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Triangulated Categories and Matrix Factorizations

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Abstract

In this thesis we study triangulated categories and look at one specific example, the homotopy category of matrix factorizations. First we define categories and functors. Then we introduce additive and triangulated categories and see that the octahedral axiom can be replaced by Neeman's mapping cone axiom. After this we look at matrix factorizations and the homotopy category of matrix factorizations, $\mathbf{HMF}(S, x)$, which leads us to one of our main results, i.e. that $\mathbf{HMF}(S, x)$ is triangulated. We prove this with both the octahedral axiom and Neeman's mapping cone theorem. Lastly we look at the homotopy category of totally acyclic complexes over a local, regular ring and see that this is equivalent to $\mathbf{HMF}(S, x)$.

Sammendrag

I denne oppgaven ser vi på triangulerte kategorier og trekker frem ett spesifikt eksempel: homotopikategorien av matrisefaktoriseringer. Vi begynner med å definere kategorier og funktorer for så å introdusere additive og triangulerte kategorier. Her viser vi at oktaederaksiomet kan erstattes med Neemans "mapping cone"-aksiom. Deretter ser vi på matrisefaktoriseringer og homotopikategorien av matrisefaktoriseringer, $\mathbf{HMF}(S, x)$, som vi viser at er triangulert. Til dette bruker vi både oktaederaksiomet og Neemans aksiom. Til slutt ser vi på homotopikategorien til totalt asykliske komplekser og viser at over en lokal, regulær ring, er denne ekvivalent med $\mathbf{HMF}(S, x)$.

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Contents

Chapter 1

Introduction

The concept of categories was introduced by Samuel Eilenberg and Saunders Mac Lane in the 1945 article "General Theory of Natural Equivalences", [18], after the authors had already touched on the subject in 1942. Categories were invented to express certain constructions in algebraic topology, but has since developed rapidly and is now a big part of for example homological algebra.

The focus in this thesis is the triangulated category $\mathbf{HMF}(S, x)$ which is the homotopy category of matrix factorizations. The notion of triangulated categories was introduced in algebraic geometry in the Ph.D. thesis of Jean-Luis Verdier, and in algebraic topology by Dieter Puppe. Verdier was looking at derived categories and observed they had some special "triangles". The axioms of the basic properties of these triangles then became the axioms of the triangulated categories. Since they were introduced, these categories have played an important role in many branches of mathematics, e.g representation theory, algebraic geometry, algebraic topology, commutative algebra and more.

Matrix factorizations were introduced some years later by Eisenbud in [10]. Here he studied free resolutions over the corresponding factor rings and showed that if we take a finitely generated maximal Cohen-Macaulay module over the factor ring $Q/(x)$, where Q is a regular local ring and x a nonzero element, then its minimal free resolution is obtained from a matrix factorization of x over Q.

In this thesis we start by introducing categories in chapter 2, and give some examples. We also look at functors and natural transformations and show that a functor is an equivalence if and only if it is full, faithful and dense. Then, in chapter 3, we introduce zero objects, products and coproducts before we look at additive categories and show that $C(\mathscr{A})$, the category of complexes over $\mathscr{A}$, is an additive category. After this we define triangulated categories and show some of their properties, and then look at Neeman's mapping cone axiom and show that it can replace the octahedral axiom. Then, in chapter 5, we introduce matrix factorizations and prove that $\mathbf{HMF}(S, x)$ is triangulated. Lastly we show that $\mathbf{HMF}(S, x)$ is equivalent with $\mathbf{K}_{\mathtt{tac}}(R)$.

Chapter 2

General categories

2.1 Categories

A category is a collection of related objects and maps between them.

Definition. A *category* $\mathscr{C}$ consists of

- a collection $\mathbf{Ob}(\mathscr{C})$ of *objects*,
- for each A and $B \in \mathbf{Ob}(\mathscr{C})$, a set $\mathbf{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* from A to B,
- for each $A, B, C \in \mathbf{Ob}(\mathscr{C})$, a function

$$\begin{aligned} \mathbf{Hom}_{\mathscr{C}}(B, C) \times \mathbf{Hom}_{\mathscr{C}}(A, B) &\to \mathbf{Hom}_{\mathscr{C}}(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

  called *composition*,
- for each $A \in \mathbf{Ob}(\mathscr{C})$, an element $1_A \in \mathbf{Hom}_{\mathscr{C}}(A, A)$ called the *identity* on A

satisfying the following

1. for each $f \in \mathbf{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathbf{Hom}_{\mathscr{C}}(B, C)$ and $h \in \mathbf{Hom}_{\mathscr{C}}(C, D)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

   i.e. *associativity* holds, and
2. for each $f \in \mathbf{Hom}_{\mathscr{C}}(A, B)$ we have

$$f \circ 1_A = f = 1_B \circ f$$

   i.e. the *identity laws* hold.

Remark. It is common to write just $A \in \mathscr{C}$ instead of $A \in \mathbf{Ob}(\mathscr{C})$ and $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in \mathbf{Hom}_{\mathscr{C}}(A, B)$. Also, it is common to write gf instead of $g \circ f$.

So, a category consists of objects and maps between them, and these maps follow the law of associativity and behave as one would expect with regards to identity elements. Now let us look at some examples of categories.

Example. (a) **Set**, the category whose objects are sets and the morphisms are just maps between them. Composition is the regular composition of maps and the identity on a set is just the identity map.

(b) **Gr**, the category whose objects are groups and the morphisms are group homomorphism with the standard composition and identity map.

(c) **Ab**, which is the same as above, but the objects are abelian groups.

(d) **Top**, where the objects are topological spaces and the morphisms are continuous maps.

This means there are categories for sets, groups and topological spaces, but these are just some examples. There are also categories of vector spaces, rings, posets and so on. And when we have one category, there is always another that is closely related.

Definition. For every category $\mathscr{C}$ we define the *opposite* category $\mathscr{C}^{\text{op}}$ by

- $\mathbf{Ob}(\mathscr{C}^{\text{op}}) = \mathbf{Ob}(\mathscr{C})$
- $\mathbf{Hom}_{\mathscr{C}^{\text{op}}}(A, B) = \mathbf{Hom}_{\mathscr{C}}(B, A)$

with

- $f \circ_{\mathscr{C}^{\text{op}}} g = g \circ_{\mathscr{C}} f$.

Now that we know what a category is and know some examples, it is natural to look at what happens between them. But before we do that, we will look at something that happens inside them, i.e. we want to know what an *isomorphism* in a category is.

Definition. Let $\mathscr{C}$ be a category. We say that a map $f : A \to B$ in $\mathscr{C}$ is an *isomorphism* if $\exists g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$. We say that A and B are *isomorphic* and write $A \cong B$.

If we relate this to the example above we see that in **Set** the isomorphisms are the bijections, in **Gr** they are the group isomorphisms and in **Top** they are the homeomorphisms. This can seem trivial at first glance, but it is not. In each case a short proof is needed to see that these are indeed the isomorphisms of the categories.

2.2 Functors

Let us look at the maps between categories.

Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. We define a *(covariant) functor* $F : \mathscr{C} \to \mathscr{D}$ by

- A function $\mathbf{Ob}(\mathscr{C}) \to \mathbf{Ob}(\mathscr{D})$ written as $C \mapsto F(C)$ or $C \mapsto FC$
- For each $C_1, C_2 \in \mathscr{C}$ a function $\mathbf{Hom}_{\mathscr{C}}(C_1, C_2) \to \mathbf{Hom}_{\mathscr{D}}(F(C_1), F(C_2))$ written $f \mapsto F(f)$ or $f \mapsto Ff$

such that

- $F(g \circ f) = F(g) \circ F(f)$ when $f : C_1 \to C_2$, $g : C_2 \to C_3$
- $F(1_C) = 1_{F(C)} \quad \forall C \in \mathscr{C}$.

So functors are maps between categories that preserve composition of maps and identities. Let us look at some examples.

Example. (a) *Forgetful functors*, e.g $F : \mathbf{Gr} \to \mathbf{Set}$. This functor "forgets" the structure of the group. That means that if A is a group, $F(A)$ is the underlying set, and if f is a group homomorphism, $F(f)$ is just the function itself.

(b) *Inclusion functors*, e.g $G : \mathbf{Ab} \to \mathbf{Gr}$. This functor "includes" the abelian groups into the category of all groups. So, if A is an abelian group and f a group homomorphism, $G(A) = A$ and $G(f) = f$. This functor is also forgetful, as it "forgets" that abelian groups are abelian, they are just groups.

Both of these were examples of what we call *covariant* functors, but there is also another kind.

Definition. A *contravariant* functor between two categories $\mathscr{C}$ and $\mathscr{D}$, is a functor $F : \mathscr{C}^{\text{op}} \to \mathscr{D}$.

Let us look at an example.

Example. Let k be a field and $\mathbf{Vect}_k$ the category of vector spaces over k. We then have a contravariant functor

$$(\quad)^* = \mathbf{Hom}(-, k) : \mathbf{Vect}_k^{\text{op}} \to \mathbf{Vect}_k$$

sending each vector space V to its dual V^*.

Like other maps, functors have different qualities. We state the following definition for covariant functors.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. F is

- *faithful* if, for each $A, B \in \mathscr{C}$, $F : \mathbf{Hom}_{\mathscr{C}}(A, B) \to \mathbf{Hom}_{\mathscr{D}}(F(A), F(B))$ is *injective*,
- *full* if, for each $A, B \in \mathscr{C}$, $F : \mathbf{Hom}_{\mathscr{C}}(A, B) \to \mathbf{Hom}_{\mathscr{D}}(F(A), F(B))$ is *surjective*,
- *dense* if, for each $D \in \mathscr{D} \quad \exists C \in \mathscr{C}$ such that $F(C) \cong D$.

Remark. Some books, for example [16], call dense functors *essentially surjective on objects*.

Now we have looked at categories and functors, so let us look at the maps between functors, what is called *natural transformations*.

2.3 Natural transformations

Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors between them.

A *natural transformation* $\eta : F \to G$ is a family $\eta_C : F(C) \to G(C) \quad \forall C \in \mathbf{Ob}(\mathscr{C})$ of morphisms in $\mathscr{D}$ such that for every map $f : C \to C'$ in $\mathscr{C}$, the following square commutes:

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\eta_C"] & F(C') \arrow[d, "\eta_{C'}"] \\
G(C) \arrow[r, "G(f)"] & G(C')
\end{tikzcd}$$

The η_C's are called the *components* of η.

A natural transformation η is called a *natural isomorphism* if all the η_C are isomorphisms in $\mathscr{D}$. And with that we get the following.

Definition. Given functors $\mathscr{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathscr{D}$, we say that

$$F(A) \cong G(A) \textit{ naturally} \text{ in A}$$

if F and G are naturally isomorphic. We often write $F \underset{\text{nat}}{\cong} G$.

This brings us to the important notion of *equivalence*.

Definition. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that

$$F \circ G \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{D}} \quad \text{and} \quad G \circ F \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{C}}.$$

This is not always so easy to check, but we have a theorem that makes it easier.

Theorem 2.3.1. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is an equivalence if and only if it is full, faithful and dense.*

Proof. $(\Rightarrow)$ Assume that F is an equivalence and let G be as in the definition. If we let $\eta : G \circ F \to \mathrm{id}_{\mathscr{C}}$ be a natural isomorphism we have, for any $f \in \mathbf{Hom}_{\mathscr{C}}(C_1, C_2)$, the commutative square

$$\begin{tikzcd}
GFC_1 \arrow[r, "\eta_{C_1}"] \arrow[d, "GFf"] & C_1 \arrow[d, "f"] \\
GFC_2 \arrow[r, "\eta_{c_2}"] & C_2.
\end{tikzcd}$$

Since this commutes we get that $f = \eta_{C_2} \circ GFf \circ \eta_{C_1}^{-1}$ is uniquely determined by Ff. Hence F is faithful.

Now, let $\zeta : F \circ G \to \mathrm{id}_{\mathscr{D}}$ be a natural isomorphism. This means that for any object $D \in \mathscr{D}$ we have $FGD \cong 1_{\mathscr{D}}D = D$ i.e. $C = GD$ is such that $FC \cong D$ and hence F is dense.

Finally we need to show that F is full so we let η and ζ be as above and let $f : FC_1 \to FC_2$ in $\mathscr{D}$. We then construct a commutative diagram

$$\begin{tikzcd}
FC_1 \arrow[d, "f"] & FGFC_1 \arrow[l, "F\eta_{C_1}"'] \arrow[r, "\zeta_{FC_1}"] \arrow[d, "g"] & FC_1 \arrow[d, "h"] \\
FC_2 & FGFC_2 \arrow[l, "F\eta_{C_2}"] \arrow[r, "\zeta_{FC_2}"] & FC_2
\end{tikzcd}$$

where g and h are the unique maps that make the squares commute. Since ζ is natural we get that $g = FGh$ and since the left hand square is commutative we get

$$\begin{aligned} f &= F\eta_{C_2} \circ FGh \circ (F\eta_{C_1})^{-1} \\ &= F(\eta_{C_2} \circ Gh \circ \eta_{C_1}^{-1}). \end{aligned}$$

This shows that f is in the image of F and hence F is full.

$(\Leftarrow)$ Assume that F is full, faithful and dense. Since F is dense, we have that for every object $D \in \mathscr{D}$ there is an object $C \in \mathscr{C}$ such that $D \cong FC$. We fix one such C and denote it by GD. We also choose and fix an isomorphism $\zeta_D : FGD \to D$. For every $f \in \mathbf{Hom}_{\mathscr{D}}(D_1, D_2)$ we use the bijection

$$\mathbf{Hom}_{\mathscr{C}}(GD_1, GD_2) \to \mathbf{Hom}_{\mathscr{D}}(FGD_1, FGD_2)$$

that is induced by F since it is full and faithful, and define Gf to be the preimage of $\zeta_{D_2}^{-1} \circ f \circ \zeta_{D_1}$. This makes G a functor from $\mathscr{D}$ to $\mathscr{C}$:

Firstly we have

$$G\mathrm{id}_D = F^{-1}(\zeta_D^{-1} \circ \mathrm{id}_D \circ \zeta_D) = F^{-1}(\mathrm{id}_{FGD}) = \mathrm{id}_{GD}$$

Secondly if we have $f : D_1 \to D_2$ and $g : D_2 \to D_3$ we get

$$\begin{aligned} G(g \circ f) &= F^{-1}(\zeta_{D_3}^{-1} \circ g \circ f \circ \zeta_{D_1}) \\ &= F^{-1}(\zeta_{D_3}^{-1} \circ g \circ \zeta_{D_2} \circ \zeta_{D_2}^{-1} \circ f \circ \zeta_{D_1}) \\ &= F^{-1}(\zeta_{D_3}^{-1} \circ g \circ \zeta_{D_2}) \circ F^{-1}(\zeta_{D_2}^{-1} \circ f \circ \zeta_{D_1}) \\ &= Gg \circ Gf. \end{aligned}$$

Hence G is a functor.

Now we claim that ζ is a natural isomorphism $F \circ G \to \mathrm{id}_{\mathscr{D}}$. Let $f \in \mathbf{Hom}_{\mathscr{D}}(D_1, D_2)$. Then we have

$$\zeta_{D_2} \circ FGf = \zeta_{D_2} \circ \zeta_{D_2}^{-1} \circ f \circ \zeta_{D_1} = f \circ \zeta_{D_1}$$

and hence ζ is a natural isomorphism.

Finally we construct a natural isomorphisms $\eta : G \circ F \to \mathrm{id}_{\mathscr{C}}$. First we observe that for any $C \in \mathscr{C}$, ζ induces mutually inverse natural isomorphisms:

$$\zeta_{FC} : F \circ G \circ FC \to FC \quad \text{and} \quad \zeta_{FC}^{-1} : F \to F \circ G \circ FC.$$

Since F is full and faithful, we can find unique morphisms $\eta_C : GFC \to C$ and $\eta_C^{'} : C \to GFC$ such that

$$\zeta_{FC} = F\eta_C \quad \text{and} \quad \zeta_{FC}^{-1} = F\eta_C^{'}$$

it follows that η is a natural transformation, with inverse $\eta^{'}$ and hence we get

$$F \circ G \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{D}} \quad \text{and} \quad G \circ F \underset{\text{nat}}{\cong} \mathrm{id}_{\mathscr{C}}.$$

Hence F is an equivalence. $\square$

Chapter 3

Additive Categories

We now begin to look at different types of categories. First we will look at *additive categories*, but before we can define them we need some more theory.

3.1 Zero objects

In a category the objects have different qualities. For instance we know that objects can have different sizes, e.g. in **Set** we have sets with only one element and we have sets like $\mathbb{Z}$ which has an infinite number of elements. Objects can also have different structure, like in **Gr** where some groups are abelian while others are not. Since categories consist of both objects and maps between them, we will now look at qualities that take into consideration both an object and the maps to or from it. This is where *initial* and *terminal* objects come in.

Definition. Let $\mathscr{C}$ be a category. An object I in $\mathscr{C}$ is called *initial* if for every $C \in \mathbf{Ob}\mathscr{C}$ there is exactly one map $I \to C$.

An object T in $\mathscr{C}$ is called *terminal* if for every $C \in \mathbf{Ob}\mathscr{C}$ there is exactly one map $C \to T$.

Remark. There is a duality between initial and terminal object. A terminal object in $\mathscr{C}$ is an initial object in $\mathscr{C}^{\mathrm{op}}$.

So an object is initial if there is exactly one map going out of it and terminal if there is exactly one map going in to it, for all objects in the category. Let us look at some examples.

Example. (a) In the category **Set** the empty set is initial and every set $\{x\}$ with only one element is terminal. That means **Set** has only one initial object but many terminal ones, but all the terminal objects are isomorphic.

(b) In the category **Gr** the group of one element is both initial and terminal and is unique up to isomorphism.

(c) In the category of categories, **Cat**, the category **0** with no objects or arrows is initial, and the category **1** with only one object and its identity map is terminal.

Notice that in (b) in the above example the trivial group is both initial and terminal. These objects have their own name:

Definition. An object in a category $\mathscr{C}$ is a *zero object* if it is both initial and terminal.

Let us look at some examples.

Example. (a) In **Set** there are no zero objects.

(b) As seen above, in **Gr** the trivial group is a zero object. Similarly for the category of vector spaces and linear transformations.

(c) For a ring R, the trivial R-module is the zero-object in **Mod R**, the category of R-modules.

Proposition 3.1.1. *Every zero object is unique up to isomorphism.*

Proof. Let A and B be zero objects, i.e. they are both initial and terminal. The diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[dr, "\mathrm{id}_A"'] & B \arrow[d, "v"] \arrow[dr, "\mathrm{id}_B"] & \\
 & A \arrow[r, "u"'] & B
\end{tikzcd}$$

commutes since each morphism originates from an initial object and hence is unique. That means v is the inverse of u and so u is an isomorphism. The same diagram also commutes because each morphism ends in a terminal object and hence is unique. Which again means v is the inverse of u and hence u is an isomorphism. ☐

3.2 Products and coproducts

Something we are used to from set theory are products and sums. These are special cases of *products* and *coproducts.*

Definition. Let A and B be two objects in a category $\mathscr{C}$. A *product* of A and B is an object P along with morphisms $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ such that:

Given any diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$ there exist a unique morphism $u : X \to P$ such that the following diagram commutes

$$\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

i.e $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$.

