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Resonances and Constructions of Fatou-Bieberbach Maps

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Abstract

We study and analyze a proof of a theorem by Rosay and Rudin on the Fatou-Bieberbach method of constructing biholomorphic images of \mathbb{C}^n in \mathbb{C}^n , starting with an automorphism with an attracting fixed point. We thoroughly investigate constructions of Fatou-Bieberbach maps. As a result, we lay much emphasis on the concept of resonances and how they affect our attempt to linearize an automorphism with an attracting fixed point, by a biholomorphic change of variables. We give several examples and some basic explanations to several concepts in order to give an in-depth and basic feel of the whole proof.

Dedication

To my lovely wife, Dinah Naadu Aidoo, I dedicate this thesis as a token of my appreciation for all her support and sacrifice. I love you.

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Chapter 1

Introduction

In the paper by Rosay and Rudin [6], they worked on an older topic in several complex variables: The Fatou-Bieberbach method of constructing biholomorphic images of \mathbb{C}^n in \mathbb{C}^n , starting with an automorphism with an attracting fixed point. In order to address the question of linearization of contractions, they gave a proof of the following related theorem:

Theorem 1.0.1. *Suppose that $F \in \text{Aut}(\mathbb{C}^n)$ fixes a point $p \in \mathbb{C}^n$ and that all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the operator $F'(p)$ satisfy $|\lambda_i| < 1$, for $1 \leq i \leq n$. Let Ω be the set of all $z \in \mathbb{C}^n$ for which $\lim_{k \rightarrow \infty} F^k(z) = p$ where $F^k = F \circ F^{k-1}$, $F^1 = F$. Then there exists a biholomorphic map Ψ from Ω onto \mathbb{C}^n .*

Based on this theorem, Rosay and Rudin constructed some new examples of Fatou-Bieberbach regions Ω in \mathbb{C}^2 . Most of these regions were ranges of biholomorphic maps $\Phi = \Psi^{-1} : \mathbb{C}^2 \rightarrow \Omega$ with the Jacobian of Φ , $J\Phi \equiv 1$. This is because most of the automorphisms used in their constructions had constant Jacobians. In this thesis, we attempt to break down the proof by Rosay and Rudin for easy understanding. Our major

concern is to clearly understand how the Fatou-Bieberbach maps or biholomorphic maps Ψ are constructed as well as the effects that resonances, if they should occur, have on these constructions.

Our choice of the biholomorphic map Ψ is obtained as a solution of the functional equation

$$(**) \quad G^{-1} \circ \Psi \circ F = \Psi$$

where G is a “normal form” for F . Thus, G is of the form $Az + \phi(z)$, where $A = F'(p)$ is linear and $\phi(z) = O(|z|^2)$ is non-linear. In other words, we say that F is formally conjugated to its normal form G . In special cases, $G = F'(p)$. Thus, we refer to the expression in $(**)$ as linearizing the map F by a biholomorphic change of variables Ψ . A solution to the functional equation $(**)$ has been proved to be given as $\Psi = \lim_{k \rightarrow \infty} G^{-k} \circ F^k$ in Reich’s paper [5, page 142]. The interesting part here is that, the sequence $\{G^{-k} \circ F^k\}$ does not necessarily have to converge, not even in the formal power series sense, and not even in some small neighborhood of the fixed point p . A counterexample given in [6, page 74], is again given in this thesis (Example 3) to confirm this assertion. Because the sequence $\{G^{-K} \circ F^k\}$ does not always converge, we shall introduce polynomial maps, say, $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ which satisfies some specific properties, and then analyze the new sequence $\{G^{-k} \circ T \circ F^k\}$. This new sequence converges on every compact subset of $\Omega \subset \mathbb{C}^n$.

We shall give thorough and more basic approaches to understanding the proofs of the related theorems and lemmas in the appendix of [6]. Like, Rosay and Rudin, the so-called *lower-triangular mappings* will be used to give the proof of theorem 1.0.1. We will throw more light on the concept of resonances, focusing carefully on the “threats” that they pose when finding a solution to the functional equation in $(**)$. The set of eigenvalues of

$F'(p)$ plays a crucial part in the construction of the biholomorphic map Ψ . If there exists any resonance relations between the eigenvalues, they may affect our construction of a solution. Particularly, they may alter our choice of triangular map needed to construct the biholomorphic map Ψ . It is therefore necessary to study their behavior and how they affect our results if they should occur. Thus, in writing out the proof of the theorem, we shall consider cases where these resonances occur as well as cases where they do not.

Chapter 2 of this thesis gives some relevant definitions and also discusses some important concepts that will guide us to fully grasp the proofs of the theorems and lemmas in this thesis. In chapter 3, we give a simplified version of the main theorem stated, by assuming that $|\lambda_1|^2 < |\lambda_n|$. This situation, as we shall see in later chapters, implies that there are no resonance relations between the eigenvalues of $F'(p)$ since $1 > |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| > 0$. In this special case, our choice of G equals the linear operator $F'(p) = A$. The simplified version of the theorem fails to hold if the assumption is violated. A counterexample is given in Example 3.

In chapter 4, we give the proof of the main theorem after we study and prove three important lemmas. Several cases about the occurrences and non-occurrences of resonances are considered during the proofs of the lemmas. We shall study some examples in \mathbb{C}^2 and in \mathbb{C}^3 to help us to understand the importance of resonances in our construction as hinted earlier on.

In recent times, many people have studied the so-called *random iterations*. The question we ask is this: When given a sequence of automorphisms $\{F_j\}$ with $F_j(p) = p$ for each $j \in \mathbb{N}$, and the modulus of the eigenvalues of each $F_j'(p)$ is strictly between 0 and 1, and we define the region of attraction $\Omega = \{z \in \mathbb{C}^n : F_j \circ \dots \circ F_1(z) \rightarrow p \text{ as } j \rightarrow \infty\}$; Is

Ω biholomorphic to \mathbb{C}^n ?

E. F. Wold in [8] showed that if the sequence $\{F_j\}$ is uniformly attracting, that is, $C\|w\| \leq \|F_j(w)\| \leq D\|w\|$ for all $j \in \mathbb{N}$ with $0 < C < D < 1$ and $D^2 < C$, then Ω is biholomorphic to \mathbb{C}^n . Also Han Peters and Iris Smit in [3] have shown that in \mathbb{C}^2 if the sequence $\{F_j\}$ is uniformly attracting and $D^{\frac{11}{5}} < C$ then Ω is biholomorphic to \mathbb{C}^2 .

Constructing Fatou-Bieberbach maps involves making the required choices of a triangular map G and a polynomial map T . When given a sequence of automorphisms $\{F_j\}$, it is much difficult to choose these desired maps. If G is either a lower or upper triangular map, it works out well for just one automorphism map, say, F . The problem however with the sequence of automorphisms $\{F_j\}$ is that we may have to switch between lower and upper triangular maps when we iterate randomly. The last chapter of this thesis shows that this method falls apart. We attempt to make the method by Rosay and Rudin more transparent in the hope that one can find something that works for the sequence $\{F_j\}$.

Chapter 2

Preliminaries

2.1 Holomorphic Mappings

We will consider holomorphic maps from \mathbb{C}^n to \mathbb{C}^n , where $n \in \mathbb{N}$. We start with some basic definitions:

Definition 2.1.1. A *holomorphic mapping* F of a domain $\Omega \subset \mathbb{C}^l$ to a domain $\Omega' \subset \mathbb{C}^k$ is a function

$$F(z) = (f_1(z), \dots, f_k(z)),$$

where f_j is a complex-valued holomorphic function for all $1 \leq j \leq k$ and $z = (z_1, \dots, z_l)$.

Holomorphic maps may also be represented by a system of linear equations

$$\eta_j = f_j(z_1, \dots, z_l) \quad j = 1, 2, \dots, k \quad \text{and} \quad \eta_j \in \mathbb{C}.$$

A *biholomorphic mapping* F is a mapping which is holomorphic, injective, surjective, and also has a holomorphic inverse F^{-1} . We are interested in the case where $l = k = n$.

If we have a biholomorphic mapping F from $\Omega \subset \mathbb{C}^n$ onto $\Omega' \subset \mathbb{C}^n$ then we say that the domains Ω and Ω' are *biholomorphically equivalent*.

When $n = 1$, then the biholomorphic mappings are simply conformal mappings. The Riemann Mapping Theorem, which says that any simply connected proper subdomain of the plane is biholomorphically equivalent to the disc, is a very important result of the case when $n = 1$. However, when $n > 1$ the same analogy fails to be true. For example, the ball and the polydiscs are not biholomorphically equivalent. An early result in several complex variables established by *Poincaré*, showed that the ball and the bidisc in \mathbb{C}^2 are not biholomorphic to each other.

Definition 2.1.2. An *automorphism* of \mathbb{C}^n is a biholomorphic mapping of \mathbb{C}^n onto itself. Put differently, F is said to be an automorphism of \mathbb{C}^n if it is holomorphic, one-to-one, and onto, and also has a holomorphic inverse F^{-1} . We denote the group of all such automorphisms by $\text{Aut}(\mathbb{C}^n)$. The operation for this group is composition. For $n = 1$, $\text{Aut}(\mathbb{C}^n)$ consists of all linear mappings (affine maps) F such that $F(z) = az + b$ with $a, b \in \mathbb{C}$, and $a \neq 0$. However, when $n \geq 2$, $\text{Aut}(\mathbb{C}^n)$ is a complicated group with infinite dimension. The group $\text{Aut}(\mathbb{C}^n)$, for $n \geq 2$ contains mappings F defined as

- (i) $F(z) = (z_1, \dots, z_{j-1}, z_j + f(z_1, \dots, \hat{z}_j, \dots, z_n), z_{j+1}, \dots, z_n)$ or
(ii) $F(z) = (z_1, \dots, z_{j-1}, z_j e^{h(z_1, \dots, \hat{z}_j, \dots, z_n)}, z_{j+1}, \dots, z_n)$

where f, h are entire functions in all of \mathbb{C}^n and \hat{z}_j means that z_j is omitted.

We call the mappings $F \in \text{Aut}(\mathbb{C}^n)$ as defined in (i) *Shears* and those in (ii) are called *Overshears*. The set consisting of all shears and overshears as well as all their compositions is dense in $\text{Aut}(\mathbb{C}^n)$ [1]. So all shears and overshears are automorphisms of \mathbb{C}^n . There are other forms of automorphisms; some of which can be found in [6].

Definition 2.1.3. A point $p \in \mathbb{C}^n$ is said to be a *fixed point* of $F \in \text{Aut}(\mathbb{C}^n)$ if $F(p) = p$. Let λ_i be the eigenvalues of the linear operator $F'(p)$ for $1 \leq i \leq n$. We say that the

automorphism F has an *attracting* fixed point if all the eigenvalues of $F'(p)$ are less than one in absolute value, that is, $|\lambda_i| < 1$.

F has a *repelling* fixed point if $|\lambda_i| > 1$ for $1 \leq i \leq n$. We say that F has a *neutral* fixed point if $|\lambda_i| = 1$ for $1 \leq i \leq n$.

Example 1. Let $n = 1$ and consider $F \in \text{Aut}(\mathbb{C})$. By definition 2.1.2, $F(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$. We want $F(p) = p$, so let's assume $b = 0$ and fix a point $p = 0$ in \mathbb{C} . Thus, $F(z) = az$ and $F(0) = 0$. So we have that $p = 0$ is a fixed point of F . Taking iterates of F gives us $F^n(z) = a^n z$. Also $F'(z) = a$ and $F'(0) = a$. Now observe the following:

(α). If $|a| < 1$ then $F^n(z) \rightarrow 0$ as $n \rightarrow \infty$ for all $z \in \mathbb{C}$.

(β). If $|a| > 1$ then $F^n(z) \rightarrow \infty$ if $z \neq 0$.

(γ). If $|a| = 1$ we can write $a = e^{i\theta}$, then $F^n(z) = e^{in\theta} z$.

From the above, we see that in (α), the point $p = 0$ is an attracting fixed point. In particular, F takes the whole \mathbb{C} to the origin. In (β), the only point in \mathbb{C} that converges to the origin is the fixed point $p = 0$. Every other point aside the origin moves further and further away from the origin with every iterate of F . So $p = 0$ is a repelling fixed point here. Lastly, in (γ), $p = 0$ is a neutral fixed point. Every point in \mathbb{C} remains the same or is rotated.

Definition 2.1.4. Let $F \in \text{Aut}(\mathbb{C}^n)$, $p \in \mathbb{C}^n$ and $F(p) = p$. Assume that the eigenvalues of $F'(p)$ satisfy $|\lambda_i| < 1$ for all $i = 1, 2, \dots, n$. Define

$$\Omega = \left\{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = p \right\}.$$

Then Ω is said to be a region attracted to a fixed point p by the automorphism F .

We say that Ω is a *Fatou-Bieberbach domain* if it is a proper subset of \mathbb{C}^n and is biholomorphically equivalent to \mathbb{C}^n . By proper subset, we mean that $\Omega \subset \mathbb{C}^n$ and $\Omega \neq \mathbb{C}^n$. Said differently, if F has more than one fixed point and Ω is biholomorphically equivalent to \mathbb{C}^n , then Ω is a Fatou-Bieberbach domain.

A *Fatou-Bieberbach map* is the biholomorphic map Ψ from Ω onto \mathbb{C}^n .

Example 2. Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be defined as $F(z, w) = (\frac{1}{2}w + z^2, \frac{1}{2}z)$.

$$\text{So } F'(0, 0) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Clearly, F is one-to-one and onto, and also there exists an inverse F^{-1} which is holomorphic. We rewrite F as $F = I \circ J \circ K$, where $I(z, w) = (w, z)$, $J(z, w) = (z, w + 4z^2)$, and $K(z, w) = (\frac{1}{2}z, \frac{1}{2}w)$. So $F^{-1} = K^{-1} \circ J^{-1} \circ I^{-1}$. Thus $F^{-1}(z, w) = (2w, 2z - 8w^2)$. The automorphism F fixes the origin and the eigenvalues of $F'(0, 0)$ are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ which in absolute value are all less than one. Also $|\lambda_1|^2 < |\lambda_2|$ and $|\lambda_2|^2 < |\lambda_1|$. Define

$$\Omega = \left\{ (z, w) \in \mathbb{C}^2 : \lim_{k \rightarrow \infty} F^k(z, w) = (0, 0) \right\}.$$

Then by theorem 1.0.1, there is a map $\Psi : \Omega \longrightarrow \mathbb{C}^2$ such that Ω and \mathbb{C}^2 are biholomorphically equivalent.

