\|w\|^2_{L^2(\mathbb{R})} + \frac{\Delta t}{6}\left(1 + C^2\|w\|^2_{L^2(\mathbb{R})}\right)\|w\|^2_{L^2(\mathbb{R})},\end{aligned}$$

which again implies

$$\frac{1}{2}\left\|u^{n+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} + \frac{2\Delta t}{3}\int_{\mathbb{R}}\left(D^{\frac{1}{2}}u^{n+\frac{1}{2}}\right)^2\varphi_x\,dx \leq \frac{1}{2}\left\|u^n\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} + C\left(\left\|u^0\right\|_{L^2(\mathbb{R})}\right)\Delta t,$$

where we in the last term have used that the L^2-norm of $w = u^{n+\frac{1}{2}}$ is bounded by the norm of u^0. By dropping the positive second term on the left hand side, summing from $n = 0$ to $n = m - 1$ and utilising that this is a telescoping sum we obtain

$$\left\|u^m\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} \leq \left\|u^0\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} + C\left(\left\|u^0\right\|_{L^2(\mathbb{R})}\right)m\Delta t.$$

Also, first summing and then dropping $\frac{1}{2}\left\|u^{m+1}\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})}$ on the left hand side yields the estimate

$$\Delta t\sum_{n=0}^{m}\int_{-R}^{R}\left(D^{\frac{1}{2}}u^{n+\frac{1}{2}}\right)^2dx \leq \frac{3}{2}\left(\frac{1}{2}\left\|u^0\sqrt{\varphi}\right\|^2_{L^2(\mathbb{R})} + C\left(\left\|u^0\right\|_{L^2(\mathbb{R})}\right)(m+1)\Delta t\right).$$

Together these estimates imply that if the initial data $u^0 \in L^2(\mathbb{R})$ then

$$u^{n+\frac{1}{2}} \in \ell^2\left([0, m\Delta t]; H^{\frac{1}{2}}([-R, R])\right)$$

which shows that the solutions of the Crank–Nicolson temporal discretised equation also exhibit the local smoothing effect of the BO equation. As said effect is the main ingredient of the convergence proof in the case of KdV, we want to show that it is present also in our fully discretised element scheme presented in the next chapter. When formulating

the fully discrete scheme we follow [12] in using test functions of the form φv, where v belongs to some finite element space. This makes (3.11) hold which will lead to a $H^{\frac{1}{2}}$-bound like the one obtained here. The problem with this form of the scheme is that one loses the a priori preservation of the L^2-norm that was obtained here by choosing the test function $u^{n+\frac{1}{2}}$. This will make it necessary to introduce some Courant–Friedrichs–Lewy (CFL) condition relating Δt and the spatial discretisation to bound the L^2-norm of the approximate solutions.

Chapter 4

Formulation of the numerical scheme

Here we formulate the Crank–Nicolson type Galerkin scheme under consideration. First we present some remarks on notation and the discretisation of time and space. Then we use the weak formulation of the problem and a Crank–Nicolson temporal discretisation to define a sequence of functions approximating the exact solution at each discrete time step. We also define an iteration scheme to solve the implicit equation for each time step and show that this has a solution.

4.1 Notation and discretisation

We start by partitioning the real line in equally sized elements in the form of intervals. First define the grid points $x_j = j\Delta x$ for $j \in \mathbb{Z}$, where Δx is the spatial discretisation parameter or step length. Then the elements can be written as $I_j = [x_{j-1}, x_j]$. Now turn to the discretisation of the time interval considered. Given a fixed time horizon $T > 0$ and a temporal discretisation parameter Δt we set $t_n = n\Delta t$ for $n \in \{0, 1, \ldots, N\}$, where $\left(N + \frac{1}{2}\right)\Delta t = T$. For convenience we also use the notation $t_{n+\frac{1}{2}} = \left(t_n + t_{n+1}\right)/2$.

Let φ be defined as in the previous section. We define the weighted L^2-inner product

$$\langle u, v\rangle_\varphi = \langle u, v\varphi\rangle\,,$$

and the associated weighted norm $\|u\|_{2,\varphi}^2 = \langle u, u\rangle_\varphi$.

4.2 Galerkin scheme

As always for the finite element method we start by deriving a weak formulation of the problem (1.1), like the one obtained in (3.8). Applying the Crank–Nicolson temporal discretisation to the weak formulation gives (3.9). Instead of looking for solutions to this equation in $H^2(\mathbb{R})$ we will look for solutions belonging to a finite-dimensional subspace $S_{\Delta x}$ of this Hilbert space.

We define the subspace $S_{\Delta x}$ as follows; assuming $r \geq 2$ is a fixed integer we denote the space of polynomials on the interval I of degree $\leq r$ by $\mathbb{P}_r(I)$. Our goal is to find an

approximation $u^{\Delta x}$ to the solution of (1.1) which for all $t \in [0,T]$ belongs to

$$S_{\Delta x} = \{v \in H^2(\mathbb{R}) \mid v \in \mathbb{P}_r(I_j), j \in \mathbb{Z}\}. \tag{4.1}$$

Now define $\mathcal{P}$ to be the L^2-orthogonal projection onto $S_{\Delta x}$. By definition, this projection of a function u satisfies

$$\int_{\mathbb{R}} (\mathcal{P}u)\, v\, dx = \int_{\mathbb{R}} uv\, dx, \quad v \in S_{\Delta x}.$$

Then we define the sequence $\{u^n\}_{n\geq 0}$ through the following procedure: Given $u^0 = \mathcal{P}u_0$, find $u^{n+1} \in S_{\Delta x}$ which satisfies

$$\left\langle u^{n+1}, \varphi v\right\rangle - \Delta t \left\langle \frac{\left(u^{n+\frac{1}{2}}\right)^2}{2}, (\varphi v)_x \right\rangle + \Delta t \left\langle \mathcal{H}\left(u^{n+\frac{1}{2}}\right)_x, (\varphi v)_x \right\rangle = \left\langle u^n, \varphi v\right\rangle, \tag{4.2}$$

for all $v \in S_{\Delta x}$ and $n \in \{0, 1, \ldots, N\}$. Clearly, (4.2) is an implicit scheme and consequently one must solve a nonlinear equation to obtain u^{n+1} from u^n. The procedure for solving this equation at each time step is described in the following section. Note also that $\left\|u^0\right\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}$, and thus from here on we will always use the L^2-norm of the initial data u_0 as an upper bound for the norm of the approximation u^0.

Before proceeding with the scheme we present some inequalities which will be instrumental in proving the existence of a solution for each time step.

Lemma 4.1 (Inverse inequalities). *Given $z \in S_{\Delta x} \subset H^2(\mathbb{R})$, the following inverse inequalities hold*

$$\|z_x\|_{L^\infty(\mathbb{R})} \leq \frac{C_1^{1/2}}{(\Delta x)^{1/2}} \|z_x\|_{L^2(\mathbb{R})} \leq \frac{C_2^{1/2}}{(\Delta x)^{3/2}} \|z\|_{L^2(\mathbb{R})}, \tag{4.3}$$

where the constants $C_1, C_2 > 0$ are independent of z and Δx. Note that the first inequality also holds for z instead of z_x.

Proof. We follow Ciarlet [6], and Brenner and Scott [4]. Note that we momentarily use the hat notation for the reference element and its associated variables and functions, instead of Fourier transformed functions. Let $K = [x_{j-1}, x_j]$ and $\hat{K} = [0,1]$, i.e. an arbitrary finite element and the reference element, with the affine, invertible transformation

$$F : \hat{K} \to K, \quad F(\hat{x}) = (\Delta x)\hat{x} + (j-1)\Delta x.$$

Then if $v \in W^{m,p}(K)$ for some integer $m \geq 0$ and some number $p \in [1,\infty]$, the function $\hat{v} := v \circ F$ belongs to $W^{m,p}(\hat{K})$. Additionally there exists a constant $C = C(m)$ such that

$$|\hat{v}|_{W^{m,p}(\hat{K})} \leq C(\Delta x)^{m-\frac{1}{p}} |v|_{W^{m,p}(K)}, \quad v \in W^{m,p}(K), \tag{4.4}$$

$$|v|_{W^{m,p}(K)} \leq C(\Delta x)^{\frac{1}{p}-m} |\hat{v}|_{W^{m,p}(\hat{K})}, \quad \hat{v} \in W^{m,p}(\hat{K}). \tag{4.5}$$

To show this we start by assuming $v \in C^m(K)$ so that $\hat{v} \in C^m(\hat{K})$. Then

$$\begin{aligned}
|\hat{v}|_{W^{m,p}(\hat{K})} &= \left(\int_{\hat{K}} |\partial^m \hat{v}(\hat{x})|^p \, d\hat{x} \right)^{\frac{1}{p}} \\
&= (\Delta x)^m \left(\int_{\hat{K}} |\partial^m v(F(\hat{x}))|^p \, d\hat{x} \right)^{\frac{1}{p}} \\
&= (\Delta x)^{m-\frac{1}{p}} \left(\int_{K} |\partial^m v(x)|^p \, dx \right)^{\frac{1}{p}} \\
&= (\Delta x)^{m-\frac{1}{p}} |v|_{W^{m,p}(K)},
\end{aligned}$$

where in the second equality we used that $\frac{d^m \hat{v}(\hat{x})}{d\hat{x}^m} = (\Delta x)^m \frac{d^m v(x)}{dx^m}$, while in the third equality we have made a straightforward change of variables. This proves (4.4) and (4.5) for $v \in C^m(K)$ and $\hat{v} \in C^m(\hat{K})$.

In the case $p < \infty$ we prove the inequalities for $W^{m,p}$ using the continuity of the linear operator $\iota : v \in C^m(K) \to \hat{v} \in W^{m,p}(\hat{K})$ with respect to the norms $\| \cdot \|_{W^{m,p}(K)}$ and $\| \cdot \|_{W^{m,p}(\hat{K})}$, the denseness of $C^m(K)$ in $W^{m,p}(K)$, and the definition of the unique extension of the mapping ι to $W^{m,p}(K)$.

In the case $p = \infty$ we have that $v \in W^{m,\infty}(K)$ belongs to $W^{m,p}(K)$ for all $p < \infty$ as K is bounded. From the preceding we then have $\hat{v} \in W^{m,p}(\hat{K})$ for all $p < \infty$, and for $p \geq 1, 0 \leq \alpha \leq m$ we have

$$|\partial^\alpha \hat{v}|_{W^{0,p}(\hat{K})} \leq C(m)(\Delta x)^\alpha \sup_{q \geq 1} \left((\Delta x)^{-\frac{1}{q}} \|v\|_{W^{m,q}(K)} \right).$$

This shows that the upper bound on the seminorm $|\partial^\alpha \hat{v}|$ is independent of p, and so for each $0 \leq \alpha \leq m$, $\partial^\alpha \hat{v} \in L^\infty(\hat{K})$ which gives that $\hat{v} \in W^{m,\infty}(\hat{K})$. To conclude, we use (4.4) together with the fact that for any $w \in L^\infty(\Omega)$ and Ω bounded we have

$$|w|_{W^{0,\infty}(\Omega)} = \lim_{p \to \infty} |w|_{W^{0,p}(\Omega)}.$$

Inequality (4.5) is proved analogously.

We now move on to the actual proof of the lemma. The space of shape functions $\hat{P}$ on the reference element is a finite subset of $\mathbb{P}_r(\hat{K})$ with $r \geq 2$ from the definition of the finite element space $S_{\Delta x}$ (4.1). As $S_{\Delta x} \subset H^2(\mathbb{R})$ we have that $\hat{P} \subset H^2(\hat{K})$, and then $\hat{P} \subset W^{l,p}(\hat{K}) \cap W^{m,q}(\hat{K})$ for suitable $0 \leq m \leq l$ and $p, q \in [1, \infty]$, e.g. $0 \leq m \leq l \leq 2$, $p = q = 2$, or $0 \leq m \leq l \leq 1$, $p = \infty$, $q = 2$. We will first prove the local inequality

$$|z|_{W^{l,p}(K)} \leq C(\Delta x)^{m-l+\frac{1}{p}-\frac{1}{q}} |z|_{W^{m,q}(K)},$$

where $C = C(l, p, q)$.

Given $z \in S_{\Delta x}$ and an element $K \in \mathscr{T}_{\Delta x}$, where $\mathscr{T}_{\Delta x}$ denotes the partition of the domain corresponding to the discretisation parameter Δx, we denote the mapping of $z\big|_K$ to the reference element $\hat{K}$ by $\hat{z}_K$.