Definition. Let A and B be two objects in a category $\mathscr{C}$. A *coproduct* of A and B is an object C together with morphisms $A \xrightarrow{c_1} C \xleftarrow{c_2} B$ such that:

Given $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : C \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "z_1"] \arrow[r, "c_1"'] & C \arrow[u, dashed, "u"'] & B \arrow[ul, "z_2"'] \arrow[l, "c_2"]
\end{tikzcd}$$

that is $z_1 = u \circ c_1$ and $z_2 = u \circ c_2$.

So a coproduct is the dual of a product, i.e. it is a product in the opposite category. We often write $A \Pi B$ for products and $A \amalg B$ for coproducts. Now let us look at an example.

Example. If we look at the category of sets, we have the cartesian product $A \Pi B$ which is the set of ordered pairs $A \Pi B = \{(a, b) \mid a \in A,\ b \in B\}$. Here we have two coordinate projections

$$A \xleftarrow{p_1} A \Pi B \xrightarrow{p_2} B$$

with

$$p_1(a,b) = a, \quad p_2(a,b) = b$$

which means that any element $c \in A \Pi B$ can be written as $c = (p_1(c), p_2(c))$. Thus we get the following diagram

$$\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[d, dashed, "{(a,b)}"] \arrow[dr, "b"] & \\
A & A \Pi B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B.
\end{tikzcd}$$

We also have the coproduct $A \amalg B$ which is the disjoint union of A and B. It can for example be constructed as $A \amalg B = \{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}$, with the maps

$$c_1(a) = (a,1), \; c_2(b) = (b,2).$$

Given any f and g as in the following commutative diagram

$$\begin{tikzcd}
 & Z & \\
A \arrow[ur, "f"] \arrow[r, "c_1"'] & A \amalg B \arrow[u, dashed, "{[f,g]}"'] & B \arrow[ul, "g"'] \arrow[l, "c_2"]
\end{tikzcd}$$

we can define

$$[f,g]\,(x,y) = \begin{cases} f(x) & \text{if} \quad y = 1 \\ g(x) & \text{if} \quad y = 2. \end{cases}$$

Then, for an h with $h \circ c_1 = f$ and $h \circ c_2 = g$ we get that for any $(x,y) \in A \amalg B$ we must have

$$h(x,y) = [f,g]\,(x,y).$$

In this example the coproduct is clearly different from the product, which is most often the case. However, there are some categories where $A \amalg B$ is isomorphic to $A \Pi B$, for example the category of abelian groups. When this holds, the common value of $A \amalg B$ and $A \Pi B$ is called a biproduct and is denoted $A \oplus B$. Another useful property of coproducts (and products) is the following.

Proposition 3.2.1. *Coproducts are unique up to isomorphism.*

Proof. Suppose $A \xrightarrow{c_1} C \xleftarrow{c_2} B$ and $A \xrightarrow{d_1} D \xleftarrow{d_2} B$ are two coproducts of A and B. Since D is a coproduct there is a unique $v : D \to C$ such that $v \circ d_1 = c_1$ and $v \circ d_2 = c_2$. And since C is a coproduct there exists a unique $u : C \to D$ such that $u \circ c_1 = d_2$ and $u \circ c_2 = d_2$. This gives us the following commutative diagrams:

$$\begin{tikzcd}
 & C & & & D & \\
A \arrow[ur, "c_1"] \arrow[r, "d_1"'] \arrow[dr, "c_1"'] & D \arrow[u, dashed, "v"'] & B \arrow[ul, "c_2"'] \arrow[l, "d_2"] \arrow[dl, "c_2"] & A \arrow[ur, "d_1"] \arrow[r, "c_1"'] \arrow[dr, "d_1"'] & C \arrow[u, dashed, "u"'] & B. \arrow[ul, "d_2"'] \arrow[l, "c_2"] \arrow[dl, "d_2"] \\
 & C \arrow[u, dashed, "u"'] & & & D \arrow[u, dashed, "v"'] &
\end{tikzcd}$$

Looking at the first diagram, we get that $v \circ u \circ c_1 = c_1$ and $v \circ u \circ c_2 = c_2$. We also have that $1_C \circ c_1 = c_1$ and $1_C \circ c_2 = c_2$ so uniqueness gives us that $v \circ u = 1_C$. Now looking at the second diagram we get that $u \circ v \circ d_1 = d_1$ and $u \circ v \circ d_2 = d_2$. From uniqueness and the fact that $1_D \circ d_1 = d_1$ and $1_D \circ d_2 = d_2$ we see that $u \circ v = 1_D$. This means that u and v are isomorphisms, hence C and D are isomorphic. □

3.3 Additive categories

Now it is time to use all this and define what an additive category is.

Definition. A category $\mathscr{A}$ is an *additive category* if the following conditions hold:

1) For every pair $X, Y \in \mathbf{Ob}(\mathscr{A})$, $\mathbf{Hom}_{\mathscr{A}}(X, Y)$ is an abelian group and the composition of morphisms is bilinear.

2) $\mathscr{A}$ contains a zero object.

3) For any pair $X, Y \in \mathbf{Ob}(\mathscr{A})$, there exists a coproduct $X \amalg Y$ in $\mathscr{A}$.

A functor $F : \mathscr{A} \to \mathscr{B}$ between two such categories is *additive* if for all $X, Y \in \mathbf{Ob}(\mathscr{A})$ and all $f, g \in \mathbf{Hom}_{\mathscr{A}}(X, Y)$ we have

$$F(f + g) = Ff + Fg,$$

i.e it induces a homomorphism of groups $\mathbf{Hom}_{\mathscr{A}}(X, Y) \to \mathbf{Hom}_{\mathscr{A}}(FX, FY)$.

Proposition 3.3.1. *Let $\mathscr{A}$ be an additive category and let $A, B \in \mathbf{Ob}(\mathscr{A})$.*

(i) Assume $A \Pi B$ exists in $\mathscr{A}$ and let $p : A \Pi B \to A$ and $q : A \Pi B \to B$ be the projections. Now let $i : A \to A \Pi B$ and $j : B \to A \Pi B$ be two morphisms such that

$$p \circ i = 1_A, \quad q \circ j = 1_B, \quad p \circ j = q \circ i = 0. \tag{3.1}$$

Then we have

$$i \circ p + j \circ q = 1_{A \Pi B}. \tag{3.2}$$

(ii) Let $P \in \mathbf{Ob}(\mathscr{A})$ and let $i : A \to P$, $j : B \to P$, $p : P \to A$ and $q : P \to B$ be morphisms satisfying (3.1) *and* (3.2). *Then P is a product of A and B by (p, q) and a coproduct by (i, j). Hence we have*

$$A \Pi B \cong A \amalg B.$$

Proof. (i) We have $p \circ (i \circ p + j \circ q) = (p \circ i) \circ p + (p \circ j) \circ q = p = p \circ 1_{A \Pi B}$ and $q \circ (i \circ p + j \circ q) = (q \circ i) \circ p + (q \circ j) \circ q = q = q \circ 1_{A \Pi B}$. Hence $i \circ p + j \circ q = 1_{A \Pi B}$.

(ii) If we let u in

$$\begin{tikzcd}
& X \arrow[ld, "x_1"'] \arrow[d, dashed, "u"] \arrow[rd, "x_2"] & \\
A & P \arrow[l, "p"] \arrow[r, "q"'] & B
\end{tikzcd}$$

be $i \circ x_1 + j \circ x_2$ we see that the diagram commutes. But to be sure we have a product we need this u to be unique. So assume we have a $\theta : X \to P$ such that $u = \theta$ makes the diagram above commute. Then we have

$$\begin{aligned}
\theta &= 1_P \circ \theta = (i \circ p + j \circ q) \circ \theta \\
&= i \circ (p \circ \theta) + j \circ (q \circ \theta) = i \circ x_1 + j \circ x_2.
\end{aligned}$$

Hence $u = i \circ x_1 + j \circ x_2$ is unique and (P, p, q) is a product.

Now if we let the v in

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[r, "c_1"'] & P \arrow[u, dashed, "v"'] & B \arrow[ul, "z_2"'] \arrow[l, "c_2"] \end{tikzcd}$$

be $c_1 \circ p + c_2 \circ q$ we get a commutative diagram. To see if this v is unique we let $\eta : P \to Z$ such that $v = \eta$ makes the above diagram commute. Then we have

$$\begin{aligned} \eta &= \eta \circ 1_P = \eta \circ (i \circ p + j \circ q) \\ &= (\eta \circ i) \circ p + (\eta \circ j) \circ q = c_1 \circ p + c_2 \circ q. \end{aligned}$$

Hence $v = c_1 \circ p + c_2 \circ q$ is unique and (P, i, j) is a coproduct. □

This means that when we want to check if a category is additive, it is not necessary to find a coproduct between objects. We can find a product or biproduct instead, because they are all the same. Let us now look at some examples of additive categories.

Example. (1) **Ab** is an additive category. The zero object is the trivial group, addition of morphisms is defined pointwise and the biproduct is given by direct sums.

(2) Let R be an associative ring with unity. Then **Mod R** is an additive category. So is **mod R**, the category of finitely generated R-modules.

Before we look at the next example we need some definitions.

Definition. Let $\mathscr{A}$ be an additive category. A *complex over* $\mathscr{A}$ is a family $X = (X_n, d_n^X)_{n\in\mathbb{Z}}$ where $X_n \in \mathbf{Ob}(\mathscr{A})$ and $d_n^X \in \mathbf{Hom}_{\mathscr{A}}(X_n, X_{n-1})$ such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A complex is often written as a sequence as follows:

$$\begin{tikzcd} \dots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}"] & X_n \arrow[r, "d_n"] & X_{n-1} \arrow[r, "d_{n-1}"] & \dots \end{tikzcd}$$

Now let $X = (X_n, d_n^X)$ and $Y = (Y_n, d_n^Y)$ be two complexes. Then $f : X \to Y$ is a *morphism of complexes* if it is a family of morphisms $f = (f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} \dots \arrow[r] & X_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & X_n \arrow[r] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r] & Y_{n+1} \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \dots \end{tikzcd}$$

commutes.

If we put this together we get a new category.

Definition. Let $\mathscr{A}$ be an additive category. Then the collection of complexes over $\mathscr{A}$ together with the morphisms of complexes form a new category called the *category of complexes over* $\mathscr{A}$, denoted $C(\mathscr{A})$.

Proposition 3.3.2. *$C(\mathscr{A})$ is an additive category.*

Proof. 1) Addition of morphisms is defined degreewise so if $f = (f_n) : X \to Y$ and $g = (g_n) : X \to Y$ we get $f + g := (f_n + g_n)_{n\in\mathbb{Z}}$. Since $\mathscr{A}$ is additive we know that $a + b = b + a$ for $a, b \in \mathbf{Hom}_{\mathscr{A}}(A, B)$, and we get that $f + g = (f_n + g_n) = (g_n + f_n) = g + f$, since $f_n, g_n \in \mathbf{Hom}_{\mathscr{A}}(X_n, Y_n)$. We also get the bilinearity from $\mathscr{A}$.

2) The zero object in $C(\mathscr{A})$ is the complex $(0_{\mathscr{A}}, d)$ where $0_{\mathscr{A}}$ is the zero object in $\mathscr{A}$ and d is the unique morphism on the zero object.

3) We need to prove that for every pair of objects $X, Y \in C(\mathscr{A})$ there exists a coproduct $X \amalg Y$ in $C(\mathscr{A})$. Let $X = (X_n, d_n^X)$ and $Y = (Y_n, d_n^Y)$ be two complexes. $X \amalg Y$ is then defined degreewise with the coproduct in $\mathscr{A}$ i.e. $X \amalg Y = (X_n \amalg Y_n, d_n)_{n\in\mathbb{Z}}$ where d is obtained by the universal property as in

$$\begin{tikzcd}
 & X_{n-1} \amalg Y_{n-1} & \\
X_n \arrow[ur, "\iota_{X_{n-1}} d_n^X"] \arrow[r, "\iota_{X_n}"'] & X_n \amalg Y_n \arrow[u, "d_n"'] & Y_n \arrow[ul, "\iota_{Y_{n-1}} d_n^Y"'] \arrow[l, "\iota_{Y_n}"]
\end{tikzcd}$$

Using uniqueness in the universal property on the following diagram

$$\begin{tikzcd}
 & X_{n-2} \amalg Y_{n-2} & \\
X_n \arrow[ur, "0"] \arrow[r, "\iota_{X_n}"'] & X_n \amalg Y_n \arrow[u, "d_{n-1}d_n"'] & Y_n \arrow[ul, "0"'] \arrow[l, "\iota_{Y_n}"]
\end{tikzcd}$$

it follows that $d_{n-1}d_n = 0$. The complex $X \amalg Y$ satisfies the properties of a coproduct in $C(\mathscr{A})$ with morphisms of complexes $\iota_X = (\iota_{X_n})_{n\in\mathbb{Z}} : X_n \to X_n \amalg Y_n$ and $\iota_Y = (\iota_{Y_n})_{n\in\mathbb{Z}} : Y_n \to X_n \amalg Y_n$. To check that the universal property holds we let Z be an arbitrary complex and let $f_X : X_n \to Z_n$ and $f_Y : Y_n \to Z_n$ be two arbitrary morphisms of complexes. The unique morphism of complexes that satisfies $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$ is $f = (f_n)_{n\in\mathbb{Z}} : X \amalg Y \to Z$ where we get f_n from the universal property in the following diagram:

$$\begin{tikzcd}
 & Z_n & \\
X_n \arrow[ur, "(f_X)_n"] \arrow[r, "\iota_{X_n}"'] & X_n \amalg Y_n \arrow[u, "f_n"'] & Y_n \arrow[ul, "(f_Y)_n"'] \arrow[l, "\iota_{Y_n}"]
\end{tikzcd}$$

Hence the coproduct exists and so $C(\mathscr{A})$ is an additive category. □

Chapter 4

Triangulated Categories

4.1 Definition

Triangulated categories are additive categories together with a functor called the suspension, and what we call triangles. Let us define these first.

Definition. Let $\mathscr{A}$ be an additive category and let $\Sigma : \mathscr{A} \to \mathscr{A}$ be an additive automorphism. A *triangle* in $\mathscr{A}$ is a sequence $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ of objects and morphisms in $\mathscr{A}$.

Let $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ and $B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \Sigma B_1$ be two triangles in $\mathscr{A}$. A *morphism of triangles* (ϕ_1, ϕ_2, ϕ_3) is a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\phi_3"] & \Sigma A_1 \arrow[d, "\Sigma\phi_1"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma B_1.
\end{tikzcd}$$

If ϕ_1, ϕ_2 and ϕ_3 are isomorphisms in $\mathscr{A}$, we say that (ϕ_1, ϕ_2, ϕ_3) is an *isomorphism of triangles*.

Now we can define triangulated categories.

Definition. Let $\mathscr{T}$ be an additive category. Then $\mathscr{T}$ together with an additive automorphism Σ and a collection Δ of what we call distinguished triangles, is called a *triangulated category* if the following hold:

(TR1) (a) If a triangle is isomorphic to a triangle in Δ it is itself in Δ.

(b) For every $A \in \mathbf{Ob}(\mathscr{T})$ the triangle $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \Sigma A$ is in Δ.

(c) For every $A_1, A_2 \in \mathbf{Ob}(\mathscr{A})$ and $\alpha \in \mathbf{Hom}_{\mathscr{T}}(A_1, A_2)$ there is a triangle in Δ of the form

$$A_1 \xrightarrow{\alpha} A_2 \longrightarrow A_3 \longrightarrow \Sigma A_1.$$

(TR2) If $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ is in Δ, then the left rotation

$$A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1 \xrightarrow{-\Sigma\alpha_1} \Sigma A_2$$

is also in Δ, and vice versa.

(TR3) If

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1 \quad \text{and} \quad B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \Sigma B_1$$

are two triangles in Δ, each commutative diagram of the form

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, "\Sigma\phi_1"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma B_1
\end{tikzcd}$$

can be completed (not necessarily uniquely) to a morphism of triangles.

(TR4) (*Octahedral axiom*) Given a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\gamma_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma A_1 \\
 & C_3 \arrow[d, "\gamma_3"] & & \\
 & \Sigma A_2 & &
\end{tikzcd}$$

where the top two rows and second column are in Δ. Then there exist morphisms $\phi_3 : A_3 \to B_3$ and $\theta_3 : B_3 \to C_3$ such that the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\phi_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\gamma_2"] & B_3 \arrow[r, "\beta_3"] \arrow[d, "\theta_3"] & \Sigma A_1 \\
 & C_3 \arrow[r, equal] \arrow[d, "\gamma_3"] & C_3 \arrow[d, "\Sigma\alpha_2\circ\gamma_3"] & \\
 & \Sigma A_2 \arrow[r, "\Sigma\alpha_2"] & \Sigma A_3 &
\end{tikzcd} \tag{4.1}$$

is commutative, the third column is in Δ, and $\gamma_3 \circ \theta_3 = \Sigma\alpha_1 \circ \beta_3$.

The last diagram can also be written as an octahedron:

$$\begin{tikzcd}
 & B_3 \arrow[ddl, "|" marking] \arrow[dr, dashed, "\theta_3"] & \\
A_3 \arrow[ur, dashed, "\phi_3"] \arrow[d, "|" marking] & & C_3 \arrow[ll, "|" marking] \arrow[ddl, "|" marking] \\
A_1 \arrow[rr, "\beta_1"] \arrow[dr, "\alpha_1"'] & & B_2 \arrow[uul, "\beta_2"'] \arrow[u, "\gamma_3"'] \\
 & A_2 \arrow[uul, "\alpha_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

where $B_3 \;{-\!\!\!|\!\!\!\rightarrow}\; A_1$ means a morphism $B_3 \to \Sigma A_1$.

4.2 Properties

Having defined triangulated categories, we now look at some elementary properties. In all the following results we assume $\mathscr{T}$ is a triangulated category with suspension Σ.

Proposition 4.2.1. *Let* $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ *be in* Δ*. Then any composition of two consecutive morphisms vanishes, i.e.* $\alpha_2 \circ \alpha_1 = 0$*,* $\alpha_3 \circ \alpha_2 = 0$ *and* $(\Sigma\alpha_1) \circ \alpha_3 = 0$.