We want to show that Ω is a Fatou-Bieberbach domain. So simply, we want to show that $\Omega \neq \mathbb{C}^2$. Now define $W = \{(z, w) \in \mathbb{C}^2 : |z| > 100 \text{ and } |z| > |w|\}$. Then it is enough to show that $F(W) \subset W$. Note that since $(0, 0) \notin W$, $F^k(z, w)$ cannot converge to $(0, 0)$ whenever $(z, w) \in W$.

Let $(z, w) \in W$. We want to show that $|\frac{1}{2}w + z^2| > 100$ and $|\frac{1}{2}w + z^2| > |\frac{1}{2}z|$.

$$|\frac{1}{2}w + z^2| \geq |z|^2 - \frac{1}{2}|w| > 100|z| - \frac{1}{2}|z| \quad \text{since } |z| > |w|$$

$$\text{So } |\frac{1}{2}w + z^2| > (100 - \frac{1}{2})|z| > |z| > 100.$$

Also $|\frac{1}{2}w + z^2| > (100 - \frac{1}{2})|z| > \frac{1}{2}|z|$. So we have shown that $F(z, w) \in W$ which also implies that $F(W) \subset W$. Thus, $\Omega \neq \mathbb{C}^2$ and so Ω is a Fatou-Bieberbach domain.

Alternatively, the map F has more than one fixed point. Particularly, $(0, 0)$ and $(\frac{3}{4}, \frac{3}{8})$ are fixed points of F . This means that Ω cannot be all of \mathbb{C}^2 .

2.2 Resonances

As stated in the introduction, finding a solution to the functional equation in (**) simply depends on the set of eigenvalues of $F'(p) = A$, commonly referred to as the *Spectrum of A*. Resonance relations between the eigenvalues of A can affect our approach to finding a desired solution. This emphasizes how relevant these resonances are in our work. Thus, we consider the following definitions:

Definition 2.2.1. A *monomial* is a polynomial with only one term. A monomial can be expressed in one variable or several variables. For example, $z^2w^4t^3u$ is a monomial in four variables. Also, *homogeneous polynomial* is a polynomial that has its nonzero terms all having the same degree. For example, $zw^2t^3 + 8z^2w^2t^2 + 3w^4t^2$ is a homogeneous polynomial of degree 6, in three variables. We see that the sum of the exponents in each term is 6.

We focus more on homogeneous polynomials and monomials in several variables, so for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, c constant, and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we have that $z^\alpha = cz_1^{\alpha_1} \dots z_n^{\alpha_n}$ is a monomial in n variables and $\sum_{|\alpha|=m} c_\alpha z^\alpha = \sum_{|\alpha|=m} c_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n}$ is a homogeneous polynomial in n variables of degree m , where $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Definition 2.2.2. Let F be a map from \mathbb{C}^n to \mathbb{C}^n and let $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ be a multi-index with $|k| \geq 2$ such that

$$\lambda^k - \lambda_j := \lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \quad \text{for some } 1 \leq j \leq n.$$

We call such a relation a *resonance of F relative to the j -th coordinate*. We call k a *resonant multi-index relative to the j -th coordinate*. As defined in [4],

$$\text{Res}_j(\lambda) := \{k \in \mathbb{N}^n \mid |k| \geq 2 \text{ and } \lambda^k = \lambda_j\}.$$

Definition 2.2.3. A *resonant monomial* is a monomial $z^k = cz_1^{k_1} \cdots z_n^{k_n}$ in the j -th coordinate with $k \in \text{Res}_j(\lambda)$, that is, $|k| \geq 2$ and $\lambda^k = \lambda_j$ and for some constant c .

Now let $H_m = (h_1, \dots, h_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a holomorphic map with its components h_i being homogeneous polynomials of degree m . We shall denote the vector space of all such holomorphic maps by \mathcal{H}_m . Let \mathcal{B} be a basis for \mathcal{H}_m , then \mathcal{B} consists of all maps H_m that have every component to be zero except for one. The only nonzero component, is a monomial with degree m . So if h_j is this nonzero component, then $h_j(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n = m$. Now we define a new monomial h_j^* as follows:

$$h_j^*(z) = z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}}$$

where $\alpha_1 + \cdots + \alpha_{j-1} = m$, for $m \geq 2$ and also $\lambda_j = \lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}}$.

The h_j^* defined here is a resonant monomial in the j -th coordinate. All elements $H_m \in \mathcal{B}$ for which the nonzero component is a resonant monomial in a specific coordinate shall be termed as *Special*. We let $X_m \subset \mathcal{H}_m$ be the subspace of \mathcal{H}_m spanned by the special basis elements. We denote by Y_m the subspace of \mathcal{H}_m spanned by the other basis elements that are not special. Consider $1 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 0$.

When m is so large that $|\lambda_1|^m < |\lambda_n|$, then no member of \mathcal{B} is *special*. Here is the reason why:

$|\lambda_1|^m < |\lambda_n|$ and so $|\lambda_i|^m < |\lambda_n|$. This implies that $|\lambda_i|^m < |\lambda_j|$ since $|\lambda_1| < 1$.

For the *special* members of \mathcal{B} , we know that $\lambda_j = \lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}}$. This implies that there is a resonance relation at the j -th coordinate. Let $\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1} = m$. Thus

$$|\lambda_j| = |\lambda_1|^{\alpha_1} \cdots |\lambda_{j-1}|^{\alpha_{j-1}} \leq |\lambda_1|^{\alpha_1} |\lambda_1|^{\alpha_2} \cdots |\lambda_1|^{\alpha_{j-1}} = |\lambda_1|^m$$

$$|\lambda_j| \leq |\lambda_1|^m < |\lambda_n| \leq |\lambda_j| \quad \text{since } |\lambda_1|^m < |\lambda_n| \text{ by assumption.}$$

This means that λ_j cannot be expressed as $\lambda_j = \lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}}$ if $|\lambda_1|^m < |\lambda_n|$ for m sufficiently large.

We shall let the map Γ_A be defined by $\Gamma_A(H) = A \circ H - H \circ A$ for $H \in \mathcal{H}_m$. So Γ_A is a linear operator on \mathcal{H}_m for each m .

2.3 Triangular Mappings

From the introduction, we stated clearly that we are interested in Lower-triangular mappings. The reason is that when constructing a Fatou-Bieberbach map, we need to find a lower triangular map whose iterates takes the whole \mathbb{C}^n to zero. In addition, the inverse of these iterates also takes any neighborhood of zero to the whole of \mathbb{C}^n . In simple terms, we need a lower triangular map that contracts uniformly and whose inverse expands uniformly. We shall therefore define this lower triangular map G , and study its properties.

Consider the holomorphic maps $G = (g_1, \dots, g_n)$ from \mathbb{C}^n to \mathbb{C}^n which is given by the system of equations of the form

$$g_1(z) = c_1 z_1$$

$$g_2(z) = c_2 z_2 + h_2(z_1)$$

$$g_3(z) = c_3 z_3 + h_3(z_1, z_2)$$

$$\vdots$$

$$g_n(z) = c_n z_n + h_n(z_1, \dots, z_{n-1})$$

where c_j 's are scalars and h_j is a holomorphic function of (z_1, \dots, z_{j-1}) which vanishes at zero for every j . This kind of map is referred to as a *Lower Triangular Map*. The Jacobian matrix of G is given by

$$G'(z) = \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} & \cdots & \frac{\partial g_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial z_1} & \frac{\partial g_n}{\partial z_2} & \cdots & \frac{\partial g_n}{\partial z_n} \end{pmatrix} = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ \frac{\partial h_2}{\partial z_1} & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial z_1} & \frac{\partial h_n}{\partial z_2} & \cdots & c_n \end{pmatrix}.$$

Therefore the matrix that represents the linear operator $G'(0)$ is also lower triangular. We also know that for a triangular matrix, its determinant is simply the product of the entries in the leading diagonal. Thus, $G'(0)$ is invertible if and only if none of the c_j 's equals zero. So G is an automorphism of \mathbb{C}^n if and only if none of the c_j 's are zero.

Now if g_1, g_2, \dots, g_n are polynomials, that is, $g_j : \mathbb{C}^n \rightarrow \mathbb{C}$ is a polynomial map in several variables, then we call the mapping G a *polynomial mapping*. G is then said to be a *Lower Triangular Polynomial Automorphism of \mathbb{C}^n* . The degree of the lower triangular map G is given as $\deg G = \max_i \deg g_i$ for $1 \leq i \leq n$. We shall define the iterates of G , as $G^k = (g_1^{(k)}, \dots, g_n^{(k)})$. Therefore the degree of the iterates G^k is given as $\deg G^k = \max_i \deg g_i^{(k)}$ for $1 \leq i \leq n$.

A similar idea still holds if we have an upper triangular map $F = (f_1, \dots, f_n)$ given by the system of equations below.

$$f_1(z) = a_1 z_1 + q_1(z_2, \dots, z_n)$$

$$f_2(z) = a_2 z_2 + q_2(z_3, \dots, z_n)$$

$$\begin{aligned} & \vdots \\ f_{n-1}(z) &= a_{n-2}z_{n-2} + q_{n-2}(z_{n-1}, z_n) \\ f_{n-1}(z) &= a_{n-1}z_{n-1} + q_{n-1}(z_n) \\ f_n(z) &= a_n z_n \end{aligned}$$

where the a_i 's are scalars and q_j is a holomorphic function of (z_{j+1}, \dots, z_n) and $q_j(0) = 0$ for every j . F is then said to be an upper triangular map. The Jacobian matrix of F is given by

$$F'(z) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix} = \begin{pmatrix} a_1 & \frac{\partial q_1}{\partial z_2} & \cdots & \frac{\partial q_1}{\partial z_n} \\ 0 & a_2 & \cdots & \frac{\partial q_2}{\partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Likewise, F is an automorphism of \mathbb{C}^n if and only if none of the a_i 's are zero. And so we can also have an *Upper Triangular Polynomial Automorphism of \mathbb{C}^n* if f_1, \dots, f_n are polynomial maps. The degree of F as well as the degree of the iterates F^k are all defined in a similar way as in the case of lower triangular maps.

Chapter 3

Proof of a simplified version of the main theorem

As we clearly stated in the introduction, theorem 1.0.1 becomes very easy to prove when the eigenvalues, $\lambda_1, \dots, \lambda_n$, of the linear operator are related such that $|\lambda_1|^2 < |\lambda_n|$, for $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. This assumption simply implies that there cannot be any resonance relations between the eigenvalues of the linear operator. We start by stating the following theorem:

Theorem 3.0.1. *Suppose $F \in \text{Aut}(\mathbb{C}^n)$, $p \in \mathbb{C}^n$, $F(p) = p$ and the eigenvalues λ_i of the matrix $A = F'(p)$ satisfy $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ and $|\lambda_1|^2 < |\lambda_n|$.*

$$\text{Define } \Omega = \left\{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = p \right\}.$$

Then Ω is a region and there is a biholomorphic map Ψ from Ω onto \mathbb{C}^n which is given by $\Psi = \lim_{k \rightarrow \infty} A^{-k} F^k$ and its convergence on every compact subset of Ω is uniform.

Proof. We begin this proof by letting $p = 0$. It is important to note that we do not lose generality here if we take $p = 0$. Also let $\alpha, \beta_1, \beta_2, \beta$ be some constants satisfying

$\beta > \beta_2 > \beta_1 > |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| > \alpha$ and $\beta^2 < \alpha$. This implies $\beta < 1$. At $p = 0$, $\exists r > 0$ such that $z \in \mathbb{B}(0, r)$ and so we have that

$$F(z) = Az + O(|z|^2)$$

$$|F(z)| = |Az + O(|z|^2)| \leq \|A\| |z| + |O(|z|^2)|$$

By the definition of $O(|z|^2)$, it means that there exists a positive real number C such that

$$|F(z)| \leq \|A\| |z| + C|z|^2 \leq \|A\| |z| + Cr|z| \quad \text{since } |z| \leq r$$

Now let $Cr < \delta$.

So $|F(z)| \leq (\|A\| + \delta)|z|$ since $\|A\| < \beta_1$ from the hypothesis of the above theorem.

$$|F(z)| \leq \beta_2|z| \quad \text{since } \|A\| + \delta < \beta_1 + \delta < \beta_2$$

By the Spectral theorem, there exists m such that for $z \in \mathbb{B}(0, r)$ we get that $F^m(z) \in \mathbb{B}(0, r)$. Thus

$$|F^m(z)| \leq \beta_2^m |z| \quad \text{for some fixed } m.$$

Let $N = jm + i$, $j = 1, 2, \dots$ and $0 \leq i \leq m - 1$. We put $K = \max \left\{ \frac{|F^i(z)|}{|z|} \right\}$ for $0 < |z| < r$. Then we have that

$$|F^N(z)| = |F^i(F^{jm}(z))| \leq K |F^{jm}(z)| \leq K \beta_2^{jm} |z|.$$

And so for N large enough we get that

$$|F^N(z)| < \beta^N \quad \text{for every } z \in \mathbb{B}(0, r).$$

Now let $q \in \Omega$ then by the definition of Ω in the theorem we can see that $F^j(q) \in \mathbb{B}(p, r)$ for some $j \in \mathbb{N}$.

$$\text{So } q \in F^{-j}(\mathbb{B}(p, r)) \quad \text{and} \quad \Omega = \bigcup_{j=0}^{\infty} F^{-j}(\mathbb{B}(p, r))$$

This shows that Ω is a region and also $F(\Omega) = \Omega$. Let E be a compact subset of Ω . Then we have that $E \subset F^{-J}(\mathbb{B}(p, r))$ for some J . Consider the sequence $\{A^{-k}F^k|_E\}$ and let $k = m + J$ and $\omega \in \mathbb{B}(p, r)$. Thus, $A^{-k}F^k(\omega) = A^{-(m+J)}F^{m+J}(\omega) = A^{-J}(A^{-m}F^m)F^J(\omega)$.

Hence for $\{A^{-k}F^k\}$ to converge it is sufficient to show that $\{A^{-m}F^m\}$ converges in $\mathbb{B}(p, r)$.

Therefore we want to show that for $z \in \mathbb{B}(p, r)$ the sequence $\{A^{-k}F^k(z)\}$ is a Cauchy sequence.