We first consider the case $m = 0$. As $\hat{P}$ is finite-dimensional we have by norm equivalence that there exists $C = C(l, p, q)$ such that

$$\|\hat{z}_K\|_{W^{l,p}(\hat{K})} \leq C\|\hat{z}_K\|_{L^q(\hat{K})}.$$

From this, (4.4) and (4.5) we then have

$$\begin{aligned}
|z|_{W^{j,p}(K)} &\leq C(\Delta x)^{-j+\frac{1}{p}}|\hat{z}_K|_{W^{j,p}(\hat{K})} \\
&\leq C(\Delta x)^{-j+\frac{1}{p}}\|\hat{z}_K\|_{W^{j,p}(\hat{K})} \\
&\leq C(\Delta x)^{-j+\frac{1}{p}}\|\hat{z}_K\|_{L^q(\hat{K})} \\
&= C(\Delta x)^{-j+\frac{1}{p}}|\hat{z}_K|_{W^{0,q}(\hat{K})} \\
&\leq C(\Delta x)^{-j+\frac{1}{p}-\frac{1}{q}}|z|_{W^{0,q}(K)},
\end{aligned}$$

for $0 \leq j \leq l$. By setting $j = l$ in the above we have proved the estimate for $m = 0$.

Now we consider $0 \leq m \leq l$. Noting that $\partial^m \hat{P}$ also is finite-dimensional, thus implying norm equivalence for functions in this space, and setting $j = l - m$ in the previous result we obtain

$$\begin{aligned}
|z|_{W^{l,p}(K)} &= |\partial^m z|_{W^{l-m,p}(K)} \\
&\leq C(\Delta x)^{-(l-m)+\frac{1}{p}-\frac{1}{q}}|\partial^m z|_{W^{0,q}(K)} \\
&= C(\Delta x)^{-(l-m)+\frac{1}{p}-\frac{1}{q}}|z|_{W^{m,q}(K)},
\end{aligned}$$

which proves the inequality for general m.

Finally we turn to the global estimates and the case $p = \infty$. Then there exists a finite element $K_0 \in \mathscr{T}_{\Delta x}$ such that

$$\begin{aligned}
|z|_{W^{l,\infty}(\mathbb{R})} &= \max_{K \in \mathscr{T}_{\Delta x}} |z|_{W^{l,\infty}(K)} \\
&= |z|_{W^{l,\infty}(K_0)} \\
&\leq C(\Delta x)^{m-l-\frac{1}{q}}|z|_{W^{m,q}(K_0)} \\
&\leq C(\Delta x)^{m-l-\frac{1}{q}}|z|_{W^{m,q}(\mathbb{R})},
\end{aligned}$$

where we have used the previous local result for K_0. Substituting $m = l = 1$ and $q = 2$ proves the first inequality of (4.3), while setting $m = l = 0$ gives the same inequality for z instead of z_x.

Consider now the case $m = 0$, $l = 1$ and $p = q = 2$. Using the local estimate we

obtain

$$
\begin{aligned}
|z|_{W^{1,2}(\mathbb{R})} &= \left(\sum_{K \in \mathscr{T}_{\Delta x}} |z|^2_{W^{1,2}(K)} \right)^{\frac{1}{2}} \\
&\leq C(\Delta x)^{0-1} \left(\sum_{K \in \mathscr{T}_{\Delta x}} |z|^2_{W^{0,2}(K)} \right)^{\frac{1}{2}} \\
&= \frac{C}{\Delta x} |z|_{W^{0,2}(\mathbb{R})},
\end{aligned}
$$

which proves the second inequality of (4.3). □

Remark 4.1. Note that in the above lemma we are allowed to use the identities

$$
|z|_{W^{l,\infty}(\mathbb{R})} = \max_{K \in \mathscr{T}_{\Delta x}} |z|_{W^{l,\infty}(K)}
$$

and

$$
|z|^p_{W^{l,p}(\mathbb{R})} = \sum_{K \in \mathscr{T}_{\Delta x}} |z|^p_{W^{l,p}(K)}
$$

as we are using conforming finite elements. That is, the shape functions of neighbouring elements agree on the boundary, which is necessary for $S_{\Delta x}$ to be a subspace of $H^2(\mathbb{R})$.

4.3 Solvability for one time step

To show the existence of a solution u^n for each time step we define the iteration scheme (4.6)

$$
\begin{cases}
\left\langle w^{\ell+1}, \varphi v \right\rangle - \frac{\Delta t}{2} \left\langle \left(\frac{w^\ell + u^n}{2} \right)^2, (\varphi v)_x \right\rangle + \Delta t \left\langle \left(\mathcal{H} \frac{w^{\ell+1} + u^n}{2} \right)_x, (\varphi v)_x \right\rangle = \left\langle u^n, \varphi v \right\rangle, \\
w^0 = u^n,
\end{cases}
$$

which is to hold for all test functions $v \in S_{\Delta x}$. We now present a lemma that guarantees the solvability of the implicit scheme (4.2). This technique is due to Simon Laumer.

Lemma 4.2 (Laumer). *Choose a constant L such that $0 < L < 1$ and set*

$$
K = \frac{7 - L}{1 - L} > 7.
$$

We consider the iteration scheme (4.6) *and assume that the following CFL condition holds,*

$$
\lambda \leq \frac{L}{2\sqrt{2}\sqrt{C_2} K \|u^n\|_{2,\varphi}}, \tag{4.7}
$$

where C_2 is defined in (4.3) *and λ is given by*

$$\lambda^2 = \frac{\Delta t^2}{\Delta x^3}. \tag{4.8}$$

Then there exists a function u^{n+1} which solves (4.2) *and $\lim_{\ell\to\infty} w^\ell = u^{n+1}$. Additionally,*

$$\|u^{n+1}\|_{2,\varphi} \le K\|u^n\|_{2,\varphi}. \tag{4.9}$$

Proof. We start by rewriting (4.6) as

$$\left\langle w^{\ell+1}, \varphi v\right\rangle + \frac{\Delta t}{4}\left\langle \left(u^n w^\ell\right)_x, \varphi v\right\rangle + \frac{\Delta t}{4}\left\langle w^\ell w_x^\ell, \varphi v\right\rangle + \frac{\Delta t}{2}\left\langle \mathcal{H} w_x^{\ell+1}, (\varphi v)_x\right\rangle = \mathcal{F}(u^n, \varphi v)$$

for all $v \in S_{\Delta x}$, where

$$\mathcal{F}(u^n, \varphi v) := \langle u^n, \varphi v\rangle + \frac{\Delta t}{8}\left\langle (u^n)^2, (\varphi v)_x\right\rangle - \frac{\Delta t}{2}\left\langle \mathcal{H} u_x^n, (\varphi v)_x\right\rangle.$$

From the above equation one derives

$$\left\langle w^{\ell+1} - w^\ell, \varphi v\right\rangle + \frac{\Delta t}{4}\left\langle \left(u^n\left(w^\ell - w^{\ell-1}\right)\right)_x, \varphi v\right\rangle + \frac{\Delta t}{4}\left\langle w^\ell w_x^\ell - w^{\ell-1} w_x^{\ell-1}, \varphi v\right\rangle$$
$$+ \frac{\Delta t}{2}\left\langle \mathcal{H}\left(w^{\ell+1} - w^\ell\right)_x, (\varphi v)_x\right\rangle = 0.$$

Now substitute $v = w^{\ell+1} - w^\ell =: w$ in the above equation to get

$$\langle w, \varphi w\rangle + \frac{\Delta t}{2}\langle \mathcal{H} w_x, (\varphi w)_x\rangle$$
$$= \underbrace{-\frac{\Delta t}{4}\left\langle \left(u^n\left(w^\ell - w^{\ell-1}\right)\right)_x, \varphi w\right\rangle}_{\mathcal{A}_1} \underbrace{-\frac{\Delta t}{4}\left\langle w^\ell w_x^\ell - w^{\ell-1} w_x^{\ell-1}, \varphi w\right\rangle}_{\mathcal{A}_2}.$$

For the term involving the Hilbert transform we estimate as before and use the fact that $\varphi \ge 1$ to obtain

$$\langle \mathcal{H} w_x, (\varphi w)_x\rangle \ge \left\|\sqrt{\varphi_x} D^{\frac{1}{2}} w\right\|_{L^2(\mathbb{R})}^2 - \widetilde{C}\|w\|_{L^2(\mathbb{R})}^2 \ge -\widetilde{C}\|w\|_{2,\varphi}^2.$$

We then estimate the term $\mathcal{A}_2$ by repeatedly applying Cauchy's inequality and using

(4.3),

$$
\begin{aligned}
\mathcal{A}_2 &= \frac{1}{4}\int_{\mathbb{R}}(-\Delta t)\left(w^\ell w_x^\ell - w^{\ell-1}w_x^{\ell-1}\right)\varphi w\,dx \\
&\le \frac{\Delta t^2}{8}\int_{\mathcal{R}}\left(w^\ell w_x^\ell - w^{\ell-1}w_x^{\ell-1}\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&= \frac{\Delta t^2}{8}\int_{\mathbb{R}}\left(\left(w^\ell - w^{\ell-1}\right)w_x^\ell + w^{\ell-1}\left(w_x^\ell - w_x^{\ell-1}\right)\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&\le \frac{\Delta t^2}{4}\int_{\mathbb{R}}\left(w^\ell - w^{\ell-1}\right)^2\left(w_x^\ell\right)^2\varphi\,dx \\
&\quad + \frac{\Delta t^2}{4}\int_{\mathbb{R}}\left(w^{\ell-1}\right)^2\left(w_x^\ell - w_x^{\ell-1}\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&\le \frac{\Delta t^2}{4}\|w_x^\ell\|_{L^\infty(\mathbb{R})}^2\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2 + \frac{\Delta t^2}{4}\|w_x^\ell - w_x^{\ell-1}\|_{L^\infty(\mathbb{R})}^2\|w^{\ell-1}\|_{2,\varphi}^2 + \frac{1}{8}\|w\|_{2,\varphi}^2 \\
&\le \frac{C_2\Delta t^2}{4\Delta x^3}\|w^\ell\|_{L^2(\mathbb{R})}^2\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2 + \frac{C_2\Delta t^2}{4\Delta x^3}\|w^\ell - w^{\ell-1}\|_{L^2(\mathbb{R})}^2\|w^{\ell-1}\|_{2,\varphi}^2 + \frac{1}{8}\|w\|_{2,\varphi}^2,
\end{aligned}
$$

which yields

$$
\mathcal{A}_2 \le \frac{1}{8}\|w\|_{2,\varphi}^2 + \frac{1}{2}C_2\lambda^2\max\{\|w^\ell\|_{2,\varphi}^2, \|w^{\ell-1}\|_{2,\varphi}^2\}\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2.
$$

Likewise we estimate $\mathcal{A}_1$,

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{1}{4}\int_{\mathbb{R}}(-\Delta t)\left(u^n\left(w^\ell - w^{\ell-1}\right)\right)_x\varphi w\,dx \\
&\le \frac{\Delta t^2}{8}\int_{\mathbb{R}}\left(\left(u^n\left(w^\ell - w^{\ell-1}\right)\right)_x\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&= \frac{\Delta t^2}{8}\int_{\mathbb{R}}\left(u_x^n\left(w^\ell - w^{\ell-1}\right) + u^n\left(w_x^\ell - w_x^{\ell-1}\right)\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&\le \frac{\Delta t^2}{4}\int_{\mathbb{R}}(u_x^n)^2\left(w^\ell - w^{\ell-1}\right)^2\varphi\,dx + \frac{\Delta t}{4}\int_{\mathbb{R}}(u^n)^2\left(w_x^\ell - w_x^{\ell-1}\right)^2\varphi\,dx + \frac{1}{8}\int_{\mathbb{R}} w^2\varphi\,dx \\
&\le \frac{\Delta t^2}{4}\|u_x^n\|_{L^\infty(\mathbb{R})}^2\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2 + \frac{\Delta t^2}{4}\|w_x^\ell - w_x^{\ell-1}\|_{L^\infty(\mathbb{R})}^2\|u^n\|_{2,\varphi}^2 + \frac{1}{8}\|w\|_{2,\varphi}^2 \\
&\le \frac{C_2\Delta t^2}{4\Delta x^3}\|u^n\|_{L^2(\mathbb{R})}^2\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2 + \frac{C_2\Delta t^2}{4\Delta x^3}\|w^\ell - w^{\ell-1}\|_{L^2(\mathbb{R})}^2\|u^n\|_{2,\varphi}^2 + \frac{1}{8}\|w\|_{2,\varphi}^2,
\end{aligned}
$$

which in turn implies

$$
\mathcal{A}_1 \le \frac{1}{8}\|w\|_{2,\varphi}^2 + \frac{1}{2}C_2\lambda^2\|u^n\|_{2,\varphi}\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2.
$$

Collecting the bounds we have the following inequality for $\ell \ge 1$,

$$
\begin{aligned}
&\|w\|_{2,\varphi}^2 - \frac{\Delta t}{2}\widetilde{C}\|w\|_{2,\varphi}^2 \\
&\le \frac{1}{4}\|w\|_{2,\varphi}^2 + C_2\lambda^2\max\{\|w^\ell\|_{2,\varphi}^2, \|w^{\ell-1}\|_{2,\varphi}^2, \|u^n\|_{2,\varphi}^2\}\|w^\ell - w^{\ell-1}\|_{2,\varphi}^2,
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2}\left(\frac{3}{2}-\widetilde{C}\Delta t\right)\|w^{\ell+1}-w^{\ell}\|_{2,\varphi}^2\\&\leq C_2\lambda^2\max\{\|w^{\ell}\|_{2,\varphi}^2,\|w^{\ell-1}\|_{2,\varphi}^2,\|u^n\|_{2,\varphi}^2\}\|w^{\ell}-w^{\ell-1}\|_{2,\varphi}^2.\end{aligned}$$