Proof. Because of the rotation axiom (TR2), we only need to show that $\alpha_2 \circ \alpha_1 = 0$. We also get from (TR2) that we have a distinguished triangle $A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1 \xrightarrow{-\Sigma\alpha_1} \Sigma A_2$. From this and from (TR1)(b) and (TR3) we can complete the following diagram to a morphism of triangles:

$$\begin{tikzcd}
A_2 \arrow[r, "\alpha_2"] \arrow[d, "\alpha_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, equal] & \Sigma A_1 \arrow[r, "-\Sigma\alpha_1"] & \Sigma A_2 \arrow[d, "\Sigma\alpha_2"] \\
A_3 \arrow[r, "1"] & A_3 \arrow[r, "0"] & 0 \arrow[r, "0"] & \Sigma A_3.
\end{tikzcd}$$

From this we see that $(\Sigma\alpha_2) \circ (-\Sigma\alpha_1) = 0$ and since Σ is an automorphism this means that $\alpha_2 \circ \alpha_1 = 0$. □

The following result shows that distinguished triangles give rise to long exact sequences.

Proposition 4.2.2. *Let* $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ *be in* Δ*. For any* $T \in \mathscr{T}$ *there is a long exact sequence of abelian groups*

$$\ldots \longrightarrow \mathit{Hom}_{\mathscr{T}}(T, \Sigma^i A_1) \xrightarrow{(\Sigma^i \alpha_1)_*} \mathit{Hom}_{\mathscr{T}}(T, \Sigma^i A_2) \xrightarrow{(\Sigma^i \alpha_2)_*} \mathit{Hom}_{\mathscr{T}}(T, \Sigma^i A_3)$$

$$\xrightarrow{(\Sigma^i \alpha_3)_*} \mathit{Hom}_{\mathscr{T}}(T, \Sigma^{i+1} A_1) \longrightarrow \ldots$$

where $f_* := \mathbf{Hom}_{\mathscr{T}}(T, f)$*, the morphism induced by* f *under the functor* $\mathbf{Hom}_{\mathscr{T}}(T, -)$.

Proof. By (TR2) we only need to show that

$$\mathrm{Hom}_{\mathscr{T}}(T,\Sigma^i A_1) \xrightarrow{(\Sigma^i\alpha_1)_*} \mathrm{Hom}_{\mathscr{T}}(T,\Sigma^i A_2) \xrightarrow{(\Sigma^i\alpha_2)_*} \mathrm{Hom}_{\mathscr{T}}(T,\Sigma^i A_3)$$

is exact. By proposition 4.2.1 we know that $(\Sigma^i\alpha_2)\circ(\Sigma^i\alpha_1) = 0$ and hence $(\Sigma^i\alpha_2)_*\circ(\Sigma^i\alpha_1)_* = 0$. This means that the image of $(\Sigma^i\alpha_1)_*$ is contained in the kernel of $(\Sigma^i\alpha_2)_*$.

To show the other inclusion we let $f \in \mathrm{Ker}(\Sigma^i\alpha_2)_*$. Consider the following diagram

$$\begin{tikzcd}
\Sigma^{-i}T \arrow[r, "0"] \arrow[d, "\Sigma^{-i}f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \Sigma^{-i+1}T \arrow[r, "1"] & \Sigma^{-i+1}T \arrow[d, "\Sigma^{-i+1}f"] \\
A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[r, "-\Sigma\alpha_1"] & \Sigma A_2
\end{tikzcd}$$

We know that the rows are distinguished triangles by (TR1)(b) and (TR2) and by our assumption on f we see that the left square commutes. From (TR3) we get that there exists an $h : \Sigma^{-i+1}T \to \Sigma A_1$ completing the diagram to a morphism of triangles. This means $\Sigma^{-i+1}f = (\Sigma\alpha_1)\circ h$ and hence, since Σ is an automorphism, $f = (\Sigma^i\alpha_1)\circ(\Sigma^{i-1}h)$ is in the image of $(\Sigma^i\alpha_1)_*$ as we wanted. ☐

Proposition 4.2.3. *Let* $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ *be in* Δ*, with* $\alpha_3 = 0$*. Then there is a* $\beta_1 : A_2 \to A_1$ *such that* $\beta_1\circ\alpha_1 = 1_{A_1}$*, and a* $\beta_2 : A_3 \to A_2$ *such that* $\alpha_2\circ\beta_2 = 1_{A_3}$*. We then say that the triangle splits.*

Proof. We first show the part about α_1. Consider the following commutative diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "0"] \arrow[d, "0"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "1"] & A_1 \arrow[r, "0"] & 0 \arrow[r, "0"] & \Sigma A_1
\end{tikzcd}$$

By (TR2) and (TR3) this can be completed to a morphism of triangles. That means there exists a $\beta_1 : A_2 \to A_1$ such that $\beta_1\circ\alpha_1 = 1_{A_1}$, which is what we wanted.

For α_2 we look at the following: From (TR1)(b) we have that $A_3 \xrightarrow{1} A_3 \to 0 \to \Sigma A_3$ is in Δ, and from (TR2) that $\Sigma^{-1}0 \to A_3 \xrightarrow{1} A_3 \to 0$ is too. Since Σ is an automorphism, $\Sigma^{-1}0 = 0$ and we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A_3 \arrow[r, "1"] & A_3 \arrow[r, "0"] \arrow[d, equal] & 0 \arrow[d, "0"] \\
A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[r, "\alpha_2"] & A_3 \arrow[r, "0"] & \Sigma A_1
\end{tikzcd}$$

As above we see that with (TR2) and (TR3) we can complete this to a morphism of triangles and hence get a $\beta_2 : A_3 \to A_2$ such that $\alpha_2\circ\beta_2 = 1_{A_3}$. ☐

4.3 Replacing the octahedral axiom

The octahedral axiom may seem big and complicated, and it is usually a lot of work to prove that it holds. Amnon Neeman therefore introduced a new axiom which replaces the octahedral axiom and is a bit more understandable. Here, instead of considering a large commutative diagram, we look at mapping cones. If we assume (TR1)-(TR3) hold we have the following:

(TR4') Given any diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, "\Sigma\phi_1"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma B_1
\end{tikzcd}$$

whose rows are distinguished triangles, there exists a $\phi_3 : A_3 \to B_3$ such that the diagram commutes and the mapping cone

$$A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \phi_2 & \beta_1 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \phi_3 & \beta_2 \end{bmatrix}} \Sigma A_1 \oplus B_3 \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\phi_1 & \beta_3 \end{bmatrix}} \Sigma A_2 \oplus \Sigma B_1$$

is a distinguished triangle.

That this axiom is equivalent to the octahedral axiom was proved by Neeman in [20] and [19], but we will instead follow the proof for n-angulated categories presented in [6]. First we need a lemma.

Lemma 4.3.1. *Suppose $\mathscr{T}$ is a category satisfying (TR1), (TR2), (TR3) and (TR4'), and let*

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma A_1
\end{tikzcd}$$

be a commutative diagram where the rows are distinguished triangles. Apply axiom (TR4') and complete the diagram to a morphism of triangles

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\phi_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma A_1
\end{tikzcd}$$

in such a way that the mapping cone is also a distinguished triangle. Then the triangle

$$A_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{[\phi_3 \ \beta_2]} B_3 \xrightarrow{\Sigma\alpha_1 \circ \beta_3} \Sigma A_2$$

is a distinguished triangle.

Proof. In the direct sum diagram

$$\begin{tikzcd}
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & A_3 \oplus B_2 \arrow[r, "{[\phi_3 \ \beta_2]}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & B_3 \arrow[r, "\Sigma\alpha_1\beta_3"] \arrow[d, "\begin{bmatrix} -\beta_3 \\ 1 \end{bmatrix}"] & \Sigma A_2 \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] \\
A_2 \oplus A_1 \arrow[r, "\begin{bmatrix} -\alpha_2 & 0 \\ \phi_2 & \beta_1 \end{bmatrix}"] \arrow[d, "{[1 \ \alpha_1]}"] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} -\alpha_3 & 0 \\ \phi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & \Sigma A_1 \oplus B_3 \arrow[r, "\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ 1 & \beta_3 \end{bmatrix}"] \arrow[d, "{[0 \ 1]}"] & \Sigma A_2 \oplus \Sigma A_1 \arrow[d, "{[1 \ \Sigma\alpha_1]}"] \\
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}"] & A_3 \oplus B_2 \arrow[r, "{[\phi_3 \ \beta_2]}"] & B_3 \arrow[r, "\Sigma\alpha_1\beta_3"] & \Sigma A_2
\end{tikzcd}$$

the middle row is the mapping cone, i.e a distinguished triangle. In Proposition 1.2.3 in [20], Neeman showed that Δ is closed under direct summands. Hence the top (and bottom) row in the above diagram is also a distinguished triangle. $\square$

Now we can show that (TR4) and (TR4') are equivalent, and as in [6] we do this in two steps.

Theorem 4.3.2. *Assume Δ is a collection of triangles satisfying axioms (TR1) and (TR2). If Δ satisfies (TR4') it also satisfies (TR3) and (TR4).*

Proof. (TR3) follows directly from (TR4'). Assume we have a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, equal] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, equal] \\
A_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma A_1
\end{tikzcd}$$

where the rows are in Δ, and in addition

$$A_2 \xrightarrow{\phi_2} B_2 \xrightarrow{\gamma_2} C_3 \xrightarrow{\gamma_3} \Sigma A_2$$

is in Δ. We want to show that we can complete this diagram to a morphism of triangles and show that the diagram (4.1) commutes, with the right column in Δ. We also need to show that $\gamma_3 \circ \theta_3 = \Sigma\alpha_1 \circ \beta_3$.

We apply (TR4') to the diagram above and complete it to a morphism $(1, \phi_2, \phi_3)$ of triangles in such a way that the mapping cone is in Δ. Then the top part of (4.1) is commutative.

Now, by Lemma 4.3.1 we know that the triangle

$$A_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{bmatrix} \phi_3 & \beta_2 \end{bmatrix}} B_3 \xrightarrow{\Sigma\alpha_1 \circ \beta_3} \Sigma A_2$$

is in Δ. That means we can apply (TR4') on the diagram

$$\begin{tikzcd}
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}"] \arrow[d, equal] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} \phi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & B_3 \arrow[r, "\Sigma\alpha_1 \circ \beta_3"] & \Sigma A_2 \arrow[d, equal] \\
A_2 \arrow[r, "\phi_2"] & B_2 \arrow[r, "\gamma_2"] & C_3 \arrow[r, "\gamma_3"] & \Sigma A_2
\end{tikzcd}$$

and complete it to a morphism

$$\begin{tikzcd}
A_2 \arrow[r, "\begin{bmatrix} -\alpha_2 \\ \phi_2 \end{bmatrix}"] \arrow[d, equal] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} \phi_3 & \beta_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & B_3 \arrow[r, "\Sigma\alpha_1 \circ \beta_3"] \arrow[d, "\theta_3"] & \Sigma A_2 \arrow[d, equal] \\
A_2 \arrow[r, "\phi_2"] & B_2 \arrow[r, "\gamma_2"] & C_3 \arrow[r, "\gamma_3"] & \Sigma A_2
\end{tikzcd} \tag{4.2}$$

of triangles so that the mapping cone

$$A_3 \oplus B_2 \oplus A_2 \xrightarrow{\begin{bmatrix} -\phi_3 & -\beta_2 & 0 \\ 0 & 1 & \phi_2 \end{bmatrix}} B_3 \oplus B_2 \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 \circ \beta_3 & 0 \\ \theta_3 & \gamma_2 \end{bmatrix}} \Sigma A_2 \oplus C_3 \xrightarrow{\begin{bmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\phi_2 & 0 \\ 1 & \gamma_3 \end{bmatrix}} \Sigma A_3 \oplus \Sigma B_2 \oplus \Sigma A_2$$

is in Δ. Note that since the diagram (4.2) is commutative, we get from the middle square that $\begin{bmatrix} 0 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \theta_3 \circ \phi_3 & \theta_3 \circ \beta_2 \end{bmatrix} \Rightarrow \gamma_2 = \theta_3 \circ \beta_2$, so the middle square in (4.1) is commutative. Note also that since the mapping cone above is in Δ, the composition of the two last arrows is zero by proposition 4.2.1. This means that

$$\begin{bmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\phi_2 & 0 \\ 1 & \gamma_3 \end{bmatrix} \begin{bmatrix} -\Sigma\alpha_1 \circ \beta_3 & 0 \\ \theta_3 & \gamma_2 \end{bmatrix} = \begin{bmatrix} -\Sigma\alpha_2 \circ \Sigma\alpha_1 \circ \beta_3 & 0 \\ \Sigma\phi_2 \circ \Sigma\alpha_1 \circ \beta_3 & 0 \\ \gamma_3 \circ \theta_3 - \Sigma\alpha_1 \circ \beta_3 & \gamma_3 \circ \gamma_2 \end{bmatrix} = 0 \Rightarrow \gamma_3 \circ \theta_3 = \Sigma\alpha_1 \circ \beta_3$$

We also have that the mapping cone is the middle row of the direct sum diagram

$$\begin{tikzcd}[column sep=large]
A_3 \arrow[r, "\phi_3"] \arrow[d, "\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}"] & B_3 \arrow[r, "\theta_3"] \arrow[d, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"] & C_3 \arrow[r, "\Sigma\alpha_2\circ\gamma_3"] \arrow[d, "\begin{bmatrix} -\gamma_3 \\ 1 \end{bmatrix}"] & \Sigma A_3 \arrow[d, "\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}"] \\
A_3 \oplus B_2 \oplus A_2 \arrow[r, "\begin{bmatrix} -\phi_3 & -\beta_2 & 0 \\ 0 & 1 & \phi_2 \end{bmatrix}"] \arrow[d, "{[-1\ 0\ \alpha_2]}"] & B_3 \oplus B_2 \arrow[r, "\begin{bmatrix} -\Sigma\alpha_1\circ\beta_3 & 0 \\ \theta_3 & \gamma_2 \end{bmatrix}"] \arrow[d, "{[1\ \beta_2]}"] & \Sigma A_2 \oplus C_3 \arrow[r, "\begin{bmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\phi_2 & 0 \\ 1 & \gamma_3 \end{bmatrix}"] \arrow[d, "{[0\ 1]}"] & \Sigma A_3 \oplus \Sigma B_2 \oplus \Sigma A_2 \arrow[d, "{[-1\ 0\ \Sigma\alpha_2]}"] \\
A_3 \arrow[r, "\phi_3"] & B_3 \arrow[r, "\theta_3"] & C_3 \arrow[r, "\Sigma\alpha_2\circ\gamma_3"] & \Sigma A_3,
\end{tikzcd}$$

which commutes by relations previously established. This implies, by Proposition 1.2.3 in [20], that the top (and bottom) row is in Δ i.e. (TR4) holds. ☐

Now let us prove the converse.

Theorem 4.3.3. *If Δ is a collection of triangles satisfying axioms (TR1)-(TR4), then it also satisfies axiom (TR4').*

Proof. Let

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\phi_2"] & A_3 \arrow[r, "\alpha_3"] & \Sigma A_1 \arrow[d, "\Sigma\phi_1"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & \Sigma B_1
\end{tikzcd}$$

be a commutative diagram where the rows are in Δ. We call these $A_\bullet$ and $B_\bullet$. We want to prove that we can complete this diagram to a morphism of triangles in such a way that the mapping cone of this morphism is in Δ.

From the diagram we are given, we build a new diagram

$$\begin{tikzcd}[column sep=large]
A_1 \oplus B_1 \arrow[r, "\begin{bmatrix} 0 & 0 \\ -\alpha_1 & 0 \\ 0 & -1 \end{bmatrix}"] \arrow[d, equal] & B_2 \oplus A_2 \oplus B_1 \arrow[r, "\begin{bmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}"] \arrow[d, "{[1\ -\phi_2\ -\beta_1]}"] & A_3 \oplus B_2 \arrow[r, "\begin{bmatrix} -\alpha_3 & 0 \\ 0 & 0 \end{bmatrix}"] & \Sigma A_1 \oplus \Sigma B_1 \arrow[d, equal] \\
A_1 \oplus B_1 \arrow[r, "{[\phi_2\circ\alpha_1\ \beta_1]}"] & B_2 \arrow[r, "\begin{bmatrix} 0 \\ -\beta_2 \end{bmatrix}"] \arrow[d, "0"] & \Sigma A_1 \oplus B_3 \arrow[r, "\begin{bmatrix} -1 & 0 \\ \Sigma\phi_1 & \beta_3 \end{bmatrix}"] & \Sigma A_1 \oplus \Sigma B_1 \\
 & \Sigma A_2 \oplus \Sigma B_1 \arrow[d, "\begin{bmatrix} -\Sigma\phi_2 & -\Sigma\beta_1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}"] & & \\
 & \Sigma B_2 \oplus \Sigma A_2 \oplus \Sigma B_1 & &
\end{tikzcd} \tag{4.3}$$

where the top left square commutes. Now let $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ be the triangles

$$X_\bullet\colon\quad B_2 \oplus A_2 \oplus B_1 \xrightarrow{[1\ -\phi_2]} B_2 \xrightarrow{0} \Sigma A_2 \oplus \Sigma B_1 \xrightarrow{\begin{bmatrix} -\Sigma\phi_2 & -\Sigma\beta_1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}} \Sigma B_2 \oplus \Sigma A_2 \oplus \Sigma B_1$$

$$Y_\bullet\colon\quad A_1 \oplus B_1 \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ -\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} B_2 \oplus A_2 \oplus B_1 \xrightarrow{\left[\begin{smallmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]} A_3 \oplus B_2 \xrightarrow{\left[\begin{smallmatrix} -\alpha_3 & 0 \\ 0 & 0 \end{smallmatrix}\right]} \Sigma A_1 \oplus \Sigma B_1$$

$$Z_\bullet\colon\quad A_1 \oplus B_1 \xrightarrow{[\,\phi_2\circ\alpha_1 \ \ \beta_1\,]} B_2 \xrightarrow{\left[\begin{smallmatrix} 0 \\ -\beta_2 \end{smallmatrix}\right]} \Sigma A_1 \oplus B_3 \xrightarrow{\left[\begin{smallmatrix} -1 & 0 \\ \Sigma\phi_1 & \beta_3 \end{smallmatrix}\right]} \Sigma A_1 \oplus \Sigma B_1$$

In order to use (TR4) we need these three to be in Δ. We can easily see that $X_\bullet$ is isomorphic to the direct product of the trivial triangle on B_2, and the left rotations of the trivial triangles on A_2 and B_1, i.e.

$$B_2 \xrightarrow{1} B_2 \to 0 \to \Sigma B_2, \quad A_2 \to 0 \to \Sigma A_2 \xrightarrow{1} \Sigma A_2 \quad \text{and} \quad B_1 \to 0 \to \Sigma B_1 \xrightarrow{1} \Sigma B_1,$$

all of which are in Δ. Similarly we see that $Y_\bullet$ is isomorphic to the direct sum of $A_\bullet$, the trivial triangle on B_1 and the right rotation of the trivial triangle on B_2 i.e.