Let $\omega \in \mathbb{B}(p, r)$

$$\begin{aligned} \omega - A^{-1}F(\omega) &= \omega - A^{-1}(A(\omega) + O(|\omega|^2)) \\ &= \omega - \omega - A^{-1}O(|\omega|^2) \quad \text{since } A^{-1} \text{ is linear} \\ &= A^{-1}(O(|\omega|^2)) \end{aligned}$$

$$|\omega - A^{-1}F(\omega)| = |A^{-1}(O(|\omega|^2))| \leq b|\omega|^2 \quad \dots\dots\dots (\star)$$

b is some constant. Let $z \in E$.

$$\begin{aligned} |A^{-N}F^N(z) - A^{-(N+1)}F^{N+1}(z)| &= |A^{-N}(F^N(z) - A^{-1}F(F^N(z)))| \\ &\leq \|A^{-N}\| |F^N(z) - A^{-1}F(F^N(z))| \quad \text{from } (\star) \\ &\leq b\|A^{-N}\| |F^N(z)|^2 \quad \text{since } F^N(z) \in \mathbb{B}(p, r) \text{ for } N \text{ large enough} \\ &< \alpha^{-N}b(\beta^N)^2 < \left(\frac{\beta^2}{\alpha}\right)^N \end{aligned}$$

Now let $\zeta_k = A^{-k}F^k$ and consider the sequence $\{\zeta_k\}$. Let $m < n$ for some $m, n \in \mathbb{N}$

$$\begin{aligned} |\zeta_m - \zeta_n| &= |\zeta_m - \zeta_{m+1} + \zeta_{m+1} - \zeta_{m+2} + \dots + \zeta_{n-1} - \zeta_n| \\ &\leq |\zeta_m - \zeta_{m+1}| + \dots + |\zeta_{n-1} - \zeta_n| \\ &\leq \sum_{j=m}^n \left(\frac{\beta^2}{\alpha}\right)^j \leq \sum_{j=m}^{\infty} \left(\frac{\beta^2}{\alpha}\right)^j = \left(\frac{\beta^2}{\alpha}\right)^m \left(\frac{1}{1 - \frac{\beta^2}{\alpha}}\right) \quad \text{for } \frac{\beta^2}{\alpha} < 1 \end{aligned}$$

Hence the sequence $\{\zeta_k\}$ is Cauchy. Therefore its limit $\Psi = \lim_{k \rightarrow \infty} \zeta_k$ is a limit of a sequence of automorphisms, hence it is holomorphic.

Claim 1. Ψ is one-to-one.

Assume $\Psi = \lim_{k \rightarrow \infty} A^{-k}F^k$ is NOT one-to-one. This means that $\exists x, y : x \neq y$ and $\Psi(x) = \Psi(y)$. We also note that the Jacobian determinant, $J\Psi = \det \Psi' \neq 0$. This means that Ψ is open. Let V, W be open neighborhoods, then $\Psi(V)$ and $\Psi(W)$ are also open.

For $x \in V$ and $y \in W$, we have that $\Psi(x) = \Psi(y)$. This implies that $\Psi(V) \cap \Psi(W) \neq \emptyset$ and so $A^{-k}F^k(V) \cap A^{-k}F^k(W) \neq \emptyset$ for k large enough.

$$A^{-k}F^k(\tilde{x}) = A^{-k}F^k(\tilde{y}) \quad \text{for some } \tilde{x} \in V \text{ and } \tilde{y} \in W.$$

Hence we have a contradiction. This proves that Ψ is one-to-one.

Claim 2. Ψ is onto.

We want to show that for $\Psi : \Omega \rightarrow \mathbb{C}^n$, $\Psi(\Omega) = \mathbb{C}^n$ for $\Psi = \lim_{k \rightarrow \infty} A^{-k}F^k$. All eigenvalues of A^{-1} are larger than 1 in absolute value. Therefore A^{-1} is an expansion. Let $\mathbb{B}(p, \varepsilon) \subset \mathbb{C}^n$, since $F \in \text{Aut}(\mathbb{C}^n)$ has an attracting fixed point p , it implies that $\mathbb{B}(p, \varepsilon) \subset \Omega$. This is because for $z \in \mathbb{B}(p, \varepsilon)$ we have that $\lim_{k \rightarrow \infty} F^k(z) = p$. So $A^{-1}(\mathbb{B}(p, \varepsilon))$ is larger than $\mathbb{B}(p, \varepsilon)$. *i.e.* $A^{-1}(\mathbb{B}(p, \varepsilon)) \supset \mathbb{B}(p, (1 + \delta)\varepsilon)$ for some small $\delta > 0$. $F(\Omega) = \Omega$, and so $F^k(\Omega) = \Omega$. $\Psi(\Omega) = \lim_{k \rightarrow \infty} A^{-k}F^k(\Omega)$ since $A^{-k}(\mathbb{B}(p, \varepsilon)) \rightarrow \mathbb{C}^n$ as $k \rightarrow \infty$.

So we have that Ψ is a biholomorphic map from Ω onto \mathbb{C}^n . □

Example 3. We shall define $F \in \text{Aut}(\mathbb{C}^2)$ by $F(z, w) = (\alpha z, \beta w + z^2)$, where

$1 > \alpha > \beta > 0$. This implies that $F(0, 0) = (0, 0)$, and $A = F'(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Now

observe the following iterations of F :

$$\begin{aligned} F^2(z, w) &= \left(\alpha^2 z, \beta^2 w + \beta \left(1 + \frac{\alpha^2}{\beta} \right) z^2 \right) \\ F^3(z, w) &= \left(\alpha^3 z, \beta^3 w + \beta^2 \left(1 + \frac{\alpha^2}{\beta} + \left(\frac{\alpha^2}{\beta} \right)^2 \right) z^2 \right) \\ &\vdots \\ F^k(z, w) &= \left(\alpha^k z, \beta^k w + \beta^{k-1} \left(1 + \frac{\alpha^2}{\beta} + \dots + \left(\frac{\alpha^2}{\beta} \right)^{k-1} \right) z^2 \right) \end{aligned}$$

$F^k(z, w) \rightarrow 0$ as $k \rightarrow \infty$ for all $(z, w) \in \mathbb{C}^2$. This is because α and β are less than 1.

$$A^{-1} = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} \text{ and } A^{-k} = \begin{pmatrix} \frac{1}{\alpha^k} & 0 \\ 0 & \frac{1}{\beta^k} \end{pmatrix}.$$

Also we have that

$$(A^{-k}F^k)(z, w) = \left(z, w + \beta^{-1} \left(1 + \frac{\alpha^2}{\beta} + \dots + \left(\frac{\alpha^2}{\beta} \right)^{k-1} \right) z^2 \right).$$

The sequence $\{A^{-k}F^k\}$ converges if $\frac{\alpha^2}{\beta} < 1$ or $\alpha^2 < \beta$. Thus, the map $\Psi : \Omega \rightarrow \mathbb{C}^2$, defined as $\Psi = \lim_{k \rightarrow \infty} A^{-k}F^k$ exists if $\frac{\alpha^2}{\beta} < 1$. The sequence however fails to converge if $\frac{\alpha^2}{\beta} \geq 1$.

The obvious conclusion here is this: the sequence $\{A^{-k}F^k\}$ in theorem 3.0.1 fails to converge if the assumption that $|\lambda_1|^2 < |\lambda_n|$ does not hold. The sequence fails to converge, not even locally, and even on the level of formal power series if this assumption is violated. It is important to note that there are two (2) difficulties that arise when this assumption is violated. The first is the presence of resonances and the second is when $|\lambda_1| \not\leq |\lambda_n|$. In the latter case, we need the polynomial map T_m instead of just the identity map Id .

If we assume that $\alpha^2 = \beta$, then the sequence $\{A^{-k}F^k\}$ will not converge as k tends to infinity. However, there is a way to fix this problem when it occurs. The first case of example 4 addresses this situation. Particularly, this situation implies that there is a resonance relation between the eigenvalues α and β . Also if $\alpha^2 > \beta$, then we know that we can no longer use the identity map but instead a polynomial map T_m for some $m \geq 2$.

Thus, in both scenarios, we find a lower triangular map $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with the desired properties and a polynomial map $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ also with specific properties such that the sequence $\{G^{-k} \circ T \circ F^k\}$ converges as k tends to infinity. In that case, $\Psi = \lim_{k \rightarrow \infty} G^{-k} \circ T \circ F^k$.

We shall explain how this G and T are chosen in chapter 4.

Chapter 4

Proof of Main Theorem

Before we prove the main theorem, we will first consider three important lemmas to help prepare us for the proof of the theorem. The first lemma gives us the necessary details we need to know about the ideal choice of the lower or upper triangular map of \mathbb{C}^n as well as its required properties.

Lemma 4.0.1. *Let G be a lower triangular polynomial automorphism of \mathbb{C}^n .*

i). *The degrees of the iterates G^k of G are then bounded, and there exists a constant $\beta < \infty$ so that*

$$G^k(U^n) \subset \beta^k U^n \quad k = 1, 2, 3, \dots$$

where U^n is the unit polydisc in \mathbb{C}^n .

ii). *If $|c_i| < 1$ for $1 \leq i \leq n$ then $G^k(z) \rightarrow 0$ uniformly on every compact subset of \mathbb{C}^n and*

$$\bigcup_{k=1}^{\infty} G^{-k}(V) = \mathbb{C}^n$$

for every neighborhood V of 0.

Proof. (i). Let $G = (g_1, g_2, \dots, g_n)$ be a lower triangular polynomial automorphism of \mathbb{C}^n and let $G^k = (g_1^{(k)}, g_2^{(k)}, \dots, g_n^{(k)})$. From the definition of G , we have that

$$\begin{aligned} g_1(z) &= c_1 z_1 \\ g_2(z) &= c_2 z_2 + h_2(z_1) \\ &\vdots \\ g_n(z) &= c_n z_n + h_n(z_1, \dots, z_{n-1}) \end{aligned}$$

We observe that for $G \circ G$, the first 3 components give us

$$\begin{aligned} g_1^{(2)}(z) &= (g_1 \circ g_1)(z) = c_1^2 z_1 \\ g_2^{(2)}(z) &= (g_2 \circ g_2)(z) = c_2^2 z_2 + c_2 h_2(z_1) + h_2(c_1 z_1) \\ g_3^{(2)}(z) &= (g_3 \circ g_3)(z) = c_3^2 z_3 + c_3 h_3(z_1, z_2) + h_3(c_1 z_1, c_2 z_2 + h_2(z_1)) \end{aligned}$$

Similarly, the first 3 components of $G \circ G \circ G$, are as follows:

$$\begin{aligned} g_1^{(3)}(z) &= c_1^3 z_1 \\ g_2^{(3)}(z) &= c_2^2 (g_2(z)) + c_2 h_2(g_1(z)) + h_2(c_1 g_1(z)) \\ &= c_2^3 z_2 + c_2^2 h_2(z_1) + c_2 h_2(c_1 z_1) + h_2(c_1^2 z_1) \\ g_3^{(3)}(z) &= c_3^2 (g_3(z)) + c_3 h_3(g_1(z), g_2(z)) + h_3(c_1 g_1(z), c_2 (g_2(z)) + h_2(g_1(z))) \\ &= c_3^3 z_3 + c_3^2 h_3(z_1, z_2) + c_3 h_3(c_1 z_1, c_2 z_2 + h_2(z_1)) + h_3(c_1^2 z_1, c_2^2 z_2 + c_2 h_2(z_1) + h_2(c_1 z_1)). \end{aligned}$$

It is easy to observe that the degree of the iterates of each of the components remains the same at some particular point in the iteration. We shall now consider the more general case.

$$\text{Let } \mu_i = \deg g_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad \text{let } d = \mu_1 \cdots \mu_n$$

We want to show that the statement $S(n, k) : \deg g_i^{(k)} \leq \mu_1 \cdots \mu_i$ holds for all $i = 1, 2, \dots, n$. We show this by induction on i and k . Now consider the following three (3) statements:

$$\begin{array}{lll}
 S(m,k) & : & \deg g_i^{(k)} \leq \mu_1 \cdots \mu_i \quad \text{for } 1 \leq i \leq m \\
 S(1,k) & : & \deg g_1^{(k)} = \mu_1 \quad \text{true for all } k \\
 S(m,1) & : & \deg g_i = \mu_i \quad \text{true for all } 1 \leq i \leq m
 \end{array}$$

We have that $G^{k+1} = G \circ G^k$, hence

$$g_i^{(k+1)} = c_i g_i^{(k)} + h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)}) \quad \text{for } 2 \leq i \leq n.$$

$$\deg g_i^{(k+1)} = \mu_i \max_{s \leq i-1} \deg g_s^{(k)} \quad \text{and so } \deg g_i^{(k+1)} \leq \mu_i \mu_1 \cdots \mu_{i-1}$$

$$\deg g_i^{(k+1)} = \mu_1 \cdots \mu_i \leq \mu_1 \cdots \mu_n = d. \quad \text{It is easy to see that } \deg G = \max_i \deg g_i.$$

Therefore we have that $\deg G^k = \max_i \deg g_i^{(k)}$ for all $i = 1, 2, 3, \dots, n$. This implies that

$\deg g_i^{(k)} \leq \mu_1 \cdots \mu_i$ for all $i = 1, 2, \dots, n$. As a result, the statement $S(n, k)$ holds for

all $i = 1, 2, \dots, n$. Which means that the degree of the iterates G^k of G are bounded.

Expressing it mathematically, we write, $\deg G^k = \max_{1 \leq i \leq n} \deg g_i^{(k)} \leq \mu_1 \cdots \mu_n = d$.

Let M be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ that have $|\alpha| \leq d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We need to place a bound on the coefficients of the g_i^k 's in order to avoid the situation whereby they tend to infinity when iterated. Therefore, we choose $C \geq 1$ so that $|g_i| \leq C$ on U^n for $1 \leq i \leq n$. Now we put $\beta = MC^d$.

Claim 3. $|g_i^{(k)}(z)| \leq \beta^k$ for $z \in U^n$, $1 \leq i \leq n$ and $k = 1, 2, \dots$

We shall prove this by induction of k . Let $g_i^{(k)}(z) = \sum_{|\alpha| \leq d} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{|\alpha| \leq d} a_\alpha z^\alpha$

The above claim holds when $k = 1$, since $|g_i(z)| \leq C \leq \beta$.