Assuming Δt small enough that $\frac{3}{2}-\widetilde{C}\Delta t\geq 1$ we obtain

$$\|w^{\ell+1}-w^{\ell}\|_{2,\varphi}^2\leq 2C_2\lambda^2\max\{\|w^{\ell}\|_{2,\varphi}^2,\|w^{\ell-1}\|_{2,\varphi}^2,\|u^n\|_{2,\varphi}^2\}\|w^{\ell}-w^{\ell-1}\|_{2,\varphi}^2. \tag{4.10}$$

We will now bound w^1, and so by setting $\ell=0$ in (4.6) we get

$$\begin{aligned}\left\langle w^1-u^n,\varphi v\right\rangle+\Delta t\left\langle\mathcal{H}\left(\frac{u^n+w^1}{2}\right)_x,(\varphi v)_x\right\rangle&=\frac{\Delta t}{2}\left\langle(u^n)^2,(\varphi v)_x\right\rangle\\&=-\Delta t\left\langle u^nu_x^n,\varphi v\right\rangle.\end{aligned}$$

Choosing $v=\frac{u^n+w^1}{2}$ gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left((w^1)^2-(u^n)^2\right)\varphi\,dx\quad&+\Delta t\int_{\mathbb{R}}\mathcal{H}\left(\frac{u^n+w^1}{2}\right)_x\left(\varphi\frac{u^n+w^1}{2}\right)_x dx\\&=-\Delta t\int_{\mathbb{R}}u^nu_x^n\frac{u^n+w^1}{2}\varphi\,dx.\end{aligned}$$

The term involving the Hilbert transform is estimated in the familiar fashion as

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{H}\left(\frac{u^n+w^1}{2}\right)_x\left(\varphi\frac{u^n+w^1}{2}\right)_x dx&\geq-\frac{\widetilde{C}}{4}\left\|u^n+w^1\right\|_{2,\varphi}^2\\&\geq-\frac{\widetilde{C}}{2}\left(\|u^n\|_{2,\varphi}^2+\left\|w^1\right\|_{2,\varphi}^2\right).\end{aligned}$$

The right hand side is estimated in two steps, using Cauchy's inequality and the inverse equality (4.3). First we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}(-\Delta t)u^nu_x^nu^n\varphi\,dx&\leq\frac{\Delta t^2}{4}\int_{\mathbb{R}}(u^n)^2(u_x^n)^2\varphi\,dx+\frac{1}{4}\int_{\mathbb{R}}(u^n)^2\varphi\,dx\\&\leq\frac{\Delta t^2}{4}\|u_x^n\|_{L^\infty(\mathbb{R})}^2\|u^n\|_{2,\varphi}^2+\frac{1}{4}\|u^n\|_{2,\varphi}^2\\&\leq\frac{C_2\Delta t^2}{4\Delta x^3}\|u^n\|_{L^2(\mathbb{R})}^2\|u^n\|_{2,\varphi}^2+\frac{1}{4}\|u^n\|_{2,\varphi}^2\\&\leq\frac{1}{4}\|u^n\|_{2,\varphi}^2+\frac{C_2}{4}\lambda^2\|u^n\|_{2,\varphi}^4.\end{aligned}$$

Similarly we get

$$\frac{1}{2}\int_{\mathbb{R}}(-\Delta t)u^nu_x^nw^1\varphi\,dx\leq\frac{1}{4}\|w^1\|_{2,\varphi}^2+\frac{C_2}{4}\lambda^2\|u^n\|_{2,\varphi}^4.$$

Combining the estimates gives

$$\begin{aligned}&\frac{1}{2}\|w^1\|_{2,\varphi}^2-\frac{1}{2}\|u^n\|_{2,\varphi}^2-\frac{\widetilde{C}\Delta t}{2}\|w^1\|_{2,\varphi}^2-\frac{\widetilde{C}\Delta t}{2}\|u^n\|_{2,\varphi}^2\\&\leq\frac{1}{4}\|u^n\|_{2,\varphi}^2+\frac{C_2}{4}\lambda^2\|u^n\|_{2,\varphi}^4+\frac{1}{4}\|w^1\|_{2,\varphi}^2+\frac{C_2}{4}\lambda^2\|u^n\|_{2,\varphi}^4,\end{aligned}$$

which is equivalent to

$$\left(\frac{1}{4}-\frac{\widetilde{C}\Delta t}{2}\right)\|w^1\|_{2,\varphi}^2\leq\left(\frac{1}{2}+\frac{1}{4}+\frac{\widetilde{C}\Delta t}{2}\right)\|u^n\|_{2,\varphi}^2+\frac{C_2}{2}\lambda^2\|u^n\|_{2,\varphi}^4.$$

Choosing Δt small enough that $\frac{1}{4}-\frac{\widetilde{C}\Delta t}{2}\geq\frac{1}{8}$ then gives

$$\|w^1\|_{2,\varphi}^2\leq 8\left(1+C_2\lambda^2\|u^n\|_{2,\varphi}^2\right)\|u^n\|_{2,\varphi}^2. \tag{4.11}$$

Now we claim that the following holds for $\ell\geq 1$,

$$\|w^{\ell+1}-w^\ell\|_{2,\varphi}\leq L\|w^\ell-w^{\ell-1}\|_{2,\varphi}, \tag{4.12a}$$

$$\|w^\ell\|_{2,\varphi}\leq K\|u^n\|_{2,\varphi}, \tag{4.12b}$$

$$\|w^1\|_{2,\varphi}\leq 5\|u^n\|_{2,\varphi}. \tag{4.12c}$$

The proof follows an induction argument. From (4.11) and (4.7) we get

$$\begin{aligned}\|w^1\|_{2,\varphi}&\leq\left(2\sqrt{2}+2\sqrt{2}\sqrt{C_2}\lambda\|u^n\|_{2,\varphi}\right)\|u^n\|_{2,\varphi}\\&\leq\left(2\sqrt{2}+\frac{L}{K}\right)\|u^n\|_{2,\varphi}\\&\leq 5\|u^n\|_{2,\varphi}\\&\leq K\|u^n\|_{2,\varphi},\end{aligned}$$

and so (4.12c) and (4.12b) hold for $\ell=1$. Setting $\ell=1$ in (4.10) while using (4.7) we obtain

$$\begin{aligned}\|w^2-w^1\|_{2,\varphi}&\leq\sqrt{2C_2}\lambda\max\{\|w^1\|_{2,\varphi},\|u^n\|_{2,\varphi}\}\|w^1-u^n\|_{2,\varphi}\\&\leq\left(\sqrt{2C_2}\lambda\,5\|u^n\|_{2,\varphi}\right)\|w^1-u^n\|_{2,\varphi}\\&\leq\frac{5L}{2K}\|w^1-u^n\|_{2,\varphi}\\&\leq L\|w^1-u^n\|_{2,\varphi},\end{aligned}$$

which shows that (4.12a) holds for $\ell=1$. Now assume that (4.12a) and (4.12b) hold for

$\ell = 1, \ldots, m$. One then has

$$\begin{aligned}
\|w^{m+1}\|_{2,\varphi} &\leq \sum_{\ell=0}^{m} \|w^{\ell+1} - w^{\ell}\|_{2,\varphi} + \|w^0\|_{2,\varphi} \\
&\leq \|w^1 - w^0\|_{2,\varphi} \sum_{\ell=0}^{m} L^{\ell} + \|w^0\|_{2,\varphi} \\
&\leq \big(\|w^1\|_{2,\varphi} + \|w^0\|_{2,\varphi}\big) \sum_{\ell=0}^{m} L^{\ell} + \|w^0\|_{2,\varphi} \\
&\leq 6\|u^n\|_{2,\varphi} \frac{1}{1-L} + \|u^n\|_{2,\varphi} \\
&= \frac{7-L}{1-L} \|u^n\|_{2,\varphi} \\
&= K\|u^n\|_{2,\varphi},
\end{aligned}$$

thus (4.12b) holds for all ℓ. This result together with (4.10) and (4.7) lead to

$$\begin{aligned}
\|w^{\ell+1} - w^{\ell}\|_{2,\varphi} &\leq \sqrt{2C_2}\lambda \max\{\|w^{\ell}\|_{2,\varphi}, \|w^{\ell-1}\|_{2,\varphi}, \|u^n\|_{2,\varphi}\} \|w^{\ell} - w^{\ell-1}\|_{2,\varphi} \\
&\leq \sqrt{2C_2}\lambda K \|u^n\|_{2,\varphi} \|w^{\ell} - w^{\ell-1}\|_{2,\varphi} \\
&\leq L\|w^{\ell} - w^{\ell-1}\|_{2,\varphi}.
\end{aligned}$$

This shows that (4.12a) holds for all ℓ as well. Since $0 < L < 1$ this shows that $\{w^{\ell}\}$ is Cauchy and hence converges, which completes the proof of Lemma 4.2. □

Chapter 5

Convergence of the scheme

In this chapter we will prove the convergence of the numerical scheme introduced in the previous section. We begin with the following important lemma.

Lemma 5.1. *Let λ, K and L be defined as in Lemma 4.2 and let u^n be the solution of the scheme* (4.2)*. Assume also that Δt satisfies*

$$\lambda \leq \frac{L}{2\sqrt{2}\sqrt{C_2}K\sqrt{y(T)}}, \tag{5.1}$$

for some $y(T)$ which only depends on $\|u_0\|_{L^2(\mathbb{R})}$. Then there exist a positive time T and a constant C, both depending only on $\|u_0\|_{L^2(\mathbb{R})}$ such that for all n satisfying $n\Delta t \leq T$ the following estimate holds

$$\|u^n\|_{L^2(\mathbb{R})} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}\right). \tag{5.2}$$

Additionally, the approximation u^n satisfies the following $H^{\frac{1}{2}}$-estimate

$$\Delta t \sum_{(n+\frac{1}{2})\Delta t \leq T} \left\| D^{\frac{1}{2}} u^{n+\frac{1}{2}} \right\|^2_{L^2([-R,R])} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}\right), \quad \left(n+\frac{1}{2}\right)\Delta t < T. \tag{5.3}$$

Proof. Starting with (4.2) it follows that (3.11) holds, and with the same estimates as in the associated chapter and the fact that $1 \leq \varphi(x) \leq 2+2R$ we obtain the inequality

$$\begin{aligned}
&\int_{\mathbb{R}} \left(u^{n+1}\right)^2 \varphi\, dx + 2\Delta t \int_{\mathbb{R}} \left(D^{\frac{1}{2}} u^{n+\frac{1}{2}}\right)^2 \varphi_x\, dx - 2\Delta t \widetilde{C} \int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi\, dx \\
&\leq \int_{\mathbb{R}} (u^n)^2 \varphi\, dx + \frac{2\Delta t}{3} \int_{\mathbb{R}} \left(D^{\frac{1}{2}} u^{n+\frac{1}{2}}\right)^2 \varphi_x\, dx + \frac{2\Delta t}{3} C_S^2 \int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi\, dx \\
&\quad + \frac{\Delta t}{3} \int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi\, dx + \frac{\Delta t}{3} C^2 \left(\int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi\, dx\right)^2,
\end{aligned}$$

which again implies

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(u^{n+1}\right)^2 \varphi \, dx + \frac{4}{3}\Delta t \int_{\mathbb{R}} \left(D^{\frac{1}{2}} u^{n+\frac{1}{2}}\right)^2 \varphi_x \, dx \\
&\leq \int_{\mathbb{R}} \left(u^n\right)^2 \varphi \, dx + \Delta t \, C \left[\int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi \, dx + \left(\int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 \varphi \, dx \right)^2 \right].
\end{aligned} \tag{5.4}
$$

Dropping the term involving the fractional derivative and writing $a_n = \int_{\mathbb{R}} \left(u^n\right)^2 \varphi \, dx$ then gives

$$
a_{n+1} \leq a_n + \Delta t f(a_{n+\frac{1}{2}}), \tag{5.5}
$$

with the function

$$
f(a) = C \left[a + a^2 \right].
$$

It is clear that $a_{n+\frac{1}{2}} \leq (a_n + a_{n+1})/2$ and so $\{a_n\}$ solves the Crank–Nicolson method for the differential inequality

$$
\frac{da}{dt} \leq f(a).
$$

Let us then consider the following ordinary differential equation

$$
\begin{cases}
\frac{dy}{dt} = f\left(\frac{K^2+1}{2} y\right), & t > 0, \\
y(0) = a_0,
\end{cases}
$$

where K comes from Lemma 4.2. It is not difficult to show that this differential equation has a unique solution y which blows up at some finite time T_∞ depending only on the initial condition, and so we choose $T = T_\infty/2$. Note that the solution of this ODE is strictly increasing and convex. We now compare this solution with (5.5) under the assumption that the CFL condition (5.1) holds. We claim that $a_n \leq y(t_n)$ for all $n \geq 0$, and argue by induction. As $y(0) = a_0$, the claim holds for $n = 0$. Now assume that it holds for $n \in \{0, 1, \ldots, m\}$. As $0 \leq a_m \leq y(T)$, (5.1) implies that (4.7) holds, and thus Lemma 4.2 gives $a_{m+1} \leq K^2 a_m$. Therefore we have

$$
a_{m+\frac{1}{2}} \leq (a_m + a_{m+1})/2 \leq \left(\frac{K^2+1}{2}\right) a_m.
$$

The convexity of f then gives

$$
\begin{aligned}
a_{m+1} &\leq a_m + \Delta t f\left(\frac{K^2+1}{2} a_m\right) \\
&\leq y(t_m) + \Delta t f\left(\frac{K^2+1}{2} y(t_m)\right) \\
&\leq y(t_m) + \Delta t \frac{dy}{dt}(t_m) \\
&\leq y(t_{m+1}),
\end{aligned}
$$

which proves the claim. Since $\varphi \geq 1$ we get the L^2-stability estimate (5.2),

$$\|u^n\|_{L^2(\mathbb{R})} \leq \sqrt{y(T)} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}\right).$$

Consequently, summing over (5.4) yields the estimate

$$\Delta t \sum_{n\Delta t \leq T} \int_{-R}^{R} \left|D^{\frac{1}{2}} u^{n+\frac{1}{2}}\right|^2 dx \leq C(\|u_0\|_{L^2(\mathbb{R})}).$$

This proves (5.3) and completes the proof of Lemma 5.1. □

5.1 Bounds on temporal derivative

We will here obtain bounds on the temporal derivative to be used later in the analysis. The following lemma will be of use.