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1, \quad B_1 \xrightarrow{1} B_1 \to 0 \to \Sigma B_1 \quad \text{and} \quad 0 \to B_1 \xrightarrow{1} B_1 \to 0,$$

which are also in Δ. Finally we notice that $Z_\bullet$ is isomorphic to the direct sum of $B_\bullet$ and the left rotation of trivial triangle on A_1 i.e.

$$B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \Sigma B_1 \quad \text{and} \quad A_1 \to 0 \to \Sigma A_1 \xrightarrow{1} \Sigma A_1,$$

which are of course both in Δ. This means that by Proposition 1.2.3 in [20] $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ are in Δ.

Now, since $X_\bullet$, $Y_\bullet$ and $Z_\bullet$ are in Δ, we can use (TR4) on the diagram (4.3). This means we can find morphisms $\sigma : A_3 \oplus B_2 \to \Sigma A_1 \oplus B_3$ and $\theta : \Sigma A_1 \oplus B_3 \to \Sigma A_2 \oplus \Sigma B_1$ such that the following holds:

(1) $(1, [\,1 \ -\phi_2 \ \beta_1\,], \sigma) : Y_\bullet \to Z_\bullet$ is a morphism of triangles,

(2) $\left[\begin{smallmatrix} -\Sigma\phi_2 & -\Sigma\beta_1 \\ 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \circ \theta = \left[\begin{smallmatrix} 0 & 0 \\ -\Sigma\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} -1 & 0 \\ \Sigma\phi_1 & \beta_3 \end{smallmatrix}\right]$ and

(3) the triangle $A_3 \oplus B_2 \xrightarrow{\sigma} \Sigma A_1 \oplus B_3 \xrightarrow{\theta} \Sigma A_2 \oplus \Sigma B_1 \xrightarrow{\left[\begin{smallmatrix} -\Sigma\alpha_2 & 0 \\ \Sigma\phi_2 & -\Sigma\beta_1 \end{smallmatrix}\right]} \Sigma A_3 \oplus \Sigma B_2$ is in Δ.

In other words we have a commutative diagram

$$\begin{tikzcd}
A_1 \oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & 0 \\ -\alpha_1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}"] \arrow[d, equal] & B_2 \oplus A_2 \oplus B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & -\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]}"] \arrow[d, "{[\,1 \ -\phi_2 \ -\beta_1\,]}"] & A_3 \oplus B_2 \arrow[r, "{\left[\begin{smallmatrix} -\alpha_3 & 0 \\ 0 & 0 \end{smallmatrix}\right]}"] \arrow[d, "\sigma"] & \Sigma A_1 \oplus \Sigma B_1 \arrow[d, equal] \\
A_1 \oplus B_1 \arrow[r, "{[\,\phi_2\circ\alpha_1 \ \ \beta_1\,]}"] & B_2 \arrow[r, "{\left[\begin{smallmatrix} 0 \\ -\beta_2 \end{smallmatrix}\right]}"] \arrow[d, "0"] & \Sigma A_1 \oplus B_3 \arrow[r, "{\left[\begin{smallmatrix} -1 & 0 \\ \Sigma\phi_1 & \beta_3 \end{smallmatrix}\right]}"] \arrow[d, "\theta"] & \Sigma A_1 \oplus \Sigma B_1 \\
 & \Sigma A_2 \oplus \Sigma B_1 \arrow[r, equal] \arrow[d, "{\left[\begin{smallmatrix} -\Sigma\phi_2 & -\Sigma\beta_1 \\ -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}"] & \Sigma A_2 \oplus \Sigma B_1 \arrow[d, "{\left[\begin{smallmatrix} \Sigma\alpha_2 & 0 \\ -\Sigma\phi_2 & -\Sigma\beta_1 \end{smallmatrix}\right]}"] & \\
 & \Sigma B_2 \oplus \Sigma A_2 \oplus \Sigma B_1 \arrow[r, "{\left[\begin{smallmatrix} 0 & -\Sigma\alpha_2 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]}"] & \Sigma A_3 \oplus \Sigma B_2 &
\end{tikzcd}$$

where the two top rows and two middle columns are in Δ. Since the diagram is commutative we get, if we let $\sigma = \left[\begin{smallmatrix}\sigma_1 & \sigma_2\\ \sigma_3 & \sigma_4\end{smallmatrix}\right]$, that

$$\left[\begin{smallmatrix}-1 & 0\\ \Sigma\phi_1 & \beta_3\end{smallmatrix}\right]\left[\begin{smallmatrix}\sigma_1 & \sigma_2\\ \sigma_3 & \sigma_4\end{smallmatrix}\right] = \left[\begin{smallmatrix}-\sigma_1 & -\sigma_2\\ \Sigma\phi_1\circ\sigma_1+\beta_3\circ\sigma_3 & \Sigma\phi_1\circ\sigma_2+\beta_3\circ\sigma_4\end{smallmatrix}\right] = \left[\begin{smallmatrix}-\alpha_3 & 0\\ 0 & 0\end{smallmatrix}\right].$$

Hence $\sigma_1 = \alpha_3$ and $\sigma_2 = 0$, and we have the relations $\Sigma\phi_1\alpha_3 = -\beta_3 \circ \sigma_3$ and $\beta_3 \circ \sigma_4 = 0$. Further we get

$$\left[\begin{smallmatrix}0\\ -\beta_2\end{smallmatrix}\right]\left[\begin{smallmatrix}-1 & -\phi_2 & -\beta_1\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0 & 0\\ -\beta_2 & \beta_2\circ\phi_2 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}\alpha_3 & 0\\ \sigma_3 & \sigma_4\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & -\alpha_2 & 0\\ 1 & 0 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0 & 0\\ \sigma_4 & -\sigma_3\circ\alpha_2 & 0\end{smallmatrix}\right]$$

i.e. $\sigma_4 = -\beta_2$ and $\beta_2 \circ \phi_2 = -\sigma_3 \circ \alpha_2$. From this we get that $\sigma_3 = -\phi_3$ for some $\phi_3 : A_3 \to B_3$ such that $\phi = (\phi_1, \phi_2, \phi_3)$ is a morphism of triangles.

Further, if we let $\theta = \left[\begin{smallmatrix}\theta_1 & \theta_2\\ \theta_3 & \theta_4\end{smallmatrix}\right]$ we get from (2) that

$$\left[\begin{smallmatrix}-\Sigma\phi_2 & -\Sigma\beta_1\\ 1 & 0\\ 0 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}\theta_1 & \theta_2\\ \theta_3 & \theta_4\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0\\ -\Sigma\alpha_1 & 0\\ 0 & -1\end{smallmatrix}\right]\left[\begin{smallmatrix}-1 & 0\\ \Sigma\phi_1 & \beta_3\end{smallmatrix}\right]$$
$$\left[\begin{smallmatrix}\Sigma\phi_2\circ\theta_1-\Sigma\beta_1\circ\theta_3 & \Sigma\phi_2\circ\theta_2-\Sigma\beta_1\circ\theta_4\\ -\theta_1 & -\theta_2\\ -\theta_3 & -\theta_4\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0\\ \Sigma\alpha_1 & 0\\ -\Sigma\phi_1 & -\beta_3\end{smallmatrix}\right]$$

and hence $\theta = \left[\begin{smallmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\phi_1 & \beta_3\end{smallmatrix}\right]$. This means that from (3) and what we have just shown, we get that the triangle

$$A_3 \oplus B_2 \xrightarrow{\left[\begin{smallmatrix}\alpha_3 & 0\\ -\phi_3 & -\beta_2\end{smallmatrix}\right]} \Sigma A_1 \oplus B_3 \xrightarrow{\left[\begin{smallmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\phi_1 & \beta_3\end{smallmatrix}\right]} \Sigma A_2 \oplus \Sigma B_1 \xrightarrow{\left[\begin{smallmatrix}\Sigma\alpha_2 & 0\\ -\Sigma\phi_2 & -\Sigma\beta_1\end{smallmatrix}\right]} \Sigma A_3 \oplus \Sigma B_2$$

is in Δ. The right rotation of this triangle is isomorphic by $(1, \left[\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right], 1)$ to the triangle

$$A_2 \oplus B_1 \xrightarrow{\left[\begin{smallmatrix}-\alpha_2 & 0\\ \phi_2 & \beta_1\end{smallmatrix}\right]} A_3 \oplus B_2 \xrightarrow{\left[\begin{smallmatrix}-\alpha_3 & 0\\ \phi_3 & \beta_2\end{smallmatrix}\right]} \Sigma A_1 \oplus B_3 \xrightarrow{\left[\begin{smallmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\phi_1 & \beta_3\end{smallmatrix}\right]} \Sigma A_2 \oplus \Sigma B_1$$

which is the mapping cone of ϕ. This means by (TR1)(a) and(TR2) that this mapping cone is in Δ and hence (TR4’) holds.

$\square$

Chapter 5

Matrix factorizations

5.1 Definition

Definition. Let S be a commutative ring and $x \in S$. A *matrix factorization* (F, G, ϕ, ψ) of x in S is a diagram

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

where F and G are finitely generated free S-modules, and ϕ and ψ are S-homomorphisms such that

$$\psi \circ \phi = x \cdot 1_F \quad \text{and} \quad \phi \circ \psi = x \cdot 1_G.$$

A morphism θ between two matrix factorizations $(F_1, G_1, \phi_1, \psi_1)$ and $(F_2, G_2, \phi_2, \psi_2)$ of x, is a pair of maps $f : F_1 \to F_2$ and $g : G_1 \to G_2$ such that the following diagram commutes:

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"] \arrow[d, "f"] & G_1 \arrow[r, "\psi_1"] \arrow[d, "g"] & F_1 \arrow[d, "f"] \\
F_2 \arrow[r, "\phi_2"] & G_2 \arrow[r, "\psi_2"] & F_2
\end{tikzcd}$$

The category of matrix factorizations is denoted by $\mathbf{MF}(S, x)$ and is an additive category with the obvious notion of a zero object and direct sums.

Definition. The *suspension* $\Sigma(F, G, \phi, \psi)$ of a matrix factorization (F, G, ϕ, ψ) is the matrix factorization

$$G \xrightarrow{-\psi} F \xrightarrow{-\phi} G$$

of x.

Definition. For the map $\theta : (F_1, G_1, \phi_1, \psi_1) \to (F_2, G_2, \phi_2, \psi_2)$ above, we define the *mapping cone* C_θ of θ to be the the diagram

$$G_1 \oplus F_2 \xrightarrow{\begin{bmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{bmatrix}} F_1 \oplus G_2 \xrightarrow{\begin{bmatrix} -\phi_1 & 0 \\ f & \psi_2 \end{bmatrix}} G_1 \oplus F_2.$$

This is also a matrix factorization of x and gives two natural maps of matrix factorizations in $\mathbf{MF}(S,x)$:

$$\begin{tikzcd}
(F_2,G_2,\phi_2,\psi_2) \arrow[d, "i_\theta"] & F_2 \arrow[r, "\phi_2"] \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1_{F_2} \end{smallmatrix}\right]"] & G_2 \arrow[r, "\psi_2"] \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1_{G_2} \end{smallmatrix}\right]"] & F_2 \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1_{F_2} \end{smallmatrix}\right]"] \\
C_\theta & G_1 \oplus F_2 \arrow[r, "\left[\begin{smallmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{smallmatrix}\right]"] & F_1 \oplus G_2 \arrow[r, "\left[\begin{smallmatrix} -\phi_1 & 0 \\ f & \psi_2 \end{smallmatrix}\right]"] & G_1 \oplus F_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_\theta \arrow[d, "\pi_\theta"] & G_1 \oplus F_2 \arrow[r, "\left[\begin{smallmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{smallmatrix}\right]"] \arrow[d, "{[1_{G_1}\ 0]}"] & F_1 \oplus G_2 \arrow[r, "\left[\begin{smallmatrix} -\phi_1 & 0 \\ f & \psi_2 \end{smallmatrix}\right]"] \arrow[d, "{[1_{F_1}\ 0]}"] & G_1 \oplus F_2 \arrow[d, "{[1_{G_1}\ 0]}"] \\
\Sigma(F_1,G_1,\phi_1,\psi_1) & G_1 \arrow[r, "-\psi_1"] & F_1 \arrow[r, "-\phi_1"] & G_1
\end{tikzcd}$$

Definition. Let $\theta,\theta' : (F_1,G_1,\phi_1,\psi_1) \to (F_2,G_2,\phi_2,\psi_2)$ be two maps in $\mathbf{MF}(S,x)$ with the same source and target, where $\theta = (f,g)$ and $\theta' = (f',g')$. We say θ and θ' are *homotopic* if there are diagonal maps s and t in the diagram

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"] \arrow[d, shift right, "f'"'] \arrow[d, shift left, "f"] & G_1 \arrow[r, "\psi_1"] \arrow[d, shift right, "g'"'] \arrow[d, shift left, "g"] \arrow[dl, "s"'] & F_1 \arrow[d, shift right, "f'"'] \arrow[d, shift left, "f"] \arrow[dl, "t"'] \\
F_2 \arrow[r, "\phi_2"] & G_2 \arrow[r, "\psi_2"] & F_2
\end{tikzcd}$$

such that

$$\begin{aligned} f - f' &= s \circ \phi_1 + \psi_2 \circ t \\ g - g' &= t \circ \psi_1 + \phi_2 \circ s. \end{aligned}$$

This defines an equivalence relation on the abelian groups of morphisms in the category $\mathbf{MF}(S,x)$, and we denote the equivalence class of a morphism θ by $[\theta]$. Homotopies are compatible with addition and composition of maps in $\mathbf{MF}(S,x)$. This means that if we have $\theta,\theta' : (F_1,G_1,\phi_1,\psi_1) \to (F_2,G_2,\phi_2,\psi_2)$ and $\eta,\eta' : (F_2,G_2,\phi_2,\psi_2) \to (F_3,G_3,\phi_3,\psi_3)$ with $\theta \sim \theta'$ and $\eta \sim \eta'$, we have $(\eta \circ \theta) \sim (\eta' \circ \theta')$. Similarly, if we have $\theta,\theta',\eta,\eta' : (F_1,G_1,\phi_1,\psi_1) \to (F_2,G_2,\phi_2,\psi_2)$ with $\theta \sim \theta'$ and $\eta \sim \eta'$ we have $(\theta+\eta) \sim (\theta'+\eta')$.

Now we can define a new category $\mathbf{HMF}(S,x)$, the *homotopy category*, which has the same objects as $\mathbf{MF}(S,x)$ but the morphism sets are the homotopy equivalence classes defined above. These sets are also abelian groups, i.e $[\theta]+[\eta] = [\eta]+[\theta]$, which means $\mathbf{HMF}(S,x)$ is an additive category with the same zero object, which now is unique only up to homotopy, and the usual direct sums.

We notice that the suspension $\Sigma(F,G,\phi,\psi)$ induces an additive automorphism on $\mathbf{HMF}(S,x)$ with $\Sigma^2 = \text{id}$. Now we define Δ to be the collection of triangles isomorphic to triangles of the form

$$(F_1,G_1,\phi_1,\psi_1) \xrightarrow{[\theta]} (F_2,G_2,\phi_2,\psi_2) \xrightarrow{[i_\theta]} C_\theta \xrightarrow{[\pi_\theta]} \Sigma(F_1,G_1,\phi_1,\psi_1)$$

which are called *standard triangles*. This brings us to the main result of this section:

Theorem 5.1.1. $\mathrm{HMF}(S,x)$ *together with the suspension* Σ *and the distinguished triangles* Δ*, is a triangulated category.*

Proof. The proof is an adaptation of the proof of Theorem 6.7 in [11] that shows the homotopy category of an additive category is triangulated. We need to show that the axioms (TR1) - (TR4) hold. The first, (TR1)(a), holds from the construction of Δ.

(TR1)(b) From the construction of the standard triangles we know that there is one on the form

$$(F,G,\phi,\psi) \xrightarrow{[\mathrm{id}]} (F,G,\phi,\psi) \longrightarrow C_{\mathrm{id}} \longrightarrow \Sigma(F,G,\phi,\psi)$$

Want to show that C_{id} is isomorphic to the zero object $0 \to 0 \to 0$ in $\mathbf{HMF}(S,x)$, and to do this we show that the identity morphism on the cone C_{id} is homotopic to the zero map. The identity map

$$\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d, "1_F"] & G \arrow[r, "\psi"] \arrow[d, "1_G"] & F \arrow[d, "1_F"] \\
F \arrow[r, "\phi"] & G \arrow[r, "\psi"] & F
\end{tikzcd}$$

has mapping cone

$$G \oplus F \xrightarrow{\left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right]} F \oplus G \xrightarrow{\left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right]} G \oplus F$$

That means we need s and t in the diagram

$$\begin{tikzcd}
G \oplus F \arrow[r, "{\left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right]}"] \arrow[d, "1_{G\oplus F}"] & F \oplus G \arrow[r, "{\left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right]}"] \arrow[d, "1_{F\oplus G}"] \arrow[dl, "s"'] & G \oplus F \arrow[d, "1_{G\oplus F}"] \arrow[dl, "t"'] \\
G \oplus F \arrow[r, "{\left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right]}"] & F \oplus G \arrow[r, "{\left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right]}"] & G \oplus F
\end{tikzcd}$$

that satisfy

$$\begin{aligned}
1_{G\oplus F} &= s \circ \left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right] + \left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right] \circ t \quad \text{and} \\
1_{F\oplus G} &= t \circ \left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right] + \left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right] \circ s.
\end{aligned}$$

Put $s = t = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. This gives us

$$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} -\psi & 0 \\ 1_G & \phi \end{smallmatrix}\right] + \left[\begin{smallmatrix} -\phi & 0 \\ 1_F & \psi \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_G & \phi \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & -\phi \\ 0 & 1_F \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_G & 0 \\ 0 & 1_F \end{smallmatrix}\right] = 1_{G\oplus F}$$

which is what we wanted. The same holds for $1_{F\oplus G}$. This means the identity morphism on C_{id} is homotopic to the zero morphism which means C_{id} must be isomorphic to the zero object in the homotopy category. Hence

$$(F,G,\phi,\psi) \xrightarrow{[\mathrm{id}]} (F,G,\phi,\psi) \longrightarrow 0 \longrightarrow \Sigma(F,G,\phi,\psi)$$

is a distinguished triangle and (TR1)(b) holds.

(TR1)(c) This follows from the construction of Δ. If we have a morphism

$$\theta : (F_1,G_1,\phi_1,\psi_1) \to (F_2,G_2,\phi_2,\psi_2)$$

we can find the mapping cone C_θ and from this find the maps i_θ and π_θ. We take the homotopy classes and get

$$(F_1,G_1,\phi_1,\psi_1) \xrightarrow{[\theta]} (F_2,G_2,\phi_2,\psi_2) \xrightarrow{[i_\theta]} C_\theta \xrightarrow{[\pi_\theta]} \Sigma(F_1,G_1,\phi_1,\psi_1)$$

which is a standard triangle and hence in Δ.