The coefficient $a_\alpha = \int_{T^n} g_i^{(k)}(z) \bar{z}^\alpha$, where T^n is the unit Torus. In terms of one variable,

we have that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\alpha\theta} e^{-i\gamma\theta} d\theta = \begin{cases} 0 & \text{if } \alpha \neq \gamma \\ 1 & \text{if } \alpha = \gamma \end{cases}$$

Claim 4. $|a_\alpha| \leq \beta^k$

Since $G^{k+1} = G^k \circ G$ we have that

$$\begin{aligned} g_i^{(k+1)} &= g_i^{(k)}(g_1, \dots, g_n) = \sum_{|\alpha| \leq d} a_\alpha g_1^{\alpha_1} \cdots g_n^{\alpha_n} \\ |g_i^{(k+1)}| &= \left| \sum_{|\alpha| \leq d} a_\alpha g_1^{\alpha_1} \cdots g_n^{\alpha_n} \right| \leq \sum_{|\alpha| \leq d} |a_\alpha| |g_1|^{\alpha_1} \cdots |g_n|^{\alpha_n} \leq M \beta^k C^{|\alpha|} \\ &\leq M \beta^k C^d = \beta^{k+1} \end{aligned}$$

We can see that claim 3 is true since k has been replaced by $k+1$. Hence claim 3 is true if claim 4 holds. We shall now verify claim 4.

$$\begin{aligned} |a_\alpha| &= \left| \int_{T^n} g_i^{(k)}(z) \bar{z}^\alpha d\theta \right| \leq \left(\frac{1}{2\pi} \right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} |g_i^{(k)}(z)| |\bar{z}|^\alpha d\theta_1 \cdots d\theta_n \\ &= \left(\frac{1}{2\pi} \right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} |g_i^{(k)}(z)| d\theta_1 \cdots d\theta_n \quad \text{since } |\bar{z}|^\alpha = 1 \\ &\leq \left(\frac{1}{2\pi} \right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \beta^k d\theta_1 \cdots d\theta_n \quad \text{since } |g_i^{(k)}(z)| \leq \beta^k \end{aligned}$$

So $|a_\alpha| \leq \beta^k$. This implies that claim 3 true. Hence $G^k(U^n) \subset \beta^k U^n$ for $k = 1, 2, \dots$

(ii). We now want to show that for $|c_i| < 1$ and $1 \leq i \leq n$, $G^k(z) \rightarrow 0$ uniformly on every compact subsets of \mathbb{C}^n and $\bigcup_{k=1}^{\infty} G^{-k}(V) = \mathbb{C}^n$.

Let $|c_i| < 1$ for $1 \leq i \leq n$ and let E be a compact subset of \mathbb{C}^n . Let $\|\cdot\|_E$ denote the supremum norm over E . So $\|g_j^{(k)}\|_E = \sup\{|g_j^{(k)}(z)| : z \in E\}$. Thus, $\|g_1^{(k)}\|_E \rightarrow 0$ as $k \rightarrow \infty$. We prove by induction on i . We assume that $1 < i \leq n$ and

$$(4.1) \quad \lim_{k \rightarrow \infty} \|g_j^{(k)}\|_E = 0 \quad \text{for } 1 \leq j < i.$$

$$(4.2) \quad \lim_{k \rightarrow \infty} \|h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)})\|_E = 0 \quad \text{since } h_i(0) = 0 \text{ and from (4.1).}$$

Let $\varepsilon > 0$. We know that $g_i^{(k+1)} = g_i \circ g_i^{(k)}$ and so

$$\begin{aligned} |g_i^{(k+1)}| &= |c_i g_i^{(k)} + h_i| \leq |c_i| |g_i^{(k)}| + |h_i| \\ |g_i^{(k+1)}| &\leq |c_i| |g_i^{(k)}| + |h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)})| < |c_i| |g_i^{(k)}| + \varepsilon \end{aligned}$$

for sufficiently large k . This follows from the fact that $g_i^{(k+1)} = c_i g_i^{(k)} + h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)})$.

Let $\limsup_{k \rightarrow \infty} \|g_i^{(k)}\|_E = a$. By taking lim sup of the above inequality, we get that $a \leq |c_i| a + \varepsilon$ and so $a \leq \frac{\varepsilon}{1 - |c_i|}$ for all $\varepsilon > 0$. So (4.1) still holds if we replace i with $i+1$. So by induction on i we conclude that $G^k(z) \rightarrow 0$ uniformly on all compact subsets of \mathbb{C}^n .

We now use induction to confirm that $\|g_i^{(k)}\|$ are bounded on the set E . Now we assume that the expression in (4.1) holds. Then it is enough to show this:

$$\begin{aligned} |g_i^{(k+1)}| &\leq |c_i| |g_i^{(k)}| + \varepsilon_k \\ &\leq |c_i|^2 |g_i^{(k-1)}| + |c_i| \varepsilon_{(k-1)} + \varepsilon_k \\ &\leq |c_i|^3 |g_i^{(k-2)}| + |c_i|^2 \varepsilon_{k-2} + |c_i| \varepsilon_{k-1} + \varepsilon_k \\ &\leq |c_i|^k |g_i| + |c_i|^{k-1} \varepsilon_1 + \dots + |c_i|^2 \varepsilon_{k-2} + |c_i| \varepsilon_{k-1} + \varepsilon_k \\ &\leq |c_i|^k |g_i| + \sum_{j=1}^{k-1} |c_i|^j B + \varepsilon_k \end{aligned}$$

where B is any positive number. So $\|g_i\|_E$ is bounded.

The second part of (ii) follows directly as a consequence of the first part. □

Lemma 4.0.2. $\mathcal{H}_m = X_m + \Gamma_A(\mathcal{H}_m)$, for $m \geq 2$, where \mathcal{H}_m , X_m , and $\Gamma_A = Y_m$ are defined as in section 2.2.

Proof. We will study the homogeneous polynomials h . Consider

$$h(z_1, \dots, z_{j-1}) = \sum_{n \leq |\alpha| \leq N} a z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}} = h_n + h_{n+1} + \dots + h_N$$

Also $h_k = \sum_{|\alpha|=k} a z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}}$ is a sum of monomials. \mathcal{H}_m is spanned by monomials z^β for $|\beta| = m$ and X_m is a subspace of \mathcal{H}_m spanned by the special monomials. From the

definition in section 2.2, $\Gamma_A(H) = A \circ H - H \circ A$, $H \in \mathcal{H}_m$. We choose coordinates so that the matrix A is lower triangular.

Claim 5. $Y_m = \Gamma_A(\mathcal{H}_m)$

Let D be a diagonal matrix which has $\lambda_1, \lambda_2, \dots, \lambda_n$ down its main diagonal. Also let $H = (0, \dots, 0, z^\alpha, 0, \dots, 0) \in \mathcal{B}$ with z^α at the j -th spot and $\lambda = (\lambda_1, \dots, \lambda_n)$. The space Y_m is the the space spanned by the elements of the basis of \mathcal{B} that are not special.

$$\begin{aligned} \Gamma_D(H) &= D \circ H - H \circ D \\ &= (0, \dots, 0, \lambda_j z^\alpha, 0, \dots, 0) - (0, \dots, 0, \lambda^\alpha z^\alpha, 0, \dots, 0) \\ &= (0, \dots, 0, (\lambda_j - \lambda^\alpha) z^\alpha, 0, \dots, 0) \\ &= (\lambda_j - \lambda^\alpha) (0, \dots, 0, z^\alpha, 0, \dots, 0) = (\lambda_j - \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}) H \end{aligned}$$

It is easy to see that if $\alpha_k > 0$ for $k = j, j+1, \dots, n$, then we have that $\lambda_j - \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n} \neq 0$.

However, if H is special then

$$\begin{aligned} \Gamma_D(H) &= D \circ H - H \circ D \\ &= (0, \dots, 0, \lambda_j z_1^{\alpha_1} \dots z_{j-1}^{\alpha_{j-1}}, 0, \dots, 0) - (0, \dots, 0, \lambda_1^{\alpha_1} z_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}} z_{j-1}^{\alpha_{j-1}}, 0, \dots, 0) \\ &= (0, \dots, 0, (\lambda_j - \lambda_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}}) z_1^{\alpha_1} \dots z_{j-1}^{\alpha_{j-1}}, 0, \dots, 0) \\ &= (\lambda_j - \lambda_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}}) H = 0 \end{aligned}$$

Note that $\lambda_j = \lambda_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}}$ whenever H is special in \mathcal{H}_m . It is therefore clear that Γ_D

“kills” precisely the elements in \mathcal{B} which are special. Let $H = (h_1, h_2, \dots, h_n) \in X_m$.

Each $h_j(z) = \sum a_n z_1^{\alpha_1} \dots z_{j-1}^{\alpha_{j-1}}$ is a linear combination of resonant monomials which are also homogeneous of degree m .

$$\begin{aligned} \Gamma_D(H) &= D \circ H - H \circ D = D(h_1, \dots, h_n) - H(Dz) \\ &= (\lambda_1 h_1, \dots, \lambda_n h_n) - (h_1(Dz), \dots, h_n(Dz)) \\ &= (\lambda_1 h_1 - h_1(Dz), \dots, \lambda_n h_n - h_n(Dz)). \end{aligned}$$

So if we consider the j -th component of $\Gamma_D(H)$ we can observe the following:

$\lambda_j h_j - h_j(\lambda_1 z_1, \dots, \lambda_n z_n) = \lambda_j h_j - \lambda_1^{\alpha_1} z_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}} z_{j-1}^{\alpha_{j-1}}$. The reason for this is because

$$h_j(\lambda_1 z_1, \dots, \lambda_n z_n) = \lambda_1^{\alpha_1} z_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}} z_{j-1}^{\alpha_{j-1}}.$$

So $\lambda_j h_j - h_j(Dz) = \lambda_j h_j - \lambda_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}} z_1^{\alpha_1} \dots z_{j-1}^{\alpha_{j-1}} = 0$. This implies that $\Gamma_D(X_m) = 0$

and so $\mathcal{H}_m = X_m + \Gamma_D(\mathcal{H}_m)$. Now the obvious question we ask ourselves is this:

Is $\Gamma_A(X_m) = \Gamma_D(X_m)$? We now focus on our given A and let $\varepsilon > 0$.

$$\text{Let } A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad S_\varepsilon = \begin{pmatrix} \varepsilon^n & 0 & \dots & 0 \\ 0 & \varepsilon^{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon \end{pmatrix} \quad \text{and}$$

$$S_\varepsilon^{-1} = \begin{pmatrix} \frac{1}{\varepsilon^n} & 0 & \dots & 0 \\ 0 & \frac{1}{\varepsilon^{n-1}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\varepsilon} \end{pmatrix}$$

$$S_\varepsilon^{-1} A S_\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon^n} & 0 & \dots & 0 \\ 0 & \frac{1}{\varepsilon^{n-1}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \varepsilon^n & 0 & \dots & 0 \\ 0 & \varepsilon^{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon \end{pmatrix}.$$

$$S_\varepsilon^{-1}AS_\varepsilon = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ \varepsilon a_{21} & a_{22} & 0 & \cdots & 0 \\ \varepsilon^2 a_{31} & \varepsilon a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{n-1} a_{n1} & \varepsilon^{n-2} a_{n2} & \varepsilon^{n-3} a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

$$\text{This implies that } S_\varepsilon^{-1}AS_\varepsilon \longrightarrow \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = D \quad \text{as } \varepsilon \longrightarrow 0.$$

Now let π be the projection in \mathcal{H}_m whose range is X_m and whose nullspace is Y_m . Then the map $\pi + \Gamma_D$ is a linear operator which is one-one and invertible on \mathcal{H}_m . For $\varepsilon > 0$ small enough, the linear operator $\pi + \Gamma_{S_\varepsilon^{-1}AS_\varepsilon}$ is also invertible. Let $S = S_\varepsilon$, to each $G \in \mathcal{H}_m$, we will correspond some $H_o \in X_m$ and some $H \in \mathcal{H}_m$ such that $S^{-1}GS \in \mathcal{H}_m$. So $S^{-1}GS = H_o + \Gamma_{S^{-1}AS}(H) = H_o + (S^{-1}AS)H - H(S^{-1}AS)$ and

$$G = SH_oS^{-1} + A(SHS^{-1}) - (SHS^{-1})A$$

Since $S = S_\varepsilon$ is a diagonal matrix then it implies that SHS^{-1} is just a scalar multiple of H , for every $H \in \mathcal{B}$. This is because

$$\begin{aligned} SHS^{-1}(z) &= S(0, \dots, 0, z^\alpha, 0, \dots, 0) \left(\frac{1}{\varepsilon^n} z_1, \dots, \frac{1}{\varepsilon} z_n \right) \\ &= S \left(0, \dots, 0, \left(\frac{1}{\varepsilon^n} z_1 \right)^{\alpha_1} \cdots \left(\frac{1}{\varepsilon} z_n \right)^{\alpha_n}, 0, \dots, 0 \right) \\ &= \left(0, \dots, 0, \varepsilon^{n-(j-1)} \left(\left(\frac{1}{\varepsilon^n} z_1 \right)^{\alpha_1} \cdots \left(\frac{1}{\varepsilon} z_n \right)^{\alpha_n} \right), 0, \dots, 0 \right). \end{aligned}$$

Also for $H_o \in X_m$, $SH_oS^{-1} \in X_m$. Let $SH_oS^{-1} = H_1$ and $SHS^{-1} = H_2$, then

$$G = SH_oS^{-1} + A(SHS^{-1}) - (SHS^{-1})A.$$

$$G = H_1 + AH_2 - H_2A, \text{ for } H_1 \in X_m, H_2 \in \mathcal{H}_m$$

So $G \in X_m + \Gamma_A(\mathcal{H}_m)$ □

We shall now study an example in preparation towards our next lemma. The aim of this example is to explain very simply the behavior of the occurrences of resonances and how they affect our choices of G and H , where G is lower triangular polynomial automorphism and $H \in \mathcal{H}_m$.

Example 4. Let $F(z, w) = (\alpha z, \beta w + z^2)$, $(z, w) \in \mathbb{C}^2$ and α, β are real positive

numbers and $1 > \alpha > \beta > 0$. $F'(0, 0) = A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$$F(z, w) = A \begin{pmatrix} z \\ w \end{pmatrix} + O(|z|^2) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + (0, z^2)$$

Now let $T_m : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map with $T_m(0) = 0$ and $T'_m(0) = Id$, where Id is the identity map. Let G be defined as in lemma 4.0.1 and consider the equation below:

$$(*) \quad G^{-1} \circ T_m \circ F - T_m = O(|z|^m) \quad m \geq 2.$$

Applying G to $(*)$ we get the following:

$$(**) \quad T_m \circ F - G \circ T_m = O(|z|^m) \quad m \geq 2$$

CASE I: We assume that $\alpha^2 = \beta$. This implies a resonance relation between the eigenvalues of A . Let $G_2 = A$ and $T_2 = Id$ for $m = 2$. Then by $(**)$ we get that;

$$(Id \circ F - G_2 \circ Id)(z, w) = F(z, w) - A \begin{pmatrix} z \\ w \end{pmatrix} = (0, z^2). \text{ Let } P_2(z, w) = (0, z^2) \in \mathcal{H}_2.$$

By our assumption, we see that the entry in the second coordinate of $(F - A)(z, w)$ is a

resonant monomial. Let $Q_2(z, w) = (0, z^2)$, then $Q_2 \in X_2 \subset \mathcal{H}_2$.

We define $G_{m+1} = G_m + Q_m$ and $T_{m+1} = T_m + H_m \circ T_m$, for $Q_m \in X_m$ and $H_m \in \mathcal{H}_m$.