Lemma 5.2. *Let $\psi \in C_c^\infty(-R,R)$. Then there exists a projection $P : C_c^\infty(-R,R) \to S_{\Delta x} \cap C_c(-R,R)$ such that*

$$\int_{\mathbb{R}} uP(\psi)\,\varphi\,dx = \int_{\mathbb{R}} u\psi\varphi\,dx, \quad u \in S_{\Delta x}.$$

Additionally, P satisfies the bounds

$$\begin{cases} \|P(\psi)\|_{L^2(\mathbb{R})} \leq C\|\psi\|_{L^2(\mathbb{R})}, \\ \|P(\psi)\|_{H^1(\mathbb{R})} \leq C\|\psi\|_{H^1(\mathbb{R})}, \\ \|P(\psi)\|_{H^2(\mathbb{R})} \leq C\|\psi\|_{H^2(\mathbb{R})}, \end{cases} \tag{5.6}$$

where the constant C is independent of Δx.

Proof. The proof is a straightforward adaptation of the L^2-projection results found in the monograph of Ciarlet [6]. □

In our upcoming estimates we also need the following inequality. Given $v \in H^1(\mathbb{R})$, we have

$$\sup_{x\in\mathbb{R}} v^2(x) \leq \frac{1}{2}\|v\|^2_{H^1(\mathbb{R})},$$

which implies

$$\|v\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|v\|_{H^1(\mathbb{R})}. \tag{5.7}$$

To see this we write

$$\begin{aligned}
v(x)^2 &= \frac{1}{2}\left(\int_{-\infty}^{x} \frac{dv^2(y)}{dy}\,dy - \int_{x}^{\infty} \frac{dv^2(y)}{dy}\,dy\right) \\
&= \int_{-\infty}^{x} vv_y\,dy - \int_{x}^{\infty} vv_y\,dy \\
&\leq \int_{-\infty}^{\infty} |v||v_y|\,dy \\
&\leq \frac{1}{2}\left(\int_{\mathbb{R}} |v|^2\,dy + \int_{\mathbb{R}} |v_y|^2\,dy\right) \\
&= \frac{1}{2}\|v\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

As mentioned in the background theory chapter, for $s > 0$, $H^{-s}(\mathbb{R})$ is the dual of $H^s(\mathbb{R})$. That is, we may define its norm as

$$\|u\|_{H^{-s}(\mathbb{R})} = \sup_{v \in C_c^\infty(\mathbb{R})} \frac{|\langle u, v\rangle|}{\|v\|_{H^s(\mathbb{R})}},$$

as $C_c^\infty(\mathbb{R})$ is dense in $H^s(\mathbb{R})$ for $s \geq 0$. Because of this density, we could have equivalently defined the norm by taking supremum over $v \in H^s(\mathbb{R})$. The above relation leads to the inequality

$$\int_{\mathbb{R}} uv\,dx \leq \|u\|_{H^{-s}(\mathbb{R})}\|v\|_{H^s(\mathbb{R})}, \tag{5.8}$$

which holds for $u \in H^{-s}(\mathbb{R})$ and $v \in H^s(\mathbb{R})$. This inequality can also be proved using the Fourier transform definition of these norms,

$$\begin{aligned}
\int_{\mathbb{R}} uv\,dx &= \int_{\mathbb{R}} \hat{u}\bar{\hat{v}}\,d\xi \\
&\leq \int_{\mathbb{R}} |\hat{u}||\hat{v}|\,d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)^{-\frac{s}{2}}|\hat{u}|(1+|\xi|^2)^{\frac{s}{2}}|\hat{v}|\,d\xi \\
&\leq \|(1+|\xi|^2)^{-\frac{s}{2}}\hat{u}\|_{L^2(\mathbb{R})}\|(1+|\xi|^2)^{\frac{s}{2}}\hat{v}\|_{L^2(\mathbb{R})} \\
&= \|u\|_{H^{-s}(\mathbb{R})}\|v\|_{H^s(\mathbb{R})},
\end{aligned}$$

where the first equality is due to Plancherel's identity (2.3), the second inequality is a standard application of Hölder's inequality, and the last equality comes from the definition (2.10). Analogously, we may define the dual norm on a bounded interval $[-R, R]$ in the particular case $s = 2$ as

$$\|u\|_{H^{-2}([-R,R])} = \sup_{v \in C_c^\infty([-R,R])} \frac{\left|\int_{-R}^{R} uv\,dx\right|}{\|v\|_{H^2([-R,R])}},$$

which leads to the inequality

$$\int_{-R}^{R} uv\,dx \leq \|u\|_{H^{-2}([-R,R])}\|v\|_{H^2([-R,R])} \tag{5.9}$$

for $u \in H^{-2}([-R,R])$ and $v \in H^2([-R,R])$.

The above relations together with Lemma 5.2 is used to prove the following lemma regarding the boundedness of the temporal derivatives of the approximate solutions.

Lemma 5.3. *Let $\{u^n\}$ be the solution of the scheme* (4.2) *and assume that the hypothesis of Lemma 5.1 holds. Then we have the following estimate*

$$\|D_t^+ u^n \varphi\|_{H^{-2}([-R,R])} \leq C(\|u_0\|_{L^2(\mathbb{R})}, R), \tag{5.10}$$

where $D_t^+ u^n$ is the forward time difference operator

$$D_t^+ u^n = \frac{u^{n+1} - u^n}{\Delta t}.$$

Proof. Start by rewriting (4.2) as

$$\left\langle D_t^+ u^n, \varphi v \right\rangle = \left\langle \frac{\left(u^{n+\frac{1}{2}}\right)^2}{2}, (\varphi v)_x \right\rangle - \left\langle \mathcal{H} u_x^{n+\frac{1}{2}}, (\varphi v)_x \right\rangle, \tag{5.11}$$

which holds for all $v \in S_{\Delta x}$. Let $\psi \in C_c^\infty(-R,R)$ and set $v = P(\psi)$, where P is the projection from Lemma 5.2, to get

$$\left\langle D_t^+ u^n, \varphi P(\psi) \right\rangle = \left\langle \frac{\left(u^{n+\frac{1}{2}}\right)^2}{2}, (\varphi P(\psi))_x \right\rangle - \left\langle \mathcal{H} u_x^{n+\frac{1}{2}}, (\varphi P(\psi))_x \right\rangle.$$

Using (5.7), (5.6) and (5.2) we estimate the first term on the right hand side as follows

$$\begin{aligned}
&\int_{\mathbb{R}} \left(u^{n+\frac{1}{2}}\right)^2 (\varphi P(\psi))_x\,dx \\
&= \int_{-R}^{R} \left(u^{n+\frac{1}{2}}\right)^2 \varphi_x P(\psi)\,dx + \int_{-R}^{R} \left(u^{n+\frac{1}{2}}\right)^2 \varphi P(\psi)_x\,dx \\
&\leq \left(\|P(\psi)\|_{L^\infty([-R,R])} + \|P(\psi)_x\|_{L^\infty([-R,R])}(2+2R)\right) \int_{-R}^{R} \left(u^{n+\frac{1}{2}}\right)^2 dx \\
&\leq \left(\|P(\psi)\|_{H^1([-R,R])} + \|P(\psi)_x\|_{H^1([-R,R])}(2+2R)\right) \left\|u^{n+\frac{1}{2}}\right\|_{L^2(\mathbb{R})}^2 \\
&\leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right) \|\psi\|_{H^2([-R,R])}.
\end{aligned}$$

The second term can be estimated as

$$\begin{aligned}
-\int_{\mathbb{R}} \mathcal{H}u_x^{n+\frac{1}{2}}(\varphi P(\psi))_x\,dx &= \int_{\mathbb{R}} \left(\mathcal{H}u^{n+\frac{1}{2}}\right)(\varphi P(\psi))_{xx}\,dx \\
&\leq \left\|\mathcal{H}u^{n+\frac{1}{2}}\right\|_{L^2(\mathbb{R})} \|(\varphi P(\psi))_{xx}\|_{L^2([-R,R])} \\
&\leq \left\|u^{n+\frac{1}{2}}\right\|_{L^2(\mathbb{R})} \|\varphi P(\psi)\|_{H^2([-R,R])} \\
&\leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right)\|\psi\|_{H^2([-R,R])},
\end{aligned}$$

where we have used (5.2), (5.6) and the L^2-isometry of the Hilbert transform.

Together this gives

$$\left|\int_{-R}^{R} D_t^+ u^n \varphi\psi\,dx\right| = \left|\int_{-R}^{R} D_t^+ u^n \varphi P(\psi)\,dx\right| \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right)\|\psi\|_{H^2([-R,R])},$$

which implies

$$\left\|D_t^+ u^n \varphi\right\|_{H^{-2}([-R,R])} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right),$$

and the estimate is proven.

If $\psi \in C_c^\infty(\mathbb{R})$ then $P(\psi) \in S_{\Delta x}$. By the exact same arguments as above, but this time on $\mathbb{R}$ instead of $[-R,R]$, we get

$$\|D_t^+ u^n \varphi\|_{H^{-2}(\mathbb{R})} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right). \tag{5.12}$$

□

5.2 Convergence to a weak solution

Prior to stating our theorem of convergence we define the weak solution of the Cauchy problem (1.1) in the following way.

Definition 5.1. *Let $Q > 0$. Then $u \in L^2\left(0,T; H^{\frac{1}{2}}(-Q,Q)\right)$ is a weak solution of* (1.1) *in the region $(-Q,Q)\times[0,T)$ if*

$$\int_0^T \int_{\mathbb{R}} \left(\phi_t u + \phi_x \frac{u^2}{2} - (\mathcal{H}\phi_{xx})\,u\right) dx\,dt + \int_{\mathbb{R}} \phi(x,0)u_0(x)\,dx = 0, \tag{5.13}$$

for all $\phi \in C_c^\infty((-Q,Q)\times[0,T))$.

The above definition is easily derived by multiplying (1.1) with the test function ϕ defined above, integrating by parts and noting that ϕ vanishes outside its support.

Now we define the approximate solution $u^{\Delta x} \in S_{\Delta x}$, which will be shown to converge to a weak solution of (1.1), by the interpolation formula

$$u^{\Delta x}(x,t) = \begin{cases} u^{n-\frac{1}{2}}(x) + (t - t_{n-\frac{1}{2}})D_t^+ u^{n-\frac{1}{2}}(x), & t \in \left[t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}\right), n \geq 1, \\ u^0(x) + 2t\frac{u^{\frac{1}{2}}(x) - u^0(x)}{\Delta t}, & t \in \left[t_0, t_{\frac{1}{2}}\right). \end{cases} \tag{5.14}$$

We then have the following convergence theorem, which is the main result of the thesis.