(TR2) We need to show that if we rotate a standard triangle, it is isomorphic to another standard triangle in $\mathbf{HMF}(S, x)$. More precisely we will show that

$$(F_2, G_2, \phi_2, \psi_2) \xrightarrow{[i_\theta]} C_\theta \xrightarrow{[\pi_\theta]} \Sigma(F_1, G_1, \phi_1, \psi_1) \xrightarrow{-\Sigma[\theta]} \Sigma(F_2, G_2, \phi_2, \psi_2)$$

is isomorphic to

$$(F_2, G_2, \phi_2, \psi_2) \xrightarrow{[i_\theta]} C_\theta \xrightarrow{\left[i_{i_\theta}\right]} C_{i_\theta} \xrightarrow{\left[\pi_{i_\theta}\right]} \Sigma(F_2, G_2, \phi_2, \psi_2)$$

We construct an isomorphism between the two triangles in the following way:

$$\begin{tikzcd}
(F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] \arrow[d, "\mathrm{id}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] \arrow[d, "\mathrm{id}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{-\Sigma[\theta]}"] \arrow[d, dashed, shift right, "{[\alpha]}"'] & \Sigma(F_2, G_2, \phi_2, \psi_2) \arrow[d, "\mathrm{id}"] \\
(F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] & C_\theta \arrow[r, "{\left[i_{i_\theta}\right]}"] & C_{i_\theta} \arrow[r, "{\left[\pi_{i_\theta}\right]}"] \arrow[u, dashed, shift right, "{[\beta]}"'] & \Sigma(F_2, G_2, \phi_2, \psi_2)
\end{tikzcd}$$

where we define the morphisms $\alpha : \Sigma(F_1, G_1, \phi_1\psi_1) \to C_{i_\theta}$ and $\beta : C_{i_\theta} \to \Sigma(F_1, G_1, \phi_1, \psi_1)$ by

$$\begin{tikzcd}[column sep=large]
\Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[d, shift right, "\alpha"'] & G_1 \arrow[r, "-\psi_1"] \arrow[d, shift right, "{\begin{bmatrix} -g \\ 1_{G_1} \\ 0 \end{bmatrix}}"'] & F_1 \arrow[r, "-\phi_1"] \arrow[d, shift right, "{\begin{bmatrix} -f \\ 1_{F_1} \\ 0 \end{bmatrix}}"'] & G_1 \arrow[d, shift right, "{\begin{bmatrix} -g \\ 1_{G_1} \\ 0 \end{bmatrix}}"'] \\
C_{i_\theta} \arrow[u, shift right, "\beta"'] & G_2 \oplus G_1 \oplus F_2 \arrow[r, "{\begin{bmatrix} -\psi_2 & 0 & 0 \\ 0 & \psi_1 & 0 \\ 1_{G_1} & g & \phi_2 \end{bmatrix}}"] \arrow[u, shift right, "{\begin{bmatrix} 0 & 1_{G_1} & 0 \end{bmatrix}}"'] & F_2 \oplus F_1 \oplus G_2 \arrow[r, "{\begin{bmatrix} -\phi_1 & 0 & 0 \\ 0 & \phi_1 & 0 \\ 1_{F_1} & f & \psi_2 \end{bmatrix}}"] \arrow[u, shift right, "{\begin{bmatrix} 0 & 1_{F_1} & 0 \end{bmatrix}}"'] & G_2 \oplus G_1 \oplus F_2 \arrow[u, shift right, "{\begin{bmatrix} 0 & 1_{G_1} & 0 \end{bmatrix}}"']
\end{tikzcd}$$

For α and β to be isomorphisms, they have to be the inverse of each other. We start by looking at $[\beta \circ \alpha]$. Here we have $\begin{bmatrix} 0 & 1_{G_1} & 0 \end{bmatrix} \begin{bmatrix} -g & 1_{G_1} & 0 \end{bmatrix}^T = 1_{G_1}$ and $\begin{bmatrix} 0 & 1_{F_1} & 0 \end{bmatrix} \begin{bmatrix} -g & 1_{F_1} & 0 \end{bmatrix}^T = 1_{F_1}$ so $[\beta \circ \alpha] = \mathrm{id}_{\Sigma(F_1, G_1, \phi_1, \psi_1)}$. The opposite gives us

$$\begin{aligned}
\begin{bmatrix} -g & 1_{G_1} & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1_{G_1} & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -g & 0 \\ 0 & 1_{G_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\begin{bmatrix} -f & 1_{F_1} & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1_{F_1} & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -f & 0 \\ 0 & 1_{F_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

This is homotopic to $\mathrm{id}_{C_{i_\theta}}$ by $s = t = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which means $\alpha \circ \beta \sim \mathrm{id}_{C_{i_\theta}}$. Hence $[\alpha]$ and $[\beta]$ are the inverse of each other.

Now we need to check that α and β induce morphisms of triangles. We look at $[\beta]$ first and see that $[\beta \circ i_{i_\theta}] = [\pi_\theta]$ by looking at the following diagram:

$$\begin{tikzcd}
C_\theta \arrow[r, "i_{i_\theta}"] & C_{i_\theta} \arrow[r, "\beta"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \\
G_1 \oplus F_2 \arrow[r, "\begin{bmatrix} 0 & 0 \\ 1_{G_1} & 0 \\ 0 & 1_{F_2} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{bmatrix}"'] & G_2 \oplus G_1 \oplus F_2 \arrow[r, "{[0\ 1_{G_1}\ 0]}"] \arrow[d, "M_1"] & G_1 \arrow[d, "-\psi_1"] \\
F_1 \oplus G_2 \arrow[r, "\begin{bmatrix} 0 & 0 \\ 1_{F_1} & 0 \\ 0 & 1_{G_2} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} -\phi_1 & 0 \\ f & \psi_2 \end{bmatrix}"'] & F_2 \oplus F_1 \oplus G_2 \arrow[r, "{[0\ 1_{G_1}\ 0]}"] \arrow[d, "M_2"] & F_1 \arrow[d, "-\phi_1"] \\
G_1 \oplus F_2 \arrow[r, "\begin{bmatrix} 0 & 0 \\ 1_{G_1} & 0 \\ 0 & 1_{F_2} \end{bmatrix}"] & G_2 \oplus G_1 \oplus F_2 \arrow[r, "{[0\ 1_{G_1}\ 0]}"] & G_1.
\end{tikzcd}$$

Next we want to show that $-\Sigma\theta \circ \beta \sim \pi_{i_\theta}$. So we first look at $-\Sigma\theta \circ \beta$.

$$\begin{tikzcd}
C_{i_\theta} \arrow[r, "\beta"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[r, "-\Sigma\theta"] & \Sigma(F_2, G_2, \phi_2, \psi_2)
\end{tikzcd}$$

$$\begin{tikzcd}
G_2 \oplus G_1 \oplus F_2 \arrow[r, "{[0\ 1_{G_1}\ 0]}"] \arrow[d, "M_1"] & G_1 \arrow[r, "-g"] \arrow[d, "-\psi_1"] & G_2 \arrow[d, "-\psi_2"] \\
F_2 \oplus F_1 \oplus G_2 \arrow[r, "{[0\ 1_{F_1}\ 0]}"] \arrow[d, "M_2"] & F_1 \arrow[r, "-f"] \arrow[d, "-\phi_1"] & F_2 \arrow[d, "-\phi_2"] \\
G_2 \oplus G_1 \oplus F_2 \arrow[r, "{[0\ 1_{G_1}\ 0]}"] & G_1 \arrow[r, "-g"] & G_2.
\end{tikzcd}$$

We want $([\,0\ {-g}\ 0\,], [\,0\ {-f}\ 0\,])$ to be homotopic with $([\,1_{G_2}\ 0\ 0\,], [\,1_{F_2}\ 0\ 0\,])$, which we get if we let $s = t = [\,0\ 0\ {-1}\,]$ in the diagram

$$\begin{tikzcd}
C_{i_\theta} \arrow[d, "-\Sigma\theta\circ\beta"', "\pi_{i_\theta}"] & G_2 \oplus G_1 \oplus F_2 \arrow[r, "\begin{bmatrix} -\psi_2 & 0 & 0 \\ 0 & -\psi_1 & 0 \\ 1_{G_2} & g & \phi_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}^T"', "\begin{bmatrix} 1_{G_2} \\ 0 \\ 0 \end{bmatrix}^T"] & F_2 \oplus F_1 \oplus G_2 \arrow[r, "\begin{bmatrix} -\phi_2 & 0 & 0 \\ 0 & -\phi_1 & 0 \\ 1_{F_2} & f & \psi_2 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix}^T"', "\begin{bmatrix} 1_{F_2} \\ 0 \\ 0 \end{bmatrix}^T"] \arrow[dl, dashed, "s"'] & G_2 \oplus G_1 \oplus F_2 \arrow[d, "\begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}^T"', "\begin{bmatrix} 1_{G_2} \\ 0 \\ 0 \end{bmatrix}^T"] \arrow[dl, dashed, "t"'] \\
\Sigma(F_2, G_2, \phi_2, \psi_2) & G_2 \arrow[r, "-\psi_2"] & F_2 \arrow[r, "-\phi_2"] & G_2.
\end{tikzcd}$$

So $[\beta]$ is an isomorphism of triangles. The fact that $[\alpha]$ is too, follows from this:

$$i_{i_\theta} = \mathrm{id}_{C_{i_\theta}} \circ i_{i_\theta} = \alpha \circ \beta \circ i_{i_\theta} = \alpha \circ \pi_\theta$$
$$-\Sigma\theta = -\Sigma\theta \circ \mathrm{id}_{\Sigma(F_1, G_1, \phi_1, \psi_1)} = -\Sigma\theta \circ \beta \circ \alpha = \pi_{i_\theta} \circ \alpha.$$

So we have two isomorphisms of triangles and hence the two triangles we started with are isomorphic and the rotation property holds.

(TR3) Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. We assume we have a diagram

$$\begin{tikzcd}
(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\theta]}"] \arrow[d, "{[\alpha]}"] & (F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] \arrow[d, "{[\beta]}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[d, "{\Sigma[\alpha]}"] \\
(F'_1, G'_1, \phi'_1, \psi'_1) \arrow[r, "{[\theta']}"] & (F'_2, G'_2, \phi'_2, \psi'_2) \arrow[r, "{[i_{\theta'}]}"] & C_{\theta'} \arrow[r, "{[\pi_{\theta'}]}"] & \Sigma(F'_1, G'_1, \phi'_1, \psi'_1)
\end{tikzcd} \tag{5.1}$$

where the left square commutes in HMF(S, x), i.e there exists maps $s : G_1 \to F'_2$ and $t : F_1 \to G'_2$ in the diagram

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"] \arrow[d, "\beta_1\circ f"', "f'\circ\alpha_1"] & G_1 \arrow[r, "\psi_1"] \arrow[d, "\beta_2\circ g"', "g'\circ\alpha_2"] \arrow[dl, "s"'] & F_1 \arrow[d, "\beta_1\circ f"', "f'\circ\alpha_2"] \arrow[dl, "t"'] \\
F'_2 \arrow[r, "\phi'_2"] & G'_2 \arrow[r, "\psi'_2"] & F'_2
\end{tikzcd}$$

such that

$$\beta_1 \circ f - f' \circ \alpha_1 = s \circ \phi_1 + \psi'_2 \circ t$$
$$\beta_2 \circ g - g' \circ \alpha_2 = t \circ \psi_1 + \phi'_2 \circ s$$

We want to complete the diagram (5.1) to a morphism of triangles. To do that we define $\gamma = (\gamma_1, \gamma_2) : C_\theta \to C_{\theta'}$ by letting

$$\gamma_1 = \begin{bmatrix} \alpha_2 & 0 \\ s & \beta_1 \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} \alpha_1 & 0 \\ t & \beta_2 \end{bmatrix}.$$

When we complete the diagram with $[\gamma]$, it commutes since $[\gamma \circ i_\theta] = [i_{\theta'} \circ \beta]$ and $[\pi_{\theta'} \circ \gamma] = [\Sigma\alpha \circ \pi_{\theta'}]$. Hence (TR3) holds.

(TR4) Assume we have the following diagram:

$$\begin{tikzcd}
(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\theta]}"] \arrow[d, equal] & (F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] \arrow[d, "{[\eta]}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[d, equal] \\
(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\eta\theta]}"] & (F_3, G_3, \phi_3, \psi_3) \arrow[r, "{[i_{\eta\theta}]}"] \arrow[d, "i_\eta"] & C_{\eta\theta} \arrow[r, "{[\pi_{\eta\theta}]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \\
 & C_\eta \arrow[d, "\pi_\eta"] & & \\
 & \Sigma B & &
\end{tikzcd}$$

where the two rows and the second column are in Δ. We want to find maps $\alpha : C_\theta \to C_{\eta\theta}$, $\beta : C_{\eta\theta} \to C_\eta$ and $\gamma : C_\eta \to \Sigma C_\theta$ such that the diagram

$$\begin{tikzcd}
(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\theta]}"] \arrow[d, equal] & (F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] \arrow[d, "{[\eta]}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] \arrow[d, "{[\alpha]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[d, equal] \\
(F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\eta\theta]}"] & (F_3, G_3, \phi_3, \psi_3) \arrow[r, "{[i_{\eta\theta}]}"] \arrow[d, "i_\eta"] & C_{\eta\theta} \arrow[r, "{[\pi_{\eta\theta}]}"] \arrow[d, "{[\beta]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \\
 & C_\eta \arrow[r, equal] \arrow[d, "\pi_\eta"] & C_\eta \arrow[d, "{[\gamma]}"] & \\
 & \Sigma B \arrow[r, "{\Sigma[i_\theta]}"] & \Sigma C_\theta &
\end{tikzcd} \tag{5.2}$$

commutes, the next to last column is in Δ and $[\pi_\eta \circ \beta] = [\Sigma\theta \circ \pi_{\eta\theta}]$. Let $\theta = (f, g)$ and $\eta = (u, v)$ and define

$$\begin{aligned} \alpha : C_\theta \to C_{\eta\theta}, &\quad \text{by} \quad \left(\begin{bmatrix} 1_{G_1} & 0 \\ 0 & u \end{bmatrix}, \begin{bmatrix} 1_{F_1} & 0 \\ 0 & v \end{bmatrix}\right) \\ \beta : C_{\eta\theta} \to C_\eta, &\quad \text{by} \quad \left(\begin{bmatrix} g & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} f & 0 \\ 0 & 1_{G_3} \end{bmatrix}\right) \\ \gamma : C_\eta \to \Sigma C_\theta, &\quad \text{by} \quad \left(\begin{bmatrix} 0 & 0 \\ 1_{G_2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1_{F_2} & 0 \end{bmatrix}\right), \quad \text{i.e} \quad \gamma = \Sigma i_\theta \circ \pi_\eta. \end{aligned}$$

These three morphisms make all the squares in (5.2) commute, and we also get

$$\pi_\eta \circ \beta = \left(\begin{bmatrix} 1_{G_2} & 0 \end{bmatrix} \begin{bmatrix} g & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 1_{F_2} & 0 \end{bmatrix} \begin{bmatrix} f & 0 \\ 0 & 1_{G_3} \end{bmatrix}\right) = \left(\begin{bmatrix} g & 0 \end{bmatrix}, \begin{bmatrix} f & 0 \end{bmatrix}\right) = \left(g \begin{bmatrix} 1_{G_1} & 0 \end{bmatrix}, f \begin{bmatrix} 1_{F_1} & 0 \end{bmatrix}\right) = \Sigma\theta \circ \pi_{\eta\theta}.$$

This means that all that remains is to prove that the next to last column in (5.2) is in Δ. To do this, we want to find an isomorphism

$$\begin{tikzcd}
C_\theta \arrow[r, "{[\alpha]}"] \arrow[d, equal] & C_{\eta\theta} \arrow[r, "{[\beta]}"] \arrow[d, equal] & C_\eta \arrow[r, "{[\gamma]}"] \arrow[d, dashed, shift right, "{[\sigma]}"'] & \Sigma C_\theta \arrow[d, equal] \\
C_\theta \arrow[r, "{[\alpha]}"] & C_{\eta\theta} \arrow[r, "{[i_\alpha]}"] & C_\alpha \arrow[r, "{[\pi_\alpha]}"] \arrow[u, dashed, shift right, "{[\tau]}"'] & \Sigma C_\theta
\end{tikzcd}$$

i.e we need to find $\sigma : C_\eta \to C_\alpha$ and $\tau : C_\alpha \to C_\eta$ to complete the isomorphism of triangles. We let

$$\sigma = \left(\begin{bmatrix} 0 & 0 \\ 1_{G_2} & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1_{F_2} & 0 \\ 0 & 0 \\ 0 & 1_{G_3} \end{bmatrix}\right), \qquad \tau = \left(\begin{bmatrix} 0 & 1_{G_2} & g & 0 \\ 0 & 0 & 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 0 & 1_{G_2} & g & 0 \\ 0 & 0 & 0 & 1_{F_3} \end{bmatrix}\right).$$

To see if these morphisms gives us the commutativity, we first look at the diagram

$$\begin{tikzcd}[ampersand replacement=\&]
C_{\eta\theta} \arrow[d, "i_\alpha"] \& G_1 \oplus F_3 \arrow[r, "{\begin{bmatrix} -\psi_1 \& 0 \\ g \& \phi_3 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 \& 0 \\ 0 \& 0 \\ 1_{G_1} \& 0 \\ 0 \& 1_{F_3} \end{bmatrix}}"] \& F_1 \oplus G_3 \arrow[r, "{\begin{bmatrix} -\phi_1 \& 0 \\ f \& \psi_3 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 \& 0 \\ 0 \& 0 \\ 1_{F_1} \& 0 \\ 0 \& 1_{G_3} \end{bmatrix}}"] \& G_1 \oplus F_3 \arrow[d, "{\begin{bmatrix} 0 \& 0 \\ 0 \& 0 \\ 1_{G_1} \& 0 \\ 0 \& 1_{F_3} \end{bmatrix}}"] \\
C_\alpha \arrow[d, "\tau"] \& F_1 \oplus G_2 \oplus G_1 \oplus F_3 \arrow[r, "N_1"] \arrow[d, "{\begin{bmatrix} 0 \& 1_{G_2} \& g \& 0 \\ 0 \& 0 \& 0 \& 1_{F_3} \end{bmatrix}}"] \& G_1 \oplus F_2 \oplus F_1 \oplus G_3 \arrow[r, "N_2"] \arrow[d, "{\begin{bmatrix} 0 \& 1_{F_2} \& g \& 0 \\ 0 \& 0 \& 0 \& 1_{G_3} \end{bmatrix}}"] \& F_1 \oplus G_2 \oplus G_1 \oplus F_3 \arrow[d, "{\begin{bmatrix} 0 \& 1_{G_2} \& g \& 0 \\ 0 \& 0 \& 0 \& 1_{F_3} \end{bmatrix}}"] \\
C_\eta \& G_2 \oplus F_3 \arrow[r, "{\begin{bmatrix} -\psi_2 \& 0 \\ g \& \phi_3 \end{bmatrix}}"'] \& F_2 \oplus G_3 \arrow[r, "{\begin{bmatrix} -\phi_2 \& 0 \\ f \& \psi_3 \end{bmatrix}}"'] \& G_2 \oplus F_3,
\end{tikzcd}$$

where $N_1 = \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ -f & -\psi_2 & 0 & 0 \\ 1_{F_1} & 0 & -\psi_1 & 0 \\ 0 & u & g & \phi_2 \end{bmatrix}$ and $N_2 = \begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ -g & -\phi_2 & 0 & 0 \\ 1_{G_1} & 0 & -\phi_1 & 0 \\ 0 & v & f & \psi_2 \end{bmatrix}$, and see that

$$\tau \circ i_\alpha = \left(\begin{bmatrix} g & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} f & 0 \\ 0 & 1_{G_3} \end{bmatrix}\right) = \beta.$$