So we get that;

$$G_3(z, w) = G_2(z, w) + Q_2(z, w) = (\alpha z, \beta w) + (0, z^2) \quad \text{and}$$

$$T_3(z, w) = (T_2 + H_2 \circ T_2)(z, w) = (z, w) + H_2(z, w).$$

We now need to find H_2 to compute T_3 . By Lemma 4.0.2, we can decompose to get

$$P_2 = A \circ H_2 - H_2 \circ A + Q_2.$$

Let $H_2(z, w) = (a_1 z^2 + a_2 w^2 + a_3 zw, b_1 z^2 + b_2 w^2 + b_3 zw)$, for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{C}$.

$$(P_2 - Q_2)(z, w) = (0, 0) = A(H_2(z, w)) - H_2(\alpha z, \beta w)$$

$$= (\alpha a_1 z^2 + \alpha a_2 w^2 + \alpha a_3 zw, \beta b_1 z^2 + \beta b_2 w^2 + \beta b_3 zw) -$$

$$(\alpha^2 a_1 z^2 + \beta^2 a_2 w^2 + \alpha \beta a_3 zw, \alpha^2 b_1 z^2 + \beta^2 b_2 w^2 + \alpha \beta b_3 zw)$$

$$= \left((\alpha - \alpha^2)a_1 z^2 + (\alpha - \beta^2)a_2 w^2 + (\alpha - \alpha\beta)a_3 zw, (\beta - \alpha^2)b_1 z^2 + (\beta - \beta^2)b_2 w^2 + (\beta - \alpha\beta)b_3 zw \right)$$

So $(\alpha - \alpha^2)a_1 = (\alpha - \beta^2)a_2 = (\alpha - \alpha\beta)a_3 = (\beta - \alpha^2)b_1 = (\beta - \beta^2)b_2 = (\beta - \alpha\beta)b_3 = 0$.

We get that $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$. Hence, $H_2(z, w) = (0, 0)$.

$G_3(z, w) = (\alpha z, \beta w + z^2)$ and $T_3(z, w) = (z, w)$ since $H_2 \circ T_2 = 0$.

$$(T_3 \circ F - G_3 \circ T_3)(z, w) = (Id \circ F - G_3 \circ Id)(z, w)$$

$$= F(z, w) - G_3(z, w)$$

$$= (\alpha z, \beta w + z^2) - (\alpha z, z^2 + \beta w) = (0, 0)$$

So we choose $G(z, w) = G_3(z, w) = (\alpha z, \beta w + z^2)$. So $G^k(z, w) \rightarrow 0$ as $k \rightarrow \infty$ for

all $(z, w) \in \mathbb{C}^2$ since $G^k(z, w) = (\alpha^k z, \beta^k w + \beta^{k-1}(1 + c + \dots + c^{k-1})z^2)$. We choose

$T = T_3 = Id$. Note that if $\alpha^2 = \beta$, it implies that $\alpha^3 < \beta$. This means that there cannot

be any resonance relations between the eigenvalues after $m = 3$. Thus, our choice of T .

We want to find $G^{-k} \circ T \circ F^k$, so we will first calculate G^{-1} .

Define J and K such that

$$J(z, w) = (\alpha z, \beta w) \quad \text{and} \quad K(z, w) = \left(z, \left(\frac{1}{\alpha} z \right)^2 + w \right).$$

Then $G = K \circ J$, $G^{-1} = J^{-1} \circ K^{-1}$, and $J^{-1} = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$. Also $J^{-1}(z, w) = \left(\frac{1}{\alpha} z, \frac{1}{\beta} w \right)$

and $K^{-1}(z, w) = \left(z, w - \left(\frac{1}{\alpha} z \right)^2 \right)$. Thus,

$$G^{-1}(z, w) = J^{-1} \left(z, w - \left(\frac{1}{\alpha} z \right)^2 \right) = \left(\frac{1}{\alpha} z, \frac{1}{\beta} w - \frac{1}{\beta} \left(\frac{1}{\alpha} z \right)^2 \right).$$

$$\begin{aligned} G^{-1} \circ T \circ F(z, w) &= G^{-1}(\alpha z, \beta w + z^2) \\ &= \left[\frac{1}{\alpha}(\alpha z), \frac{1}{\beta}(\beta w + z^2) - \frac{1}{\beta} \left(\frac{1}{\alpha}(\alpha z) \right)^2 \right] \\ &= \left(z, w + \frac{1}{\beta} z^2 - \frac{1}{\beta} z^2 \right) = (z, w) \end{aligned}$$

So $G^{-1} \circ F = Id$.

Let $\Psi : \Omega \rightarrow \mathbb{C}^2$ be defined as $\Psi = \lim_{k \rightarrow \infty} G^{-k} \circ T \circ F^k$. Then $\Psi = Id$.

$\Omega = \left\{ (z, w) \in \mathbb{C}^2 : \lim_{k \rightarrow \infty} F^k(z, w) = (0, 0) \right\} = \mathbb{C}^2$ and so we see that $\Psi(\Omega) = Id(\mathbb{C}^2) = \mathbb{C}^2$.

Alternative

$G^{-1}(z, w) = \left(\frac{1}{\alpha} z, \frac{1}{\beta} w - \frac{1}{\beta \alpha^2} z^2 \right)$ and so we get that

$$G^{-k}(z, w) = \left[\frac{1}{\alpha^k} z, \frac{1}{\beta^k} w - \frac{1}{\alpha^2 \beta^k} \left(1 + \frac{\beta}{\alpha^2} + \dots + \left(\frac{\beta}{\alpha^2} \right)^{k-1} \right) z^2 \right]. \quad \text{Thus, for } T = Id,$$

$$\begin{aligned} G^{-k} \circ Id \circ F^k(z, w) &= G^{-k} \left(F^k(z, w) \right) \\ &= \left[z, w + \frac{1}{\beta} \left(1 + \frac{\alpha^2}{\beta} + \dots + \left(\frac{\alpha^2}{\beta} \right)^{k-1} - \left(\frac{\alpha^2}{\beta} \right)^{k-1} \left(1 + \frac{\beta}{\alpha^2} + \dots + \left(\frac{\beta}{\alpha^2} \right)^{k-1} \right) \right) z^2 \right] \\ &= \left[z, w + \frac{1}{\beta} (k - k) \right] = (z, w) \end{aligned}$$

Remark. In certain situations, calculating the T_m 's for $m \geq 2$ can go on forever as long as there are higher order terms. The ideal choice of T is made based on the relation between the smallest ($|\lambda_n|$) and largest ($|\lambda_1|$) eigenvalues in absolute values. i.e. If there exists an m_0 large enough such that $|\lambda_1|^{m_0} < |\lambda_n|$, then we choose $T = T_{m_0}$.

H_m cannot contain resonant monomials. Therefore the polynomial map T_m cannot contain resonant monomials.

CASE II: We assume that $\alpha^2 < \beta$. This implies that there are no resonances. This also means that $Q_m \in X_m \subset \mathcal{H}_m$ is equal to zero for all $m \geq 2$. Hence

$$G_2 = G_3 = \dots = G_m = A, \quad T_2 = Id \text{ and } P_2 = A \circ H_2 - H_2 \circ A.$$

$$F(z, w) = A \begin{pmatrix} z \\ w \end{pmatrix} + (0, z^2), \quad \text{where } A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \text{ and } P_2(z, w) = (0, z^2).$$

Let $H_2 \in \mathcal{H}_2$ be as in case I. Then by similar computation we solve for H_2 in the equation $P_2 = A \circ H_2 - H_2 \circ A$. So $a_1 = a_2 = a_3 = b_2 = b_3 = 0$ and $b_1(\beta - \alpha^2) = 1$. Thus, $b_1 = \frac{1}{\beta - \alpha^2}$. Hence $H_2(z, w) = \left(0, \frac{1}{\beta - \alpha^2} z^2\right)$. It is easy to see that when $\alpha^2 = \beta$ we run into a problem.

$$T_3 = T_2 + H_2 \circ T_2 \implies T_3(z, w) = \left(z, w + \frac{1}{\beta - \alpha^2} z^2\right).$$

Now $(T_3 \circ F - A \circ T_3)(z, w) = (0, 0)$. So we choose $G = A$ and $T = T_2 = Id$. Thus, $\Psi(z, w) = \lim_{k \rightarrow \infty} A^{-k} F^k(z, w) = \left(z, w + \frac{1}{\beta - \alpha^2} z^2\right)$.

Remark. We can observe that since $G^{-k}(z, w) = \left(\frac{1}{\alpha^k} z, \frac{1}{\beta^k} w\right)$, then $G^{-k} \circ T_3 \circ F^k = \left(z, w + \frac{1}{\beta} (1 + c + \dots + c^{k-1} + \frac{c^k}{1-c}) z^2\right) = \left(z, w + \frac{1}{\beta - \alpha^2} z^2\right)$. Thus, $\Psi(z, w) = \left(z, w + \frac{1}{\beta - \alpha^2} z^2\right)$. So if we should choose $T = T_3$ we still get Ψ as desired but it is not necessary since the assumption that $\alpha^2 < \beta$ is enough to make us choose $T = T_2$.

We shall consider another example in \mathbb{C}^3 . We begin by stating the following theorem in [7]:

Theorem 4.0.3. Let $P = (P_1, \dots, P_n)$ be any polynomial mapping from \mathbb{C}^n to itself with $P'(0)$ invertible. Let $\max_i (\deg(P_i)) \leq d$. Then there exists an automorphism ψ of \mathbb{C}^n such that $\psi(z) - P(z) = O(|z|^{d+1})$.

Example 5. Let $\tilde{F}(\tau) = (\alpha z + t^5, \beta w + z^2 + z^6, \gamma t + w^2 + z^2 w)$, for $\tau = (z, w, t) \in \mathbb{C}^3$, be a germ of an automorphism of \mathbb{C}^3 . $\tilde{F}'(0)$ is also invertible. So by theorem 4.0.3, the germ of automorphism \tilde{F} of \mathbb{C}^3 can be realized as an automorphism F defined as

$$F(\tau) = (\alpha z, \beta w + z^2, \gamma t + w^2 + z^2 w), \text{ for } 1 > |\alpha| \geq |\beta| \geq |\gamma| > 0. \text{ So } F(0, 0, 0) = (0, 0, 0)$$

and $A = F'(0, 0, 0) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$

CASE I: Assume that there are no resonance relations between the eigenvalues of A .

$$F(\tau) = A\tau + O(|\tau|^2) = (\alpha z, \beta w, \gamma t) + (0, z^2, w^2 + z^2 w).$$

Consider the expression below:

$$G_m^{-1} \circ T_m \circ F - T_m = O(|\tau|^m) \quad \text{for } m \geq 2.$$

We need to find G , lower triangular map in \mathbb{C}^3 and T , a polynomial map in \mathbb{C}^3 , such that the sequence $\{G^{-k} \circ T \circ F^k\}_k$ converges as k tends to infinity. Let $G_2 = A$ and $T_2 = Id$ for $m = 2$. We get that,

$$(T_2 \circ F - G_2 \circ T_2)(\tau) = F(\tau) - A\tau = (0, z^2, w^2 + z^2 w). \text{ So } P_2(\tau) = (0, z^2, w^2) \in \mathcal{H}_2.$$

$F(\tau) - A\tau - P_2(\tau) = (0, 0, z^2 w)$. Also $P_2 = Q_2 + A \circ H_2 - H_2 \circ A = A \circ H_2 - H_2 \circ A$ since $Q_2 = 0$.

Now let $G_{m+1} = G_m + Q_m$ and $T_{m+1} = T_m + H \circ T_m$, $m \geq 2$. So $G_m = A$ for $m \geq 2$ and $T_3 = T_2 + H_2 \circ T_2 = Id + H_2 \circ Id$. We shall then find H_2 by considering $P_2 = A \circ H_2 - H_2 \circ A$.

Let $h_i(\tau) = a_1^i z^2 + a_2^i w^2 + a_3^i t^2 + a_4^i z w + a_5^i z t + a_6^i w t$, where $a_j^i \in \mathbb{C}$ for $i = 1, 2, 3$ and $1 \leq j \leq 6$. Now let $H_2(\tau) = (h_1(\tau), h_2(\tau), h_3(\tau))$. Substituting this into P_2 and solving similarly as in example 4, we get the following:

$a_j^1 = a_k^2 = a_l^3 = 0$ for $1 \leq j \leq 6$, $2 \leq k \leq 6$ and $l = 1, 3, 4, 5, 6$. So $a_1^2 = \frac{1}{\beta - \alpha^2}$ and $a_2^3 = \frac{1}{\gamma - \beta^2}$. Thus, $H_2(\tau) = \left(0, \frac{1}{\beta - \alpha^2}z^2, \frac{1}{\gamma - \beta^2}w^2\right)$.