Theorem 5.1. *Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of functions defined by the scheme* (4.2) *and assume that $\|u_0\|_{L^2(\mathbb{R})}$ is finite. Assume furthermore that $\Delta t = \mathcal{O}(\Delta x^2)$. Then there exist a positive time T and a constant C, depending only on T, R, and $\|u_0\|_{L^2(\mathbb{R})}$ such that*

$$\left\|u^{\Delta x}\right\|_{L^\infty(0,T;L^2([-R,R]))} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right), \tag{5.15}$$

$$\left\|u^{\Delta x}\right\|_{L^2(0,T;H^{\frac{1}{2}}([-R,R]))} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right), \tag{5.16}$$

$$\left\|\partial_t u^{\Delta x}\varphi\right\|_{L^2(0,T;H^{-2}([-R,R]))} \leq C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right), \tag{5.17}$$

where $u^{\Delta x}$ is given by (5.14)*. Additionally, there exists a sequence $\{\Delta x_j\}_{j=1}^\infty$ and a function $u \in L^2\left(0,T;L^2([-R,R])\right)$ such that*

$$u^{\Delta x_j} \to u \text{ strongly in } L^2\left(0,T;L^2([-R,R])\right), \tag{5.18}$$

as $j \to \infty$. The function u is a weak solution of the Cauchy problem for (1.1)*, which is to say that it satisfies* (5.13) *with $Q = R - 1$.*

Proof. Assume for simplicity that $T = (N + \frac{1}{2})\Delta t$ for some $N \in \mathbb{N}$. For $t \in \left[t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}\right)$ we have

$$u^{\Delta x}(x,t) = (1-\alpha_n(t))u^{n-\frac{1}{2}}(x) + \alpha_n(t)u^{n+\frac{1}{2}}(x),$$

where $\alpha_n = (t - t_{n-\frac{1}{2}})/\Delta t \in [0,1]$. For $t \in \left[t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}\right) =: T_n, n = 1,2,\ldots,N$ one then has

$$\begin{aligned}
\left\|u^{\Delta x}\right\|_{L^2(\mathbb{R})} &\leq \|1-\alpha_n(t)\|_{L^\infty(T_n)}\|u^{n-\frac{1}{2}}\|_{L^2(\mathbb{R})} + \|\alpha_n\|_{L^\infty(T_n)}\|u^{n+\frac{1}{2}}\|_{L^2(\mathbb{R})} \\
&\leq \frac{1}{2}\left(\|u^{n-1}\|_{L^2(\mathbb{R})} + \|u^n\|_{L^2(\mathbb{R})} + \|u^n\|_{L^2(\mathbb{R})} + \|u^{n+1}\|_{L^2(\mathbb{R})}\right) \\
&\leq C(\|u_0\|_{L^2(\mathbb{R})}),
\end{aligned}$$

while for $t \in \left[0, t_{\frac{1}{2}}\right) =: T_0$ we have

$$\begin{aligned}
\left\|u^{\Delta x}\right\|_{L^2(\mathbb{R})} &\leq \|1-(2t)/\Delta t\|_{L^\infty(T_0)}\|u^0\|_{L^2(\mathbb{R})} + \|(2t)/\Delta t\|_{L^\infty(T_0)}\|u^{\frac{1}{2}}\|_{L^2(\mathbb{R})} \\
&\leq \frac{3}{2}\|u^0\|_{L^2(\mathbb{R})} + \frac{1}{2}\|u^1\|_{L^2(\mathbb{R})} \\
&\leq C(\|u_0\|_{L^2(\mathbb{R})}),
\end{aligned}$$

which proves (5.15).

Next we have

$$\begin{aligned}
\int_0^T \left\| D^{\frac{1}{2}} u^{\Delta x} \right\|^2_{L^2([-R,R])} dt &\le 2 \left\| D^{\frac{1}{2}} u^0 \right\|^2_{L^2([-R,R])} \int_0^{t_{\frac{1}{2}}} \left(1 - \frac{2t}{\Delta t}\right)^2 dt \\
&\quad + 2 \left\| D^{\frac{1}{2}} u^{\frac{1}{2}} \right\|^2_{L^2([-R,R])} \int_0^{t_{\frac{1}{2}}} \left(\frac{2t}{\Delta t}\right)^2 dt \\
&\quad + 2 \sum_{n=1}^N \left\| D^{\frac{1}{2}} u^{n-\frac{1}{2}} \right\|^2_{L^2([-R,R])} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} (1 - \alpha_n(t))^2 \, dt \\
&\quad + 2 \sum_{n=1}^N \left\| D^{\frac{1}{2}} u^{n+\frac{1}{2}} \right\|^2_{L^2([-R,R])} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} (\alpha_n(t))^2 \, dt \\
&\le \Delta t \, C \left\| u^0 \right\|^2_{H^1([-R,R])} + 2\Delta t \sum_{n=0}^N \left\| D^{\frac{1}{2}} u^{n+\frac{1}{2}} \right\|^2_{L^2([-R,R])},
\end{aligned}$$

where in the last inequality we have used that $\left\| u^0 \right\|_{H^{\frac{1}{2}}([-R,R])} \le C \left\| u^0 \right\|_{H^1([-R,R])}$ due to Sobolev embedding. By applying the inverse equality (4.3) and (5.3) to the last line we conclude that (5.16) holds.

For the third relation, note that

$$\begin{aligned}
u_t^{\Delta x} &= \begin{cases} D_t^+ u^{n-\frac{1}{2}}, & (x,t) \in \mathbb{R} \times [t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}), \\ \frac{u^{\frac{1}{2}} - u^0}{\Delta t/2}, & (x,t) \in \mathbb{R} \times [t_0, t_{\frac{1}{2}}), \end{cases} \\
&= \begin{cases} \frac{D_t^+ u^n + D_t^+ u^{n-1}}{2}, & (x,t) \in \mathbb{R} \times [t_{n-\frac{1}{2}}, t_{n+\frac{1}{2}}), \\ D_t^+ u^0, & (x,t) \in \mathbb{R} \times [t_0, t_{\frac{1}{2}}). \end{cases}
\end{aligned}$$

Then from (5.10) we clearly have

$$\begin{aligned}
\int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|^2_{H^{-2}([-R,R])} dt &= \int_{t_0}^{t_{\frac{1}{2}}} \left\| D_t^+ u^0 \varphi \right\|^2_{H^{-2}([-R,R])} dt \\
&\quad + \sum_{n=1}^N \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \left\| \frac{D_t^+ u^n + D_t^+ u^{n-1}}{2} \varphi \right\|^2_{H^{-2}([-R,R])} dt \\
&\le \int_{t_0}^{t_{\frac{1}{2}}} C \left(\left\| u^0 \right\|_{L^2(\mathbb{R})}, R \right) dt \\
&\quad + \sum_{n=1}^N \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} C \left(\left\| u_0 \right\|_{L^2(\mathbb{R})}, R \right) dt \\
&= C \left(\left\| u_0 \right\|_{L^2(\mathbb{R})}, R \right),
\end{aligned}$$

which proves (5.17). If one instead uses (5.12) and considers the whole real line $\mathbb{R}$, the above argument leads to

$$\left\| \partial_t u^{\Delta x} \varphi \right\|_{L^2(0,T;H^{-2}(\mathbb{R}))} \le C \left(\left\| u_0 \right\|_{L^2(\mathbb{R})}, R \right). \tag{5.19}$$

Also, as φ is a positive and bounded function, (5.15) and (5.16) give

$$\|\varphi u^{\Delta x}\|_{L^\infty(0,T;L^2([-R,R]))} \le C(\|u_0\|_{L^2(\mathbb{R})}, R), \tag{5.20a}$$

$$\|\varphi u^{\Delta x}\|_{L^2(0,T;H^{\frac{1}{2}}([-R,R]))} \le C(\|u_0\|_{L^2(\mathbb{R})}, R). \tag{5.20b}$$

Based on the bounds (5.20a), (5.20b) and (5.17) we apply Lemma 2.2 to the set $\{\varphi u^{\Delta x}\}$ to prove the existence of a sequence $\{\Delta x_j\}_{j\in\mathbb{N}}$ such that $\Delta x_j \to 0$, and a function $\tilde{u}$ such that

$$\varphi u^{\Delta x_j} \to \tilde{u} \text{ strongly in } L^2(0,T;L^2([-R,R])), \tag{5.21}$$

as j approaches infinity. As $\varphi \ge 1$, (5.21) implies that there exists u such that (5.18) holds. The strong convergence allows passage to the limit in the nonlinearity.

Now it remains to prove that u is a weak solution of (1.1). In the following we consider the standard L^2-projection of a function ψ with $k+1$ continuous derivatives into the space $S_{\Delta x}$, denoted by $\mathcal{P}$, that is,

$$\int_{\mathbb{R}} (\mathcal{P}\psi(x) - \psi(x))\, v(x)\, dx = 0, \quad v \in S_{\Delta x}.$$

For the above projection we have

$$\|\psi(x) - \mathcal{P}\psi(x)\|_{H^k(\mathbb{R})} \le C\Delta x \|\psi\|_{H^{k+1}(\mathbb{R})}, \tag{5.22}$$

where the constant C is independent of Δx. For a proof of the above estimate, see Ciarlet [6].

Note also that from (5.11) we have for $v \in S_{\Delta x}$ and $n \ge 1$,

$$\left\langle D_t^+ u^n, \varphi v\right\rangle - \left\langle \frac{\left(u^{n+\frac{1}{2}}\right)^2}{2}, (\varphi v)_x \right\rangle + \left\langle \mathcal{H} u_x^{n+\frac{1}{2}}, (\varphi v)_x \right\rangle = 0,$$

$$\left\langle D_t^+ u^{n-1}, \varphi v\right\rangle - \left\langle \frac{\left(u^{n-\frac{1}{2}}\right)^2}{2}, (\varphi v)_x \right\rangle + \left\langle \mathcal{H} u_x^{n-\frac{1}{2}}, (\varphi v)_x \right\rangle = 0.$$

Averaging the two relations gives

$$\mathcal{F}_n(\varphi v) := \left\langle D_t^+ u^{n-\frac{1}{2}}, \varphi v \right\rangle - \frac{1}{2}\left\langle \frac{\left(u^{n+\frac{1}{2}}\right)^2 + \left(u^{n-\frac{1}{2}}\right)^2}{2}, (\varphi v)_x \right\rangle + \left\langle \mathcal{H}\left(\frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2}\right)_x, (\varphi v)_x \right\rangle = 0.$$

We will start by showing that

$$\int_0^T \int_{\mathbb{R}} \left(u_t^{\Delta x} \varphi v - \frac{\left(u^{\Delta x}\right)^2}{2} (\varphi v)_x - \left(\mathcal{H} u^{\Delta x}\right)(\varphi v)_{xx} \right) dx\, dt = \mathcal{O}(\Delta x), \tag{5.23}$$

for any test function $v \in C_c^\infty((-R+1, R-1) \times [0,T))$ and φ as previously defined. We proceed as follows.

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(u_t^{\Delta x} \varphi v - \frac{\left(u^{\Delta x}\right)^2}{2} (\varphi v)_x - \left(\mathcal{H} u^{\Delta x}\right) (\varphi v)_{xx} \right) dx\, dt \\
&= \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left(u_t^{\Delta x} \varphi v - \frac{\left(u^{\Delta x}\right)^2}{2} (\varphi v)_x - \left(\mathcal{H} u^{\Delta x}\right) (\varphi v)_{xx} \right) dx\, dt \\
&\quad + \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(u_t^{\Delta x} \varphi v - \frac{\left(u^{\Delta x}\right)^2}{2} (\varphi v)_x - \left(\mathcal{H} u^{\Delta x}\right) (\varphi v)_{xx} \right) dx\, dt \\
&=: I + II.
\end{aligned}
$$

Now let $v^{\Delta x} = \mathcal{P} v$ and observe that we may write

$$
\begin{aligned}
I = &\int_0^{t_{\frac{1}{2}}} \underbrace{\int_{\mathbb{R}} \left(\left(D_t^+ u^0\right) \varphi v^{\Delta x} - \frac{\left(u^{\frac{1}{2}}\right)^2}{2} \left(\varphi v^{\Delta x}\right)_x + \left(\mathcal{H} u^{\frac{1}{2}}\right)_x \left(\varphi v^{\Delta x}\right)_x \right) dx\, dt}_{= 0 \text{ by (4.2)}} \\
&+ \underbrace{\int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left(D_t^+ u^0\right) \varphi \left(v - v^{\Delta x}\right) dx\, dt}_{\mathcal{C}_1^{\Delta x}} \\
&\underbrace{- \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \frac{\left(u^{\frac{1}{2}}\right)^2}{2} \left(\varphi (v - v^{\Delta x})\right)_x dx\, dt}_{\mathcal{C}_2^{\Delta x}} \\
&\underbrace{- \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left[\frac{\left(u^{\Delta x}\right)^2}{2} - \frac{\left(u^{\frac{1}{2}}\right)^2}{2} \right] (\varphi v)_x\, dx\, dt}_{\mathcal{C}_3^{\Delta x}} \\
&\underbrace{- \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left(\mathcal{H} u^{\frac{1}{2}}\right) \left(\varphi (v - v^{\Delta x})\right)_{xx} dx\, dt}_{\mathcal{C}_4^{\Delta x}} \\
&\underbrace{- \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left(\mathcal{H} \left(u^{\Delta x} - u^{\frac{1}{2}}\right)\right) (\varphi v)_{xx}\, dx\, dt}_{\mathcal{C}_5^{\Delta x}},
\end{aligned}
$$

and

$$
\begin{aligned}
II = &\sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \underbrace{\mathcal{F}_n(\varphi v^{\Delta x})}_{=0} \, dt \\
&\underbrace{+ \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(D_t^+ u^{n-\frac{1}{2}} \right) \varphi \left(v - v^{\Delta x} \right) \, dx \, dt}_{\mathcal{E}_1^{\Delta x}} \\
&\underbrace{- \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \frac{1}{2} \frac{\left(u^{n+\frac{1}{2}} \right)^2 + \left(u^{n-\frac{1}{2}} \right)^2}{2} \left(\varphi(v - v^{\Delta x}) \right)_x \, dx \, dt}_{\mathcal{E}_2^{\Delta x}} \\
&\underbrace{- \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \frac{1}{2} \left[\left(u^{\Delta x} \right)^2 - \frac{\left(u^{n+\frac{1}{2}} \right)^2 + \left(u^{n-\frac{1}{2}} \right)^2}{2} \right] (\varphi v)_x \, dx \, dt}_{\mathcal{E}_3^{\Delta x}} \\
&\underbrace{- \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(\mathcal{H} \frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2} \right) \left(\varphi(v - v^{\Delta x}) \right)_{xx} \, dx \, dt}_{\mathcal{E}_4^{\Delta x}} \\
&\underbrace{- \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(\mathcal{H} \left(u^{\Delta x} - \frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2} \right) \right) (\varphi v)_{xx} \, dx \, dt}_{\mathcal{E}_5^{\Delta x}} .
\end{aligned}
$$