Next we see from

$$\begin{tikzcd}
C_\eta \arrow[d, "\sigma"] & G_2 \oplus F_3 \arrow[r, "{\left[\begin{smallmatrix} -\psi_2 & 0 \\ g & \psi_3 \end{smallmatrix}\right]}"] \arrow[d, "{\begin{bmatrix} 0 & 0 \\ 1_{G_2} & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix}}"] & F_2 \oplus G_3 \arrow[r, "{\left[\begin{smallmatrix} -\phi_2 & 0 \\ f & \psi_3 \end{smallmatrix}\right]}"] \arrow[d, "{\begin{bmatrix} 0 & 0 \\ 1_{F_2} & 0 \\ 0 & 0 \\ 0 & 1_{G_3} \end{bmatrix}}"] & G_2 \oplus F_3 \arrow[d, "{\begin{bmatrix} 0 & 0 \\ 1_{G_2} & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix}}"] \\
C_\alpha \arrow[d, "\pi_\alpha"] & F_1 \oplus G_2 \oplus G_1 \oplus F_3 \arrow[r, "N_1"] \arrow[d, "{\begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 1_{G_2} & 0 & 0 \end{bmatrix}}"] & G_1 \oplus F_2 \oplus F_1 \oplus G_3 \arrow[r, "N_2"] \arrow[d, "{\begin{bmatrix} 1_{G_1} & 0 & 0 & 0 \\ 0 & 1_{F_2} & 0 & 0 \end{bmatrix}}"] & F_1 \oplus G_2 \oplus G_1 \oplus F_3 \arrow[d, "{\begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 1_{G_2} & 0 & 0 \end{bmatrix}}"] \\
\Sigma C_\theta & F_1 \oplus G_2 \arrow[r, "{\left[\begin{smallmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{smallmatrix}\right]}"] & G_1 \oplus F_2 \arrow[r, "{\left[\begin{smallmatrix} -\phi_1 & 0 \\ f & \psi_2 \end{smallmatrix}\right]}"] & F_1 \oplus G_2
\end{tikzcd}$$

that $\pi_\alpha \circ \sigma = (\left[\begin{smallmatrix} 0 & 0 \\ 1_{G_2} & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1_{F_2} & 0 \end{smallmatrix}\right]) = \gamma$.

The last two commutativity relations only hold up to homotopy. For the first one we claim that $i_\alpha - \sigma \circ \beta$ is homotopic to zero, and for the second we claim $\pi_\alpha - \gamma \circ \tau$ is also homotopic to zero. This gives us $[i_\alpha] = [\sigma \circ \beta]$ and $[\pi_\alpha] = [\gamma \circ \tau]$ which is what is needed for the diagram to commute. First we note that $i_\alpha - \sigma \circ \beta = (\begin{bmatrix} 0 & 0 \\ 1_{G_1} & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ g & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1_{F_1} & 0 \\ 0 & 0 \\ 0 & 1_{G_3} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ f & 0 \\ 0 & 0 \\ 0 & 1_{G_3} \end{bmatrix}) = (\begin{bmatrix} 0 & 0 \\ -g & 0 \\ 1_{G_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -f & 0 \\ 1_{F_1} & 0 \\ 0 & 0 \end{bmatrix})$. With the homotopy maps $s = t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ we get

$$\begin{bmatrix} 0 & 0 \\ -g & 0 \\ 1_{G_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\psi_1 & 0 \\ g & \phi_2 \end{bmatrix} + \begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ -g & -\phi_2 & 0 & 0 \\ 1_{G_1} & 0 & -\phi_1 & 0 \\ 0 & v & f & \psi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\psi_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \psi_1 & 0 \\ -g & 0 \\ 1_{G_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -g & 0 \\ 1_{G_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The same holds for $\begin{bmatrix} 0 & 0 \\ -f & 0 \\ 1_{F_1} & 0 \\ 0 & 0 \end{bmatrix}$ and we have $[i_\alpha] = [\sigma \circ \beta]$. Next we note that $\pi_\alpha - \gamma \circ \tau = (\begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 1_{G_2} & 0 & 0 \end{bmatrix} - \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{G_2} & g & 0 \end{smallmatrix}\right], \begin{bmatrix} 1_{G_1} & 0 & 0 & 0 \\ 0 & 1_{F_2} & 0 & 0 \end{bmatrix} - \left[\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{F_2} & f & 0 \end{smallmatrix}\right]) = (\begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 0 & -g & 0 \end{bmatrix}, \begin{bmatrix} 1_{G_1} & 0 & 0 & 0 \\ 0 & 0 & -f & 0 \end{bmatrix})$. With the homotopy maps $s = t = \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ we get

$$\begin{aligned} \begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 0 & -g & 0 \end{bmatrix} &= \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ -f & -\psi_2 & 0 & 0 \\ 1_{F_1} & 0 & -\psi_1 & 0 \\ 0 & u & g & \phi_2 \end{bmatrix} + \left[\begin{smallmatrix} \psi_1 & 0 \\ -g & -\phi_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} 1_{F_1} & 0 & -\psi_1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 & \psi_1 & 0 \\ 0 & 0 & -g & 0 \end{smallmatrix}\right] = \begin{bmatrix} 1_{F_1} & 0 & 0 & 0 \\ 0 & 0 & -g & 0 \end{bmatrix}. \end{aligned}$$

The same holds for $\begin{bmatrix} 1_{G_1} & 0 & 0 & 0 \\ 0 & 0 & -f & 0 \end{bmatrix}$ and we have $[\pi_\alpha] = [\gamma \circ \tau]$ which means the diagram commutes and $[\sigma]$ and $[\tau]$ are morphisms of triangles.

What remains now is to check that they are in fact isomorphisms of triangles. We see that $\tau \circ \sigma = (\begin{bmatrix} 0 & 1_{G_2} & g & 0 \\ 0 & 0 & 0 & 1_{F_3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1_{G_2} & 0 \\ 0 & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 0 & 1_{F_2} & g & 0 \\ 0 & 0 & 0 & 1_{G_3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1_{F_2} & 0 \\ 0 & 0 \\ 0 & 1_{G_3} \end{bmatrix}) = (\begin{bmatrix} 1_{G_2} & 0 \\ 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 1_{F_2} & 0 \\ 0 & 1_{G_3} \end{bmatrix}) = \mathrm{id}_{C_\eta}$. The composition $\sigma \circ \tau$ is given by

$$(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{G_2} & g & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{F_3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{F_2} & f & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{G_3} \end{bmatrix}).$$

If we define the homotopy maps $s = t = \left[\begin{smallmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$ we get

$$\begin{aligned} \begin{bmatrix} -1_{F_1} & 0 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -1_{G_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &= \left[\begin{smallmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] \begin{bmatrix} \phi_1 & 0 & 0 & 0 \\ -f & -\psi_2 & 0 & 0 \\ 1_{F_1} & 0 & -\psi_1 & 0 \\ 0 & u & g & \phi_2 \end{bmatrix} + \begin{bmatrix} \psi_1 & 0 & 0 & 0 \\ -g & -\phi_2 & 0 & 0 \\ 1_{G_1} & 0 & -\phi_1 & 0 \\ 0 & v & f & \psi_2 \end{bmatrix} \left[\begin{smallmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] \\ &= \begin{bmatrix} -1_{F_1} & 0 & \psi_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\psi_1 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -1_{G_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1_{F_1} & 0 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -1_{G_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

which means $\sigma \circ \tau - \mathrm{id}_{C_\alpha}$ is homotopic to zero, i.e. $[\sigma \circ \tau] = \mathrm{id}_{C_\alpha}$ in HMF(S, x). This means both $[\sigma]$ and $[\tau]$ are isomorphisms and hence we have proved the octahedral axiom for HMF(S, x), which means HMF(S, x) is a triangulated category. □

In Chapter 4 we introduced Neeman's mapping cone axiom and saw that we can use this instead of the octahedral axiom. This means we could have used (TR4') in the proof above:

Proof. (TR4') Assume we have a commutative diagram

$$\begin{tikzcd} (F_1, G_1, \phi_1, \psi_1) \arrow[r, "{[\theta]}"] \arrow[d, "{[\alpha]}"] & (F_2, G_2, \phi_2, \psi_2) \arrow[r, "{[i_\theta]}"] \arrow[d, "{[\beta]}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] & \Sigma(F_1, G_1, \phi_1, \psi_1) \arrow[d, "{\Sigma[\alpha]}"] \\ (F_1', G_1', \phi_1', \psi_1') \arrow[r, "{[\theta']}"] & (F_2', G_2', \phi_2', \psi_2') \arrow[r, "{[i_{\theta'}]}"] & C_{\theta'} \arrow[r, "{[\pi_{\theta'}]}"] & \Sigma(F_1', G_1', \phi_1', \psi_1') \end{tikzcd}$$

as in (TR3). We want to complete this to a morphism of triangles in such a way that the mapping cone is in Δ. To simplify notation we let $A = (F_1, G_1, \phi_1, \psi_1), A' = (F_1', G_1', \phi_1', \psi_1'), B = (F_2, G_2, \phi_2, \psi_2)$ and $B = (F_2', G_2', \phi_2', \psi_2')$ so we get the diagram

$$\begin{tikzcd} A \arrow[r, "{[\theta]}"] \arrow[d, "{[\alpha]}"] & B \arrow[r, "{[i_\theta]}"] \arrow[d, "{[\beta]}"] & C_\theta \arrow[r, "{[\pi_\theta]}"] & \Sigma A \arrow[d, "{\Sigma[\alpha]}"] \\ A' \arrow[r, "{[\theta']}"] & B' \arrow[r, "{[i_{\theta'}]}"] & C_{\theta'} \arrow[r, "{[\pi_{\theta'}]}"] & \Sigma A' \end{tikzcd} \tag{5.3}$$

Let $[\gamma] : C_\theta \to C_{\theta'}$ be defined by $\gamma = \left[\begin{smallmatrix} \alpha_2 & 0 \\ s & \beta_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} \alpha_1 & 0 \\ t & \beta_2 \end{smallmatrix}\right]$. We saw in the proof of (TR3) that this is a morphism of matrix factorizations and that it completes (5.3) to a morphism of triangles. Now, the mapping cone of (5.3) is given by

$$B \oplus A' \xrightarrow{\left[\begin{smallmatrix} -i_\theta & 0 \\ \beta & \theta' \end{smallmatrix}\right]} C_\theta \oplus B' \xrightarrow{\left[\begin{smallmatrix} -\pi_\theta & 0 \\ \gamma & i_{\theta'} \end{smallmatrix}\right]} \Sigma A \oplus C_{\theta'} \xrightarrow{\left[\begin{smallmatrix} -\Sigma\theta & 0 \\ \Sigma\alpha & \pi_{\theta'} \end{smallmatrix}\right]} \Sigma B \oplus \Sigma A'.$$

To prove that this is in Δ we need to show that it is isomorphic to a standard triangle. If we let $\sigma = \left[\begin{smallmatrix} -i_\theta & 0 \\ \beta & \theta' \end{smallmatrix}\right]$, $\delta = \left[\begin{smallmatrix} -\pi_\theta & 0 \\ \gamma & i_{\theta'} \end{smallmatrix}\right]$ and $\varepsilon = \left[\begin{smallmatrix} -\Sigma\theta & 0 \\ \Sigma\alpha & \pi_{\theta'} \end{smallmatrix}\right]$ we can look at the diagram

$$\begin{tikzcd} B \oplus A' \arrow[r, "{[\sigma]}"] \arrow[d, equal] & C_\theta \oplus B' \arrow[r, "{[\delta]}"] \arrow[d, equal] & \Sigma A \oplus C_{\theta'} \arrow[r, "{[\varepsilon]}"] \arrow[d, dashed, "{[\tau]}"] & \Sigma B \oplus \Sigma A' \arrow[d, equal] \\ B \oplus A' \arrow[r, "{[\sigma]}"] & C_\theta \oplus B' \arrow[r, "{[i_\sigma]}"] & C_\sigma \arrow[r, "{[\pi_\sigma]}"] & \Sigma B \oplus \Sigma A'. \end{tikzcd} \tag{5.4}$$

Here we have three new elements:

- C_σ :

$$G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2 \xrightarrow{A} F_2 \oplus F'_1 \oplus F_1 \oplus G_2 \oplus G'_2 \xrightarrow{B} G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2$$

where $A = \begin{bmatrix} -\psi_2 & 0 & 0 & 0 & 0 \\ 0 & -\psi'_1 & 0 & 0 & 0 \\ 0 & 0 & -\psi_1 & 0 & 0 \\ -1 & 0 & g & \phi_2 & 0 \\ \beta_2 & g' & 0 & 0 & \phi'_2 \end{bmatrix}$ and $B = \begin{bmatrix} -\phi_2 & 0 & 0 & 0 & 0 \\ 0 & -\phi'_1 & 0 & 0 & 0 \\ 0 & 0 & -\phi_1 & 0 & 0 \\ -1 & 0 & f & \psi_2 & 0 \\ \beta_1 & f' & 0 & 0 & \psi'_2 \end{bmatrix}$

- $i_\sigma : C_\theta \oplus B' \to C_\sigma$

$$\begin{tikzcd}
G_1 \oplus F_2 \oplus F'_2 \arrow[r, "{\begin{bmatrix} -\psi_1 & 0 & 0 \\ g & \phi_2 & 0 \\ 0 & 0 & \phi'_2 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}}"] & F_1 \oplus G_2 \oplus G'_2 \arrow[r, "{\begin{bmatrix} -\phi_1 & 0 & 0 \\ f & \psi_2 & 0 \\ 0 & 0 & \psi'_2 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}}"] & G_1 \oplus F_2 \oplus F'_2 \arrow[d, "{\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}}"] \\
G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2 \arrow[r, "A"] & F_2 \oplus F'_1 \oplus F_1 \oplus G_2 \oplus G'_2 \arrow[r, "B"] & G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2
\end{tikzcd}$$

and

- $\pi_\sigma : C_\sigma \to \Sigma B \oplus \Sigma A'$

$$\begin{tikzcd}
G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2 \arrow[r, "A"] \arrow[d, "{\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}}"] & F_2 \oplus F'_1 \oplus F_1 \oplus G_2 \oplus G'_2 \arrow[r, "B"] \arrow[d, "{\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}}"] & G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2 \arrow[d, "{\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}}"] \\
G_2 \oplus G'_1 \arrow[r, "{\begin{bmatrix} -\psi_2 & 0 \\ 0 & -\psi'_1 \end{bmatrix}}"] & F_2 \oplus F'_1 \arrow[r, "{\begin{bmatrix} -\phi_2 & 0 \\ 0 & -\phi'_1 \end{bmatrix}}"] & G_2 \oplus G'_1
\end{tikzcd}$$

First we need to find $\tau : \Sigma a \oplus C_{\theta'} \to C_\sigma$ such that (5.4) commutes. We have

$$\begin{tikzcd}
G_1 \oplus G'_1 \oplus F'_2 \arrow[r, "{\begin{bmatrix} -\psi_1 & 0 & 0 \\ 0 & -\psi'_1 & 0 \\ 0 & g' & \phi'_2 \end{bmatrix}}"] \arrow[d, "\tau_1"] & F_1 \oplus F'_1 \oplus G'_2 \arrow[r, "{\begin{bmatrix} -\phi_1 & 0 & 0 \\ 0 & -\phi'_1 & 0 \\ 0 & f' & \psi'_2 \end{bmatrix}}"] \arrow[d, "\tau_2"] & G_1 \oplus G'_1 \oplus F'_2 \arrow[d, "\tau_1"] \\
G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2 \arrow[r, "A"] & F_2 \oplus F'_1 \oplus F_1 \oplus G_2 \oplus G'_2 \arrow[r, "B"] & G_2 \oplus G'_1 \oplus G_1 \oplus F_2 \oplus F'_2
\end{tikzcd}$$

where we define

$$\tau_1 = \begin{bmatrix} -g & 0 & 0 \\ \alpha_2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tau_2 = \begin{bmatrix} -f & 0 & 0 \\ \alpha_1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 1 \end{bmatrix}.$$

Now,

$$A \circ \tau_1 = \begin{bmatrix} \psi_2 \circ g & 0 & 0 \\ -\psi'_1 \circ \alpha_2 & -\psi'_1 & 0 \\ \psi_1 & 0 & 0 \\ g - g & 0 & 0 \\ -\beta_2 \circ g + g' \circ \alpha_2 + \phi'_2 \circ s & g' & \phi_2 \end{bmatrix} = \begin{bmatrix} f \circ \psi_1 & 0 & 0 \\ -\alpha_1 \circ \psi_1 & -\psi'_1 & 0 \\ \phi_1 & 0 & 0 \\ 0 & 0 & 0 \\ -t \circ \psi_1 & g' & \phi_2 \end{bmatrix} = \tau_2 \circ \begin{bmatrix} -\psi_1 & 0 & 0 \\ 0 & -\psi'_1 & 0 \\ 0 & g' & \phi'_2 \end{bmatrix}$$

and

$$B \circ \tau_2 = \begin{bmatrix} \phi_2 \circ f & 0 & 0 \\ -\phi'_1 \circ \alpha_1 & -\phi'_1 & 0 \\ \phi_1 & 0 & 0 \\ f - f & 0 & 0 \\ -\beta_1 \circ f + f' \circ \alpha_1 + \psi'_2 \circ t & f' & \psi_2 \end{bmatrix} = \begin{bmatrix} g \circ \phi_1 & 0 & 0 \\ -\alpha_2 \circ \phi_1 & -\phi'_1 & 0 \\ \phi_1 & 0 & 0 \\ 0 & 0 & 0 \\ -s \circ \phi_1 & f' & \psi_2 \end{bmatrix} = \tau_1 \circ \begin{bmatrix} -\phi_1 & 0 & 0 \\ 0 & -\phi'_1 & 0 \\ 0 & f' & \psi'_2 \end{bmatrix}$$

which means τ is a morphism of matrix factorizations. Next we need to show that τ completes (5.4) to a morphism of triangles, i.e that $[\pi_\sigma]\circ[\tau]=[\varepsilon]$ and $[\tau]\circ[\delta]=[i_\sigma]$. First we see that

$$\pi_\sigma\circ\tau=\left(\left[\begin{smallmatrix}1&0&0&0&0\\0&1&0&0&0\end{smallmatrix}\right]\begin{bmatrix}-g&0&0\\\alpha_2&1&0\\-1&0&0\\0&0&0\\s&0&1\end{bmatrix},\left[\begin{smallmatrix}1&0&0&0&0\\0&1&0&0&0\end{smallmatrix}\right]\begin{bmatrix}-f&0&0\\\alpha_1&1&0\\-1&0&0\\0&0&0\\t&0&1\end{bmatrix}\right)=\left(\left[\begin{smallmatrix}-g&0&0\\\alpha_2&1&0\end{smallmatrix}\right],\left[\begin{smallmatrix}-f&0&0\\\alpha_1&1&0\end{smallmatrix}\right]\right)=\varepsilon$$

so the first relation holds. Next we look at $\tau\circ\delta$.