So $T_3(\tau) = (z, w, t) + \left(0, \frac{1}{\beta - \alpha^2}z^2, \frac{1}{\gamma - \beta^2}w^2\right) = \left(z, w + \frac{1}{\beta - \alpha^2}z^2, t + \frac{1}{\gamma - \beta^2}w^2\right)$.

$$\begin{aligned}
 (T_3 \circ F - G_3 \circ T_3)(\tau) &= T_3 \circ F(\tau) - A \circ T_3(\tau) \\
 &= \left[0, 0, \left(1 + \frac{2\beta}{\gamma - \beta^2}\right)z^2w + \frac{1}{\gamma - \beta^2}z^4\right].
 \end{aligned}$$

So $P_3(\tau) = \left[0, 0, \left(1 + \frac{2\beta}{\gamma - \beta^2}\right)z^2w\right]$ and $P_3 = A \circ H_3 - H_3 \circ A$, since $Q_3 = 0$. Now

we simply assume that $H_3(\tau) = (az^3, bw^3, ct^3 + dz^2w)$, for $a, b, c, d \in \mathbb{C}$. Substituting H_3 into P_3 and solving, we get that:

$$a = b = c = 0 \quad \text{and} \quad d = \frac{1}{\gamma - \alpha^2\beta} + \frac{2\beta}{(\gamma - \alpha^2\beta)(\gamma - \beta^2)}. \quad \text{So } H_3(\tau) = (0, 0, dz^2w).$$

$$T_4(\tau) = \left[z, w + \frac{1}{\beta - \alpha^2}z^2, t + \frac{1}{\gamma - \beta^2}w^2 + dz^2w + \frac{d}{\beta - \alpha^2}z^4\right].$$

$$\begin{aligned}
 (T_4 \circ F - G_4 \circ T_4)(\tau) &= T_4 \circ F(\tau) - A \circ T_4(\tau) \\
 &= \left[0, 0, \left(\frac{1}{\gamma - \beta^2} + \left(\alpha^2 + \frac{\alpha^4 - \gamma}{\beta - \alpha^2}\right)d\right)z^4\right].
 \end{aligned}$$

Then $P_4(\tau) = (0, 0, \mu z^4)$, where $\mu = \frac{1}{\gamma - \beta^2} + \left(\alpha^2 + \frac{\alpha^4 - \gamma}{\beta - \alpha^2}\right)d$. Similarly, we simply, let $H_4(\tau) = (ez^4, fw^4, gz^4)$. Computing $P_4 = A \circ H_4 - H_4 \circ A$ gives the following results:

$$e = f = 0 \quad \text{and} \quad g = \frac{\mu}{\gamma - \alpha^4}. \quad \text{So } H_4(\tau) = (0, 0, gz^4).$$

$$\text{So } T_5(\tau) = \left[z, w + \frac{1}{\beta - \alpha^2}z^2, t + \frac{1}{\gamma - \beta^2}w^2 + dz^2w + \frac{1}{\gamma - \alpha^4}\left(\frac{1}{\gamma - \beta^2} + \alpha^2d\right)z^4\right].$$

$$(T_5 \circ F - G_5 \circ T_5)(\tau) = T_5 \circ F(\tau) - A \circ T_5(\tau) = (0, 0, 0).$$

We therefore choose $G(\tau) = A\tau = (\alpha z, \beta w, \gamma t)$. Hence $G^{-1}(\tau) = \left(\frac{1}{\alpha}z, \frac{1}{\beta}w, \frac{1}{\gamma}t\right)$ and

$G^{-k}(\tau) = \left(\frac{1}{\alpha^k}z, \frac{1}{\beta^k}w, \frac{1}{\gamma^k}\right)$. Let t^* be the expression below:

$$\sum_{j=2}^k \left(\left[\beta^{j-1}w + \beta^{j-2} \left(\sum_{i=2}^j \left(\frac{\alpha^2}{\beta} \right)^{i-2} \right) z^2 \right]^2 + (\alpha^2)^{j-1} z^2 \left[\beta^{j-1}w + \beta^{j-2} \left(\sum_{i=2}^j \left(\frac{\alpha^2}{\beta} \right)^{i-2} \right) z^2 \right] \right) \gamma^{k-j}.$$

Then $F^k(\tau) = \left[\alpha^k z, \beta^k w + \beta^{k-1} \left(1 + \frac{\alpha^2}{\beta} + \dots + \left(\frac{\alpha^2}{\beta} \right)^{k-1} \right) z^2, \gamma^k t + \gamma^{k-1} (w^2 + z^2 w) + t^* \right]$.

Clearly $F^k(\tau) \rightarrow 0$ as $k \rightarrow \infty$. So we shall find $\Psi = \lim_{k \rightarrow \infty} G^{-k} \circ T \circ F^k$.

Observations:

1. If $\alpha^j < \gamma$, we choose $T = T_j$ for $1 \leq j \leq 4$ and so $\Psi(\tau) = \lim_{k \rightarrow \infty} A^{-k} \circ T_j \circ F^k(\tau) = T_5(\tau)$.

Clearly, $T_5 \circ F - G_5 \circ T_5 = 0$. This implies that $\Psi(\tau) = T_5(\tau)$.

2. For $\alpha^m < \gamma$, $m \geq 5$, then we choose $T = T_m = T_5$ since $T_q = T_5$ for $m \geq 6$. Also

$$\Psi(\tau) = \lim_{k \rightarrow \infty} A^{-k} \circ T_m \circ F^k = T_5(\tau).$$

CASE II: Assume $\beta^2 = \gamma$ and $\alpha^5 = \gamma$. These are resonance of F relative to the third coordinate. We can simply conclude that $\alpha^6 < \gamma$. In a similar way, we need to find G and T such that

$$G^{-1} \circ T_m \circ F - T_m = O(|z|^m), \quad m \geq 2$$

and also the sequence $\{G^{-k} \circ T \circ F^k\}_k$ converges as k tends to infinity. By letting $G_2 = A$ and $T_2 = Id$ for $m = 2$, we consider $(T_2 \circ F - G_2 \circ T_2)(\tau)$. We get $P_2(\tau) = (0, z^2, w^2) \in \mathcal{H}_2$.

$P_2 = Q_2 + A \circ H_2 - H_2 \circ A$, $H_2 \in \mathcal{H}_2$ and $Q_2 \in X_2$. The entry in the third coordinate of P_2 together with the assumption that $\beta^2 = \gamma$ implies a resonance monomial. Thus, $Q_2(\tau) = (0, 0, w^2) \in X_2$.

Let $G_{m+1} = G_m + Q_m$ and $T_{m+1} = T_m + H_m \circ T_m$, for $m \geq 2$.

$G_3 = G_2 + Q_2$ and so $G_3(\tau) = (\alpha z, \beta w, \gamma t + w^2)$ and $T_3 = Id + H_2 \circ Id$. We then find H_2 by solving $P_2 - Q_2 = A \circ H_2 - H_2 \circ A$. If we were to solve $P_2 = A \circ H_2 - H_2 \circ A$ for H_2 , we will end up getting $0 = 1$ in one of the equations. So by subtracting Q_2 , we can now

solve for H_2 . Let H_2 be as in *case I*, then we get that $H_2(\tau) = \left[0, \frac{1}{\beta - \alpha^2}z^2, 0\right]$.

$$\text{So } T_3(\tau) = (z, w, t) + \left(0, \frac{1}{\beta - \alpha^2}z^2, 0\right) = \left(z, w + \frac{1}{\beta - \alpha^2}z^2, t\right).$$

This implies that $(T_3 \circ F - G_3 \circ T_3)(\tau) = \left[0, 0, \left(1 - \frac{2}{\beta - \alpha^2}\right)z^2w - \frac{1}{(\beta - \alpha^2)^2}z^4\right]$.

$$\text{So } P_3(\tau) = \left[0, 0, \left(1 - \frac{2}{\beta - \alpha^2}\right)z^2w\right]. \text{ Let } \lambda = 1 - \frac{2}{\beta - \alpha^2}. P_3 = A \circ H_3 - H_3 \circ A$$

since $Q_3 = 0$. Now, by simply letting $H_3(\tau) = (a'z^3, b'w^3, c't^3 + d'z^2w)$ and solving $P_3 = A \circ H_3 - H_3 \circ A$, we get $d' = \frac{\lambda}{\gamma - \alpha^2\beta}$. $\therefore H_3(\tau) = (0, 0, d'z^2w)$.

$$T_4(\tau) = \left[z, w + \frac{1}{\beta - \alpha^2}z^2, t + d'z^2w + \frac{d'}{\beta - \alpha^2}z^4\right] \text{ and } G_4(\tau) = G_3(\tau). \text{ We get that}$$

$$(T_4 \circ F - G_4 \circ T_4)(\tau) = \left[0, 0, \left(d'\alpha^2 + \frac{d'\alpha^4}{\beta - \alpha^2} - \frac{d'\gamma}{\beta - \alpha^2} - \frac{1}{(\beta - \alpha^2)^2}\right)z^4\right].$$

$$\text{So } P_4(\tau) = (0, 0, \mu z^4), \text{ where } \mu = d'\alpha^2 + \frac{d'\alpha^4}{\beta - \alpha^2} - \frac{d'\gamma}{\beta - \alpha^2} - \frac{1}{(\beta - \alpha^2)^2}. \text{ Also } Q_4 = 0$$

and $G_5(\tau) = G_3(\tau)$. Simply let $H_4(\tau) = (k_1z^4 + k_2w^4, l_1z^4 + l_2w^4, m_1z^4 + m_2w^4)$ and consider $P_4 = A \circ H_4 - H_4 \circ A$. We get that

$$k_1 = k_2 = l_1 = l_2 = m_2 = 0 \text{ and } m_1 = \frac{\mu}{\gamma - \alpha^4}. \text{ Thus, } H_4(\tau) = (0, 0, m_1z^4).$$

$$\text{So } T_5(\tau) = \left[z, w + \frac{1}{\beta - \alpha^2}z^2, t + d'z^2w + \left(\frac{d'}{\beta - \alpha^2} + m_1\right)z^4\right].$$

Hence $(T_5 \circ F - G_5 \circ T_5)(\tau) = (0, 0, 0)$.

So we choose $G(\tau) = G_3(\tau) = (\alpha z, \beta w, \gamma t + w^2)$ and $T = T_6 = T_5$ since $Q_5 = 0$. Thus, we have that $G^{-1}(\tau) = \left(\frac{1}{\alpha}z, \frac{1}{\beta}w, \frac{1}{\gamma}t - \frac{1}{\beta^2\gamma}w^2\right)$ and

$$G^{-k}(\tau) = \left[\frac{1}{\alpha^k}z, \frac{1}{\beta^k}w, \frac{1}{\gamma^k}t - \frac{1}{\beta^2\gamma^k}\left(1 + \frac{\gamma}{\beta^2} + \dots + \left(\frac{\gamma}{\beta^2}\right)^{k-1}\right)w^2\right] \text{ and so we have that}$$

$$\Psi(\tau) = \lim_{k \rightarrow \infty} G^{-k} \circ T \circ F^k(\tau) = T_5(\tau).$$

Lemma 4.0.4. *Suppose that V is a neighborhood of 0 in \mathbb{C}^n , $F : V \rightarrow \mathbb{C}^n$ is holomorphic, $F(0) = 0$, and that all eigenvalues λ_i of $A = F'(0)$ satisfy $0 < |\lambda_i| < 1$. Then there exists*

i). a lower triangular polynomial automorphism G of \mathbb{C}^n with $G(0) = 0$, $G'(0) = A$, and

ii). polynomial maps $T_m : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $T_m(0) = 0$, $T'_m(0) = Id$, so that

$$G^{-1} \circ T_m \circ F - T_m = O(|z|^m) \quad m = 2, 3, \dots$$

Proof. We start by choosing coordinates so that A is lower triangular with eigenvalues λ_i such that $|\lambda_1| \geq \dots \geq |\lambda_n|$. Suppose that $T_m : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $T_m(0) = 0$, $T'_m(0) = Id$, G_m is a lower triangular polynomial automorphism of \mathbb{C}^n with $G'_m(0) = A$ and for $m \geq 2$, we get that

$$T_m \circ F - G_m \circ T_m = O(|z|^m) \tag{4.1}$$

If for $m = 2$ we put $G_2 = A$ and $T_2 = Id$, then (4.1) is true. This is because

$$T_2 \circ F - G_2 \circ T_2 = O(|z|^2) \quad \text{and so} \quad (Id \circ F - A \circ Id)(z) = F(z) - Az = O(|z|^2).$$

Let $P_m \in \mathcal{H}_m$. P_m is defined as all the m th order terms of $T_m \circ F - G_m \circ T_m$ and Q_m represents all the m th order terms of $T_m \circ F - G_m \circ T_m$ spanned by the *special* basis elements. By lemma 4.0.2 we can decompose P_m as $P_m = Q_m + A \circ H_m - H_m \circ A$ for some $Q_m \in X_m$ and $H_m \in \mathcal{H}_m$. We therefore rewrite (4.1) as :

$$T_m \circ F - G_m \circ T_m - P_m = O(|z|^{m+1}) \tag{4.2}$$

The above expression, (4.2), means that the power series expansion on the left side has terms of degree $m + 1$ or more. This is so because from (4.1) we know that the power

series expansion of its left side has terms of degree m or more and since $P_m \in \mathcal{H}_m$, has terms of degree exactly m , the power series expansion of the left side of (4.2) has terms of degree $m + 1$ or more. Here is why

$$T_m \circ F - G_m \circ T_m = \left(\dots, \sum_{|\alpha|=m} a_\alpha z^\alpha + \sum_{|\alpha| \geq m+1} a_\alpha z^\alpha, \dots \right) \quad \text{and}$$

$$P_m = \left(\dots, \sum_{|\alpha|=m} a_\alpha z^\alpha, \dots \right).$$

We get that $T_m \circ F - G_m \circ T_m - P_m = \left(\dots, \sum_{|\alpha| \geq m+1} a_\alpha z^\alpha, \dots \right) = O(|z|^{m+1})$.

Now we define the following:

$$G_{m+1} = G_m + Q_m \quad \text{and} \quad T_{m+1} = T_m + H_m \circ T_m.$$

We want to show that (4.1) holds for $m + 1$. We shall prove this by also considering the idea of *resonance* as explained in section 2.2. In the first case, we assume that there is no occurrence of resonance and prove that (4.1) holds for $m + 1$. Secondly, we shall consider the case where there is resonance.

CASE I: We assume that there are no resonances. This implies that $Q_m \in X_m = 0$, thus, $Q_m = 0$. Therefore $P_m = Q_m + A \circ H_m - H_m \circ A = A \circ H_m - H_m \circ A$. By the given definition, $G_{m+1} = G_m = A$ and $T_{m+1} = T_m + H_m \circ T_m$.

$$\begin{aligned} T_{m+1} \circ F - G_{m+1} \circ T_{m+1} &= T_{m+1} \circ F - A \circ T_{m+1} \\ &= (H_m \circ T_m + T_m) \circ F - A \circ (H_m \circ T_m + T_m) \\ &= T_m \circ F + H_m \circ T_m \circ F - A \circ H_m \circ T_m - A \circ T_m \\ &= T_m \circ F - A \circ T_m + H_m \circ (I + O(|z|^2)) \circ (A + O(|z|^2)) - A \circ H_m \circ (I + O(|z|^2)) \\ &= T_m \circ F - A \circ T_m + H_m \circ I \circ A + O(|z|^{m+1}) - A \circ H_m \circ I + O(|z|^{m+1}) \\ &= T_m \circ F - A \circ T_m + (H_m \circ A - A \circ H_m) + O(|z|^{m+1}) \\ &= T_m \circ F - A \circ T_m - P_m + O(|z|^{m+1}) = O(|z|^{m+1}) \quad \text{from (4.2)} \end{aligned}$$

We therefore need to find an H such that $P_m = A \circ H_m - H_m \circ A$

CASE II: We assume that there are resonances. So $P_m = Q_m + A \circ H_m - H_m \circ A$.