Now we estimate the preceding terms. From (5.9), (5.22) and (5.17) we obtain

$$
\begin{aligned}
\mathcal{C}_1^{\Delta x} + \mathcal{E}_1^{\Delta x} &= \int_0^T \int_{-R}^{R} \partial_t u^{\Delta x} \varphi \left(v - v^{\Delta x} \right) \, dx \, dt \\
&\leq \int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|_{H^{-2}([-R,R])} \left\| v - v^{\Delta x} \right\|_{H^2([-R+1,R-1])} \, dt \\
&\leq \int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|_{H^{-2}([-R,R])} C \Delta x \, \|v\|_{H^3([-R+1,R-1])} \, dt \\
&\leq C \Delta x \left\| \partial_t u^{\Delta x} \varphi \right\|_{L^2(0,T;H^{-2}([-R,R]))} \|v\|_{L^2(0,T;H^3([-R+1,R-1]))} \\
&\leq \Delta x \, C \left(\|u_0\|_{L^2(\mathbb{R})}, R \right) \|v\|_{L^2(0,T;H^3([-R+1,R-1]))} \xrightarrow{\Delta x \to 0} 0.
\end{aligned}
$$

From (5.2), (5.7) and (5.22) we get

$$
\begin{aligned}
\mathcal{C}_2^{\Delta x} + \mathcal{E}_2^{\Delta x} \leq{}& \frac{1}{2}\int_0^{t_{\frac{1}{2}}}\int_{-R+1}^{R-1}\left|u^{\frac{1}{2}}\right|^2\left|\left(\varphi(v-v^{\Delta x})\right)_x\right|\,dx\,dt \\
&+\frac{1}{4}\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1}\left|u^{n+\frac{1}{2}}\right|^2\left|\left(\varphi(v-v^{\Delta x})\right)_x\right|\,dx\,dt \\
&+\frac{1}{4}\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1}\left|u^{n-\frac{1}{2}}\right|^2\left|\left(\varphi(v-v^{\Delta x})\right)_x\right|\,dx\,dt \\
\leq{}& (2+2R)\int_0^{t_{\frac{1}{2}}}\left\|\left(v-v^{\Delta x}\right)_x\right\|_{L^\infty([-R+1,R-1])}\left\|u^{\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
&+\int_0^{t_{\frac{1}{2}}}\left\|v-v^{\Delta x}\right\|_{L^\infty([-R+1,R-1])}\left\|u^{\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
&+(2+2R)\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|\left(v-v^{\Delta x}\right)_x\right\|_{L^\infty([-R+1,R-1])}\left\|u^{n+\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
&+\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|v-v^{\Delta x}\right\|_{L^\infty([-R+1,R-1])}\left\|u^{n+\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
&+(2+2R)\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|\left(v-v^{\Delta x}\right)_x\right\|_{L^\infty([-R+1,R-1])}\left\|u^{n-\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
&+\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|v-v^{\Delta x}\right\|_{L^\infty([-R+1,R-1])}\left\|u^{n-\frac{1}{2}}\right\|^2_{L^2([-R,R])}\,dt \\
\leq{}& C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\int_0^T\left\|\left(v-v^{\Delta x}\right)_x\right\|_{H^1([-R+1,R-1])}\,dt \\
&+C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\int_0^T\left\|v-v^{\Delta x}\right\|_{H^1([-R+1,R-1])}\,dt \\
\leq{}& C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\int_0^T\left\|v-v^{\Delta x}\right\|_{H^2([-R+1,R-1])}\,dt \\
\leq{}& C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\int_0^T C\Delta x\,\|v\|_{H^3([-R+1,R-1])}\,dt \\
\leq{}& \Delta x\,C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\|v\|_{L^\infty(0,T;H^3([-R+1,R-1]))}\xrightarrow{\Delta x\to 0}0.
\end{aligned}
$$

The next terms may be rewritten as

$$
\begin{aligned}
\mathcal{C}_3^{\Delta x} + \mathcal{E}_3^{\Delta x} &= \int_0^{t_{\frac{1}{2}}} \int_{-R+1}^{R+1} \left(-\frac{\left(u^{\Delta x}\right)^2}{2} + \frac{\left(u^{\frac{1}{2}}\right)^2}{2} \right) (\varphi v)_x \, dx \, dt \\
&\quad + \frac{1}{2} \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} \left(-\left(u^{\Delta x}\right)^2 + \frac{\left(u^{n+\frac{1}{2}}\right)^2 + \left(u^{n-\frac{1}{2}}\right)^2}{2} \right) (\varphi v)_x \, dx \, dt \\
&= \int_0^{t_{\frac{1}{2}}} \int_{-R+1}^{R+1} \frac{1}{2} \left(-\left(u^0\right)^2 + \left(u^{\frac{1}{2}}\right)^2 \right) (\varphi v)_x \, dx \, dt \\
&\quad - \int_0^{t_{\frac{1}{2}}} \int_{-R+1}^{R+1} \left(u^0 t \left(D_t^+ u^0\right) (\varphi v)_x + \frac{1}{2} t^2 (D_t^+ u^0)^2 (\varphi v)_x \right) dx \, dt \\
&\quad + \frac{1}{2} \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} \frac{1}{2} \left(-\left(u^{n-\frac{1}{2}}\right)^2 + \left(u^{n+\frac{1}{2}}\right)^2 \right) (\varphi v)_x \, dx \, dt \\
&\quad - \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} u^{n-\frac{1}{2}} (t - t_{n-\frac{1}{2}}) \left(D_t^+ u^{n-\frac{1}{2}}\right) (\varphi v)_x \, dx \, dt \\
&\quad - \frac{1}{2} \sum_{n=0}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} (t - t_{n-\frac{1}{2}})^2 (D_t^+ u^{n-\frac{1}{2}})^2 (\varphi v)_x \, dx \, dt \\
&= \frac{\Delta t}{4} \int_0^{t_{\frac{1}{2}}} \int_{-R+1}^{R+1} \left(u^{\frac{1}{2}} + u^0 \right) \left(D_t^+ u^0\right) (\varphi v)_x \, dx \, dt \\
&\quad - \int_0^{t_{\frac{1}{2}}} \int_{-R+1}^{R+1} \left(u^0 t \left(D_t^+ u^0\right) (\varphi v)_x + \frac{1}{2} t^2 \left(D_t^+ u^0\right)^2 (\varphi v)_x \right) dx \, dt \\
&\quad + \frac{\Delta t}{4} \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} \left(u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}} \right) \left(D_t^+ u^{n-\frac{1}{2}}\right) (\varphi v)_x \, dx \, dt \\
&\quad - \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} u^{n-\frac{1}{2}} (t - t_{n-\frac{1}{2}}) \left(D_t^+ u^{n-\frac{1}{2}}\right) (\varphi v)_x \, dx \, dt \\
&\quad - \frac{1}{2} \sum_{n=0}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{-R+1}^{R-1} (t - t_{n-\frac{1}{2}})^2 (D_t^+ u^{n-\frac{1}{2}})^2 (\varphi v)_x \, dx \, dt.
\end{aligned}
$$

Using (5.9), (5.10), (4.3) and (5.2) we have the estimate

$$
\begin{aligned}
&\left|\int_0^{t_{\frac{1}{2}}}\int_{-R+1}^{R-1} u^0(D_t^+u^0)(\varphi v)_x\,dx\,dt\right| + \left|\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1} u^{n-\frac{1}{2}}(D_t^+u^{n-\frac{1}{2}})(\varphi v)_x\,dx\,dt\right| \\
&\le \int_0^{t_{\frac{1}{2}}} \left\|u^0\right\|_{L^\infty([-R+1,R-1])} \left\|D_t^+u^0\varphi\right\|_{H^{-2}([-R,R])} \left\|\varphi v\right\|_{H^3([-R+1,R-1])}\,dt \\
&\quad + \sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \left\|u^{n-\frac{1}{2}}\right\|_{L^\infty([-R+1,R-1])} \left\|D_t^+u^{n-\frac{1}{2}}\varphi\right\|_{H^{-2}([-R,R])} \left\|\varphi v\right\|_{H^3([-R+1,R-1])}\,dt \\
&\le C\left(\left\|u_0\right\|_{L^2(\mathbb{R})},R\right)\int_0^{t_{\frac{1}{2}}} \frac{C}{\sqrt{\Delta x}}\left\|u^0\right\|_{L^2(\mathbb{R})}\left\|\varphi v\right\|_{H^3([-R+1,R-1])}\,dt \\
&\quad + C\left(\left\|u_0\right\|_{L^2(\mathbb{R})},R\right)\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \frac{C}{\sqrt{\Delta x}}\left\|u^{n-\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|\varphi v\right\|_{H^3([-R+1,R-1])}\,dt \\
&\le \frac{1}{\sqrt{\Delta x}}C\left(\left\|u_0\right\|_{L^2(\mathbb{R})},R\right)\int_0^T \left\|\varphi v\right\|_{H^3([-R+1,R-1])}\,dt \\
&\le \frac{1}{\sqrt{\Delta x}}C\left(\left\|u_0\right\|_{L^2(\mathbb{R})},R\right)\left\|v\right\|_{L^\infty(0,T;H^3([-R+1,R-1]))}.
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
&\left|\int_0^{t_{\frac{1}{2}}}\int_{-R+1}^{R-1} u^{\frac{1}{2}}(D_t^+u^0)(\varphi v)_x\,dx\,dt\right| + \left|\sum_{n=1}^{N}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1} u^{n+\frac{1}{2}}(D_t^+u^{n-\frac{1}{2}})(\varphi v)_x\,dx\,dt\right| \\
&\le \frac{1}{\sqrt{\Delta x}}C\left(\left\|u_0\right\|_{L^2(\mathbb{R})},R\right)\left\|v\right\|_{L^\infty(0,T;H^3([-R+1,R-1]))}.
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
&\left|\int_0^{t_{\frac{1}{2}}}\int_{-R+1}^{R+1} u^0 t\left(D_t^+u^0\right)(\varphi v)_x\,dx\,dt\right| \\
&\quad + \left|\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1} u^{n-\frac{1}{2}}(t-t_{n-\frac{1}{2}})\left(D_t^+u^{n-\frac{1}{2}}\right)(\varphi v)_x\,dx\,dt\right| \\
&\le \frac{\Delta t}{2}\int_0^{t_{\frac{1}{2}}} \left\|u^0\right\|_{L^\infty([-R+1,R-1])}\int_{-R+1}^{R-1}\left|D_t^+u^0\right|\left|(\varphi v)_x\right|\,dx\,dt \\
&\quad + \Delta t\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \left\|u^{n-\frac{1}{2}}\right\|_{L^\infty([-R+1,R-1])}\int_{-R+1}^{R-1}\left|D_t^+u^{n-\frac{1}{2}}\right|\left|(\varphi v)_x\right|\,dx\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^{t_{\frac{1}{2}}}\int_{-R+1}^{R-1} t^2\left(D_t^+u^0\right)^2(\varphi v)_x\,dx\,dt\right| \\
&\quad+\left|\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\int_{-R+1}^{R-1}(t-t_{n-\frac{1}{2}})^2(D_t^+u^{n-\frac{1}{2}})^2(\varphi v)_x\,dx\,dt\right| \\
&\le \frac{\Delta t}{2}\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|u^{\frac{1}{2}}-u^0\right\|_{L^\infty([-R+1,R-1])}\int_{-R+1}^{R-1}\left|D_t^+u^0\right|\left|(\varphi v)_x\right|dx\,dt \\
&\quad+\Delta t\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|u^{n+\frac{1}{2}}-u^{n-\frac{1}{2}}\right\|_{L^\infty([-R+1,R-1])}\int_{-R+1}^{R-1}\left|D_t^+u^{n-\frac{1}{2}}\right|\left|(\varphi v)_x\right|dx\,dt.
\end{aligned}
$$