$$\tau\circ\delta=\left(\begin{bmatrix}-g&0&0\\\alpha_2&1&0\\-1&0&0\\0&0&0\\s&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\\alpha_2&0&0\\s&\beta_1&1\end{bmatrix},\begin{bmatrix}-f&0&0\\\alpha_1&1&0\\-1&0&0\\0&0&0\\t&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\\alpha_1&0&0\\t&\beta_2&1\end{bmatrix}\right)=\left(\begin{bmatrix}g&0&0\\-\alpha_2+\alpha_2&0&0\\1&0&0\\0&0&0\\-s+s&\beta_1&1\end{bmatrix},\begin{bmatrix}f&0&0\\-\alpha_1+\alpha_1&0&0\\1&0&0\\0&0&0\\-t+t&\beta_2&1\end{bmatrix}\right)\neq i_\sigma$$

This means we need to find homotopy maps k and l in the diagram

$$\begin{tikzcd}
G_1\oplus F_2\oplus F_2' \arrow[r, "{\begin{bmatrix}-\psi_1&0&0\\g&\phi_2&0\\0&0&\phi_2'\end{bmatrix}}"] \arrow[d, "{\begin{bmatrix}g&0&0\\0&0&0\\1&0&0\\0&0&0\\0&\beta_1&1\end{bmatrix}}"', "{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}}"] & F_1\oplus G_2\oplus G_2' \arrow[r, "{\begin{bmatrix}-\phi_1&0&0\\f&\psi_2&0\\0&0&\psi_2'\end{bmatrix}}"] \arrow[dl, "k"] \arrow[d, "{\begin{bmatrix}f&0&0\\0&0&0\\1&0&0\\0&0&0\\0&\beta_2&1\end{bmatrix}}"', "{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}}"] & G_1\oplus F_2\oplus F_2' \arrow[dl, "l"] \arrow[d, "{\begin{bmatrix}g&0&0\\0&0&0\\1&0&0\\0&0&0\\0&\beta_1&1\end{bmatrix}}"', "{\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}}"] \\
G_2\oplus G_1'\oplus G_1\oplus F_2\oplus F_2' \arrow[r, "A"] & F_2\oplus F_1'\oplus F_1\oplus G_2\oplus G_2' \arrow[r, "B"] & G_2\oplus G_1'\oplus G_1\oplus F_2\oplus F_2'
\end{tikzcd}$$

such that

$$\begin{aligned}\begin{bmatrix}g&0&0\\0&0&0\\0&0&0\\0&-1&0\\0&\beta_1&0\end{bmatrix}&=k\circ\begin{bmatrix}-\psi_1&0&0\\g&\phi_2&0\\0&0&\phi_2'\end{bmatrix}+B\circ l\\\begin{bmatrix}f&0&0\\0&0&0\\0&0&0\\0&-1&0\\0&\beta_2&0\end{bmatrix}&=l\circ\begin{bmatrix}-\phi_1&0&0\\f&\psi_2&0\\0&0&\psi_2'\end{bmatrix}+A\circ k.\end{aligned}$$

This holds for $k=l=\left[\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right]$ which means $[\tau]\circ[\delta]=[i_\sigma]$ and hence τ completes (5.4) to a morphism of triangles.

What remains now is to show that τ is an isomorphism, i.e. that it has an inverse. To do this we we define $\omega: C_\sigma\to\Sigma A\oplus C_{\theta'}$,

$$\begin{tikzcd}
G_2\oplus G_1'\oplus G_1\oplus F_2\oplus F_2' \arrow[r, "A"] \arrow[d, "\omega_1"] & F_2\oplus F_1'\oplus F_1\oplus G_2\oplus G_2' \arrow[r, "B"] \arrow[d, "\omega_2"] & G_2\oplus G_1'\oplus G_1\oplus F_2\oplus F_2' \arrow[d, "\omega_1"] \\
G_1\oplus G_1'\oplus F_2' \arrow[r, "{\begin{bmatrix}-\psi_1&0&0\\0&-\psi_1'&0\\0&g'&\phi_2'\end{bmatrix}}"] & F_1\oplus F_1'\oplus G_2' \arrow[r, "{\begin{bmatrix}-\phi_1&0&0\\0&-\phi_1'&0\\0&f'&\psi_2'\end{bmatrix}}"] & G_1\oplus G_1'\oplus F_2',
\end{tikzcd} \tag{5.5}$$

by

$$\omega=\left(\begin{bmatrix}0&0&-1&0&0\\0&1&\alpha_2&0&0\\0&0&s&\beta_1&1\end{bmatrix},\begin{bmatrix}0&0&-1&0&0\\0&1&\alpha_1&0&0\\0&0&t&\beta_2&1\end{bmatrix}\right).$$

This makes (5.5) commute since

$$\begin{bmatrix}-\psi_1&0&0\\0&-\psi_1'&0\\0&g'&\phi_2'\end{bmatrix}\circ\omega_1=\begin{bmatrix}0&0&\psi_1&0&0\\0&-\psi_1'&-\psi_1'\circ\alpha_2&0&0\\0&g'&g'\circ\alpha_2+\phi_2'\circ s&\phi_2'\circ\beta_1&\phi_2'\end{bmatrix}=\begin{bmatrix}0&0&\psi_1&0&0\\0&-\psi_1'&-\alpha_1\circ\psi_1&0&0\\-\beta_2+\beta_2&g'&-t\circ\psi_1+\beta_2\circ g&\beta_2\circ\phi_2&\phi_2'\end{bmatrix}=\omega_2\circ A$$

and

$$\begin{bmatrix} -\phi_1 & 0 & 0 \\ 0 & -\phi_1' & 0 \\ 0 & f' & \psi_2' \end{bmatrix} \circ \omega_2 = \begin{bmatrix} 0 & 0 & \phi_1 & 0 & 0 \\ 0 & -\phi_1' & -\phi_1' \circ \alpha_1 & 0 & 0 \\ 0 & f' & g' \circ \alpha_1 + \psi_2' \circ t & \psi_2' \circ \beta_2 & \psi_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \phi_1 & 0 & 0 \\ 0 & -\phi_1' & -\alpha_2 \circ \phi_1 & 0 & 0 \\ -\beta_1 + \beta_1 & f' & -s \circ \phi_1 + \beta_1 \circ f & \beta_1 \circ \psi_2 & \psi_2' \end{bmatrix} = \omega_1 \circ B.$$

To see that $[\omega]$ is the inverse of $[\tau]$ we first look at $\omega \circ \tau$:

$$\omega \circ \tau = \left(\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & \alpha_2 & 0 & 0 \\ 0 & 0 & s & \beta_1 & 1 \end{bmatrix} \begin{bmatrix} -g & 0 & 0 \\ \alpha_2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & t & \beta_2 & 1 \end{bmatrix} \begin{bmatrix} -f & 0 & 0 \\ \alpha_1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 1_{\Sigma A \oplus C_{\theta'}}$$

so $[\omega] \circ [\tau] = \left[1_{\Sigma A \oplus C_{\theta'}} \right]$. Next we see that

$$\tau \circ \omega = \left(\begin{bmatrix} -g & 0 & 0 \\ \alpha_2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ s & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & \alpha_2 & 0 & 0 \\ 0 & 0 & s & \beta_1 & 1 \end{bmatrix}, \begin{bmatrix} -f & 0 & 0 \\ \alpha_1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & t & \beta_2 & 1 \end{bmatrix} \right) = \left(\begin{bmatrix} 0 & 0 & g & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & f & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & 1 \end{bmatrix} \right)$$

which means we need to find maps m and n in the diagram

$$\begin{tikzcd}
G_2 \oplus G_1' \oplus G_1 \oplus F_2 \oplus F_2' \arrow[r, "A"] \arrow[d, "{\begin{bmatrix} -1 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \beta_1 & 0 \end{bmatrix}}"] & F_2 \oplus F_1' \oplus F_1 \oplus G_2 \oplus G_2' \arrow[r, "B"] \arrow[dl, "m"] \arrow[d, "{\begin{bmatrix} -1 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 \end{bmatrix}}"] & G_2 \oplus G_1' \oplus G_1 \oplus F_2 \oplus F_2' \arrow[dl, "n"] \arrow[d, "{\begin{bmatrix} -1 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \beta_1 & 0 \end{bmatrix}}"] \\
G_2 \oplus G_1' \oplus G_1 \oplus F_2 \oplus F_2' \arrow[r, "A"] & F_2 \oplus F_1' \oplus F_1 \oplus G_2 \oplus G_2' \arrow[r, "B"] & G_2 \oplus G_1' \oplus G_1 \oplus F_2 \oplus F_2'
\end{tikzcd}$$

such that

$$\begin{bmatrix} -1 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \beta_1 & 0 \end{bmatrix} = m \circ A + B \circ n \quad \text{and} \quad \begin{bmatrix} -1 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 \end{bmatrix} = n \circ B + A \circ m$$

We get this if we let $m = n = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and hence $[\tau] \circ [\omega] = [1_{C_\sigma}]$ and τ is an isomorphism, which means (TR4') holds. ☐

5.2 Equivalence with the homotopy category of totally acyclic complexes

An important result concerning $\mathbf{HMF}(S, x)$ has to do with long exact sequences or acyclic complexes. More precisely the homotopy category where the objects are acyclic complexes $\mathbb{P} : ... \to P_1 \to P_0 \to P_{-1} \to ...$ where the P_i are finitely generated free R-modules for the commutative ring $R = S/(x)$. We denote this category $\mathbf{K_{ac}}(R)$. This is also a triangulated category. With $\Sigma : \mathbf{K_{ac}}(R) \to \mathbf{K_{ac}}(R)$ defined as a shift to the left, i.e. the usual shifting of complexes, and the distinguished triangles the triangles isomorphic to standard triangles using mapping cones, we can use the same proof as with $\mathbf{HMF}(S, x)$ with only small adjustments.

Before we present the result we need some preliminaries.

Definition. Let R be a ring and let $x \in R$ be different from zero. Then x is a *non-zerodivisor* if for any $y \in R$

$$xy = 0 \Rightarrow y = 0.$$

Instead of looking at the whole of $\mathbf{K_{ac}}(R)$ we will look at a subcategory:

Definition. Let R be a commutative ring. A complex $\mathbb{P}$ of finitely generated projective R-modules is *totally acyclic* if both $\mathbb{P}$ and $\mathbf{Hom}_R(\mathbb{P}, R)$ are acyclic. These complexes form a homotopy category $\mathbf{K_{tac}}(R)$, which is a triangulated subcategory of $\mathbf{K_{ac}}$.

The fact that some complexes are acyclic but not totally acyclic is studied by Jorgensen and Şega in [14] and by Iyengar and Krause in [12].

Lemma 5.2.1. *Let R be a commutative ring and*

$$\begin{tikzcd}
\dots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] \arrow[dl, "s_0"'] & F_{-1} \arrow[r, "f_{-1}"] \arrow[d, "\phi_{-1}"] \arrow[dl, "s_{-1}"'] & F_{-2} \arrow[r] \arrow[d, "\phi_{-2}"] & \dots \\
\dots \arrow[r] & G_2 \arrow[r, "g_2"] & G_1 \arrow[r, "g_1"] & G_0 \arrow[r, "g_0"] & G_{-1} \arrow[r, "g_{-1}"] & G_{-2} \arrow[r] & \dots
\end{tikzcd}$$

a diagram in which the rows are totally acyclic complexes of free R-modules. Moreover, suppose that $\phi = (..., \phi_1, \phi_0, \phi_{-1}, ...)$ is a chain map and that the two diagonal maps s_0 and s_{-1} satisfy

$$\phi_0 = g_1 \circ s_0 + s_{-1} \circ f_0.$$

Then s_0 and s_{-1} can be completed to a nullhomotopy $s = (..., s_1, s_0, s_{-1}, ...)$ on ϕ.

Proof. To complete the nullhomotopy on ϕ we need to find maps s_i such that $\phi_i = g_{i+1} \circ s_i + s_{i-1} \circ f_i$. Let $\theta : F_1 \to G_1$ be defined by $\theta = \phi_1 - s_0 \circ f_1$. Since

$$\begin{aligned}
g_1 \circ \theta &= g_1 \circ \phi_1 - g_1 \circ s_0 \circ f_1 \\
&= \phi_0 \circ f_1 - g_1 \circ s_0 \circ f_1 \\
&= (g_1 \circ s_0 + s_{-1} \circ f_0) \circ f_1 - g_1 \circ s_0 \circ f_1 \\
&= g_1 \circ s_0 \circ f_1 + s_{-1} \circ f_0 \circ f_1 - g_1 \circ s_0 \circ f_1 = 0
\end{aligned}$$

we have $\operatorname{Im} \theta \subseteq \operatorname{Ker} g_1 = \operatorname{Im} g_2$. This means there exists an $s_1 : F_1 \to G_2$ in

$$\begin{tikzcd}
 & F_1 \arrow[dl, dashed, "s_1"'] \arrow[d, "\theta"] & \\
G_2 \arrow[r, "g_2"] & \operatorname{Im} g_2 \arrow[r] & 0
\end{tikzcd}$$

with $\theta = g_2 \circ s_1$ which means $\phi_1 = s_0 \circ f_1 + g_1 \circ s_1$. In this way we can find $s_2, s_3, ...$ and so on, i.e. all the maps to the left of s_0.

Now we need to find the maps to the right. Apply $\operatorname{Hom}_R(-, R)$ and write $(-)^*$ for $\operatorname{Hom}_R(-, R)$. Then we get the diagram

$$\begin{tikzcd}
\dots \arrow[r] & G^*_{-3} \arrow[r, "g^*_{-2}"] \arrow[d, "\phi^*_{-3}"] & G^*_{-2} \arrow[r, "g^*_{-1}"] \arrow[d, "\phi^*_{-2}"] & G^*_{-1} \arrow[r, "g^*_0"] \arrow[d, "\phi^*_{-1}"] & G^*_0 \arrow[r, "g^*_1"] \arrow[d, "\phi^*_0"] \arrow[dl, "s^*_{-1}"'] & G^*_1 \arrow[r] \arrow[d, "\phi^*_1"] \arrow[dl, "s^*_0"'] & \dots \\
\dots \arrow[r] & F^*_{-3} \arrow[r, "f^*_{-2}"] & F^*_{-2} \arrow[r, "f^*_{-1}"] & F^*_{-1} \arrow[r, "f^*_0"] & F^*_0 \arrow[r, "f^*_1"] & F^*_1 \arrow[r] & \dots
\end{tikzcd}$$

Here we have

$$\phi^*_0 = (g_1 \circ s_0)^* + (s_{-1} \circ f_0)^* = s^*_0 \circ g^*_1 + f^*_0 \circ s^*_{-1}$$

We know that for a free R-module Q, Q^* is also free, so we can use the same method as above to find t_{-2}, t_{-3}, t_{-4} and so on. We apply $\operatorname{Hom}_R(-, R)$ on this new diagram and since $(-)^{**}$ is an equivalence on the category of free modules we get the original diagram. Let $s_i = t_i^*$ for $i = -2, -3, -4, ...$, then we have a nullhomotopy

$$(..., s_{-3}, s_{-2}, s_{-1}, s_0, s_1, ...)$$

which is what we wanted. $\square$

Now we can present the first result.

Theorem 5.2.2. *Let S be a commutative ring, $x \in S$ a non-zerodivisor and R the factor ring $R = S/(x)$. To a matrix factorization*

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

in $\mathbf{MF}(S, x)$ we assign the sequence

$$... \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} ...$$

which is a complex of free R-modules, and for morphisms in $\mathbf{MF}(S, x)$ we assign the obvious morphisms of complexes. This induces a fully faithful triangle functor

$$T : \mathbf{HMF}(S, x) \to \mathbf{K_{tac}}(R).$$

Proof. We need to prove three things: that T is a triangle functor, that it is faithful and that it is full. We follow the proof in [5] and begin with showing that T is a triangle functor.

We start by reducing a matrix factorization (F, G, ϕ, ψ) modulo x and get

$$F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF.$$

This is exact: Firstly we have $\overline{\psi} \circ \overline{\phi} = x \cdot 1_F = 0$ so $\operatorname{Im} \overline{\phi} \subset \operatorname{Ker} \overline{\psi}$. Now let $a \in G$ and assume $\overline{a} = a + xG \in \operatorname{Ker} \overline{\psi}$ i.e.

$$a + xG \mapsto 0 + xF.$$

We also have

$$a + xG \mapsto \psi(a) + xF$$

so we know that $\psi(a) \in xF$ i.e. $\psi(a) = x \cdot f$ for some $f \in F$. From this we get

$$\begin{aligned} \psi(a) = x \cdot f &= \psi \circ \phi(f) \\ \psi(a - \phi(f)) &= 0 \\ \phi \circ \psi(a - \phi(f)) &= 0 \\ x \cdot (a - \phi(f)) &= 0 \end{aligned}$$

and since x is a non-zerodivisor this means that $a = \phi(f)$. Hence $a \in \operatorname{Im} \phi$ and $\overline{a} \in \operatorname{Im} \overline{\phi}$. This means that $\operatorname{Ker} \overline{\psi} \subset \operatorname{Im} \overline{\phi}$ and the sequence is exact.

From this we get an acyclic complex

$$\mathbb{M} : \quad ... \longrightarrow F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \longrightarrow ...$$

of finitely generated free R-modules. We need to show that $\mathrm{Hom}_R(\mathbb{M}, R)$ is acyclic too.