Then we have that

$$Q_m \circ T_{m+1} - Q_m = O(|z|^{m+1}) \quad (4.3)$$

$$T_{m+1} - T_m - H_m = O(|z|^{m+1}) \quad (4.4)$$

Proof of 3.5

Let $m = 2, T_2 = Id$, and $G_2 = A$. So we get that;

$$T_3 = T_2 + H_2 \circ Id \quad H_2 \in \mathcal{H}_2$$

$$\text{And so } Q_2 \circ T_3 - Q_2 = Q_2 \circ (Id + H \circ Id) - Q_2 \quad Q_2 \in X_2$$

$$= Q_2 + Q_2 \circ H_2 - Q_2 = Q_2 \circ H_2 = O(|z|^3)$$

Even more generally, we have that

$$Q_m \circ T_{m+1} = Q_m \circ (Id + S_2) \quad \text{where } S_2 \text{ represents the higher order terms}$$

$$= \left(\sum_{|\alpha|=m} a_\alpha z^\alpha \right) \circ (Id + S_2)$$

$$= \sum_{|\alpha|=m} [(Id + S_2)(z)]^\alpha = \sum_{|\alpha|=m} a_\alpha z^\alpha + O(|z|^{m+1})$$

$$Q_m \circ T_{m+1} - Q_m = \sum_{|\alpha|=m} a_\alpha z^\alpha + O(|z|^{m+1}) - \sum_{|\alpha|=m} a_\alpha z^\alpha = O(|z|^{m+1}) \quad \square$$

Proof of 3.6

$$T_{m+1} - T_m - H_m = H_m \circ T_m - H_m$$

$$= H_m \circ (Id + O(|z|^2)) - H_m = O(|z|^{m+1}) \quad \square$$

Now we want to show that (4.1) is true for $m + 1$. Said differently, we shall show that

$T_{m+1} \circ F - G_{m+1} \circ T_{m+1} = O(|z|^{m+1})$ is also true.

Let $\mathbf{I} = T_{m+1} \circ F - G_{m+1} \circ T_{m+1}$ and $\mathbf{II} = T_m \circ F - G_m \circ T_m - P_m$

Claim 6. $\mathbf{I} - \mathbf{II} = O(|z|^{m+1})$

If this claim is true, then $\mathbf{I} = \mathbf{II} + O(|z|^{m+1}) = O(|z|^{m+1})$ since from equation 4.2 we know that $\mathbf{II} = O(|z|^{m+1})$. Now by substituting the definitions of G_{m+1} and T_{m+1} into \mathbf{I} and \mathbf{II} , we arrive at the following:

$$\begin{aligned} \mathbf{I} - \mathbf{II} &= (T_{m+1} \circ F - G_{m+1} \circ T_{m+1}) - (T_m \circ F - G_m \circ T_m - P_m) \\ &= (T_m + H_m \circ T_m) \circ F - (G_m + Q_m) \circ T_{m+1} - T_m \circ F + G_m \circ T_m + P_m \\ &= T_m \circ F + H_m \circ T_m \circ F - G_m \circ T_{m+1} - Q_m \circ T_{m+1} - T_m \circ F + G_m \circ T_m + P_m \end{aligned}$$

$$\mathbf{I} - \mathbf{II} = H_m \circ T_m \circ F + G_m \circ T_m - G_m \circ T_{m+1} - Q_m \circ T_{m+1} + Q_m + A \circ H_m - H_m \circ A$$

$$\text{So } \mathbf{I} - \mathbf{II} = \underbrace{H_m \circ T_m \circ F - H_m \circ A}_{\alpha} + \underbrace{G_m \circ T_m - G_m \circ T_{m+1} + A \circ H_m}_{\beta} + \underbrace{Q_m - Q_m \circ T_{m+1}}_{\gamma}$$

We will show that α , β , and γ are all equal to $O(|z|^{m+1})$ and we will be done.

Proof of α

$$\begin{aligned} H_m \circ T_m \circ F - H_m \circ A &= H_m \circ (Id + O(|z|^2)) \circ (A + O(|z|^2)) - H_m \circ A \\ &= H_m \circ Id \circ A + O(|z|^{m+1}) - H_m \circ A = O(|z|^{m+1}) \quad \square \end{aligned}$$

Proof of β

$$G_2 = A$$

$$G_3 = G_2 + Q_2 = A + Q_2, \quad Q_2 \in X_2$$

$$G_4 = G_3 + Q_3 = A + Q_2 + Q_3, \quad Q_3 \in X_3$$

\vdots

$$\begin{aligned}
 G_m &= A + \sum_{j=2}^{m-1} Q_j, \quad Q_j \in X_j \\
 G_m \circ T_m - G_m \circ T_{m+1} + A \circ H_m &= A \circ T_m + \sum_{j=2}^{m-1} Q_j \circ T_m - (A + \sum_{j=2}^{m-1} Q_j)(T_m + H_m \circ T_m) + A \circ H_m \\
 &= A \circ T_m + \sum_{j=2}^{m-1} Q_j \circ T_m - A \circ (T_m + H_m \circ T_m) - \sum_{j=2}^{m-1} Q_j \circ (T_m + H_m \circ T_m) + A \circ H_m \\
 &= A \circ T_m - A \circ T_m - A \circ H_m \circ T_m + A \circ H_m + \sum_{j=2}^{m-1} [Q_j \circ T_m - Q_j \circ (T_m + H_m \circ T_m)] \\
 &= A \circ H_m - A \circ H_m \circ T_m + \sum_{j=2}^{m-1} [Q_j \circ T_m - Q_j \circ (T_m + H_m \circ T_m)] \\
 &= A \circ H_m - A \circ H_m \circ (Id + O(|z|^2)) + \sum_{j=2}^{m-1} [Q_j \circ T_m - Q_j \circ (T_m + H_m \circ T_m)] \\
 &= A \circ H_m - A \circ H_m + O(|z|^{m+1}) + \sum_{j=2}^{m-1} (Q_j \circ T_m - Q_j \circ T_m + O(|z|^{m+1})) \\
 &= O(|z|^{m+1}) + O(|z|^{m+1})
 \end{aligned}$$

Hence $G_m \circ T_m - G_m \circ T_{m+1} + A \circ H_m = O(|z|^{m+1})$. □

Remark. $A \circ H_m$ is of degree m . In the above proof of β we can simply take $Q_j(z) = z^2$. This implies that $Q_j(T_m) - Q_j(T_m + H_m \circ T_m) = T_m^2 - T_m^2 + O(|z|^{m+1})$. Also if B, C, D , and E are maps, then the following holds: $(B + C + D) \circ E = B \circ E + C \circ E + D \circ E$. However, $E \circ (B + C + D) \neq E \circ B + E \circ C + E \circ D$. Equality will hold in this situation if only E is linear. T_m consists of polynomials of degree at most $(m - 1)!$ or less.

Proof of γ

β is the exact negation of (4.3) and so $Q_m - Q_m \circ T_{m+1} = O(|z|^{m+1})$. □

So we have that $\mathbf{I} - \mathbf{II} = O(|z|^{m+1})$ which further implies that $\mathbf{I} = \mathbf{II} + O(|z|^{m+1})$. Hence equation 4.1 is true for $m + 1$.

Now if m grows large enough, $X_m = 0$, which means that $Q \in X_m = 0$ and so $G_{m+1} = G_m$. This therefore implies that we have the G as defined in part (i) of lemma

4.0.2 which satisfies

$$T_m \circ F - G \circ T_m = O(|z|^m) \quad \text{for all } m \geq 2.$$

If we then apply G^{-1} to the above equation we get that $G^{-1} \circ T_m \circ F - T_m = O(|z|^m)$. \square

We are now well prepared for the main proof of the theorem since we have stated and proved the above three (3) lemmas. We state the following theorem:

Theorem 4.0.5. *Suppose that $F \in \text{Aut}(\mathbb{C}^n)$, $F(0) = 0$, and all eigenvalues λ_i of $F'(0)$ satisfy $|\lambda_i| < 1$ for $i = 1, 2, \dots, n$. Then there exists a biholomorphic map Φ from \mathbb{C}^n onto the region*

$$\Omega = \left\{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = 0 \right\}.$$

Moreover, Φ can be chosen so that $J\Phi \equiv 1$ if JF is constant and also $\Phi = \Psi^{-1}$.

Proof. We shall again choose coordinates so that $A = F'(0)$ is lower diagonal, and $|\lambda_1| \geq \dots \geq |\lambda_n|$. As a result of the proof of Lemma 4.0.2, we can find a diagonal operator S such that $A_0 = S^{-1}AS$ is close to being diagonal and that $|A_0 z| \leq c|z|$ holds for some constant $c < 1$ and all $z \in \mathbb{C}^n$. This is true because of the assumption that $|\lambda_1| < 1$. Now if we put $F_0 = S^{-1}FS$ and prove the above theorem for F_0 to obtain Φ_0 and Ω_0 , then it also holds for F , with $\Phi = S\Phi_0 S^{-1}$ and $\Omega = S(\Omega_0)$.

We now assume further that $\|A\| < 1$. We fix α such that $\|A\| < \alpha < 1$. So at the point 0, $\exists r > 0$ such that for $z \in \mathbb{B}(0, r)$ we have that

$$(1) \quad |F(z)| \leq \alpha|z| \quad \text{for } |z| \leq r$$

It is obvious that $\mathbb{B}(0, r) \subset \Omega$ and that Ω is a region, and $F(\Omega) = \Omega$. The proof is similar to that of theorem 3.0.1. We shall consider three steps here. First, we will associate G to F as in lemma 4.0.4. Secondly, we will apply lemma 4.0.1 (i) to G^{-1} in

place of G and lastly we shall apply the Schwarz's lemma to G^{-1} . Then there is a constant $\gamma < \infty$ such that

$$(2) \quad |G^{-k}(\omega) - G^{-k}(\omega')| \leq \gamma^k |\omega - \omega'| \quad k = 1, 2, \dots$$

for all ω, ω' with $|\omega| \leq \frac{1}{2}$, $|\omega'| \leq \frac{1}{2}$.

From part (i) of lemma 4.0.1 (by replacing G with G^{-1}) we have that $G^{-k}(U^n) \subset \gamma^k U^n$ for $k = 1, 2, \dots$ and so $\gamma > \frac{1}{|\lambda_n|}$. i.e. $\gamma > \frac{1}{|\lambda_n|} \geq \dots \geq \frac{1}{|\lambda_1|}$. Also we know by assumption that $\|A\| < \alpha < 1$ and the fact that we associate G to F in lemma 4.0.4, we get that $\alpha > |\lambda_1| \geq \dots \geq |\lambda_n|$. Recall that we can always find m positive such that $\alpha^m < |\lambda_n|$. This means that $|\lambda_1|^m < \alpha^m < \frac{1}{\gamma} < |\lambda_n|$. So for a fixed positive integer m , we have that $\alpha^m < \frac{1}{\gamma}$.

By lemma 4.0.4, we can find a polynomial map $T = T_m$, with $T(0) = 0$, $T'(0) = I$ such that $G^{-1} \circ T \circ F - T = O(|\omega|^m)$ for $m = 2, 3, \dots$. This means that there exists $\delta > 0$ and a positive number M_1 such that

$$(3) \quad |G^{-1}TF(\omega) - T(\omega)| \leq M_1|\omega|^m \quad \text{for } |\omega| < \delta.$$

Let E be a compact subset of Ω . Then we have that $F^s(E) \subset \mathbb{B}(0, r)$ for some integer s . Let $k' = s + k$, then $F^{k'}(E) = F^{s+k}(E) \subset F^k(\mathbb{B}(0, r)) \subset \alpha^k \mathbb{B}(0, r)$, for all $k \geq 0$ by (1).

It implies that we can find a positive number M_2 such that

$$(4) \quad |F^k(z)| \leq M_2 \alpha^k < \delta$$

for all $z \in E$. So for $|F^k(z)| < \delta$ we have that

$$(5) \quad \begin{aligned} |G^{-1}TF^{k+1}(z) - TF^k(z)| &= |G^{-1}TF(F^k(z)) - TF^k(z)| \\ &\leq M_1|F^k(z)|^m \leq M_1M_2^m \alpha^{mk} \end{aligned}$$

Since m is fixed and $\alpha < 1$, for k large enough, $|TF^k(z)| < \frac{1}{2}$ and $|G^{-1}TF^k(z)| < \frac{1}{2}$, for all $z \in E$. Now let $\omega = G^{-1}TF^{k+1}(z)$ and $\omega' = TF^k(z)$. Therefore by applying (2) to (5)

we get that

$$\begin{aligned}
 |G^{-k}(\omega) - G^{-k}(\omega')| &= |G^{-k}(G^{-1}TF^{k+1}(z)) - G^{-k}(TF^k(z))| \\
 &\leq \gamma^k |G^{-1}TF^{k+1}(z) - TF^k(z)| \\
 (6) \quad |G^{-(k+1)}TF^{k+1}(z) - G^{-k}TF^k(z)| &\leq \gamma^k M_1 M_2^m \alpha^{mk} = M_1 M_2^m (\gamma \alpha^m)^k
 \end{aligned}$$

We now put $\Psi_k = G^{-k} \circ T \circ F^k(z)$ and consider the sequence $\{\Psi_k\}_k$. We want to show that $\{\Psi_k\}_k$ is a Cauchy sequence. Let $\varepsilon > 0$ and $k_1 < k_2$ for some $k_1, k_2 \in \mathbb{N}$ with $k_1, k_2 > N \in \mathbb{N}$.

$$\begin{aligned}
 |\Psi_{k_2} - \Psi_{k_1}| &= |\Psi_{k_2} - \Psi_{k_2-1} + \Psi_{k_2-1} - \Psi_{k_2-2} + \cdots + \Psi_{k_1+1} - \Psi_{k_1}| \\
 &\leq |\Psi_{k_1+1} - \Psi_{k_1}| + \cdots + |\Psi_{k_2} - \Psi_{k_2-1}| \\
 &\leq \sum_{k_1=m}^{k_2-1} Dt^j \leq \sum_{k_1=m}^{\infty} Dt^j = D \frac{t^m}{1-t} < \varepsilon
 \end{aligned}$$

where $t^j = (\gamma \alpha^m)^j$ and $D = (k_2 - m)M_1 M_2^m$. The sequence $\{\Psi_k\}_k$ converges since E is a complete space. Thus, its limit

$$\Psi(z) = \lim_{k \rightarrow \infty} \Psi_k = \lim_{k \rightarrow \infty} (G^{-k} \circ T \circ F^k)(z)$$

exists uniformly on every compact subset of Ω . It also defines a map $\Psi : \Omega \rightarrow \mathbb{C}^n$ which is holomorphic and also satisfies $\Psi(0) = 0$, $\Psi'(0) = Id$ as well as the functional equation

$$(7) \quad G^{-1} \circ \Psi \circ F = \Psi$$

Claim 7. $\Psi : \Omega \rightarrow \mathbb{C}^n$ is onto.