Thus these terms can be estimated like the preceding ones, and as $\Delta t/\sqrt{\Delta x}=\mathcal{O}(\Delta x)$ we obtain

$$\mathcal{C}_3^{\Delta x}+\mathcal{E}_3^{\Delta x}\xrightarrow{\Delta x\to 0}0.$$

Using the L^2-isometry of the Hilbert transform, (5.2) and (5.22) we obtain

$$
\begin{aligned}
\mathcal{C}_4^{\Delta x}+\mathcal{E}_4^{\Delta x} &\le \int_0^{t_{\frac{1}{2}}}\left\|\mathcal{H}u^{\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|\left(\varphi(v-v^{\Delta x})\right)_{xx}\right\|_{L^2([-R+1,R-1])}\,dt \\
&\quad+\frac{1}{2}\sum_{n=1}^N\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|\mathcal{H}u^{n+\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|\left(\varphi(v-v^{\Delta x})\right)_{xx}\right\|_{L^2([-R+1,R-1])}\,dt \\
&\quad+\frac{1}{2}\sum_{n=1}^N\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|\mathcal{H}u^{n-\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|\left(\varphi(v-v^{\Delta x})\right)_{xx}\right\|_{L^2([-R+1,R-1])}\,dt \\
&\le C(R)\int_0^{t_{\frac{1}{2}}}\left\|u^{\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|v-v^{\Delta x}\right\|_{H^2([-R+1,R-1])}\,dt \\
&\quad+C(R)\sum_{n=1}^N\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|u^{n+\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|v-v^{\Delta x}\right\|_{H^2([-R+1,R-1])}\,dt \\
&\quad+C(R)\sum_{n=1}^N\int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\left\|u^{n-\frac{1}{2}}\right\|_{L^2(\mathbb{R})}\left\|v-v^{\Delta x}\right\|_{H^2([-R+1,R-1])}\,dt \\
&\le C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\int_0^T\left\|v-v^{\Delta x}\right\|_{H^2([-R+1,R-1])}\,dt \\
&\le C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\Delta x\int_0^T\|v\|_{H^3([-R+1,R-1])}\,dt \\
&\le \Delta x\,C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)T\,\|v\|_{L^2(0,T;H^3([-R+1,R-1]))} \\
&= \Delta x\,C\left(\|u_0\|_{L^2(\mathbb{R})},R\right)\|v\|_{L^2(0,T;H^3([-R+1,R-1]))}\xrightarrow{\Delta x\to 0}0.
\end{aligned}
$$

Finally, from (5.8), the H^2-isometry of the Hilbert transform and (5.19) we have the estimate

$$\begin{aligned}
\mathcal{C}_5^{\Delta x} + \mathcal{E}_5^{\Delta x} &= \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \left(u^{\Delta x} - u^{\frac{1}{2}}\right) \mathcal{H}(\varphi v)_{xx} \, dx \, dt \\
&\quad + \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \left(u^{\Delta x} - \frac{u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}}{2}\right) \mathcal{H}(v\varphi)_{xx} \, dx \, dt \\
&\leq \int_0^{t_{\frac{1}{2}}} \int_{\mathbb{R}} \Delta t \left| -\frac{1}{2} + \frac{t}{\Delta t} \right| \left| D_t^+ u^0 \right| \left| \mathcal{H}(\varphi v)_{xx} \right| \, dx \, dt \\
&\quad + \sum_{n=1}^{N} \int_{t_{n-\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \int_{\mathbb{R}} \Delta t \left| -\frac{1}{2} + \frac{t - t_{n-\frac{1}{2}}}{\Delta t} \right| \left| D_t^+ u^{n-\frac{1}{2}} \right| \left| \mathcal{H}(\varphi v)_{xx} \right| \, dx \, dt \\
&\leq \frac{\Delta t}{2} \int_0^T \int_{\mathbb{R}} \left| \partial_t u^{\Delta x} \varphi \right| \left| \mathcal{H}(\varphi v)_{xx} \right| \, dx \, dt \\
&\leq \frac{\Delta t}{2} \int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|_{H^{-2}(\mathbb{R})} \left\| \mathcal{H}(\varphi v)_{xx} \right\|_{H^2(\mathbb{R})} \, dt \\
&= \frac{\Delta t}{2} \int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|_{H^{-2}(\mathbb{R})} \left\| (\varphi v)_{xx} \right\|_{H^2(\mathbb{R})} \, dt \\
&\leq \frac{\Delta t}{2} \int_0^T \left\| \partial_t u^{\Delta x} \varphi \right\|_{H^{-2}(\mathbb{R})} \left\| \varphi v \right\|_{H^4([-R+1,R-1])} \, dt \\
&\leq \frac{\Delta t}{2} \left\| \partial_t u^{\Delta x} \varphi \right\|_{L^2(0,T;H^{-2}(\mathbb{R}))} \left\| \varphi v \right\|_{L^2(0,T;H^4([-R+1,R-1]))} \\
&\leq \Delta t \, C\left(\|u_0\|_{L^2(\mathbb{R})}, R\right) \left\| \varphi v \right\|_{L^2(0,T;H^4([-R+1,R-1]))} \xrightarrow{\Delta t \to 0} 0.
\end{aligned}$$

We combine the preceding estimates to conclude that (5.23) holds. Also, observe that by passing $\Delta x \to 0$ we obtain

$$\int_0^T \int_{\mathbb{R}} \left(u_t \varphi v - \frac{u^2}{2} (\varphi v)_x - (\mathcal{H}u)\,(\varphi v)_{xx} \right) dx \, dt = 0, \tag{5.24}$$

for any test function $v \in C_c^\infty\left([-R+1, R-1] \times [0,T)\right)$. Now we choose $v = \phi/\varphi$ in (5.24) with $\phi \in C_c^\infty\left([-R+1, R-1] \times [0,T)\right)$ and integrate by parts to conclude that (5.13) holds, that is

$$\int_0^T \int_{\mathbb{R}} \left(\phi_t u + \phi_x \frac{u^2}{2} - (\mathcal{H}\phi_{xx})\, u \right) dx \, dt + \int_{\mathbb{R}} \phi(x,0) u_0(x) \, dx = 0.$$

This finishes the proof of Theorem 5.1. □

Chapter 6

Numerical experiments

Here we present some numerical experiments to illustrate the convergence proved in the preceding chapter for the fully discrete scheme (4.2). Our examples are analytical solutions to (1.1) and (1.2) in the form of localised waves found in [31]. Inspired by [12] we let $S_{\Delta x}$ consist of piecewise cubic splines defined in the following way. Lef f and g be the functions

$$f(y) = \begin{cases} 1 + y^2(2|y| - 3), & |y| \leq 1, \\ 0, & |y| > 1, \end{cases}$$

$$g(y) = \begin{cases} y(y+1)^2, & -1 \leq y \leq 0, \\ y(y-1)^2, & 0 < y \leq 1, \\ 0, & |y| > 1. \end{cases}$$

For $j \in \mathbb{Z}$ we define the basis functions

$$v_{2j} = f\left(\frac{x - x_j}{\Delta x}\right),$$

$$v_{2j+1} = g\left(\frac{x - x_j}{\Delta x}\right),$$

where $x_j = j\Delta x$. Then $\{v_j\}_{-M}^{M}$ spans a $4M + 2$ dimensional subspace of $H^2(\mathbb{R})$. In the following we define $N := 2M$, which is the number of elements used in the approximation. We have used periodic boundary conditions in our examples, and for the double soliton example the exact solution and numerical approximation are close to zero at the boundary for all times t under consideration, and so the periodic boundary condition will approximate the decay at infinity on the full line. For our weight function φ we have chosen this depending on the domain of each example, but in each case this is a linear and positive function on the domain considered and its derivative is 1.

In our experiments we have chosen to set $\Delta t = \mathcal{O}(\Delta x)$ contrary to the assertion $\Delta t = \mathcal{O}(\Delta x^2)$ from the theory, as smaller time steps did not lead to significant improvement in the accuracy of the approximations. In the iteration to obtain u^{n+1}, (4.6), we chose

the stopping condition $\|w^{\ell+1} - w^{\ell}\|_{L^2} \leq 0.002\Delta x\|u^n\|_{L^2}$, which typically required 4–7 iterations for the cruder discretisations and 2–3 iterations for the finer ones. The integrals involved in the Hilbert transforms were computed with seven and eight point Gauss–Legendre quadrature rules respectively for the inner Cauchy principal value integral and the outer integral appearing in the inner product.

For $t = n\Delta t$, we set $u_{\Delta x}(x,t) = u^n(x,t) = \sum_{j=-M}^{M} u_j^n v_j(x)$. We have measured the relative error of the numerical approximation compared to the exact solution u,

$$E := \frac{\|u_{\Delta x} - u\|_{L^2}}{\|u\|_{L^2}},$$

where the L^2-norms were computed with the trapezoidal rule in the points x_j. Additionally we computed the relative change in L^2-norm for the approximation,

$$I := \frac{\|u_{\Delta x}(t)\|_{L^2} - \|u_{\Delta x}(0)\|_{L^2}}{\|u_{\Delta x}(0)\|_{L^2}},$$

to see how well the quantity (2.26) was conserved, as numerical methods which preserve more of the conserved quantities for completely integrable PDEs have been observed to give more accurate approximations than the ones preserving fewer.

We have also included the rate of convergence for E and I, respectively defined for each intermediate step between N and $N+1$ as

$$\frac{\ln(E(N)) - \ln(E(N+1))}{\ln(N+1) - \ln(N)}, \quad \frac{\ln(|I(N)|) - \ln(|I(N+1)|)}{\ln(N+1) - \ln(N)},$$

where we now see E and I as functions of the element number N.

6.1 Double soliton

In our first example we consider the Cauchy problem for BO on the real line (1.1). A solution to this problem is the double soliton given by

$$u_{s2}(x,t) = \frac{4c_1c_2\left(c_1\lambda_1^2 + c_2\lambda_2^2 + (c_1+c_2)^2c_1^{-1}c_2^{-1}(c_1-c_2)^{-2}\right)}{\left(c_1c_2\lambda_1\lambda_2 - (c_1+c_2)^2(c_1-c_2)^{-2}\right)^2 + \left(c_1\lambda_1 + c_2\lambda_2\right)^2}, \tag{6.1}$$

where $\lambda_1 := x - c_1t - d_1$ and $\lambda_2 := x - c_2t - d_2$. When $c_2 > c_1$ and $d_1 > d_2$, this equation represents a tall soliton overtaking a smaller one while moving to the right.

We applied the fully discrete scheme with initial data $u_0(x) = u_{s2}(x,0)$ and parameters $c_1 = 0.3$, $c_2 = 0.6$, $d_1 = -30$, and $d_2 = -55$. The time step was set to $\Delta t = 0.5\|u_0\|_{L^\infty}^{-1}\Delta x$ and the numerical solutions were computed for $t = 90$ and $t = 180$, that is during and after the taller soliton overtaking the smaller one. To approximate the full line problem we set the domain to $[-100, 100]$ with the aforementioned periodic boundary condition, and based on this domain we chose the weight function $\varphi(x) = 120 + x$. The results are presented in Table 6.1 and a comparison between the approximation for $N = 256$ and the exact solution is shown in Figure 6.1.

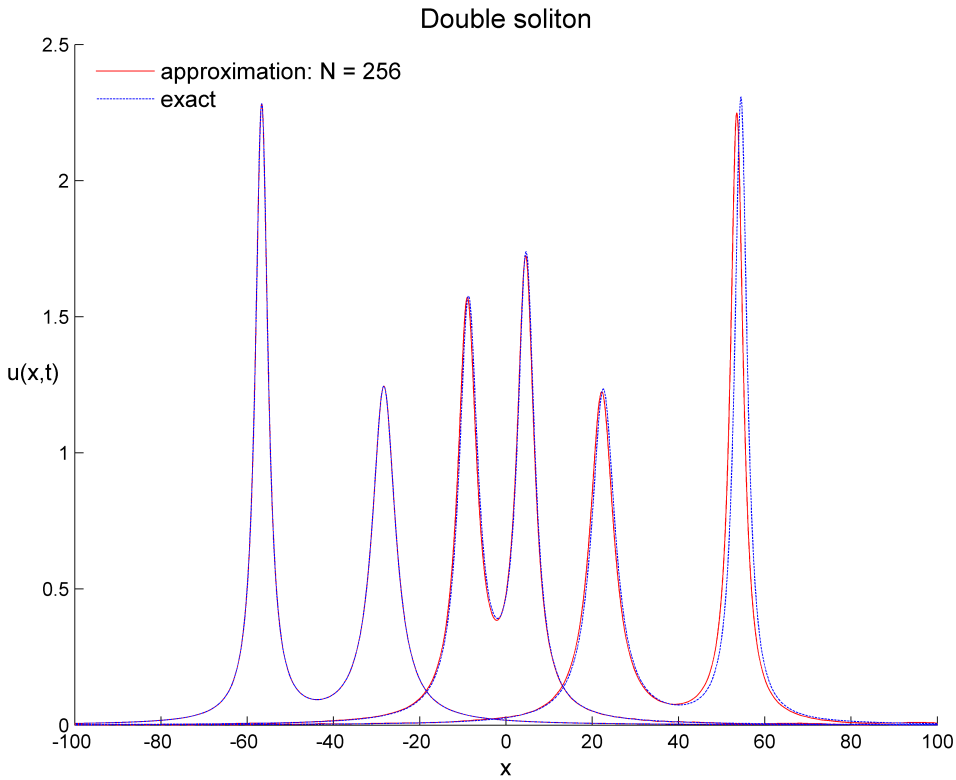

Figure 6.1: Numerical approximation for $N = 256$ and exact solution for $t = 0$, 90 and 180, respectively positioned from left to right in the plot, for full line problem with periodic boundary conditions.