We fix bases for F and G and view ϕ and ψ as matrices with elements in S. Applying $\mathrm{Hom}_S(-, S)$ to (F, G, ϕ, ψ) we get

$$\mathrm{Hom}_S(F, S) \xleftarrow{\phi^*} \mathrm{Hom}_S(G, S) \xleftarrow{\psi^*} \mathrm{Hom}_S(F, S)$$

which is a new matrix factorization in $\mathbf{MF}(S, x)$. Using the canonical isomorphism $\mathrm{Hom}(P, L) \xrightarrow{\sim} P$ for P a finitely generated free S-module, we see that the matrix factorization above is isomorphic to

$$F \xleftarrow{\phi^T} G \xleftarrow{\psi^T} F$$

in $\mathbf{MF}(S, x)$. From what we saw earlier we get a new acyclic complex by reducing modulo x:

$$\mathbb{N}: \quad ... \longleftarrow F/xF \xleftarrow{\overline{\phi^T}} G/xG \xleftarrow{\overline{\psi^T}} F/xF \xleftarrow{\overline{\phi^T}} G/xG \longleftarrow ...$$

of free R-modules.

Now, we consider the complex $\mathbb{M}$. Here the maps are matrices with entries in R and hence the arguments above show that

$$... \longleftarrow F/xF \xleftarrow{(\overline{\phi})^T} G/xG \xleftarrow{(\overline{\psi})^T} F/xF \xleftarrow{(\overline{\phi})^T} G/xG \longleftarrow ...$$

is isomorphic to the complex $\mathrm{Hom}_R(\mathbb{M}, R)$. Furthermore, since $(\overline{y})^T = \overline{y^T}$ for any matrix y over S we see that $\mathrm{Hom}_R(\mathbb{M}, R)$ is isomorphic to $\mathbb{N}$ and hence $\mathbb{M}$ is totally acyclic.

When we reduce a morphism of matrix factorizations in $\mathbf{MF}(S, x)$ modulo x we get a morphism of totally acyclic complexes. And when we reduce a homotopy between two morphisms in $\mathbf{MF}(S, x)$ we get a homotopy between two morphisms of totally acyclic complexes. This means that T is a functor and from the similarity of the constructions of standard triangles in the two categories, it is clear that T is a triangle functor, i.e. is a functor that sends triangles to triangles.

Next we want to show that T is faithful. We do this by showing that the kernel of T on the Hom-sets is equal to zero, i.e. it is injective on the Hom-sets. Let $\theta : (F_1, G_1, \phi_1, \psi_1) \to (F_2, G_2, \phi_2, \psi_2)$ be a morphism

$$\begin{tikzcd}
F_1 \arrow[r, "\phi_1"] \arrow[d, "f"] & G_1 \arrow[r, "\psi_1"] \arrow[d, "g"] & F_1 \arrow[d, "f"] \\
F_2 \arrow[r, "\phi_2"] & G_2 \arrow[r, "\psi_2"] & F_2
\end{tikzcd}$$

in $\mathbf{MF}(S, x)$. Assume that $T([\theta]) = 0$. This means that the morphism of totally acyclic complexes we get when reducing θ modulo x is nullhomotopic over R. Look at a section of such a nullhomotopy:

$$\begin{tikzcd}
F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}"] & G_1/xG_1 \arrow[r, "\overline{\psi}_1"] \arrow[d, "\overline{g}"] \arrow[dl, "\overline{s}_1"'] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}"] \arrow[dl, "\overline{t}"'] & G_1/xG_1 \arrow[d, "\overline{g}"] \arrow[dl, "\overline{s}_2"'] \\
F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r, "\overline{\psi}_2"] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2
\end{tikzcd}$$

where $\overline{s}_1$ does not necessarily equal $\overline{s}_2$. We choose liftings of these diagonal maps to S-homomorphisms

$$s_1 : G_1 \to F_2, \quad t : F_1 \to G_2, \quad s_2 : G_1 \to F_2.$$

The nullhomotopy gives us that for every $a \in F_1$, there is a $b_a \in F_2$ such that

$$f(a) - s_2 \circ \phi_1(a) - \psi_2 \circ t(a) = x \cdot b_a$$

This b_a is unique because x is a non-zerodivisor. Similarly, for every $u \in G_1$ we have a $v_u \in G_2$ such that

$$g(u) - t \circ \psi_1(u) - \phi_2 \circ s_2(u) = x \cdot v_u$$

where v_u is unique. This means that if we define maps $p : F_1 \to F_2$ by $a \mapsto b_a$ and $q : G_1 \to G_2$ by $u \mapsto v_u$, they are well-defined S-homomorphisms, and we get the equalities

$$\begin{aligned} f - s_2 \circ \phi_1 - \psi_2 \circ t &= x \cdot p \\ g - t \circ \psi_1 - \phi_2 \circ s_2 &= x \cdot q. \end{aligned}$$

We want to find a nullhomotopy on θ so we modify t to a new map $t' : F_1 \to G_1$ defined by

$$t' = t + \phi_2 \circ p.$$

So we want to show that (s_2, t') is a nullhomotopy on

$$\begin{tikzcd} F_1 \arrow[r, "\phi_1"] \arrow[d, "f"] & G_1 \arrow[r, "\psi_1"] \arrow[d, "g"] \arrow[dl, "s_2"'] & F_1 \arrow[d, "f"] \arrow[dl, "t'"'] \\ F_2 \arrow[r, "\phi_2"] & G_2 \arrow[r, "\psi_2"] & F_2. \end{tikzcd}$$

From the definition of t we get

$$\begin{aligned} f - s_2 \circ \phi_1 - \psi_2 \circ t' &= f - s_2 \circ \phi_1 - \psi_2 \circ (t + \phi_2 \circ p) \\ &= f - s_2 \circ \phi_1 - \psi_2 \circ t + \psi_2 \circ \phi_2 \circ p) \\ &= f - s_2 \circ \phi_1 - \psi_2 \circ t + x \cdot p) \\ &= x \cdot p - x \cdot p = 0, \end{aligned}$$

so the first part of the homotopy holds. Now, using the equality $g - t \circ \psi_1 - \phi_2 \circ s_1 = x \cdot q$ we see that composing $f - s_2 \circ \phi_1 - \psi_2 \circ t = x \cdot p$ with ψ_1 gives us

$$\begin{aligned} x \cdot p \circ \psi_1 &= f \circ \psi_1 - s_2 \circ \phi_1 \circ \psi_1 - \psi_2 \circ t \circ \psi_1 \\ &= \psi_2 \circ g - x \cdot s_2 - \psi_2 \circ t \circ \psi_1 \\ &= \psi_2 \circ (g - t \circ \psi_1) - x \cdot s_2 \\ &= \psi_2 \circ (\phi_2 \circ s_1 + x \cdot q) - x \cdot s_2 \\ &= x \cdot (s_1 + \psi_2 \circ q - s_2). \end{aligned}$$

Since x is a non-zerodivisor we see from the above that $p \circ \psi_1 = s_1 + \psi_2 \circ q - s_2$ which gives us

$$\begin{aligned} g - t' \circ \psi_1 - \phi_2 \circ s_2 &= g - (t + \phi_2 \circ p) \circ \psi_1 - \phi_2 \circ s_2 \\ &= g - t \circ \psi_1 - \phi_2 \circ (p \circ \psi_1 + s_2) \\ &= g - t \circ \psi_1 - \phi_2 \circ (s_1 - s_2 + \psi_2 \circ q + s_2) \\ &= g - t \circ \psi_1 - \phi_2 \circ s_1 - x \cdot q = x \cdot q - x \cdot q = 0. \end{aligned}$$

This means that (s_2, t') is a nullhomotopy on θ and hence $[\theta] = 0$ in $\mathbf{HMF}(S, x)$ and T is faithful.

Lastly we want to show that T is full, i.e. that T is surjective on the Hom-sets. So, let $(F_1, G_1, \phi_1, \psi_1)$ and $(F_2, G_2, \phi_2, \psi_2)$ be two matrix factorizations in $\mathbf{MF}(S, x)$ and let η be a chain map

$$\begin{tikzcd}
... \arrow[r] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}_1"] & G_1/xG_1 \arrow[r, "\overline{\psi}_1"] \arrow[d, "\overline{g}_1"] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}_0"] & G_1/xG_1 \arrow[r] \arrow[d, "\overline{g}_0"] & ... \\
... \arrow[r] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r, "\overline{\psi}_2"] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r] & ...
\end{tikzcd}$$

of totally acyclic complexes over R. This is the representation of a morphism $[\eta]$ in $\mathbf{K_{tac}}$. We choose a section and lift it to S which gives us a diagram

$$\begin{tikzcd}
G_1 \arrow[r, "\psi_1"] \arrow[d, "g_1"] & F_1 \arrow[r, "\phi_1"] \arrow[d, "f_0"] & G_1 \arrow[d, "g_0"] \\
G_2 \arrow[r, "\psi_2"] & F_2 \arrow[r, "\phi_2"] & G_2
\end{tikzcd}$$

where the vertical maps are chosen liftings in S. Now, let $a \in F_1$ and $u \in G_1$. We know that there exists $b_a \in G_2$ and $v_u \in F_2$ such that

$$\phi_2 \circ f_0(a) - g_0 \circ \phi_1(a) = x \cdot b_a$$
$$\psi_2 \circ g_1(a) - f_0 \circ \psi_1(a) = x \cdot v_u$$

because the diagram commutes when we reduce modulo R. Since x is a non-zerodivisor these elements are unique and hence gives us well defined S-homomorphisms defined by

$$\alpha : F_1 \to G_2, \quad a \mapsto b_a$$
$$\beta : G_1 \to F_2, \quad u \mapsto v_u,$$

which gives us the equalities

$$\phi_2 \circ f_0 - g_0 \circ \phi_1 = x \cdot \alpha$$
$$\psi_2 \circ g_1 - f_0 \circ \psi_1 = x \cdot \beta.$$

Here, the first equality gives us

$$x \cdot \psi_2 \circ \alpha \circ \psi_1 = \psi_2 \circ (\phi_2 \circ f_0 - g_0 \circ \phi_1) \circ \psi_1 = x \cdot f_0 \circ \psi_1 - x \cdot \psi_2 \circ g_0$$

which means, because x is a non-zerodivisor, that $\psi_2 \circ \alpha \circ \psi_1 = f_0 \circ \psi_1 - \psi_2 \circ g_0$.

Now, let the vertical maps in the diagram

$$\begin{tikzcd}
G_1 \arrow[r, "\psi_1"] \arrow[d, "g"] & F_1 \arrow[r, "\phi_1"] \arrow[d, "f"] & G_1 \arrow[d, "g"] \\
G_2 \arrow[r, "\psi_2"] & F_2 \arrow[r, "\phi_2"] & G_2
\end{tikzcd}$$

be defined by

$$f = f_0 - \psi_2 \circ \alpha + \beta \circ \phi_1$$
$$g = g_0 + \phi_2 \circ \beta.$$

Using the equalities from above we get

$$\begin{aligned}\psi_2 \circ g &= \psi_2 \circ g_0 + \psi_2 \circ \phi_2 \circ \beta = (f_0 \circ \psi_1 - \psi_2 \circ \alpha \circ \psi_1) + x \cdot \beta \\ &= f_0 \circ \psi_1 - \psi_2 \circ \alpha \circ \psi_1 + \beta \circ \phi_1 \circ \psi_1 \\ &= (f_0 - \psi_2 \circ \alpha + \beta \circ \phi_1) \circ \psi_1 = f \circ \psi_1\end{aligned}$$

and

$$\begin{aligned}\phi_2 \circ f &= \phi_2 \circ f_0 + \phi_2 \circ \psi_2 \circ \alpha + \phi_2 \circ \beta \circ \phi_1 \\ &= (\phi_2 \circ f_0 - x \cdot \alpha) + \phi_2 \circ \beta \circ \phi_1 \\ &= g_0 \circ \phi_1 + \phi_2 \circ \beta \circ \phi_1 = (g_0 + \phi_2 \circ \beta) \circ \phi_1 = g \circ \phi_1\end{aligned}$$

which shows that the diagram commutes. This means that $\theta = (f, g)$ is a morphism of matrix factorizations in $\mathbf{MF}(S, x)$ and we now want to show that $T([\theta]) = [\eta]$.

$T([\theta])$ in $\mathbf{K_{tac}}$ is represented by

$$\begin{tikzcd}
... \arrow[r] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}"] & G_1/xG_1 \arrow[r, "\overline{\psi}_1"] \arrow[d, "\overline{g}"] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}"] & G_1/xG_1 \arrow[r] \arrow[d, "\overline{g}"] & ... \\
... \arrow[r] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r, "\overline{\psi}_2"] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r] & ...
\end{tikzcd}$$

which is a two-periodic chain map of totally acyclic complexes. We need to show that this is homotopic to η. So consider the diagram

$$\begin{tikzcd}
... \arrow[r] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}-\overline{f_1}"] & G_1/xG_1 \arrow[r, "\overline{\psi}_1"] \arrow[d, "\overline{g}-\overline{g_1}"] & F_1/xF_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, "\overline{f}-\overline{f_0}"] \arrow[dl, "-\overline{\alpha}"'] & G_1/xG_1 \arrow[r] \arrow[d, "\overline{g}-\overline{g_0}"] \arrow[dl, "\overline{\beta}"'] & ... \\
... \arrow[r] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r, "\overline{\psi}_2"] & F_2/xF_2 \arrow[r, "\overline{\phi}_2"] & G_2/xG_2 \arrow[r] & ...
\end{tikzcd}$$

Here we get

$$\overline{f} - \overline{f_0} = -\overline{\psi}_2 \circ \overline{\alpha} + \overline{\beta} \circ \overline{\phi}_1$$

from the definition of f and so the diagram displays the "zeroth part" of a possible nullhomotopy. From Lemma 5.2.1 we know that we can complete this to a nullhomotopy and hence $T([\theta]) = [\eta]$ which means T is full. $\square$

Before we can prove the last result we need some more preliminaries.

Definition. (1) Let S be a ring with exactly one maximal ideal. Then S is a *local ring*.

(2) Let S be a Noetherian local ring with a maximal ideal m. Let $m = (a_1, a_2, ..., a_n)$ where n is chosen as small as possible. Then S is *regular* if $\dim S = n$

The standard example of a local regular ring is $S = k\,[[x_1, x_2, ..., x_n]]$, where k is a field. We also have that every field k is a regular local ring, with dimension 0.

An equivalent definition of a local regular ring is that a local Noetherian ring S is regular if it has finite global dimension. This was proven by Auslander and Buchsbaum in [1] and [2], and by Serre in [23]. Auslander-Buchsbaum also has another useful result which is called the Auslander-Buchsbaum theorem. It states that regular local rings are unique factorization domains and was proven in [3].

Now, let S be a regular local ring and let $0 \neq x \in S$. Since S is a UFD it is also an integral domain, so x is automatically a non-zerodivisor. This means Theorem 5.2.2 holds. This brings us to our last result, which was proved by Orlov in [21].

Theorem 5.2.3. *Let S be a regular local ring and $0 \neq x \in S$. Let $R = S/(x)$. Then the functor*

$$T : \mathbf{HMF}(S, x) \to \mathbf{K_{tac}}(R)$$

from Theorem *5.2.2 is an equivalence.*

Proof. We know that T is fully faithful so what remains is to prove that it is dense. Consider the ring R. In [9] it was proven that $\mathbf{K_{tac}}(R)$ is equivalent with $\underline{\mathbf{MCM}}(R)$; the stable category of maximal Cohen-Macaulay modules. Furthermore, in [10] it was shown that every maximal Cohen-Macaulay R-module is two-periodic, i.e. has a two-periodic free resolution. So every object in $\mathbf{K_{tac}}(R)$ is isomorphic to a two-periodic totally acyclic complex

$$\begin{tikzcd} ... \arrow[r] & P \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & P \arrow[r, "\alpha"] & Q \arrow[r] & ... \end{tikzcd}$$

and what is left to prove is that there exists a matrix factorization (F, G, ϕ, ψ) in $\mathbf{HMF}(S, x)$ such that T sends (F, G, ϕ, ψ) to the complex above.

Let

$$\begin{tikzcd} \mathbb{A} : ... \arrow[r] & P \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & P \arrow[r, "\alpha"] & Q \arrow[r] & ... \end{tikzcd}$$

in $\mathbf{K_{tac}}(R)$ be indecomposable and minimal. Let $M = \operatorname{Im} \alpha$. This is an R-module with a minimal free resolution

$$\begin{tikzcd} \mathbb{F}_1 : ... \arrow[r] & P \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & P \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

Since M is a maximal Cohen-Macaulay module, we have from commutative algebra that $\dim M = \dim R = \dim S - 1$. This and the Auslander-Buchsbaum-Serre theorem gives us

$$\operatorname{pd}_S M = \dim S - \operatorname{depth}_S M = 1$$

which means there exists a minimal free resolution over S

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\phi"] & G \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

Since M is an R-module, we know that $xM = 0$. We also have that $M \simeq G/\operatorname{Im} \phi$ which means $xG \subseteq \operatorname{Im} \phi$. Look at

$$\begin{tikzcd} & G \arrow[dl, dashed, "\exists \psi"'] \arrow[d, "\cdot x"] & \\ F \arrow[r, "\phi"] & \operatorname{Im} \phi \arrow[r] & 0 \end{tikzcd}$$

From the construction, this gives $\phi \circ \psi = x \cdot 1_G$. From this we get

$$\phi \circ \psi \circ \phi = (x \cdot 1_G) \circ \phi.$$

The map ϕ is injective so we know that for every $a \in F$ we have

$$\phi \circ \psi \circ \phi(a) = \phi(xa) \Rightarrow \phi(\psi \circ \phi(a) - xa) = 0 \Rightarrow \psi \circ \phi(a) = xa \quad \forall a \in F$$

which means that $\psi \circ \phi = x \cdot 1_F$. Hence

$$F \xrightarrow{\phi} G \xrightarrow{\psi} F$$

is a matrix factorization.

Reduce modulo x and get

$$... \longrightarrow F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \longrightarrow ...$$

Since this is exact we know that $\mathrm{Ker}\,\overline{\psi} = \mathrm{Im}\,\overline{\phi}$ and from before we know that $M \simeq G/\mathrm{Im}\,\phi$ and $xG \subseteq \mathrm{Im}\,\phi$. Together this gives

$$\begin{aligned} \mathrm{Im}\,\overline{\psi} &\simeq (G/xG)/\mathrm{Ker}\,\overline{\psi} \\ &= (G/xG)/\mathrm{Im}\,\overline{\phi} \\ &= (G/xG)/(\mathrm{Im}\,\psi/xG) \\ &\simeq G/\mathrm{Im}\,\phi \simeq M. \end{aligned}$$

This means we get a free resolution of M over R:

$$\mathbb{F}_2 : ... \longrightarrow F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \longrightarrow M \longrightarrow 0$$

From the construction this is also minimal. This means that $\mathbb{F}_1 \simeq \mathbb{F}_2$, so

$$\mathbb{M} \simeq (\ ... \longrightarrow F/xF \xrightarrow{\overline{\phi}} G/xG \xrightarrow{\overline{\psi}} F/xF \xrightarrow{\overline{\phi}} G/xG \longrightarrow ...\ )$$

and the functor is dense. □

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