We know that $F(\Omega) = \Omega$ and so by (7) we can see that the range of Ψ and that of $G^{-1} \circ \Psi$ are equal. So we have that

$$\begin{aligned}
 G^{-1}(\Psi(\Omega)) &= G^{-1}(\Psi(F(\Omega))) = \Psi(\Omega) \\
 G^{-2}(\Psi(\Omega)) &= G^{-1}(\Psi(\Omega)) = \Psi(\Omega) \\
 G^{-3}(\Psi(\Omega)) &= G^{-2}(\Psi(\Omega)) = \Psi(\Omega)
 \end{aligned}$$

$$\begin{array}{c} \vdots \\ G^{-k}(\Psi(\Omega)) = \Psi(\Omega) \\ \vdots \end{array}$$

Since $\mathbb{B}(0, r) \subset \Omega$, it means that $\Psi(\Omega)$ contains a neighborhood of 0. Hence by lemma 4.0.1 we can conclude that $\Psi(\Omega) = \mathbb{C}^n = G^{-k}(\Psi(\Omega))$. Hence Ψ is surjective.

Claim 8. Ψ is one-to-one.

We assume that $\Psi(x) = \Psi(y)$ for $x, y \in \Omega$. The functional equation in (7) can be rewritten as

$$(\star\star) \quad \Psi \circ F = G \circ \Psi$$

by applying G^{-1} to both sides of (7). Therefore by $(\star\star)$ and the assumption that $\Psi(x) = \Psi(y)$, we have that

$$(\star\star\star) \quad \Psi(F(x)) = G(\Psi(x)) = G(\Psi(y))$$

$$\begin{aligned} \text{So } \Psi(F^2(x)) &= \Psi(F(F(x))) = G(\Psi(F(x))) \\ &= G(\Psi(F(y))) \quad \text{by } (\star\star\star) \\ &= \Psi(F(F(y))) = \Psi(F^2(y)) \end{aligned}$$

Continuing similarly, we get that $\Psi(F^k(x)) = \Psi(F^k(y))$ for every positive k . When k is sufficiently large, $F^k(x)$ and $F^k(y)$ are in a neighborhood of 0 in which Ψ is one-to-one. Thus $F^k(x) = F^k(y)$, hence $x = y$. Therefore Ψ is one-to-one.

So we have shown that Ψ is a biholomorphic map from Ω onto \mathbb{C}^n and since $\Phi = \Psi^{-1}$, we are done.

Claim 9. $J\Phi \equiv 1$ if JF is constant.

Assume that JF is constant. If G is a polynomial automorphism of \mathbb{C}^n then the polynomial JG has no zero in \mathbb{C}^n . Particularly, we have that $JG = JF$ since $G'(0) = F'(0)$, thus,

$\det G'(0) = \det F'(0)$.

Now we apply the chain rule to $(\star\star)$ to get

$$(J\Psi)(F(z))(JF)(z) = (JG)(\Psi(z))(J\Psi)(z)$$

$$(J\Psi)(z) = (J\Psi)(F(z)) \quad \text{since } (JF)(z) = (JG)(\Psi(z))$$

$$= (J\Psi)(F^2(z))$$

$$\vdots$$

$$= (J\Psi)(F^k(z))$$

$$\vdots$$

since $F^k(z) \rightarrow 0$ as $k \rightarrow \infty$ we get that $(J\Psi)(z) = (J\Psi)(0) = 1$.

Therefore $(J\Psi)(0) = \det(\Psi'(0)) = \det I = 1$, for all $z \in \Omega$. Hence $J\Psi \equiv 1$ on \mathbb{C}^n if JF is constant.

□

Chapter 5

Constructing Fatou-Bieberbach

Maps

Given a sequence of automorphisms $\{F_j\}$, we attempt to construct Fatou-Bieberbach maps. Choosing the desired triangular map as well as the right polynomial map, becomes quite difficult here as compared to the case of just one automorphism map, say F . In making the ideal choice of a triangular map, we realize that we sometimes will have to switch between lower and upper triangular maps when dealing with a sequence of automorphisms. When this happens, the method fails to work out.

The degree of the random iterates of lower and upper triangular maps is not bounded, which makes it impossible for our choice of a triangular map to be a composition of lower and upper triangular maps. We give an example to verify that the random iterates of lower and upper triangular maps does not uniformly expand.

However, if we have random iterates of only lower triangular or only upper triangular maps the method still works out well. We explain this by starting with the theorem below.

The theorem is also true for a family of upper triangular maps.

Theorem 5.0.1. *Let $\{G_j\}_j$ be a family of lower triangular maps with $G_j(0) = 0$. Suppose that the modulus of the eigenvalues of $G'_j(0)$ is between 0 and 1 for each $j \in \mathbb{N}$. Also suppose that there exists a constant m such that $\deg G_j \leq m$ for all $j \in \mathbb{N}$. Then*

$$\bigcup_{N=1}^{\infty} G_N^{-1} \circ \cdots \circ G_1^{-1}(\mathbb{B}(0, r)) = \mathbb{C}^n, \text{ for } r > 0$$

where $\mathbb{B}(0, r)$ is a ball in \mathbb{C}^n centered at 0 with radius r .

Proof. Let $G_j = (g_{(1,j)}, \dots, g_{(n,j)})$ be a lower triangular polynomial automorphism of \mathbb{C}^n for $j \in \mathbb{N}$. So we define each G_j by the system of equations below.

$$\begin{aligned} g_{(1,j)}(z) &= c_{(1,j)}z_1 \\ g_{(2,j)}(z) &= c_{(2,j)}z_2 + h_{(2,j)}(z_1) \\ g_{(3,j)}(z) &= c_{(3,j)}z_3 + h_{(3,j)}(z_1, z_2) \\ &\vdots \\ g_{(n,j)}(z) &= c_{(n,j)}z_n + h_{(n,j)}(z_1, \dots, z_{i-1}). \end{aligned}$$

By definition of each G_j , we know that $c_{(i,j)}$'s are scalars such that $0 < |c_{(i,j)}| < 1$ and $h_{(i,j)}(0) = 0$ for $1 \leq i \leq n$ and for all $j \in \mathbb{N}$. Following a similar approach as with the proof of part (i) of lemma 4.0.1 we see that the degrees of the random iterates of the family of lower triangular maps $\{G_j\}$ are also bounded. We shall therefore show that the random iterates of the family of lower triangular maps converges to zero on every compact subset of \mathbb{C}^n .

Now let $E \subset \mathbb{C}^n$ be compact. For $G_N \circ \cdots \circ G_1(z)$, we get that the first coordinate is $g_{(1,N)} \circ \cdots \circ g_{(1,1)}(z) = \left(\prod_{k=1}^N c_{(1,N+1-k)} \right) z_1$.

We should however note that if $c_{(1,n)} = a_n = \left(1 - \frac{1}{2^n}\right)$ then $\prod_{n=1}^{\infty} a_n = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right)$ is greater than some positive number $t > 0$. Thus, $\|g_{(1,N)} \circ \cdots \circ g_{(1,1)}\|_E \rightarrow 0$ as $N \rightarrow \infty$.

So we need to place a bound on the eigenvalues of $G'_j(0)$ to avoid running into a situation like the one given above. In that case, we appropriately choose some constants p and q such that $0 < p < |c_{(i,j)}| < q < 1$, for all $j \in \mathbb{N}$ and $1 \leq i \leq n$. For instance, if we choose p and q such that $q^2 < p$, then $\|g_{(1,N)} \circ \cdots \circ g_{(1,1)}\|_E \rightarrow 0$ as $N \rightarrow \infty$. So we assume that the constants B and D have been appropriately chosen such that $0 < B < |c_{(i,j)}| < D < 1$. Then $\|g_{(1,N)} \circ \cdots \circ g_{(1,1)}\|_E \rightarrow 0$ as $N \rightarrow \infty$.

Assume now that $1 < i \leq n$ and that

$$(5.1) \quad \lim_{N \rightarrow \infty} \|g_{(j,N)} \circ \cdots \circ g_{(j,1)}\|_E = 0 \quad \text{for } 1 \leq j \leq i-1.$$

We will show that (5.1) still holds if $i-1$ is replaced by i . Consider $G_{N+1} \circ G_N \circ \cdots \circ G_1$, then we have that

$$\begin{aligned} g_{(i,N+1)} \circ g_{(i,N)} \circ \cdots \circ g_{(i,1)} &= c_{(i,N+1)} (g_{(i,N)} \circ \cdots \circ g_{(i,1)}) + \\ &\quad h_{(i,N+1)}(g_{(1,N)} \circ \cdots \circ g_{(1,1)}, \dots, g_{(i-1,N)} \circ \cdots \circ g_{(i-1,1)}). \end{aligned}$$

It follows from (5.1) that

$$(5.2) \quad \lim_{N \rightarrow \infty} \|h_{(i,N+1)}(g_{(1,N)} \circ \cdots \circ g_{(1,1)}, \dots, g_{(i-1,N)} \circ \cdots \circ g_{(i-1,1)})\| = 0$$

since $h_{(i,j)}(0) = 0$ for all $j \in \mathbb{N}$ and $1 \leq i \leq n$. So given $\varepsilon > 0$, we get that

$$\begin{aligned} |g_{(i,N+1)} \circ \cdots \circ g_{(i,1)}| &\leq |c_{(i,N+1)}| |g_{(i,N)} \circ \cdots \circ g_{(i,1)}| + \\ &\quad |h_{(i,N+1)}(g_{(1,N)} \circ \cdots \circ g_{(1,1)}, \dots, g_{(i-1,N)} \circ \cdots \circ g_{(i-1,1)})|. \\ &\leq |c_{(i,N+1)}| |g_{(i,N)} \circ \cdots \circ g_{(i,1)}| + \varepsilon(1 - |c_{(i,N+1)}|) \quad \text{from (5.2)}. \end{aligned}$$

Let $b = \limsup_{N \rightarrow \infty} \|g_{(i,N)} \circ \cdots \circ g_{(i,1)}\|_E$. Then

$$b \leq |c_{(i,N+1)}| b + \varepsilon(1 - |c_{(i,N+1)}|) \quad \text{and so } \limsup_{N \rightarrow \infty} \|g_{(i,N)} \circ \cdots \circ g_{(i,1)}\|_E \leq \varepsilon.$$

Hence $\lim_{N \rightarrow \infty} \|g_{(i,N)} \circ \cdots \circ g_{(i,1)}\|_E = 0$ for $1 < i \leq n$. So (5.1) holds if $i-1$ is replaced by i . Therefore $G_N \circ \cdots \circ G_1(z) \rightarrow 0$ uniformly on every compact set E of \mathbb{C}^n . As an immediate consequence of this result, we see that when given any neighborhood of 0, say

$\mathbb{B}(0, r)$, we conclude that

$$\bigcup_{N=1}^{\infty} G_N^{-1} \circ \cdots \circ G_1^{-1}(\mathbb{B}(0, r)) = \mathbb{C}^n. \quad \square$$

Example 6. There exists a family of triangular maps $\{G_j\}_j$ with $G_j(0) = 0$ and with the eigenvalues of $G'_j(0)$, in absolute value, greater than $1 + \varepsilon$ for all $j \in \mathbb{N}$ and for a fixed $\varepsilon > 0$ such that for all $r_o > r > 0$, we get that $\bigcup_{N=1}^{\infty} G_N \circ \cdots \circ G_1(\mathbb{B}(0, r)) \neq \mathbb{C}^2$.

Proof. Let $G_1(z, w) = (z + w, w)$, $G_2(z, w) = (z, w - z)$ and $G_3(z, w) = (\alpha z, -\alpha w + z^2)$, for $|\alpha| \geq 1$. So G_1 is an upper triangular map. G_2 and G_3 are both lower triangular maps. Now

$$G_2 \circ G_1(z, w) = (z + w, -z)$$

$$G_1 \circ G_2 \circ G_1(z, w) = (w, -z)$$

$$G_3 \circ G_1 \circ G_2 \circ G_1(z, w) = (\alpha w, \alpha z + w^2). \text{ Let } G_3 \circ G_1 \circ G_2 \circ G_1 = H.$$

So $H^{-1}(z, w) = \left(\frac{1}{\alpha} w - \frac{1}{\alpha} \left(\frac{1}{\alpha} z \right)^2, \frac{1}{\alpha} z \right)$ and $H^{-1} \in \text{Aut}(\mathbb{C}^2)$, with $H^{-1}(0) = 0$. The eigenvalues of $(H^{-1})'(0)$ are $\pm \frac{1}{\alpha}$.

We now define $\Omega = \left\{ (z, w) \in \mathbb{C}^2 : \lim_{k \rightarrow \infty} H^{-k}(z, w) = (0, 0) \right\}$.

However, $\Omega \neq \mathbb{C}^2$. Simply, the fixed point $(\alpha(1 - \alpha^2), 1 - \alpha^2) \notin \Omega$ and so Ω cannot be the whole of \mathbb{C}^2 . Thus, $\bigcup_{k=1}^{\infty} H^k(\mathbb{B}(0, r)) \neq \mathbb{C}^2$. \square

This example confirms that whenever we have random iterations of lower and upper triangular maps our method of constructing a suitable triangular map with the required properties fails.

We already know from the proof of the main theorem that when we have an automorphism F of \mathbb{C}^n and $F'(0) = 0$, with its eigenvalues in absolute values, strictly between 0 and 1, then $\Omega = \left\{ z \in \mathbb{C}^n : \lim_{k \rightarrow \infty} F^k(z) = 0 \right\}$ is biholomorphic to \mathbb{C}^n . Likewise, when given

a sequence of automorphisms $\{F_j\}$ of \mathbb{C}^n , we find out whether or not Ω is biholomorphic to \mathbb{C}^n , where $\Omega = \{z \in \mathbb{C}^n : F_N \circ \cdots \circ F_1(z) \rightarrow 0\}$.

B. Stensønes and J. E. Fornæss in [2] showed that when given a sequence of automorphisms $\{F_j\}$ of a complex manifold M of dimension n , with $F_j(0) = 0$ and with the modulus of the eigenvalues of $F_j'(0)$ strictly between 0 and 1, then Ω is biholomorphic to some domain of \mathbb{C}^n .

Let $F_j : M \rightarrow M$ be as defined above. Let $A_j = F_j'(0)$. Also let U be a small neighborhood of 0 in M . Then the projection map π from U to the tangent space of M at 0 T_0M , is one to one. Let $\Omega \subset M$. Then for sufficiently large N , we get that

$$K_N = \{z \in \Omega : F_N \circ \cdots \circ F_1(z) \in U\} \text{ for } K_N \text{ compact in } \Omega, \text{ and also}$$

$$A_1^{-1} \circ \cdots \circ A_N^{-1} \circ \pi \circ F_N \circ \cdots \circ F_1(K_N) = B_N \subset B_{N+1}, \text{ where } B_N \subset \mathbb{C}^n.$$

Now for N sufficiently large, let $\bigcup_{j=N}^{\infty} B_j = D \subset \mathbb{C}^n$. Then Ω is biholomorphic to the domain D of \mathbb{C}^n .

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