The latter shows that the shape and position of the numerical approximation for $N = 256$ agrees quite well with the exact solution. Still, we observe that for $t = 180$ there is a visible error in the height of the tallest soliton, which has introduced a small phase error in the approximation. Since the soliton is very narrow this causes a relative error of nearly 30%, as seen from Table 6.1. This error is actually larger than the error for $N = 128$ where the approximate solution has pronounced oscillations, especially near the boundary, as emphasised in Figure 6.2. Still, the phase error for $N = 128$ is much smaller than for $N = 256$, and since the solitons are so narrow this has a much larger impact on the relative L^2-norm of the error. Thus an element number of $N = 128$ or less seems to be too small to get consistent reduction in the errors for increasing N. We see that for N greater than or equal to 256, the relative L^2-error at both $t = 90$ and $t = 180$ decreases, but not in a systematic fashion. The same is observed for the relative change in L^2-norm for the approximation. This lack of systematic reduction in the norms is probably caused by our approximation of the full line problem by a periodic one on a finite interval. It should also be pointed out that this is a complicated example, as one

t	N	E	rate	I	rate
90	128	0.01844	-1.45	$-7.0540 \cdot 10^{-4}$	-2.45
	256	0.05021	1.58	$-3.8527 \cdot 10^{-3}$	1.95
	512	0.01678	0.68	$-9.9617 \cdot 10^{-4}$	1.46
	1024	0.01044	1.16	$3.6244 \cdot 10^{-4}$	3.12
	2048	0.00467	0.08	$4.1568 \cdot 10^{-5}$	3.34
	4096	0.00442		$4.1170 \cdot 10^{-6}$	
180	128	0.11959	-1.32	$-6.4459 \cdot 10^{-4}$	-3.60
	256	0.29755	1.75	$-7.7958 \cdot 10^{-3}$	1.69
	512	0.08869	0.74	$-2.4158 \cdot 10^{-3}$	1.39
	1024	0.05295	2.35	$9.2166 \cdot 10^{-4}$	2.99
	2048	0.01040	0.89	$1.1612 \cdot 10^{-4}$	2.99
	4096	0.00561		$1.4617 \cdot 10^{-5}$	

Table 6.1: Relative L^2-error at $t = 90$ and $t = 180$ for full line problem with initial data u_{s2} and periodic boundary conditions.

has to approximate the nonlinear interaction between two passing solitons.

6.2 Periodic single wave

In our second example we consider the Cauchy problem for the $2L$-periodic BO equation[1] (1.2). In this case there exists a $2L$-periodic single wave solution that tends to a single soliton as the period goes to infinity, given by

$$u_{p1}(x,t) = \frac{2c\delta}{1 - \sqrt{1-\delta^2}\cos\left(c\delta(x-ct)\right)}, \quad \delta = \frac{\pi}{cL}.$$

We applied the scheme with initial data $u_0(x) = u_{p1}(x,0)$ with parameters $c = 0.25$ and $L = 15$. Based on this value for L, we chose the weight function $\varphi(x) = 20 + x$. The time step was set to $\Delta t = 0.5\Delta x$ and the approximate solution was computed for $t = 480$, which is four periods for the exact solution. A visualisation of the results for $N = 16, 32$ and 64 is shown in Figure 6.3.

The plot clearly shows that the approximations close in on the exact solution as the element number increases. This observation is confirmed by the relative L^2-norm of the errors presented in Table 6.2, which shows that we have a convergence rate of about two. Additionally, for the lowest numbers of N the change in norm for the approximation has a rate of about three, but for $N = 512$ and $N = 1024$ this rate increases significantly. From this it is clear that we have rapid convergence for this periodic problem, and the errors are consistently reduced. This better behaviour compared to the previous example

[1] We only proved convergence for the full line BO equation, but we include this example of the method applied to the periodic problem for the sake of completeness.

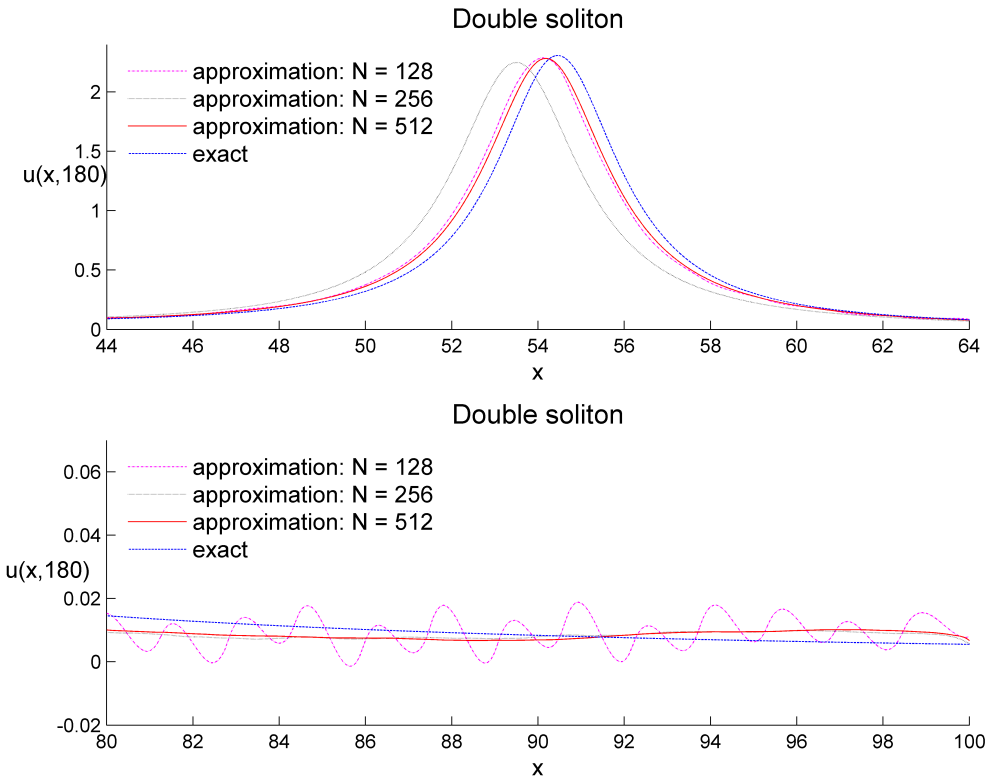

Figure 6.2: Exact and numerical solutions at $t = 180$ for $N = 128$, 256 and 512 for full line problem with periodic boundary conditions. In the upper plot only the rightmost soliton is shown for a better visualisation of the phase error. In the lower plot only the rightmost end of the domain is shown to better visualise the oscillations occurring for $N = 128$.

N	E	rate	I	rate
16	$1.5141 \cdot 10^{-1}$		$6.8073 \cdot 10^{-3}$	
		2.43		2.96
32	$2.8063 \cdot 10^{-2}$		$8.7663 \cdot 10^{-4}$	
		2.28		3.00
64	$5.7793 \cdot 10^{-3}$		$1.0966 \cdot 10^{-4}$	
		2.16		3.02
128	$1.2918 \cdot 10^{-3}$		$1.3514 \cdot 10^{-5}$	
		2.08		3.09
256	$3.0517 \cdot 10^{-4}$		$1.5914 \cdot 10^{-6}$	
		2.03		3.36
512	$7.4825 \cdot 10^{-5}$		$1.5486 \cdot 10^{-7}$	
		1.98		6.92
1024	$1.8907 \cdot 10^{-5}$		$-1.2785 \cdot 10^{-9}$	

Table 6.2: Relative L^2-error at $t = 480$ for periodic problem with initial data u_{p1}.

could be expected, as the problem is periodic and so there is no need to approximate an

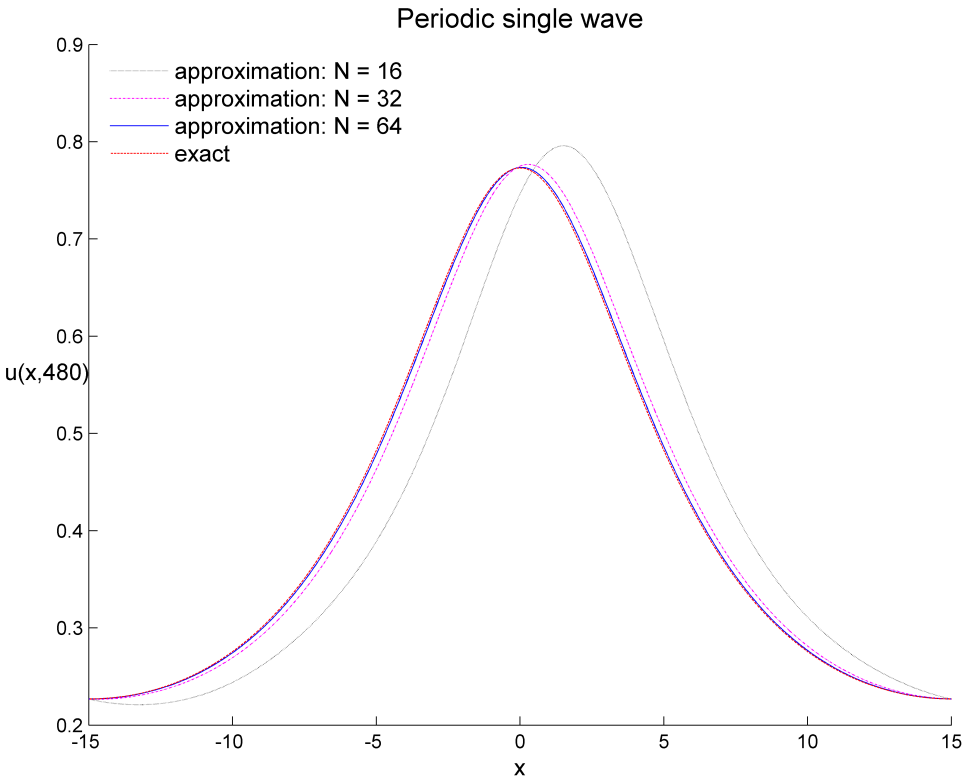

Figure 6.3: Exact and numerical solutions of the $2L$-periodic problem at $t = 480$ for element numbers $N = 16$, 32 and 64, with $L = 15$ and initial data u_{p1}.

unbounded domain, in addition to that we are simply approximating the translation of a single wave. The final example suggests that the scheme should be convergent also for the periodic BO equation.

Chapter 7

Concluding remarks

We have studied a Crank–Nicolson type Galerkin scheme for the Benjamin–Ono equation. For the full line initial value problem (1.1) we proved convergence to a weak solution for initial data in L^2, where the main ingredient of the proof was a smoothing effect which bounded the approximate solutions locally in $H^{\frac{1}{2}}$-norm. The first step in the proof was to prove the existence of a solution for the next time step in the iterative scheme used to handle the nonlinearity for this implicit method, and this was done by assuming a CFL condition relating the spatial and temporal discretisation, combined with inverse inequalities applicable to our finite element method. By using the smoothing effect and additionally bounding the temporal derivative of the approximate solution in H^{-2}-norm we were able to invoke the Aubin–Simon compactness lemma to deduce the existence of a sequence of approximations which converges to a weak solution of the equation.

Finally we presented some numerical experiments to demonstrate the method. An example involving two interacting solitons on the real line illustrated the method in the case for which we have proved convergence, and in agreement with theory we observed that the L^2-error decreased for sufficiently small and decreasing spatial step lengths Δx. We also presented an example of the method applied to a single periodic wave in the case of the periodic BO equation for which we have not proved convergence, but the example was included for the sake of completeness. In this case the errors were systematically reduced and relatively few elements were needed for satisfying results, even after four periods for the exact solution. The latter gives a good indication of that the scheme should be convergent also in the case of the periodic BO equation, and it is an interesting question whether this is true.

A natural objective for further work would then be to try and prove that the method is convergent for (1.2). As many of our intermediate estimates presented here also hold for bounded domains and periodic functions, the principal distinction is related to the commutator estimates, Sobolev inequalities and the cut-off function φ. If one in proving these estimates is to proceed in the same fashion as here one has to consider the Fourier transform and Sobolev inequalities on the torus, and likewise one would have to define an appropriate cut-off function on the torus instead of the real line.

Another possible way of expanding upon this work is to compare the accuracy and

performance of this method with other numerical methods which have been applied to the Benjamin–Ono equation, such as the finite difference scheme presented in [9], operator splitting schemes [10] or spectral methods.

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